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Arithmetic degrees of special cycles and derivatives of Siegel Eisenstein series

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Abstract. Let V be a rational quadratic space of signature $(m, 2)$. A conjecture of Kudla relates the arithmetic degrees of top degree special cycles on an integral model of a Shimura variety associated with $\mathrm{SO}(V)$ to the coefficients of the central derivative of an incoherent Siegel Eisenstein series of genus $m + 1$. We prove this conjecture for the coefficients of non-singular index T when T is not positive definite. We also prove it when T is positive definite and the corresponding special cycle has dimension 0. To obtain these results, we establish new local arithmetic Siegel–Weil formulas at the archimedean and non-archimedean places.

Keywords. Shimura variety, orthogonal group, Siegel–Weil formula, Kudla program, Whittaker function, special cycle, Green current

1. Introduction

The classical Siegel–Weil formula connects the arithmetic of quadratic forms with Eisenstein series for symplectic groups [Si], [We], [KRal]. In particular, it yields explicit formulas for the representation numbers of integers by the genus of a quadratic form in terms of generalized divisor sum functions.

The Siegel–Weil formula also has important geometric applications. For instance, it leads to formulas for the degrees of special cycles on orthogonal Shimura varieties in terms of Fourier coefficients of Eisenstein series. To describe this, we let (V, Q) be a rational quadratic space of signature $(m, 2)$. To simplify the exposition, we assume throughout the introduction that m is even; the general case is treated in the body of the paper. Denote by $H = \mathrm{SO}(V)$ the special orthogonal group of V , and let \mathcal{D} be the corresponding hermitian symmetric space, realized as the Grassmannian of oriented negative definite planes in $V(\mathbb{R})$. For a compact open subgroup $K \subset H(\mathbb{A}_f)$ we consider the Shimura variety

$$X_K = H(\mathbb{Q}) \backslash (\mathcal{D} \times H(\mathbb{A}_f)) / K.$$

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It is a quasi-projective variety of dimension m , which has a canonical model over \mathbb{Q} . Every positive definite subspace $U \subset V$ of dimension n induces an embedding of groups $\mathrm{SO}(U^\perp) \rightarrow H$ and thereby a special cycle $Z(U)$ of codimension n on X_K . Moreover, for every positive definite symmetric matrix $T \in \mathrm{Sym}_n(\mathbb{Q})$ and every K -invariant Schwartz function $\varphi \in S(V^n(\mathbb{A}_f))$ there is a composite codimension n cycle

$$Z(T, \varphi)$$

on X_K , which is a certain linear combination of the $Z(U)$ for which U has Gram matrix $2T$. The classes of these cycles in the cohomology $H^{2n}(X_K, \mathbb{C})$ and in the Chow group $\mathrm{Ch}^n(X_K)$ are important geometric invariants. Kudla and Millson also defined cycle classes $Z(T, \varphi)$ for positive semi-definite $T \in \mathrm{Sym}_n(\mathbb{Q})$. They showed that the generating series of the cohomology classes of these cycles is a Siegel modular form of genus n and weight $1 + m/2$ (see [KM2], [Ku3]), generalizing the celebrated work of Hirzebruch–Zagier for Hilbert modular surfaces [HZ]. The analogous statement for the classes in the Chow group was conjectured by Kudla and recently proved in [Zh], [BW].

It is natural to ask for more precise information about the automorphic properties of the generating series. For the special cycles of maximal codimension, that is, for $n = m$, this question can be answered by means of the Siegel–Weil formula. If X_K has r connected components, the compactly supported cohomology $H_c^{2m}(X_K, \mathbb{C})$ is isomorphic to \mathbb{C}^r via the degree maps on the connected components. If X_K is compact, Kudla showed [Ku3, Theorem 10.3] that the generating series of the degrees of the special cycles is given by a Siegel Eisenstein series of genus n and weight $\kappa = 1 + m/2$, that is,

$$\sum_T \deg(Z(T, \varphi)) \cdot q^T = C \cdot E(\tau, 1/2, \lambda(\varphi) \otimes \Phi_\kappa). \tag{1.1}$$

Here $\tau = u + iv$ is a variable in the Siegel upper half-plane \mathbb{H}_n , and C is an explicit normalizing constant which is independent of φ . Moreover, $\lambda(\varphi)$ denotes a certain section of the induced representation $I(s, \chi_V)$ of $\mathrm{Sp}_n(\mathbb{A}_f)$ associated with φ , and Φ_κ denotes the standard section of weight κ of the corresponding induced representation of $\mathrm{Sp}_n(\mathbb{R})$ (see Section 2.1). If X_K is non-compact, the Eisenstein series is usually non-holomorphic and the treatment of the non-holomorphic contributions needs extra care (see e.g. [Fu], [FM2]).

The proof of this result involves the Schwartz forms $\varphi_{KM}^n(x, z) \in S(V^n(\mathbb{R})) \otimes A^{2n}(\mathcal{D})$ constructed by Kudla–Millson [KM2], which are Poincaré dual to special codimension n -cycles. Since they transform with weight κ under the maximal compact subgroup $U(n) \subset \mathrm{Sp}_n(\mathbb{R})$, the theta series

$$\theta_{KM}^n(\tau, \varphi, z, h) = \det(v)^{-\kappa/2} \sum_{x \in V^n(\mathbb{Q})} \varphi(h^{-1}x) \cdot (\omega(g_\tau)\varphi_{KM}^n(x, z))$$

is a smooth (non-holomorphic) Siegel modular form of weight κ in τ . Here $h \in H(\mathbb{A}_f)$, ω denotes the Weil representation of Sp_n and $g_\tau = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R})$ with $a^t a = v$. The T -th Fourier coefficient of this theta series represents the de Rham cohomology class

of $Z(T, \varphi)$. When $n = m$, the generating series for the degrees is obtained by integrating $\theta_{KM}^n(\tau, \varphi, z, h)$ over X_K . This can be evaluated by means of the Siegel–Weil formula, leading to (1.1). As a consequence, it can be shown that the intersection number of two special cycles $Z(T_1, \varphi_1)$ and $Z(T_2, \varphi_2)$ of complementary codimensions n_1 and n_2 is given by the corresponding Fourier coefficient of the block diagonal restriction to $\mathbb{H}_{n_1} \times \mathbb{H}_{n_2}$ of the Eisenstein series (see [Ku3, Section 10]). By means of the doubling method, connections to special values of L -functions can be obtained.

Kudla initiated a program connecting the Arakelov geometry of special cycles on integral models of orthogonal (and unitary) Shimura varieties to Siegel (hermitian) modular forms (see e.g. [Ku2], [KRY2]). In particular, in this setting arithmetic degrees of special cycles are conjecturally connected to derivatives of Siegel Eisenstein series. We describe some aspects of this program which are important for the present paper.

We consider arithmetic cycles in the sense of Gillet–Soulé (see [GiSo], [SABK]), which are given by pairs consisting of a cycle on an integral model of X_K and a Green current for the cycle. For $x \in V(\mathbb{R})$, Kudla constructed a Green function

$$z \mapsto \xi_0(x, z)$$

on \mathcal{D} . It has a logarithmic singularity along the special divisor determined by x (see (5.1)). More generally, if $x = (x_1, \dots, x_n) \in V^n(\mathbb{R})$ is such that the moment matrix $T = Q(x) = \frac{1}{2}((x_i, x_j))_{i,j}$ is invertible, one obtains a Green current for the special codimension n cycle $\mathcal{D}_x = \{z \in \mathcal{D} \mid (z, x_1) = \dots = (z, x_n) = 0\}$ by taking the star product

$$\xi_0^n(x, z) = \xi_0(x_1, z) * \dots * \xi_0(x_n, z).$$

It satisfies the current equation

$$dd^c[\xi_0^n(x, z)] + \delta_{\mathcal{D}_x} = [\varphi_{KM,0}^n(x, z)],$$

where $\varphi_{KM,0}^n(x, z) = \varphi_{KM}^n(x, z) \cdot e^{2\pi \operatorname{tr} Q(x)}$ is essentially the Poincaré dual form considered above, and $\delta_{\mathcal{D}_x}$ is the Dirac current given by integration over \mathcal{D}_x . For the rest of this introduction we assume that $T \in \operatorname{Sym}_n(\mathbb{Q})$ is invertible. Then we obtain a Green current for the cycle $Z(T, \varphi)$ on X_K by

$$G(T, \varphi, v, z, h) = \sum_{\substack{x \in V^n(\mathbb{Q}) \\ Q(x)=T}} \varphi(h^{-1}x) \cdot \xi_0^n(xv^{1/2}, z),$$

where $z \in \mathcal{D}$ and $h \in H(\mathbb{A}_f)$.

To describe the integral models of X_K and the special cycles we are working with, we assume for convenience that V contains a unimodular even lattice L . This assumption can and will be relaxed when considering local integral models later on (see Remark 1.3). We let $K = \operatorname{SO}(\hat{L})$ be the stabilizer of $\hat{L} = L \otimes \hat{\mathbb{Z}}$ in $H(\mathbb{A}_f)$, and let $\varphi = \varphi_L = \operatorname{char}(\hat{L}^n)$ be the characteristic function of \hat{L}^n . By work of Kisin, Vasiu, and Madapusi Pera, the Shimura variety X_K has a canonical integral model \mathcal{X}_K , which is a smooth stack over \mathbb{Z} (see [Kis], [MP], [AGHM, Theorem 4.2.2]). There is a polarized abelian scheme A of

dimension 2^{m+1} over \mathcal{X}_K , which is equipped with an action of the Clifford algebra $C(L)$ of L . For an S -valued point of \mathcal{X}_K there is a space of special endomorphisms

$$V(A_S) \subset \text{End}_{C(L)}(A_S)$$

on the pull-back A_S of A , which is endowed with a positive definite even quadratic form Q (see [AGHM, Section 4]). It can be used to define an integral model of $\mathcal{Z}(T, \varphi)$ of $\mathcal{Z}(T, \varphi)$ as the substack of \mathcal{X}_K whose S -valued points have an $x \in V(A_S)^n$ with $Q(x) = T$. The pair

$$\widehat{\mathcal{Z}}(T, \varphi, v) = (\mathcal{Z}(T, \varphi), G(T, \varphi, v))$$

determines a class in an arithmetic Chow group. Through the Green current it depends on $v = \mathfrak{S}(\tau)$. In analogy with the geometric situation described earlier, we would like to understand the classes of these cycles and their relations.

As before we focus on the case of top degree cycles, which is here the case $n = m + 1$. If T is not positive definite, then $\mathcal{Z}(T, \varphi)$ vanishes, but the arithmetic cycle $\widehat{\mathcal{Z}}(T, \varphi, v)$ has non-trivial current part. On the other hand, if T is positive definite, then $\widehat{\mathcal{Z}}(T, \varphi, v)$ has trivial current part, and the cycle is entirely supported in positive characteristic. In fact, if it is non-trivial then it is supported in the fiber above one single prime p . The dimensions of the irreducible components were recently determined by Soylu [So]. In particular, he showed that $\mathcal{Z}(T, \varphi)(\overline{\mathbb{F}}_p)$ is finite if and only if the reduction of T modulo p is of rank $n - 1$, $n - 2$, or of rank $n - 3$ (plus a technical condition). We refer to Theorem 6.3 for details. Throughout this paper we consider the cases when either T is not positive definite, or T is positive definite and $\mathcal{Z}(T, \varphi)$ has dimension 0.

According to [MP, Theorem 7.4], there exists a regular toroidal compactification $\overline{\mathcal{X}}_K$ of \mathcal{X}_K with generic fiber \overline{X}_K . The cycle $\widehat{\mathcal{Z}}(T, \varphi, v)$ defines a class in $\widehat{\text{Ch}}_C^n(\overline{\mathcal{X}}_K)$. Recall that there exists an arithmetic degree map

$$\widehat{\text{deg}} : \widehat{\text{Ch}}_C^n(\overline{\mathcal{X}}_K) \rightarrow \mathbb{C}$$

which is given as a sum of local degrees

$$\widehat{\text{deg}}(\mathcal{Z}, G) = \sum_{p \leq \infty} \widehat{\text{deg}}_p(\mathcal{Z}, G),$$

where for an arithmetic cycle (\mathcal{Z}, G) the local degrees are defined as

$$\widehat{\text{deg}}_p(\mathcal{Z}, G) = \begin{cases} \sum_{x \in \mathcal{Z}(\overline{\mathbb{F}}_p)} \frac{\text{ht}_p(x)}{|\text{Aut}(x)|} \cdot \log p & \text{if } p < \infty, \\ \frac{1}{2} \int_{\overline{X}_K(\mathbb{C})} G & \text{if } p = \infty. \end{cases}$$

Here $\text{ht}_p(x)$ denotes the length of the étale local ring $\mathcal{O}_{\mathcal{Z}, x}$ of \mathcal{Z} at the point x . Kudla conjectured the following description of the arithmetic degrees of special cycles in terms of derivatives of Siegel Eisenstein series of genus n (see [Ku2], [Ku5]).

Conjecture 1.1 (Kudla). *Assume that $n = m + 1$ and $T \in \text{Sym}_n(\mathbb{Q})$ is invertible. Then*

$$\widehat{\text{deg}}(\widehat{\mathcal{Z}}(T, \varphi, v)) \cdot q^T = \hat{C} \cdot E'_T(\tau, 0, \lambda(\varphi) \otimes \Phi_\kappa),$$

where \hat{C} denotes a constant which is independent of T and φ (see Theorem 7.1), $E_T(\tau, s, \Phi)$ denotes the T -th Fourier coefficient of a Siegel Eisenstein series $E(\tau, s, \Phi)$, and the derivative is taken with respect to s .

Note that for T positive definite with $\mathcal{Z}(T, \varphi)$ of higher dimension the arithmetic degree has to be defined more carefully as in [Te], but we do not consider this here. The conjecture can be further generalized to include the cases where T is singular, leading to an identity between the generating series of the arithmetic degrees of the $\widehat{\mathcal{Z}}(T, \varphi, v)$ and the central derivative of the Eisenstein series $E(\tau, s, \lambda(\varphi) \otimes \Phi_\kappa)$ analogous to (1.1), which can be viewed as an *arithmetic Siegel–Weil formula*. The full conjecture is known for $m = 0$ and for the $m = 1$ case of Shimura curves (see [KRY1], [KRY2]).

To state our results on Conjecture 1.1, we let $\mathcal{C} = \bigotimes_{p \leq \infty} \mathcal{C}_p$ be the incoherent quadratic space over \mathbb{A} for which $\mathcal{C}_f = \bigotimes_{p < \infty} \mathcal{C}_p \cong V(\mathbb{A}_f)$ and \mathcal{C}_∞ is positive definite of dimension $m + 2$. The Eisenstein series appearing in Conjecture 1.1 is naturally associated with the Schwartz function on $S(\mathcal{C}^n)$ given by the tensor product of φ and the Gaussian on \mathcal{C}_∞^n via the intertwining operator λ . Hence it is incoherent and vanishes at $s = 0$. The conjecture gives a formula for the leading term of the Taylor expansion in s at $s = 0$. Following Kudla [Ku2], define the Diff set associated with \mathcal{C} and T as

$$\text{Diff}(\mathcal{C}, T) = \{p \leq \infty \mid \mathcal{C}_p \text{ does not represent } T\}. \tag{1.2}$$

Then $\text{Diff}(\mathcal{C}, T)$ is a non-empty finite set, and $\infty \in \text{Diff}(\mathcal{C}, T)$ if and only if T is not positive definite.

Theorem 1.2. *Assume that $T \in \text{Sym}_n(\mathbb{Q})$ is invertible. Then Conjecture 1.1 holds in the following cases:*

- (1) *if $|\text{Diff}(\mathcal{C}, T)| > 1$; in this case both sides of the equality vanish;*
- (2) *if $\text{Diff}(\mathcal{C}, T) = \{\infty\}$; in this case T is not positive definite, and the only contribution comes from the archimedean place, i.e.,*

$$\widehat{\text{deg}}(\widehat{\mathcal{Z}}(T, \varphi, v)) \cdot q^T = \widehat{\text{deg}}_\infty(\widehat{\mathcal{Z}}(T, \varphi, v)) \cdot q^T = \hat{C} \cdot E'_T(\tau, 0, \lambda(\varphi) \otimes \Phi_\kappa);$$

- (3) *if $\text{Diff}(\mathcal{C}, T) = \{p\}$ for a finite prime $p \neq 2$ and $\mathcal{Z}(T, \varphi)(\overline{\mathbb{F}}_p)$ has dimension 0; in this case, the only contribution comes from the prime p , i.e.,*

$$\widehat{\text{deg}}(\widehat{\mathcal{Z}}(T, \varphi, v)) \cdot q^T = \widehat{\text{deg}}_p(\widehat{\mathcal{Z}}(T, \varphi, v))q^T = \hat{C} \cdot E'_T(\tau, 0, \lambda(\varphi) \otimes \Phi_\kappa).$$

Remark 1.3. Since the cycle $\widehat{\mathcal{Z}}(T, \varphi, v)$ is supported in a single fiber, all assertions of Theorem 1.2 can be reformulated in terms of ‘local’ models of \overline{X}_K . We will actually prove the local analogues in much greater generality.

Generalizing (2) we will show that if $\infty \in \text{Diff}(\mathcal{C}, T)$ then

$$\widehat{\text{deg}}_\infty(\widehat{\mathcal{Z}}(T, \varphi, v)) \cdot q^T = \hat{C} \cdot E'_T(\tau, 0, \lambda(\varphi) \otimes \Phi_\kappa). \tag{1.3}$$

Since this is an assertion only about the complex fiber \overline{X}_K , we will be able to prove it for any compact open subgroup $K \subset H(\mathbb{A}_f)$ (in particular V does not have to contain an even unimodular lattice) and any K -invariant Schwartz function $\varphi \in S(V(\mathbb{A}_f)^n)$ (see Theorem 7.1).

To generalize (3) we consider the canonical integral model $\mathcal{X}_{K,(p)}$ of \overline{X}_K over the localization $\mathbb{Z}_{(p)}$. In this setting we will show that if $p \in \text{Diff}(\mathcal{C}, T)$ for a finite prime $p \neq 2$ and $\mathcal{Z}(T, \varphi)$ is 0-dimensional, then

$$\widehat{\text{deg}}_p(\widehat{\mathcal{Z}}(T, \varphi, v)) \cdot q^T = \hat{C} \cdot E'_T(\tau, 0, \lambda(\varphi) \otimes \Phi_\kappa). \tag{1.4}$$

This will be proved under the assumption that K is the stabilizer of a \mathbb{Z}_p -unimodular lattice $L \subset V$ and for $\varphi = \text{char}(\hat{L}^n)$ (see Theorem 7.3).

To prove Theorem 1.2, we decompose the Fourier coefficients of the Eisenstein series into local factors. If $\Phi = \otimes_v \Phi_v$ is a factorizable section of the induced representation, then

$$E_T(g, s, \Phi) = \prod_{v \leq \infty} W_{T,v}(g, s, \Phi_v),$$

where $W_{T,v}(g, s, \Phi_v)$ is the local Whittaker function given by (2.1). It is a basic fact that the local Whittaker function $W_{T,p}(g_p, 0, \lambda_p(\varphi_p))$ vanishes for every $p \in \text{Diff}(\mathcal{C}, T)$.

This implies assertion (1) of Theorem 1.2 in a rather direct way. Indeed, if $|\text{Diff}(\mathcal{C}, T)| > 1$ then the right hand side of the conjectured identity is automatically zero. To prove that the left hand side also vanishes, we consider for a prime $p \leq \infty$ the neighboring global quadratic space $V^{(p)}$ at p associated with \mathcal{C} , which is the quadratic space over \mathbb{Q} with local components $V_q^{(p)} \cong \mathcal{C}_q$ for all $q \neq p$ and such that $V_p^{(p)}$ and \mathcal{C}_p have the same dimension and quadratic character but different Hasse invariants (for $p = \infty$ we also require that $V_\infty^{(\infty)}$ has signature $(m, 2)$, and hence $V^{(\infty)} = V$).

If $\mathcal{Z}(T, \varphi)(\overline{\mathbb{F}}_p)$ is non-empty for a prime $p < \infty$, then one can show (see the proof of Proposition 7.2 for example) that $V^{(p)}$ represents T . This implies $\text{Diff}(\mathcal{C}, T) = \{p\}$. Similarly, the proof of Theorem 7.1 shows that if the Green current $G(T, \varphi, v)$ is non-vanishing, then $V^{(\infty)}$ represents T and hence $\text{Diff}(\mathcal{C}, T) = \{\infty\}$.

In the situation of part (2) of Theorem 1.2, when $\text{Diff}(\mathcal{C}, T) = \{\infty\}$, the local Whittaker function $W_{T,\infty}(g, 0, \Phi_\kappa)$ vanishes, and hence

$$E'_T(\tau, 0, \lambda(\varphi) \otimes \Phi_\kappa) = \prod_{p < \infty} W_{T,p}(g, 0, \lambda(\varphi_p)) \times W'_{T,\infty}(g, 0, \Phi_\kappa).$$

The derivative of the archimedean Whittaker function is given by the following *arithmetic local Siegel–Weil formula* for the archimedean local height function:

$$\text{ht}_\infty(x) = \frac{1}{2} \int_{\mathcal{D}} \xi_0^n(x, z)$$

on $V^n(\mathbb{R})$, which is our second main result (see also Theorem 5.2). The contributions from the non-archimedean places can be computed by means of the *local Siegel–Weil formula* (see Propositions 2.2 and 2.3).

Theorem 1.4. *Let $x \in V^n(\mathbb{R})$ be such that the $Q(x) = T$ is invertible. Then*

$$\text{ht}_\infty(xv^{1/2}) \cdot q^T = -B_{n,\infty} \det(v)^{-\kappa/2} \cdot W'_{T,\infty}(g_\tau, 0, \Phi_\kappa), \tag{1.5}$$

where

$$B_{n,\infty} = \frac{e^{\left(\frac{n^2+n-4}{8}\right)}(n-1)! \prod_{k=1}^{n-1} \Gamma\left(\frac{n-k}{2}\right)}{2^{n-2}(2\pi)^{n(n+3)/4}}.$$

In the special case $m = 0$ Theorem 1.4 was proved in [KRY1], for $m = 1$ in [Ku2], and for $m = 2$ in [YZZ]. For the related case of Shimura varieties associated to unitary groups of signature $(m, 1)$ it was proved in [Liu]. But the argument of [Liu] does not seem to generalize to the case of orthogonal groups considered in the present paper. Recently, Garcia and Sankaran [GaSa] employed Quillen’s theory of super-connections to obtain a different proof of Theorem 1.4.

In all these works it is first noticed that because of the equivariance of $\xi_0^n(x, z)$ with respect to the action of $H(\mathbb{R})$, the local height function $\text{ht}_\infty(x)$ only depends on $T = Q(x)$. Then a crucial step consists in proving that $\text{ht}_\infty(T) := \text{ht}_\infty(x)$ is invariant under the action of $\text{SO}(n)$ on $\text{Sym}_n(\mathbb{R})$ (respectively $\text{U}(n)$ on $\text{Herm}_n(\mathbb{C})$) by conjugation. Hence it suffices to prove the claimed identity for diagonal T . In this case the star product reduces to a single integral, which can be related to the derivative of the Whittaker function by a direct (but rather involved) computation.

Our approach is different. For general non-singular T , we consider the recursive formula for the star product (see (5.5)) and compute its ‘main term’ by means of the classical archimedean local Siegel–Weil formula (see Theorem 5.4). The result turns out to be the sum of a main term, which is the desired right hand side of (1.5), plus a boundary term, given by the derivative of a genus $n - 1$ Whittaker function. By an inductive argument, the boundary term cancels against the remaining terms of the star product. This approach does not require proving $\text{SO}(n)$ -invariance of the local height function at the outset. We obtain this invariance a posteriori from the obvious invariance of the Whittaker function.

Finally, we describe our approach to part (3) of Theorem 1.2. When $\text{Diff}(\mathcal{C}, T) = \{p\}$ for a finite prime $p \neq 2$, the local Whittaker function $W_{T,p}(g, 0, \Phi_\kappa)$ vanishes, and hence

$$E'_T(\tau, 0, \lambda(\varphi) \otimes \Phi_\kappa) = W'_{T,p}(1, 0, \lambda(\varphi_p)) \times \prod_{\substack{q < \infty \\ q \neq p}} W_{T,q}(1, 0, \lambda(\varphi_q)) \times W_{T,\infty}(\tau, 0, \Phi_\kappa).$$

The derivative of the local Whittaker function at p is given by the following *arithmetic local Siegel–Weil formula*, which is our third main result paralleling Theorem 1.4. The terms away from p can again be computed by means of the local Siegel–Weil formula. Recall that $\varphi = \varphi_L \in S(V^n(\mathbb{A}_f))$ is the characteristic function of \hat{L}^n .

Theorem 1.5. *Let $p \neq 2$ be a prime number and assume that $\mathcal{Z}(T, \varphi)(\bar{\mathbb{F}}_p)$ is finite. Then for $x \in \mathcal{Z}(T, \varphi)(\bar{\mathbb{F}}_p)$, the local height $\text{ht}_p(x)$ is independent of the choice of x and is given by*

$$\text{ht}_p(x) \cdot \log p = \frac{W'_{T,p}(1, 0, \lambda(\varphi))}{W_{T^u,p}(1, 0, \lambda(\varphi))},$$

where T^u is any unimodular matrix in $\text{Sym}_n(\mathbb{Z}_p)$ (i.e., $\det T^u \in \mathbb{Z}_p^\times$).

This theorem will be restated and proved as Theorem 6.9. As in the archimedean case the proof is given by an inductive argument. According to Soylu's condition mentioned above, $\mathcal{Z}(T, \varphi)(\overline{\mathbb{F}}_p)$ being finite implies that T is \mathbb{Z}_p -equivalent to $\text{diag}(T_1, T_2)$ where T_1 is \mathbb{Z}_p -unimodular of rank $n - 3$. On the Whittaker function side, we will prove the following recursive formula (see Proposition 6.8):

$$\frac{W'_{T,p}(1, 0, \lambda(\varphi_{L_p}))}{W_{T^u,p}(1, 0, \lambda(\varphi_{L_p}))} = \frac{W'_{T_2,p}(1, 0, \lambda(\varphi_{L_{2,p}}))}{W_{T_2^u,p}(1, 0, \lambda(\varphi_{L_{2,p}}))}. \quad (1.6)$$

Here T^u and T_2^u are unimodular symmetric \mathbb{Z}_p matrices of order n and 3 respectively, and $L_{2,p}$ is a unimodular \mathbb{Z}_p -quadratic lattice of rank 4 with

$$L_p = M_{1,p} \oplus L_{2,p}$$

for a unimodular \mathbb{Z}_p -quadratic lattice $M_{1,p}$ whose quadratic matrix is given by T_1 .

This suggests a similar recursion for the local height function $\text{ht}_p(x)$. Soylu proved that the abelian variety associated with x is supersingular. The local height function depends only on the associated p -divisible group, and it can be computed using the p -adic uniformization of the supersingular locus by a Rapoport–Zink space (see Section 6). The required recursion formula for the local height function is proved by employing recent work of Li and Zhu ([LZ, Lemma 3.1.1], see Corollary 6.5).

By the recursion formulas, the proof of Theorem 1.5 is reduced to the case $n = 3$ in the local situation, where $L = L_{2,p}$ is a unimodular \mathbb{Z}_p -lattice of rank 4 and $T = T_{2,p}$ is a symmetric \mathbb{Z}_p -matrix of rank 3, and $x = (x_1, x_2, x_3)$ determines a point in $\mathcal{Z}(T, \varphi)(\overline{\mathbb{F}}_p)$ on the associated Rapoport–Zink space. This turns out to be exactly the local case considered by Kudla and Rapoport in their work on (twisted) Hilbert modular surfaces [KRap1].

This paper is organized as follows. Section 2 contains some preliminaries and basic facts about Whittaker functions. Moreover, we state the classical local Siegel–Weil formula with an explicit formula for the constant of proportionality. In Section 3 we derive a variant of the archimedean local Siegel–Weil formula for integrals of certain Schwartz functions over the hermitian symmetric space of the orthogonal group, again with explicit constant of proportionality. The main result, Theorem 3.1, is one of the key ingredients in the proof of Theorem 1.4. In Section 4 we investigate the asymptotic behavior of the archimedean Whittaker function as one of the radial parameters goes to infinity. Our analysis relies on Shimura's work on confluent hypergeometric functions [Shi]. The main result, Theorem 4.8, which is of independent interest, is the second main ingredient in the proof of Theorem 1.4. Section 5 is devoted to the proof of the archimedean arithmetic Siegel–Weil formula, Theorem 1.4. In Section 6 we recall some facts about the Rapoport–Zink space for GSpin groups from [HP] and [So] and prove the non-archimedean local arithmetic Siegel–Weil formula, Theorem 1.5. Finally, Section 7 contains the proofs of our main global results, Theorem 1.2 and the refinements described in Remark 1.3.

2. The local Siegel–Weil formula

In this section we introduce the basic local set-up and recall the local Siegel–Weil formula (see Theorem 2.1). We make the constant involved explicit in Proposition 2.2.

2.1. The basic local set-up and local Whittaker functions

Let F be a local field or the ring of adèles of a number field, and let ψ be a non-trivial additive character of F (or adèle class character). Let $P = NM$ be the standard Siegel parabolic subgroup of the symplectic group $\mathrm{Sp}_n(F)$ given by

$$M = \left\{ m(a) = \begin{pmatrix} a & & \\ & \iota_a^{-1} & \\ & & 1 \end{pmatrix} \mid a \in \mathrm{GL}_n(F) \right\},$$

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b & \\ & 1 & \\ & & 1 \end{pmatrix} \mid b \in \mathrm{Sym}_n(F) \right\}.$$

We also denote

$$w = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Let $\mathrm{Mp}_{n,F}$ be the metaplectic cover of $\mathrm{Sp}_n(F)$, identified with $\mathrm{Mp}_{n,F} = \mathrm{Sp}_n(F) \times \{\pm 1\}$ via the normalized Rao cocycle $c_R(g_1, g_2)$ given in [Rao] (with the minor correction in [Kul, p. 379]):

$$[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 g_2, \epsilon_1 \epsilon_2 c_R(g_1, g_2)].$$

For $g \in \mathrm{Sp}_n(F)$, we will simply denote $g = [g, 1]$.

Let (V, Q) be a non-degenerate quadratic space over F of dimension l . Then there is a Weil representation $\omega = \omega_{V, \psi}$ of $\mathrm{Mp}_{n,F}$ on $S(V^n)$ given by [Kul, p. 400]. In particular,

$$\omega(n(b))\phi(x) = \psi(\mathrm{tr}(Q(x)b))\phi(x),$$

$$\omega(m(a))\phi(x) = \chi_V(\det a)\gamma(\det a, \frac{1}{2}\psi)^{-1}(\det a, -1)_F^{l(l-1)/2} |\det a|^{l/2} \phi(xa),$$

$$\omega(w)\phi(x) = \gamma(V^n) \int_{V^n} \phi(y)\psi(-\mathrm{tr}(x, y)) d_\psi y,$$

where $d_\psi y$ is the self-dual Haar measure on V with respect to ψ , and $\gamma(V^n) = \gamma(\psi \circ \det Q)^{-n}$. Here $\gamma(\psi)$ and $\gamma(a, \psi)$ (for $a \in F^\times$) are the local Weil indices defined in [Rao, Appendix], and

$$\det Q = 2^{-l} \det V = \det(\frac{1}{2}(e_i, e_j)) \in F^\times / (F^\times)^2,$$

for an F -basis $\{e_1, \dots, e_l\}$ of V . Finally,

$$\chi_V(a) = (a, (-1)^{l(l-1)/2} \det V)_F$$

is the quadratic character associated to V . It is well-known that the Weil representation factors through $\mathrm{Sp}_n(F)$ when l is even. Since

$$\gamma(a, \frac{1}{2}\psi)^2 = (a, -1)_F,$$

the formula for $\omega(m(a))\phi$ above works for both even and odd l . From now on, let $G = \mathrm{Sp}_n(F)$ or $\mathrm{Mp}_{n,F}$ depending on whether n is odd or even, and let P be the standard Siegel parabolic subgroup or the preimage of the standard Siegel parabolic subgroup. If $F = \mathbb{R}$, we let $K_G \subset G$ be the maximal compact subgroup given by either

$$\left\{ k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid \mathbf{k} = a + ib \in \mathrm{U}(n) \right\} \cong \mathrm{U}(n)$$

or the inverse image of $\mathrm{U}(n)$ under the covering map (when $G = \mathrm{Mp}_{n,\mathbb{R}}$).

For a character χ of F^\times , let $I(s, \chi) = \mathrm{Ind}_P^G \chi |\det|^s$ be the induced representation. A section $\Phi \in I(s, \chi)$ satisfies

$$\Phi(n(b)m(a)g, s) = \begin{cases} \chi(\det a) |\det a|^{s+\rho_n} \Phi(g, s) & \text{if } G = \mathrm{Sp}_n(F), \\ \chi(\det a) \gamma(\det a, \frac{1}{2}\psi)^{-1} |\det a|^{s+\rho_n} \Phi(g, s) & \text{if } G = \mathrm{Mp}_{n,F}, \end{cases}$$

where

$$\rho_n = (n + 1)/2.$$

For a symmetric matrix $T \in \mathrm{Sym}_n(F)$, the Whittaker function of Φ with respect to T is defined to be

$$W_T(g, s, \Phi) = \int_{\mathrm{Sym}_n(F)} \Phi(w_n(b)g, s) \psi(-\mathrm{tr}(Tb)) d_\psi b, \tag{2.1}$$

where $d_\psi b$ is the self-dual Haar measure on $\mathrm{Sym}_n(F)$ with respect to the pairing $(b_1, b_2) \mapsto \psi(\mathrm{tr}(b_1 b_2))$. It has the transformation behavior

$$\begin{aligned} &W_T(n(b)m(a)g, s, \Phi) \\ &= \psi(\mathrm{tr}(Tb)) \chi(a)^{-1} |a|^{\rho_n - s} \cdot \begin{cases} W_{\tau_a T a}(g, s, \Phi) & \text{if } G = \mathrm{Sp}_n(F), \\ \gamma(a, \frac{1}{2}\psi) W_{\tau_a T a}(g, s, \Phi) & \text{if } G = \mathrm{Mp}_{n,F}. \end{cases} \end{aligned} \tag{2.2}$$

Here we have shortened $\gamma(a, \frac{1}{2}\psi) = \gamma(\det a, \frac{1}{2}\psi)$ (and similarly for $\chi(a)$). We remark that $\gamma(a, \frac{1}{2}\psi) = 1$ when $\det a > 0$ and $F = \mathbb{R}$.

Let $s_{l,n} = (l - n - 1)/2$. Then there is a G -equivariant map

$$\lambda : S(V^n) \rightarrow I(s_{l,n}, \chi_V), \quad \lambda(\phi)(g) = (\omega(g)\phi)(0). \tag{2.3}$$

We will also denote by $\lambda(\phi)$ the associated standard section in $I(s, \chi_V)$ with $\lambda(\phi)(g, s_{l,n}) = \lambda(\phi)(g)$. Assume that $l = n + 1$. Then a formal unfolding suggests that there is a Haar measure dh on $H = \mathrm{SO}(V)$ such that for all $\phi \in S(V^n)$,

$$O_T(\omega(g)\phi, dh) = C \cdot W_T(g, 0, \lambda(\phi)),$$

where C is some constant which is independent of T and ϕ , and

$$O_T(\phi, dh) = \int_{H(F)} \phi(h^{-1}x) dh$$

if there is $x \in V^n$ with $Q(x) = T$ (otherwise set the orbital integral to be zero). This is the content of the so-called local Siegel–Weil formula which we will describe in the next two subsections. In particular, we will determine the constant C .

2.2. Kudla’s local Siegel–Weil formula

In this subsection we review the local Siegel–Weil formula given in [KRY2, Section 5.3], following a general result in [Ra, Chapter 4]. Let the notation be as in Section 2.1, and assume $\dim V = n + 1$. Let

$$Q : V^n \rightarrow \text{Sym}_n(F), \quad Q(x) = \frac{1}{2}((x_i, x_j)), \tag{2.4}$$

be the moment map. Let V_{reg}^n be the subset of $x \in V^n$ with $\det Q(x) \neq 0$, and let $\text{Sym}_n^{\text{reg}}(F)$ be the subset of $T \in \text{Sym}_n(F)$ with $\det T \neq 0$. Then Q induces a regular map from V_{reg}^n to $\text{Sym}_n^{\text{reg}}(F)$.

Put $a(n) = n(n + 1)/2$. We let α be a gauge form on V^n , that is, a generator of $(\wedge^{a(n)} V^n)^*$ (a top level differential of the topological vector space V^n), and let β be a gauge form on $\text{Sym}_n(F)$, i.e., a generator of $(\wedge^{a(n)}(\text{Sym}_n(F)))^*$.

Fix an $x = (x_1, \dots, x_n) \in V_{\text{reg}}^n$ with $Q(x) = T$. If we identify the tangent space $T_x(V_{\text{reg}}^n)$ with V^n , then the differential dQ_x is given by

$$dQ_x(v) = \frac{1}{2}((x, v) + (v, x)) \in \text{Sym}_n(F), \quad v \in V^n.$$

Let

$$j_x : \text{Sym}_n(F) \rightarrow V^n, \quad j_x(u) = \frac{1}{2}xQ(x)^{-1}u. \tag{2.5}$$

Then $dQ_x \circ j_x(u) = u$, and we have the decomposition

$$T_x(V_{\text{reg}}^n) = \text{Im}(j_x) \oplus \ker(dQ_x).$$

Now choose any $u = (u_1, \dots, u_{a(n)}) \in (\text{Sym}_n(F))^{a(n)}$ with $\beta(u) \neq 0$. We define an $a(n)$ -form $v \in (\wedge^{a(n)} V^n)^*$ on V_{reg}^n as follows: for any $t = (t_1, \dots, t_{a(n)}) \in (V^n)^{a(n)}$, we put

$$v(t) = \alpha(j_x(u), t)\beta(u)^{-1}. \tag{2.6}$$

This quantity is independent of the choice of u . Then [KRY2, Lemma 5.3.1] asserts that

$$\alpha = Q^*(\beta) \wedge v, \tag{2.7}$$

$$v = (h, g)^*v, \tag{2.8}$$

for $h \in \text{SO}(V)$ and $g \in \text{GL}_n$, where $\text{SO}(V) \times \text{GL}_n$ acts on V^n via $(h, g)x = hxg^{-1}$. Moreover, v defines a gauge form on $Q^{-1}(T)$ if we identify $\ker dQ_x$ with the tangent space $T_x(Q^{-1}(T))$ of $Q^{-1}(T)$. Finally, using the isomorphism

$$i_x : \text{SO}(V) \rightarrow Q^{-1}(T), \quad i_x(h) = hx \tag{2.9}$$

(here $\dim V = n + 1$ is critical to ensure that the pointwise stabilizer H_x of x is trivial), we obtain a gauge form $i_x^*(v)$ on $\text{SO}(V)$, which we will still denote by v for simplicity. The key point (see [KRY2, Lemma 5.3.2]) is that this gauge form v does not depend on T or x , which can be seen by (2.8).

This gauge form v gives a Haar measure $dh = d_v h$ on $\text{SO}(V)$. Let $d_\alpha x$ be the Haar measure on V^n associated to α and $d_\beta T$ be the Haar measure on $\text{Sym}_n(F)$ associated

to β , and let $d_\psi x$ and $d_\psi T$ be the self-dual Haar measures on V^n and $\text{Sym}_n(F)$ with respect to ψ , respectively. Then there are constants $c(\alpha, \psi)$ and $c(\beta, \psi)$ such that

$$d_\alpha x = c(\alpha, \psi)d_\psi x, \quad d_\beta T = c(\beta, \psi)d_\psi T. \tag{2.10}$$

Finally, we can state Kudla’s local Siegel–Weil formula, which is [KRY2, Proposition 5.3.3] (although only stated for $n = 2$ there, the proof goes through for general n without any change).

Theorem 2.1 (Local Siegel–Weil formula). *Given a gauge form α on V^n and a gauge form β on $\text{Sym}_n(F)$, let $d_\nu h$ be the Haar measure on $H(F)$ associated to α and β as above. Then for any $\phi \in S(V^n)$, $T \in \text{Sym}_n^{\text{reg}}(F)$, and $g \in G$,*

$$O_T(\omega(g)\phi, d_\nu h) = C(V, \alpha, \beta, \psi) \cdot W_T(g, 0, \lambda(\phi)),$$

where

$$C(V, \alpha, \beta, \psi) = \frac{c(\alpha, \psi)}{\gamma(V^n)c(\beta, \psi)},$$

and $\gamma(V^n) = \gamma(V)^n$ by [Kul, Lemma 3.4].

We remark that our $C(V, \alpha, \beta, \psi)$ is the reciprocal of the same notation in [KRY2].

2.3. Explicit construction

Let $\underline{e} = (e_1, \dots, e_{n+1})$ be an ordered basis of V and put $J = Q(\underline{e}) = \frac{1}{2}((e_i, e_j)) \in \text{Sym}_{n+1}(F)$. When F is p -adic, let $L = \bigoplus \mathcal{O}_F e_j$ be the associated \mathcal{O}_F -lattice. Using this basis, we identify V with F^{n+1} (column vectors) and V^n with $M_{n+1,n}$.

Let E_{ij} denote a matrix whose (ij) -entry is one and all other entries are zero (we do not specify the size of the matrix). Then $\{E_{ij} \mid 1 \leq i \leq n + 1, 1 \leq j \leq n\}$ is a basis of V^n . Let de_{ij} be its dual basis, and let $\alpha = \bigwedge_{ij} de_{ij}$ be the gauge form on V^n (up to sign, which does not affect the associated Haar measure) with

$$\alpha((E_{ij})) = \alpha(E_{11}, E_{12}, \dots, E_{n+1,n}) = 1. \tag{2.11}$$

Notice that $Y_{ij} = E_{ij} + E_{ji}$ is a basis of $\text{Sym}_n(F)$ ($1 \leq i \leq j \leq n$), and let dy_{ij} be its dual basis. Let $\beta = \bigwedge_{ij} dy_{ij}$. Then (up to sign)

$$\beta(Y_{11}, Y_{12}, \dots, Y_{n,n}) = 1. \tag{2.12}$$

Proposition 2.2. *Let J , α , and β be as above, and let $d_\nu h$ be the associated Haar measure on $H(F) = \text{SO}(V)(F)$. Take $\psi(x) = e(x) = e^{2\pi i x}$ when $F = \mathbb{R}$ and assume that ψ is unramified when F is p -adic. Then*

$$O_T(\omega(g)\phi, d_\nu h) = C(J) \cdot W_T(g, 0, \lambda(\phi))$$

for all $\phi \in S(V^n)$ and $g \in G$. Here

$$C(J) = \gamma(V^n)^{-1} |2|_F^{n+n(n-1)/4} |\det(2J)|_F^{-n/2}.$$

Finally, when $F = \mathbb{R}$ and V has signature (p, q) , then $\gamma(V^n) = e(n(q - p)/8)$.

By the proposition, we see that $d_\nu h$ depends only on $|\det(2J)|_F$. For this reason, we will sometime denote $d_\nu h$ by $d_J h$ or $d_L h$ in the p -adic case. We also write $C(L) = C(J)$ in the p -adic case as $\det(2J) = \det L$.

Proof of Proposition 2.2. First assume that F is p -adic. Let \mathcal{O}_F be the ring of integers of F . Let $L = \bigoplus \mathcal{O}_F e_i = \mathcal{O}_F^n \subset V = F^n$ and $f = \text{char}(L^n) = \text{char}(M_{n+1,n}(\mathcal{O}_F)) \in S(V^n)$. Then the Fourier transforms of f with respect to $d_\alpha x$ and $d_\psi x$ are given by

$$\begin{aligned} \hat{f}_\alpha(X) &= \int_{M_{n+1,n}(\mathcal{O}_F)} \psi(-\text{tr}(2^t X J Z)) \prod dz_{ij} = \text{char}(L'^n)(X), \\ \hat{f}_\psi(X) &= \int_{M_{n+1,n}(\mathcal{O}_F)} \psi(-\text{tr}(2^t X J Z)) d_\psi Z = \text{char}(L'^n)(X) \text{vol}(L^n, d_\psi x), \end{aligned}$$

where L' is the dual lattice of L with respect to ψ . Since $d_\psi x$ is the self-dual Haar measure on V with respect to ψ , one has

$$\text{vol}(L, d_\psi x) = |\det(2J)|_F^{1/2}.$$

Consequently, $c(\alpha, \psi) = |\det(2J)|_F^{-n/2}$.

Next, for $t = (t_{ij}) \in \text{Sym}_n(F)$, $d_\beta t = |2|_F^{-n} \prod dt_{ij}$. Let $f = \text{char}(\text{Sym}_n(\mathcal{O}_F))$. Then

$$\begin{aligned} \hat{f}_\beta(y) &= \int_{\text{Sym}_n(\mathcal{O}_F)} \psi(-\text{tr}(yt)) |2|_F^{-n} \prod_i \psi(-y_{ii} t_{ii}) dt_{ii} \prod_{i < j} \psi(-2y_{ij} t_{ij}) dt_{ij} \\ &= |2|_F^{-n} \prod_i \text{char}(\mathcal{O}_F)(y_{ii}) \prod_{i < j} \text{char}(\frac{1}{2}\mathcal{O}_F)(y_{ij}). \end{aligned}$$

On the other hand, if $d_\psi t$ is the self-dual Haar measure on $\text{Sym}_n(F)$ with respect to ψ , then

$$\text{vol}(\text{Sym}_n(\mathcal{O}_F), d_\psi t) = |2|_F^{n(n-1)/4},$$

and

$$\begin{aligned} \hat{f}_\psi(y) &= \int_{\text{Sym}_n(\mathcal{O}_F)} \psi(-\text{tr}(yt)) d_\psi t \\ &= \text{vol}(\text{Sym}_n(\mathcal{O}_F), d_\psi t) \prod \text{char}(\mathcal{O}_F)(y_{ii}) \prod_{i < j} \text{char}(\frac{1}{2}\mathcal{O}_F)(y_{ij}). \end{aligned}$$

So $c(\beta, \psi) = |2|_F^{-n-n(n-1)/4}$. Now it is clear that $C(V, \alpha, \beta, \psi) = C(J)$ as claimed.

Now assume that $F = \mathbb{R}$ and $\psi(x) = e(x)$. To compute the quantity $c(\alpha, \psi)$, we write $J = {}^t P \text{diag}(a_1, \dots, a_{n+1}) P$ and denote $|J| = {}^t P \text{diag}(|a_1|, \dots, |a_{n+1}|) P$. We consider the Schwartz function on $M_{n+1,n}(\mathbb{R})$ given by

$$f(x) = e^{-2\pi \text{tr}(x|J|x)} = e^{-2\pi \sum |a_i| \tilde{x}_{ij}^2},$$

where we write $Px = (\tilde{x}_{ij})$. Then its Fourier transform with respect to $d_\alpha x = \prod dx_{ij}$ is

$$\hat{f}_\alpha(x) = \int_{M_{n+1,n}(\mathbb{R})} f(y) e(\text{tr}(2^t x J y)) \prod dy_{ij} = 2^{-n(n+1)/2} |\det J|^{-n/2} f(x),$$

and so $c(\alpha, \psi) = |\det(2J)|^{-n/2}$ as claimed.

To compute $c(\beta, \psi)$, notice that $d_\beta T = 2^{-n} \prod_i dt_{ii} \prod_{i < j} dt_{ij}$ for $T = (t_{ij}) \in \text{Sym}_n(\mathbb{R})$, and consider the Schwartz function on $\text{Sym}_n(\mathbb{R})$ given by

$$f(T) = e^{-\pi(\sum_i t_{ii}^2 + 2\sum_{i < j} t_{ij}^2)}.$$

Then its Fourier transform with respect to $d_\beta T$ is

$$\begin{aligned} \hat{f}_\beta(b) &= \int_{\text{Sym}_n(\mathbb{R})} f(T)\psi(-\text{tr}(Tb)) d_\beta T \\ &= 2^{-n} \prod_j e^{-\pi b_{jj}^2} \int_{\mathbb{R}} e^{-\pi(t_{jj} + ib_{jj})^2} dt_{jj} \prod_{j < k} e^{-2\pi b_{jk}^2} \int_{\mathbb{R}} e^{-2\pi(t_{jk} + ib_{jk})^2} dt_{jk} \\ &= 2^{-n-n(n-1)/4} f(b). \end{aligned}$$

This shows the equality $c(\beta, \psi) = 2^{-n-n(n-1)/4}$. We again have $C(V, \alpha, \beta, \psi) = C(J)$ as claimed. The formula for $\gamma(V^n)$ is given by $\beta_V(w)$ in [Kul, (3.4)]. □

The following proposition shows how to compute the Haar measure $d_v h$ in some cases and will be used in Section 7.

Proposition 2.3. *Let F be a p -adic local field with $p \neq 2$ and a uniformizer π , and let ψ be an unramified additive character of F . For a lattice L over \mathcal{O}_F , let $K_L = \text{SO}(L)$ be the stabilizer of L in $\text{SO}(V)(F)$, where $V = L \otimes_{\mathcal{O}_F} F$. Let $d_L h$ be the Haar measure on $H(F) = \text{SO}(V)(F)$ defined above.*

(1) *When L is unimodular of rank $n + 1$, we have*

$$\frac{\text{vol}(K_L, d_L h)}{C(L)} = W_T(1, 0, \lambda(\phi_L))$$

for any unimodular symmetric matrix $T \in \text{Sym}_n(\mathcal{O}_F)$. Here $\phi_L = \text{char}(L^n)$.

(2) *Assume $L = L_1 \oplus L_0$ with L_1 unimodular of rank $n - 1$ and $L_0 = (\mathcal{O}_E, \pi \mathbf{N}_{E/F})$, where E is the unique unramified quadratic field extension. Let $T = \text{diag}(T_1, \pi)$ with $T_1 = \frac{1}{2}((e_i, e_j))$ for some \mathcal{O}_F -basis $\{e_1, \dots, e_{n-1}\}$ of L_1 . Then*

$$\frac{\text{vol}(K_L, d_L h)}{C(L)} = W_T(1, 0, \lambda(\phi_L)).$$

In both cases, $C(L) = C(J)$ is given by Proposition 2.2.

Proof. We prove (2) using Proposition 2.2 with $\phi = \phi_L$, and leave the slightly easier (1) to the reader. Choose a basis $\{e_n, e_{n+1}\}$ of L_0 so that $Q(ae_n + be_{n+1}) = \pi(a^2 + \epsilon b^2)$ for some $\epsilon \in \mathcal{O}_F^\times$. Let $e = (e_1, \dots, e_n) \in L^n$. Then $Q(e) = T$. We claim that

$$K_T := \{h \in H(F) \mid he \in L^n\} = K_L.$$

Clearly, $K_L \subset K_T$. We just need to prove that $he_{n+1} \in L$ for $h \in K_T$. In this case, $hL_1 \subset L$ is unimodular, so $L = hL_1 \oplus M_0$, where $M_0 = L \cap (hL_1)^\perp$ is a rank 2 lattice with $\det M_0 = \det L_0 = \epsilon\pi^2$. Write

$$M_0 = \mathcal{O}_F \tilde{e}_n + \mathcal{O}_F \tilde{e}_{n+1}, \quad Q(x\tilde{e}_n + y\tilde{e}_{n+1}) = \epsilon_1 \pi^{a_1} x^2 + \epsilon_2 \pi^{a_2} y^2$$

with $\epsilon_1\epsilon_2 = \epsilon$ and non-negative integers a_i satisfying $a_1 + a_2 = 2$. Since

$$he_n = x\tilde{e}_n + y\tilde{e}_{n+1} \in L \cap (hL_1)^\perp = M_0,$$

we have $x, y \in \mathcal{O}_F$ and

$$\pi = N(e_n) = N(he_n) = \epsilon_1\pi^{a_1}x^2 + \epsilon_2\pi^{a_2}y^2,$$

which implies $a_1 = a_2 = 1$. Now write

$$he_{n+1} = a\tilde{e}_n + b\tilde{e}_{n+1}, \quad a, b \in F.$$

Then

$$\epsilon\pi = N(e_{n+1}) = N(he_{n+1}) = \pi(\epsilon_1a^2 + \epsilon_2b^2),$$

i.e.,

$$\epsilon_1\epsilon = N_{E/F}(\epsilon_1a + \sqrt{-\epsilon}b),$$

which implies $\epsilon_1a + \sqrt{-\epsilon}b \in E^1$, which is integral over \mathcal{O}_F . So $a, b \in \mathcal{O}_F$ and $h(e_{n+1}) \in L$. This proves $K_T = K_L$. Applying the local Siegel–Weil formula to $\phi = \phi_L$, we have

$$\int_{H(F)} \phi_L(h^{-1}e) d_Lx = C(L) \cdot W_T(1, 0, \lambda(\phi_L).$$

The left hand side is equal to $\text{vol}(K_T, d_Lx)$. So we have

$$\text{vol}(K_L, d_Lh) = C(L) \cdot W_T(1, 0, \lambda(\phi_L)$$

as claimed. □

We remark that the Whittaker functions involved in the above proposition have explicit formulas: see Section 7.

Now we describe ν and $d_\nu h = d_J h$ more explicitly by choosing the basis \underline{e} and thus J nicely, i.e., we assume $J = \text{diag}(a_1, \dots, a_{n+1})$. It will be used in the next section.

Let $\mathfrak{h} = \mathfrak{so}(V)$ be the Lie algebra of $\text{SO}(V)$. In terms of coordinates with respect to the basis \underline{e} , one has $X = (x_{ij}) \in \mathfrak{h}$ if and only if ${}^tXJ + JX = 0$, i.e., $a_i x_{ij} + a_j x_{ji} = 0$. Hence we have the following lemma.

Lemma 2.4. *Let $X_{ij} = a_j E_{ij} - a_i E_{ji}$ for $1 \leq i < j \leq n + 1$. Then $\{X_{ij}\}$ gives a basis of \mathfrak{h} as an F -vector space.*

Proposition 2.5. *Let the notation be as above. Then*

$$\nu(X_{12}, X_{13}, \dots, X_{n,n+1}) = \pm 1.$$

Proof. We choose

$$x = (e_1, \dots, e_n) = \begin{pmatrix} I_n \\ 0 \end{pmatrix} \in V^n$$

and $T = Q(x) = \text{diag}(a_1, \dots, a_n)$. Then

$$j_x(Y_{ij}) = \frac{1}{2}xT^{-1}Y_{ij} = \frac{1}{2}(a_i^{-1}E_{ij} + a_j^{-1}Y_{ji}).$$

Recall that $i_x : H \rightarrow Q^{-1}(T)$, $h \mapsto hx$. Hence the associated map on the tangent spaces, $di_x : \mathfrak{h} \rightarrow V^{a(n)}$, is given by

$$di_x(X_{ij}) = X_{ij}x = \begin{cases} X_{ij} & \text{if } j \leq n, \\ -a_i E_{n+1,i} & \text{if } j = n + 1. \end{cases}$$

Therefore,

$$\left(\bigwedge_{1 \leq i < j \leq n+1} (di_x)(X_{ij}) \right) \wedge \left(\bigwedge_{1 \leq j \leq n} j_x(Y_{ij}) \right) = \pm \bigwedge_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n+1}} E_{ij},$$

and

$$\begin{aligned} v(\wedge X_{ij}) &= \alpha \left(\left(\bigwedge_{1 \leq i < j \leq n+1} (di_x)(X_{ij}) \right) \wedge \left(\bigwedge_{1 \leq j \leq n} j_x(Y_{ij}) \right) \right) \beta(\wedge Y_{ij})^{-1} \\ &= \pm \alpha \left(\bigwedge_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n+1}} E_{ij} \right) = \pm 1. \end{aligned}$$

This concludes the proof of the proposition. □

3. The local Siegel–Weil formula on a hermitian symmetric domain

Let V be a quadratic space over \mathbb{R} of signature $(m, 2)$, and let $H = \text{SO}(V)$. Let \mathcal{D} be the corresponding hermitian domain, which we realize as the Grassmannian of oriented negative 2-planes in V . The purpose of this section is to prove Proposition 3.1, a variant of the archimedean local Siegel–Weil formula involving an integral over \mathcal{D} . Throughout this section we fix the additive character $\psi(x) = e(x)$ of \mathbb{R} and assume that $n = m + 1$. Recall that $\rho_n = (n + 1)/2$.

Let $e, f \in V$ be isotropic vectors such that $(e, f) = 1$, and let $V_0 = (\mathbb{R}e + \mathbb{R}f)^\perp \subset V$. Then V_0 has signature $(m - 1, 1)$ and we have the Witt decomposition $V = V_0 + \mathbb{R}e + \mathbb{R}f$. The hermitian symmetric domain \mathcal{D} can also be realized as the tube domain

$$\mathcal{H} = \{z = x + iy \in V_{0,\mathbb{C}} \mid Q(y) < 0\} \tag{3.1}$$

via the isomorphism

$$\mathcal{H} \rightarrow \mathcal{D}, \quad z \mapsto \mathbb{R}\Re(w(z)) + \mathbb{R}\Im(w(z)),$$

where

$$w(z) = z + e - Q(z)f \in V_{\mathbb{C}}.$$

Then $H(\mathbb{R})$ acts on \mathcal{H} by linear fractional transformations, characterized by

$$hw(z) = j(h, z) \cdot w(hz)$$

where $j(h, z)$ denotes the automorphy factor $H(\mathbb{R}) \times \mathcal{H} \rightarrow \mathbb{C}^\times$, $j(h, z) = (hw(z), f)$.

The map $z \mapsto w(z)$ can be viewed as a section of the tautological bundle over \mathcal{D} . The Petersson norm of this section is $-\frac{1}{2}(w(z), \bar{w}(z)) = -(y, y)$. Hence

$$\Omega = dd^c \log(-(y, y)) \tag{3.2}$$

defines an invariant $(1, 1)$ -form on $\mathcal{H} \cong \mathcal{D}$, the first Chern form of the dual of the tautological bundle on \mathcal{D} equipped with the Petersson metric. Here $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$. According to [Ku4, Proposition 4.11], in the coordinates of \mathcal{H} , it is given by

$$\Omega = dd^c \log \left(-\frac{1}{2}(\omega(z), \omega(\bar{z})) \right) = -\frac{1}{2\pi i} \left(-\frac{(y, dz) \wedge (y, d\bar{z})}{(y, y)^2} + \frac{(dz, d\bar{z})}{2(y, y)} \right). \tag{3.3}$$

Moreover, it can be obtained from the Kudla–Millson form $\varphi_{KM}(x, z)$ (see (5.2)) by

$$\Omega = \varphi_{KM}(0, z), \tag{3.4}$$

an identity which we will only need in Section 5. Notice that $-\Omega$ is a Kähler form, and therefore $(-\Omega)^m$ is a positive invariant top degree form on \mathcal{H} .

Proposition 3.1 (Local Siegel–Weil formula on \mathcal{D}). *Let $\phi_\infty(x, z) \in S(V_{\mathbb{R}}^n) \otimes C^\infty(\mathcal{D})$ with $\phi_\infty(hx, hz) = \phi_\infty(x, z)$ for all $z \in \mathcal{D}$, $x \in V_{\mathbb{R}}^n$ and $h \in H(\mathbb{R})$. Then $\lambda(\phi_\infty)$ is independent of z , and*

$$\int_{\mathcal{D}} \phi_\infty(x, z) \Omega^m = B_{n,\infty} \cdot W_T(1, 0, \lambda(\phi_\infty))$$

with $T = Q(x)$ and

$$B_{n,\infty} = \frac{e^{\left(\frac{n^2+n-4}{8}\right)}(n-1)! \prod_{k=1}^{n-1} \Gamma\left(\frac{n-k}{2}\right)}{2^{n-2}(2\pi)^{n(n+3)/4}}.$$

In particular, $B_{2,\infty} = \frac{i}{4\sqrt{2}\pi^2}$ and

$$\frac{B_{n,\infty}}{B_{n-1,\infty}} = i^n \frac{\Gamma(\rho_n)}{(2\pi)^{\rho_n}}. \tag{3.5}$$

The basic idea of the proof is simple and natural: we relate the gauge form on the tangent space \mathfrak{p} of \mathbb{D} to the differential form Ω^m precisely. The actual calculation is a little long and technical, and can be skipped on first reading. We will also provide an alternative proof in Section 5.5.

3.1. The differential Ω^m and the gauge form v

Let $\underline{e} = (e_1, e_2, \dots, e_m, e_n, e_{n+1})$ be an ordered basis of V with quadratic matrix

$$J = \frac{1}{2}((e_i, e_j)) = \begin{pmatrix} I_m & 0 \\ 0 & -I_2 \end{pmatrix}.$$

We write V_+ (respectively V_-) for the subspace generated by the e_i with $1 \leq i \leq m$ (respectively $i = n, n + 1$). Let $K_\pm = \text{SO}(V_\pm)$. Then $K_\infty = K_+ \times K_-$ is a maximal connected compact subgroup of $H(\mathbb{R})$. In the notation of the last section, $a_i = 1$ for $1 \leq i \leq m$ and $a_i = -1$ for $i = n, n + 1$.

Let $\mathfrak{h} = \mathfrak{so}(V)$ be the Lie algebra of $\mathrm{SO}(V)$. Then $X \in \mathfrak{h}$ if and only if $X = \begin{pmatrix} X_1 & X_2 \\ {}^tX_2 & X_3 \end{pmatrix}$ with ${}^tX_1 = -X_1 \in M_m$, ${}^tX_3 = -X_3 \in M_2$, and $X_2 \in M_{m,2}$. In other words, one has a decomposition

$$\mathfrak{h} = \mathfrak{k}_+ \oplus \mathfrak{k}_- \oplus \mathfrak{p},$$

where \mathfrak{k}_\pm is the Lie algebra of $K_\pm = \mathrm{SO}(V_\pm)$ given by matrices satisfying ${}^tX_1 = -X_1$ (respectively ${}^tX_3 = -X_3$).

It is easy to see that the gauge form ν given in Proposition 2.5 has the following decomposition (up to sign):

$$\nu = \nu_+ \wedge \nu_- \wedge \nu_{\mathfrak{p}}, \tag{3.6}$$

where ν_+ , ν_- , and $\nu_{\mathfrak{p}}$ are the gauge forms on K_+ , K_- , and $H(\mathbb{R})/K_\infty$, which are characterized by

$$\begin{aligned} \nu_+(X_{12}, \dots, X_{m-1,m}) &= 1, \\ \nu_-(X_{n,n+1}) &= 1, \\ \nu_{\mathfrak{p}}(X_{1n}, X_{1,n+1}, \dots, X_{mn}, X_{m,n+1}) &= 1. \end{aligned} \tag{3.7}$$

Now we deal with the relation between Ω^m and $\nu_{\mathfrak{p}}$. We use a tube domain realization for \mathcal{D} as above. To this end we define a different basis \underline{e}' of V as follows. Let $e = \frac{1}{2}(e_1 + e_{n+1})$, $f = \frac{1}{2}(e_1 - e_{n+1})$, and $\underline{e}' = (e_n, e_2, \dots, e_m, e, f)$. Its associated matrix is

$$J' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & I_{m-1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

We put

$$\begin{aligned} V_0 &= \bigoplus_{i=2}^n \mathbb{R}e_i \cong \mathbb{R}^m, \\ z &= \sum z_i e_i = {}^t(z_n, z_2, \dots, z_m) \end{aligned} \tag{3.8}$$

with quadratic form $Q(z) = \sum \epsilon_i z_i^2$ with $\epsilon_i = \pm 1$ depending on whether $i < n$ or $i \geq n$. Then $V = V_0 \oplus \mathbb{R}e \oplus \mathbb{R}f$ is a Witt decomposition as considered before. We write \mathcal{H} for the corresponding tube domain realization of \mathcal{D} as in (3.1). We will also identify V with \mathbb{R}^{n+1} and V^n with $M_{n+1,n}(\mathbb{R})$ with respect to the basis \underline{e}' :

$$\begin{aligned} v &= \sum_{2 \leq i \leq n} z_i e_i + v_0 e + v_1 f = {}^t(z_n, z_2, \dots, z_m, v_0, v_1) = [v]_{\underline{e}'}, \\ x &= (\tilde{x}_1, \dots, \tilde{x}_n) = [x]_{\underline{e}'} \in M_{n+1,n}(\mathbb{R}). \end{aligned}$$

Similarly, we will use $[v]_{\underline{e}}$ and $[x]_{\underline{e}}$ to denote the coordinates of v and x with respect to the basis \underline{e} when necessary. For $\gamma \in H(\mathbb{R})$, we denote $[\gamma]_{\underline{e}}$ and $[\gamma]_{\underline{e}'}$ for its coordinates

with respect to the bases \underline{e} and \underline{e}' respectively. Then one has

$$[v]_{\underline{e}'} = A[v]_{\underline{e}}, \quad [\gamma]_{\underline{e}'} = A[\gamma]_{\underline{e}}A^{-1}, \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & I_{m-1} & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{3.9}$$

We now compute the action of $H(\mathbb{R})$ on \mathcal{H} more explicitly. For $h \in H(\mathbb{R})$, write

$$[h]_{\underline{e}'} = \begin{pmatrix} \tilde{h}_{11} & \tilde{H}_{12} & \tilde{h}_{13} & \tilde{h}_{14} \\ \tilde{H}_{21} & \tilde{H}_{22} & \tilde{H}_{23} & \tilde{H}_{24} \\ \tilde{h}_{31} & \tilde{H}_{32} & \tilde{h}_{33} & \tilde{h}_{34} \\ \tilde{h}_{41} & \tilde{H}_{42} & \tilde{h}_{43} & \tilde{h}_{44} \end{pmatrix},$$

where all the \tilde{H}_{ij} are matrices, all \tilde{h}_{ij} s are numbers, and \tilde{H}_{22} is a square matrix of order $m - 1$. Then for $z = {}^t(z_n, z_2, \dots, z_m) = \begin{pmatrix} z_n \\ \underline{z} \end{pmatrix} \in \mathcal{H}$, we have

$$h(z) = j(h, z)^{-1} \begin{pmatrix} z_n \tilde{h}_{11} + \tilde{H}_{12} \underline{z} + \tilde{h}_{13} - \tilde{h}_{14} Q(z) \\ z_n \tilde{H}_{21} + \tilde{H}_{22} \underline{z} + \tilde{H}_{23} - \tilde{H}_{24} Q(z) \underline{z} \end{pmatrix},$$

$$j(h, z) = z_n \tilde{h}_{31} + \tilde{H}_{32} \underline{z} + \tilde{h}_{33} - \tilde{h}_{34} Q(z).$$

Fixing the base point $z = ie_n \in \mathcal{H}$, we have the isomorphism

$$l_z : H(\mathbb{R})/K_\infty \cong \mathcal{H}, \quad h \mapsto h(z) = \begin{pmatrix} \tilde{h}_{11}i + \tilde{h}_{13} - \tilde{h}_{14} \\ \tilde{H}_{21}i + \tilde{H}_{23} - \tilde{H}_{24} \end{pmatrix} (\tilde{h}_{31} + \tilde{h}_{33}i - \tilde{h}_{34})^{-1}. \tag{3.10}$$

This induces an isomorphism between \mathfrak{p} and the tangent space $T_z(\mathcal{H}) \cong V_{0,\mathbb{C}}$ (extending to the tangent bundle, too):

$$dl_z : \mathfrak{p} \cong V_{0,\mathbb{C}}, \quad X \mapsto X(z) = \begin{pmatrix} \tilde{x}_{13} - \tilde{x}_{14} + \tilde{x}_{31} \\ \tilde{X}_{23} - \tilde{X}_{24} \end{pmatrix} + i \begin{pmatrix} \tilde{x}_{11} - \tilde{x}_{33} + \tilde{x}_{34} \\ \tilde{X}_{21} \end{pmatrix}, \tag{3.11}$$

where \tilde{x}_i and \tilde{X}_{ij} are the coordinates of X with respect to \underline{e}' just as for h . In terms of the coordinates with respect to \underline{e} , one has

$$[X]_{\underline{e}} = \begin{pmatrix} 0 & 0 & x_{13} & x_{14} \\ 0 & 0_{m-1,m-1} & X_{23} & X_{24} \\ x_{13} & X_{23}^t & 0 & 0 \\ x_{14} & X_{24}^t & 0 & 0 \end{pmatrix} \in \mathfrak{p},$$

and by a direct direct calculation using (3.9) we obtain

$$dl_z(X) = \begin{pmatrix} -i(x_{14} + ix_{13}) \\ X_{24} + iX_{23} \end{pmatrix}. \tag{3.12}$$

So we have proved the following lemma.

Lemma 3.2. *The isomorphism dl_z induces*

$$(dl_z)^*(dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n) = \pm \nu_p,$$

where $(dx_j + idy_j)$ is the dual \mathbb{C} -basis of the basis $(e_j)_{2 \leq j \leq n}$ of $V_{0,\mathbb{C}}$.

Recall the formula (3.3) for the $H(\mathbb{R})$ -invariant $(1, 1)$ -form Ω .

Lemma 3.3. *Using the above notation, we have*

$$(-\Omega)^m = \frac{m!}{(2\pi)^m} \frac{\bigwedge_{j=2}^n dx_j \wedge dy_j}{(-Q(y))^m}.$$

Proof. Using the coordinates of (3.8), one sees

$$\begin{aligned} \Omega &= -\frac{1}{2\pi i} \left(-\frac{1}{Q(y)^2} \sum \epsilon_i \epsilon_j y_i y_j dz_i \wedge d\bar{z}_j + \frac{1}{2Q(y)} \sum \epsilon_i dz_i \wedge d\bar{z}_i \right) \\ &= -\frac{1}{2\pi i} \left(-\frac{1}{Q(y)^2} \alpha + \frac{1}{2Q(y)} \beta \right), \end{aligned}$$

where α and β have the obvious meanings. Notice that

$$\alpha^2 = \sum_{i,j,k,l=2}^n \epsilon_i \epsilon_j \epsilon_k \epsilon_l y_i y_j y_k y_l \cdot \alpha(i, j, k, l)$$

with

$$\alpha(i, j, k, l) = dz_i \wedge d\bar{z}_j \wedge dz_k \wedge d\bar{z}_l.$$

Since $\alpha(i, j, k, l) = -\alpha(k, j, i, l)$, we have $\alpha^2 = 0$. This implies

$$\Omega^m = \left(-\frac{1}{4\pi i Q(y)} \right)^m \left(\beta^m - \frac{2m}{Q(y)} \alpha \wedge \beta^{m-1} \right).$$

It is easy to check that

$$\beta^m = -m! \bigwedge_{j=2}^n dz_j \wedge d\bar{z}_j, \quad \beta^{m-1} = -(m-1)! \sum_{l=2}^n \epsilon_l \beta_l,$$

where β_l is $\bigwedge_{j=2}^n dz_j \wedge d\bar{z}_j$ with $dz_l \wedge d\bar{z}_l$ missing. So

$$\alpha \wedge \beta^{m-1} = -(m-1)! \sum_{l=2}^n \epsilon_l y_l^2 \bigwedge_{j=2}^n dz_j \wedge d\bar{z}_j = -(m-1)! Q(y) \bigwedge_{j=2}^n dz_j \wedge d\bar{z}_j,$$

and therefore

$$\Omega^m = \left(-\frac{1}{4\pi i Q(y)} \right)^m m! \bigwedge_{j=2}^n dz_j \wedge d\bar{z}_j = \frac{m!}{(2\pi Q(y))^m} \bigwedge_{j=2}^n dx_j \wedge dy_j$$

as claimed. □

It is well-known that $\frac{\bigwedge_{j=2}^n dx_j \wedge dy_j}{(-Q(y))^m}$ is the $H(\mathbb{R})$ -invariant Haar measure on \mathcal{H} associated to $\bigwedge_{j=2}^n dx_j \wedge dy_j$. So we obtain the following proposition from the above two lemmas.

Proposition 3.4. *Let the notation be as above and $z = ie_n \in \mathcal{H}$. Then*

$$(dl_z)^*(-\Omega)^m = \pm \frac{m!}{(2\pi)^m} \nu_p.$$

Proof of Proposition 3.1. First, let $z = ie_n \in \mathcal{H}$ as in Proposition 3.4, let $\nu = \nu_+ \wedge \nu_- \wedge \nu_p$ be as in (3.6), and let $d_\nu h, dh_+, dh_-,$ and $d_p h$ be the associated Haar measures. Then, by Proposition 3.4,

$$\begin{aligned} \int_{H(\mathbb{R})} \phi_\infty(h^{-1}x, z) d_\nu h &= \text{vol}(K_+, dh_+) \text{vol}(K_-, dh_-) \int_{H(\mathbb{R})/K_\infty} \phi_\infty(x, hz) d_p h \\ &= \text{vol}(K_+, dh_+) \text{vol}(K_-, dh_-) \frac{(2\pi)^m}{m!} \int_{\mathcal{H}} \phi_\infty(x, z) (-\Omega)^m \\ &= (-1)^m \text{vol}(K_+, dh_+) \text{vol}(K_-, dh_-) \frac{(2\pi)^m}{m!} \int_{\mathcal{D}} \phi_\infty(x, z) \Omega^m. \end{aligned}$$

On the other hand, Proposition 2.2 gives

$$\int_{H(\mathbb{R})} \phi_\infty(h^{-1}x, z) d_\nu h = C(\text{diag}(I_m, -I_2)) \cdot W_{T,\infty}(1, 0, \lambda(\phi_\infty)).$$

Consequently,

$$B_{n,\infty} = \frac{(-1)^m m! C(\text{diag}(I_m, -I_2))}{(2\pi)^m \text{vol}(K_+, dh_+) \text{vol}(K_-, dh_-)}.$$

Applying Proposition 2.2 to $K_+ = \text{SO}(V_+)$ and $\phi_\infty = e^{-2\pi \text{tr } Q_+(x)}$, one sees by Proposition 4.3 that

$$\begin{aligned} \text{vol}(K_+, dh_+) e^{-2\pi(m-1)} &= C(I_m) W_{I_{m-1},\infty}(1, 0, \Phi_{m/2}) \\ &= C(I_m) \frac{(-2\pi i)^{m(m-1)/2}}{\Gamma_{m-1}(m/2)} e^{-2\pi(m-1)}, \end{aligned}$$

where $\Gamma_n(s)$ is given by (4.1). We obtain

$$\text{vol}(K_+, dh_+) = C(I_m) \frac{(-2\pi i)^{m(m-1)/2}}{\Gamma_{m-1}(m/2)} = \frac{2^{m-1} \pi^{m(m+1)/4}}{\prod_{k=0}^{m-1} \Gamma\left(\frac{m-k}{2}\right)}.$$

Similarly, $\text{vol}(K_-, dh_-) = 2\pi$. Plugging these formulas into that for $B_{n,\infty}$, one proves Proposition 3.1. □

We remark that the above calculation of $\text{vol}(K_+, dh_+)$ has the following well-known formula as a consequence.

Corollary 3.5. *Let $l \geq 1$ be an integer, and let*

$$\text{SO}_l(\mathbb{R}) = \{g \in \text{GL}_l(\mathbb{R}) \mid g^t g = I_n, \det g = 1\}$$

be the standard special orthogonal group. Let ν_l be the gauge form defined as ν_+ for $K_+ = \text{SO}_m(\mathbb{R})$, and let dh_l be the associated Haar measure on $\text{SO}_l(\mathbb{R})$. Then

$$\text{vol}(\text{SO}_l(\mathbb{R}), dh_l) = \frac{2^{l-1} \pi^{l(l+1)/4}}{\prod_{k=0}^{l-1} \Gamma\left(\frac{l-k}{2}\right)}.$$

4. Asymptotic properties of Whittaker functions

Throughout this section we consider the local field $F = \mathbb{R}$, the additive character $\psi(x) = e(x)$, and the group $G = \mathrm{Sp}_n(\mathbb{R})$ or $\mathrm{Mp}_{n,\mathbb{R}}$. We investigate the asymptotic behavior of the archimedean Whittaker function for G as one of the radial parameters of the Levi subgroup M goes to ∞ . The main results are Theorem 4.8 and Corollary 4.11. Our analysis is based on Shimura’s work on confluent hypergeometric functions [Shi]. We fix a quadratic character χ of \mathbb{R}^\times and a half-integer $\kappa \in \rho_n + \mathbb{Z}$ (not necessarily equal to ρ_n) satisfying the compatibility condition

$$\begin{cases} (-1)^\kappa = \chi(-1) & \text{if } n \equiv 1 \pmod{2}, \\ \kappa \equiv \frac{1}{2}\chi(-1) & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

We also fix a matrix $T \in \mathrm{Sym}_n(\mathbb{R})$.

4.1. Basic properties of archimedean Whittaker functions

Let $\Phi = \Phi_\kappa \in I_n(s, \chi)$ be the weight κ standard section, that is, the unique function in $I_n(s, \chi)$ whose restriction to K_G is the character $\det(\mathbf{k})^\kappa$. Then the Whittaker integral (2.1) can be expressed in terms of Shimura’s confluent hypergeometric function. As in [Ku2, Lemma 9.3], the following result can be proved.

Lemma 4.1. *Assume that $\det(T) \neq 0$. If $a \in \mathrm{GL}_n^+(\mathbb{R})$ and $y = a^t a$, then*

$$W_T(m(a), s, \Phi_\kappa) = C_{n,\infty} \cdot |a|^{s+\rho_n} \xi(y, T, \alpha, \beta),$$

where

$$\xi(y, T, \alpha, \beta) = \int_{\mathrm{Sym}_n(\mathbb{R})} \det(x + iy)^{-\alpha} \det(x - iy)^{-\beta} e(-\mathrm{tr}(Tx)) \, dx$$

denotes Shimura’s confluent hypergeometric function of matrix argument [Shi, (1.25)] with

$$\alpha = \frac{1}{2}(s + \rho_n + \kappa), \quad \beta = \frac{1}{2}(s + \rho_n - \kappa).$$

Here $dx = \bigwedge_{i \leq j} dx_{ij}$ is the Lebesgue measure on $\mathrm{Sym}_n(\mathbb{R}) \cong \mathbb{R}^{n(n+1)/2}$, and $C_{n,\infty} = 2^{n(n-1)/4}$.

The normalizing factor $C_{n,\infty}$ comes from comparing the measures dn and dx . Recall that the Siegel gamma function of genus n is defined by

$$\Gamma_n(s) := \int_{\substack{x \in \mathrm{Sym}_n(\mathbb{R}) \\ x > 0}} e^{-\mathrm{tr}(x)} \det(x)^{s-\rho_n} \, dx = \pi^{n(n-1)/4} \prod_{k=0}^{n-1} \Gamma(s - k/2). \tag{4.1}$$

Following Shimura, we define another special function by

$$\eta(y, T, \alpha, \beta) = \int_{\substack{u \in \text{Sym}_n(\mathbb{R}) \\ u > -T \\ u > T}} e^{-\text{tr}(uy)} \det(u + T)^{\alpha - \rho_n} \det(u - T)^{\beta - \rho_n} du. \tag{4.2}$$

For all regular T , by [Shi, Remark 4.3], the integral converges when $\Re(\alpha) > \rho_n - 1$ and $\Re(\beta) > \rho_n - 1$. According to [Shi, (1.29)], we have

$$\xi(y, T, \alpha, \beta) = \frac{i^{n(\beta - \alpha)} 2^{-n(\rho_n - 1)} (2\pi)^{n\rho_n}}{\Gamma_n(\alpha)\Gamma_n(\beta)} \cdot \eta(2y, \pi T, \alpha, \beta), \tag{4.3}$$

and therefore

$$W_T(m(a), s, \Phi_\kappa) = c_n(\alpha, \beta) \cdot |a|^{s + \rho_n} \cdot \eta(2y, \pi T, \alpha, \beta), \quad \text{where} \tag{4.4}$$

$$c_n(\alpha, \beta) = C_{n,\infty} \cdot \frac{i^{n(\beta - \alpha)} 2^{-n(\rho_n - 1)} (2\pi)^{n\rho_n}}{\Gamma_n(\alpha)\Gamma_n(\beta)}.$$

Lemma 4.2. *If $S \in \text{GL}_n^+(\mathbb{R})$, we have*

$$\eta({}^t S g S, h, \alpha, \beta) = |S|^{2(\rho_n - \alpha - \beta)} \eta(g, Sh {}^t S, \alpha, \beta),$$

$$W_T(m(a), s, \Phi_\kappa) = |S|^{\rho_n - s} W_{t_S T S}(m(S^{-1}a), s, \Phi_\kappa).$$

Proof. The first assertion follows from [Shi, (3.1.K)]. The second assertion follows from this by means of (4.4). □

The special values of Eisenstein series and Whittaker functions at $s = 0$ will be of particular interest. Here we collect the facts that we will require.

Proposition 4.3. *Assume that $\det(T) \neq 0$ and $\kappa = \rho_n$.*

(i) *If $\text{sig}(T) = (n - j, j)$ with $0 \leq j \leq n$, then*

$$\text{ord}_{s=0} W_T(m(a), s, \Phi_\kappa) \geq \left\lfloor \frac{j + 1}{2} \right\rfloor.$$

(ii) *If $\text{sig}(T) = (n, 0)$, then*

$$W_T(m(a), 0, \Phi_\kappa) = \frac{(-2\pi i)^{n\kappa} 2^{-n(n-1)/4}}{\Gamma_n(\kappa)} (\det y)^{\kappa/2} e^{-2\pi \text{tr} T y}.$$

Proof. According to [Shi, Theorem 4.2], the function

$$\Gamma_{n-j} \left(\beta - \frac{j}{2} \right)^{-1} \Gamma_j \left(\alpha - \frac{n-j}{2} \right)^{-1} \eta(2y, \pi T, \alpha, \beta)$$

is holomorphic for $(\alpha, \beta) \in \mathbb{C}^2$. Hence, in view of (4.4), $W_T(m(a), s, \Phi_\kappa)$ is equal to a holomorphic function in a neighborhood of $s = 0$ times the gamma factor

$$\frac{\Gamma_{n-j}(\beta - j/2)}{\Gamma_n(\beta)}.$$

Therefore, the first assertion follows from (4.1) by working out the vanishing order of this gamma factor.

To prove (ii), we use [Shi, (4.35.K), (4.12.K), (4.6.K)] to see that for $\text{sig}(T) = (n, 0)$ we have

$$\eta(g, h, \rho_n, \beta) = \Gamma_n(\beta) \det(g)^{-\beta} e^{-\text{tr}(gh)}.$$

By means of (4.4) we get

$$\begin{aligned} W_T(m(a), 0, \Phi_\kappa) &= c_n(\rho_n, \beta) |a|^{s+\rho_n} \eta(2y, \pi T, \rho_n, \beta)|_{s=0} \\ &= \frac{(-2\pi i)^{n\rho_n} 2^{-n(n-1)/4}}{\Gamma_n(\rho_n)} (\det y)^{\rho_n/2} e^{-2\pi \text{tr} T y}, \end{aligned}$$

where $y = a^t a$. This proves the proposition. □

Remark 4.4. Assume that $\text{sig}(T) = (n, 0)$ and $\kappa = \rho_n$. Then with the constant $B_{n,\infty}$ of Proposition 3.1 we have

$$B_{n,\infty} \cdot W_T(1, 0, \Phi_\kappa) = -2e^{-2\pi \text{tr} T}.$$

Later we will also need the following lemmas.

Lemma 4.5. *If f is a measurable function on $\text{Sym}_n(\mathbb{R})$ and $a \in \text{GL}_n(\mathbb{R})$, then*

$$\int_{\text{Sym}_n(\mathbb{R})} f(ab^t a) db = |a|^{-2\rho_n} \int_{\text{Sym}_n(\mathbb{R})} f(b) db.$$

Lemma 4.6. *If $S \in \text{Sym}_n(\mathbb{R})$ is positive definite, then*

$$\int_{\mathbb{R}^n} e^{-^t v S v} dv = \pi^{n/2} \det(S)^{-1/2}.$$

Here dv denotes the Lebesgue measure on \mathbb{R}^n .

Lemma 4.7. *Let $u = \begin{pmatrix} u_1 & u_{12} \\ {}^t u_{12} & u_2 \end{pmatrix}$ be a symmetric block matrix. Then the following are equivalent:*

- (1) $u > 0$,
- (2) $u_1 > 0$ and $u_2 > {}^t u_{12} u_1^{-1} u_{12}$,
- (3) $u_2 > 0$ and $u_1 > u_{12} u_2^{-1} {}^t u_{12}$.

In this case

$$\det(u) = \det(u_1) \det(u_2 - {}^t u_{12} u_1^{-1} u_{12}) = \det(u_2) \det(u_1 - u_{12} u_2^{-1} {}^t u_{12}).$$

Proof. This is a direct consequence of the Jacobi decompositions

$$\begin{aligned} \begin{pmatrix} u_1 & u_{12} \\ {}^t u_{12} & u_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ {}^t(u_1^{-1} u_{12}) & 1 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 - {}^t u_{12} u_1^{-1} u_{12} \end{pmatrix} \begin{pmatrix} 1 & u_1^{-1} u_{12} \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} u_1 & u_{12} \\ {}^t u_{12} & u_2 \end{pmatrix} &= \begin{pmatrix} 1 & u_{12} u_2^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 - u_{12} u_2^{-1} {}^t u_{12} & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ {}^t(u_{12} u_2^{-1}) & 1 \end{pmatrix}, \end{aligned}$$

whenever the inverses make sense. See also [Shi, Lemma 2.1]. □

4.2. Asymptotic properties

Here we investigate the asymptotic behavior of the Whittaker function $W_T(g, s, \Phi_\kappa)$. We assume that $T \in \text{Sym}_n(\mathbb{R})$ with $\det(T) \neq 0$, and $a \in \text{GL}_n^+(\mathbb{R})$. We put $y = a^t a$. Recall that

$$\eta(y, T, \alpha, \beta) = \int_{\substack{u \in \text{Sym}_n(\mathbb{R}) \\ u > -T \\ u > T}} e^{-\text{tr}(uy)} \det(u + T)^{\alpha - \rho_n} \det(u - T)^{\beta - \rho_n} du.$$

We write T and the variable of integration u in block form as

$$T = \begin{pmatrix} T_1 & T_{12} \\ {}^t T_{12} & T_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 & u_{12} \\ {}^t u_{12} & u_2 \end{pmatrix} \tag{4.5}$$

with $T_1 \in \mathbb{R}$, $T_2 \in \text{Sym}_{n-1}(\mathbb{R})$, and $T_{12} \in \mathbb{R}^{1 \times (n-1)}$, and analogously for u .

Theorem 4.8. *Let $y = \begin{pmatrix} y_1 & y_{12} \\ {}^t y_{12} & y_2 \end{pmatrix} \in \text{Sym}_n(\mathbb{R})$ be a positive definite matrix in block form as in (4.5). If $T_1 \leq 0$ we have*

$$\lim_{y_1 \rightarrow \infty} e^{T_1 y_1} y_1^\beta \cdot \eta^{(n)}(y, T, \alpha, \beta) = 0.$$

If $T_1 > 0$ we have

$$\begin{aligned} &\lim_{y_1 \rightarrow \infty} e^{T_1 y_1} y_1^\beta \cdot \eta^{(n)}(y, T, \alpha, \beta) \\ &= e^{-2T_1 {}^t y_{12} + \text{tr}(\tilde{T}_2 - T_2)y_2} \cdot \Gamma(\beta + 1 - \rho_n) \pi^{(n-1)/2} (2T_1)^{\alpha - \rho_n} \eta^{(n-1)}(y_2, \tilde{T}_2, \alpha - 1/2, \beta), \end{aligned}$$

where

$$\tilde{T} = \begin{pmatrix} \tilde{T}_1 & 0 \\ 0 & \tilde{T}_2 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 - {}^t T_{12} T_1^{-1} T_{12} \end{pmatrix}.$$

Here we have added a superscript to η to indicate in which genus it is considered.

Remark 4.9. In the case $n = 1$ the function $\eta^{(0)}$ is to be interpreted as the constant function with value 1. Then the theorem states

$$\lim_{y_1 \rightarrow \infty} e^{T_1 y_1} y_1^\beta \cdot \eta^{(1)}(y, T, \alpha, \beta) = \begin{cases} 0 & \text{if } T_1 < 0, \\ \Gamma(\beta) (2T_1)^{\alpha - \rho_1} & \text{if } T_1 > 0. \end{cases} \tag{4.6}$$

On the other hand, for $T \in \mathbb{R}^\times$ and $y \in \mathbb{R}_{>0}$ we have

$$\eta^{(1)}(y, T, \alpha, \beta) = e^{-|T|y} \cdot |2T|^{\alpha + \beta - 1} \cdot \begin{cases} \Gamma(\alpha) U(\alpha, \alpha + \beta, 2|T|y) & \text{if } T < 0, \\ \Gamma(\beta) U(\beta, \alpha + \beta, 2|T|y) & \text{if } T > 0, \end{cases}$$

where $U(a, b, z)$ denotes Kummer’s confluent hypergeometric function [AS, (13.1.3)]. The asymptotic behavior of the Kummer function $U(a, b, y) = y^{-a} + O(y^{-a-1})$ as $y \rightarrow \infty$ (see e.g. [AS, (13.5.2)]) matches (4.6).

Proof of Theorem 4.8. Step 1. We first consider the case where $\Re(\alpha) > \rho_n$ and $\Re(\beta) > \rho_n - 1/2$. We put

$$y' = \begin{pmatrix} y_1^{-1/2} & \\ & 1 \end{pmatrix} y \begin{pmatrix} y_1^{-1/2} & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & y_{12}/y_1^{1/2} \\ {}_t y_{12}/y_1^{1/2} & y_2 \end{pmatrix}.$$

By means of Lemma 4.5 we rewrite the integral as follows:

$$\begin{aligned} \eta(y, T, \alpha, \beta) &= e^{-\text{tr}Ty} \int_{\substack{u \in \text{Sym}_n(\mathbb{R}) \\ u+2T > 0 \\ u > 0}} e^{-\text{tr}(uy)} |u + 2T|^{\alpha-\rho_n} |u|^{\beta-\rho_n} du \\ &= y_1^{-\beta} e^{-\text{tr}Ty} \\ &\quad \times \int_{\substack{\begin{pmatrix} u_1/y_1 & u_{12}/y_1^{1/2} \\ {}_t u_{12}/y_1^{1/2} & u_2 \end{pmatrix} + 2T > 0 \\ u > 0}} e^{-\text{tr}(uy')} \left| \begin{pmatrix} u_1/y_1 & u_{12}/y_1^{1/2} \\ {}_t u_{12}/y_1^{1/2} & u_2 \end{pmatrix} + 2T \right|^{\alpha-\rho_n} \cdot |u|^{\beta-\rho_n} du. \end{aligned}$$

Here and throughout the proof we briefly write $|u|$ for the determinant of u . In view of Lemma 4.7, we obtain

$$\begin{aligned} \eta(y, T, \alpha, \beta) &= y_1^{-\beta} e^{-\text{tr}Ty} \int_{\substack{u_2+2T_2 > 0 \\ u > 0}} \chi(u, y_1, T) e^{-\text{tr}(uy')} \left| \begin{pmatrix} u_1/y_1 & u_{12}/y_1^{1/2} \\ {}_t u_{12}/y_1^{1/2} & u_2 \end{pmatrix} + 2T \right|^{\alpha-\rho_n} \cdot |u|^{\beta-\rho_n} du, \end{aligned} \tag{4.7}$$

where $\chi(u, y_1, T)$ denotes the characteristic function

$$\chi(u, y_1, T) = \begin{cases} 1 & \text{if } \frac{u_1}{y_1} + 2T_1 - \left(\frac{u_{12}}{\sqrt{y_1}} + 2T_{12}\right)(u_2 + 2T_2)^{-1} {}_t \left(\frac{u_{12}}{\sqrt{y_1}} + 2T_{12}\right) > 0, \\ 0 & \text{if } \frac{u_1}{y_1} + 2T_1 - \left(\frac{u_{12}}{\sqrt{y_1}} + 2T_{12}\right)(u_2 + 2T_2)^{-1} {}_t \left(\frac{u_{12}}{\sqrt{y_1}} + 2T_{12}\right) \leq 0. \end{cases}$$

We now compute the desired limit as $y_1 \rightarrow \infty$ assuming that the integration can be interchanged with the limit. After that we will come back to the justification of the interchange. We have

$$\begin{aligned} &\lim_{y_1 \rightarrow \infty} e^{T_1 y_1} y_1^\beta \cdot \eta^{(n)}(y, T, \alpha, \beta) \\ &= e^{-2T_{12}' y_{12} - \text{tr} T_2 y_2} \\ &\quad \times \int_{\substack{u_2+2T_2 > 0 \\ u > 0}} \lim_{y_1 \rightarrow \infty} \chi(u, y_1, T) e^{-\text{tr}(uy')} \left| \begin{pmatrix} u_1/y_1 & u_{12}/y_1^{1/2} \\ {}_t u_{12}/y_1^{1/2} & u_2 \end{pmatrix} + 2T \right|^{\alpha-\rho_n} \cdot |u|^{\beta-\rho_n} du \\ &= e^{-2T_{12}' y_{12} - \text{tr} T_2 y_2} \int_{\substack{\begin{pmatrix} 0 & 0 \\ 0 & u_2 \end{pmatrix} + 2T > 0 \\ u > 0}} e^{-u_1 - \text{tr} u_2 y_2} \left| \begin{pmatrix} 0 & 0 \\ 0 & u_2 \end{pmatrix} + 2T \right|^{\alpha-\rho_n} \cdot |u|^{\beta-\rho_n} du. \end{aligned}$$

If $T_1 \leq 0$, then the domain of integration is empty and the integral vanishes as claimed.

If $T_1 > 0$ and $n = 1$, then the remaining integral reduces to the Euler integral for the Gamma function, which implies the assertion in this case. If $T_1 > 1$ and $n > 1$, then we use Lemma 4.7 again (but now the other of the two formulas) to write

$$\begin{aligned} \left| \begin{pmatrix} 0 & 0 \\ 0 & u_2 \end{pmatrix} + 2T \right|^{\alpha - \rho_n} &= 2T_1 \cdot |u_2 + 2T_2 - {}^t(2T_{12})(2T_1)^{-1}(2T_{12})| \\ &= 2\tilde{T}_1 \cdot |u_2 + 2\tilde{T}_2|. \end{aligned}$$

Inserting this in the integral, we obtain

$$\begin{aligned} &\lim_{y_1 \rightarrow \infty} e^{T_1 y_1} y_1^\beta \cdot \eta^{(n)}(y, T, \alpha, \beta) \\ &= e^{-2T_{12} {}^t y_{12} - \text{tr } T_2 y_2} (2T_1)^{\alpha - \rho_n} \int_{\substack{u_2 + 2\tilde{T}_2 > 0 \\ u > 0}} e^{-u_1 - \text{tr } u_2 y_2} |u_2 + 2\tilde{T}_2|^{\alpha - \rho_n} \cdot |u|^{\beta - \rho_n} du \\ &= e^{-2T_{12} {}^t y_{12} - \text{tr } T_2 y_2} (2T_1)^{\alpha - \rho_n} \\ &\quad \times \int_{\substack{u_2 + 2\tilde{T}_2 > 0 \\ u_2 > 0 \\ u_1 - u_{12} u_2^{-1} {}^t u_{12} > 0}} e^{-u_1 - \text{tr } u_2 y_2} |u_2 + 2\tilde{T}_2|^{\alpha - \rho_n} \cdot |u_2|^{\beta - \rho_n} (u_1 - u_{12} u_2^{-1} {}^t u_{12})^{\beta - \rho_n} du. \end{aligned} \tag{4.8}$$

We carry out the u_1 -integration and employ Lemma 4.6 to get

$$\begin{aligned} &\lim_{y_1 \rightarrow \infty} e^{T_1 y_1} y_1^\beta \cdot \eta^{(n)}(y, T, \alpha, \beta) \\ &= e^{-2T_{12} {}^t y_{12} - \text{tr } T_2 y_2} (2T_1)^{\alpha - \rho_n} \Gamma(\beta - \rho_n + 1) \\ &\quad \times \int_{\substack{u_2 + 2\tilde{T}_2 > 0 \\ u_2 > 0}} e^{-\text{tr } u_2 y_2} \cdot |u_2 + 2\tilde{T}_2|^{\alpha - \rho_n} \cdot |u_2|^{\beta - \rho_n} \int_{u_{12} \in \mathbb{R}^{1 \times (n-1)}} e^{-u_{12} u_2^{-1} {}^t u_{12}} du_{12} du_2 \\ &= e^{-2T_{12} {}^t y_{12} - \text{tr } T_2 y_2} (2T_1)^{\alpha - \rho_n} \pi^{(n-1)/2} \Gamma(\beta - \rho_n + 1) \\ &\quad \times \int_{\substack{u_2 + 2\tilde{T}_2 > 0 \\ u_2 > 0}} e^{-\text{tr } u_2 y_2} \cdot |u_2 + 2\tilde{T}_2|^{\alpha - \rho_n} \cdot |u_2|^{\beta + 1/2 - \rho_n} du_2. \end{aligned}$$

Shifting the variable of integration, we get

$$\begin{aligned} &\lim_{y_1 \rightarrow \infty} e^{T_1 y_1} y_1^\beta \cdot \eta^{(n)}(y, T, \alpha, \beta) \\ &= \exp(-2T_{12} {}^t y_{12} - \text{tr } T_2 y_2 + \text{tr } \tilde{T}_2 y_2) (2T_1)^{\alpha - \rho_n} \pi^{(n-1)/2} \Gamma(\beta - \rho_n + 1) \\ &\quad \times \int_{\substack{u_2 + \tilde{T}_2 > 0 \\ u_2 - \tilde{T}_2 > 0}} e^{-\text{tr } u_2 y_2} \cdot |u_2 + \tilde{T}_2|^{\alpha - \rho_n - 1/2} \cdot |u_2 - \tilde{T}_2|^{\beta - \rho_n - 1} du_2. \end{aligned}$$

Since the latter integral is $\eta^{(n-1)}(y_2, \tilde{T}_2, \alpha - 1/2, \beta)$, we obtain the claimed formula.

Step 2. We now justify the interchange of the integral and the limit in (4.7). To this end we bound the integrand in

$$\int_{\substack{u_2+2T_2>0 \\ u>0}} \chi(u, y_1, T) e^{-\text{tr}(uy')} \left| \begin{pmatrix} u_1/y_1 & u_{12}/y_1^{1/2} \\ {}^t u_{12}/y_1^{1/2} & u_2 \end{pmatrix} + 2T \right|^{\alpha-\rho_n} \cdot |u|^{\beta-\rho_n} du \quad (4.9)$$

by an integrable function which is independent of y_1 . On the domain of integration and where $\chi(u, y, T)$ is non-zero, the quantity

$$\begin{aligned} & \left| \begin{pmatrix} u_1/y_1 & u_{12}/y_1^{1/2} \\ {}^t u_{12}/y_1^{1/2} & u_2 \end{pmatrix} + 2T \right| \\ &= |u_2 + 2T_2| \cdot \left(\frac{u_1}{y_1} + 2T_1 - \left(\frac{u_{12}}{\sqrt{y_1}} + 2T_{12} \right) (u_2 + 2T_2)^{-1} {}^t \left(\frac{u_{12}}{\sqrt{y_1}} + 2T_{12} \right) \right) \end{aligned}$$

is bounded by

$$|u_2 + 2T_2| \cdot (u_1 + 2|T_1|)$$

from above when $y_1 > 1$. It is bounded by 0 from below. Moreover, for $y_1 > 4y_{12}y_2^{-1} {}^t y_{12}$ we have

$$y' > \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & y_2 \end{pmatrix}$$

and therefore

$$e^{-\text{tr}(uy')} \leq e^{-u_1/2 - \text{tr}(u_2y_2)}.$$

Hence, for such y_1 the integrand in (4.9) is bounded by

$$e^{-u_1/2 - \text{tr}(u_2y_2)} (u_1 + 2|T_1|)^{\Re(\alpha) - \rho_n} |u_2 + 2T_2|^{\Re(\alpha) - \rho_n} \cdot |u|^{\Re(\beta) - \rho_n}$$

on the domain of integration. Here we have also used $\Re(\alpha) > \rho_n$. There exists a constant $C = C(\alpha) > 0$ such that this is bounded by

$$C \cdot e^{-u_1/4 - \text{tr}(u_2y_2)} |u_2 + 2T_2|^{\Re(\alpha) - \rho_n} \cdot |u|^{\Re(\beta) - \rho_n}$$

locally uniformly in α . Note that this function is independent of y_1 . Consequently, by the dominated convergence theorem, the interchange of the integral and the limit in (4.7) follows if

$$I(y_2, T_2, \alpha, \beta) = \int_{\substack{u_2+2T_2>0 \\ u>0}} e^{-u_1/4 - \text{tr}(u_2y_2)} |u_2 + 2T_2|^{\Re(\alpha) - \rho_n} \cdot |u|^{\Re(\beta) - \rho_n} du \quad (4.10)$$

converges. But this integral is of the same form as the one on the right hand side of (4.8). The same computation shows that the integral is a constant times

$$e^{\frac{1}{2} \text{tr}(y_2T_2)} \cdot \eta^{(n-1)} \left(\frac{1}{2} y_2, T_2, \Re(\alpha) - \frac{1}{2}, \Re(\beta) \right).$$

This shows the convergence for $\Re(\alpha) > \rho_n$ and $\Re(\beta) > \rho_n - 1/2$.

Step 3. We now show that the formulas of the theorem actually hold for all $\alpha, \beta \in \mathbb{C}$. If T is positive definite, we use the functional equation of η under $(\alpha, \beta) \mapsto (\rho_n - \beta, \rho_n - \alpha)$ and argue as in [Shi, p. 281]. For general T , we then apply the integral representation (4.24) in [Shi, p. 289] to deduce the assertion. \square

If $a \in \mathbb{R}^\times$, we let

$$d(a) = \begin{pmatrix} a & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \text{GL}_n(\mathbb{R}). \tag{4.11}$$

Theorem 4.8 implies the following asymptotic behavior of the Whittaker function.

Corollary 4.10. *Let $\kappa = \rho_n$. For general invertible T we have*

$$\begin{aligned} \lim_{a \rightarrow \infty} e^{2\pi(T_1 a^2 + T_1^{-1} T_{12}^t T_{12})} a^{-\rho_n} \cdot W_T(m(d(a)), s, \Phi_\kappa) \\ = \begin{cases} \frac{i^{-n} (2\pi)^{\rho_n}}{\Gamma(s/2 + \rho_n)} (\pi T_1)^{s/2} \cdot W_{\tilde{T}_2}(1, s, \Phi_{\kappa-1/2}) & \text{if } T_1 > 0, \\ 0 & \text{if } T_1 \leq 0. \end{cases} \end{aligned}$$

Here the Whittaker function on the left hand side is in genus n and the one on the right hand side is in genus $n - 1$ (which is clear from the size of the matrices in the subscripts).

Proof. Let $a > 0$. According to (4.4), we have

$$\begin{aligned} a^{-\rho_n} \cdot W_T(m(d(a)), s, \Phi_\kappa) &= c_n(\alpha, \beta) \cdot a^s \cdot \eta^{(n)}(2d(a)^2, \pi T, \alpha, \beta), \\ c_n(\alpha, \beta) &= \frac{i^{n(\beta-\alpha)} 2^{-n(\rho_n-1)/2} (2\pi)^{n\rho_n}}{\Gamma_n(\alpha)\Gamma_n(\beta)}. \end{aligned}$$

If $T_1 \leq 0$, by means of Theorem 4.8 we obtain

$$\lim_{a \rightarrow \infty} e^{2\pi(T_1 a^2 + T_1^{-1} T_{12}^t T_{12})} a^{-\rho_n} \cdot W_T(m(d(a)), s, \Phi_\kappa) = 0.$$

If $T_1 > 0$, we find

$$\begin{aligned} \lim_{a \rightarrow \infty} e^{2\pi(T_1 a^2 + T_1^{-1} T_{12}^t T_{12})} a^{-\rho_n} \cdot W_T(m(d(a)), s, \Phi_\kappa) \\ = c_n(\alpha, \beta) \Gamma(\beta + 1 - \rho_n) \pi^{(n-1)/2} (2\pi T_1)^{\alpha - \rho_n} 2^{-s/2} \eta^{(n-1)}(2, \pi \tilde{T}_2, \alpha - 1/2, \beta). \end{aligned}$$

Using (4.4) in genus $n - 1$, we see that

$$W_{\tilde{T}_2}(1, s, \Phi_{\kappa-1/2}) = c_{n-1}(s/2 + \rho_{n-1}, s/2) \cdot \eta^{(n-1)}(2, \pi \tilde{T}_2, \alpha - 1/2, \beta).$$

Inserting this we get

$$\begin{aligned} \lim_{a \rightarrow \infty} e^{2\pi(T_1 a^2 + T_1^{-1} T_{12}' T_{12})} a^{-\rho_n} \cdot W_T(m(d(a)), s, \Phi_\kappa) \\ = \frac{c_n(\alpha, \beta) \Gamma(\beta + 1 - \rho_n)}{c_{n-1}(s/2 + \rho_{n-1}, s/2)} \pi^{(n-1)/2} (\pi T_1)^{s/2} \cdot W_{\tilde{T}_2}(1, s, \Phi_{\kappa-1/2}). \end{aligned}$$

Employing the relations

$$\Gamma_{n-1}(\beta) \Gamma_1(\beta + 1 - \rho_n) = \pi^{-(n-1)/2} \Gamma_n(\beta), \tag{4.12}$$

$$\Gamma_{n-1}(\beta + \rho_{n-1}) \Gamma_1(\beta + \rho_n) = \pi^{-(n-1)/2} \Gamma_n(\beta + \rho_n), \tag{4.13}$$

we find

$$\frac{c_n(\alpha, \beta) \Gamma(\beta + 1 - \rho_n)}{c_{n-1}(s/2 + \rho_{n-1}, s/2)} = \frac{i^{-n} 2^{\rho_n} \pi}{\Gamma(s/2 + \rho_n)}.$$

Consequently,

$$\begin{aligned} \lim_{a \rightarrow \infty} e^{2\pi(T_1 a^2 + T_1^{-1} T_{12}' T_{12})} a^{-\rho_n} \cdot W_T(m(d(a)), s, \Phi_\kappa) \\ = \frac{i^{-n} (2\pi)^{\rho_n}}{\Gamma(s/2 + \rho_n)} (\pi T_1)^{s/2} \cdot W_{\tilde{T}_2}(1, s, \Phi_{\kappa-1/2}). \end{aligned}$$

This concludes the proof of the corollary. □

Corollary 4.11. *Let $\kappa = \rho_n$. If T has signature $(n - j, j)$ with $j \geq 1$, then*

$$\begin{aligned} \lim_{a \rightarrow \infty} e^{2\pi(T_1 a^2 + T_1^{-1} T_{12}' T_{12})} a^{-\rho_n} \cdot W'_T(m(d(a)), 0, \Phi_\kappa) \\ = \begin{cases} \frac{i^{-n} (2\pi)^{\rho_n}}{\Gamma(\rho_n)} \cdot W'_{\tilde{T}_2}(1, 0, \Phi_{\kappa-1/2}) & \text{if } T_1 > 0, \\ 0 & \text{if } T_1 \leq 0. \end{cases} \end{aligned}$$

5. The archimedean arithmetic Siegel–Weil formula

Here we use the archimedean local Siegel–Weil formula (see Proposition 3.1), the asymptotic behavior of Theorem 4.8, and some computations in the induced representation to prove Theorem 1.4.

We use the same set-up and notation as in Section 3. In particular, V is a quadratic space over \mathbb{R} of signature $(m, 2)$, and $\kappa = (m + 2)/2$. Moreover, \mathcal{D} is the hermitian domain associated with $H = \text{SO}(V)$, realized as the Grassmannian of oriented negative 2-planes in V .

5.1. Green currents and local heights

For $z \in \mathcal{D}$ the orthogonal complement z^\perp is positive definite of dimension m . If $x \in V$, we denote the orthogonal projection of x to z and z^\perp by x_z and x_{z^\perp} , respectively. The quadratic form

$$(x, x)_z = (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z)$$

is positive definite on V and called the *majorant* associated with z . We also put

$$R(x, z) = -(x_z, x_z)$$

so that $(x, x)_z = (x, x) + 2R(x, z)$. For $0 \neq x \in V$, we define

$$\mathcal{D}_x = \{z \in \mathcal{D} \mid z \perp x\} = \{z \in \mathcal{D} \mid R(x, z) = 0\}.$$

Then \mathcal{D}_x is a non-trivial analytic divisor of \mathcal{D} if $Q(x) > 0$, and it is empty if $Q(x) \leq 0$ (which we will view as the zero divisor). Following [Ku2] we define the Kudla Green function

$$\xi(x, z) = -\text{Ei}(-2\pi R(x, z)) \cdot e^{-\pi(x,x)}, \tag{5.1}$$

where $\text{Ei}(u) = \int_{-\infty}^u e^t \frac{dt}{t}$ is the exponential integral [AS, Chapter 5]. If $x \in V$ is fixed, then $\xi(x, z)$ is a smooth function on $\mathcal{D} \setminus \mathcal{D}_x$ with a logarithmic singularity along \mathcal{D}_x . It has the equivariance property $\xi(gx, gz) = \xi(x, z)$ for $g \in H(\mathbb{R})$. The differential form

$$\varphi_{KM}(x, z) = dd^c \xi(x, z) \tag{5.2}$$

extends to a smooth $(1, 1)$ -form on all of \mathcal{D} , where $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$. It is the Kudla–Millson Schwartz form which is Poincaré dual to the cycle $e^{-\pi(x,x)}\mathcal{D}_x$ [KM1], [Ku2]. More precisely, as currents on \mathcal{D} , we have the identity

$$dd^c[\xi(x)] + e^{-\pi(x,x)}\delta_{\mathcal{D}_x} = [\varphi_{KM}(x)].$$

Because of the $H(\mathbb{R})$ -equivariance of $\xi(x, z)$, the $(1, 1)$ -form

$$\Omega = \varphi_{KM}(0, z)$$

is $H(\mathbb{R})$ -invariant on \mathcal{D} . In fact, it is equal to the invariant differential form defined earlier in (3.2).

For $x = (x_1, \dots, x_n) \in V^n$ and $z \in \mathcal{D}$ we also define the Kudla–Millson Schwartz form in genus n as

$$\varphi_{KM}^n(x, z) = \varphi_{KM}(x_1, z) \wedge \dots \wedge \varphi_{KM}(x_n, z).$$

With respect to the action of G through the Weil representation it transforms under the maximal compact subgroup K_G with the character $\det(\mathbf{k})^k$. If $Q(x) \in \text{Sym}_n(\mathbb{R})$ is invertible, the form $\varphi_{KM}^n(x, z)$ is Poincaré dual to the special cycle $e^{-\pi \text{tr}(x,x)}\mathcal{D}_x$, where

$$\mathcal{D}_x = \{z \in \mathcal{D} \mid z \perp x_i \text{ for } i = 1, \dots, n\}$$

(see [KM1]). We define a Green current for the cycle $e^{-\pi \operatorname{tr}(x,x)} \mathcal{D}_x$ by taking the star product

$$\xi^n(x, z) = \xi(x_1, z) * \cdots * \xi(x_n, z) \tag{5.3}$$

in the sense of [GiSo]. As a current on compactly supported smooth differential forms it satisfies the equation

$$dd^c[\xi^n(x)] + e^{-\pi \operatorname{tr}(x,x)} \cdot \delta_{\mathcal{D}_x} = [\varphi_{KM}^n(x)]. \tag{5.4}$$

When \mathcal{D}_x is compact, it follows from the growth estimates in [KM1, Section 6] that $\xi^n(x, z)$ is rapidly decaying and extends to a current on forms of moderate growth with (5.4) still holding. A recursive formula for the star product is given by

$$\xi^n(x, z) = \xi(x_1, z) \wedge \varphi_{KM}^{n-1}((x_2, \dots, x_n), z) + e^{-\pi(x_1, x_1)} \delta_{\mathcal{D}_{x_1}} \wedge \xi^{n-1}((x_2, \dots, x_n), z), \tag{5.5}$$

where φ_{KM}^0 has to be interpreted as 1 and ξ^0 as 0. The current $\xi^n(x, z)$ is invariant under permutations of the components of x .

Sometimes it is convenient to put

$$\xi_0^n(x, z) = \xi^n(x, z) \cdot e^{\pi \operatorname{tr}(x,x)}, \quad \varphi_{KM,0}^n(x, z) = \varphi_{KM}^n(x, z) \cdot e^{\pi \operatorname{tr}(x,x)}.$$

Then the current equation becomes

$$dd^c[\xi_0^n(x)] + \delta_{\mathcal{D}_x} = [\varphi_{KM,0}^n(x)].$$

Note that the current equation (5.4) together with Proposition 4.3 implies the following geometric local Siegel–Weil formula, which is the local archimedean version of (1.1).

Proposition 5.1. *Assume that $n = m$ and $T = Q(x)$ is invertible. Then*

$$\int_{\mathcal{D}} \varphi_{KM}^n(x, z) = \begin{cases} 2e^{-2\pi \operatorname{tr} T} & \text{if } T \text{ is positive definite,} \\ 0 & \text{if } T \text{ is not positive definite.} \end{cases}$$

Moreover, in both cases this is equal to

$$2^{1-\frac{n}{2}(\kappa+3/2)} \pi^{-n\kappa} i^{n\kappa} \Gamma_n(\kappa) |T|^{-1/2} W_T(1, 1/2, \Phi_\kappa),$$

where $\Phi_\kappa \in I_n(s, \chi_V)$ is the weight κ standard section, that is, the unique standard section whose restriction to K_G is the character $\det(\mathbf{k})^\kappa$.

Proof. The first statement is a direct consequence of the current equation (5.4) applied to the constant function 1. The second statement can be deduced from the first one by means of the formulas of [Shi]. Since we do not need it here, we omit the proof. □

Throughout the rest of this subsection we assume that $n = m + 1$. Then $\xi_0^n(x, z)$ is a top degree current, which can be evaluated at the constant function 1. For $x \in V^n$ we define the archimedean local height function by

$$\text{ht}_\infty(x) = \frac{1}{2} \int_{\mathcal{D}} \xi_0^n(x, z). \tag{5.6}$$

In this section we prove the archimedean arithmetic local Siegel–Weil formula, Theorem 1.4, relating $\text{ht}_\infty(x)$ to the derivative of a Whittaker function in genus n . It can be viewed as an arithmetic analogue of Proposition 5.1. We restate the theorem for convenience.

Theorem 5.2. *Let $x \in V^n(\mathbb{R})$ be such that the moment matrix $T = Q(x)$ is invertible. Then*

$$\text{ht}_\infty(xv^{1/2}) \cdot q^T = -B_{n,\infty} \det(v)^{-\kappa/2} \cdot W'_T(g_\tau, 0, \Phi_\kappa), \tag{5.7}$$

where $B_{n,\infty}$ is the constant in Proposition 3.1, and $\Phi_\kappa \in I_n(s, \chi_V)$ is the weight κ standard section, that is, the unique standard section whose restriction to K_G is the character $\det(\mathbf{k})^\kappa$. The derivative of the Whittaker function is taken with respect to s .

Let $x \in V^n$ and assume that $T = Q(x)$ is invertible. To prove Theorem 5.2 we employ the recursive formula (5.5) for the star product. It implies that

$$\text{ht}_\infty(x) = \text{ht}_\infty^{\text{main}}(x) + \frac{1}{2} \int_{\mathcal{D}_{x_1}} \xi_0^{n-1}((x_2, \dots, x_n), z), \tag{5.8}$$

where we write

$$\text{ht}_\infty^{\text{main}}(x) = \frac{1}{2} \int_{\mathcal{D}} \xi_0(x_1, z) \wedge \varphi_{KM,0}^{n-1}((x_2, \dots, x_n), z) \tag{5.9}$$

for the main term of the local height function. The second summand on the right hand side of (5.8) vanishes when $Q(x_1) \leq 0$, in which case \mathcal{D}_{x_1} is empty. When $Q(x_1) > 0$, this quantity is a local height function in genus $n - 1$ for the quadratic space $V_1 = x_1^\perp \subset V$ of signature $(m - 1, 2)$. The divisor \mathcal{D}_{x_1} is naturally isomorphic to the Grassmannian of V_1 . Let

$$\text{pr} : V \rightarrow V_1, \quad \text{pr}(y) = y - \frac{(y, x_1)}{(x_1, x_1)} x_1,$$

be the orthogonal projection and put $\tilde{x} = (\text{pr}(x_2), \dots, \text{pr}(x_n)) \in V_1^{n-1}$. If we write T in block form as in (4.5) then the moment matrix of \tilde{x} is

$$\tilde{T}_2 = Q(\tilde{x}) = T_2 - {}^t T_{12} T_1^{-1} T_{12}. \tag{5.10}$$

Lemma 5.3. *Assume the above notation. If $z \in \mathcal{D}_{x_1}$, then*

$$\xi_0^{n-1}((x_2, \dots, x_n), z) = \xi_0^{n-1}(\tilde{x}, z)$$

and

$$\frac{1}{2} \int_{\mathcal{D}_{x_1}} \xi_0^{n-1}((x_2, \dots, x_n), z) = \text{ht}_\infty(\tilde{x}).$$

Here the height function on the right hand side is for the tuple $\tilde{x} \in V_1^{n-1}$.

Proof. If $z \in \mathcal{D}_{x_1}$ and $y \in V$, then $R(y, z) = R(\text{pr}(y), z)$. Hence the assertion is a direct consequence of the definition of $\xi_0(y, z)$. □

The following result gives a formula for the main term of the local height function.

Theorem 5.4. *Assume that $n = m + 1$. Let $x \in V^n$ be such that $T = Q(x)$ is invertible, and put $\tilde{T}_2 = Q(\tilde{x})$ as in (5.10). Then*

$$e^{-2\pi \text{tr} T} \cdot \text{ht}_\infty^{\text{main}}(x) = -B_{n,\infty} \cdot W'_T(1, 0, \Phi_\kappa) + B_{n-1,\infty} e^{-2\pi(\text{tr} T - \text{tr} \tilde{T}_2)} \cdot W'_{\tilde{T}_2}(1, 0, \Phi_{\kappa-1/2}).$$

If $n = 1$, the second summand on the right hand side is interpreted as 0.

The proof of Theorem 5.4 will be given in the next three subsections.

Proof that Theorem 5.4 implies Theorem 5.2. Recall that for $\tau \in \mathbb{H}_n$ we have put $g_\tau = \begin{pmatrix} 1 & u \\ & a \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in \text{Sp}_n(\mathbb{R})$ with $a \in \text{GL}_n^+(\mathbb{R})$ and $a^t a = v$. Using the transformation behavior (2.2) of the Whittaker function, we find that

$$\det(v)^{-\kappa/2} W_T(g_\tau, s, \Phi_\kappa) = e^{2\pi i \text{tr}(Tu)} |a|^{-s} \cdot W_{i_a T a}(1, s, \Phi_\kappa). \tag{5.11}$$

Since the signature of V is $(m, 2)$ and $n = m + 1$, the matrix T must have negative eigenvalues. By Proposition 4.3, the Whittaker function $W_T(g, s, \Phi_\kappa)$ vanishes at $s = 0$. Employing (5.11), we see that (5.7) is equivalent to

$$\text{ht}_\infty(x a) \cdot e^{-2\pi \text{tr} {}^t a T a} = -B_{n,\infty} \cdot W'_{i_a T a}(1, 0, \Phi_\kappa).$$

Consequently, it suffices to prove (5.7) for $\tau = i 1_n$, that is

$$\text{ht}_\infty(x) \cdot e^{-2\pi \text{tr} T} = -B_{n,\infty} \cdot W'_T(1, 0, \Phi_\kappa). \tag{5.12}$$

We show (5.12) by induction on n . If $n = 1$, then $\text{ht}_\infty^{\text{main}}(x) = \text{ht}_\infty(x)$, and we have nothing to show. Assume now that $n > 1$. According to (5.8) and Lemma 5.3 we have

$$\text{ht}_\infty(x) = \text{ht}_\infty^{\text{main}}(x) + \text{ht}_\infty(\tilde{x}).$$

By Theorem 5.4, we obtain

$$e^{-2\pi \text{tr} T} \cdot \text{ht}_\infty(x) = -B_{n,\infty} \cdot W'_T(1, 0, \Phi_\kappa) + B_{n-1,\infty} e^{-2\pi(\text{tr} T - \text{tr} \tilde{T}_2)} \cdot W'_{\tilde{T}_2}(1, 0, \Phi_{\kappa-1/2}) + e^{-2\pi \text{tr} T} \cdot \text{ht}_\infty(\tilde{x}).$$

If we use Theorem 5.2 in genus $n - 1$ to compute the last term on the right hand side, we get the assertion. □

5.2. The main term of the local height

In this subsection we assume again that $n = m + 1$. We give a first formula for the main term of the local height in terms of a certain Whittaker function. We begin by rewriting the Green function $\xi(x, z)$ defined in (5.1) in terms of the Gaussian

$$\varphi_G(x, z) = e^{-\pi(x,x)z} \in S(V), \tag{5.13}$$

$$\varphi_{G,0}(x, z) = \varphi_G(x, z) \cdot e^{\pi(x,x)} = e^{-2\pi R(x,z)}. \tag{5.14}$$

Lemma 5.5. *If $x \in V$ and $z \in \mathcal{D}$, then*

$$\xi_0(x, z) = \int_{t=1}^{\infty} \varphi_{G,0}(\sqrt{t}x, z) \frac{dt}{t}.$$

Proof. The statement follows from the integral representation

$$-\text{Ei}(-z) = \int_1^{\infty} \frac{e^{-zt}}{t} dt$$

by inserting the definitions of $\xi_0(x, z)$ and $R(x, z)$. □

By our assumption on m , the Schwartz form φ_{KM}^{n-1} is a top degree differential form on \mathcal{D} . We write it as

$$\varphi_{KM}^{n-1}(y, z) = \varphi_{KM}^{n-1,*}(y, z) \cdot \Omega^{n-1}.$$

For $x \in V^n$, we define a Schwartz function¹ by

$$\psi_{11}^*(x, z) = \varphi_G(x_1, z) \cdot \varphi_{KM}^{n-1,*}((x_2, \dots, x_n), z). \tag{5.15}$$

Proposition 5.6. *Let $x \in V^n$, put $T = Q(x)$, and write $T = \begin{pmatrix} T_1 & T_{12} \\ T_{12} & T_2 \end{pmatrix}$ as in (4.5). The main term of the local height function is given by*

$$\text{ht}_{\infty}^{\text{main}}(x) = B_{n,\infty} e^{2\pi \text{tr} T_2} \int_1^{\infty} W_{d(a)T d(a)}(1, 0, \lambda(\psi_{11}^*)) \cdot e^{2\pi Q(x_1 a)} \frac{da}{a},$$

where $B_{n,\infty}$ denotes the constant in Proposition 3.1 and $d(a)$ is given by (4.11).

Proof. Using Lemma 5.5 and (5.15), we see that

$$\begin{aligned} \text{ht}_{\infty}^{\text{main}}(x) &= \frac{1}{2} \int_{\mathcal{D}} \xi_0(x_1, z) \wedge \varphi_{KM,0}^{n-1}((x_2, \dots, x_n), z) \\ &= \frac{1}{2} \int_{t=1}^{\infty} \int_{\mathcal{D}} \varphi_{G,0}(\sqrt{t}x_1, z) \varphi_{KM,0}^{n-1}((x_2, \dots, x_n), z) \frac{dt}{t} \\ &= e^{2\pi \text{tr} T_2} \cdot \int_{a=1}^{\infty} \int_{\mathcal{D}} \psi_{11}^*(x d(a), z) \Omega^{n-1} \cdot e^{2\pi Q(x_1 a)} \frac{da}{a}. \end{aligned}$$

By the local Siegel–Weil formula, Proposition 3.1, we have

$$\int_{\mathcal{D}} \psi_{11}^*(x, z) \Omega^{n-1} = B_{n,\infty} \cdot W_T(1, 0, \lambda(\psi_{11}^*)).$$

Inserting this, we obtain the assertion. □

¹ Later we will also define a $\text{Sym}_n(\mathbb{R})$ -valued Schwartz function ψ whose $(1, 1)$ -component will be ψ_{11} .

5.3. Some Lie algebra computations

In this subsection, we temporarily drop the assumption that $n = m + 1$. We compute the Whittaker function $W_T(1, s, \lambda(\psi_{11}^*))$ more explicitly. We begin by recalling from [BFK1, Section 5] some facts about the Lie algebra of G . Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_+ + \mathfrak{p}_-$$

be the Harish-Chandra decomposition of $\mathfrak{g} = \text{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$. Let $S = \text{Sym}_n(\mathbb{R})$. Then there are isomorphisms

$$p_{\pm} : S_{\mathbb{C}} \rightarrow \mathfrak{p}_{\pm}, \quad X \mapsto p_{\pm}(X) = \frac{1}{2} \begin{pmatrix} X & \pm iX \\ \pm iX & -X \end{pmatrix}. \tag{5.16}$$

The group K_G acts on \mathfrak{g} by the adjoint representation, $\text{Ad}(k)g = kgk^{-1}$, and on $S_{\mathbb{C}}$ by $\mathbf{k}.X = \mathbf{k}X^t\mathbf{k}$ for $k \in K_G$. For the isomorphism (5.16) we have

$$\text{Ad}(k)p_+(X) = p_+(\mathbf{k}.X), \tag{5.17}$$

$$\text{Ad}(k)p_-(X) = p_-(\bar{\mathbf{k}}.X). \tag{5.18}$$

The trace pairing

$$\langle p_+(X), p_-(Y) \rangle = \text{tr}(XY)$$

is invariant under the action of K_G , and therefore $\mathfrak{p}_{\pm}^* \cong \mathfrak{p}_{\mp}$ as K_G -modules. Let (e_{α}) be a basis of S , and write (e_{α}^{\vee}) for the dual basis with respect to the trace form. Then $(p_-(e_{\alpha}^{\vee}))$ is a basis of \mathfrak{p}_- , and we write (η'_{α}) for the dual basis of \mathfrak{p}_-^* . We identify \mathfrak{p}_-^* with $S_{\mathbb{C}}$ by the map

$$\psi = \sum_{\alpha} \psi_{\alpha} \eta'_{\alpha} \mapsto \sum_{\alpha} \psi_{\alpha} e_{\alpha}. \tag{5.19}$$

Recall that the Lie algebra $\mathfrak{gl}_n(\mathbb{C}) \cong \text{Mat}_n(\mathbb{C})$ is isomorphic to \mathfrak{k} via the map

$$k : \text{Mat}_n(\mathbb{C}) \rightarrow \mathfrak{k}, \quad Y \mapsto k(Y) = \frac{1}{2} \begin{pmatrix} Y - {}^tY & -i(Y + {}^tY) \\ i(Y + {}^tY) & Y - {}^tY \end{pmatrix}. \tag{5.20}$$

Let $E_{jk} \in \text{Mat}_n(\mathbb{C})$ be the elementary matrix having 1 at position (j, k) and all other entries 0. Then the matrices

$$Y_{jk} = k(E_{jk}) = \frac{1}{2} \begin{pmatrix} E_{jk} - E_{kj} & -i(E_{jk} + E_{kj}) \\ i(E_{jk} + E_{kj}) & E_{jk} - E_{kj} \end{pmatrix}, \tag{5.21}$$

for $1 \leq j, k \leq n$, form a basis of \mathfrak{k} .

We denote by $\mathbb{C}(\ell)$ the K_G -module given by the action of K_G on \mathbb{C} by multiplication with $\det(\mathbf{k})^{\ell}$. Recall that the space $A^{p,q}(\mathbb{H}_n)$ of differential forms on \mathbb{H}_n can be described by the isomorphism

$$A^{p,q}(\mathbb{H}_n) \rightarrow [C^{\infty}(G) \otimes \wedge^p(\mathfrak{p}_+^*) \otimes \wedge^q(\mathfrak{p}_-^*)]^{K_G}.$$

Here, the operator corresponding to $\bar{\partial}$ on $A^{p,q}(\mathbb{H}_n)$ is given by

$$D = \sum_{\alpha} p_{-}(e_{\alpha}^{\vee}) \otimes \eta'_{\alpha}, \tag{5.22}$$

where η'_{α} acts on $\wedge^{\cdot}(\mathfrak{p}^*)$ by exterior multiplication.

The following result, which describes the action of K_G on ψ_{11} , is taken from the unpublished manuscript [BFK2]. We thank Jens Funke and Steve Kudla for allowing us to include it here.

Proposition 5.7. *Identify \mathfrak{p}^* with $S_{\mathbb{C}}$ as in (5.19). There is a Schwartz form*

$$\psi \in [S(V^n) \otimes A^{n-1,n-1}(\mathcal{D}) \otimes \mathfrak{p}^* \otimes \mathbb{C}(-\kappa)]^{K_G}$$

with diagonal components

$$\psi_{rr}(x) = \varphi_G(x_r) \cdot \varphi_{KM}(x_1) \wedge \cdots \wedge \widehat{\varphi_{KM}(x_r)} \wedge \cdots \wedge \varphi_{KM}(x_n),$$

which satisfies $\psi(0) = \Omega^{n-1} \cdot 1_n$ and

$$\omega(k)\psi(x) = \det(\mathbf{k})^{\kappa} \cdot \mathbf{k}^{-1}\psi(x) {}^t\mathbf{k}^{-1} \quad \text{for } k \in K_G.$$

Proof. To prove this result we use the Fock model realization of the Weil representation as described in the appendix of [FM1]. Let $\mathcal{F} = \mathcal{F}(\mathbb{C}^{(m+2) \times n})$ be the space of polynomial functions on $V_{\mathbb{C}}^n \cong \mathbb{C}^{(m+2) \times n}$. As in [FM1] we denote the variables by $z_{\alpha j}$, $z_{\mu j}$, where $\alpha = 1, \dots, m$, $\mu = m + 1, m + 2$, and $j = 1, \dots, n$. The Lie algebra $\mathfrak{g} \times \mathfrak{so}(V)_{\mathbb{C}}$ acts on \mathcal{F} via the Weil representation.

Let $\mathfrak{so}(V)_{\mathbb{C}} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ be the Cartan decomposition as in Section 3.1. Let $X_{\alpha\mu}$ be the standard basis of \mathfrak{p}_H and denote by $\omega_{\alpha\mu}$ the corresponding dual basis of \mathfrak{p}_H^* . The Kudla–Millson Schwartz forms can be viewed as elements of

$$[\mathcal{F} \otimes \wedge^{\cdot} \mathfrak{p}_H^*]^{K_H}.$$

We have

$$\varphi_{KM}^n = \varphi_{KM,(1)} \wedge \cdots \wedge \varphi_{KM,(n)},$$

where

$$\varphi_{KM,(j)} = -\frac{1}{8\pi^2} \cdot \sum_{\alpha,\beta=1}^m z_{\alpha j} z_{\beta j} \otimes \omega_{\alpha,m+1} \wedge \omega_{\beta,m+2}.$$

The Gaussian $\varphi_{G,(j)}$ corresponds to the constant polynomial 1 for every j .

We define the Schwartz form $\psi = (\psi_{jk})$ in the Fock model by putting

$$\begin{aligned} \psi_{kk} &= \varphi_{G,(k)} \wedge \prod_{l \neq k} \varphi_{KM,(l)}, \\ \psi_{jk} &= -\frac{1}{8\pi^2} \left(-\frac{1}{2} \sum_{\alpha,\beta=1}^m (z_{\alpha j} z_{\beta k} + z_{\alpha k} z_{\beta j}) \otimes \omega_{\alpha,m+1} \wedge \omega_{\beta,m+2} \right) \\ &\quad \wedge \prod_{l \neq j,k} \varphi_{KM,(l)} \quad \text{for } j \neq k. \end{aligned}$$

This has the desired diagonal components. Using the intertwining operator between the Schrödinger and the Fock model of the Weil representation, it is easily checked that $\psi_{jk}(0) = 0$ for $j \neq k$. On the other hand, by (3.4), we have $\psi_{jj}(0) = \Omega^{n-1}$, and therefore $\psi(0) = \Omega^{n-1} \cdot 1_n$.

To verify the transformation law under K_G , we compute the action of the Lie algebra \mathfrak{k} under the Weil representation. Recall that the basis element Y_{jk} defined in (5.21) acts by

$$\omega(Y_{jk}) = \frac{1}{2}(m-2)\delta_{jk} + \sum_{\alpha=1}^m z_{\alpha j} \frac{\partial}{\partial z_{\alpha k}} - \sum_{\mu=m+1}^{m+2} z_{\mu k} \frac{\partial}{\partial z_{\mu j}}.$$

In fact, since the element Y_{jk} corresponds to $\frac{1}{2i}(w'_k \circ w''_j)$ in the notation of [FM1], this claim follows from [FM1, Lemma A.1].

Now a direct computation shows

$$\begin{aligned} \omega(Y_{jj})\psi_{ll} &= \begin{cases} (\kappa - 2) \cdot \psi_{ll} & \text{if } j = l, \\ \kappa \cdot \psi_{ll} & \text{if } j \neq l, \end{cases} \\ \omega(Y_{jk})\psi_{ll} &= \begin{cases} -2\psi_{kl} & \text{if } j = l \text{ and } j \neq k, \\ 0 & \text{if } j \neq l \text{ and } j \neq k, \end{cases} \\ \omega(Y_{jk})\psi_{jk} &= -\psi_{kk} \quad \text{if } j \neq k. \end{aligned}$$

This implies that the ψ_{jk} generate an irreducible representation of K_G , which has ψ_{nn} as a highest weight vector, and which is isomorphic to $\det^\kappa \otimes \text{Sym}^2(\mathbb{C}^n)^\vee$. Hence, we obtain the claimed transformation law. □

The intertwining operator $\lambda : S(V^n) \rightarrow I(s_0, \chi_V)$ (see (2.3)) induces a map

$$\begin{aligned} [S(V^n) \otimes A^{n-1, n-1}(\mathcal{D}) \otimes \mathfrak{p}_-^* \otimes \mathbb{C}(-\kappa)]^{K_G} \\ \rightarrow [I(s_0, \chi_V) \otimes A^{n-1, n-1}(\mathcal{D}) \otimes \mathfrak{p}_-^* \otimes \mathbb{C}(-\kappa)]^{K_G}, \end{aligned}$$

which we also denote by λ . We define $\Psi \in [I(s_0, \chi_V) \otimes \mathfrak{p}_-^* \otimes \mathbb{C}(-\kappa)]^{K_G}$ by

$$\Psi \cdot \Omega^{n-1} = \lambda(\psi), \tag{5.23}$$

and write $\Psi(g, s)$ for the corresponding extension to a standard section.

Corollary 5.8. *Identify \mathfrak{p}_-^* with $S_{\mathbb{C}}$ as in (5.19). For $k \in K_G$ and $g \in G$ we have*

$$\Psi(gk, s) = \det(\mathbf{k})^\kappa \cdot \mathbf{k}^{-1} \Psi(g, s) {}^t \mathbf{k}^{-1}.$$

Moreover, $\Psi(1, s) = 1_n$.

This corollary characterizes Ψ uniquely. We now use the action of \mathfrak{p}_- in the induced representation to find a different expression for Ψ .

Proposition 5.9. *Let D be the operator defined in (5.22), and let*

$$r(D) : I(s, \chi_V) \rightarrow I(s, \chi_V) \otimes \mathfrak{p}^*$$

be the induced operator on the induced representation. Then

$$r(D)\Phi_\kappa(g, s) = \frac{1}{2}(s + \rho_n - \kappa)\Psi(g, s).$$

Proof. We first show that $r(D)\Phi_\kappa(g, s)$ has the same K_G -type as Ψ . Via the isomorphism (5.19), the operator D induces an operator

$$\tilde{D} : I(s, \chi_V) \rightarrow I(s, \chi_V) \otimes S_{\mathbb{C}}, \quad \tilde{D} = \sum_{\alpha} p_{-}(e_{\alpha}^{\vee}) \otimes e_{\alpha}.$$

It satisfies

$$\text{Ad}(k)\tilde{D} = \mathbf{k}^{-1} \cdot \tilde{D}$$

for $k \in K_G$, where the action on the left hand side is on the first factor of the tensor product and the action on the right hand side on the second factor. In fact, we have

$$\begin{aligned} \text{Ad}(k)\tilde{D} &= \sum_{\alpha} p_{-}(\bar{\mathbf{k}} \cdot e_{\alpha}^{\vee}) \otimes e_{\alpha} = \sum_{\alpha} \sum_{\beta} \text{tr}(\bar{\mathbf{k}} \cdot e_{\alpha}^{\vee} e_{\beta}) \cdot p_{-}(e_{\beta}^{\vee}) \otimes e_{\alpha} \\ &= \sum_{\beta} p_{-}(e_{\beta}^{\vee}) \otimes \sum_{\alpha} \text{tr}(e_{\alpha}^{\vee} ({}^t \bar{\mathbf{k}} \cdot e_{\beta})) \cdot e_{\alpha} = \sum_{\beta} p_{-}(e_{\beta}^{\vee}) \otimes \mathbf{k}^{-1} \cdot e_{\beta} \end{aligned}$$

(see also [BFK1, Lemma 5.1]). But this implies, again using the identification (5.19), that $r(D)\Phi_\kappa$ has the transformation law

$$r(k)r(D)\Phi_\kappa(g, s) = r(\text{Ad}(k)D)r(k)\Phi_\kappa(g, s) = \det(\mathbf{k})^\kappa \cdot \mathbf{k}^{-1}\Phi_\kappa(g, s) {}^t \mathbf{k}^{-1}.$$

In other words, it has the same K_G -type as Ψ .

Since the different K_G -types in $I(s, \chi_V)$ have multiplicity 1, there exists a constant $c(s)$ such that

$$r(D)\Phi_\kappa(g, s) = c(s)\Psi(g, s). \tag{5.24}$$

To determine the constant, we evaluate at the unit element. According to Corollary 5.8, we have $\Psi(1, s) = 1$. We now consider $r(D)\Phi_\kappa$. For $X \in S_{\mathbb{C}}$ we compute $r(p_{-}(X))\Phi_\kappa$. In the Lie algebra \mathfrak{g} we write

$$p_{-}(X) = \frac{1}{2} \begin{pmatrix} X & -iX \\ -iX & -X \end{pmatrix} = \frac{1}{2} \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix} - i \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}. \tag{5.25}$$

We compute the actions of the three summands individually. We have

$$\begin{aligned} \frac{1}{2} dr \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \Phi_\kappa(1, s) &= \frac{1}{2} \frac{d}{dt} \Phi_\kappa(m(e^{tX}), s) \Big|_{t=0} \\ &= \frac{1}{2} \frac{d}{dt} \det(e^{tX})^{s+\rho_n} \Phi_\kappa(1, s) \Big|_{t=0} \\ &= \frac{1}{2}(s + \rho_n) \text{tr } X. \end{aligned}$$

Next, we compute, using the action of K_G ,

$$\begin{aligned} \frac{i}{2}dr\begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}\Phi_\kappa(1, s) &= \frac{i}{2}\frac{d}{dt}\Phi_\kappa\left(\exp t\begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, s\right)\Big|_{t=0} \\ &= \frac{i}{2}\frac{d}{dt}\Phi_\kappa\left(\begin{pmatrix} \cos(tX) & \sin(tX) \\ -\sin(tX) & \cos(tX) \end{pmatrix}, s\right)\Big|_{t=0} \\ &= \frac{i}{2}\frac{d}{dt}\det(e^{itX})^\kappa\Phi_\kappa(1, s)\Big|_{t=0} = -\frac{1}{2}\kappa\operatorname{tr}(X). \end{aligned}$$

Finally, we notice that

$$\begin{aligned} -idr\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\Phi_\kappa(1, s) &= -i\frac{d}{dt}\Phi_\kappa\left(\exp t\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, s\right)\Big|_{t=0} = -i\frac{d}{dt}\Phi_\kappa(n(tX), s)\Big|_{t=0} \\ &= 0. \end{aligned}$$

Putting the terms together, we obtain

$$r(p_-(X))\Phi_\kappa(1, s) = \frac{1}{2}(s + \rho_n - \kappa)\operatorname{tr} X, \quad r(D)\Phi_\kappa(g, s) = \frac{1}{2}(s + \rho_n - \kappa) \cdot 1_n.$$

This shows that the constant $c(s)$ in (5.24) is equal to $\frac{1}{2}(s + \rho_n - \kappa)$. □

Corollary 5.10. *Let $e_{11} \in S$ be the matrix whose upper left entry is 1 and whose other entries are all 0. Then*

$$\begin{aligned} r(p_-(e_{11}))\Phi_\kappa(g, s) &= \frac{1}{2}(s + \rho_n - \kappa)\lambda(\psi_{11}^*)(g, s), \\ r(p_-(e_{11}))W_T(g, s, \Phi_\kappa) &= \frac{1}{2}(s + \rho_n - \kappa)W_T(g, s, \lambda(\psi_{11}^*)). \end{aligned}$$

Proof. The first equality is a direct consequence of Proposition 5.9. It implies the second equality, since the Whittaker integral is an intertwining map of (\mathfrak{g}, K) -modules. □

Proposition 5.11. *For $a \in \operatorname{GL}_n(\mathbb{R})$ we have*

$$\begin{aligned} r(p_-(e_{11}))W_T(m(a), s, \Phi_\kappa) &= \left(2\pi\operatorname{tr}(Tae_{11}{}^t a) - \frac{\kappa}{2} + \frac{1}{2}\sum_{i=1}^n a_{i1} \cdot \frac{\partial}{\partial a_{i1}}\right)W_T(m(a), s, \Phi_\kappa). \end{aligned}$$

Proof. For the proof we put $X = e_{11}$ and split $p_-(X)$ as in (5.25). We compute the action of the three terms individually. We have

$$\begin{aligned} \frac{1}{2}dr\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}W_T(m(a), s, \Phi_\kappa) &= \frac{1}{2}\frac{d}{dt}W_T(m(a)m(e^{tX}), s, \Phi_\kappa)\Big|_{t=0} \\ &= \frac{1}{2}\frac{d}{dt}W_T(m(a + taX), s, \Phi_\kappa)\Big|_{t=0} \\ &= \frac{1}{2}\sum_{i=1}^n a_{i1} \cdot \frac{\partial}{\partial a_{i1}}W_T(m(a), s, \Phi_\kappa). \end{aligned}$$

Next, we compute, using the action of K_G ,

$$\begin{aligned} \frac{i}{2}dr \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix} W_T(m(a), s, \Phi_\kappa) &= \frac{i}{2} \frac{d}{dt} W_T \left(m(a) \exp t \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, s, \Phi_\kappa \right) \Big|_{t=0} \\ &= \frac{i}{2} \frac{d}{dt} \det(e^{itX})^\kappa W_T(m(a), s, \Phi_\kappa) \Big|_{t=0} \\ &= -\frac{\kappa}{2} \cdot W_T(m(a), s, \Phi_\kappa). \end{aligned}$$

Finally, we notice that

$$\begin{aligned} -idr \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} W_T(m(a), s, \Phi_\kappa) &= -i \frac{d}{dt} W_T(m(a)n(tX), s, \Phi_\kappa) \Big|_{t=0} \\ &= -i \frac{d}{dt} W_T(n(taX^t a)m(a), s, \Phi_\kappa) \Big|_{t=0} \\ &= -i \frac{d}{dt} e^{(t \operatorname{tr}(T a X^t a))} \Big|_{t=0} W_T(m(a), s, \Phi_\kappa) \\ &= 2\pi \operatorname{tr}(T a e_{11}^t a) \cdot W_T(m(a), s, \Phi_\kappa). \end{aligned}$$

Putting the terms together, we obtain

$$\begin{aligned} r(p_-(e_{11}))W_T(m(a), s, \Phi_\kappa) &= \left(2\pi \operatorname{tr}(T a e_{11}^t a) - \frac{\kappa}{2} + \frac{1}{2} \sum_{i=1}^n a_{i1} \cdot \frac{\partial}{\partial a_{i1}} \right) W_T(m(a), s, \Phi_\kappa). \quad \square \end{aligned}$$

Corollary 5.12. *Assume that $n = m + 1$ and $\det(T) \neq 0$. Write T in block form as in (4.5), and recall the definition (4.11) of $d(a)$. For $a \in \mathbb{R}_{>0}$ we have*

$$W_T(m(d(a)), 0, \lambda(\psi_{11}^*)) = 2 \left(2\pi T_1 a^2 - \frac{\kappa}{2} + \frac{a}{2} \frac{\partial}{\partial a} \right) W'_T(m(d(a)), 0, \Phi_\kappa).$$

Proof. Using Corollary 5.10 and Proposition 5.11, we see that

$$\begin{aligned} W_T(m(d(a)), s, \lambda(\psi_{11}^*)) &= 2(s + \rho_n - \kappa)^{-1} \cdot \left(2\pi T_1 a^2 - \frac{\kappa}{2} + \frac{a}{2} \frac{\partial}{\partial a} \right) W_T(m(d(a)), s, \Phi_\kappa). \end{aligned}$$

Since $n = m + 1$, we have $\rho_n = \kappa$. Moreover, because of the signature of V , the matrix T is not positive definite. Hence, according to Proposition 4.3, the Whittaker function on the right hand side vanishes at $s = 0$. This implies the assertion. \square

5.4. The main term of the local height revisited

Here we combine the results of the previous two subsections with the asymptotic properties of Whittaker functions derived in Section 4.2.

Proof of Theorem 5.4. Recall that $n = m + 1$, $x \in V^n$, and $T = Q(x)$ is invertible. We have to show that

$$e^{-2\pi \operatorname{tr} T} \cdot \operatorname{ht}_\infty^{\operatorname{main}}(x) = -B_{n,\infty} \cdot W'_T(1, 0, \Phi_\kappa) + B_{n-1,\infty} e^{-2\pi(\operatorname{tr} T - \operatorname{tr} \tilde{T}_2)} \cdot W'_{\tilde{T}_2}(1, 0, \Phi_{\kappa-1/2}),$$

where \tilde{T}_2 is defined by (5.10). According to Proposition 5.6 we know that

$$\operatorname{ht}_\infty^{\operatorname{main}}(x) = B_{n,\infty} e^{2\pi \operatorname{tr} T_2} \int_1^\infty W_{d(a)Td(a)}(1, 0, \lambda(\psi_{11}^*)) \cdot e^{2\pi Q(x_1 a)} \frac{da}{a}.$$

Inserting (2.2) and the formula of Corollary 5.12, we obtain

$$\begin{aligned} \operatorname{ht}_\infty^{\operatorname{main}}(x) &= 2B_{n,\infty} e^{2\pi \operatorname{tr} T_2} \\ &\times \int_1^\infty a^{-\rho_n} \left(\left(2\pi T_1 a^2 - \frac{\kappa}{2} + \frac{a}{2} \frac{\partial}{\partial a} \right) W'_T(m(d(a)), 0, \Phi_\kappa) \right) e^{2\pi Q(x_1 a)} \frac{da}{a}. \end{aligned}$$

Noticing that

$$\begin{aligned} 2a^{-\rho_n-1} e^{2\pi Q(x_1 a)} \left(2\pi T_1 a^2 - \frac{\kappa}{2} + \frac{a}{2} \frac{\partial}{\partial a} \right) W'_T(m(d(a)), 0, \Phi_\kappa) \\ = \frac{\partial}{\partial a} (W'_T(m(d(a)), 0, \Phi_\kappa) e^{2\pi Q(x_1 a)} a^{-\rho_n}), \end{aligned}$$

we find

$$\begin{aligned} \operatorname{ht}_\infty^{\operatorname{main}}(x) &= B_{n,\infty} e^{2\pi \operatorname{tr} T_2} \int_1^\infty \frac{\partial}{\partial a} (W'_T(m(d(a)), 0, \Phi_\kappa) e^{2\pi Q(x_1 a)} a^{-\rho_n}) da \\ &= B_{n,\infty} e^{2\pi \operatorname{tr} T_2} \left(-W'_T(1, 0, \Phi_\kappa) e^{2\pi Q(x_1)} + \lim_{a \rightarrow \infty} W'_T(m(d(a)), 0, \Phi_\kappa) e^{2\pi Q(x_1 a)} a^{-\rho_n} \right) \\ &= -B_{n,\infty} e^{2\pi \operatorname{tr} T} \cdot W'_T(1, 0, \Phi_\kappa) \\ &\quad + B_{n,\infty} e^{2\pi \operatorname{tr} T_2} \left(\lim_{a \rightarrow \infty} W'_T(m(d(a)), 0, \Phi_\kappa) e^{2\pi Q(x_1 a)} a^{-\rho_n} \right). \end{aligned}$$

We now employ Corollary 4.11 to evaluate the limit on the right hand side. We obtain

$$\begin{aligned} \operatorname{ht}_\infty^{\operatorname{main}}(x) &= -B_{n,\infty} e^{2\pi \operatorname{tr} T} \cdot W'_T(1, s_0, \Phi_\kappa) \\ &\quad + \begin{cases} \frac{B_{n,\infty} i^{-n} (2\pi)^{\rho_n}}{\Gamma(\rho_n)} \cdot e^{2\pi(\operatorname{tr} T_2 - {}^t T_{12} T_1^{-1} T_{12})} W'_{\tilde{T}_2}(1, 0, \Phi_{\kappa-1/2}) & \text{if } T_1 > 0, \\ 0 & \text{if } T_1 \leq 0. \end{cases} \end{aligned}$$

Hence the claim follows from (3.5). □

5.5. An alternative proof of Proposition 3.1

Here we use Corollary 5.10 and Proposition 5.1 to give an alternative way of computing the constant $B_{n,\infty}$ appearing in Proposition 3.1. Assume that $n = m + 1$. Let $\phi_\infty(x, z) \in S(V^n) \otimes C^\infty(\mathcal{D})$ with $\phi_\infty(hx, hz) = \phi_\infty(x, z)$ for all $z \in \mathcal{D}$, $x \in V^n$ and $h \in H(\mathbb{R})$. Then by Theorem 2.1 we know that

$$\int_{\mathcal{D}} \phi_\infty(x, z) \Omega^m = B_{n,\infty} \cdot W_T(1, 0, \lambda(\phi_\infty)) \tag{5.26}$$

for some non-zero constant $B_{n,\infty}$, which is independent of $\phi_\infty(x, z)$ and $T = Q(x)$. To compute $B_{n,\infty}$ we pick the special Schwartz function

$$\psi_{11}^*(x, z) = \varphi_G(x_1, z) \cdot \varphi_{KM}^{n-1,*}((x_2, \dots, x_n), z)$$

as in (5.15). Evaluating (5.26) in the limit $x_1 \rightarrow 0$ and using the fact that $\varphi_G(0, z) = 1$, we obtain

$$\int_{\mathcal{D}} \varphi_{KM}^m((x_2, \dots, x_n), z) = B_{n,\infty} \cdot W_T(1, 0, \lambda(\psi_{11}^*)), \tag{5.27}$$

where $T = \begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix}$ and $T_2 = Q((x_2, \dots, x_n))$. The left hand side of (5.27) is given by Proposition 5.1. If $T_2 > 0$ we have

$$\int_{\mathcal{D}} \varphi_{KM}^m((x_2, \dots, x_n), z) = 2e^{-2\pi \operatorname{tr} T_2}. \tag{5.28}$$

We now compute the right hand side of (5.27).

Lemma 5.13. For $T = \begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix}$ with $T_2 > 0$ as above, we have

$$W_T(1, 0, \lambda(\psi_{11}^*)) = -\frac{2^{\frac{n}{2}(\rho_n+1)}}{\Gamma_n(\rho_n)} \left(\frac{\pi}{i}\right)^{n\rho_n} e^{-2\pi \operatorname{tr} T_2}.$$

Proof. We first show that for $T = \begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix}$, we have

$$W_T(1, s, \lambda(\psi_{11}^*)) = -W_T(1, s, \Phi_\kappa). \tag{5.29}$$

In fact, using the notation of Corollary 5.10 and Proposition 5.11, we have

$$\begin{aligned} W_T(1, s, \lambda(\psi_{11}^*)) &= \frac{2}{s} \cdot (r(p-(e_{11}))W_T(g, s, \Phi_\kappa))\Big|_{g=1} \\ &= \frac{2}{s} \cdot \left(2\pi \operatorname{tr}(Te_{11}) - \frac{\kappa}{2} + \frac{1}{2} \frac{\partial}{\partial a_{11}}\right) W_T(m(a), s, \Phi_\kappa)\Big|_{a=1}. \end{aligned}$$

Here we have also used the fact that $n = m + 1$ and therefore $\rho_n = \kappa$. Taking into account the special form of T and the transformation law (2.2), we deduce

$$\begin{aligned} W_T(1, s, \lambda(\psi_{11}^*)) &= \frac{2}{s} \cdot \left(-\frac{\kappa}{2} + \frac{1}{2} \frac{\partial}{\partial a_{11}}\right) (a_{11}^{\rho_n-s} W_T(1, s, \Phi_\kappa))\Big|_{a_{11}=1} \\ &= -W_T(1, s, \Phi_\kappa). \end{aligned}$$

Next we compute $W_T(1, 0, \Phi_\kappa)$ for $T = \begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix}$. According to [Shi, (4.6.K)] and (4.4) the function $\omega(g, h, \alpha, \beta)$ of [Shi] satisfies

$$\begin{aligned} \omega(2 \cdot 1_n, \pi T, \alpha, \beta) &= 2^{-(n-1)\alpha} \frac{|2\pi T_2|^{\rho_n - \alpha} |2 \cdot 1_n|^{\alpha + \beta - \rho_n}}{\Gamma_{n-1}(\beta - 1/2)\Gamma_1(\alpha + \beta - \rho_n)} \eta(2 \cdot 1_n, \pi T, \alpha, \beta) \\ &= 2^{-(n-1)\alpha} \frac{|2\pi T_2|^{\rho_n - \alpha} |2 \cdot 1_n|^{\alpha + \beta - \rho_n}}{c_n(\alpha, \beta)\Gamma_{n-1}(\beta - 1/2)\Gamma_1(\alpha + \beta - \rho_n)} W_T(1_n, s, \Phi_\kappa) \\ &= 2^{-\rho_n(n/2+1)+n\beta-n/2+\alpha(2-n)} \pi^{\alpha(1-n)-1} i^{n(\alpha-\beta)} |T_2|^{\rho_n - \alpha} \\ &\quad \times \frac{\Gamma(\beta)\Gamma_n(\alpha)}{\Gamma(\alpha + \beta - \rho_n)} W_T(1_n, s, \Phi_\kappa). \end{aligned}$$

Here, in the latter equality, we have also used (4.12). We find

$$\omega(2, \pi T, \rho_n, 0) = 2^{-\frac{3}{2}n\rho_n + \rho_n - n/2} \pi^{\rho_n(1-n)-1} i^{n\rho_n} \Gamma_n(\rho_n) W_T(1, 0, \Phi_\kappa).$$

On the other hand, by [Shi, Theorem 4.2 and (4.35.K)], we have

$$\omega(2, \pi T, \rho_n, 0) = \omega(2, \pi T, \rho_n + 1/2, 1/2) = 2^{-(n-1)\rho_n} \pi^{(n-1)/2} e^{-2\pi \operatorname{tr} T},$$

and therefore

$$e^{-2\pi \operatorname{tr} T} = 2^{-\frac{n}{2}(\rho_n+1)} \pi^{-n\rho_n} i^{n\rho_n} \Gamma_n(\rho_n) W_T(1, 0, \Phi_\kappa). \tag{5.30}$$

Putting this identity into (5.29), we obtain the assertion. □

Combining (5.27), (5.28), and Lemma 5.13, we find

$$B_{n,\infty} = -2^{\frac{4-n^2-3n}{4}} \left(\frac{i}{\pi}\right)^{n\rho_n} \Gamma_n(\rho_n).$$

In particular, $B_{n,\infty}/B_{n-1,\infty} = i^n \frac{\Gamma(\rho_n)}{(2\pi)^{\rho_n}}$, $B_{1,\infty} = \frac{1}{\pi i}$, and $B_{2,\infty} = \frac{i}{4\sqrt{2}\pi^2}$.

6. The local arithmetic Siegel–Weil formula at an odd prime p

In this section we assume that $p \neq 2$ is a prime. Let $W = W(\bar{\mathbb{F}}_p)$ be the Witt ring of $\bar{\mathbb{F}}_p$ and $\mathbb{K} = W_{\mathbb{Q}}$ be the fraction field of W , which is the completion of the maximal unramified extension of \mathbb{Q}_p . Let σ be the Frobenius of W (such that its reduction to $\bar{\mathbb{F}}_p$ is the Frobenius $x \mapsto x^p$).

Let L be a unimodular quadratic lattice over \mathbb{Z}_p of rank $n + 1$ and put $V = L_{\mathbb{Q}_p}$. Let $C(L)$ be the Clifford algebra of L , and let $D(L) = \operatorname{Hom}(C(L), \mathbb{Z}_p)$ be its dual. We write $\tilde{H} = \operatorname{GSpin}(L)$ for the general Spin group over \mathbb{Z}_p , and notice that $\tilde{H}(\mathbb{Z}_p) \subset C(L)^\times$ acts on $C(L)$ via left multiplication and thus acts on D_L . Let ι be the main involution on $C(V)$ which fixes V pointwise. If $\delta \in C(V)^\times$ with $\delta^\iota = -\delta$, then $\psi_\delta(x, y) = \operatorname{tr}(x\delta y^\iota)$ defines a non-degenerate symplectic form on $C(V)$. We will require that $\delta \in C(L)$ and $\delta\delta^\iota \in \mathbb{Z}_p^\times$, which implies that $C(L)$ is unimodular under this symplectic form. This induces an embedding

$$i = i_\delta : \tilde{H} \rightarrow \operatorname{GSp}(C(L)). \tag{6.1}$$

It is also known that \tilde{H} is ‘cut out’ by a family of tensors (s_α) , $s_\alpha \in C(L)^\otimes$, in the sense that for any \mathbb{Z}_p -algebra R we have

$$\tilde{H}(R) = \{h \in \mathrm{GSp}(C(L))(R) \mid hs_\alpha = s_\alpha\}.$$

6.1. *The local unramified Shimura datum and the Rapoport–Zink space associated to \tilde{H}*

Here we set up some notation for the rest of this section. We recall the construction of an unramified local Shimura datum for \tilde{H} due to Howard and Pappas, and the associated Rapoport–Zink space. We refer to [HP] for details.

Choose a \mathbb{Z}_p -basis $e = \{e_1, \dots, e_{n+1}\}$ of L with Gram matrix

$$((e_i, e_j)) = \mathrm{diag}\left(I_{n-2}, \epsilon_L, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \tag{6.2}$$

with $\epsilon_L = -\det L$. Define

$$\mu : \mathbb{G}_m \rightarrow \tilde{H}, \quad t \mapsto \mu(t) = t^{-1}e_n e_{n+1} + e_{n+1}e_n \in \tilde{H}, \tag{6.3}$$

$$b = e_{n-1}(p^{-1}e_n + e_{n+1}) \in \tilde{H}(\mathbb{Q}_p) \subset \tilde{H}(\mathbb{K}). \tag{6.4}$$

Then $(\tilde{H}, [b], \{\mu\}, C(L))$ is the local unramified Shimura datum constructed by Howard and Pappas in [HP, Section 4] for \tilde{H} . Here $\{\mu\}$ is the conjugacy class of the cocharacter μ under $\tilde{H}(\mathbb{K})$, and $[b]$ is the σ -conjugacy class of the basic element b , i.e., the set of elements $h^\sigma b h^{-1}$ with $h \in \tilde{H}(\mathbb{K})$. Associated to b there are two isocrystals

$$(V_{\mathbb{K}} = V \otimes_{\mathbb{Q}_p} \mathbb{K}, b \circ \sigma) \quad \text{and} \quad (D_{\mathbb{K}} = D(L) \otimes_{\mathbb{Z}_p} \mathbb{K}, b \circ \sigma).$$

Let

$$\mathbb{V} = V_{\mathbb{K}}^{b \circ \sigma} \quad \text{and} \quad \mathbb{L} = (L \otimes_{\mathbb{Z}_p} W)^{b \circ \sigma}.$$

A direct calculation shows that \mathbb{L} has a \mathbb{Z}_p -basis $e' = \{e'_1, \dots, e'_{n+1}\}$ with Gram matrix

$$((e'_i, e'_j)) = \mathrm{diag}(I_{n-2}, \epsilon_{\mathbb{L}}, p, -pu) \tag{6.5}$$

where $u \in \mathbb{Z}_p^\times$ with $(p, u) = -1$ and $-u\epsilon_{\mathbb{L}} = \epsilon_L$. We can actually take $e'_i = e_i$ for $i \leq n - 2$. In particular, $\mathbb{V} = \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a quadratic space over \mathbb{Q}_p with the same dimension and the same determinant as V , but with opposite Hasse invariant.

According to [HP, Lemma 2.2.5], there is a unique p -divisible group

$$\mathbb{X}_0 = \mathbb{X}_0(\tilde{H}, [b], \{\mu\}, C(L))$$

over $\bar{\mathbb{F}}_p$ whose contravariant Dieudonné module is $\mathbb{D}(\mathbb{X}_0)(W) \cong D_W = D(L) \otimes_{\mathbb{Z}_p} W$ with Frobenius $F = b \circ \sigma$. Moreover, the Hodge filtration on $\mathbb{D}(\mathbb{X}_0)(\bar{\mathbb{F}}_p)$ is induced by $\mu_{\bar{\mathbb{F}}_p}$ (up to conjugation). The symplectic form ψ_δ induces a principal polarization λ_0 on \mathbb{X}_0 .

Let $\mathrm{RZ}(\mathbb{X}_0, \lambda_0)$ be the Rapoport–Zink space associated to $\mathrm{GSp}(C(L), \psi_\delta)$ (see [RZ] and [HP, Section 2.3]). It is a smooth formal scheme over $\mathrm{Spf}(W)$ representing the moduli

problem over Nilp_W of triples $(X, \lambda, \rho)/S$, where S is a formal scheme over W on which p is Zariski locally nilpotent, (X, λ) is a p -divisible group with principal polarization λ , and ρ is a quasi-isogeny

$$\rho : \mathbb{X}_0 \times_{\mathbb{F}_p} \bar{S} \dashrightarrow X \times_S \bar{S}, \quad \bar{S} = S \times_W \bar{\mathbb{F}}_p,$$

which respects polarization up to a scalar, in the sense that Zariski locally on \bar{S} , we have

$$\rho^\vee \circ \lambda \circ \rho = c(\rho)^{-1} \lambda_0, \quad c(\rho) \in \mathbb{Q}_p^\times.$$

Let $\text{RZ} = \text{RZ}(\tilde{H}, [b], \{\mu\}, C(L))$ be the GSpin Rapoport–Zink space constructed in [HP, Section 4]. This space comes with a closed immersion $\text{RZ} \rightarrow \text{RZ}(\mathbb{X}_0, \lambda_0)$, and by restricting the universal object one obtains a universal triple $(X^{\text{univ}}, \lambda^{\text{univ}}, \rho^{\text{univ}})$ over RZ . The universal quasi-isogeny preserves the polarization only up to a scalar, which induces a decomposition of RZ as a union of open and closed formal subschemes,

$$\text{RZ} = \bigsqcup_l \text{RZ}^{(l)}, \tag{6.6}$$

where $\text{RZ}^{(l)} \subset \text{RZ}$ is cut out by the condition $\text{ord}_p c(\rho^{\text{univ}}) = l \in \mathbb{Z}$. According to [HP, Section 7] (see also Section 7.3 here), RZ can be used to uniformize the supersingular locus at p of some Shimura variety associated with (\tilde{H}, \mathcal{D}) .

Notice that $V_{\mathbb{K}}$ acts on $C(V)_{\mathbb{K}}$ via right multiplication, which induces an action on the isocrystal $D_{\mathbb{K}}$. This gives an embedding $V_{\mathbb{K}} \subset \text{End}(D_{\mathbb{K}})$. Moreover, $\mathbb{V} = V_{\mathbb{K}}^{b \circ \sigma} \subset \text{End}(D_{\mathbb{K}})$ commutes with the Frobenius $F = b \circ \sigma$. Since $D_{\mathbb{K}} \cong \mathbb{D}(\mathbb{X}_0)(\mathbb{K})$, we obtain an embedding $\mathbb{V} \subset \text{End}^0(\mathbb{X}_0)$. Following [HP] we call \mathbb{V} the *special endomorphism space* of \mathbb{X}_0 .

Let $\tilde{\mathbb{H}}$ be the algebraic group $\text{GSpin}(\mathbb{V})$. Then $\tilde{\mathbb{H}}(\mathbb{Q}_p) = \{h \in \tilde{H}(\mathbb{K}) \mid hb = b\sigma(h)\}$ acts by automorphisms on $D_{\mathbb{K}}$, giving rise to a quasi-action on \mathbb{X}_0 . This quasi-action has the property

$$c(h\rho) = \mu_{\tilde{\mathbb{H}}}(h)c(\rho),$$

where $\mu_{\tilde{\mathbb{H}}}$ is the spin character of $\tilde{\mathbb{H}}$. So $h \in \tilde{\mathbb{H}}(\mathbb{Q}_p)$ induces an isomorphism $\text{RZ}^{(l)} \cong \text{RZ}^{(l + \text{ord}_p \mu_{\tilde{\mathbb{H}}}(h))}$. In particular,

$$p^{\mathbb{Z}} \backslash \text{RZ} \cong \text{RZ}^{(0)} \sqcup \text{RZ}^{(1)}. \tag{6.7}$$

According to [She, Corollary 7.8], $\overline{\text{RZ}} = p^{\mathbb{Z}} \backslash \text{RZ}$ is exactly the Rapoport–Zink space of H associated to the basic local unramified Shimura datum induced from the datum $(\tilde{H}, [b], \{\mu\}, C(L))$.

Finally, let $J \subset \mathbb{V}$ be an integral \mathbb{Z}_p -submodule of rank $1 \leq r \leq n$. We define the special cycle $\mathcal{Z}(J)$, following Soylu [So], as the formal subscheme of RZ cut out by the condition

$$\rho \circ J \circ \rho^{-1} \subset \text{End}(X). \tag{6.8}$$

Here, for an S -point $\alpha : S \rightarrow \text{RZ}$, $X = \alpha^*(X^{\text{univ}})$ and $\rho = \alpha^*(\rho^{\text{univ}})$ are the pull-backs of the universal objects.

If J has an ordered \mathbb{Z}_p -basis $x = (x_1, \dots, x_r) \in \mathbb{V}^r$, we also denote $\mathcal{Z}(J) = \mathcal{Z}(x)$. The moment matrix $T = Q(x) = \frac{1}{2}((x_i, x_j))$ in $\text{Sym}_r(\mathbb{Q}_p)$ is determined by J up to

\mathbb{Z}_p -equivalence. Soylu gave an explicit formula for the dimension of the reduced scheme $\mathcal{Z}(J)^{\text{red}}$ underlying $\mathcal{Z}(J)$ in terms of T and L [So, Section 4.2].

The purpose of this section is to prove a local arithmetic Siegel–Weil formula for $\mathcal{Z}(J)$. We show that when $\mathcal{Z}(J)$ is 0-dimensional, the local height of each point $P \in \mathcal{Z}(J)$ depends only on T , not on the choice of the point P , and is equal to the central derivative of some local Whittaker function (Theorem 6.9).

6.2. Dual vertex lattices and decomposition of the Rapoport–Zink space

A \mathbb{Z}_p -lattice $\Lambda \subset \mathbb{V} = V_{\mathbb{K}}^{b\circ\sigma}$ is called a *dual vertex lattice* if $p\Lambda' \subset \Lambda \subset \Lambda'$, i.e., its dual Λ' is a vertex lattice in the sense of [HP]. Let

$$\Omega_{\Lambda} = \Lambda' / \Lambda, \quad Q_{\Lambda}(x) = pQ(x) \pmod{\mathbb{Z}_p},$$

be the associated quadratic space over \mathbb{F}_p . Then $t_{\Lambda} = \dim_{\mathbb{F}_p} \Omega_{\Lambda}$ is called the *type number* of Λ . Let $\bar{\Omega}_{\Lambda} = \Omega_{\Lambda} \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$. According to [HP, Section 5.3], there is a projective variety S_{Λ} over $\bar{\mathbb{F}}_p$ such that

$$S_{\Lambda}(\bar{\mathbb{F}}_p) = \{\mathfrak{L} \subset \bar{\Omega}_{\Lambda} \mid \text{maximal isotropic and } \dim(\mathfrak{L} + \text{Frob}(\mathfrak{L})) = t_{\Lambda}/2 + 1\}.$$

Moreover, $S_{\Lambda} = S_{\Lambda}^{+} \cup S_{\Lambda}^{-}$ has two connected components, both smooth and projective of dimension $t_{\Lambda}/2 - 1$.

For a dual vertex lattice Λ of \mathbb{V} , let RZ_{Λ} be the closed formal subscheme of RZ defined by the condition

$$\rho \circ \Lambda \circ \rho^{-1} \subset \text{End}(X).$$

The following theorem summarizes some of the basic properties of RZ_{Λ} and RZ . Assertions (1), (3), (4) are due to Howard and Pappas [HP, Proposition 5.1.2, Section 6.5, Remark 6.5.7]. The second assertion is due to Li and Zhu [LZ, Theorem 4.2.11].

Theorem 6.1. (1) For a dual vertex lattice Λ , t_{Λ} is even and

$$t_{\Lambda} \leq t_{\max} = \begin{cases} n & \text{if } n \text{ is even,} \\ n - 1 & \text{if } n \text{ is odd, } \det L = (-1)^{(n+1)/2}, \\ n + 1 & \text{if } n \text{ is odd, } \det L \neq (-1)^{(n+1)/2}. \end{cases}$$

Moreover, Ω_{Λ} is the unique non-split space over \mathbb{F}_p of dimension t_{Λ} , and every dual vertex lattice contains a ‘minimal’ dual vertex lattice with $t_{\Lambda} = t_{\max}$. Moreover, $\text{RZ}_{\Lambda}^{\text{red}}$ is of dimension $t_{\Lambda}/2 - 1$.

(2) The formal scheme RZ_{Λ} is reduced.

(3) One has

$$\text{RZ} = \bigcup_{t_{\Lambda}=t_{\max}} \text{RZ}_{\Lambda},$$

and

$$\text{RZ}^{\text{red}} = \bigsqcup_{\Lambda} \text{BT}_{\Lambda}, \quad \text{BT}_{\Lambda} = \text{RZ}_{\Lambda} - \bigcup_{\Lambda_1 \subsetneq \Lambda} \text{RZ}_{\Lambda_1}.$$

(4) Let $\bar{\text{RZ}}_{\Lambda} = p^{\mathbb{Z}} \backslash \text{RZ}_{\Lambda}$. Then $\bar{\text{RZ}}_{\Lambda} \cong S_{\Lambda}$ as projective varieties over $\bar{\mathbb{F}}_p$.

Proposition 6.2. *Up to \mathbb{Z}_p -isomorphism, there is a unique dual vertex lattice $\Lambda(t)$ of type number t for every even integer $1 < t = 2r \leq t_{\max}$, which is given by $\Lambda(t) = \bigoplus \mathbb{Z}_p f_i$ with Gram matrix*

$$((f_i, f_j)) = \text{diag}(I_{n-t}, \alpha, pI_{t-1}, p\beta) \tag{6.9}$$

with $\alpha, \beta \in \mathbb{Z}_p^\times / (\mathbb{Z}_p^\times)^2$ satisfying

$$(p, (-1)^r \beta) = -1 \quad \text{and} \quad \alpha\beta = \det L \pmod{(\mathbb{Z}_p^\times)^2}.$$

In particular, $\Lambda(2)$ is the lattice given by (6.5).

Proof. Since $p\Lambda' \subset \Lambda \subset \Lambda'$, we see that $\Lambda = \bigoplus \mathbb{Z}_p e_i$ with

$$Q\left(\sum x_i e_i\right) = \sum \alpha_i p^{a_i} x_i^2$$

with $\alpha_i \in \mathbb{Z}_p^\times$ and $0 \leq a_1 \leq \dots \leq a_{n+1} \leq 1$. The condition $t_\Lambda = \dim_{\mathbb{F}_p} \Lambda' / \Lambda = t$ implies $a_1 = \dots = a_{n-t+1} = 0$, and $a_{n-t+2} = \dots = a_{n+1} = 1$. So we can change the basis to make (6.9) true. Since L is unimodular, V has Hasse invariant 1, and hence \mathbb{V} has Hasse invariant -1 , i.e.,

$$-1 = (2p, (2p)^{t-1} \beta)(2p, (2p)^{t-2} \beta) \cdots (2p, 2p\beta) = (2p, (2p)^r \beta) = (p, (-1)^r \beta).$$

In particular, $E = \mathbb{Q}_p(\sqrt{(-1)^r \beta})$ is the unique unramified quadratic field extension of \mathbb{Q}_p , and β is uniquely determined up to a square by this condition. On the other hand, $\det \mathbb{V} = \det V$ gives

$$\alpha\beta = \det L \pmod{(\mathbb{Z}_p^\times)^2},$$

which then determines α uniquely up to a square. □

6.3. Special cycles and local heights

Recall the definition of the special cycle $\mathcal{Z}(J)$ at the end of Section 6.1. It is not hard to see [So, Section 4.2] that

$$\mathcal{Z}(J)^{\text{red}} = \bigcup_{\substack{\Lambda \text{ dual vertex lattice} \\ J \subset \Lambda}} \text{RZ}_\Lambda^{\text{red}}.$$

The following theorem is part of [So, Theorems 4.13, 4.16 and Proposition 4.15]. (Recall our convention that $\dim V = n + 1$ and notice that our $2T$ is Soylu’s T .)

Theorem 6.3 (Soylu). *Let $\mathcal{Z}(J)^{\text{red}}$ be the reduced scheme of $\mathcal{Z}(J)$ and assume that $J = J(x_1, \dots, x_n) \subset \mathbb{V}$ has rank n and is integral. Assume that $T = Q(x)$ is \mathbb{Z}_p -equivalent to $\text{diag}(T_1, T_2)$ where T_1 is unimodular of rank $r = r(T)$ (which is also the rank of $T \pmod{p}$ over \mathbb{F}_p), and $T_2 \in p \text{Sym}_{n-r}(\mathbb{Z}_p)$. Then $\mathcal{Z}(J)^{\text{red}}$ is 0-dimensional if and only if one of the following conditions holds:*

- (1) $r(T) = n - 1, n - 2$.
- (2) $r(T) = n - 3$ and $\det(2T_1) = \det L$.

In that case,

$$\mathcal{Z}(J)^{\text{red}} = \bigsqcup_{\substack{J \subset \Lambda \\ t_\Lambda = 2}} \text{RZ}_\Lambda = \bigsqcup_{\substack{J \subset \Lambda \\ \Lambda \cong \Lambda(2)}} \text{RZ}_\Lambda.$$

Proof. We give a sketch of the proof in this special case to give a rough idea what is involved in the general theorems of Soylu [So, Section 4]. Choose a \mathbb{Z}_p -basis $e = \{e_1, \dots, e_n\}$ of J with $\frac{1}{2}(e_i, e_j) = \text{diag}(T_1, T_2)$, and let M_1 be the submodule of J generated by e_1, \dots, e_r , which is unimodular. To have $J \hookrightarrow \Lambda$, one has to have $t_\Lambda \leq n - r + 1$. In the case $r = n - 1, n - 2$, one has $t_\Lambda = 2$, and $\text{RZ}_\Lambda = \text{RZ}_\Lambda^{\text{red}}$ is reduced of dimension 0. So

$$\mathcal{Z}(J)^{\text{red}} = \bigsqcup_{\substack{\Lambda \supset J \\ t_\Lambda=2}} \text{RZ}_\Lambda.$$

In the case $r(T) = n - 3$, one might have $t_\Lambda = 2$ or 4. If $t_\Lambda = 4$, then (as M_1 is unimodular)

$$\Lambda \cong M_1 \oplus \Lambda_2,$$

where Λ_2 has a \mathbb{Z}_p -basis with Gram matrix $p \text{diag}(1, 1, 1, \epsilon)$. Since the Hasse invariant of \mathbb{V} is -1 , we see $(p, -\epsilon) = -1$. On the other hand, $\det \mathbb{V} = \det V$ forces

$$\epsilon = \det M_1 \det L = \det(2T_1) \det L, \quad \text{i.e.,} \quad \det(2T_1) \neq \det L.$$

Therefore, if $\det(2T_1) = \det L$, we cannot embed J into a dual vertex lattice Λ with $t_\Lambda = 4$, and thus $\mathcal{Z}(J)$ is 0-dimensional and reduced as argued above.

When $\det(2T_1) \neq \det L$, Soylu proved that there is indeed some embedding $J \subset \Lambda$ with $t_\Lambda = 4$. We refer to [So, Section 4] for the details. \square

Let M_1 be a unimodular quadratic \mathbb{Z}_p -lattice of rank $r < n - 2$, and assume that there are isometric embeddings $M_1 \subset L$ and $M_1 \subset \Lambda$, where $\Lambda \subset \mathbb{V}$ is a dual vertex lattice. Write

$$L = M_1 \oplus L_2, \quad \Lambda = M_1 \oplus \Lambda_2. \tag{6.10}$$

Notice that, choosing proper bases of M_1 and L , the data b and μ defined in (6.3) and (6.4) still make sense for the unimodular lattice L_2 , so we have a local unramified Shimura datum $(\tilde{H}(r), [b], \{\mu\}, C(L_2))$ and its associated Rapoport–Zink space $\text{RZ}(r)$. Here $\tilde{H}(r) = \text{GSpin}(L_2)$. Moreover, one can easily check that $\mathbb{V}_2 = L_2^{\text{bo}\sigma}_{\mathbb{K}}$ is a direct summand of \mathbb{V} , and Λ_2 is a dual vertex lattice of \mathbb{V}_2 . The embedding $L_2 \subset L$ induces a closed immersion

$$i(r) : \text{RZ}(r) \hookrightarrow \text{RZ}. \tag{6.11}$$

The following proposition is a direct consequence of [LZ, Lemma 3.1.1].

Proposition 6.4 (Li–Zhu). *Let the notation be as above and assume $r \leq n - 3$.*

- (1) $i(r)\text{RZ}(r) = \mathcal{Z}(M_1)$.
- (2) Assume that $J = M_1 \oplus J_2$ is a \mathbb{Z}_p -submodule of Λ . Then $i(r)\mathcal{Z}_{\text{RZ}(r)}(J_2) = \mathcal{Z}_{\text{RZ}}(J)$.

Proof. Assume that $\{x_1, \dots, x_r\}$ is a basis of M_1 with Gram matrix $\text{diag}(\alpha_1, \dots, \alpha_r)$ and $\alpha_i \in \mathbb{Z}_p^\times$. Applying [LZ, Lemma 3.1.1] r times, we obtain the above proposition. Notice that the cited lemma still holds with the same proof when the norm of x_n is a unit in \mathbb{Z}_p (not necessarily equal to 1). \square

From now on, we assume that $\mathcal{Z}(J)$ is 0-dimensional. For $P \in \mathcal{Z}(J)$, its local height index is defined to be

$$\text{ht}_p(P) = \text{the length of the formal complete local ring } \widehat{\mathcal{O}}_{\mathcal{Z}(J), P}. \tag{6.12}$$

By Theorem 6.3, we have $r(T) \geq n - 3$. There is a decomposition

$$J = M_1 \oplus J_2$$

with M_1 unimodular of rank $n - 3$. Furthermore we can choose bases of M_1 and J_2 so that the Gram matrix of J becomes $2T$ with $T = \text{diag}(T_1, T_2)$ where $2T_1$ is the Gram matrix of M_1 and $T_2 = \text{diag}(\alpha_1 p^{a_1}, \alpha_2 p^{a_2}, \alpha_3 p^{a_3})$ is the matrix of J_2 with $\alpha_i \in \mathbb{Z}_p^\times$ and $0 \leq a_1 \leq a_2 \leq a_3$. We can always embed M_1 into L . Assume that $\mathcal{Z}(J)(\overline{\mathbb{F}}_p)$ is not empty. Then there is an embedding $M_1 \subset J \subset \Lambda$ for some dual vertex lattice Λ .

Corollary 6.5. *Let the notation and hypotheses be as above (in particular $\mathcal{Z}(J)$ is 0-dimensional). For $P \in \mathcal{Z}(J)$ write $P^* = i(n - 3)^{-1}P \in \text{RZ}(n - 3)$. Then*

$$\text{ht}_p(P) = \text{ht}_p(P^*).$$

The local height $\text{ht}_p(P^*)$ has been studied in [KRap1] (the case $a_1 = 0$ actually follows from [KRY2] with $n - 3$ replaced by $n - 2$). Assume the decompositions (6.10). Notice that L_2 is unimodular of rank 4. There are two cases: either

- (i) $\det L_2 = 1$ and $L_2 \cong M_2(\mathbb{Z}_p)$ with $Q(x) = \det x$, or
- (ii) $\det L_2 = u \in \mathbb{Z}_p^\times$ where $E = \mathbb{Q}_p(\sqrt{u})$ is the unique unramified quadratic field extension of \mathbb{Q}_p , i.e., $(p, u) = -1$.

In the second case, L_2 is \mathbb{Z}_p -equivalent to \mathbb{Z}_p^4 with the quadratic form $Q(x) = x_1x_2 + x_3^2 - ux_4^2$, or more conceptually

$$L_2 \cong \left\{ A = \begin{pmatrix} \alpha & a \\ b & \alpha' \end{pmatrix} \mid a, b \in \mathbb{Z}_p, \alpha \in \mathcal{O}_E \right\}, \quad Q(A) = \det A,$$

where α' is the Galois conjugate of α . The second case only occurs when $r(T) \geq n - 2$, i.e., $a_1 = 0$. Indeed, if $a_1 > 0$, i.e., $r(T) = n - 3$, then we would have

$$\det(2T_1) = \det L = \det M_1 \det L_2 = \det(2T_1) \det L_2,$$

which implies $1 = \det L_2 = u$, a contradiction. The condition $a_1 = 0$ is exactly the condition given in [KRap1, Theorem 2] for $\mathcal{Z}_{\text{RZ}(n-3)}(J_2)(\overline{\mathbb{F}}_p)$ to be finite. So in both cases, $\text{RZ}(n - 3)$ is associated to the supersingular locus at p of the Hilbert modular surface over a real quadratic field F with p split or inert in F , $F_p = \mathbb{Q}_p \times \mathbb{Q}_p$ or E . In [KRap1], Kudla and Rapoport considered twisted Hilbert modular surfaces to avoid issues with the boundary. But their localization at p , considered in [KRap1, Sections 6–12], is for our p the same as for a regular Hilbert modular surface, and hence their local results apply. We restate it here as the following theorem for convenience. In case (ii), $a_1 = 0$, it is [KRap1, Proposition 6.2]. In case (i), it is [GK, Proposition 5.4], restated in [KRap1, Proposition 11.2] with a minor mistake (it should not assume $a_1 = 0$ in this case).

Proposition 6.6 (Kudla–Rapoport). *Let the notation and hypotheses be as above, in particular $\mathcal{Z}_{\text{RZ}(n-3)}(J_2)$ is 0-dimensional, and let $P^* \in \mathcal{Z}_{\text{RZ}(n-3)}(J_2)$. Recall that T_2 is \mathbb{Z}_p -equivalent to $\text{diag}(\alpha_1 p^{a_1}, \alpha_2 p^{a_2}, \alpha_3 p^{a_3})$ with $0 \leq a_1 \leq a_2 \leq a_3$, and $\alpha_i \in \mathbb{Z}_p^\times$. Then $\text{ht}_p(P^*) = v_p(T_2)$, where $v_p(T_2)$ is given as follows:*

(1) *When $a_2 \equiv a_1 \pmod{2}$, $v_p(T_2)$ is equal to*

$$\sum_{i=0}^{a_1-1} (i+1)(a_1+a_2+a_3-3i)p^i + \sum_{i=a_1}^{(a_1+a_2)/2-1} (a_1+1)(2a_1+a_2+a_3-4i)p^i + \frac{a_1+1}{2}(a_3-a_2+1)p^{(a_1+a_2)/2}.$$

(2) *When $a_2 \not\equiv a_1 \pmod{2}$, $v_p(T_2)$ is equal to*

$$\sum_{i=0}^{a_1-1} (i+1)(a_1+a_2+a_3-3i)p^i + \sum_{i=a_1}^{(a_1+a_2-1)/2} (a_1+1)(2a_1+a_2+a_3-4i)p^i.$$

6.4. Local Whittaker functions and the local arithmetic Siegel–Weil formula

Let $\psi = \psi_p$ be the ‘canonical’ unramified additive character of \mathbb{Q}_p used in this paper. Let L be an integral quadratic lattice over \mathbb{Z}_p of rank l , and let $\chi_L = ((-1)^{l(l-1)/2} \det L, \cdot)_p$ be the associated quadratic character. For every integer $r \geq 0$ we also consider the lattice $L^{(r)} = L \oplus H^r$, where $H = \mathbb{Z}_p^2$ is the standard hyperbolic plane with the quadratic form $Q(x, y) = xy$. We temporarily allow L to be non-unimodular.

Let $T \in \text{Sym}_n(\mathbb{Z}_p)$ be non-singular with $n \leq l$. Then according to [Ku2, Appendix] and [Ya], there is a local density polynomial $\alpha_p(X, T, L)$ of X such that for every integer $r \geq 0$, one has

$$\alpha_p(p^{-r}, T, L) = \int_{\text{Sym}_n(\mathbb{Q}_p)} \int_{L^{(r),n}} \psi(\text{tr } b(Q(x) - T)) \, dx \, db,$$

where dx and db are the standard Haar measures with $\text{vol}(L, dx) = \text{vol}(L^{(r)}, dx) = 1$ and $\text{vol}(\text{Sym}_n(\mathbb{Z}_p), db) = 1$. We write $\varphi_L = \text{char}(L^n) \in S(L^n_{\mathbb{Q}_p})$ for the characteristic function of L . Then it is easy to see that

$$W_{T,p}(1, s, \lambda(\varphi_L)) = \left(\frac{\gamma(L)}{\sqrt{[L':L]}} \right)^n \alpha_p(p^{-s}, T, L).$$

Here $\gamma(L) = \gamma(L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is the local Weil index. We also recall [Ya, Section 2] that $\alpha_p(p^{-r}, T, L)$ is the local representation density $\alpha_p(M_T, L^{(r)}) = \beta_p(M_T, L^{(r)})$ studied in Kitaoka’s book [Kit, Section 5.6]. Here $M_T = \mathbb{Z}_p^n$ is the quadratic lattice associated to T , i.e., with the quadratic form $Q(x) = {}^t x T x$. For a unimodular lattice L of rank l , define

$$\delta_L = \begin{cases} 0 & \text{if } l \equiv 1 \pmod{2}, \\ \chi_L(p) & \text{if } l \equiv 0 \pmod{2}. \end{cases} \tag{6.13}$$

Lemma 6.7. *Let $T \in \text{Sym}_n(\mathbb{Z}_p)$ with $\det T \in \mathbb{Z}_p^\times$, i.e., T is unimodular.*

- (1) *Assume that $L = L_1 \oplus L_0$ is an integral lattice over \mathbb{Z}_p such that $Q(x) \in p\mathbb{Z}_p$ for every $x \in L_0$. Then $\alpha_p(X, T, L) = \alpha_p(X, T, L_1)$.*
- (2) *Assume that L is \mathbb{Z}_p -unimodular of rank $l \geq n$. Then*

$$\begin{aligned} \alpha_p(X, T, L) &= (1 - \delta_L p^{-l/2} X)(1 + \delta_{M_T \oplus L^-} p^{-(l-n)/2} X) \prod_{(l-n+1)/2 \leq e \leq (l-1)/2} (1 - p^{-2e} X^2). \end{aligned}$$

Here L^- denotes the lattice L with the rescaled quadratic form $Q^-(x) = -Q(x)$.

Proof. Let $\tilde{L} = L/pL$ with the \mathbb{F}_p -valued quadratic form $\tilde{Q}(x) = Q(x) \pmod p$ for an integral quadratic \mathbb{Z}_p -lattice L . Replacing L by $L^{(r)}$, we may assume $X = 1$ in the proof.

For (1), write l and l_i for the rank of L and L_i respectively with $l = l_0 + l_1$. Notice that \tilde{L}_0 is a zero quadratic space of dimension l_0 . Every isometry from \tilde{M}_T to \tilde{L} splits into the sum of an isometry from \tilde{M}_T to \tilde{L}_1 and a homomorphism from \tilde{M}_T to \tilde{L}_0 . So [Kit, p. 99, exercise] gives

$$\begin{aligned} \alpha_p(1, T, L) &= p^{n(n+1)/2-nl} |\{\text{isometries from } \tilde{M}_T \text{ to } \tilde{L}\}| \\ &= p^{n(n+1)/2-nl} |\{\text{isometries from } \tilde{M}_T \text{ to } \tilde{L}_1\}| \cdot |\text{Hom}(\tilde{M}_T, \tilde{L}_0)| \\ &= p^{n(n+1)/2-nl_1} |\{\text{isometries from } \tilde{M}_T \text{ to } \tilde{L}_1\}| \\ &= \alpha_p(1, T, L_1). \end{aligned}$$

For (2), [Kit, Theorem 1.3.2] and the formula in [Kit, p. 99, exercise] imply

$$\alpha_p(1, T, L) = (1 - \chi(\tilde{L}) p^{-l/2})(1 + \chi(\tilde{M}_T \oplus \tilde{L}^-) p^{-(l-n)/2}) \prod_{\substack{l-n+1 \leq e \leq l-1 \\ e \text{ even}}} (1 - p^{-e}),$$

where $\chi(\tilde{M})$ for a unimodular quadratic \mathbb{Z}_p -lattice \tilde{M} is defined as follows. When $l = \dim \tilde{M}$ is odd, $\chi(\tilde{M}) = 0$. When l is even, $\chi(\tilde{M})$ is ± 1 depending on whether \tilde{M} is equivalent to a direct sum of hyperbolic planes or not. Assume $l = 2r + 2$ is even. Since M is unimodular, M is equivalent to $H^r \oplus M_0$ with $M_0 = \mathbb{Z}_p^2$ with $Q(x, y) = x^2 - \epsilon y^2$ for some $\epsilon \in \mathbb{Z}_p^\times$. Then $\chi(\tilde{M}) = 1$ if and only if \tilde{M}_0 is a hyperbolic plane, which is the same as saying that ϵ is a square in \mathbb{F}_p , i.e., $(\epsilon, p) = 1$. On the other hand, it is easy to check that

$$\chi_M(x) = ((-1)^{r+1} \det M, x) = (\epsilon, x).$$

So $\chi(\tilde{M}) = \chi_M(p)$ in this case. This proves (2). □

Proposition 6.8. *Assume T is \mathbb{Z}_p -equivalent to $\text{diag}(T_1, T_2)$ with T_1 being unimodular of rank $n - 3$ and $T_2 = \text{diag}(\alpha_1 p^{a_1}, \alpha_2 p^{a_2}, \alpha_3 p^{a_3})$ with $\alpha_i \in \mathbb{Z}_p^\times$ and $0 \leq a_1 \leq a_2 \leq a_3$. Let L be a unimodular lattice of rank $n + 1$. Let M_1 be the unimodular quadratic lattice with Gram matrix $2T_1$, and fix an embedding $M_1 \hookrightarrow L$, which results in a decomposition*

$L = M_1 \oplus L_2$. Then $W_{T,p}(1, 0, \lambda(\varphi_L)) = 0$ if and only if $W_{T_2,p}(1, 0, \lambda(\varphi_{L_2})) = 0$. In that case, we have

$$\frac{W'_{T,p}(1, 0, \lambda(\varphi_L))}{W_{T^u,p}(1, 0, \lambda(\varphi_L))} = \frac{W'_{T_2,p}(1, 0, \lambda(\varphi_{L_2}))}{W_{T_2^u,p}(1, 0, \lambda(\varphi_{L_2}))} = v_p(T_2).$$

Here T^u and T_2^u denote any unimodular symmetric matrices over \mathbb{Z}_p of order n and 3 , respectively, and $v_p(T_2)$ is given in Proposition 6.6.

Proof. By Lemma 6.7, $W_{T^u,p}(1, 0, \lambda(\varphi_L)) = \gamma(V_p)^n \alpha(1, T^u, L)$ does not depend on the choice of the p -unimodular T^u . We choose $T^u = \text{diag}(T_1, T_2^u)$ with T_2^u being unimodular. Now [Kit, Corollary 5.6.1] implies $(X = p^{-s})$

$$\begin{aligned} \frac{W_{T,p}(1, s, \lambda(\varphi_L))}{W_{T^u,p}(1, s, \lambda(\varphi_L))} &= \frac{\alpha_p(X, T, L)}{\alpha_p(X, T^u, L)} = \frac{\alpha_p(X, T_1, L)\alpha_p(X, T_2, L_2)}{\alpha_p(X, T_1, L)\alpha_p(X, T_2^u, L_2)} \\ &= \frac{W_{T_2,p}(1, s, \lambda(\varphi_{L_2}))}{W_{T_2^u,p}(1, s, \lambda(\varphi_{L_2}))}. \end{aligned}$$

This proves the first identity and also the claim about the vanishing at $s = 0$. Assume $W_{T,p}(1, 0, \lambda(\varphi_L)) = 0$. By [KRap1, Propositions 11.5 and 7.2], we have

$$W'_{T_2,p}(1, 0, \lambda(\varphi_{L_2})) = \gamma(V_2^3)(1 - p^{-2})(1 - \chi_{L_2}(p)p^{-2})v_p(T_2).$$

On the other hand, Lemma 6.7 gives

$$W_{T_2^u,p}(1, 0, \lambda(\varphi_{L_2})) = \gamma(V_2^3)(1 - p^{-2})(1 - \chi_{L_2}(p)p^{-2}).$$

Now the second identity is clear. □

Combining Propositions 6.6 and 6.8 and Corollary 6.5, we obtain the following local arithmetic Siegel–Weil formula.

Theorem 6.9. *Let L be a unimodular quadratic \mathbb{Z}_p -lattice of rank $n + 1$ with $p \neq 2$, and let RZ be the Rapoport–Zink space as in Section 6.1. Let $T \in \text{Sym}_n(\mathbb{Z}_p)$ be of rank n and let $J \subset \mathbb{V}$ be a \mathbb{Z}_p -sublattice of rank n which has a basis with Gram matrix $2T$. Assume $\mathcal{Z}(J)$ is 0-dimensional and let $P \in \mathcal{Z}(J)$. Then*

$$\text{ht}_p(P) \log p = \frac{W'_{T,p}(1, 0, \lambda(\varphi_L))}{W_{T^u,p}(1, 0, \lambda(\varphi_L))},$$

where T^u is a unimodular matrix in $\text{Sym}_n(\mathbb{Z}_p)$ (i.e., $\det T^u \in \mathbb{Z}_p^\times$).

7. Arithmetic Siegel–Weil formulas

In this section, we will prove the arithmetic Siegel–Weil formulas as stated in Theorem 1.2 and Remark 1.3. Throughout, let V be a quadratic space over \mathbb{Q} of signature $(m, 2)$, and let $H = \text{SO}(V)$.

7.1. Vanishing of coefficients of Eisenstein series

Let $n = m + 1$ and let $\mathcal{C} = \bigotimes_{p \leq \infty} \mathcal{C}_p$ be the incoherent quadratic space over \mathbb{A} defined in the introduction. Recall the $G_{\mathbb{A}}$ -equivariant map

$$\lambda = \otimes \lambda_p : S(\mathcal{C}^n) \rightarrow I(0, \chi_V), \quad \lambda(\phi)(g) = \omega(g)\phi(0).$$

For simplicity, we also write $\lambda(\phi)$ for the associated standard section in $I(s, \chi_V)$. Let $\phi_{\infty}^{\mathcal{C}}(x) = e^{-\pi \operatorname{tr}(x,x)} \in S(\mathcal{C}_{\infty}^n)$, then $\lambda_{\infty}(\phi_{\infty}) = \Phi_{\kappa} \in I(s, \chi_V)$ is the standard section of weight $\kappa = (m + 2)/2$. Recall that for a standard factorizable section $\Phi = \prod \Phi_p \in I(s, \chi_V)$, the Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in P_{\mathbb{Q}} \backslash \operatorname{Sp}_n(\mathbb{Q})} \Phi(\gamma g, s)$$

has a meromorphic continuation to the whole complex s -plane and is holomorphic at $s = 0$. It has a Fourier expansion of the form

$$E(g, s, \Phi) = \sum_{T \in \operatorname{Sym}_n(\mathbb{Q})} E_T(g, s, \Phi).$$

When $T \in \operatorname{Sym}_n(\mathbb{Q})$ is non-singular, the T -th Fourier coefficient factorizes,

$$E_T(g, s, \Phi) = \prod_{p \leq \infty} W_{T,p}(g_p, s, \Phi_p),$$

into a product of local Whittaker functions (see (2.1)). For every $\phi \in S(V(\mathbb{A}_f)^n) = S(\mathcal{C}_{\mathbb{A}_f}^n)$, we define the Siegel–Eisenstein series of weight κ on the Siegel upper half-plane \mathbb{H}_n as

$$E(\tau, s, \lambda(\phi) \otimes \Phi_{\kappa}) = \det(v)^{-\kappa/2} \cdot E(g_{\tau}, s, \lambda(\phi) \otimes \Phi_{\kappa}), \tag{7.1}$$

where we write $g_{\tau} = n(u)m(a) \in G_{\mathbb{R}}$ with $u = \mathfrak{R}(\tau) \in \operatorname{Sym}_n(\mathbb{R})$ and $a \in \operatorname{GL}_n(\mathbb{R})$ such that $a^t a = v$ as usual. In particular, we have $g_{\tau}(i1_n) = \tau$. We could choose for a the positive symmetric square root of v but we do not have to. The Eisenstein series vanishes automatically at $s = 0$ due to incoherence. The arithmetic Siegel–Weil formula, envisioned by Kudla, aims to give an arithmetic meaning to its central derivative at $s = 0$. From now on, assume $T = \operatorname{Sym}_n(\mathbb{Q})$ is non-singular, and let

$$\operatorname{Diff}(\mathcal{C}, T) = \{p \leq \infty \mid \mathcal{C}_p \text{ does not represent } T\} \tag{7.2}$$

be Kudla’s Diff set defined in the introduction. Then $\operatorname{Diff}(\mathcal{C}, T)$ is a non-empty finite set, and $\infty \in \operatorname{Diff}(\mathcal{C}, T)$ if and only if T is not positive definite. Moreover, if $p \in \operatorname{Diff}(\mathcal{C}, T)$, then $W_{T,p}(g_p, 0, \lambda_p(\phi_p)) = 0$. So

$$\operatorname{ord}_{s=0} E_T(g, s, \lambda(\phi) \otimes \Phi_{\kappa}) \geq |\operatorname{Diff}(\mathcal{C}, T)| \tag{7.3}$$

for every $\phi \in S(V(\mathbb{A}_f)^n)$.

7.2. The arithmetic Siegel–Weil formula at infinity

Here we prove Theorem 1.2(2). We begin by recalling the global set-up.

For a compact open subgroup $K \subset H(\mathbb{A}_f)$ we consider the Shimura variety X_K whose associated complex space is

$$X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash \mathcal{D} \times H(\mathbb{A}_f) / K.$$

It is a quasi-projective variety of dimension m , which has a canonical model over \mathbb{Q} .

Given $x = (x_1, \dots, x_n) \in V(\mathbb{Q})^n$ with $Q(x) = \frac{1}{2}(x, x) = \frac{1}{2}((x_i, x_j)) > 0$, let H_x be the stabilizer of x in H . For $h \in H(\mathbb{A}_f)$, let $K_{h,x} = H_x(\mathbb{A}_f) \cap hKh^{-1}$ be the corresponding compact open subgroup of $H_x(\mathbb{A}_f)$. Then

$$H_x(\mathbb{Q}) \backslash \mathcal{D}_x \times H_x(\mathbb{A}_f) / K_{h,x} \rightarrow X_K, \quad [z, h_1] \mapsto [z, h_1 h],$$

gives rise to a cycle $Z(h, x)$ in X_K of codimension n . More generally, given a positive definite $T \in \text{Sym}_n(\mathbb{Q})$ and any K -invariant Schwartz function $\varphi \in S(V^n(\mathbb{A}_f))$, Kudla [Ku3] defines a weighted cycle as follows: If there exists an $x \in V^n(\mathbb{Q})$ with $Q(x) = T$, put

$$Z(T, \varphi) = \sum_{h \in H_x(\mathbb{A}_f) \backslash H(\mathbb{A}_f) / K} \varphi(h^{-1}x) Z(h, x) \in Z^n(X_K).$$

If there is no such x , set $Z(T, \varphi) = 0$. These weighted cycles behave well under pull-back (for varying K). Moreover, if $T \in \text{Sym}_n(\mathbb{Q})$ is regular but not positive definite, we put $Z(T, \varphi) = 0$.

If $T \in \text{Sym}_n(\mathbb{Q})$ is regular, we define a Green current for the cycle $Z(T, \varphi)$ by

$$G(T, \varphi, v, z, h) = \sum_{\substack{x \in V^n(\mathbb{Q}) \\ Q(x)=T}} \varphi(h^{-1}x) \cdot \xi_0^n(xa, z),$$

where $z \in \mathcal{D}$, $h \in H(\mathbb{A}_f)$, and $a^t a = v = \mathfrak{S}(\tau)$. The pair

$$\widehat{Z}(T, \varphi, v) = (Z(T, \varphi), G(T, \varphi, v)) \in \widehat{\text{Ch}}_C^n(X_K)$$

defines an arithmetic cycle, which depends on v . For the rest of this section we assume that $n = m + 1$. In this case, the cycles $Z(T, \varphi)$ are all trivial (in the generic fiber) for signature reasons. However, for indefinite T , the arithmetic cycles $\widehat{Z}(T, \varphi, v)$ typically have a non-trivial current part. We are interested in their archimedean arithmetic degree

$$\widehat{\text{deg}}_\infty(\widehat{Z}(T, \varphi, v)) = \frac{1}{2} \int_{X_K(\mathbb{C})} G(T, \varphi, v).$$

We are now ready to prove Theorem 1.2(2) of the introduction, which we restate here in a version which also gives an explicit value for the constant of proportionality.

Theorem 7.1. *Assume that $T \in \text{Sym}_n(\mathbb{Q})$ is of signature $(n - j, j)$ with $j > 0$ and that $\varphi \in S(V(\mathbb{A}_f)^n)$ is K -invariant. Then the arithmetic Siegel–Weil formula holds for T , i.e.,*

$$\widehat{\text{deg}}_\infty \widehat{Z}(T, \varphi, v) \cdot q^T = C_{n,\infty} \cdot E'_T(\tau, 0, \lambda(\varphi) \otimes \Phi_\kappa),$$

where the constant $C_{n,\infty}$ is given as follows. Let $L \subset V$ be an integral lattice, and let $d_L h = \prod_{p < \infty} d_{L_p} h$ be the associated Haar measure on $H(\mathbb{A}_f)$, and $C(L) = \prod_{p < \infty} C(L_p)$ be the associated constant given in Proposition 2.3 (with respect to the unramified additive character ψ_f of \mathbb{A}_f). Then

$$C_{n,\infty} = -B_{n,\infty} \frac{C(L)}{\text{vol}(K, d_L h)}.$$

Proof. The archimedean arithmetic degree is given by

$$\begin{aligned} \widehat{\text{deg}}_\infty(\widehat{Z}(T, \varphi, v)) &= \frac{1}{2} \int_{X_K(\mathbb{C})} G(T, \varphi, v) \\ &= \frac{1}{2} \int_{H(\mathbb{Q}) \backslash \mathcal{D} \times H(\mathbb{A}_f) / K} \sum_{\substack{x \in V(\mathbb{Q})^n \\ Q(x)=T}} \varphi(h_f^{-1}x) \cdot \xi_0^n(xa, z) dh_f. \end{aligned}$$

This quantity vanishes if $V(\mathbb{Q})$ does not represent T . Then, by the Hasse principle, there is at least one finite prime p such that $V(\mathbb{Q}_p)$ does not represent T , i.e., $p \in \text{Diff}(\mathcal{C}, T)$. As $\infty \in \text{Diff}(\mathcal{C}, T)$, we see that $|\text{Diff}(\mathcal{C}, T)| > 1$ and $E'_T(g, 0, \lambda(\varphi) \otimes \Phi_\kappa) = 0$. Hence the theorem holds trivially.

We now assume that there exists an $x_0 \in V(\mathbb{Q})^n$ with $Q(x_0) = T$. Then, by Witt’s theorem, any other $x \in V(\mathbb{Q})^n$ with $Q(x) = T$ is an $H(\mathbb{Q})$ -translate of x_0 . Let dh_f be any prefixed Haar measure on $H(\mathbb{A}_f)$. Notice also that the pointwise stabilizer $H_{x_0}(\mathbb{Q})$ of x_0 is trivial since $n = m + 1$. By unfolding, the above integral is equal to

$$\begin{aligned} \widehat{\text{deg}}_\infty(\widehat{Z}(T, \varphi, v)) &= \frac{1}{2} \text{vol}(K, dh_f)^{-1} \int_{H_{x_0}(\mathbb{Q}) \backslash \mathcal{D} \times H(\mathbb{A}_f)} \varphi(h_f^{-1}x_0) \cdot \xi_0^n(x_0a, z) dh_f \\ &= \frac{1}{2} \text{vol}(K, dh_f)^{-1} \int_{H(\mathbb{A}_f)} \varphi(h_f^{-1}x_0) dh_f \cdot \int_{\mathcal{D}} \xi_0^n(x_0a, z). \end{aligned}$$

By Theorem 5.2, the archimedean integral is equal to

$$\frac{1}{2} \int_{\mathcal{D}} \xi_0^n(x_0a, z) = \text{ht}_\infty(x_0a) = -B_{n,\infty} \det(v)^{-\kappa/2} \cdot W'_{T,\infty}(g_\tau, s_0, \Phi_\kappa) \cdot q^{-T}.$$

On the other hand, the quantity

$$\text{vol}(K, dh_f)^{-1} \int_{H(\mathbb{A}_f)} \varphi(h_f^{-1}x_0) dh_f$$

is clearly independent of the choice of the product Haar measure dh_f . We choose $dh_f = d_L h$; then the local Siegel–Weil formula, Proposition 2.2, gives

$$\text{vol}(K, dh_f)^{-1} \int_{H(\mathbb{A}_f)} \varphi(h_f^{-1}x_0) dh_f = \frac{C(L)}{\text{vol}(K, d_L h)} W_{T,f}(1, 0, \lambda(\varphi)).$$

This implies the assertion. □

7.3. *The arithmetic Siegel–Weil formula at a finite prime*

Assume that $n = m + 1$. Let $p \neq 2$ be a prime number. Let $L \subset V$ be a p -unimodular lattice. Let $H = \text{SO}(L)$ and put $\tilde{H} = \text{GSpin}(L)$. Let $K = K_p K^p$ be a compact open subgroup of $H(\mathbb{A}_f)$ fixing L with

$$K_p = H(\mathbb{Z}_p) = \{h \in H(\mathbb{Q}_p) \mid hL_p = L_p\}.$$

For convenience, we assume that there is a compact open subgroup $\tilde{K} \subset \tilde{H}(\mathbb{A}_f)$ which contains $\hat{\mathbb{Z}}^\times$ and which maps onto K . Such a \tilde{K} always exists if K is contained in the discriminant kernel subgroup of some even lattice in V (see Remark 7.4). Under this assumption, the Shimura variety X_K associated to (H, K) is the same as the Shimura variety associated to (\tilde{H}, \tilde{K}) . The associated complex spaces are both equal to

$$\tilde{H}(\mathbb{Q}) \backslash \mathbb{D} \times \tilde{H}(\mathbb{A}_f) / \tilde{K} \cong H(\mathbb{Q}) \backslash \mathbb{D} \times H(\mathbb{A}_f) / K.$$

Let $e, f \in V$ be orthogonal vectors of negative length in $\mathbb{Z}_{(p)}^\times$. Then $\delta = ef \in C(V)$ with $\delta^t = -\delta$ and $N(\delta) = \delta\delta^t \in \mathbb{Z}_{(p)}^\times$. This determines a symplectic form $\psi_\delta(x, y) = \text{tr}(x\delta y^t)$ on $C(V)$, for which the lattice $C(L)$ is p -unimodular. We obtain an embedding

$$\tilde{H} \rightarrow \text{GSp}(C(V))$$

and a morphism of Shimura varieties over \mathbb{Q} from X_K to the Siegel Shimura variety determined by the symplectic space $(C(V), \psi_\delta)$ and a suitable compact open subgroup.

The integral model of the Siegel Shimura variety induces then an integral model $\mathcal{X} = \mathcal{X}_K$ of X_K [Kis], [AGHM, Section 4]. Kisin showed that \mathcal{X} is smooth over $\mathbb{Z}_{(p)}$ if the compact open subgroup $K^p \subset H(\mathbb{A}_f^p)$ is sufficiently small.

By pulling back the universal abelian scheme, we obtain a polarized abelian scheme $(\mathcal{A}^{\text{KS}}, \lambda^{\text{KS}}, \eta^{\text{KS}})$ with level structure over \mathcal{X} , the *Kuga–Satake abelian scheme*. It is equipped with a right $C(L)$ -action.

Given a $\mathbb{Z}_{(p)}$ -scheme S and an S -point $\alpha : S \rightarrow \mathcal{X}$, we obtain a triple $\mathbf{A}_\alpha = (A, \lambda, \eta) = \alpha^*(\mathcal{A}^{\text{KS}}, \lambda^{\text{KS}}, \eta^{\text{KS}})$ by pulling back the Kuga–Satake scheme. In particular, η is a \tilde{K}^p -level structure

$$\eta : \mathbf{H}_{\mathbb{A}_f^p} := \bigotimes_{\substack{l < \infty \\ l \neq p}} (H_l^1(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \xrightarrow{\sim} C(V) \otimes_{\mathbb{Q}} \mathbb{A}_f^p,$$

sending $\mathbf{V}_{\mathbb{A}_f^p}$ (the étale realization of the motive associated to the representation of \tilde{H} on V) onto $V \otimes \mathbb{A}_f^p$. Let $V(\mathbf{A}_\alpha) \subset \text{End}_{C(L)}(A)_{(p)}$ be the space of special endomorphisms of \mathbf{A}_α defined in [So, Definition 3.3].

Given $T \in \text{Sym}_n(\mathbb{Q})$ with $\det T \neq 0$, the special cycle $\mathcal{Z}(T) \rightarrow \mathcal{X}$ is defined as the stack over \mathcal{X} with functor of points

$$\begin{aligned} \mathcal{Z}(T)(S) &= \{(\alpha, x) \mid \alpha \in \mathcal{X}(S), x = (x_1, \dots, x_n) \in V(\mathbf{A}_\alpha)^n, \\ &\quad Q(x) = T, \eta \circ x_j \circ \eta^{-1} \in \hat{L}^{(p)}\}, \end{aligned}$$

where $\hat{L}^{(p)} = \prod_{l \neq p} L_l$, and $\hat{L} = \prod_l L_l$. In this subsection we drop the Schwartz function φ from the notation of $\mathcal{Z}(T, \varphi)$, since we only consider it here for the characteristic function of \hat{L}^n .

Soylu [So, Proposition 3.7] showed that the image of the forgetful map

$$\mathcal{Z}(T) \rightarrow \mathcal{X}$$

sends $\mathcal{Z}(T)(\bar{\mathbb{F}}_p)$ into the supersingular locus $\mathcal{X}_{ss} \subset \mathcal{X}(\bar{\mathbb{F}}_p)$. According to [HP, Proposition 7.2.3], there exists an $\alpha_0 \in \mathcal{X}_{ss}$ such that the p -divisible group (X_0, λ_0) associated to \mathbf{A}_{α_0} is equal the p -divisible group $(\mathbb{X}_0, \lambda_0)$ considered in Section 6.1.

According to [HP, Theorem 7.2.4] or [She, Theorem 1.2], there is an isomorphism of formal schemes

$$\Theta : \mathbb{H}(\mathbb{Q}) \backslash \overline{\mathbb{RZ}} \times \mathbb{H}(\mathbb{A}_f^p) / K^p \cong \tilde{\mathbb{H}}(\mathbb{Q}) \backslash \mathbb{RZ} \times \tilde{\mathbb{H}}(\mathbb{A}_f^p) / \tilde{K}^p \cong (\hat{\mathcal{X}}_W)_{/\mathcal{X}_{ss}}, \tag{7.4}$$

where $(\hat{\mathcal{X}}_W)_{/\mathcal{X}_{ss}}$ is the completion of \mathcal{X}_W along the supersingular locus \mathcal{X}_{ss} , and $\overline{\mathbb{RZ}} = p^{\mathbb{Z}} \backslash \mathbb{RZ}$. The above discussion implies that for every $(\alpha, x) \in \mathcal{Z}(T)(\bar{\mathbb{F}}_p)$ the space of special endomorphisms satisfies

$$V(\mathbf{A}_\alpha) \otimes \mathbb{Q} \cong \mathbb{V},$$

where \mathbb{V} is the neighboring quadratic space over \mathbb{Q} associated with \mathcal{C} at p .

Proposition 7.2. *Let \mathbb{L} be a fixed lattice of \mathbb{V} such that \mathbb{L}_p is a dual vertex lattice in \mathbb{V}_p of type 2 as in Section 6.2 and $\hat{\mathbb{L}}_q \cong \hat{\mathbb{L}}_q$ for $q \neq p$. Let $\varphi_{\mathbb{L}} = \text{char}(\hat{\mathbb{L}}^n)$. Let $T \in \text{Sym}_n(\mathbb{Q})$ and assume that it satisfies the conditions of Theorem 6.3 at the prime p . Then*

$$|\mathcal{Z}(T)(\bar{\mathbb{F}}_p)| := \sum_{x \in \mathcal{Z}(T)(\bar{\mathbb{F}}_p)} \frac{1}{|\text{Aut}(x)|} = 2 \frac{C(\mathbb{L})}{\text{vol}(\mathbb{K}, d_{\mathbb{L}}h)} W_{T,f}(1, 0, \lambda(\varphi_{\mathbb{L}})) < \infty.$$

Here $\mathbb{K} = \mathbb{K}_p K^p$ is the compact open subgroup of $\mathbb{H}(\mathbb{A}_f)$ with $\mathbb{K}_p = \text{SO}(\mathbb{L}_p) \subset \mathbb{H}(\mathbb{Q}_p)$. In particular, if $\mathcal{Z}(T)(\bar{\mathbb{F}}_p)$ is not empty, then $\text{Diff}(\mathcal{C}, T) = \{p\}$.

Proof. Let $\pi : \mathcal{Z}(T) \rightarrow \mathcal{X}$ be the forgetful map, and identify via (7.4)

$$\mathcal{X}_{ss}(\bar{\mathbb{F}}_p) = \mathbb{H}(\mathbb{Q}) \backslash \overline{\mathbb{RZ}} \times \mathbb{H}(\mathbb{A}_f^p) / K^p,$$

where $\mathcal{X}_{ss}(\bar{\mathbb{F}}_p)$ denotes the supersingular locus of $\mathcal{X}(\bar{\mathbb{F}}_p)$. By a result of Soyly [So, Proposition 3.7] the image of $\mathcal{Z}(T)(\bar{\mathbb{F}}_p)$ lies in $\mathcal{X}_{ss}(\bar{\mathbb{F}}_p)$. Notice that $(\mathbf{A}, x) \in \mathcal{Z}(T)(\bar{\mathbb{F}}_p)$ implies that the p -divisible group X of \mathbf{A} belongs to $\mathcal{Z}(J(x))(\bar{\mathbb{F}}_p)$, where $J(x)$ is the sublattice of \mathbb{V}_p generated by the p -adic components of x (recall that the stabilizer of x in \hat{H}_p is trivial).

By Proposition 6.2, we have

$$\overline{\mathcal{Z}(J(x))}(\bar{\mathbb{F}}_p) = \bigsqcup_{\substack{t_\Lambda=2 \\ x \in \Lambda^n}} S_\Lambda(\bar{\mathbb{F}}_p) = \bigsqcup_{\substack{h_p \in \mathbb{H}(\mathbb{Q}_p)/\mathbb{K}_p \\ x \in h_p \mathbb{L}_p}} S_{h_p \mathbb{L}_p}(\bar{\mathbb{F}}_p),$$

where $\overline{\mathcal{Z}(J)}$ is the image of $\mathcal{Z}(J)$ in $\overline{\mathbb{R}\mathbb{Z}}$. Recall that $|S_\Lambda(\overline{\mathbb{F}}_p)| = 2$ for any dual vertex lattice $\Lambda \subset \mathbb{V}_p$ of type 2. So we find

$$\begin{aligned} \sum_{x \in \mathcal{Z}(T)(\overline{\mathbb{F}}_p)} \frac{1}{|\text{Aut}(x)|} &= 2 \sum_{\substack{x \in \mathbb{V}^n \\ Q(x)=T}} \sum_{h \in \mathbb{H}(\mathbb{Q}) \backslash \mathbb{H}(\mathbb{A}_f) / \mathbb{K}} \frac{1}{|\Gamma_h|} \varphi_{\mathbb{L}}(h^{-1}x) \\ &= \frac{2}{\text{vol}(\mathbb{K}, d_{\mathbb{L}}h)} \int_{\mathbb{H}(\mathbb{A}_f)} \varphi_{\mathbb{L}}(h^{-1}x) d_{\mathbb{L}}h, \end{aligned}$$

if there is an $x \in \mathbb{V}^n$ with $Q(x) = T$ (otherwise, it is zero). Here $d_{\mathbb{L}}$ is the Haar measure on $\mathbb{H}(\mathbb{A}_f)$ associated to the lattice \mathbb{L} , and $|\Gamma_h| = h^{-1}H(\mathbb{Q})h \cap K$. Now applying the local Siegel–Weil formula, we obtain the proposition. \square

Recall that the arithmetic degree of $\mathcal{Z}(T)$ at p is defined as

$$\widehat{\text{deg}}_p(\mathcal{Z}(T)) = \sum_{x \in \mathcal{Z}(T)(\overline{\mathbb{F}}_p)} \frac{\text{ht}_p(x)}{|\text{Aut}(x)|} \cdot \log p \tag{7.5}$$

where $\text{ht}_p(x)$ is the length of the étale local ring $\mathcal{O}_{\mathcal{Z}(T),x}$ of $\mathcal{Z}(T)$ at the point x . The following result is a refinement of Theorem 1.2(3).

Theorem 7.3. *Fix a prime number $p \neq 2$. Let $L \subset V$ be a p -unimodular lattice. Let $T \in \text{Sym}_n(\mathbb{Q})$ be such that T_p satisfies the conditions in Theorem 6.3. Then the arithmetic Siegel–Weil formula holds for T with*

$$\widehat{\text{deg}}_p(\mathcal{Z}(T)) \cdot q^T = C_{n,p} \cdot E'_T(\tau, 0, \lambda(\varphi_L) \otimes \Phi_\kappa),$$

where

$$C_{n,p} = -B_{n,\infty} \frac{C(L)}{\text{vol}(K, d_L h)}.$$

In particular, $C_{n,p} = C_{n,\infty}$.

Proof. By Theorem 6.9 and Propositions 7.2, 6.8, and 2.3 we have

$$\begin{aligned} \widehat{\text{deg}}_p(\mathcal{Z}(T)) &= \sum_{x \in \mathcal{Z}(T)(\overline{\mathbb{F}}_p)} \frac{1}{|\text{Aut}(x)|} \cdot \text{ht}_p(x) \log p \\ &= \frac{2C(\mathbb{L})}{\text{vol}(\mathbb{K}, d_{\mathbb{L}}h)} W_{T,f}(1, 0, \lambda(\varphi_{\mathbb{L}})) \cdot \nu_p(T_2) \log p \\ &= \frac{2C(L)}{\text{vol}(K, d_L h)} \frac{\text{vol}(K_p, d_{L_p})}{C(L_p)} \frac{C(\mathbb{L}_p)}{\text{vol}(\mathbb{K}_p, d_{\mathbb{L}_p}h)} W_{T,p}(1, 0, \lambda(\varphi_{\mathbb{L}_p})) \\ &\quad \cdot \left(\prod_{q \nmid p} W_{T,q}(1, s, \lambda(\varphi_{L_q})) \right) \Big|_{s=0} \cdot \frac{W'_{T,p}(1, 0, \lambda(\varphi_{L_p}))}{W_{\tilde{T},p}(1, 0, \lambda(\varphi_{L_p}))} \\ &= \frac{2C(L)}{\text{vol}(K, d_L h)} \cdot \frac{E'_T(1, 0, \lambda(\varphi_L) \otimes \Phi_\kappa)}{W_{T,\infty}(1, 0, \Phi_\kappa)}. \end{aligned}$$

Here $\tilde{T} \in \text{Sym}_n(\mathbb{Z}_p)$ is any p -unimodular matrix. Remark 4.4 gives

$$-\frac{1}{2} B_{n,\infty} \cdot W_{T,\infty}(1, 0, \Phi_\kappa) = e^{-2\pi \text{tr } T} \tag{7.6}$$

for any positive definite $n \times n$ matrix T . So

$$-\frac{1}{2} B_{n,\infty} \cdot (\det v)^{-\kappa/2} W_{T,\infty}(g_\tau, 0, \Phi_\kappa) = q^T.$$

Hence we obtain the claimed formula. \square

Remark 7.4. In this subsection we have assumed for convenience that φ is the characteristic function of \hat{L} , and that there is a compact open subgroup $\tilde{K} \subset \tilde{H}(\mathbb{A}_f)$ containing $\tilde{\mathbb{Z}}^\times$ and mapping onto K . Both can be relaxed. First, we can naturally modify the definition $\mathcal{Z}(T)$ in [So] to include $\mathcal{Z}(T, \varphi)$ for all $\varphi = \varphi_p \varphi^p \in S(V(\mathbb{A}_f)^n)^K$ with $\varphi_p = \text{char}(L_p^n)$. The proof of Proposition 7.2 goes through without any change. As already mentioned, the assumption on K is always fulfilled if there exists an even lattice $M \subset V$ which is stabilized by K and such that K acts trivially on M'/M . In other words, every ‘sufficiently small’ compact open subgroup K satisfies the condition. Finally, we indicate how the results can be modified to hold for general K . Take a compact open subgroup \tilde{K}_1 of $\tilde{H}(\mathbb{A}_f)$ containing $\tilde{\mathbb{Z}}^\times$ such that its image K_1 in $H(\mathbb{A}_f)$ is contained in K . Then there is a natural projection $\mathcal{X}_{K_1} \rightarrow \mathcal{X}_K$ and an analogous projection of Rapoport–Zink spaces. The p -adic uniformization identity (7.4) still holds according to [She, Theorem 1.2]. For $\varphi \in S(V(\mathbb{A}_f)^n)^K$, the special cycle $\mathcal{Z}_{K_1}(T, \varphi)$ is K -invariant and descends to a special cycle $\mathcal{Z}_K(T, \varphi)$ on \mathcal{X}_K .

Remark 7.5. Assume that $\text{Diff}(\mathcal{C}, T) = \{p\}$. We observe the following variant of the local arithmetic Siegel–Weil formula:

$$\begin{aligned} \text{ht}_p(x) \log p &= \frac{W'_{T,p}(1, 0, \lambda(\varphi_L))}{W^L_{\tilde{T},p}(1, 0, \lambda(\varphi_L))}, & p < \infty, \text{ where } \tilde{T} \text{ is } p\text{-unimodular,} \\ \frac{1}{2} \text{ht}_\infty(x) &= \frac{W'_{T,\infty}(1, 0, \Phi_\kappa)}{W^L_{\tilde{T},\infty}(1, 0, \Phi_\kappa)}, & p = \infty, \text{ where } \tilde{T} \text{ is positive definite with } \text{tr } \tilde{T} = \text{tr } T. \end{aligned}$$

Here the extra $\frac{1}{2}$ makes sense as we take the integral over the whole symmetric domain \mathcal{D} instead of its connected component \mathcal{D}^+ , while at a finite prime p , we did it at each individual point (connected component). This reinterpretation is different from the previous ones used in [KRY2], [KRap1], [KRap2], and [HY] among others.

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