J. Eur. Math. Soc. 23, 1797–1857 (2021) DOI 10.4171/JEMS/1044

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Pollicott–Ruelle spectrum and Witten Laplacians

Received April 5, 2018

Abstract. We study the asymptotic behavior of eigenvalues and eigenmodes of the Witten Laplacian on a smooth compact Riemannian manifold without boundary. We show that they converge to the Pollicott–Ruelle spectral data of the corresponding gradient flow acting on appropriate anisotropic Sobolev spaces. As an application of our methods, we also construct a natural family of quasimodes satisfying the Witten–Helffer–Sjöstrand tunneling formulas and the Fukaya conjecture on Witten deformation of the wedge product.

Keywords. Semiclassical analysis, hyperbolic dynamical systems, gradient flow, anisotropic Sobolev space, tunneling formula

1. Introduction

Let M be a smooth ($\mathcal{C}^\infty$), compact, oriented, boundaryless manifold of dimension $n \geq 1$. Let $f : M \to \mathbb{R}$ be a smooth Morse function whose set of critical points is denoted by $\mathrm{Crit}(f)$. In [60], Witten introduced the following semiclassical deformation of the de Rham coboundary operator:

$$\forall h > 0, \quad d_{f,h} := e^{-f/h} d e^{f/h} = d + \frac{df}{h} \wedge : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$$

where $\Omega^\bullet(M)$ denotes the smooth differential forms on M. Then, fixing a smooth Riemannian metric g on M, he considered the adjoint of this operator with respect to the induced scalar product on the space $L^2(M, \Lambda(T^*M))$ of L^2 forms:

$$\forall h > 0, \quad d_{f,h}^* = d^* + \frac{\iota_{V_f}}{h} : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M),$$

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Mathematics Subject Classification (2020): Primary 58J50; Secondary 37D15, 35P20, 35S35, 58J10

where V_f is the gradient vector field associated with the pair (f, g), i.e. the unique vector field satisfying

$$\forall x \in M, \quad df(x) = g_x(V_f(x), \cdot).$$

The operator $d_{f,h} + d_{f,h}^*$ is the analog of a Dirac operator and its square is usually defined to be the Witten Laplacian [60]. In the present paper, we take a different convention and we choose to rescale it by a factor $h/2$. Hence, the *Witten Laplacian* will be defined as

$$\boxed{W_{f,h} := \frac{h}{2}(d_{f,h}d_{f,h}^* + d_{f,h}^*d_{f,h}) = e^{-f/h}(\mathcal{L}_{V_f} + h\Delta_g/2)e^{f/h}},$$

where $\mathcal{L}_{V_f}$ is the Lie derivative along the gradient vector field. This defines a self-adjoint, elliptic operator whose principal symbol coincides with the principal symbol of the Hodge–de Rham Laplace operator acting on forms. It has a discrete spectrum on $L^2(M, \Lambda^k(T^*M))$ that we denote, for every $0 \leq k \leq n$, by

$$0 \leq \lambda_1^{(k)}(h) \leq \lambda_2^{(k)}(h) \leq \cdots \leq \lambda_j^{(k)}(h) \to +\infty \quad \text{as } j \to +\infty.$$

It follows from the works of Witten [60] and Helffer–Sjöstrand [43] that there exists a constant $\epsilon_0 > 0$ such that, for every $0 \leq k \leq n$ and for every $h > 0$ small enough, there are exactly $c_k(f)$ eigenvalues inside the interval $[0, \epsilon_0]$, where $c_k(f)$ is the number of critical points of index k—see e.g. the recent proof of Michel and Zworski [49, Prop. 1]. Building on the strategy initiated by Witten, Helffer and Sjöstrand also showed that one can associate to these low energy eigenmodes an orientation complex whose Betti numbers are the same as the Betti numbers of the manifold [43, Th. 0.1]. Another approach to this question was developed by Bismut and Zhang in their works on the Reidemeister torsion [5, 6, 62]: following Laudenbach [47], they interpreted the Morse complex in terms of currents.

The aim of our article is to describe the convergence of *all the spectral data* (meaning both eigenvalues and eigenmodes) of the Witten Laplacian. This will be achieved by using microlocal techniques that were developed in the context of dynamical systems [17, 18]. Note that *part of these results* could probably be obtained by more classical methods in the spirit of the works of Simon [55] and Helffer–Sjöstrand [43] on harmonic oscillators. We refer the reader to the book of Helffer and Nier [42] for a detailed account of the state of the art on these aspects. Regarding the convergence of the spectrum, Frenkel, Losev and Nekrasov [29] did very explicit computations of Witten's spectrum for the case of the height function on the sphere, and they implicitly connect this spectrum to a dynamical spectrum as we shall do here. They also give a strategy to derive asymptotic expansions for dynamical correlators of holomorphic gradient flows acting on compact Kähler manifolds. Yet, unlike [29], we attack the problem from the dynamical viewpoint rather than from the semiclassical perspective. Also, we work in the $\mathcal{C}^\infty$ case instead of the compact Kähler case and we make use of tools from microlocal analysis to replace tools from complex geometry.

The main purpose of the present work is to propose an approach to these problems having a more dynamical flavour than these references. We stress that our study of the limit operator is self-contained and that it does not make use of the tools developed in the above references. It is more inspired by the study of the so-called transfer operators in dynamical systems [37, 2, 65], and this dynamical perspective allows us to make some explicit connection between the spectrum of the Witten Laplacian and the dynamical results from [17, 19].

Conventions. All along the article, we denote by $\Omega^k(M)$ the set of $\mathcal{C}^\infty$ differential forms of degree k, i.e. smooth sections $M \to \Lambda^k(T^*M)$. The topological dual of $\Omega^{n-k}(M)$ is the set of currents of degree k and it will be denoted by $\mathcal{D}'^k(M)$, meaning differential k-forms with coefficients in the set of distributions [54].

2. Main results

2.1. Semiclassical versus dynamical convergence and a question by Harvey–Lawson

In order to illustrate our results, we let φ_f^t be the flow induced by the gradient vector field V_f, and, given any critical point a of f of index k, we introduce its unstable manifold

$$W^u(a) := \left\{x \in M : \lim_{t\to-\infty} \varphi_f^t(x) = a\right\}.$$

Recall from the works of Smale [56] that this defines a smooth embedded submanifold of M whose dimension is equal to $n-k$ and whose closure is a union of unstable manifolds under the so-called Smale transversality assumption. Then, among other results, we shall prove the following theorem:

Theorem 2.1 (Semiclassical versus dynamical convergence). *Let f be a smooth Morse function and g be a smooth Riemannian metric such that V_f is $\mathcal{C}^1$-linearizable near every critical point and such that V_f satisfies the Smale transversality assumption. Let $0 \leq k \leq n$. Then, for every $a \in \mathrm{Crit}(f)$ of index k, there exists (U_a, S_a) in $\mathcal{D}'^k(M)\times\mathcal{D}'^{n-k}(M)$ such that the support of U_a is equal to $\overline{W^u(a)}$ and*

$$\mathcal{L}_{V_f}(U_a) = 0.$$

Moreover, there exists $\epsilon_0 > 0$ small enough such that, for every $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$ and every $0 < \epsilon < \epsilon_0$,

$$\begin{aligned}\lim_{t\to+\infty} \int_M \varphi_f^{-t*}(\psi_1) \wedge \psi_2 &= \lim_{h\to 0^+} \int_M \mathbf{1}_{[0,\epsilon]}(W_{f,h}^{(k)})(e^{-f/h}\psi_1) \wedge (e^{f/h}\psi_2) \\ &= \sum_{a:\,\dim W^u(a)=n-k} \int_M \psi_1 \wedge S_a \int_M U_a \wedge \psi_2, \qquad (1)\end{aligned}$$

where $\mathbf{1}_{[0,\epsilon]}(W_{f,h}^{(k)})$ is the spectral projector on $[0,\epsilon]$ for the self-adjoint elliptic operator $W_{f,h}^{(k)}$.

Remark 2.2. The Smale transversality assumption means that the stable and unstable manifolds satisfy some transversality conditions [56]—see §3.1.1 for a brief reminder. Recall that, given a Morse function f, this property is satisfied by a dense open set of Riemannian metrics thanks to the Kupka–Smale Theorem [46, 57]. The hypothesis of being $\mathcal{C}^1$-linearizable near every critical point means that, near every a in $\mathrm{Crit}(f)$, one can find a $\mathcal{C}^1$-chart such that the vector field can be written locally as $V_f(x) = L_f(a)x\partial_x$, where $L_f(a)$ is the unique (symmetric) matrix satisfying $d^2 f(a) = g_a(L_f(a)\cdot, \cdot)$. By fixing a finite number of nonresonance conditions on the eigenvalues of $L_f(a)$, the Sternberg–Chen Theorem [51] ensures that, for a given f, one can find an open and dense subset of Riemannian metrics satisfying this property.

Let us now comment on the several statements contained in this theorem. First, as we shall see in Lemma 5.10, the current U_a coincides with the current of integration $[W^u(a)]$ when restricted to the open set $M \setminus \partial W^u(a)$ with $\partial W^u(a) = \overline{W^u(a)} \setminus W^u(a)$. Hence, the first part of the theorem shows how one can extend $[W^u(a)]$ to a globally defined current which still satisfies the transport equation $\mathcal{L}_{V_f}(U_a) = 0$. This extension was produced by Laudenbach in the case of locally flat metrics in [47] by carefully analyzing the structure of the boundary $\partial W^u(a)$. Here, we make this extension for more general metrics via a spectral method and the analysis of the structure of the boundary is in some sense hidden in the construction of our spectral framework [17, 18]. We emphasize that Laudenbach's construction shows that these extensions are currents of *finite mass* while our method does not say a priori anything about that aspect.

The second part of the theorem shows that several quantities that appeared in previous analytical works on Morse theory coincide. In the case of a locally flat metric, the fact that the first and the third quantity in (1) are equal was shown by Harvey and Lawson [41]. In [17], we showed how to prove this equality when the flow satisfies some (smooth) linearization properties more general than the ones appearing in [47, 41]. Here, we will extend the argument from [17] to show that this equality remains true under the rather weak assumptions of Theorem 2.1. The last equality tells us that the low eigenmodes of the Witten Laplacian converge in a weak sense to the same quantities. In particular, it recovers the fact that the number of small eigenvalues in degree k is equal to the number of critical points of index k.

In a nutshell, our theorem identifies a certain semiclassical limit of scalar product of quasimodes for the Witten Laplacian with a large time limit of some dynamical correlation for the gradient flow which converges to equilibrium:

$$\boxed{\lim_{h\to 0^+} \underbrace{\langle \mathbf{1}_{[0,\epsilon]}(W_{f,h}^{(k)})(e^{-f/h}\psi_1), e^{f/h}\psi_2\rangle_{L^2}}_{\text{quantum object}} = \lim_{t\to+\infty} \underbrace{\langle \varphi_f^{-t*}(\psi_1), \psi_2\rangle_{L^2}}_{\text{dynamical object}}}$$

for every $(\psi_1, \psi_2) \in \Omega^k(M)^2$. From this point of view, this theorem gives some insight on a question raised by Harvey and Lawson in [40, Intro.] who asked about the connection between their approach to Morse theory and Witten's approach.

Remark 2.3. In order to get another intuition on the content of this theorem, let us write formally that

$$\lim_{t\to+\infty}\varphi_f^{-t*} = \lim_{t\to+\infty} e^{-t\mathcal{L}_{V_f}} = \lim_{t\to+\infty}\lim_{h\to 0^+} e^{-t(\mathcal{L}_{V_f}+h\Delta_g/2)}$$
$$= \lim_{t\to+\infty}\lim_{h\to 0^+} e^{f/h}e^{-tW_{f,h}}e^{-f/h}.$$

It is then tempting to exchange the two limits, and Theorem 2.1 shows that intertwining these two limits requires taking into account the small eigenvalues of the Witten Laplacian.

Proving the second part of this theorem amounts to *determining the limit of the spectral projectors* of the Witten Laplacian (after conjugation by $e^{f/h}$) viewed as operators from $\Omega^k(M)$ to $\mathcal{D}'^k(M)$. Recall that Helffer–Sjöstrand [43, §1] and Bismut–Zhang [6, Def. 6.6] constructed explicit bases for the bottom of the spectrum of the Witten Laplacian. Using the approach of these references, we would have to verify that these quasimodes (after renormalization by $e^{f/h}$) converge to the currents constructed by Laudenbach [47]—see [13, Ch. 9] for a related discussion. As far as we know, this question has not been addressed explicitly in the literature. This convergence will come out naturally of our spectral analysis. We will in fact show the convergence of *all* the spectral projectors (not only at the bottom of the spectrum) and identify their limits in terms of dynamical quantities—see Theorem 2.4 below.

2.2. *Asymptotics of Witten spectral data*

Before stating our results on the convergence of the spectral data of the Witten Laplacian, we need to describe a dynamical question which was studied in great detail in [17] in the case of Morse–Smale gradient flows—see also [3, 22] for earlier related results. Recall that a classical question in dynamical systems is to study the asymptotic behavior of the correlation function

$$\boxed{\forall 0\le k\le n,\ \forall(\psi_1,\psi_2)\in\Omega^k(M)\times\Omega^{n-k}(M),\qquad C_{\psi_1,\psi_2}(t) := \int_M \varphi_f^{-t*}(\psi_1)\wedge\psi_2,}$$

which already appeared in the statement of Theorem 2.1. Following [52, 53], it will in fact be simpler to consider the Laplace transform of $t\mapsto\varphi_f^{-t*}$, i.e. for $\mathrm{Re}(z)$ large enough,

$$\boxed{\widehat{R}_k(z) = (z+\mathcal{L}_{V_f}^{(k)})^{-1} := \int_0^{+\infty} e^{-tz}\varphi_f^{-t*}\,dt : \Omega^k(M)\to\mathcal{D}'^k(M).}$$

One of the consequences of our results from [17, 18] is that this Laplace transform admits a meromorphic extension from $\mathrm{Re}(z) > C_0$ (with $C_0 > 0$ large enough) to $\mathbb{C}$ under the assumptions of Theorem 2.1. In [17, 19], we also gave an explicit description of the poles and residues of this function under $\mathcal{C}^\infty$-linearization properties of the vector field V_f.

These assumptions were for instance satisfied when infinitely many nonresonance assumptions are satisfied. We shall explain in Theorem 5.1 how to recover this result under the weaker assumptions of Theorem 2.1.

Proving such a meromorphic extension is part of the study of Pollicott–Ruelle resonances in the theory of hyperbolic dynamical systems. We refer for instance to the book of Baladi [2] or to the survey article of Gouëzel [37] for detailed accounts and references related to these dynamical questions. More specifically, we used a microlocal approach to deal with these spectral problems. We also refer to the survey of Zworski [65] for the relation of these questions to scattering theory from the microlocal viewpoint. Coming back to dynamical systems, the Pollicott–Ruelle resonances are interpreted as the spectrum of $-\mathcal{L}_{V_f}$ on appropriate Banach spaces of currents. In the following, we shall denote by $\mathcal{R}_k$ the poles of the meromorphic continuation of $\widehat{R}_k(z)$, and by $\pi_{z_0}^{(k)}$ the residue at each $z_0 \in \mathbb{C}$. These poles are the so-called Pollicott–Ruelle resonances, while the range of the residues are the resonant states. They correspond to the spectral data of $-\mathcal{L}_{V_f}$ on appropriate anisotropic Sobolev spaces of currents and they describe in some sense the structure of the long time dynamics of the gradient flow. Our main spectral result shows that the spectral data of the Witten Laplacian converges to this Pollicott–Ruelle spectrum. More precisely, one has

Theorem 2.4 (Convergence of Witten spectral data). *Suppose that the assumptions of Theorem 2.1 are satisfied. Let $0 \leq k \leq n$. Then*

(1) *for every $j \geq 1$, $-\lambda_j^{(k)}(h)$ converges as $h \to 0^+$ to some $z_0 \in \mathcal{R}_k$,*

(2) *conversely, any $z_0 \in \mathcal{R}_k$ is the limit of a sequence $(-\lambda_j^{(k)}(h))_{h\to 0^+}$.*

Moreover, for any $z_0 \in \mathbb{R}$, there exists $\epsilon_0 > 0$ small enough such that, for every $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$,

$$\forall 0 < \epsilon \leq \epsilon_0, \qquad \lim_{h\to 0^+} \int_M \mathbf{1}_{[z_0-\epsilon, z_0+\epsilon]}(-W_{f,h}^{(k)})(e^{-f/h}\psi_1) \wedge (e^{f/h}\psi_2) = \int_M \pi_{z_0}^{(k)}(\psi_1) \wedge \psi_2.$$

Following Theorem 5.1 below, this result shows that the Witten eigenvalues converge, as $h \to 0$, to integer combinations of Lyapunov exponents. Small eigenvalues are known to be exponentially small in terms of h [43, 42, 49] but our proof does not say a priori anything about this aspect of the Witten–Helffer–Sjöstrand result. The convergence of Witten eigenvalues could be recovered from the techniques of [43, 55] but the convergence of spectral projectors would be more subtle to prove with these kind of semiclassical methods as one would first need to identify the limit. In fact, this theorem also tells us that, up to renormalization by $e^{f/h}$, the spectral projectors of the Witten Laplacian converge to the residues of the dynamical correlation function. In Section 5, we will describe more precisely the properties of these limiting spectral projectors.

2.3. Witten–Helffer–Sjöstrand's instanton formulas

Following [20], we can verify that $((U_a)_{a\in \mathrm{Crit}(f)}, d)$ generate a finite-dimensional complex which is nothing other than the Thom–Smale–Witten complex [60, Eq. (2.2)]. In

Section 6, we will explain how to prove some topological statement which complements what was proved in [20]:

Theorem 2.5 (Witten's instanton formula). *Suppose that the assumptions of Theorem 2.1 are satisfied. Then, for every pair of critical points (a, b) with $\mathrm{ind}(b) = \mathrm{ind}(a)+1$, there exists*[1] $n_{ab} \in \mathbb{Z}$ *such that*

$$\forall a \in \mathrm{Crit}(f), \qquad dU_a = \sum_{b:\,\mathrm{ind}(b)=\mathrm{ind}(a)+1} n_{ab} U_b \tag{2}$$

where n_{ab} counts algebraically the number of instantons connecting a and b. In particular, the complex $((U_a)_{a\in Crit(f)}, d)$ can be defined over $\mathbb{Z}$ and realizes in the space of currents the Morse homology over $\mathbb{Z}$.

In the case of locally flat metrics, this relation between the formula for the boundary of unstable currents and Witten's instanton formula follows for instance from the work of Laudenbach [47], and his proof could probably be revisited to deal with more general metrics. Yet, the proof we will give of this result is of completely different nature and it will be based on our spectral approach to the problem. The main difference with [17, 20] is that, in these references, we were able to prove that the complex $((U_a)_{a\in\mathrm{Crit}(f)}, d)$ forms a subcomplex of the de Rham complex of currents which was quasi-isomorphic to the de Rham complex $(\Omega^\bullet(M), d)$ but we worked in the (co)homology with coefficients in $\mathbb{R}$. The instanton formula (2) allows us to actually consider $((U_a)_{a\in\mathrm{Crit}(f)}, d)$ as a $\mathbb{Z}$-module and directly relate it to the famous Morse complex defined over $\mathbb{Z}$ appearing in the literature whose integral homology groups contain more information than those with real coefficients [35, p. 620].

Coming back to the bottom of the spectrum of the Witten Laplacian, we can define an analogue of Theorem 2.5 at the semiclassical level. For that purpose, we need to introduce analogues of Helffer–Sjöstrand WKB states for the Witten Laplacian [43, 42]. We fix $\epsilon_0 > 0$ small enough so that the range of $\mathbf{1}_{[0,\epsilon_0)}(W_{f,h}^{(k)})$ in every degree k is equal to the number of critical points of index k. Then, for $h > 0$ small enough, we define the following *WKB states*:

$$\boxed{U_a(h) := \mathbf{1}_{[0,\epsilon_0)}(W_{f,h}^{(k)})(e^{\frac{f(a)-f}{h}} U_a) \in \Omega^k(M),} \tag{3}$$

where k is the index of the point a. We will show in Proposition 7.5 that, for every critical point a, the sequence $(e^{\frac{f-f(a)}{h}} U_a(h))_{h\to 0^+}$ converges to U_a in $\mathcal{D}'(M)$. As a corollary of Theorem 2.5, these WKB states also satisfy the following *exact* tunneling formulas:

Corollary 2.6 (Witten–Helffer–Sjöstrand tunneling formula). *Suppose that the assumptions of Theorem 2.1 are satisfied. Then, for every critical point a of f and for every $h > 0$ small enough,*

$$d_{f,h} U_a(h) = \sum_{b:\,\mathrm{ind}(b)=\mathrm{ind}(a)+1} n_{ab} e^{\frac{f(a)-f(b)}{h}} U_b(h),$$

[1] An explicit expression is given in §6.1.

where n_{ab} is as in Theorem 2.5.

The formula we obtain may seem slightly different from the one appearing in [43, Eq. (3.27)]—see also[2] [6, Th. 6.12] when f is a Morse function satisfying $f(a) = \mathrm{ind}(a)$ for every critical point a. This is mostly due to the choice of normalization, and we will compare our quasimodes more precisely with the ones of Helffer–Sjöstrand in Section 8.

2.4. *A conjecture by Fukaya*

As a last application of our analysis, we would like to show that our family $(U(h))_{h\to 0^+}$ of WKB states also satisfies Fukaya's asymptotic formula for Witten's deformation of the wedge product [32, Conj. 4.1]. This approach could probably be adapted to treat the case of higher order products. Yet, this would be at the expense of a more delicate combinatorial work that would be beyond the scope of the present article and we shall discuss this elsewhere. Recall that this conjecture was recently solved by Chan–Leung–Ma [12] via WKB approximation methods [43] which are different from our approach.

Let us now precisely describe the framework of Fukaya's conjecture for products of order 2 which corresponds to the classical wedge product $\wedge$—see also §7.6 for more details. Consider three smooth real valued functions (f_1, f_2, f_3) on M, and consider their differences:

$$f_{12} = f_2 - f_1, \quad f_{23} = f_3 - f_2, \quad f_{31} = f_1 - f_3.$$

We assume that the functions (f_{12}, f_{23}, f_{31}) are Morse. To every such pair (ij), we associate a Riemannian metric g_{ij}, and we make the assumption that the corresponding gradient vector fields $V_{f_{ij}}$ satisfy the Morse–Smale property[3] and that they are $\mathcal{C}^1$-linearizable. In particular, they are amenable to the above spectral analysis, and, for any critical point a_{ij} of f_{ij} and for every $0 < h \leq 1$, we can consider a WKB state $U_{a_{ij}}(h)$. From elliptic regularity, these are smooth differential forms on M and Fukaya predicted that the integral

$$\int_M U_{a_{12}}(h) \wedge U_{a_{23}}(h) \wedge U_{a_{31}}(h)$$

has a nice asymptotic formula whenever the intersection $W^u(a_{12}) \cap W^u(a_{23}) \cap W^u(a_{31})$ consists of finitely many points. Note that, for this integral to make sense, we implicitly suppose that

$$\dim W^u(a_{12}) + \dim W^u(a_{23}) + \dim W^u(a_{31}) = 2n. \tag{4}$$

Let us explain the difficulty behind this question. After renormalization, a way to solve this conjecture amounts first to proving the convergence of the family of smooth differential forms

$$\tilde{U}_{a_{ij}}(h) := e^{\frac{f_{ij} - f_{ij}(a_{ij})}{h}} U_{a_{ij}}(h)$$

in the *space of currents* as $h \to 0^+$. As already said, we are not aware of a place in the literature where this convergence of (renormalized) Witten quasimodes is handled, as this

[2] Note that the proof from this reference makes use of Laudenbach's construction [47] while the one from [43] is self-contained.

[3] This means that the $V_{f_{ij}}$ satisfy the Smale transversality assumptions.

is not the approach followed in [43, 6, 12] to prove tunneling formulae. Our construction shows that these smooth forms indeed converge to $U_{a_{ij}}$ in the space of currents. The additional difficulty one has to treat in order to answer Fukaya's question is the following. Even if the currents $\lim_{h\to 0^+} \tilde{U}_{a_{ij}}(h)$ exist and even if the wedge product of their limits makes sense, it is not clear that we can interchange the limits as follows:

$$\int_M \lim_{h\to 0^+} \tilde{U}_{a_{12}}(h)\wedge \lim_{h\to 0^+} \tilde{U}_{a_{23}}(h)\wedge \lim_{h\to 0^+} \tilde{U}_{a_{31}}(h) = \lim_{h\to 0^+} \int_M \tilde{U}_{a_{12}}(h)\wedge\tilde{U}_{a_{23}}(h)\wedge\tilde{U}_{a_{31}}(h).$$

In order to justify this, the second difficulty of Fukaya's question is to show that convergence holds in the appropriate topology involving control of the wavefront set of the currents.

Without additional assumptions, there is no reason why all this would be true. Fukaya thus requires that the triple (f_{12}, g_{12}), (f_{23}, g_{23}) and (f_{31}, g_{31}) satisfy the *generalized Morse–Smale property* [45, §6.8]. That is, for every

$$(a_{12}, a_{23}, a_{31}) \in \mathrm{Crit}(f_{12}) \times \mathrm{Crit}(f_{23}) \times \mathrm{Crit}(f_{31}),$$

one has, for every $x \in W^u(a_{12}) \cap W^u(a_{23}) \cap W^u(a_{31})$,

$$T_x M = (T_x W^u(a_{12}) \cap T_x W^u(a_{23})) + T_x W^u(a_{31}), \tag{5}$$

and similarly for any permutation of (12, 23, 31). Note that, to every Morse function, we associate a priori a different metric. As for the Morse–Smale property, the Kupka–Smale method [46, 57] applies: the generalized Morse–Smale property is satisfied in an open and dense subset of smooth functions (f_1, f_2, f_3) and of smooth metrics (g_{12}, g_{23}, g_{31}). Our last result shows that the WKB states we have constructed satisfy Fukaya's conjecture under this geometric assumption:

Theorem 2.7 (Fukaya's instanton formula). *In the above notation, let* $(V_{f_{12}}, V_{f_{23}}, V_{f_{31}})$ *be a family of Morse–Smale gradient vector fields which are* $\mathcal{C}^1$*-linearizable, and which have the generalized Morse–Smale property. Then, for every*

$$(a_{12}, a_{23}, a_{31}) \in \mathrm{Crit}(f_{12}) \times \mathrm{Crit}(f_{23}) \times \mathrm{Crit}(f_{31})$$

such that $\dim W^u(a_{12}) + \dim W^u(a_{23}) + \dim W^u(a_{31}) = 2n$,

$$U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}}$$

defines an element of $\mathcal{D}'^n(M)$ *satisfying* $\int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}} \in \mathbb{Z}$*, and*

$$\lim_{h\to 0^+} e^{-\frac{f_{12}(a_{12})+f_{23}(a_{23})+f_{31}(a_{31})}{h}} \int_M U_{a_{12}}(h) \wedge U_{a_{23}}(h) \wedge U_{a_{31}}(h) = \int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}}.$$

Recall that the integers $\int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}}$ defined by triple intersections of unstable currents have a deep geometrical meaning. On the one hand, these integers actually *count* the number of Y-shaped gradient flow trees [30, p. 8] as described in §7.5. On the other hand, they give a representation of the cup-product in Morse cohomology at the cochain level—see §7.6 for a brief reminder. Hence, the second part of this theorem shows that if we define an analogue of the cup-product at the semiclassical level, then it converges, up to some renormalization factors, to the usual cup-product in Morse cohomology. We emphasize that the main new property in this theorem is really the asymptotic formula as $h \to 0^+$. Up to some normalization factors, this is exactly the asymptotic formula conjectured by Fukaya for the WKB states of Helffer–Sjöstrand [43]. Here, our states are constructed in a slightly different manner. Yet, they belong to the same eigenspaces as the ones from [43]—see Section 8 for a comparison. Finally, we note that, going through the details of the proof, we would find that the rate of convergence is in fact of order $\mathcal{O}(h)$. However, for simplicity of exposition, we do not keep track of this aspect in our argument.

2.5. *Lagrangian intersections*

We would like to recall the nice symplectic interpretation of the exponential prefactors appearing in Theorems 2.6 and 2.7. Let us start with the case of Theorem 2.6 where we only consider a pair $(f, 0)$ of functions where $f - 0 = f$ is Morse. We can consider the pair of exact Lagrangian submanifolds $\Lambda_f := \{(x; d_x f) : x \in M\} \subset T^*M$ and $\underline{0} \subset T^*M$.[4] Given (a, b) in $\text{Crit}(f)^2$, we can define a disk D whose boundary $\partial D \subset \Lambda_f \cup \underline{0}$ is a 2-gon made up of the union of two smooth curves e_1 and e_2 joining a and b, contained in the Lagrangian submanifolds Λ_f and $\underline{0}$ respectively. Denote by θ the Liouville one-form and by $\omega = d\theta$ the canonical symplectic form on T^*M. Then by the Stokes formula,

$$\int_D \omega = \int_{\partial D} \theta = \int_{e_1} df = f(a) - f(b),$$

where we choose e_1 to be oriented from b to a. Hence, the exponents in the asymptotic formula of Theorem 2.6 can be interpreted as the symplectic area of the disk D defined by Λ_f and the zero section of T^*M.[5] In the semiclassical terminology of [42, §15], this quantity is controlled by the Agmon distance associated with the potential $\|d_x f\|^2_{g^*(x)}$. Yet, it does not seem to have an interpretation as the action along some Hamiltonian trajectory.

A similar geometric interpretation holds in the case of Theorem 2.7. We consider a triangle (3-gon) T inside T^*M with vertices $(v_{12}, v_{23}, v_{31}) \in (T^*M)^3$ whose projections on M are (a_{12}, a_{23}, a_{31}). The edges (e_1, e_2, e_3) are contained in the three Lagrangian submanifolds Λ_{f_1}, Λ_{f_2} and Λ_{f_3}. To go from v_{23} to v_{12}, we follow some smooth curve e_2 in Λ_{f_2}, from v_{31} to v_{23} we follow some line e_3 in Λ_{f_3} and from v_{12} to v_{31}, we follow some line e_1 in Λ_{f_1}. These three lines define the triangle T and we can compute

$$\int_T \theta = \sum_{j=1}^{3} \int_{e_j} df_j = -f_1(a_{12}) + f_1(a_{31}) - f_2(a_{23}) + f_2(a_{12}) - f_3(a_{31}) + f_3(a_{23}),$$

which is (up to sign) the term appearing in the exponential factor of Theorem 2.7. Note that the triangle T does not necessarily bound a disk.

[4] Λ_f is the graph of df, whereas the zero section is the graph of 0.

[5] Hence the name *disk instantons*.

2.6. *Convergence of the Witten Laplacian to the gradient vector field*

The *key observation* used to prove several of our results is the following exact relation [29, Eq. (3.6)]:

$$\boxed{e^{f/h}W_{f,h}e^{-f/h} = \mathcal{L}_{V_f} + h\Delta_g/2,} \tag{6}$$

where $\mathcal{L}_{V_f}$ is the Lie derivative along the gradient vector field and $\Delta_g = dd^* + d^*d \geq 0$ is the Laplace–Beltrami operator. Indeed, one has [29, Eqs. (3.4), (3.5)]

$$e^{f/h}W_{f,h}e^{-f/h} = \frac{h}{2}e^{f/h}(d_{f,h} + d_{f,h}^*)^2 e^{-f/h} = \frac{h}{2}(d + d_{2f,h}^*)^2 = \frac{h}{2}(dd_{2f,h}^* + d_{2f,h}^* d),$$

which yields (6) thanks to the Cartan formula. Hence, the rough idea is to prove that the spectrum of the Witten Laplacian converges to the spectrum of the Lie derivative, provided that it makes sense. This kind of strategy was used by Frenkel, Losev and Nekrasov [29] to compute the spectrum of $\mathcal{L}_{V_f}$ in the case of the height function on the canonical 2-sphere. However, their strategy is completely different from ours. They compute explicitly the spectrum of the Witten Laplacian and show how to take the limit as $h \to 0^+$. Here, we will instead compute the spectrum of the limit operator explicitly and show without explicit computations why the spectrum of the Witten Laplacian should converge to the limit spectrum. In particular, our proof makes no explicit use of the classical results of Helffer and Sjöstrand on the Witten Laplacian [43].

Our first step will be to define an appropriate functional framework where one can study the spectrum of $\mathcal{L}_{V_f} + h\Delta_g/2$ for $0 \leq h \leq h_0$. Recall that, following the microlocal strategy of Faure and Sjöstrand [28] for the study of the analytical properties of hyperbolic dynamical systems, we constructed in [17] some families of anisotropic Sobolev spaces $\mathcal{H}^{m_\Lambda}(M)$ indexed by a parameter $\Lambda > 0$ and such that

$$-\mathcal{L}_{V_f} : \mathcal{H}^{m_\Lambda}(M) \to \mathcal{H}^{m_\Lambda}(M)$$

has discrete spectrum on the half-plane $\{\mathrm{Re}(z) > -\Lambda\}$. This spectrum is intrinsic and it turns out to be the correlation spectrum appearing in Theorem 5.1. For an Anosov vector field V, Dyatlov and Zworski proved that the correlation spectrum is in fact the limit of the spectrum of an operator of the form $\mathcal{L}_V + h\Delta_g/2$ [25]—see also [7, 27, 64, 21] for related questions. We will thus show how to adapt the strategy of Dyatlov and Zworski to our framework. It means that we will prove that the family of operators

$$(\widehat{H}_h := -\mathcal{L}_{V_f} - h\Delta_g/2)_{h\in[0,+\infty)}$$

has nice spectral properties on the anisotropic Sobolev spaces $\mathcal{H}^{m_\Lambda}(M)$ constructed in [17]. This will be the object of Section 3. Once these properties are established, we will verify in which sense the spectrum of $\widehat{H}_h$ converges to the spectrum of $\widehat{H}_0$ in the semiclassical limit $h \to 0^+$—see Section 4 for details. In [17], we computed explicitly the spectrum of $\widehat{H}_0$ on these anisotropic Sobolev spaces. Under some (generic) smooth linearization properties, we obtained an explicit description of the eigenvalues and a fairly explicit description of the generalized eigenmodes. Here, we generalize the results of [17] by relaxing these smoothness assumptions and by computing the spectrum under the more general assumptions of Theorem 2.1. For that purpose, we will make crucial use of some earlier results of Baladi and Tsujii [3] on hyperbolic diffeomorphisms in order to compute the eigenvalues. Compared to [17, 19], we will however get a somewhat less precise description of the corresponding eigenmodes. This will be achieved in Section 5. Then, in Section 6, we combine these results to prove Theorems 2.1 to 2.6. In Section 7, we describe the wavefront set of the generalized eigenmodes and we show how to use this information to prove Theorem 2.7. Finally, in Section 8, we briefly compare our quasimodes with the ones appearing in [43].

The article ends with two appendices. Appendix A shows how to prove the holomorphic extension of the dynamical Ruelle determinant in our framework. Appendix B contains the proof of a technical lemma needed for our analysis of wavefront sets.

2.7. Conventions

In all this article, φ_f^t is a Morse–Smale gradient flow which is $\mathcal{C}^1$-linearizable acting on a smooth, compact, oriented and boundaryless manifold of dimension $n \geq 1$.

3. Anisotropic Sobolev spaces and Pollicott–Ruelle spectrum

In [17, 18], we have shown how one can build a proper spectral theory for the operator $-\mathcal{L}_{V_f}$. In other words, we constructed some anisotropic Sobolev spaces of currents on which we could prove that the spectrum of $-\mathcal{L}_{V_f}$ is discrete in a certain half-plane $\text{Re}(z) > -C_0$. The corresponding discrete eigenvalues are intrinsic and are the so-called Pollicott–Ruelle resonances. Our construction was based on a microlocal approach that was initiated by Faure and Sjöstrand [28] in the framework of Anosov flows and further developed by Dyatlov and Zworski [23]. As already explained in §2.6, we will try to relate the spectrum of the Witten Laplacian to the spectrum of $-\mathcal{L}_{V_f}$ by the use of the relation (6). Hence, our first step will be to show that our construction from [17] can be adapted to fit (in a uniform manner) the operator

$$\boxed{\widehat{H}_h := -\mathcal{L}_{V_f} - h\Delta_g/2.}$$

Note that we changed the sign so that φ_f^{-t*} will correspond to the propagator in positive times of $\widehat{H}_0$. In the case of Anosov flows, this perturbation argument was introduced by Dyatlov and Zworski [25]. As their spectral construction is slightly different from the one of Faure and Sjöstrand [28] and as our proof of the meromorphic extension of $\hat{C}_{\psi_1,\psi_2}$ in [17] is closer to [28] than to [25], we need to slightly revisit some of the arguments given in [28, 17] to fit the framework of [25]. This is the purpose of this section where we will recall the definition of anisotropic Sobolev spaces and of the corresponding Pollicott–Ruelle resonances. More precisely, among other useful things, we will prove

Proposition 3.1. *There exists some constant $C_0 > 0$ such that, for every $0 \leq h \leq 1$, the Schwartz kernel of $(\widehat{H}_h - z)^{-1}$ is holomorphic on $\text{Re}(z) > C_0$. Moreover, it has a meromorphic extension from $\text{Re}(z) > C_0$ to $\mathbb{C}$ whose poles coincide with the Witten eigenvalues for $h > 0$.*

The poles of this meromorphic extension are called the *resonances* of the operator $\widehat{H}_h$ and, for $h = 0$, they are called *Pollicott–Ruelle resonances*.

3.1. *Anisotropic Sobolev spaces*

In [17, 18], one of the key difficulties is the construction of an order function adapted to the Morse–Smale dynamics induced by the flow φ_f^t. Before defining our anisotropic Sobolev spaces, we recall some of the properties proved in that reference and we also recall along the way some properties of Morse–Smale gradient flows. We refer to [59] for a detailed introduction to that topic.

3.1.1. Stable and unstable manifolds. Similarly to the unstable manifold $W^u(a)$, we can define, for every $a \in \mathrm{Crit}(f)$,

$$W^s(a) := \Big\{x \in M : \lim_{t\to+\infty} \varphi_f^t(x) = a\Big\}.$$

A remarkable property of gradient flows is that, given any x in M, there exists a unique (a, b) in $\mathrm{Crit}(f)^2$ such that $f(a) \leq f(b)$ and

$$x \in W^u(a) \cap W^s(b).$$

Equivalently, the unstable manifolds form a partition of M. It is known from the work of Smale [56] that these submanifolds are embedded in M [59, p. 134] and that their dimension is $n - r(a)$ where $r(a)$ is the Morse index of a. The *Smale transversality assumption* is the requirement that, given any x in M, one has

$$T_xM = T_xW^u(a) + T_xW^s(b).$$

Equivalently, it says that the intersection of

$$\Gamma_+ = \Gamma_+(V_f) := \bigcup_{a\in\mathrm{Crit}(f)} N^*(W^s(a)) \quad \text{and} \quad \Gamma_- = \Gamma_-(V_f) := \bigcup_{a\in\mathrm{Crit}(f)} N^*(W^u(a))$$

is empty, where $N^*(\mathcal{W}) \subset T^*M \setminus \underline{0}$ denotes the conormal of the manifold $\mathcal{W}$. In the proofs of Section 3, an important role is played by the Hamiltonian vector field generated by

$$H_f(x;\xi) := \xi(V_f(x)).$$

Recall that the corresponding Hamiltonian flow can be written

$$\Phi_f^t(x;\xi) := \big(\varphi_f^t(x), (d\varphi^t(x)^T)^{-1}\xi\big),$$

and that it induces a flow on the unit cotangent bundle S^*M by setting

$$\tilde{\Phi}_f^t(x;\xi) := \left(\varphi_f^t(x), \frac{(d\varphi^t(x)^T)^{-1}\xi}{\|(d\varphi^t(x)^T)^{-1}\xi\|_{g^*\circ\varphi^t(x)}}\right).$$

The corresponding vector fields are denoted by X_{H_f} and $\tilde{X}_{H_f}$.

3.1.2. Escape function. In all this subsection, V_f satisfies the assumption of Theorem 2.1. We recall the following result [28, Lemma 2.1]:

Lemma 3.2. *Let V^u and V^s be small open neighborhoods of $\Gamma_+ \cap S^*M$ and $\Gamma_- \cap S^*M$ respectively, and let $\epsilon > 0$. Then there exist $\mathcal{W}^u \subset V^u$ and $\mathcal{W}^s \subset V^s$, $\tilde{m}$ in $\mathcal{C}^\infty(S^*M, [0,1])$ and $\eta > 0$ such that $\tilde{X}_{H_f}.\tilde{m} \geq 0$ on S^*M, $\tilde{X}_{H_f}.\tilde{m} \geq \eta > 0$ on $S^*M - (\mathcal{W}^u \cup \mathcal{W}^s)$, $\tilde{m}(x;\xi) > 1 - \epsilon$ for $(x;\xi) \in \mathcal{W}^s$ and $\tilde{m}(x;\xi) < \epsilon$ for $(x;\xi) \in \mathcal{W}^u$.*

This lemma was proved by Faure and Sjöstrand [28] in the case of Anosov flows and its extension to gradient flows requires some results on the Hamiltonian dynamics that were obtained in [17, Sect. 3]—see also [18, Sect. 4] in the more general framework of Morse–Smale flows.

As we have a function $\tilde{m}(x;\xi)$ defined on S^*M, we introduce a smooth function m defined on T^*M which satisfies

$$m(x;\xi) = N_1\tilde{m}(x, \xi/\|\xi\|_x) - N_0(1 - \tilde{m}(x, \xi/\|\xi\|_x)) \quad \text{for } \|\xi\|_x \geq 1,$$

and

$$m(x;\xi) = 0 \quad \text{for } \|\xi\|_x \leq 1/2.$$

We set the order function of our escape function to be

$$m_{N_0,N_1}(x;\xi) = -f(x) + m(x;\xi).$$

It was shown in [17, Lemma 4.1] that it has the following properties (for V^u, V^s and $\epsilon > 0$ small enough[6]):

Lemma 3.3 (Escape function). *Let $s \in \mathbb{R}$ and $N_0, N_1 > 4(\|f\|_{\mathcal{C}^0} + |s|)$. Then there exists $c_0 > 0$ (depending on (M,g) but not on s, N_0 or N_1) such that $m_{N_0,N_1}(x;\xi) + s$*

- *takes values in $[-2N_0, 2N_1]$,*
- *is 0-homogeneous for $\|\xi\|_x \geq 1$,*
- *is $\leq -N_0/2$ on a conic neighborhood of Γ_- (for $\|\xi\|_x \geq 1$),*
- *is $\geq N_1/2$ on a conic neighborhood of Γ_+ (for $\|\xi\|_x \geq 1$),*

and such that there exists $R_0 > 0$ for which the escape function

$$G^s_{N_0,N_1}(x;\xi) := (m_{N_0,N_1}(x;\xi) + s)\log(1 + \|\xi\|_x^2)$$

satisfies, for every $(x;\xi)$ in T^*M with $\|\xi\|_x \geq R_0$,

$$X_{H_f}.(G^s_{N_0,N_1})(x;\xi) \leq -C_N := -c_0 \min\{N_0, N_1\}.$$

6 In particular, $V^u \cap V^s = \emptyset$.

3.1.3. The order function. We can now define our anisotropic Sobolev space in the terminology of [27, 28]. First of all, such spaces require the existence of an order function $m_{N_0,N_1}(x;\xi)$ in $\mathcal{C}^\infty(T^*M)$ with bounded derivatives which is adapted to the dynamics of φ_f^t. Once we are given an escape function by Lemma 3.3, we set

$$A_{N_0,N_1}(x;\xi) := \exp G^0_{N_0,N_1}(x;\xi),$$

where $G^0_{N_0,N_1}(x;\xi) := m_{N_0,N_1}(x;\xi)\log(1+\|\xi\|_x^2)$ belongs to the class of symbols $S^\epsilon(T^*M)$ for every $\epsilon > 0$. We shall denote this property by writing $G^0_{N_0,N_1} \in S^{+0}(T^*M)$. We emphasize that the construction below will require dealing with symbols of variable order m_{N_0,N_1} whose pseudodifferential calculus was described in [27, Appendix].

3.1.4. Anisotropic Sobolev currents. Let us now define the spaces we shall work with. Let $0 \leq k \leq n$. We consider the vector bundle $\Lambda^k(T^*M) \to M$ of exterior k-forms. We define $\mathbf{A}^{(k)}_{N_0,N_1}(x;\xi) := A_{N_0,N_1}(x;\xi)\mathbf{Id} \subset \Gamma(T^*M, \mathrm{End}(\Lambda^k(T^*M)))$, which is the product of the weight $A_{N_0,N_1} \in C^\infty(T^*M)$ with the canonical identity section $\mathbf{Id}$ of the endomorphism bundle $\mathrm{End}(\Lambda^k(T^*M)) \to M$. We fix the canonical inner product $\langle\cdot,\cdot\rangle^{(k)}_{g^*}$ on $\Lambda^k(T^*M)$ induced by the metric g on M. This allows us to define the Hilbert space $L^2(M, \Lambda^k(T^*M))$ and to introduce an anisotropic Sobolev space of currents by setting

$$\mathcal{H}_k^{m_{N_0,N_1}}(M) = \mathrm{Op}(\mathbf{A}^{(k)}_{N_0,N_1})^{-1} L^2(M, \Lambda^k(T^*M)),$$

where $\mathrm{Op}(\mathbf{A}^{(k)}_{N_0,N_1})$ is a formally self-adjoint pseudodifferential operator with principal symbol $\mathbf{A}^{(k)}_m$. We refer to [23, App. C.1] for a brief reminder on pseudodifferential operators with values in vector bundles—see also [4]. In particular, adapting the proof of [27, Cor. 4] to the vector bundle valued framework, one can verify that $\mathbf{A}^{(k)}_{N_0,N_1}$ is an elliptic symbol, and thus $\mathrm{Op}(\mathbf{A}^{(k)}_{N_0,N_1})$ can be chosen to be invertible. More precisely, up to adding a smoothing operator, $\mathrm{Op}(\mathbf{A}^{(k)}_{N_0,N_1})^{-1}$ is equal to $\mathrm{Op}((\mathbf{A}^{(k)}_{N_0,N_1})^{-1}(1+q))$, where $q \in S^{-1+0}(T^*M, \Lambda^k(T^*M))$. Mimicking the proofs of [27], we can deduce some properties of these spaces of currents. First of all, they are endowed with a Hilbert structure inherited from the L^2-structure on M. The space

$$\mathcal{H}_k^{m_{N_0,N_1}}(M)' = \mathrm{Op}(\mathbf{A}^{(k)}_{N_0,N_1}) L^2(M, \Lambda^k(T^*M))$$

is the topological dual of $\mathcal{H}_k^{m_{N_0,N_1}}(M)$. We also note that $\mathcal{H}_k^{m_{N_0,N_1}}(M)$ can be identified with $\mathcal{H}_0^{m_{N_0,N_1}}(M) \otimes_{\mathcal{C}^\infty(M)} \Omega^k(M)$. Finally,

$$\Omega^k(M) \subset \mathcal{H}_k^{m_{N_0,N_1}}(M) \subset \mathcal{D}'^{,k}(M),$$

where the injections are continuous.

3.1.5. Hodge star and duality for anisotropic Sobolev currents. Recall now that the Hodge star operator [4, Part I.4] is the unique isomorphism $\star_k : \Lambda^k(T^*M) \rightarrow \Lambda^{n-k}(T^*M)$ such that, for every ψ_1 in $\Omega^k(M)$ and ψ_2 in $\Omega^{n-k}(M)$,

$$\int_M \psi_1 \wedge \psi_2 = \int_M \langle \psi_1, \star_k^{-1} \psi_2 \rangle_{g^*(x)}^{(k)} \omega_g(x),$$

where $\langle \cdot, \cdot \rangle_{g^*(x)}^{(k)}$ is the induced Riemannian metric on $\Lambda^k(T^*M)$ and where ω_g is the Riemannian volume form. In particular, $\star_k$ induces an isomorphism from $\mathcal{H}_k^{m_{N_0,N_1}}(M)'$ to $\mathcal{H}_{n-k}^{-m_{N_0,N_1}}(M)$, whose Hilbert structure is given by the scalar product

$$\mathcal{H}_{n-k}^{-m_{N_0,N_1}}(M)^2 \ni (\psi_1, \psi_2) \mapsto \langle \star_k^{-1} \psi_1, \star_k^{-1} \psi_2 \rangle_{\mathcal{H}_k^m(M)'}.$$

Thus, the topological dual of $\mathcal{H}_k^{m_{N_0,N_1}}(M)$ can be identified with $\mathcal{H}_{n-k}^{-m_{N_0,N_1}}(M)$, where, for every ψ_1 in $\Omega^k(M)$ and ψ_2 in $\Omega^{n-k}(M)$, one has the following duality relation:

$$\begin{aligned}\langle \psi_1, \psi_2 \rangle_{\mathcal{H}_k^m \times \mathcal{H}_{n-k}^{-m}} &= \int_M \psi_1 \wedge \psi_2 = \langle \mathrm{Op}(\mathbf{A}_{N_0,N_1}^{(k)}) \psi_1, \mathrm{Op}(\mathbf{A}_{N_0,N_1}^{(k)})^{-1} \star_k^{-1} \overline{\psi_2} \rangle_{L^2} \\ &= \langle \psi_1, \star_k^{-1} \psi_2 \rangle_{\mathcal{H}_k^m \times (\mathcal{H}_k^m)'}.\end{aligned}$$

3.2. *Pollicott–Ruelle resonances*

Now that we have defined the appropriate spaces, we have to explain the spectral properties of $\widehat{H}_h := -\mathcal{L}_{V_f} - h\Delta_g/2$ acting on $\mathcal{H}_k^{m_{N_0,N_1}}(M)$. Following Faure and Sjöstrand [28], we introduce the following conjugation of the operator $i\widehat{H}_h$:

$$\widehat{P}_h = \widehat{A}_N \widehat{H}_h \widehat{A}_N^{-1}, \tag{7}$$

where we write $\widehat{A}_N$ instead of $\mathrm{Op}(\mathbf{A}_{N_0,N_1}^{(k)})$ for simplicity. Similarly, we often write G_N^0 instead of G_{N_0,N_1}^0, etc. For similar reasons, we also omit the dependence on k.

In any case, the spectral properties of $\widehat{H}_h$ acting on the anisotropic Sobolev space $\mathcal{H}_k^{m_{N_0,N_1}}(M)$ are the same as those of the operator $\widehat{P}_h$ acting on the simpler Hilbert space $L^2(M, \Lambda^k(T^*M))$. We will now apply the strategy of Faure and Sjöstrand in order to derive some spectral properties of the above operators. Along the way, we keep track of the dependence on h which is needed to apply the arguments from Dyatlov and Zworski [25] on the convergence of the spectrum. In all this section, we follow closely the proofs from [28, Sect. 3] and we emphasize the differences.

3.2.1. The conjugation argument. The first step in Faure–Sjöstrand's proof consists in computing the symbol of the operator $\widehat{P}_h$. Starting from this operator, we decompose it into two terms

$$\widehat{P}_h = \underbrace{-\widehat{A}_N \mathcal{L}_{V_f} \widehat{A}_N^{-1}}_{:=\widehat{Q}_1,\ \text{hyperbolic part}} \quad \underbrace{-h\widehat{A}_N \frac{\Delta_g}{2} \widehat{A}_N^{-1}}_{:=h\widehat{Q}_2,\ \text{elliptic perturbation}}$$

which we will treat separately for the sake of simplicity. The key ingredient of [28] is the following lemma:

Lemma 3.4 ([28, Lemma 3.2]). *The operator $\widehat{Q}_1 + \mathcal{L}_{V_f}$ is a pseudodifferential operator in $\Psi^{+0}(M, \Lambda^k(T^*M))$ whose symbol in any given system of coordinates is of the form*[7]

$$(X_{H_f}.G_N^0)(x;\xi)\mathbf{Id} + \mathcal{O}(S^0) + \mathcal{O}_m(S^{-1+0}),$$

where X_{H_f} is the Hamiltonian vector field generating the characteristic flow of $\mathcal{L}_{V_f}$ in T^*M whose definition is recalled in §3.1.1. The operator $\widehat{Q}_2$ is a pseudodifferential operator in $\Psi^2(M, \Lambda^k(T^*M))$ whose symbol in any given system of coordinates is of the form

$$-\frac{\|\xi\|^2_{g^*(x)}}{2}\mathbf{Id} + \mathcal{O}_m(S^{1+0}).$$

Note that, compared to [28], we study $\widehat{H}_h$ rather than $i\widehat{H}_h$. In this lemma, the notation $\mathcal{O}(\cdot)$ means that the remainder is independent of the order function m_{N_0,N_1}, while $\mathcal{O}_m(\cdot)$ means that it depends on m_{N_0,N_1}. As all the principal symbols are proportional to $\mathbf{Id}_{\Lambda^k(T^*M)}$, the proof of [28] can be adapted almost verbatim to encompass the case of a general vector bundle and of the term corresponding to the Laplace–Beltrami operator. Hence, we shall omit it and refer to [28] for a detailed proof. Recall that more general symbols with values in $\Lambda^k(T^*M)$ do not commute and the composition formula does not work as in the scalar case for more general symbols.

In particular, the lemma says that $\widehat{Q}_1$ is an element in $\Psi^1(M, \Lambda^k(T^*M))$. We can consider that it acts on the domain $\Omega^k(M)$ which is dense in $L^2(M, \Lambda^k(T^*M))$. In particular, according to [28, Lemma A.1], it has a unique closed extension as an unbounded operator on $L^2(M, \Lambda^k(T^*M))$. For $\widehat{Q}_2$, this property comes from the fact that the symbol is elliptic [61, Ch. 13, p. 125]. In other words, for $h > 0$, the domain of $\widehat{P}_h$ is the domain of $\widehat{Q}_2$ (namely $H^2(M, \Lambda^k T^*M)$), while it is given by the domain of $\widehat{Q}_1$ for $h = 0$. The same properties also hold for the adjoint operator.

3.2.2. The adjoint part of the operator and its symbol. We now verify that this operator has a discrete spectrum in a certain half-plane in $\mathbb{C}$. Following [28], this will be done by arguments from analytic Fredholm theory. Compared to that reference, one aspect of our proof is simpler beause, in the Anosov case, the escape function does not decay in the flow direction and one has to use the ellipticity of the symbol in that direction. Here, the escape function decays everywhere. Hence this extra difficulty does not appear. Recall

[7] Observe that the $\mathcal{O}(S^0)$ term comes from the subprincipal symbol of $-\mathcal{L}_{V_f}$.

that the strategy from [28] consists in studying the properties of the adjoint part of the operator

$$\widehat{P}_{\mathrm{Re}}(h) := \tfrac{1}{2}(\widehat{P}_h^* + \widehat{P}_h) = \tfrac{1}{2}(\widehat{Q}_1^* + \widehat{Q}_1) + \frac{h}{2}(\widehat{Q}_2^* + \widehat{Q}_2), \quad (8)$$

whose symbol (for every $h > 0$) is, according to Lemma 3.4, given in any given system of coordinates by

$$P_{\mathrm{Re}}(x;\xi) = X_{H_f}.G_N^0(x;\xi)\mathbf{Id} + \mathcal{O}(S^0) + \mathcal{O}_m(S^{-1+0}) - h(\|\xi\|_x^2/2 + \mathcal{O}_m(S^{1+0})),$$

where the first three terms correspond to the contribution of $\widehat{Q}_1$ and the last two terms to the contribution of $\widehat{Q}_2$. Here the remainder $\mathcal{O}(S^0)$ comes from Lemma 3.4 and more precisely from the subprincipal symbol of $-\mathcal{L}_{V_f}$ in our choice of quantization. We already note that, according to Lemma 3.3, there exists some constant $C > 0$ independent of m_{N_0,N_1} such that, in the sense of quadratic forms,

$$X_{H_f}.G_N^0(x;\xi)\mathbf{Id} \leq (-C_N + C)\mathbf{Id} + \mathcal{O}_m(S^{-1+0}), \quad (9)$$

where C_N is the constant defined in Lemma 3.3.

We can now follow the proof of [28]. First of all, arguing as in [28, Lemma 3.3], we can show that $\widehat{P}_h$ has empty spectrum for $\mathrm{Re}(z) > C_0$, where C_0 is some positive constant that may depend on m but which can be made uniform in $h \in [0,1)$. In other words, the resolvent

$$(\widehat{P}_h - z)^{-1} : L^2(M, \Lambda^k(T^*M)) \to L^2(M, \Lambda^k(T^*M))$$

defines a bounded operator for $\mathrm{Im}(z) > C_0$. In particular, this shows the first part of Proposition 3.1. Now, we will show how to extend it meromorphically to some half-plane

$$\{z : \mathrm{Re}(z) \geq (C - C_N)/2\},$$

for any choice of $N_0,\ N_1$ large enough.

3.2.3. From resolvent to semigroup. Before doing that, we already note that the proof of [28, Lemma 3.3] implicitly shows that, for every z in $\mathbb{C}$ satisfying $\mathrm{Re}(z) > C_0$,

$$\|(\widehat{P}_0 - z)^{-1}\|_{L^2(M,\Lambda^k(T^*M))\to L^2(M,\Lambda^k(T^*M))} \leq \frac{1}{\mathrm{Re}(z) - C_0}, \quad (10)$$

which will allow us to relate the spectrum of the generator to the spectrum of the corresponding semigroup φ_f^{-t*}. In particular, combining this observation with [26, Cor. 3.6, p. 76], we know that, for $t \geq 0$,

$$\varphi_f^{-t*} : \mathcal{H}_k^{m_{N_0,N_1}}(M) \to \mathcal{H}_k^{m_{N_0,N_1}}(M)$$

generates a strongly continuous semigroup whose norm satisfies

$$\forall t \geq 0, \quad \|\varphi_f^{-t*}\|_{\mathcal{H}_k^{m_{N_0,N_1}}(M)\to\mathcal{H}_k^{m_{N_0,N_1}}(M)} \leq e^{tC_0}. \quad (11)$$

3.2.4. Resolvent construction and meromorphic continuation. We fix some large integer $L > \dim(M)/2$ to ensure that the operator $(1+\Delta_g)^{-L}$ is *trace class*. As a first step towards our proof of meromorphic continuation, we show the following lemma:

Lemma 3.5. *There exists some* $R > 0$ *such that if we set*

$$\widehat{\chi}_R := -R(1+\Delta_g)^{-L},$$

then

$$(\widehat{P}_h + \widehat{\chi}_R - z)^{-1} : L^2(M, \Lambda^k(T^*M)) \to L^2(M, \Lambda^k(T^*M))$$

defines a bounded operator for $\mathrm{Re}(z) > (C - C_N)/2$ *and*

$$\|(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}\|_{L^2\to L^2} \leq \frac{1}{\mathrm{Re}(z) - (C - C_N)/2}.$$

At this point of our argument, the fact that the operators are trace class is not that important but it will be useful later on when we will consider determinants.

Proof of Lemma 3.5. For every u in $\mathcal{C}^\infty(M)$ and for every $0 \leq h \leq 1$, we combine (8) and (9) with the sharp Gårding inequality. This yields

$$\mathrm{Re}\,\langle \widehat{P}_h u, u\rangle \leq (C - C_N)\|u\|_{L^2}^2 + C_m\|u\|_{H^{-1/4}}^2 - \frac{h}{2}\|u\|_{H^1}^2 + C_m\|u\|_{H^{3/4}}^2.$$

Hence,

$$\mathrm{Re}\,\langle(\widehat{P}_h + C_N - C)u, u\rangle \leq -h/2\|u\|_{H^1}^2 + C_m(\|u\|_{H^{-1/4}} + h\|u\|_{H^{3/4}}^2).$$

Now, observe that, for every $\epsilon > 0$, there exists some constant $C_\epsilon > 0$ such that

$$\|u\|_{H^{-1/4}}^2 \leq \epsilon\|u\|_{L^2}^2 + C_\epsilon\|u\|_{H^{-2L}}^2 \quad \text{and} \quad \|u\|_{H^{3/4}}^2 \leq \epsilon\|u\|_{H^1}^2 + C_\epsilon\|u\|_{H^{-2L}}^2.$$

Taking $0 < C_m\epsilon < \min\{1/2, (C_N - C)/2\}$, one obtains

$$\mathrm{Re}\left\langle\left(\widehat{P}_h + \frac{C_N - C}{2}\right)u, u\right\rangle \leq C_m C_\epsilon\|u\|_{H^{-2L}}.$$

For $R = C_m C_\epsilon$, we now set

$$\widehat{\chi}_R := -R(1+\Delta_g)^{-L},$$

and we find

$$\mathrm{Re}\left\langle\left(\widehat{P}_h + \frac{C_N - C}{2}\right)u, u\right\rangle \leq -\mathrm{Re}\,\langle\widehat{\chi}_R u, u\rangle. \tag{12}$$

We can now argue as in [28, Lemma 3.3] to conclude that $\widehat{P}_h + \widehat{\chi}_R - z$ is invertible for $\mathrm{Re}(z) > (C - C_N)/2$. In fact, set $\delta = \mathrm{Re}(z) - (C - C_N)/2$ in order to get

$$\mathrm{Re}\,\langle(\widehat{P}_h + \widehat{\chi}_R - z)u, u\rangle \leq -\delta\|u\|^2.$$

Applying the Cauchy–Schwarz inequality, we find that

$$\|(\widehat{P}_h + \widehat{\chi}_R - z)u\| \, \|u\| \geq |\mathrm{Re}\, \langle(\widehat{P}_h + \widehat{\chi}_R - z)u, u\rangle| \geq \delta\|u\|^2.$$

This implies that $\widehat{P}_h + \widehat{\chi}_R - z$ is injective. We can argue similarly for the adjoint operator to obtain

$$\|(\widehat{P}_h^* + \widehat{\chi}_R - \overline{z})u\| \, \|u\| \geq \delta\|u\|^2,$$

from which we can infer that $\widehat{P}_h + \widehat{\chi}_R - z$ is surjective [10, Th. II.19]. Hence,

$$(\widehat{P}_h + \widehat{\chi}_R - z)^{-1} : L^2(M, \Lambda^k(T^*M)) \to L^2(M, \Lambda^k(T^*M))$$

defines a bounded operator for $\mathrm{Re}(z) > (C - C_N)/2$ and its operator norm satisfies

$$\|(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}\|_{L^2 \to L^2} \leq \frac{1}{\mathrm{Re}(z) - (C - C_N)/2}. \qquad \square$$

We can now write that

$$\mathrm{Re}(z) > \frac{C - C_N}{2} \Longrightarrow \widehat{P}_h - z = \big(\mathrm{Id} - \widehat{\chi}_R(P_h + \widehat{\chi}_R - z)^{-1}\big)(\widehat{P}_h + \widehat{\chi}_R - z). \tag{13}$$

Note that $\widehat{\chi}_R \in \Psi^{-2L}(M)$ is by definition a trace class operator for L large enough (at least $> \dim(M)/2$ [65, Prop. B.20]). This implies that the operator

$$\widehat{\chi}_R(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}$$

is trace class for every $h \geq 0$ as the composition of a trace class operator and a bounded one. Moreover, it depends holomorphically on z in the domain $\{\mathrm{Re}(z) > -(C_N - C)/2\}$, implying that $\widehat{P}_h - z$ is a holomorphic family of Fredholm operators for z in that domain. Finally, we can apply arguments from analytic Fredholm theory to $\widehat{P}_h - z$ [63, Th. D.4, p. 418], which yields the analytic continuation of $(\widehat{P}_h - z)^{-1}$ as a meromorphic family of Fredholm operators for $z \in \{\mathrm{Re}(z) > (C - C_N)/3\}$. Arguing as in [28, Lemma 3.5], we can conclude that $\widehat{P}_h$ has discrete spectrum with finite multiplicity on $\mathrm{Re}(z) > (C - C_N)/2$. To summarize, one has

Lemma 3.6. *The operator*

$$(\widehat{P}_h - z)^{-1} : L^2(M, \Lambda^k(T^*M)) \to L^2(M, \Lambda^k(T^*M))$$

has a meromorphic continuation from $\mathrm{Re}(z) > C_0$ *to* $\mathrm{Re}(z) > (C - C_N)/2$.

Since $\widehat{P}_h$ is conjugate to $\widehat{H}_h$, the above discussion implies that $\widehat{H}_h$ has a discrete spectrum with finite multiplicity on $\mathrm{Re}(z) > (C - C_N)/2$ as an operator acting on $\mathcal{H}_k^{m_{N_0,N_1}}(M)$. In particular, this shows the meromorphic continuation of the Schwartz kernel (in the sense of distributions in $\mathcal{D}'(M \times M)$) of $(\widehat{H}_h - z)^{-1}$ from $\mathrm{Re}(z) > C_0$ to $\mathrm{Re}(z) > (C - C_N)/2$—see [28, Sect. 4] for more details. In the case $h > 0$, the poles of this meromorphic continuation are exactly the Witten eigenvalues. In particular, they are of the form

$$0 \geq -\lambda_1^{(k)}(h) \geq -\lambda_2^{(k)}(h) \geq \cdots \geq -\lambda_j^{(k)}(h) \to -\infty \quad \text{as } j \to +\infty.$$

Our next step will be to show that this Witten spectrum indeed converges to the Pollicott–Ruelle spectrum.

3.2.5. Convention. In the following, we shall denote this intrinsic discrete spectrum by $\mathcal{R}_k(h)$. Its elements correspond to the eigenvalues of $\widehat{H}_h$ acting on an appropriate Sobolev space of currents of degree k. When $h > 0$, these are the Witten eigenvalues (up to a factor -1), while for $h = 0$ they represent the correlation spectrum of the gradient flow, which is often referred to as the Pollicott–Ruelle spectrum. Given z_0 in $\mathcal{R}_k(0)$, we will denote by $\pi_{z_0}^{(k)}$ the spectral projector associated with the eigenvalue z_0, which can be viewed as a finite rank linear map from $\Omega^k(M)$ to $\mathcal{D}'^k(M)$. Recall from [28, Sect. 4] that this operator is intrinsic.

3.2.6. Boundedness on standard Sobolev spaces. Denote by $H^s(M, \Lambda^k(T^*M))$ the standard Sobolev space of index $s > 0$, i.e.

$$H^s(M, \Lambda^k(T^*M)) := (1 + \Delta_g^{(k)})^{-s/2} L^2(M, \Lambda^k(T^*M)).$$

The above construction shows that

$$(\widehat{P}_h + \widehat{\chi}_R - z)^{-1} : L^2(M, \Lambda^k(T^*M)) \to L^2(M, \Lambda^k(T^*M))$$

defines a bounded operator for $\mathrm{Re}(z) > (C - C_N)/2$ which depends holomorphically on z. We will in fact need something slightly stronger:

Lemma 3.7. *Let $s_0 > 0$ and $N_0, N_1 > 4(\|f\|_{C^0} + s_0)$. Then there exists $R > 0$ such that, for $\mathrm{Re}(z) > -(C_N - C)/2$ and for every $s \in [-s_0, s_0]$, the resolvent*

$$(P_h + \widehat{\chi}_R - z)^{-1} \tag{14}$$

exists as a holomorphic function from $\{\mathrm{Re}(z) > -(C_N - C)/2\}$ to bounded operators $H^{2s}(M, \Lambda^k(T^*M)) \to H^{2s}(M, \Lambda^k(T^*M))$. Moreover, for every $0 \leq h \leq 1$,

$$\|(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}\|_{H^{2s} \to H^{2s}} \leq \frac{1}{\mathrm{Re}(z) + (C_N - C)/2}.$$

The argument is the same as before except that the order function has to be replaced by $m_{N_0,N_1} + s$, and a direct inspection of the proof shows that all the constants can be made uniform for s in some fixed interval $[-s_0, s_0]$.

3.3. Pollicott–Ruelle resonances as zeros of a Fredholm determinant

From expression (13), we know that, for $\mathrm{Re}(z) > (C - C_N)/3$, z belongs to the spectrum of $\widehat{P}_h$ if and only if the operator $\mathrm{Id} + \widehat{\chi}_R(P_h - \widehat{\chi}_R - z)^{-1}$ is not invertible. As we have shown that

$$\widehat{\chi}_R(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}$$

is a trace class operator on $L^2(M, \Lambda^k(T^*M))$, this is equivalent to saying that z is a zero of the Fredholm determinant [24, Prop. B.25]

$$D_{m_{N_0,N_1}}(h, z) := \det\nolimits_{L^2}\big(\mathrm{Id} - \widehat{\chi}_R(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}\big).$$

Moreover, the multiplicity of z as an eigenvalue of $\widehat{P}_h$ coincides with the multiplicity of z as a zero of $D_m(h, z)$ [24, Prop. B.29].

4. From the Witten spectrum to the Pollicott–Ruelle spectrum

Now that we have recalled the precise notion of resonance spectrum for the limit operator $-\mathcal{L}_{V_f}$, we would like to explain how the Witten spectrum converges to the resonance spectrum of the Lie derivative. This will be achieved by an argument due to Dyatlov and Zworski [25] in the context of *Anosov* flows—see also [64]. In this section, we briefly recall their proof adapted to our framework.

Remark 4.1. In [25], Dyatlov and Zworski prove something slightly stronger as they obtain smoothness in h. Here, we are aiming at something simpler and we shall not prove smoothness which would require more work, beyond the scope of the present article—see [25] for details in the Anosov case.

4.1. Convergence of the eigenvalues

We fix $N_0, N_1, s_0 > 2$ and R as in the statement of Lemma 3.7. Using the conventions of Section 3, we start by studying the regularity of the operator

$$[0,1] \ni h \mapsto K_m(h) := \widehat{\chi}_R(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}.$$

Recall that $K_m(h)$ is a holomorphic map on $\{\mathrm{Re}(z) > (C - C_N)/3\}$ with values in the space of trace class operators on L^2. For $h, h' \in [0,1]$, we now write

$$(\widehat{P}_h + \widehat{\chi}_R - z) - (\widehat{P}_{h'} + \widehat{\chi}_R - z) = (h - h')\widehat{Q}_2 : H^2 \to L^2$$

where we recall that

$$\widehat{Q}_2 = -\widehat{A}_N \frac{\Delta_g}{2} \widehat{A}_N^{-1}.$$

Applying Lemma 3.7 with $s_0 > 2$, we can compose $\widehat{Q}_2$ with the two resolvents to get

$$\frac{(\widehat{P}_h + \widehat{\chi}_R - z)^{-1} - (\widehat{P}_{h'} + \widehat{\chi}_R - z)^{-1}}{h - h'} = -(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}\widehat{Q}_2(\widehat{P}_{h'} + \widehat{\chi}_R - z)^{-1}. \quad (15)$$

Still from Lemma 3.7 with $s_0 > 2$, we find that (15) is bounded for $\mathrm{Re}(z) > (C - C_N)/3$ and uniformly for $h \in [0,1]$ as an operator from L^2 to H^{-2}. Hence, we have verified that

$$h \mapsto (\widehat{P}_h + \widehat{\chi}_R - z)^{-1}$$

defines a *Lipschitz (thus continuous)* map in h with values in the set $\mathrm{Hol}(\{\mathrm{Re}(z) > (C - C_N)/3\}, \mathcal{B}(L^2, H^{-2}))$ of holomorphic functions in z valued in the Banach space $\mathcal{B}(L^2, H^{-2})$ of bounded operators from L^2 to H^{-2}. Recall now that

$$\widehat{\chi}_R = -R(1 + \Delta_g)^{-L}$$

is trace class from H^{-2} to L^2 for L large enough (more precisely,[8] $L > \dim(M)/2 + 1$). Denote by $\mathcal{L}^1(H^{-2}(M), L^2(M)) \subset \mathcal{B}(H^{-2}(M), L^2(M))$ the set of trace class operators acting on these spaces [24, Sect. B.4]. By continuity of the composition map

[8] This follows from Weyl's law.

$\mathcal{L}^1(H^{-2}, L^2) \times \mathcal{B}(L^2, H^{-2}) \ni (A, B) \mapsto AB \in \mathcal{L}^1(L^2, L^2)$ [24, Eq. (B.4.6)], the operator

$$K_m(h) = \underbrace{\widehat{\chi}_R}_{\text{trace class}} \underbrace{(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}}_{\text{Lipschitz in } \mathcal{B}(L^2, H^{-2})}$$

is the composition of a Lipschitz operator valued in the set $\mathrm{Hol}(\{\mathrm{Re}(z) > (C - C_N)/3\}, \mathcal{B}(L^2, H^{-2}))$ with the fixed trace class operator $\widehat{\chi}_R \in \mathcal{L}^1$. Therefore K_m must be a Lipschitz map in $h \in [0, 1]$ valued in $\mathrm{Hol}(\{\mathrm{Re}(z) > (C - C_N)/3\}, \mathcal{L}^1(L^2, L^2))$. We have thus shown the following Lemma:

Lemma 4.2. *Let $N_0, N_1 > 4(\|f\|_{\mathcal{C}^0} + 2)$ and let $R > 0$ be as in the statement of Lemma 3.7 with $s_0 = 2$. Then the map $h \mapsto K_m(h)$ is Lipschitz (hence continuous) from $[0, 1]$ to the space of holomorphic functions on $\{\mathrm{Re}(z) > (C - C_N)/3\}$ with values in the space of trace class operators on L^2.*

Remark 4.3. Note that, for simplicity, we have omitted the dependence on the degree k in that statement.

Let us now draw some consequences from this lemma. From [24, Sect. B.5, p. 426], the determinant map

$$D_{m_{N_0,N_1}}(h, \cdot) : \left\{ \mathrm{Im}(z) > \frac{C - C_N}{3} \right\} \ni z \mapsto \det_{L^2}\big(\mathrm{Id} - \widehat{\chi}_R(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}\big)$$

is holomorphic. Moreover, one knows from [24, Prop. B.26] that

$$|D_{m_{N_0,N_1}}(h, z) - D_{m_{N_0,N_1}}(h', z)| \leq \|K_m(h, z) - K_m(h', z)\|_{\mathrm{Tr}} e^{1 + \|K_m(h,z)\|_{\mathrm{Tr}} + \|K_m(h',z)\|_{\mathrm{Tr}}},$$

which, combined with Lemma 4.2, implies that $h \mapsto D_{m_{N_0,N_1}}(h, \cdot)$ is a continuous map from $[0, 1]$ to the space of holomorphic functions on $\{\mathrm{Re}(z) > (C - C_N)/3\}$.

Fix now an eigenvalue z_0 of $\widehat{P}_0$ lying in the half-plane $\{\mathrm{Re}(z) > (C - C_N)/3\}$ and having algebraic multiplicity m_{z_0}. This corresponds to a zero of multiplicity m_{z_0} of the determinant map $D_{m_{N_0,N_1}}(0, \cdot)$ evaluated at $h = 0$. As the spectrum of $\widehat{P}_0$ is discrete with finite multiplicity on this half-plane, we can find a small enough $r_0 > 0$ such that the closed disk centered at z_0 of radius r_0 contains only the eigenvalue z_0. The map $h \mapsto D_{m_{N_0,N_1}}(h, \cdot) \in \mathrm{Hol}(\{\mathrm{Re}(z) > (C - C_N)/3\})$ being continuous, we know that, for all $0 < r_1 \leq r_0$, for $h \geq 0$ small enough (which depends on z_0 and on r_1) and for $|z - z_0| = r_1$,

$$|D_{m_{N_0,N_1}}(h, z) - D_{m_{N_0,N_1}}(0, z)| < \min_{z' : |z' - \lambda_0| = r_0} |D_{m_{N_0,N_1}}(0, z')| \leq |D_{m_{N_0,N_1}}(0, z)|.$$

Hence, from the Rouché Theorem and for $h \geq 0$ small enough, the number of zeros counted with multiplicity of $D_{m_{N_0,N_1}}(h)$ lying in the disk $\{z : |z - z_0| \leq r_1\}$ equals m_{z_0}. Since, for $h > 0$, the Witten eigenvalues lie on the real axis, we have shown the following theorem:

Theorem 4.4. *Let $0 \leq k \leq n$. Then the set $\mathcal{R}_k = \mathcal{R}_k(0)$ of Pollicott–Ruelle resonances of $-\mathcal{L}_{V_f}^{(k)}$ is contained inside $(-\infty, 0]$. Moreover, given any z_0 in $(-\infty, 0]$, there exists $r_0 > 0$ such that, for every $0 < r_1 \leq r_0$, for $h > 0$ small enough (depending on z_0 and r_1), the number of elements (counted with algebraic multiplicity) inside*

$$\mathcal{R}_k(h) \cap \{z : |z - z_0| \leq r_1\}$$

is constant and equal to the algebraic multiplicity of z_0 as an eigenvalue of $-\mathcal{L}_{V_f}^{(k)}$.

As expected, this theorem shows that the Witten eigenvalues converge to the Pollicott–Ruelle resonances of $-\mathcal{L}_{V_f}$. Yet, for the moment, it does not say anything on the precise values of Pollicott–Ruelle resonances and we shall come back to this question in Section 5.

4.2. Convergence of spectral projectors

Now we can prove the convergence of the spectral projectors of the Witten Laplacian to the operators $\pi_{z_0}^{(k)}$ that were defined in §3.2.5 as the spectral projectors of $-\mathcal{L}_{V_f}$:

Theorem 4.5. *Let $0 \leq k \leq n$ and $z_0 \in \mathbb{R}$.*[9] *Then there exists $r_0 > 0$ such that, for every $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$,*

$$\forall 0 < r_1 \leq r_0, \qquad \lim_{h \to 0^+} \int_M \mathbf{1}_{[z_0 - r_1, z_0 + r_1]}(-W_{f,h}^{(k)})(e^{-f/h}\psi_1) \wedge (e^{f/h}\psi_2) = \int_M \pi_{z_0}^{(k)}(\psi_1) \wedge \psi_2.$$

In fact, the result also holds for any (ψ_1, ψ_2) in $\mathcal{H}_k^{m_{N_0,N_1}}(M) \times \mathcal{H}_{n-k}^{-m_{N_0,N_1}}(M)$.

Together with Theorem 4.4, this theorem shows that all the spectral data of the Witten Laplacian converge to the ones of $-\mathcal{L}_{V_f}$. In particular, this concludes the proof of Theorem 2.4.

Proof of Theorem 4.5. Using Theorem 4.4, it is enough to show the existence of r_0 and to prove convergence for $r_1 = r_0$. As before, it is also enough to prove this result for the conjugated operators

$$\widehat{P}_h = -\widehat{A}_N \mathcal{L}_{V_f} \widehat{A}_N^{-1} - h\widehat{A}_N \frac{\Delta_g}{2} \widehat{A}_N^{-1}$$

acting on the standard Hilbert space $L^2(M, \Lambda^k(T^*M))$. Fix z_0 in $\mathbb{R}$ and N_0, N_1 large enough to ensure that $\mathrm{Re}(z_0) > (C - C_N)/3$. The spectral projector[10] associated with z_0 can be written as [24, Th. C.6]

$$\Pi_{z_0}^{(k)} := \frac{1}{2i\pi} \int_{\mathcal{C}(z_0, r_0)} (z - \widehat{P}_0)^{-1} dz$$

9 For $z_0 \notin \mathcal{R}_k$, one has $\pi_{z_0}^{(k)} = 0$.

10 Note that this is eventually 0 if $z_0 \notin \mathcal{R}_k$.

where $\mathcal{C}(z_0, r_0)$ is a small circle of radius r_0 centered at z_0 such that z_0 is *the only eigenvalue* of $\widehat{P}_0$ inside the closed disk surrounded by $\mathcal{C}(z_0, r_0)$. When z_0 is not an eigenvalue, we choose the disk small enough to ensure that there are no eigenvalues inside it. If we denote by m_{z_0} the algebraic multiplicity of z_0 (which can be 0 if $z_0 \notin \mathcal{R}_k$), then, for h small enough, the spectral projector associated to $\widehat{P}_h$,

$$\Pi_{z_0}^{(k)}(h) := \frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r_0)} (z - \widehat{P}_h)^{-1} \, dz,$$

has rank m_{z_0} from Theorem 4.4. We can now argue as in [25, Prop. 5.3] to show that, for every ψ_1 in $\Omega^k(M)$ and every ψ_2 in $\Omega^{n-k}(M)$,

$$\lim_{h \to 0^+} \int_M \Pi_{z_0}^{(k)}(h)(\psi_1) \wedge \psi_2 = \int_M \Pi_{z_0}^{(k)}(\psi_1) \wedge \psi_2. \tag{16}$$

Once this equality is proved, we will be able to conclude the proof by recalling that the generalized eigenmodes are independent of the choice of the order function m_{N_0,N_1} used to define $\widehat{A}_N$ and by observing that

$$\pi_{z_0}^{(k)} = -\widehat{A}_N^{-1} \Pi_{z_0}^{(k)} \widehat{A}_N$$

and

$$e^{f/h} \mathbf{1}_{[z_0 - r_0, z_0 + r_0]}(-W_{f,h}^{(k)}) e^{-f/h} = -\widehat{A}_N^{-1} \Pi_{z_0}^{(k)}(h) \widehat{A}_N.$$

Hence, it remains to prove (16). For that purpose, we use the conventions of Lemma 3.7 and write

$$(\widehat{P}_h - z)^{-1} = (\widehat{P}_h + \widehat{\chi}_R - z)^{-1} + (\widehat{P}_h - z)^{-1} \widehat{\chi}_R (\widehat{P}_h + \widehat{\chi}_R - z)^{-1}.$$

By construction of the compact operator $\widehat{\chi}_R$, the family $(\widehat{P}_h + \widehat{\chi}_R - z)^{-1}$ is holomorphic and *has no poles* in some neighborhood of z_0 as $z_0 > (C - C_N)/3$. Therefore, only the term $(\widehat{P}_h - z)^{-1} \widehat{\chi}_R (\widehat{P}_h + \widehat{\chi}_R - z)^{-1}$ contributes to the contour integral defining the spectral projector $\Pi_{\lambda_0}^{(k)}(h)$:

$$\Pi_{z_0}^{(k)}(h) = \frac{-1}{2i\pi} \int_{\mathcal{C}(z_0,r_0)} (\widehat{P}_h - z)^{-1} \widehat{\chi}_R (\widehat{P}_h + \widehat{\chi}_R - z)^{-1} \, dz.$$

From Theorem 4.4, we know that, for $|z - z_0| = r_0$ and for h small enough, the operator $(\widehat{P}_h - z)^{-1}$ is uniformly bounded as an operator in $\mathcal{B}(L^2(M), L^2(M))$. Moreover, we have seen that the map

$$[0, 1] \ni h \mapsto \left(z \mapsto \widehat{\chi}_R (\widehat{P}_h + \widehat{\chi}_R - z)^{-1} \right)$$

is continuous (in fact Lipschitz) with values in the set $\mathrm{Hol}(\{\mathrm{Re}(z) > (C - C_N)/3\}, \mathcal{L}^1)$ of holomorphic functions with values in trace class operators on L^2. This implies that, for every ψ_1 in $L^2(M, \Lambda^k(T^*M))$,

$$\Pi_{z_0}^{(k)}(h)(\psi_1) = \frac{-1}{2i\pi} \int_{\mathcal{C}(z_0,r_0)} (\widehat{P}_h - z)^{-1} \widehat{\chi}_R (\widehat{P}_0 + \widehat{\chi}_R - z)^{-1}(\psi_1) \, dz + o(1),$$

as $h \to 0^+$. Then, we write

$$(\widehat{P}_h - z)^{-1}\widehat{\chi}_R = (\widehat{P}_0 - z)^{-1}\widehat{\chi}_R + h\,\underbrace{(\widehat{P}_h - z)^{-1}\widehat{Q}_2(\widehat{P}_0 - z)^{-1}\widehat{\chi}_R}.$$

The underbraced term being uniformly bounded as an operator from L^2 to L^2 for $|z - z_0| = r_0$ (as $h \to 0$), we finally find that, for every ψ_1 in $L^2(M, \Lambda^k(T^*M))$,

$$\lim_{h\to 0^+} \|(\Pi_{z_0}^{(k)}(h) - \Pi_{z_0}^{(k)})(\psi_1)\|_{L^2} = 0,$$

which concludes the proof of (16). □

4.3. Properties of the semigroup

We would now like to relate the spectral properties of $-\mathcal{L}_{V_f}$ to the properties of the propagator $\varphi_f^{-t*} = e^{-t\mathcal{L}_{V_f}}$. To that end, the classical approach is to prove some resolvent estimate and to use some contour integral to write the inverse Laplace transform. Here, we proceed slightly differently (due to the specific nature of our problem) and we rather study the spectral properties of the time-one[11] map φ^{-1*} acting on the anisotropic Sobolev spaces that we have defined. More precisely, using the conventions of Section 3, one has:

Proposition 4.6. *Let* $0 \le k \le n$. *The operator*

$$\varphi_f^{-1*} : \mathcal{H}_k^{m_{N_0,N_1}} \to \mathcal{H}_k^{m_{N_0,N_1}}$$

is a bounded operator whose essential spectral radius is $\le e^{(C-C_N)/2}$. *The eigenvalues* λ *of* φ_f^{-1*} *with* $|\lambda| > e^{(C-C_N)/2}$ *are given by*

$$\{e^{z_0} : z_0 \in \mathcal{R}_k = \mathcal{R}_k(0) \text{ and } \mathrm{Re}(z_0) > (C - C_N)/2\}.$$

Moreover, the spectral projector of $\lambda = e^{z_0}$ *is given by the projector* $\pi_{z_0}^{(k)}$ *defined in* §3.2.5.

Recall that $\pi_{z_0}^{(k)}$ corresponds to the spectral projector of $-\mathcal{L}_{V_f}^{(k)}$ associated with the eigenvalue z_0 and that it is intrinsic (i.e. independent of the choice of the order function m_{N_0,N_1}) as its Schwartz kernel corresponds to the residue at z_0 of the meromorphic continuation of the Schwartz kernel of $(-\mathcal{L}_{V_f} - z)^{-1}$—see also [28, Th. 1.5].

Proof of Proposition 4.6. Rather than studying the time-one map of the flow, we will study the spectral properties of the hyperbolic diffeomorphism $\varphi_q := \varphi_f^{-1/q}$ for every fixed $q \ge 1$. The reason is that we aim at relating the spectral data of φ_f^{-1*} to the ones of the generator $-\mathcal{L}_{V_f}$—see below.

In the rest of the proof, we verify that φ_q^* has discrete spectrum, with arguments similar to those used for the generator. More precisely, we follow the arguments of [27, Th. 1]

[11] The choice of time 1 is rather arbitrary and this is the only thing that will be needed in our analysis.

applied to the hyperbolic diffeomorphism φ_q. Precisely, following this reference, we can verify that the order function m_{N_0,N_1} from Lemma 3.2 satisfies the assumptions of [27, Lemma 2]. Then, following [27, Section 3.2] almost verbatim, we can deduce that the transfer operator

$$\varphi_q^* : \psi \in \mathcal{H}_k^{m_{N_0,N_1}}(M) \to \varphi_f^{(-1/q)*}\psi \in \mathcal{H}_k^{m_{N_0,N_1}}(M)$$

defines a bounded operator on the anisotropic space $\mathcal{H}_k^{m_{N_0,N_1}}(M)$ which can be decomposed as

$$\varphi_q^* = \hat{r}_{m,q} + \hat{c}_{m,q}, \tag{17}$$

where $\hat{c}_{m,q}$ is a compact operator and the remainder $\hat{r}_{m,q}$ has small operator norm: $\|\hat{r}_{m,q}\| \leq e^{(C-C_N/q)/2}$ (for some uniform C that may be slightly larger than before). Taking $q = 1$, this shows the first part of the proposition.

Note that, for every $q \in \mathbb{N}$, we can make $\|\hat{r}_{m,q}\|$ arbitrarily small by choosing N large enough. Again, we can verify that the discrete spectrum is intrinsic, i.e. independent of the choice of the order function. This is because the eigenvalues and associated spectral projectors correspond to the poles and residues of a *discrete resolvent* defined as an operator from $\Omega^k(M)$ to $\mathcal{D}'^{,k}(M)$ as follows. Consider the series $\sum_{l=0}^{+\infty} e^{-lz}\varphi_q^{l*}$. Then, by the direct bound

$$\Big\|\sum_{l=0}^{+\infty} e^{-lz}\varphi_q^{l*}\psi\Big\|_{\mathcal{H}_k^{m_{N_0,N_1}}(M)} \leq \sum_{l=0}^{+\infty} e^{-l\,\mathrm{Re}(z)}\|\varphi_q^*\|^l\|\psi\|_{\mathcal{H}_k^{m_{N_0,N_1}}(M)},$$

we deduce that, for $\mathrm{Re}(z)$ large enough, the series $\sum_{l=0}^{+\infty} e^{-lz}\varphi_q^{l*}\psi$ converges absolutely in $\mathcal{H}_k^{m_{N_0,N_1}}(M)$ for every test form $\psi \in \Omega^k(M)$. Hence, by the continuous injections $\Omega^k(M) \hookrightarrow \mathcal{H}_k^{m_{N_0,N_1}}(M) \hookrightarrow \mathcal{D}'^{,k}(M)$, the identity

$$(\mathrm{Id} - e^{-z}\varphi_q^*)^{-1} = \sum_{l=0}^{+\infty} e^{-lz}\varphi_q^{l*} : \Omega^k(M) \to \mathcal{D}'^{,k}(M)$$

holds true for $\mathrm{Re}(z)$ large enough. A consequence of the decomposition (17) is that the resolvent of φ_q^*,

$$(\lambda - \varphi_q^*)^{-1} : \Omega^k(M) \to \mathcal{D}'^{,k}(M),$$

has a meromorphic extension from $|\lambda| > e^{C_0}$ to $\lambda \in \mathbb{C}$ with poles of finite multiplicity which correspond to the eigenvalues of the operator φ_q^* [27, Corollary 1]. In other words, $(\mathrm{Id} - e^{-z}\varphi_q^*)^{-1} : \Omega^k(M) \to \mathcal{D}'^{,k}(M)$ has a meromorphic extension from $\mathrm{Re}(z) > C_0$ (with $C_0 > 0$ large enough) to $z \in \mathbb{C}$ with poles of finite multiplicity. Denote by $\tilde{\pi}_{\lambda,q}^{(k)}$ the spectral projector of φ_q^* associated to the eigenvalue λ which is obtained from the contour integral formula

$$\tilde{\pi}_{\lambda,q}^{(k)} = \frac{1}{2i\pi}\int_\gamma (\mu - \varphi_q^*)^{-1}\,d\mu,$$

where γ is a small circle around λ. This corresponds to the residues of the discrete resolvent at $e^z = \lambda$.

We will now verify the second part of the proposition, that the spectral data of the diffeomorphism coincide with the ones of the generator. As φ_q^* commutes with $-\mathcal{L}_{V_f}^{(k)}$, we can deduce that the range of $\tilde{\pi}_{\lambda,q}^{(k)}$ is preserved by $-\mathcal{L}_{V_f}^{(k)}$. In particular, any eigenvalue z_0 of $-\mathcal{L}_{V_f}^{(k)}$ on that space must satisfy $e^{z_0/q} = \lambda$. As we know that the Pollicott–Ruelle spectrum of $-\mathcal{L}_{V_f}^{(k)}$ is real, we can deduce that the poles of $(\mathrm{Id} - e^{-z/q}\varphi_q^*)^{-1}$ belong to $\mathcal{R}_k \subset \mathbb{R}$ modulo $2i\pi\mathbb{Z}$. In particular, taking $q = 1$, this shows that the eigenvalues of φ_f^{-1*} are exactly given by the expected set. Take now z_0 in $\mathcal{R}_k$; it remains to show that

$$\tilde{\pi}_{e^{z_0},1}^{(k)} = \pi_{z_0}^{(k)}, \tag{18}$$

i.e. the spectral projectors are the same for both problems. It is here that we will crucially use the fact that q is arbitrary. Note that, as $\varphi_q^q = \varphi_1$, one has $\tilde{\pi}_{\lambda,q}^{(k)} = \tilde{\pi}_{\lambda^q,1}^{(k)}$ for every $q \geq 1$. Recall also that the eigenvalues were shown to be real for every $q \geq 1$. Hence, $\tilde{\pi}_{e^{z_0/q},q}^{(k)} = \tilde{\pi}_{e^{z_0},1}^{(k)}$. We now decompose the resolvent $(z + \mathcal{L}_{V_f}^{(k)})^{-1}$ as follows:

$$(z + \mathcal{L}_{V_f}^{(k)})^{-1} = \sum_{l=0}^{+\infty} e^{-z/q}\varphi_q^* \int_0^{1/q} e^{-zt}\varphi_f^{-t*}dt = (\mathrm{Id} - e^{-z/q}\varphi_q^*)^{-1} \int_0^{1/q} e^{-zt}\varphi_f^{-t*}\,dt.$$

For $\mathrm{Re}(z)$ large enough, this expression makes sense viewed as an operator from $\Omega^k(M)$ to $\mathcal{D}'^{,k}(M)$. We have seen that it can be meromorphically continued to $\mathbb{C}$ by using the fact that we have built a proper spectral framework and that we may pick N_0 and N_1 arbitrarily large. Consider now a small contour γ around z_0 containing no other elements of $\mathcal{R}_k$. Integrating over this contour tells us that, for every $q \geq 1$,

$$\pi_{z_0}^{(k)} = \tilde{\pi}_{e^{z_0/q},q}^{(k)}\, q \int_0^{1/q} e^{-z_0 t}\varphi_f^{-t*}dt = \tilde{\pi}_{e^{z_0},1}^{(k)} \int_0^1 e^{-tz_0/q}\varphi_f^{(-t/q)*}\,dt.$$

As an operator on $\Omega^k(M)$, we can observe that $\int_0^1 e^{-tz_0/q}\varphi_f^{(-t/q)*}\,dt$ converges to the identity as $q \to +\infty$. Hence, $\pi_{z_0}^{(k)} = \tilde{\pi}_{e^{z_0},1}^{(k)}$ as expected. □

As a direct corollary of Proposition 4.6, we get the following result on the asymptotics of the correlation function of the time-one map φ_f^{-1*}:

Corollary 4.7. *Let $0 \leq k \leq n$. Then for any $z_0 \in \mathcal{R}_k = \mathcal{R}_k(0)$, there is an integer $d_{z_0}^{(k)} \geq 1$ such that for any $\Lambda > 0$, there exist N_0, N_1 large enough such that for every $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$ and for every integer $p \geq 0$,*

$$\int_M \varphi_f^{-p*}(\psi_1) \wedge \psi_2 = \sum_{z_0 \in \mathcal{R}_k:\, z_0 \geq -\Lambda} e^{pz_0} \sum_{l=0}^{d_{z_0}^{(k)}-1} \frac{p^l}{l!} \int_M (\mathcal{L}_{V_f}^{(k)} + z_0)^l (\pi_{z_0}^{(k)}(\psi_1)) \wedge \psi_2$$
$$+ \mathcal{O}(e^{-\Lambda p} \|\psi_1\|_{\mathcal{H}_k^{m_{N_0,N_1}}} \|\psi_2\|_{\mathcal{H}_{n-k}^{-m_{N_0,N_1}}}),$$

where $\pi_{z_0}^{(k)} : \Omega^k(M) \to \mathcal{D}'^k(M)$ *is a finite rank continuous linear map defined in* §3.2.5. *In fact, the result also holds for any* ψ_1 *in* $\mathcal{H}_k^{m_{N_0,N_1}}(M)$.

Note that together with (11), this expansion could be applied to more general times t which are not necessarily in $\mathbb{Z}_+$.

5. Computation of the Pollicott–Ruelle resonances

In [17], we gave a full description of the Pollicott–Ruelle spectrum of a Morse–Smale gradient flow under certain nonresonance assumptions. Our proof was based on an explicit construction of the generalized eigenmodes and we shall now give a slightly different proof based on the works of Baladi and Tsujii on Axiom A diffeomorphisms [3, 2]. In order to state the result, we define the *dynamical Ruelle determinant* [2, pp. 65–68], for every $0 \leq k \leq n$, as

$$\boxed{\zeta_R^{(k)}(z) := \exp\left(-\sum_{l=1}^{+\infty} \frac{e^{-lz}}{l} \sum_{a \in \mathrm{Crit}(f)} \frac{\mathrm{Tr}(\Lambda^k(d\varphi_f^{-l}(a)))}{|\det(\mathrm{Id} - d\varphi_f^{-l}(a))|}\right).}$$

This quantity is related to the notion of distributional determinants [39, p. 313]. This function is well defined for $\mathrm{Re}(z)$ large enough, and, from Appendix A, it has a holomorphic extension to $\mathbb{C}$. The zeros of this holomorphic extension can be explicitly described in terms of the Lyapunov exponents of the flow φ_f^t at the critical points of f:

$$\forall a \in \mathrm{Crit}(f), \qquad \chi_1(a) \leq \cdots \leq \chi_r(a) < 0 < \chi_{r+1}(a) \leq \cdots \leq \chi_n(a),$$

where $(\chi_j(a))_{j=1}^n$ are the eigenvalues of $L_f(a)$, the unique (symmetric) matrix satisfying $d^2 f(a) = g_a(L_f(a)\cdot, \cdot)$. Using the conventions of Section 2, one has

Theorem 5.1. *Suppose that the assumptions of Theorem 2.1 are satisfied. Then, for every* $0 \leq k \leq n$, *the set of Pollicott–Ruelle resonances is given by*

$$\mathcal{R}_k = \{z_0 \in \mathbb{R} : \zeta_R^{(k)}(z_0) = 0\}.$$

Moreover, for every $z_0 \in \mathbb{R}$, *the rank of the spectral projector*

$$\pi_{z_0}^{(k)} : \Omega^k(M) \to \mathcal{D}'^k(M)$$

is equal to the multiplicity[12] *of* z_0 *as a zero of* $\zeta_R^{(k)}(z)$.

Among other things, this result shows that the correlation spectrum depends only on the Lyapunov exponents of the flow. In other words, the global correlation spectrum of a gradient flow depends only on the 0-jet of the metric at the critical points. Thus, this result gives in some sense some insights into Bowen's first problem in [9] from the perspective of the global dynamics of the flow instead of the local one. Note that if we were interested

[12] When $z_0 \notin \mathcal{R}_k$, one has $\pi_{z_0}^{(k)} = 0$.

in the local dynamics near critical points, this could be recovered from the results of Baladi and Tsujii [3]. Still regarding Bowen's question, we will also verify below that the range of the residues are generated by families of currents carried by the unstable manifolds of the gradient flows. Besides this support property, we do not say much on the structure of these residues except in the case $z_0 = 0$—see Lemma 5.10.

Regarding Section 3, the only thing left to prove is that the eigenvalues and their algebraic multiplicities are given by the zeros of the Ruelle dynamical determinant. As already mentioned, this result was already proved in [17, 19] under stronger linearization assumptions. Our new proof will only make use of the assumptions that the gradient flow is $\mathcal{C}^1$-linearizable, which is necessary to construct our anisotropic Sobolev space but not for the results from [3]. Yet, in some sense, it will be less self-contained as we shall use the results of [3] as a "black-box", while in the proof of [17] we determined the spectrum by hand even if under more restrictive assumptions. Another advantage of the proof from [17] was that it gave an explicit local form of the eigenmodes and some criteria under which we do not have Jordan blocks—see also [19] for slightly more precise results. The key idea compared with [17, 19] is to use the localized results of Baladi–Tsujii to guess the global resonance spectrum from the one near each critical point. To go from local to global, we will use the geometry of the stratification by unstable manifolds to *glue together*, in some sense, these *local spectra* and make them into a global spectrum.

Before starting our proof, let us recall the following classical result of Smale which will be useful to organize our induction arguments [56]—see [18] for a brief reminder on Smale's work:

Theorem 5.2 (Smale partial order relation). *Suppose that φ_f^t is a Morse–Smale gradient flow. Then, for every a in* $\mathrm{Crit}(f)$*, the closure of the unstable manifold $W^u(a)$ is the union of unstable manifolds $W^u(b)$ for some critical points in* $\mathrm{Crit}(f)$*. Moreover, we write $b \preceq a$ (resp. $b \prec a$) if $W^u(b)$ is contained in the closure of $W^u(a)$ (resp. $W^u(b) \subset \overline{W^u(a)}$, $W^u(b) \neq W^u(a)$). Then $\preceq$ is a* ***partial order relation*** *on* $\mathrm{Crit}(f)$*. Finally if $b \prec a$, then* $\dim W^u(b) < \dim W^u(a)$.

In this section, we use the results of Baladi and Tsujii [3]. For that purpose, we treat near every critical point the time-1 map $\varphi_1 := \varphi_f^{-1}$ of the flow φ_f^t as a hyperbolic diffeomorphism with only one fixed point. Recall that φ_f^t is a Morse–Smale gradient flow which is $\mathcal{C}^1$-linearizable, hence amenable to the analysis of the previous sections.

5.1. Local spectra from the work of Baladi–Tsujii

We start by recalling the results of [3]. Fix $0 \leq k \leq n$, the degree of the differential forms we are going to consider and a critical point a of f. Note that the reference [3] mostly deals with 0-forms, i.e. functions on M, which corresponds to $k = 0$. General results for transfer operators acting on vector bundles are given in [3, Section 2] and [2, §6.4]. In this subsection, we consider the transfer operator acting on sections of the bundle $\Lambda^k T^*M \to M$ of k-forms on M by pullback: $\Gamma(M, \Lambda^k T^*M) \ni h \mapsto \varphi_0^* u \in \Gamma(M, \Lambda^k T^*M)$. For any open subset $U \subset M$, we will denote by $\Omega_c^\bullet(U)$ the differential forms with compact support in U. Then one can find a small enough open neighborhood V_a of a in M such that, for every $(\psi_1, \psi_2) \in \Omega_c^k(V_a) \times \Omega_c^{n-k}(V_a)$, the map

$$\hat{c}_{\psi_1,\psi_2,a} : z \mapsto \sum_{l=1}^{+\infty} e^{-lz} \int_M \varphi_1^{l*}(\psi_1) \wedge \psi_2$$

has a meromorphic extension to $\mathbb{C}$. This is a straightforward consequence of[13] [3, Th. 2.1] and [2, Th. 6.12, p. 178], once we note that smooth differential forms are contained in the Banach spaces of distributional sections of $\Lambda^k T^*M$ used in these references. The result of [3] is in fact much more general as it holds for any Axiom A diffeomorphism provided that the observables are supported in the neighborhood of a basic set (which is here reduced to the critical point a). Note that this result could also be deduced from the analysis in [38]. Moreover in [3, Th. 2.2] (see also [2, Th. 6.13, p. 179]), Baladi and Tsujii proved the stronger result that the poles of $\hat{c}_{\psi_1,\psi_2,a}$ where ψ_1, ψ_2 run over $\Omega_c^k(V_a) \times \Omega_c^{n-k}(V_a)$ are exactly equal (with multiplicities) to the *real zeros* of some *dynamical Ruelle determinant* [2, p. 179]:

$$\boxed{\zeta_{R,a}^{(k)}(z) := \exp\left(-\sum_{l=1}^{+\infty} \frac{e^{-lz}}{l} \frac{\mathrm{Tr}(\Lambda^k(d\varphi_1^l(a)))}{|\det(\mathrm{Id} - d\varphi_1^l(a))|}\right).}$$

Recall that we show in Appendix A that these functions are holomorphic in $\mathbb{C}$ and that we can compute their zeros. Actually, it has been proved in the literature [48, 36, 23, 22] for various classes of dynamical systems that the poles of dynamical correlations correspond to the zeros of the dynamical Ruelle determinant. Moreover, for any such pole z_0, one can find a continuous linear map

$$\pi_{a,z_0}^{(k)} : \Omega_c^k(V_a) \to \mathcal{D}'^k(V_a),$$

which is of finite rank equal to the multiplicity of z_0 as a zero of $\zeta_{R,a}^{(k)}$ and such that the residue of $\hat{c}_{\psi_1,\psi_2,a}(z)$ at $z = z_0$ is equal to

$$\int_M \pi_{a,z_0}^{(k)}(\psi_1) \wedge \psi_2.$$

Again, $\pi_{a,z_0}^{(k)}$ corresponds to the spectral projector of φ_1^* acting on a certain anisotropic Banach space of currents in $\mathcal{D}'^k(V_a)$. Now the key observation is that the spectral projector $\pi_{a,z_0}^{(k)} : \Omega_c^k(V_a) \to \mathcal{D}'^k(V_a)$, whose existence follows from [3], is just the localized version of the global spectral projector $\pi_{z_0}^{(k)}$ defined in §3.2.5. Indeed, using Corollary 4.7, we find that, for $\psi_1 \in \Omega_c^k(V_a)$,

$$\forall \psi_1 \in \Omega_c^k(V_a), \quad \pi_{z_0}^{(k)}(\psi_1) = \pi_{a,z_0}^{(k)}(\psi_1), \tag{19}$$

[13] In that reference, the authors allow diffeomorphisms with low regularity. Here, everything is smooth and we can make the essential spectral radius arbitrarily small by letting $r \to +\infty$ in that reference.

where *equality holds in the sense of currents in* $\mathcal{D}'^k(V_a)$. To see this, one sums over $p \geq 1$ in Corollary 4.7 in order to recover the local correlation function $\hat{c}_{\psi_1,\psi_2,a}$ and to identify its residues at $z = z_0$.

The above means that every element of $\{z_0 \in \mathbb{R} : \zeta_{R,a}^{(k)}(z_0) = 0\}$ contributes to the set $\mathcal{R}_k$ of Pollicott–Ruelle resonances of the transfer operator acting on k-forms. The objective is to show that there are no other contributions to $\mathcal{R}_k$. More precisely, we shall prove that $\mathcal{R}_k$ exactly equals the union over $\text{Crit}(f)$ of local spectra:

$$\mathcal{R}_k = \bigcup_{a \in \text{Crit}(f)} \{z_0 : \zeta_{R,a}^{(k)}(z_0) = 0\},$$

where the zeros are counted with multiplicity.

5.2. *Gluing local spectra*

The main purpose of this section is to prove the following statement from which Theorem 5.1 follows:

Proposition 5.3. *Let* $0 \leq k \leq n$ *and let* $z_0 \in \mathbb{R}$. *Then*

$$\text{Rk}(\pi_{z_0}^{(k)}) = \sum_{a \in \text{Crit}(f)} \text{Rk}(\pi_{a,z_0}^{(k)}).$$

In particular, as already explained, one can deduce from [3, 2] that $\text{Rk}(\pi_{z_0}^{(k)})$ is equal to the multiplicity of z_0 as a zero of $\zeta_R^{(k)} = \prod_{a \in \text{Crit}(f)} \zeta_{R,a}^{(k)}$. Note that this may be 0 if z_0 does not belong to the set $\mathcal{R}_k$ of resonances.

5.2.1. *Construction of a "good" basis of Pollicott–Ruelle resonant states.*

Let $z_0 \in \mathcal{R}_k$. We fix a basis $(U_j)_{j=1}^{m_{z_0}}$ of the range of $\pi_{z_0}^{(k)}$. These are generalized eigenstates of eigenvalue z_0 for $-\mathcal{L}_{V_f}$ acting on a suitable anisotropic Sobolev space of currents of degree k. We aim at showing that we can choose this family in such a way that $\text{supp}(U_j) \subset \overline{W^u(a)}$ for some critical point a of f (depending on j). Intuitively, we are looking for a "good" basis of generalized eigencurrents with *minimal possible support* which by some propagation argument should be at least the closure of an unstable manifold.

We also warn the reader that the notion of linear independence we need for our basis is a bit subtle and depends on the open subset in which we consider our current. Indeed, we may have some currents which are linearly independent as elements in $\mathcal{D}'^{,k}(M)$ but become dependent when we restrict them to smaller open subsets $U \subset M$. We define:

Definition 5.4 (Independent germs at a given point). A family $(u_i)_{i \in I}$ of currents in $\mathcal{D}'^{,k}(M)$ consists of *linearly independent germs* at $a \in M$ if for each open neighborhood V_a of a, $(u_i)_{i \in I}$ are linearly independent as elements of $\mathcal{D}'^{,k}(V_a)$.

With this definition in mind, we want to prove

Lemma 5.5. *Let* $0 \leq k \leq n$ *and* $z_0 \in \mathcal{R}_k$. *For every* $a \in \text{Crit}(f)$*, there exists an integer* $m_a^{(k)}(z_0) \geq 0$ *together with a corresponding basis of generalized eigencurrents*

$$\{U_j(b, z_0) : b \in \text{Crit}(f),\ 1 \leq j \leq m_b^{(k)}(z_0)\}$$

of the range of $\pi_{z_0}^{(k)}$ *such that*

$$\forall a \in \text{Crit}(f), \forall 1 \leq j \leq m_a^{(k)}(z_0), \quad \text{supp}(U_j(a, z_0)) \subset \overline{W^u(a)},$$

and, for all $a \in \text{Crit}(f)$*, the family* $(U_j(a, z_0))_{j=1}^{m_a^{(k)}(z_0)}$ *consists of independent germs at* a.

We denote by

$$\{S_j(a, z_0) : a \in \text{Crit}(f),\ 1 \leq j \leq m_a^{(k)}(z_0)\}$$

the dual basis for the adjoint operator $-\mathcal{L}_{V_f}^{(n-k)}$ acting on $\mathcal{H}_{n-k}^{-m_{N_0,N_1}}(M)$. In particular, the spectral projector $\pi_{z_0}^{(k)}$ can be written as follows:

$$\forall \psi_1 \in \Omega^k(M), \quad \pi_{z_0}^{(k)}(\psi_1) = \sum_{a \in \text{Crit}(f)} \sum_{j=1}^{m_a^{(k)}(z_0)} \left(\int_M \psi_1 \wedge S_j(a, z_0) \right) U_j(a, z_0). \qquad (20)$$

The currents $(S_j(a, z_0))_{j,a,z_0}$ are generalized eigenmodes for the dual operator $(-\mathcal{L}_{V_f}^{(k)})^\dagger = -\mathcal{L}_{V_{-f}}^{(n-k)}$ acting on the anisotropic Sobolev space $\mathcal{H}_{n-k}^{-m}(M)$. Also, from the definition of the dual basis, one has, for any critical points (a, b), any indices (j, k) and every (z, z') in $\mathcal{R}_k$,

$$\langle U_k(b, z'), S_j(a, z) \rangle = \int_M U_k(b, z') \wedge S_j(a, z) = \delta_{jk} \delta_{zz'} \delta_{ab}. \qquad (21)$$

The purpose of this section is now to prove Lemma 5.5. To that end, we begin with the following preliminary

Lemma 5.6. *Let* $U_1 \in \mathcal{D}'^{,k}(M)$ *be inside the range of* $\pi_{z_0}^{(k)}$. *Then, for every* $a \in \text{Crit}(f)$*, there exists* $\tilde{U}_1(a)$ *inside the range of* $\pi_{z_0}^{(k)}$ *such that*

$$U_1 = \sum_{a \in \text{Crit}(f)} \tilde{U}_1(a),$$

where each $\tilde{U}_1(a)$ *is supported in* $\overline{W^u(a)}$.

Proof. By [19, Lemma 7.7], which is a propagation lemma aimed at controlling supports of generalized eigencurrents, we know that if a current U_1 is identically 0 on a certain open set V then this vanishing property propagates along the flow and U_1 vanishes identically on $\bigcup_{t \in \mathbb{R}} \varphi_f^t(V)$. We let $\text{Max}(U_1)$ be the set of critical points a of f such that $U_1 \in \text{Ran}\, \pi_{z_0}^{(k)}$ is not identically zero near a and, for every $b \succ a$, U_1 identically vanishes near b. In particular, this means that, for every a in $\text{Max}(U_1)$, the current U_1 is supported by $W^u(a)$ in a neighborhood of a by [19, Lemma 7.8] which gives control on the support of generalized eigencurrents near maximal elements of $\text{Crit}(f)$.

Remark 5.7. We will implicitly use the fact that anisotropic Sobolev spaces of currents are $\mathcal{C}^\infty(M)$-modules, which can be seen as follows: $u \in \mathcal{H}_k^{m_{N_0,N_1}} \Leftrightarrow \widehat{A}_N u \in L^2(M)$. Hence,

$$\forall \psi \in \mathcal{C}^\infty(M), \qquad \widehat{A}_N(\psi u) = \underbrace{\widehat{A}_N \psi \widehat{A}_N^{-1}}_{\in \Psi^0(M)} \underbrace{\widehat{A}_N u}_{\in L^2} \in L^2(M)$$

where we use the composition for pseudodifferential operators [27, Th. 8, p. 39] and elements in $\Psi^0(M)$ are bounded in L^2.

Let us now decompose U_1 into currents with minimal support. For every critical point a, we set χ_a to be a smooth cut-off function which is identically 1 near a and χ_a vanishes away from a. Then, for every a in $\text{Max}(U_1)$, we define

$$\tilde{U}_1(a) := \pi_{z_0}^{(k)}(\chi_a U_1),$$

and we want to verify that $\tilde{U}_1(a)$ is supported in $\overline{W^u(a)}$ and that it is equal to U_1 near a. To that end, we apply Proposition 4.6 to the test current $\chi_a U_1$ (belonging to $\mathcal{H}_k^{m_{N_0,N_1}}$ for N_0, N_1 large enough) and to some test form ψ_2 in $\Omega^{n-k}(M)$:

$$\int_M \varphi_f^{-p*}(\chi_a \psi_1) \wedge \psi_2 = \sum_{z_0 \in \mathcal{R}_k : z_0 > -\Lambda} e^{p z_0} \sum_{l=0}^{d_{z_0}^{(k)}-1} \frac{p^l}{l!} \int_M (\mathcal{L}_{V_f}^{(k)} + z_0)^l (\pi_{z_0}^{(k)}(\chi_a \psi_1)) \wedge \psi_2$$
$$+ \mathcal{O}(e^{-\Lambda p} \|\chi_a \psi_1\|_{\mathcal{H}_k^{m_{N_0,N_1}}} \|\psi_2\|_{\mathcal{H}_{n-k}^{-m_{N_0,N_1}}}).$$

On the other hand, if we choose ψ_2 compactly supported in $M - \overline{W^u(a)}$, then we can verify that

$$\forall p \geq 0, \qquad \int_M \varphi_f^{-p*}(\chi_a U_1) \wedge \psi_2 = 0.$$

In particular, we find that

$$\forall \psi_2 \text{ with } \text{supp}(\psi_2) \cap \overline{W^u(a)} = \emptyset, \qquad \int_M \pi_{z_0}^{(k)}(\chi_a U_1) \wedge \psi_2 = 0.$$

This implies that $\tilde{U}_1(a)$ is supported by $\overline{W^u(a)}$. If we now choose ψ_2 to be compactly supported in the neighborhood of a where $\chi_a = 1$, then

$$\int_M \varphi_f^{-p*}(\chi_a U_1) \wedge \psi_2 = \int_M \varphi_f^{-p*}(U_1) \wedge \psi_2,$$

where we use the fact that U_1 is supported by $\overline{W^u(a)}$. Applying the asymptotic expansion of Proposition 4.6 one more time to the left hand side of the above equality, we find that $\tilde{U}_1(a) = \pi_{z_0}^{(k)}(\chi_a U_1)$ is equal to $U_1 = \pi_{z_0}^{(k)}(U_1)$ in a neighborhood of a. We can now define

$$\tilde{U}_1 = U_1 - \sum_{a \in \text{Max}(U_1)} \tilde{U}_1(a),$$

which by construction still belongs to the range of $\pi_{z_0}^{(k)}$ and which is now identically 0 in a neighborhood of each b satisfying $b \succeq a$ for every a in $\text{Max}(U_1)$. Then either $\tilde{U}_1 = 0$, in which case $U_1 = \sum_a \tilde{U}_1(a)$ is decomposed with this minimal support property and we are done; otherwise, we repeat the above argument with $\tilde{U}_1$ instead of U_1 and deal with critical points which are smaller with respect to Smale's partial order relation. As there are only a finite number of critical points to exhaust, this procedure will end after a finite number of steps and we will find that

$$U_1 = \sum_{a \in \text{Crit}(f)} \tilde{U}_1(a),$$

where the support of $\tilde{U}_1(a)$ is contained in $\overline{W^u(a)}$ and some of the $\tilde{U}_1(a)$ may be taken equal to 0. □

We can now turn to the proof of Lemma 5.5. Thanks to Lemma 5.6, we obtain $m_a^{(k)}(z_0) \in \mathbb{N}$, $a \in \text{Crit}(f)$ and some family $(U_{j,a}(z_0))_{a \in \text{Crit}(f), 1 \le j \le m_a^{(k)}(z_0)}$ of *nontrivial* currents which spans the image of $\pi_{z_0}^{(k)}$ and each $U_{j,a}(z_0)$ is supported in $\overline{W^u(a)}$. Note that our family of currents may not be linearly independent and we can extract a subfamily to make it into a basis of $\text{Ran}(\pi_{z_0}^{(k)})$. However, we recall that we are aiming at some stronger linear independence property than linear independence in $\mathcal{D}'^k(M)$. To fix this problem, we start from a critical point a such that $(U_j(a, z_0))_{j=1}^{m_a^{(k)}(z_0)}$ are not independent germs at a and, for every $b \succ a$, $(U_j(b, z_0))_{j=1}^{m_b^{(k)}(z_0)}$ are linearly independent germs at b. We next define a method to *localize linear dependence* near a as follows.

Definition 5.8 (Local rank of germs at some point). Consider the family of currents $(U_j(a, z_0))_{j=1}^{m_a^{(k)}(z_0)}$. Define a sequence $B_a(l)$ of balls of radius $1/l$ around a. Consider the sequence $r_l = \text{Rk}(U_j(a, z_0)|_{B_a(l)})_{j=1}^{m_a^{(k)}(z_0)}$ where each $U_j(a, z_0)|_{B_a(l)} \in \mathcal{D}'^{,k}(B_a(l))$ is the restriction of $U_j(a, z_0) \in \mathcal{D}'^{,k}(M)$ to the ball $B_a(l)$. We call $\lim_{l \to +\infty} r_l$ the *rank* of the germs $(U_j(a, z_0))_{j=1}^{m_a^{(k)}(z_0)}$ at a.

If $\lim_{l \to +\infty} r_l < m_a^{(k)}(z_0)$, then there exists an open neighborhood V_a of a such that the currents $(U_j(a, z_0)|_{V_a})_{j=1}^{m_a^{(k)}(z_0)}$ are linearly dependent in $\mathcal{D}'^{,k}(V_a)$, and the open subset V_a is optimal as one cannot find a smaller open subset around a on which one could write new linear relations among $(U_j(a, z_0))_{j=1}^{m_a^{(k)}(z_0)}$. This means that one can find some j (say $j = 1$) such that, on the open set V_a,

$$U_1(a, z_0) = \sum_{j=2}^{m_a^{(k)}(z_0)} \alpha_j U_j(a, z_0).$$

Then we set

$$\tilde{U}(z_0) = U_1(a, z_0) - \sum_{j=2}^{m_a^{(k)}(z_0)} \alpha_j U_j(a, z_0),$$

which is 0 near a. Hence, by propagation [19, Lemma 7.7], $\tilde{U}(z_0)$ is supported inside $\overline{W^u(a)} \setminus W^u(a)$. Thus, proceeding by induction on Smale's partial order relation, we can without loss of generality suppose that, for every critical point a, the currents $(U_j(a, z_0))_{j=1}^{m_a^{(k)}(z_0)}$ are linearly independent germs at a and not only as elements of $\mathcal{D}'^{,k}(M)$. This concludes the proof of Lemma 5.5.

5.2.2. Support of the dual basis. We would like to show that the dual basis

$$\{S_j(a, z_0) : a \in \text{Crit}(f),\ 1 \leq j \leq m_a^{(k)}(z_0)\}$$

defined above contains only currents with minimal support. In fact, we will prove

Lemma 5.9. *For all $z_0 \in \mathcal{R}_k$, the above dual basis satisfies the condition*

$$\boxed{\forall a \in \text{Crit}(f),\ \forall 1 \leq j \leq m_a^{(k)}(z_0), \quad \text{supp}(S_j(a, z_0)) \subset \overline{W^s(a)}.}$$

The above bound on the support of the dual basis actually shows that

$$\text{supp}(S_j(a, z_0)) \cap \text{supp}(U_j(a, z_0)) = \{a\}. \tag{22}$$

Proof of Lemma 5.9. Let $0 \leq k \leq n$ and let $z_0 \in \mathcal{R}_k$. We shall argue by induction on Smale's partial order relation $\succeq$. In that manner, it is sufficient to prove that, for every $a \in \text{Crit}(f)$ such that the conclusion of the lemma holds for all[14] $b \succ a$, one has

$$\forall 1 \leq j \leq m_a^{(k)}(z_0), \quad \text{supp}(S_j(a, z_0)) \subset \overline{W^s(a)}.$$

Fix such a critical point a and ψ_1 compactly supported in $M \setminus \overline{W^s(a)}$. Then we consider a small enough neighborhood V_a of a which does not interesect the support of ψ_1 and we fix ψ_2 in $\Omega_c^k(V_a)$. From [18, Remark 4.5, p. 17], we know that if V_a is chosen small enough, then $\varphi_f^{-t}(V_a)$ remains inside the complement of $\text{supp}(\psi_1)$ for $t \geq 0$. In particular, for every $t \geq 0$, $\varphi_f^{-t*}(\psi_1) \wedge \psi_2 = 0$. Applying the asymptotic expansion of Proposition 4.6, we then find that

$$\int_M \pi_{z_0}^{(k)}(\psi_1) \wedge \psi_2 = 0.$$

Hence, combining this with (20), we have proved that

$$\forall \psi_2 \in \Omega_c^k(V_a), \qquad \sum_{b \in \text{Crit}(f)} \sum_{j=1}^{m_b^{(k)}(z_0)} \left(\int_M \psi_1 \wedge S_j(b, z_0) \right) \left(\int_M U_j(b, z_0) \wedge \psi_2 \right) = 0.$$

[14] Note that a may be a minimum and, in that case, there is no such b.

As V_a is a small neighborhood of a and as $U_j(b, z_0)$ is carried by $\overline{W^s(b)}$, we can apply Smale's Theorem 5.2 to verify that only the points b such that $b \succeq a$ contribute to the above sum, i.e.

$$\forall \psi_2 \in \Omega_c^k(V_a), \qquad \sum_{b\in\mathrm{Crit}(f):\, b\succeq a} \sum_{j=1}^{m_b^{(k)}(z_0)} \left(\int_M \psi_1 \wedge S_j(b, z_0)\right)\left(\int_M U_j(b, z_0) \wedge \psi_2\right) = 0.$$

We can now use our inductive assumption on a and the fact that $\overline{W^s(b)} \subset \overline{W^s(a)}$ for $b \succeq a$ to get

$$\forall \psi_2 \in \Omega_c^k(V_a), \qquad \sum_{j=1}^{m_a^{(k)}(z_0)} \left(\int_M \psi_1 \wedge S_j(a, z_0)\right)\left(\int_M U_j(a, z_0) \wedge \psi_2\right) = 0.$$

As the germs of currents are independent at a, we can deduce that $\int_M \psi_1 \wedge S_j(a, z_0) = 0$ for every $1 \leq j \leq m_a^{(k)}(z_0)$, which concludes the proof. □

5.3. *Proof of Proposition 5.3*

We can now conclude the proof of Proposition 5.3. With the above conventions, it is sufficient to show that $m_a^{(k)}(z_0) = \mathrm{Rk}(\pi_{z_0,a}^{(k)})$. Hence, we fix a critical point a and thanks to (19), we can write, for every ψ_1 in $\Omega_c^k(V_a)$,

$$\pi_{z_0,a}^{(k)}(\psi_1)|_{V_a} = \sum_{b\in\mathrm{Crit}(f)} \sum_{j=1}^{m_b^{(k)}(z_0)} \left(\int_M \psi_1 \wedge S_j(b, z_0)\right) U_j(b, z_0)|_{V_a}.$$

We will now verify that all the terms corresponding to $b \neq a$ cancel. To that end, we choose V_a small enough around a such that $V_a \cap \overline{W^u(b)} = \emptyset$ (resp. $V_a \cap \overline{W^s(b)} = \emptyset$) unless $b \succeq a$ (resp. unless $b \preceq a$). Then $S_j(b, z_0) \wedge \psi_1 = 0$ unless $b \preceq a$ because $\mathrm{supp}(S_j(b, z_0)) \subset \overline{W^s(b)}$ does not meet V_a hence $\mathrm{supp}(\psi_1)$. In the same manner, $U_j(b, z_0)|_{V_a} = 0$ unless $b \succeq a$ since $\mathrm{supp}(U_j(b, z_0)) \subset \overline{W^u(b)}$ does not meet V_a unless $b \succeq a$. Therefore, all these cancellations imply that $\sum_{b\in\mathrm{Crit}(f)} \sum_{j=1}^{m_b^{(k)}(z_0)} (\int_M \psi_1 \wedge S_j(b, z_0)) U_j(b, z_0)|_{V_a} = \sum_{j=1}^{m_a^{(k)}(z_0)} (\int_M \psi_1 \wedge S_j(a, z_0)) U_j(a, z_0)|_{V_a}$, yielding

$$\pi_{z_0,a}^{(k)}(\psi_1) = \sum_{j=1}^{m_a^{(k)}(z_0)} \left(\int_M \psi_1 \wedge S_j(a, z_0)\right) U_j(a, z_0)|_{V_a}. \tag{23}$$

Thanks to Lemma 5.5, we know that the currents $U_j(a, z_0)|_{V_a}$ are linearly independent in $\mathcal{D}'^k(V_a)$. Using (22) and the fact that $S_j(a, z_0)$ is the dual basis of $U_j(a, z_0)$, we can verify that the $S_j(a, z_0)$ are also independent germs at a. Hence, one can verify that the range of $\pi_{z_0,a}^{(k)}$ is spanned by the currents $(U_j(a, z_0)|_{V_a})_{j=1}^{m_a^{(k)}(z_0)}$, which concludes the proof of Proposition 5.3.

5.4. No Jordan blocks for $z_0 = 0$

Let $0 \leq k \leq n$. Thanks to Remark A.1, we know that the multiplicity of 0 as a zero of $\zeta_R^{(k)}(z) = \prod_a \zeta_{R,a}^{(k)}(z)$ is equal to the number of critical points of index k. On the other hand, given a critical point a of index l, if we use Baladi–Tsujii's local result relating the zeros of $\zeta_{R,a}^{(k)}(z)$ to the eigenvalues of φ^{-1*} near a [3, Th. 2.2], we know that

$$m_a^{(k)}(0) = \begin{cases} 1 & \text{if } \dim W^s(a) = k = l, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if we use (23) combined with Proposition 5.3, we can deduce that, for $z_0 = 0$, one can find a basis of generalized eigencurrents for $\mathrm{Ker}(\mathcal{L}_{V_f}^{(k)})^N$ (for some large enough N):

$$\{U_a : \dim W^s(a) = k\},$$

whose support is equal to $\overline{W^u(a)}$. We would now like to verify that we can pick $N = 1$, or equivalently that there is no Jordan blocks in the kernel. Suppose by way of contradiction that we have a nontrivial Jordan block, i.e. there exist $u_0 \neq 0$ and $u_1 \neq 0$ such that

$$\mathcal{L}_{V_f}^{(k)} u_0 = 0 \quad \text{and} \quad \mathcal{L}_{V_f}^{(k)} u_1 = u_0.$$

We fix a to be a critical point of index k such that u_0 is not 0 near a. Such a point exists as u_0 is a linear combination of the $(U_b)_{b:\dim W^s(b)=k}$. Recall from Smale's theorem that, for every b in $\mathrm{Crit}(f)$, $\overline{W^u(b)} \setminus W^u(b)$ is the union of unstable manifolds whose dimension is $< \dim W^u(b)$. Hence, as u_1 is also a linear combination of the $(U_b)_{b:\dim W^s(b)=k}$, it follows that u_1 is proportional to U_a near a. In a neighborhood of a, we then have $u_0 = \alpha_0 U_a$ (with $\alpha_0 \neq 0$) and $u_1 = \alpha_1 U_a$. If we use the eigenvalue equation, we find that, in a neighborhood of a,

$$\alpha_0 \mathcal{L}_{V_f}^{(k)} U_a = 0 \quad \text{and} \quad \alpha_1 \mathcal{L}_{V_f}^{(k)} U_a = \alpha_0 U_a.$$

As U_a is not identically 0 near a, we find the expected contradiction.

We next prove the following lemma on the local structure of eigencurrents in $\mathrm{Ker}(\mathcal{L}_{V_f})$ near critical points:

Lemma 5.10. *Let y_0 be a point inside $W^u(a)$. Then one can find a local system of coordinates $(x_1, \ldots, x_n)$ such that $W^u(a)$ is given locally near y_0 by $\{x_1 = \cdots = x_r = 0\}$, where r is the index of a and the current $[W^u(a)] = \delta_0(x_1, \ldots, x_r) dx_1 \wedge \cdots \wedge dx_r$ coincides with U_a near y_0. Similarly, $S_a = [W^s(a)]$ near a.*

Proof. Recall from [56, 59] that $W^u(a)$ is an embedded submanifold inside M. Then there is a local system of coordinates $(x_1, \ldots, x_n)$ such that $W^u(a)$ is given locally near y_0 by $\{x_1 = \cdots = x_r = 0\}$, where r is the index of a. The current of integration on $W^u(a)$, for the choice of orientation given by $[dx_1 \wedge \cdots \wedge dx_r]$ (see [16, Appendix D] for a discussion of orientations for integration currents), reads in this system of coordinates $[W^u(a)] = \delta_0(x_1, \ldots, x_r) dx_1 \wedge \cdots \wedge dx_r$ by [16, Cor. D.4]. Moreover, for every test form ω whose support does not meet the boundary $\partial W^u(a) = \overline{W^u(a)} \setminus W^u(a)$, one has for all $t \in \mathbb{R}$ the identity $\langle \varphi_f^{-t*}[W^u(a)], \omega\rangle = \int_{W^u(a)} \varphi_f^{t*}\omega = \int_{\varphi_f^{-t}(W^u(a))=W^u(a)} \omega = \langle [W^u(a)], \omega\rangle$ since $\varphi_f^t : M \to M$ is an orientation preserving diffeomorphism which leaves $W^u(a)$ invariant. This implies that in the weak sense $\varphi_f^{-t*}[W^u(a)] = [W^u(a)]$ for all $t \in \mathbb{R}$, hence $\mathcal{L}_{V_f}([W^u(a)]) = 0$. Near a, $[W^u(a)]$ belongs to the anisotropic Sobolev space $\mathcal{H}_r^{m_{N_0,N_1}}(M)$ for N_0, N_1 large enough. Hence, if we fix a smooth cutoff function χ_a near a, we can verify, by a propagation argument similar to the ones used to prove Lemma 5.5, that U_a can be chosen equal to $\pi_0^{(r)}(\chi_a[W^u(a)])$, and one has $U_a = [W^u(a)]$ near a. Similarly, $S_a = [W^s(a)]$ near a. □

To end this section and as a consequence of (11), Corollary 4.7 and Lemma 5.10, let us record the following improvement of the results from [41, 17]:

Theorem 5.11 (Vacuum states). *Suppose that the assumptions of Theorem 2.1 are satisfied and fix $0 \le k \le n$. Then, for every*

$$0 < \Lambda < \min\{|\chi_j(a)| : 1 \le j \le n,\ a \in \mathrm{Crit}(f)\}$$

and every $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$,

$$\boxed{\int_M \varphi_f^{-t*}(\psi_1) \wedge \psi_2 = \sum_{a:\, \dim W^u(a)=n-k} \int_M \psi_1 \wedge S_a \int_M U_a \wedge \psi_2 + \mathcal{O}_{\psi_1,\psi_2}(e^{-\Lambda t}).}$$

6. Proofs of Theorems 2.1 to 2.6

In this section, we make use of the information collected so far to prove the main statements of the introduction except for Theorem 2.7 that will be proved in Section 7.

6.1. *Proof of Theorem 2.5*

Regarding the limit operator, it now remains to show Witten's instanton formula of Theorem 2.5. For that purpose, we first discuss some orientation issues for curves connecting a pair (a, b) of critical points of f. Choosing some orientation of every unstable manifolds $(W^u(a))_{a\in\mathrm{Crit}(f)}$ defines a local germ of current $[W^u(a)]$ near every critical point a and some integration current in $\mathcal{D}'^{,\bullet}(M \setminus \partial W^u(a))$. Both Theorem 5.11 and Lemma 5.10 show us that each germ $[W^u(a)]$ extends to a globally well defined current U_a on M which coincides with $[W^u(a)]$ on $M \setminus \partial W^u(a)$. As M is oriented, the orientation of $W^u(a)$ induces a canonical *coorientation* on $W^s(a)$ so that the intersection pairing at the level of currents gives $\int_M \chi[W^u(a)] \wedge [W^s(a)] = \chi(a)$ for every $a \in \mathrm{Crit}(f)$ and for all smooth χ compactly supported near a. Given any two critical points (a, b) satisfying $\mathrm{ind}(a) = \mathrm{ind}(b) + 1$, recall from [59, Prop. 3.6] that there exist finitely many flow lines connecting a and b. These curves are called *instantons* and we shall denote them by γ_{ab}. Such a curve is naturally oriented by the gradient vector field V_f, hence defines a current of integration of degree $n - 1$, $[\gamma_{ab}] \in \mathcal{D}'^{,n-1}(M)$.

Definition 6.1. We define an orientation coefficient $\sigma(\gamma_{ab}) \in \{\pm 1\}$ by

$$\boxed{[\gamma_{ab}] = \sigma(\gamma_{ab})[W^u(a)] \wedge [W^s(b)]} \tag{24}$$

in the neighborhood of some $x \in \gamma_{ab}$ where x differs from both (a, b).

Let us verify that this definition makes sense. From the Smale transversality assumption (see §3.1.1), one finds, for $x \in \gamma_{ab} \setminus \{a, b\}$, that the intersection of the conormals $N^*(W^u(a))$ and $N^*(W^s(b))$ is empty. Hence, according to [44, p. 267] (see also [11] or Section 7), it makes sense to consider the wedge product $[W^u(a)] \wedge [W^s(b)]$ near such a point x. Moreover, it defines, near x, the germ of integration current along γ_{ab} using the next lemma:

Lemma 6.2. *Let X, Y be two tranverse submanifolds of M whose intersection is a submanifold denoted by Z. Then choosing an orientation of X, Y, M induces a canonical orientation of Z such that near every point of Z, we have a local equation in the sense of currents $[Z] = [X] \wedge [Y]$.*

Proof. Thanks to the transversality assumption, we can use local coordinates (x, y, h) where locally $X = \{x = 0\}$, $Y = \{y = 0\}$ and $Z = \{x = 0, y = 0\}$ Hence,

$$[X] \wedge [Y] = \delta_{\{0\}}^{\mathbb{R}^p}(x)dx \wedge \delta_{\{0\}}^{\mathbb{R}^q}(y)dy = \delta_{\{0\}}^{\mathbb{R}^{p+q}}(x, y)dx \wedge dy = [Z]$$

by definition of integration currents. □

Altogether, this shows that the coefficient $\sigma(\gamma_{ab})$ is well defined. In fact, using the flow, we see that the formula

$$[\gamma_{ab}] = \sigma(\gamma_{ab})[W^u(a)] \wedge [W^s(b)]$$

holds true on $M \setminus \{a, b\}$. We are now ready to prove Theorem 2.5 by setting

$$n_{ab} = (-1)^n \sum_{\gamma_{ab}} \sigma(\gamma_{ab}),$$

where the sum runs over instantons from a to b. In other words, the integer n_{ab} counts with sign the number of instantons connecting a and b. We first recall that, as d commutes with $\mathcal{L}_{V_f}$ and as the currents $(U_a)_{a \in \mathrm{Crit}(f)}$ are elements in[15] $\mathrm{Ker}(\mathcal{L}_{V_f})$, we already know that $dU_a \in \mathrm{Ker}(\mathcal{L}_{V_f})$. Hence,

$$dU_a = \sum_{b:\, \mathrm{ind}(b) = \mathrm{ind}(a)+1} n'_{ab} U_b,$$

where the coefficients n'_{ab} are a priori real numbers. The goal is to prove that they are indeed equal to the integer coefficients n_{ab} we have just defined. Let a be some critical point of f of index k. Choose some arbitrary cutoff function χ such that $\chi = 1$ in a small

[15] Recall also that this spectrum is intrinsic, i.e. independent of the choice of the anisotropic Sobolev space.

neighborhood of a and $\chi = 0$ outside some slightly bigger neighborhood of a. Then the following identity holds true in the sense of currents:

$$d(\chi[W^u(a)]) = d(\chi U_a) = d\chi \wedge U_a + \chi \wedge dU_a = d\chi \wedge [W^u(a)],$$

where we use the fact that $[W^u(a)] = U_a$ on the support of χ, Smale's Theorem 5.2 and the fact that $\chi \wedge dU_a = 0$ since dU_a is a linear combination of the U_b with $\text{ind}(b) = \text{ind}(a) + 1$. In other words, we use the fact that the current dU_a is supported by $\partial W^u(a)$.

Choose now some critical point b such that $\text{ind}(b) = \text{ind}(a) + 1$. Then, for a small open neighborhood O of $\{a\} \cup \partial W^u(a)$, we have the following identity in the sense of currents in $\mathcal{D}'(M \setminus O)$:

$$[W^u(a)] \wedge [W^s(b)]|_{M\setminus O} = \sum_{\gamma_{ab}} \sigma(\gamma_{ab})[\gamma_{ab}]|_{M\setminus O}, \tag{25}$$

where the sum runs over instantons γ_{ab} connecting a and b. Recall from the above that the wedge product makes sense thanks to Smale's transversality assumption. We choose O in such a way that O does not meet the support of $d\chi$. Then the following identity holds true:

$$\begin{aligned}\langle d(\chi[W^u(a)]), [W^s(b)]\rangle &= \int_M d\chi \wedge [W^u(a)] \wedge [W^s(b)] \\ &= (-1)^{(n-1)} \sum_{\gamma_{ab}} \sigma(\gamma_{ab}) \int_M [\gamma_{ab}] \wedge d\chi \\ &= (-1)^{n-1} \sum_{\gamma_{ab}} \sigma(\gamma_{ab}) \int_{\gamma_{ab}} d\chi \\ &= (-1)^{n-1} \sum_{\gamma_{ab}} \sigma(\gamma_{ab}) \underbrace{(\chi(b) - \chi(a))}_{0-1} = n_{ab}.\end{aligned}$$

We have just proved that, for any function χ such that $\chi = 1$ near a and $\chi = 0$ outside some slightly bigger neighborhood of a, one has

$$\langle d(\chi[W^u(a)]), [W^s(b)]\rangle = n_{ab}.$$

Note that this equality remains true for any χ such that $\chi = 1$ near a and $\chi = 0$ in some neighborhood of $\partial W^u(a) = \overline{W^u(a)} \setminus W^u(a)$. In particular, it applies to the pullback $\varphi_f^{-t*}(\chi)$ for all $t \geq 0$. Recall in fact that $\varphi_f^{-t*}[W^u(a)] = [W^u(a)]$ on the support of $\varphi_f^{-t*}(\chi)$ by Lemma 5.10. Still from this lemma, one knows that $S_b = [W^s(b)]$ on the support of $d(\varphi_f^{-t*}(\chi))$. Therefore, one also has

$$\forall t \geq 0, \quad \langle d\varphi_f^{-t*}(\chi[W^u(a)]), S_b\rangle = \langle d\varphi_f^{-t*}(\chi[W^u(a)]), [W^s(b)]\rangle = n_{ab}.$$

Still from Lemma 5.10 and as χ is compactly supported near a, we know that, for an appropriate choice of integers N_0, N_1, the current $\chi[W^u(a)]$ belongs to the anisotropic Sobolev space $\mathcal{H}_k^{m_{N_0,N_1}}(M)$ (where $k = \text{ind}(a)$) and the spectrum of $-\mathcal{L}_{V_f}^{(k)}$ is discrete on some half-plane $\mathrm{Re}(z) > -c_0$ with $c_0 > 0$. Thanks to Proposition 4.6 and to the fact that there are no Jordan blocks, we can conclude that, in the Sobolev space $\mathcal{H}_k^{m_{N_0,N_1}}(M)$,

$$\varphi_f^{-t*}(\chi[W^u(a)]) \to \sum_{a'\in\mathrm{Crit}(f):\,\mathrm{ind}(a')=\mathrm{ind}(a)+1} \left(\int_M (\chi[W^u(a)]) \wedge S_{a'}\right) U_{a'} \quad \text{as } t \to +\infty.$$

For every smooth test $(n-k)$-form ψ_2 compactly supported in $M \setminus \overline{W^u(a)}$, we can verify that

$$\forall t \geq 0, \quad \varphi_f^{-t*}(\chi[W^u(a)]) \wedge \psi_2 = 0,$$

which implies that the above reduces to

$$\varphi_f^{-t*}(\chi[W^u(a)]) \to \underbrace{\left(\int_M (\chi[W^u(a)]) \wedge S_a\right)}_{=\langle U_a, S_a\rangle = 1} U_a = U_a \quad \text{as } t \to +\infty,$$

since $\chi(a) = 1$, $\mathrm{supp}(S_a) \cap \mathrm{supp}(\chi[W^u(a)]) = \{a\}$ by (22) and $S_a = [W^s(a)]$ near a. Then it follows from the continuity of $d : \mathcal{H}_k^{m_{N_0,N_1}}(M) \to \mathcal{H}_{k+1}^{m_{N_0,N_1}-1}(M)$ that $d\varphi^{-t*}(\chi[W^u(a)]) \to dU_a$ in $\mathcal{H}_{k+1}^{m_{N_0,N_1}-1}(M)$. Finally, by continuity of the duality pairing $\mathcal{H}_{k+1}^{m_{N_0,N_1}-1}(M) \times \mathcal{H}_{n-(k+1)}^{1-m_{N_0,N_1}}(M) \ni (u,v) \mapsto \langle u, v\rangle$, we deduce that

$$n_{ab} = \lim_{t\to+\infty} \langle S_b, d\varphi^{-t*}(\chi[W^u(a)])\rangle = \langle S_b, dU_a\rangle.$$

This shows that the complex $(\mathrm{Ker}(\mathcal{L}_{V_f}), d)$ generated by the currents $(U_a)_{a\in\mathrm{Crit}(f)}$ is well defined as a $\mathbb{Z}$-module. Then, we note that tensoring the above complex with $\mathbb{R}$ yields a complex $(\mathrm{Ker}(\mathcal{L}_{V_f}), d) \otimes_{\mathbb{Z}} \mathbb{R}$ which is quasi-isomorphic to the de Rham complex $(\Omega^\bullet(M), d)$ of smooth forms by [20, Th. 2.1] as a consequence of the *chain homotopy equation* [20, §4.2]:

$$\boxed{\exists R : \Omega^\bullet(M) \to \mathcal{D}'^{,\bullet-1}(M), \quad \mathrm{Id} - \pi_0 = d \circ R + R \circ d.} \tag{26}$$

This ends our proof of Theorem 2.5.

6.2. *Proof of the results on the Witten Laplacian*

First of all, we note that the result from Theorem 2.1,

$$\lim_{h\to 0^+} \int_M \mathbf{1}_{[0,\epsilon]}(W_{f,h}^{(k)})(e^{-f/h}\psi_1) \wedge (e^{f/h}\psi_2) = \lim_{t\to+\infty} \int_M \varphi_f^{-t*}(\psi_1) \wedge \psi_2,$$

is a direct consequence of Theorem 2.4 which yields convergence of spectral projectors, $\lim_{h\to 0^+} \int_M \mathbf{1}_{[0,\epsilon]}(W_{f,h}^{(k)})(e^{-f/h}\psi_1) \wedge (e^{f/h}\psi_2) = \int_M \pi_0^{(k)}(\psi_1) \wedge \psi_2$ combined with Theorem 5.11 where the limit term $\lim_{t\to+\infty} \int_M \varphi_f^{-t*}(\psi_1) \wedge \psi_2$ is identified with the term $\int_M \pi_{z_0}^{(k)}(\psi_1) \wedge \psi_2$ coming from the spectral projector corresponding to the eigenvalue 0—see Section 5. Hence, it now remains to recall that Theorem 2.4, which claims that the spectral data of the Witten Laplacian converge to the spectral data of $-\mathcal{L}_{V_f}$, follows straightforwardly from Theorems 4.4 and 4.5.

We now prove Corollary 2.6 about the Witten–Helffer–Sjöstrand tunneling formula for our WKB states, which becomes a direct corollary of Theorem 2.5. Indeed, our WKB states were defined by using the spectral projector on the small eigenvalues of the Witten Laplacian, i.e.

$$U_a(h) = \mathbf{1}_{[0,\epsilon_0]}(W_{f,h}^{(k)})(e^{\frac{f(a)-f}{h}} U_a),$$

where k is the index of the critical point. Thanks to Theorem 2.5, we already know

$$d_{f,h}(e^{\frac{f(a)-f}{h}} U_a) = \sum_{b:\,\mathrm{ind}(b)=\mathrm{ind}(a)+1} n_{a,b} e^{-\frac{f(b)-f(a)}{h}} e^{\frac{f(b)-f}{h}} U_b. \tag{27}$$

Recall now that $d_{f,\hbar} W_{f,\hbar} = W_{f,\hbar} d_{f,\hbar}$ and the spectral projector has the following integral expression:

$$\mathbf{1}_{[0,\epsilon_0]}(W_{f,h}^{(k)}) = \frac{1}{2i\pi} \int_{\mathcal{C}(0,\epsilon_0)} (z - W_{f,h})^{-1}\, dz.$$

Hence, $d_{f,h}$ commutes with $\mathbf{1}_{[0,\epsilon_0]}(W_{f,h}^{(\bullet)})$. It is then sufficient to apply the spectral projector to both sides of (27) to conclude the proof.

Remark 6.3. Note that the family $(U_a(h))_{a\in\mathrm{Crit}(f)}$ is made up of linearly independent currents for $h > 0$ small enough. Indeed, set $\tilde{U}_a(h) := e^{\frac{f-f(a)}{h}} U_a(h)$, which converges to U_a in the anisotropic Sobolev space thanks to Theorem 4.5, and write, for every critical point a of index k,

$$\pi_0^{(k)}(\tilde{U}_a(h)) = \sum_{b:\,\mathrm{ind}(b)=k} \left(\int_M \tilde{U}_a(h) \wedge S_b \right) U_b = \sum_{b:\,\mathrm{ind}(b)=k} (\delta_{ab} + o(1)) U_b.$$

Hence, the $(\tilde{U}_a(h))_{a\in\mathrm{Crit}(f)}$ are linearly independent for $h > 0$ small enough as the $(U_a)_{a\in\mathrm{Crit}(f)}$ are. After multiplication by $e^{-f/h}$, the same holds for the family $(U_a(h))_{a\in\mathrm{Crit}(f)}$. Note that the linear independence would also follow from the arguments of Section 8 below but our argument here is independent of the Helffer–Sjöstrand construction of quasimodes. Finally, it seems to us that determining the limit of the Helffer–Sjöstrand quasimodes would probably be a delicate task via the semiclassical methods from [43]—see Remark 8.1 below.

7. Proof of Theorem 2.7

In this section, we give the proof of Theorem 2.7 which states that our WKB states satisfy Fukaya's instanton formula. Using the conventions of Theorem 2.7, we start with the following observation:

$$\begin{aligned} U_{a_{ij}}(h) &= \mathbf{1}_{[0,\epsilon_0]}(W_{f_{ij},h})(e^{\frac{f_{ij}(a_{ij})-f_{ij}(x)}{h}} U_{a_{ij}}) \\ &= e^{\frac{f_{ij}(a_{ij})-f_{ij}(x)}{h}} \mathbf{1}_{[0,\epsilon_0]}(\mathcal{L}_{V_{f_{ij}}} + h\Delta_{g_{ij}}/2)(U_{a_{ij}}), \end{aligned}$$

where $\epsilon_0 > 0$ is small enough and where ij belongs to $\{12, 23, 31\}$. Hence, we can deduce that

$$U_{a_{12}}(h) \wedge U_{a_{23}}(h) \wedge U_{a_{31}}(h) = e^{\frac{f_{12}(a_{12})+f_{23}(a_{23})+f_{31}(a_{31})}{h}} \tilde{U}_{a_{12}}(h) \wedge \tilde{U}_{a_{23}}(h) \wedge \tilde{U}_{a_{31}}(h),$$

where, for every ij and for $h > 0$,

$$\tilde{U}_{a_{ij}}(h) := \mathbf{1}_{[0,\epsilon_0]}(\mathcal{L}_{V_{f_{ij}}} + h\Delta_{g_{ij}}/2)(U_{a_{ij}}),$$

while $\tilde{U}_a(0) := U_a$. Hence, the proof of Theorem 2.7 consists in showing that

$$\int_M \tilde{U}_{a_{12}}(h) \wedge \tilde{U}_{a_{23}}(h) \wedge \tilde{U}_{a_{31}}(h)$$

converges as $h \to 0^+$ to $\int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}}$, and that this limit is an integer. In particular, we will have to justify that $U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}}$ is well defined. The proof will be in two steps. First, we will show that $(\tilde{U}_{a_{ij}}(h))_{h\to 0^+}$ defines a bounded sequence in some space of currents $\mathcal{D}'_{\Gamma_{ij}}(M)$ with prescribed wavefront sets. Then, we will apply theorems on the continuity of wedge products for currents with transverse wavefront sets.

7.1. Background on Fukaya's conjecture

Before proving Fukaya's conjecture on Witten Laplacians, we start with a brief overview of the context in which they appear. These problems are related to symplectic topology and Morse theory, and it goes without saying that the reader is strongly advised to consult the original papers of Fukaya for further details [30, 31, 32]. In symplectic topology, one would like to attach invariants to symplectic manifolds, in particular to Lagrangian submanifolds since they play a central role in symplectic geometry. Motivated by Arnold's conjectures on Lagrangian intersections, Floer constructed an infinite-dimensional generalization of Morse homology named Lagrangian Floer homology, which is the homology of some chain complex $(CF(L_0, L_1), \partial)$ associated to pairs of Lagrangians (L_0, L_1) and generated by the intersection points of L_0 and L_1 [1, Def. 1.4, Th. 1.5]. Then, for several Lagrangians satisfying precise geometric assumptions, it is possible to define some product operations on the corresponding Floer complexes [1, Sect. 2], and the collection of all these operations and the relations among them form a so called A_∞ *structure* first described by Fukaya. The important result is that the A_∞ structure, up to some natural equivalence relation, does not depend on the various choices that were made to define it, in the same way as the Hodge–de Rham cohomology theory of a compact Riemannian manifold does not depend on the choice of metric.

Let us briefly motivate the notion of A_∞ structure by discussing a simple example. On a given smooth compact manifold M, consider the de Rham complex $(\Omega^\bullet(M), d)$ with the corresponding de Rham cohomology $H^\bullet(M) = \text{Ker}(d)/\text{Ran}(d)$. From classical results of differential topology, if N is another smooth manifold diffeomorphic to M, then we have a quasi-isomorphism between $(\Omega^\bullet(M), d)$ and $(\Omega^\bullet(N), d)$, which implies that the corresponding cohomologies are isomorphic, $H^\bullet(M) \simeq H^\bullet(N)$. This means that the space of cocycles is an invariant of our space. However, there are manifolds which have the same cohomology groups, hence the same homology groups by Poincaré duality, and which are not homeomorphic, hence (co)homology is not enough to specify the topology of a given manifold. One direction to get more invariants would be to give some information on relations among (co)cycles. Recall that $(\Omega^\bullet(M), d, \wedge)$ is a differential graded algebra where the algebra structure comes from the wedge product $\wedge$, and the fact that $\wedge$ satisfies the Leibniz rule with respect to the differential d readily implies that $\wedge : \Omega^\bullet(M) \times \Omega^\bullet(M) \to \Omega^\bullet(M)$ induces on cohomology a bilinear map $\mathfrak{m}_2 : H^\bullet(M) \times H^\bullet(M) \to H^\bullet(M)$ called the cup-product. By Poincaré duality, this operation on cohomology geometrically encodes intersection-theoretic information among cycles and gives more information than the usual (co)homology groups. Algebras of A_∞ type are a far reaching generalization of differential graded algebras where the wedge product is replaced by a sequence of k-multilinear products for all $k \geq 2$ with relations among them generalizing the Leibniz rule [58].

In perfect analogy with symplectic topology, Fukaya introduced A_∞ structures in Morse theory [30, Chapter 1]. In that case, the role of Lagrangian pairs (L_0, L_1) is played by a pair of smooth functions (f_0, f_1) such that $f_0 - f_1$ is Morse. Note that it is not a priori possible to endow the Morse complex with the wedge product $\wedge$ of currents since currents carried by the same unstable manifold cannot be intersected because of the lack of transversality. The idea is to perturb the Morse functions to create transversality. Thus, we should deal with several pairs of smooth functions. In that context, Fukaya formulated conjectures [32, Sect. 4.2] related to the A_∞ structure associated with the Witten Laplacian. He predicted that the WKB states of Helffer and Sjöstrand should verify more general asymptotic formulas than the tunneling formulas associated with the action of the twisted coboundary operator $d_{f,h}$ [32, Conj. 4.1 and 4.2]. Indeed, after twisting the de Rham coboundary operator d and getting tunneling formulas for $d_{f,h}$, the next natural idea is to find some twisted version of Cartan's exterior product $\wedge$ and see if one can find some analogue of the tunneling formulas for twisted products. At the semiclassical limit $h \to 0^+$, Fukaya conjectured that this twisted product should converge to the Morse-theoretical analogue of the wedge product modulo some exponential corrections related to *disk instantons* [33, 34]. Hence, as for the coboundary operator, the cup-product in Morse cohomology would appear in the asymptotics of the Helffer–Sjöstrand WKB states. The purpose of the next subsections is to show that our quasimodes also satisfy the asymptotic formula conjectured by Fukaya for the wedge product.

7.2. *Wavefront set of eigencurrents*

In this subsection, we fix a smooth Morse–Smale gradient vector field V_f which is $\mathcal{C}^1$-linearizable. Fix $0 \leq k \leq n$ and $\Lambda > 0$. Then, following Section 3, choose some large enough integers N_0, N_1 to ensure that for every $0 \leq h < h_0$, the operator

$$-\mathcal{L}_{V_f} - h\Delta_g/2 : \mathcal{H}_k^{m_{N_0,N_1}}(M) \supset \Omega^k(M) \to \mathcal{H}_k^{m_{N_0,N_1}}(M)$$

has discrete spectrum with finite multiplicity on the domain $\text{Re}(z) > -\Lambda$. Recall from [28, Th. 1.5] that the eigenmodes are intrinsic and that they do not depend on the choice of the order function. Recall also from Section 3 that, up to some uniform constants, the parameter Λ has to be smaller than $c_0 \min\{N_0, N_1\}$, which is the quantity appearing in Lemma 3.3. Hence, if we choose $N_1' \geq N_1$, we do not change the spectrum on $\mathrm{Re}(z) > -\Lambda$. In particular, any generalized eigenmode $U \in \mathcal{H}_k^{m_{N_0,N_1}}(M)$ associated with an eigenvalue z_0 belongs to any anisotropic Sobolev space $\mathcal{H}_k^{m_{N_0,N_1'}}(M)$ with $N_1' \geq N_1$. We also note from the proof of Theorem 4.5 that, for every $N_1' \geq N_1$,

$$\|\tilde{U}_a(h) - U_a\|_{\mathcal{H}_k^{m_{N_0,N_1'}}(M)} \to 0 \quad \text{as } h \to 0. \tag{28}$$

Remark 7.1. Note that the proof in Section 4 shows that the convergence rate is of order $\mathcal{O}(h)$ but we omit this information for simplicity of exposition.

We now have to recall a few facts on the topology of the space $\mathcal{D}'^{,k}_{\Gamma_-(V_f)}(M)$ of currents whose wavefront set is contained in the closed conic set $\Gamma_-(V_f) = \bigcup_{a\in\mathrm{Crit}(f)} N^*(W^u(a)) \subset T^*M \setminus \underline{0}$ which is defined in §3.1.1. Note that we temporarily omit the dependence on V_f as we only deal with one Morse function for the moment. Recall that on some vector space E, given some family of seminorms P, we can define a topology on E which makes it a locally convex topological vector space. A neighborhood basis of the origin is defined by the subsets $\{x \in E : P(x) < A\}$ with $A > 0$ and with P a seminorm. In the particular case of currents, we will use the strong topology:

Definition 7.2 (Strong topology and bounded subsets). The strong topology of $\mathcal{D}'^{,k}(M)$ for M compact is defined by the following seminorms. Choose some bounded set B in $\Omega^{n-k}(M)$. Then we define a seminorm P_B as $P_B(u) = \sup_{\varphi\in B} |\langle u, \varphi\rangle|$. A subset B of currents is bounded iff it is weakly bounded, which means that for every test form $\varphi \in \Omega^{n-k}(M)$, $\sup_{t\in B} |\langle t, \varphi\rangle| < +\infty$ [54, Ch. 3, p. 72]. This is equivalent to B being bounded in some Sobolev space $H^s(M, \Lambda^k(T^*M))$ of currents by an application of the uniform boundedness principle [14, Sect. 5, Lemma 23].

We can now define the normal topology in the space of currents essentially following [11, Sect. 3]:

Definition 7.3 (Normal topology on the space of currents). For every closed conic subset $\Gamma \subset T^*M \setminus \underline{0}$, the topology of $\mathcal{D}'^{,k}_{\Gamma}(M)$ is defined as the weakest topology which makes continuous the seminorms of the strong topology of $\mathcal{D}'^{,k}(M)$ and the seminorms

$$\|u\|_{N,C,\chi,\alpha,U} = \|(1 + \|\xi\|)^N \mathcal{F}(u_\alpha\chi)(\xi)\|_{L^\infty(C)}, \tag{29}$$

where χ is supported on some chart U where $u = \sum_{|\alpha|=k} u_\alpha dx^\alpha$; α is a multi-index; $\mathcal{F}$ is the Fourier transform calculated in the local chart; and C is a closed cone such that $(\mathrm{supp}\, \chi \times C) \cap \Gamma = \emptyset$. A subset $B \subset \mathcal{D}'^{,k}_{\Gamma}$ is called bounded in $\mathcal{D}'^{k}_{\Gamma}$ if it is bounded in $\mathcal{D}'^{k}$ and if all seminorms $\|\cdot\|_{N,C,\chi,\alpha,U}$ are bounded on B.

We emphasize that this definition is given purely in terms of local charts without loss of generality. The above topology is in fact *intrinsic as a consequence of the continuity of the pullback* [11, Prop. 5.1, p. 211] as emphasized by Hörmander [44, p. 265]. Note that it is the same to consider currents or distributions when we define the relevant topologies since currents are just elements of the form $\sum u_{i_1,\dots,i_k} dx^{i_1}\wedge\dots\wedge dx^{i_k}$ in local coordinates $(x^1,\dots,x^n)$ where the coefficients $u_{i_1,\dots,i_k}$ are distributions.

Note from (28) that $(\tilde{U}_a(h))_{0\leq h<1}$ is a bounded family in the anisotropic Sobolev space $\mathcal{H}_k^{m_{N_0,N_1}}(M)$ and is thus bounded in $H^{-s}(M,\Lambda^k(T^*M))$ for s large enough. In particular, from Definition 7.2, it is a bounded family in $\mathcal{D}^{\prime,k}(M)$. We would now like to verify that it is a bounded family in $\mathcal{D}^{\prime,k}_{\Gamma_-}(M)$ which converges in the normal topology as h goes to 0 in order to apply the results from [11]. For that purpose, we can already observe that, for some s large enough, $\|\tilde{U}_a(h)-U_a\|_{H^{-s}(M,\Lambda^k(T^*M))}\to 0$ as $h\to 0$. In particular, it converges for the strong topology in $\mathcal{D}^{\prime,k}(M)$. Hence, it remains to discuss the boundedness and the convergence with respect to the seminorms $\|\cdot\|_{N,C,\chi,\alpha,U}$. We note that these seminorms involve the L^∞ norm while the anisotropic spaces we deal with so far are built from L^2 norms. This problem is handled by the following lemma

Lemma 7.4 (L^2 vs L^∞). *Let N, $\tilde{N}$ be some positive integers and let W_0 be a closed cone in $\mathbb{R}^{n*}$. Then, for every closed conic neighborhood W of W_0, one can find a constant $C=C(N,\tilde{N},W)>0$ such that, for every u in $\mathcal{C}^\infty_c(B_{\mathbb{R}^n}(0,1))$,*

$$\sup_{\xi\in W_0}(1+|\xi|)^N|\widehat{u}(\xi)|\leq C\big(\|(1+|\xi|)^N\widehat{u}(\xi)\|_{L^2(W)}+\|u\|_{H^{-\tilde{N}}}\big).$$

We postpone the proof of this lemma to Appendix B and we show first how to use it in our context. We consider the family $(\tilde{U}_a(h))_{0\leq h<h_0}$ of currents in $\mathcal{D}^{\prime,k}(M)$ and we would like to show that it is a bounded family in $\mathcal{D}^{\prime,k}_{\Gamma_-}(M)$ and that $\tilde{U}_a(h)$ converges to U_a in the normal topology we have just defined. Recall that this family is bounded and that we have convergence in every anisotropic Sobolev space $\mathcal{H}_k^{m_{N_0,N_1'}}(M)$ with N_1' large enough. Fix $(x_0,\xi_0)\notin\Gamma_-$. Fix some $N>0$. Note that, up to shrinking the neighborhood used to define the order function in §3.1.2 and up to increasing N_1, we can suppose that $m_{N_0,N_1'}(x;\xi)>N/2$ for any N_1' and for every (x,ξ) in a small conical neighborhood W of (x_0,ξ_0).

Fix now a smooth test function χ supported near x_0 and a closed cone W_0 which is strictly contained in the conical neighborhood W we have just defined. Thanks to Lemma 7.4 and to the Plancherel equality, the norm we have to estimate is

$$\begin{aligned}\|(1+\|\xi\|)^N\mathcal{F}(\chi\tilde{U}_a(h))\|_{L^\infty(W_0)}&\\ &\leq C\big(\|\chi_1(\xi)(1+\|\xi\|)^N\mathcal{F}(\chi\tilde{U}_a(h))\|_{L^2}+\|\tilde{U}_a(h)\|_{H^{-\tilde{N}}}\big)\\ &\leq C\big(\|\mathrm{Op}(\chi_1(\xi)(1+\|\xi\|)^N\chi)\tilde{U}_a(h)\|_{L^2}+\|\tilde{U}_a(h)\|_{H^{-\tilde{N}}}\big),\end{aligned}$$

where $\chi_1\in\mathcal{C}^\infty$ is identically 1 on the conical neighborhood W and equal to 0 outside a slightly bigger neighborhood. For $\tilde{N}$ large enough, we can already observe that the second term $\|\tilde{U}_a(h)\|_{H^{-\tilde{N}}}$ in the upper bound is uniformly bounded as $\tilde{U}_a(h)$ is uniformly bounded in some fixed anisotropic Sobolev space. Hence, it remains to estimate

$$\|\mathrm{Op}(\chi_1(\xi)(1+\|\xi\|)^N\chi)\,\mathrm{Op}(\mathbf{A}^{(k)}_{N_0,N_1'})^{-1}\,\mathrm{Op}(\mathbf{A}^{(k)}_{N_0,N_1'})\tilde{U}_a(h))\|_{L^2(M)}.$$

By composition of pseudodifferential operators and as we chose N_1' large enough to ensure that the order function m_{N_0,N_1} is larger than $N/2$ on $\operatorname{supp}(\chi_1)$, we can deduce that this quantity is bounded (up to some constant) by $\|\operatorname{Op}(\mathbf{A}^{(k)}_{N_0,N_1'})\tilde{U}_a(h)\|_{L^2}$, which is exactly the norm on the anisotropic Sobolev space. To summarize, this argument shows

Proposition 7.5. *Let V_f be a Morse–Smale gradient flow which is $\mathcal{C}^1$-linearizable. Then there exists $h_0 > 0$ such that, for every $0 \leq k \leq n$ and every $a \in \operatorname{Crit}(f)$ of index k, the family $(\tilde{U}_a(h))_{0\leq h<h_0}$ is bounded in $\mathcal{D}'^{,k}_{\Gamma_-(V_f)}(M)$. Moreover, $\tilde{U}_a(h)$ converges to U_a for the normal topology in $\mathcal{D}'^{,k}_{\Gamma_-(V_f)}(M)$ as $h \to 0^+$.*

This proposition is the key ingredient we need in order to apply the theoretical results from [11]. Before doing that, we can already observe that if we come back to the framework of Theorem 2.7, then the generalized Morse–Smale assumptions ensure that the wavefront sets of the three families of currents are transverse. In particular, we can define the wedge product even for[16] $h = 0$, i.e. $U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}}$ defines an element in $\mathcal{D}'^{,n}(M)$—see below.

7.3. *Convergence of products*

Given two closed conic sets (Γ_1, Γ_2) which have empty intersection, the usual wedge product of smooth forms

$$\wedge : \Omega^k(M) \times \Omega^l(M) \ni (\varphi_1, \varphi_2) \mapsto \varphi_1 \wedge \varphi_2 \in \Omega^{k+l}(M)$$

extends uniquely as a hypocontinuous map for the normal topology [11, Th. 6.1],

$$\wedge : \mathcal{D}'^{k}_{\Gamma_1}(M) \times \mathcal{D}'^{l}_{\Gamma_2}(M) \ni (\varphi_1, \varphi_2) \mapsto \varphi_1 \wedge \varphi_2 \in \mathcal{D}'^{k+l}_{s(\Gamma_1,\Gamma_2)}(M),$$

with $s(\Gamma_1, \Gamma_2) = \Gamma_1 \cup \Gamma_2 \cup (\Gamma_1 + \Gamma_2)$. The notion of hypocontinuity is a strong notion of continuity adapted to bilinear maps $E \times F \to G$ where E, F, G are locally convex spaces [11, pp. 204–205]. It is weaker than joint continuity but implies that the bilinear map is separately continuous in each factor uniformly in the other factor in a bounded subset, which is enough for our purposes.[17]

Remark 7.6. The proof in [11] was given for product of distributions and it extends to currents as $\mathcal{D}'^{,k}_{\Gamma}(M) = \mathcal{D}'_{\Gamma}(M) \otimes_{\mathcal{C}^\infty(M)} \Omega^k(M)$. Recall that the fact that $\wedge$ is hypocontinuous means that, for every neighborhood $W \subset \mathcal{D}'^{,k+l}_{s(\Gamma_1,\Gamma_2)}(M)$ of zero and for every bounded set $B_2 \subset \mathcal{D}'^{,l}_{\Gamma_2(M)}$, there is some open neighborhood $U_1 \subset \mathcal{D}'^{,k}_{\Gamma_1}$ of zero such that $\wedge(U_1 \times B_2) \subset W$. The same holds true if we invert the roles of 1 and 2. We note that hypocontinuity implies boundedness, in the sense that any bounded subset of $\mathcal{D}'^{k}_{\Gamma_1}(M) \times \mathcal{D}'^{l}_{\Gamma_2}(M)$ is sent to a bounded subset of $\mathcal{D}'^{k+l}_{s(\Gamma_1,\Gamma_2)}(M)$. This follows from the observation that a set B is bounded iff for every open neighborhood U of 0, B can be rescaled by multiplication by $\lambda > 0$ so that $\lambda B \subset U$.

16 For $h \neq 0$, there is no problem as the eigenmodes are smooth by elliptic regularity.

17 The tensor product of distributions for the strong topology is hypocontinuous but not continuous [11, p. 205].

Let us now come back to the proof of Theorem 2.7. This is where we will crucially use the generalized transversality assumptions (5) introduced before Theorem 2.7. We start by considering two points a_{12} and a_{23}. In order to make the wedge product of $U_{a_{12}}$ and $U_{a_{23}}$, one needs to verify that $\Gamma_-(V_{f_{12}}) \cap \Gamma_-(V_{f_{23}}) = \emptyset$. To see this, recall that $\Gamma_-(V_{f_{12}}) \cap \Gamma_-(V_{f_{23}})$ is equal to

$$\bigcup_{(a,b)\in \mathrm{Crit}(f_{12})\times \mathrm{Crit}(f_{23})} N^*W^u(a) \cap N^*W^u(b),$$

which is a subset of

$$\bigcup_{(a,b,c)\in \mathrm{Crit}(f_{12})\times \mathrm{Crit}(f_{23})\times \mathrm{Crit}(f_{31})} (TW^u(a) \cap TW^u(c) + TW^u(b))^\perp = \emptyset,$$

where the last equality is the content of our generalized Morse–Smale transversality assumption. Combining Proposition 7.5 with the hypocontinuity of the wedge product, we find that $(\tilde{U}_{a_{12}}(h) \wedge \tilde{U}_{a_{23}}(h))_{0\leq h<h_0}$ is a bounded family in $\mathcal{D}'^{,k+l}_{s(\Gamma_-(V_{f_{12}}),\Gamma_-(V_{f_{23}}))}(M)$, where k is the index of a_{12} and l is that of a_{23}. Moreover, as $h \to 0$,

$$\tilde{U}_{a_{12}}(h) \wedge \tilde{U}_{a_{23}}(h) \to \tilde{U}_{a_{12}}(0) \wedge \tilde{U}_{a_{23}}(0) = U_{a_{12}} \wedge U_{a_{23}}$$

for the normal topology of $\mathcal{D}'^{,k+l}_{s(\Gamma_-(V_{f_{12}}),\Gamma_-(V_{f_{23}}))}(M)$. Recall that $s(\Gamma_-(V_{f_{12}}), \Gamma_-(V_{f_{23}}))$ is equal to $\Gamma_-(V_{f_{12}}) \cup \Gamma_-(V_{f_{23}}) \cup \Gamma_-(V_{f_{12}}) + \Gamma_-(V_{f_{23}})$, which is equal to

$$\bigcup_{(a,b)\in \mathrm{Crit}(f_{12})\times \mathrm{Crit}(f_{23})} (TW^u(a) \cap TW^u(b))^\perp \setminus \underline{0}.$$

Then, as our three vector fields satisfy the generalized Morse–Smale assumptions (5), we can repeat this argument with the spaces $\mathcal{D}'^{,n-(k+l)}_{\Gamma_-(V_{f_{31}})}(M)$ and $\mathcal{D}'^{,k+l}_{s(\Gamma_-(V_{f_{12}}),\Gamma_-(V_{f_{23}}))}(M)$. Hence we get, as $h \to 0$,

$$\tilde{U}_{a_{12}}(h) \wedge \tilde{U}_{a_{23}}(h) \wedge \tilde{U}_{a_{31}}(h) \to U_{a_{12}} \wedge U_{a_{23}} \wedge U_{31}$$

in $\mathcal{D}'^{,s}_{s(s(\Gamma_-(V_{f_{12}}),\Gamma_-(V_{f_{23}})),\Gamma_-(V_{f_{31}}))}(M)$. Finally, testing against the smooth form 1 in $\Omega^0(M)$, we find that, as $h \to 0$,

$$\int_M \tilde{U}_{a_{12}}(h) \wedge \tilde{U}_{a_{23}}(h) \wedge \tilde{U}_{a_{31}}(h) \to \int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{31},$$

which concludes the proof of Theorem 2.7 up to the fact that we need to verify that $\int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{31}$ is an integer.

7.4. End of the proof

In order to conclude the proof of Theorem 2.7, we will show that $\overline{W^u(a_{12})}$, $\overline{W^u(a_{23})}$ and $\overline{W^u(a_{31})}$ intersect transversally at finitely many points belonging to $W^u(a_{12}) \cap W^u(a_{23}) \cap W^u(a_{31})$. Then $\int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{31}$ is an integer in view of Lemma 6.2.

Let us start by showing that any point in $\overline{W^u(a_{12})} \cap \overline{W^u(a_{23})} \cap \overline{W^u(a_{31})}$ must belong to $W^u(a_{12}) \cap W^u(a_{23}) \cap W^u(a_{31})$. For contradiction, suppose that x belongs to $\overline{W^u(a_{12})} \cap \overline{W^u(a_{23})} \cap \overline{W^u(a_{31})}$ but not to $W^u(a_{12}) \cap W^u(a_{23}) \cap W^u(a_{31})$. From Smale's Theorem 5.2, there exist critical points b_1, b_2 and b_3 such that $x \in W^u(b_1) \cap W^u(b_2) \cap W^u(b_3)$ with $W^u(b_i) \subset \overline{W^u(a_{ij})}$ and *at least one* i satisfies $\dim(W^u(b_i)) < \dim(W^u(a_{ij}))$. This implies that

$$\dim(W^u(b_1)) + \dim(W^u(b_2)) + \dim(W^u(b_3)) < 2n.$$

Then, on account of our transversality assumption, we have

$$\dim(W^u(b_1) \cap W^u(b_2) \cap W^u(b_3)) = \dim(W^u(b_1) \cap W^u(b_2)) + \dim(W^u(b_3)) - n.$$

Using transversality one more time, we get

$$\begin{aligned}\dim(W^u(b_1) \cap W^u(b_2) \cap W^u(b_3)) &= \dim(W^u(b_1)) + \dim(W^u(b_2)) + \dim(W^u(b_3)) - 2n \\ &< 0,\end{aligned}$$

which contradicts $W^u(b_1) \cap W^u(b_2) \cap W^u(b_3) \neq \emptyset$. Hence, we have already shown that

$$\overline{W^u(a_{12})} \cap \overline{W^u(a_{23})} \cap \overline{W^u(a_{31})} = W^u(a_{12}) \cap W^u(a_{23}) \cap W^u(a_{31}),$$

and it remains to show that this intersection consists of finitely many points. For that purpose, observe that as the intersection $W^u(a_{12}) \cap W^u(a_{23}) \cap W^u(a_{31})$ is transverse, it defines a 0-dimensional submanifold of M. Thus, x belonging to $W^u(a_{12}) \cap W^u(a_{23}) \cap W^u(a_{31})$ is an isolated point inside $W^u(a_{ij})$ for the induced topology by the embedding of $W^u(a_{ij})$ in M, for every ij in $\{12, 23, 31\}$. Moreover, as this intersection coincides with its closure, we can deduce there can only be finitely many points in it, and this concludes the proof of Theorem 2.7.

7.5. Morse gradient trees

Now that we have shown that the limit in Theorem 2.7 is an integer, let us give its geometric interpretation in terms of counting gradient flow trees. From the above proof, we count with orientation the number of points in $W^u(a_{12}) \cap W^u(a_{23}) \cap W^u(a_{31})$. In dynamical terms, such a point x_0 corresponds to the intersection of three flow lines starting from a_{12}, a_{23} and a_{31} and passing through x_0. This represents a one-dimensional submanifold having the form of a Y-shaped tree whose edges are gradient lines. Hence, $\int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}}$ counts the number of such Y-shaped gradient trees given by a triple of Morse–Smale gradient flows.

7.6. *Cup-products*

These triple products can be interpreted in terms of the cup-products appearing in Morse theory [30, 31]. Indeed, we can naturally define a bilinear map

$$\mathfrak{m}_2^{(k,l)} : \mathrm{Ker}(-\mathcal{L}_{V_{f_{12}}}^{(k)}) \times \mathrm{Ker}(-\mathcal{L}_{V_{f_{23}}}^{(l)}) \to \mathrm{Ker}(-\mathcal{L}_{V_{f_{13}}}^{(k+l)}),$$

where $f_{13} = -f_{31}$. This can be done as follows. The coefficients of $\mathfrak{m}_2^{(k,l)}$ in the basis $(U_{a_{12}}, U_{a_{23}}, U_{a_{13}})$ are given by $\int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}} \in \mathbb{Z}$. Note that, as $f_{13} = -f_{31}$, one has[18] $U_{a_{31}} = S_{a_{13}}$. Hence, these coefficients can be written in a more standard way as

$$\int_M U_{a_{12}} \wedge U_{a_{23}} \wedge S_{a_{13}} \in \mathbb{Z}.$$

As we defined a map on all the generators of the Morse complex, this endows the whole Morse complex with a product which is defined, for every $(U_1, U_2) \in \mathrm{Ker}(-\mathcal{L}_{V_{f_{12}}}^{(k)}) \times \mathrm{Ker}(-\mathcal{L}_{V_{f_{23}}}^{(l)})$, as

$$\mathfrak{m}_2^{(k,l)}(U_1, U_2) = \sum_{a_{13} \in \mathrm{Crit}(f_{13})} \left(\int_M U_1 \wedge U_2 \wedge S_{a_{13}} \right) U_{a_{13}}.$$

Note that, compared with the classical theory where these maps are defined in an algebraic manner [31], our formulation is purely analytical. Thanks to the remark in §7.5, one can verify that these algebraic and analytical maps are exactly the same. It is already known that the map $\mathfrak{m}_2$ induces a cup-product on the cohomology. Let us re-prove this fact using our analytic approach. Recall from [17, 20] that the Morse complex is quasi-isomorphic to the de Rham complex $(\Omega(M), d)$ via the spectral projector associated with the eigenvalue 0. Hence, it is sufficient to show that $\mathfrak{m}_2$ induces a well defined map on the cohomology of the Morse complex. To see this, we fix (U_1, U_2) in $\mathrm{Ker}(-\mathcal{L}_{V_{f_{12}}}^{(k)}) \times \mathrm{Ker}(-\mathcal{L}_{V_{f_{23}}}^{(l)})$, and we write, using the Stokes formula,

$$\begin{aligned} A &:= \mathfrak{m}_2^{(k+1,l)}(dU_1, U_2) + (-1)^k \mathfrak{m}_2^{(k,l+1)}(U_1, dU_2) \\ &= (-1)^{k+l+1} \sum_{a_{13} \in \mathrm{Crit}(f_{13})} \left(\int_M U_1 \wedge U_2 \wedge dU_{a_{31}} \right) U_{a_{13}}. \end{aligned}$$

Then, we recall that $dU_{a_{31}}$ is in $\mathrm{Ker}(-\mathcal{L}_{V_{f_{31}}})$. Thus, we can decompose it in the basis $(U_{b_{31}})_{b_{31}:\,\mathrm{ind}(b_{31}) = \mathrm{ind}(a_{31})+1}$:

$$\begin{aligned} dU_{a_{31}} &= \sum_{b_{31}:\,\mathrm{ind}(b_{31}) = \mathrm{ind}(a_{31})+1} \left(\int_M S_{b_{31}} \wedge dU_{a_{31}} \right) U_{b_{31}} \\ &= (-1)^{k+l+1} \sum_{b_{13}:\,\mathrm{ind}(b_{13})+1 = \mathrm{ind}(a_{13})} \left(\int_M dU_{b_{13}} \wedge S_{a_{13}} \right) U_{b_{31}}, \end{aligned}$$

[18] Stable manifolds of f are unstable manifolds of $-f$.

where we use the Stokes formula one more time to write the second equality. Intertwining the sums over a_{13} and b_{13} in the expression of A yields

$$\begin{aligned} A &= \sum_{b_{13}\in\mathrm{Crit}(f_{13})}\left(\int_M U_1\wedge U_2\wedge S_{b_{13}}\right)\sum_{a_{13}:\,\mathrm{ind}(b_{13})+1=\mathrm{ind}(a_{13})}\left(\int_M S_{a_{13}}\wedge dU_{b_{13}}\right)U_{a_{13}} \\ &= \sum_{b_{13}\in\mathrm{Crit}(f_{13})}\left(\int_M U_1\wedge U_2\wedge S_{b_{13}}\right)dU_{b_{13}}, \end{aligned}$$

where we use as above the fact that $dU_{b_{13}}\in\mathrm{Ker}(-\mathcal{L}^{(k+l+1)}_{V_{f_{13}}})$ to write the second equality. This implies

$$\mathfrak{m}_2^{(k+1,l)}(dU_1,U_2)+(-1)^k\mathfrak{m}_2^{(k,l+1)}(U_1,dU_2)=d(\mathfrak{m}_2^{(k,l)}(U_1,U_2)).$$

This relation shows that $\mathfrak{m}_2$ is a cochain map for the Morse complexes $(\mathrm{Ker}(-\mathcal{L}_{V_{f_{ij}}}),d)$, hence induces a cup-product in Morse cohomology. In other terms, the map $\mathfrak{m}_2$ is a (spectral) realization in terms of currents of the algebraic cup-product coming from Morse theory [31].

Fukaya's conjecture states that, up to some exponential factors involving the Liouville period over certain triangles defined by Lagrangian submanifolds, this algebraic cup-product can be recovered by computing triple products of Witten quasimodes [32, Conj. 4.1]. To summarize this section, by giving this analytical interpretation of the Morse cup-product, we have been able to obtain Fukaya's instanton formula by considering the limit $h\to 0^+$ in appropriate Sobolev spaces where both $-\mathcal{L}_{V_f}$ and $W_{f,h}$ have nice spectral properties.

Remark 7.7. Note that $\mathfrak{m}_1=d$ and $\mathfrak{m}_2=\wedge$ are the first two operations of the Morse A_∞-category discovered by Fukaya [30, 31]. Our analysis shows that these algebraic maps can be interpreted in terms of analysis as Witten deformations of the coboundary operator and exterior products. An A_∞-category is in fact endowed with graded maps $(\mathfrak{m}_k)_{k\geq 1}$ of algebraic nature, and it is natural to think that all these algebraic maps can also be given analytic interpretations by considering appropriate Witten deformations which is the content of Fukaya's general conjectures [32, Conj. 4.2]. However, this is at the expense of a more subtle combinatorial work and we shall discuss this issue elsewhere.

Remark 7.8. Note that the approach we have developed here would also give the following formulation of the Witten–Helffer–Sjöstrand tunneling formulas. For (a,b) in $\mathrm{Crit}(f)$ such that $\mathrm{ind}(b)=\mathrm{ind}(a)+1$, write

$$\begin{aligned} \int_M d_{f,h}(U_a(h))\wedge S_b(h) &= e^{\frac{f(b)-f(a)}{h}}\int_M d(\tilde{U}_a(h))\wedge\tilde{S}_b(h) \\ &= \sum_{b':\,\mathrm{ind}(b')=\mathrm{ind}(a)+1} n_{ab'}e^{\frac{f(b)-f(a)}{h}}\int_M \tilde{U}_{b'}(h)\wedge S_b(h) \\ &= \sum_{b':\,\mathrm{ind}(b')=\mathrm{ind}(a)+1} n_{ab'}e^{\frac{f(b)-f(a)}{h}}\delta_{bb'}(1+o(1)) \\ &= n_{ab}e^{\frac{f(b)-f(a)}{h}}(1+o(1)). \end{aligned}$$

Under this form, the formulation of the instanton formula for products of order 1 is closer to [43, Eq. (3.27)]—see Section 8 below for a discussion on the difference between the normalization factors. Note that going through our proof would yield a remainder of order $\mathcal{O}(h)$.

8. Comparison with the Helffer–Sjöstrand quasimodes

In [43, Eq. (1.37)], Helffer and Sjöstrand also constructed a natural basis for the bottom of the spectrum of the Witten Laplacian. For the sake of completeness,[19] we will compare our family of quasimodes with theirs and show that they are equal at leading order. In order to apply the results of [43], we remark that the dynamical assumptions (H1) and (H2) from that reference are automatically satisfied as soon as the gradient flow satisfies the Smale transversality assumption. For (H1), this follows from Smale's Theorem 5.2, while (H2) was for instance proved in [59, Prop. 3.6].

We denote the Helffer–Sjöstrand's quasimodes by $(U_a^{\mathrm{HS}}(h))_{a\in\mathrm{Crit}(f)}$. By construction, they belong to the same eigenspaces as our quasimodes $(U_a(h))_{a\in\mathrm{Crit}(f)}$. Fix a critical point a of index k. These quasimodes do not form an orthonormal family. Yet, if $V^{(k)}(h)$ is the matrix whose coefficients are given by $\langle U_b^{\mathrm{HS}}(h), U_{b'}^{\mathrm{HS}}(h)\rangle_{L^2}$, then one knows from [43, Eq. (1.43)] that

$$V^{(k)}(h) = \mathrm{Id} + \mathcal{O}_{\Omega^k(M)}(e^{-C_0/h})$$

for some positive constant $C_0 > 0$ depending only on (f, g). Hence, if we transform this family into an orthonormal family $(\tilde{U}_{b'}^{\mathrm{HS}}(h))_{b':\,\mathrm{ind}(b')=k}$, then we get

$$\tilde{U}_{b'}^{\mathrm{HS}}(h) = \sum_{b:\,\mathrm{ind}(b)=k} (\delta_{bb'} + \mathcal{O}_{\Omega^k(M)}(e^{-C_0/h}))U_b^{\mathrm{HS}}(h).$$

In particular, the spectral projector can be written as

$$\mathbf{1}_{[0,\epsilon]}(W_{f,h}^{(k)})(x, y, dx, dy) = \sum_{b'\in\mathrm{Crit}(f):\,\mathrm{ind}(b')=k} \tilde{U}_{b'}^{\mathrm{HS}}(h)(x, dx)\tilde{U}_{b'}^{\mathrm{HS}}(h)(y, dy).$$

Hence, from the definition of our WKB state $U_a(h)$,

$$U_a(h) = \sum_{b'\in\mathrm{Crit}(f):\,\mathrm{ind}(b')=k} \int_M U_a \wedge \star_k(e^{-\frac{f-f(a)}{h}}\tilde{U}_{b'}^{\mathrm{HS}}(h))\tilde{U}_{b'}^{\mathrm{HS}}(h),$$

which can be expanded as follows:

$$U_a(h) = \sum_{b'\in\mathrm{Crit}(f):\,\mathrm{ind}(b')=k}\ \sum_{b:\,\mathrm{ind}(b)=k} \left(\int_M U_a \wedge \star_k(\delta_{bb'} + \mathcal{O}_{\Omega^k(M)}(e^{-C_0/h}))(e^{-\frac{f-f(a)}{h}}U_b^{\mathrm{HS}}(h))\right) \times \tilde{U}_{b'}^{\mathrm{HS}}(h).$$

19 We note that, except for this section, our results are self-contained and they do not rely on [43].

Everything now boils down to the calculation of

$$\alpha_{abb'}(h) = \int_M U_a \wedge \star_k(\delta_{bb'} + \mathcal{O}_{\Omega^k(M)}(e^{-C_0/h}))(e^{-\frac{f-f(a)}{h}} U_b^{\text{HS}}(h)).$$

More precisely, if we are able to prove that

$$\alpha_{abb'}(h) = \delta_{ab}\delta_{bb'}\alpha_a(h)(1+\mathcal{O}(h)) + \mathcal{O}(e^{-C_0/h}) \tag{30}$$

for a certain $\alpha_a(h) \neq 0$ depending polynomially on h (which has to be determined), then, after gathering all the equalities, we will find that

$$U_a(h) = \alpha_a(h)(1+\mathcal{O}(h))U_a^{\text{HS}}(h) + \sum_{b\neq a\in\text{Crit}(f):\,\text{ind}(b)=\text{ind}(a)} \mathcal{O}(e^{-C_0/h})U_b^{\text{HS}}(h), \tag{31}$$

showing that our quasimodes are at leading order equal to the ones of Helffer and Sjöstrand (up to some normalization factor). Let us now prove (30) by making use of the results from [43]. First of all, we write

$$\alpha_{abb'}(h) = \int_M U_a \wedge \star_k(\delta_{bb'} + \mathcal{O}_{\Omega^k(M)}(e^{-C_0/h}))(e^{-\frac{f-f(a)}{h}} U_b^{\text{HS}}(h)).$$

According to [43, Eq. (1.38)], we know that

$$U_b^{\text{HS}}(h) = \Psi_b(h) + \mathcal{O}_{\Omega^k(M)}(e^{-C_0/h}),$$

where $\Psi_b(h)$ is a certain "Gaussian state" centered at b defined by [43, Eq. (1.35)] and C_0 is some positive constant. Thus, as $f(x) \geq f(a)$ on the support of U_a, we have

$$\alpha_{abb'}(h) = \delta_{bb'} \int_M U_a \wedge \star_k(e^{-\frac{f-f(a)}{h}} \Psi_b(h)) + \mathcal{O}(e^{-C_0/h})$$

with $C_0 > 0$ which is slightly smaller than before. We now introduce a smooth cutoff function χ_a which is equal to 1 in a neighborhood of a and we write

$$\begin{aligned}\alpha_{abb'}(h) = \delta_{bb'} &\int_M U_a \wedge \star_k(\chi_a e^{-\frac{f-f(a)}{h}} \Psi_b(h)) \\ &+ \delta_{bb'} \int_M U_a \wedge \star_k((1-\chi_a) e^{-\frac{f-f(a)}{h}} \Psi_b(h)) + \mathcal{O}(e^{-C_0/h}).\end{aligned}$$

Thanks to [43, Th. 1.4] and to the fact that the support of U_a is equal to $\overline{W^u(a)}$, we know that the second term, which corresponds to the points which are far from a, is also exponentially small. Hence

$$\alpha_{abb'}(h) = \delta_{bb'} \int_M U_a \wedge \star_k(e^{-\frac{f-f(a)}{h}} \chi_a \Psi_b(h)) + \mathcal{O}(e^{-C_0/h}).$$

Thanks to Lemma 5.10, this can be rewritten as

$$\alpha_{abb'}(h) = \delta_{bb'} \int_{W^u(a)} \star_k(e^{-\frac{f-f(a)}{h}} \chi_a \Psi_b(h)) + \mathcal{O}(e^{-C_0/h}).$$

Using [43, Th. 1.4], we find that, for $a \neq b$ or $a \neq b'$, one has

$$\alpha_{abb'}(h) = \mathcal{O}(e^{-C_0/h}).$$

It remains to treat the case $a = b = b'$. In that case, we can use [43, Ths. 1.4 and 2.5] to show

$$\alpha_{abb'}(h) = \alpha_a(\pi h)^{(n-2k)/4}(1 + \mathcal{O}(h))$$

for a certain positive constant $\alpha_a \neq 0$ which depends only on the Lyapunov exponents at the critical point a (and not on h). More precisely,

$$|\alpha_a| = \left(\frac{\prod_{j=1}^{k} |\chi_j(a)|}{\prod_{j=k+1}^{n} |\chi_j(a)|}\right)^{1/4}.$$

This shows that our eigenmodes are not a priori normalized in L^2. To fix this, we would need to set, for every critical point a of f,

$$\mathbf{U}_a(h) := \frac{1}{|\alpha_a|(\pi h)^{(n-2k)/4}} U_a(h).$$

With this renormalization, the tunneling formula of Theorem 2.6 can be rewritten as

$$h d_{f,h} \mathbf{U}_a(h) = \left(\frac{h}{\pi}\right)^{1/2} \sum_{b:\,\mathrm{ind}(b)=\mathrm{ind}(a)+1} n_{ab} \left(\frac{e^{f(a)/h}}{|\alpha_a|}\right)\left(\frac{e^{f(b)/h}}{|\alpha_b|}\right)^{-1} \mathbf{U}_b(h).$$

In this form, we now recognize exactly the tunneling formula of [43, Eq. (3.27)]—see also [6, §6] in the case of a self-indexing Morse function. Concerning Fukaya's instanton formula, we observe that it can be rewritten as

$$\lim_{h\to 0^+} \frac{|\alpha_{a_{12}}\alpha_{a_{23}}\alpha_{a_{31}}|(\pi h)^{n/4}}{e^{(f_{12}(a_{12})+f_{23}(a_{23})+f_{31}(a_{31}))/h}} \int_M \mathbf{U}_{a_{12}}(h) \wedge \mathbf{U}_{a_{23}}(h) \wedge \mathbf{U}_{a_{31}}(h) = \int_M U_{a_{12}} \wedge U_{a_{23}} \wedge U_{a_{31}}.$$

Remark 8.1. We proved that, for every a in $\mathrm{Crit}(f)$, the currents $\tilde{U}_a(h) := e^{\frac{f-f(a)}{h}} U_a(h)$ converge to U_a as $h \to 0^+$. We emphasize that the above argument does not allow one to conclude that $(e^{\frac{f-f(a)}{h}} U_a^{\mathrm{HS}}(h))_{h\to 0^+}$ also converges to U_a. This does not seem obvious and it would require going more precisely through the analysis performed in [43].

Appendix A. Holomorphic continuation of the Ruelle determinant

In this appendix, we consider a Morse–Smale gradient flow φ_f^t. We fix $0 \leq k \leq n$ and $a \in \mathrm{Crit}(f)$. We recall how to prove that the local Ruelle determinant

$$\zeta_{R,a}^{(k)}(z) := \exp\Big(-\sum_{l=1}^{+\infty} \frac{e^{-lz}}{l} \frac{\mathrm{Tr}(\Lambda^k(d\varphi_f^{-l}(a)))}{|\det(\mathrm{Id} - d\varphi_f^{-l}(a))|}\Big)$$

has a holomorphic extension to $\mathbb{C}$, and we compute explicitly its zeros in terms of the Lyapunov exponents $(\chi_j(a))_{1\leq j\leq n}$. Recall that the dynamical Ruelle determinant from the introduction is given by

$$\zeta_R^{(k)}(z) = \prod_{a\in\mathrm{Crit}(f)} \zeta_{R,a}^{(k)}(z).$$

By definition of the Lyapunov exponents, we also recall that $d\varphi_f^{-1}(a) = \exp(-L_f(a))$ where $L_f(a)$ is a symmetric matrix whose eigenvalues are given by the $(\chi_j(a))_{1\leq j\leq n}$. If a is of index r, we use the convention

$$\chi_1(a) \leq \cdots \leq \chi_r(a) < 0 < \chi_{r+1}(a) \leq \cdots \leq \chi_n(a).$$

In order to show this holomorphic continuation, we start by observing that, in terms of the Lyapunov exponents,

$$\begin{aligned}|\det(\mathrm{Id} - d\varphi_f^{-l}(a))|^{-1} &= \prod_{j=1}^{r}(e^{-l\chi_j(a)} - 1)^{-1} \prod_{j=r+1}^{n}(1 - e^{-l\chi_j(a)})^{-1} \\ &= e^{l\sum_{j=1}^r \chi_j(a)} \prod_{j=1}^{n}(1 - e^{-l|\chi_j(a)|})^{-1} = e^{l\sum_{j=1}^r \chi_j(a)} \sum_{\alpha\in\mathbb{N}^n} e^{-l\alpha.|\chi(a)|},\end{aligned}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $|\chi(a)| = (|\chi_j(a)|)_{1\leq j\leq n}$. We now compute the trace

$$\mathrm{Tr}(\Lambda^k(d\varphi_f^{-l}(a))) = \sum_{J\subset\{1,\ldots,n\}:\,|J|=k} \exp\Big(-l\sum_{j\in J}\chi_j(a)\Big),$$

which implies that

$$\frac{\mathrm{Tr}(\Lambda^k(d\varphi_f^{-l}(a)))}{|\det(\mathrm{Id} - d\varphi_f^{-l}(a))|}$$

is equal to

$$\sum_{J\subset\{1,\ldots,n\}:\,|J|=k}\ \sum_{\alpha\in\mathbb{N}^n} \exp\Big(-l\Big(\sum_{j\in J\cap\{r+1,\ldots,n\}}|\chi_j(a)| + \sum_{j\in J^c\cap\{1,\ldots,r\}}|\chi_j(a)| + \alpha.|\chi(a)|\Big)\Big).$$

Under this form, one can verify that $\zeta_{R,a}^{(k)}(z)$ has a holomorphic extension to $\mathbb{C}$ whose zeros are given (modulo $2i\pi\mathbb{Z}$) by the set

$$\mathcal{R}_k(a) := \Big\{-\sum_{j\in J\cap\{r+1,\ldots,n\}}|\chi_j(a)| - \sum_{j\in J^c\cap\{1,\ldots,r\}}|\chi_j(a)| - \alpha.|\chi(a)| : |J| = k \text{ and } \alpha\in\mathbb{N}^n\Big\}.$$

Moreover, the multiplicity of z_0 in $\mathcal{R}_k(a)$ is the number of couples (α, J) such that

$$\operatorname{Re}(z_0) = -\Big(\sum_{j\in J\cap\{r+1,\dots,n\}} |\chi_j(a)| + \sum_{j\in J^c\cap\{1,\dots,r\}} |\chi_j(a)| + \alpha.|\chi(a)|\Big).$$

Remark A.1. In particular, we note that $z_0 = 0$ is a zero of $\zeta_{R,a}^{(k)}(z)$ if and only if the index of a (meaning the dimension of $W^s(a)$) is equal to k. In that case, the zero is of multiplicity 1. This implies that the multiplicity of 0 as a zero of $\zeta_R^{(k)}(z)$ is equal to the number of critical points of index k.

Appendix B. Proof of Lemma 7.4

In this appendix, we give the proof of Lemma 7.4. Up to minor modifications due to the fact that we are dealing with L^2 norms, we follow the lines of [15, p. 58]. We fix N, $\tilde{N}$, W_0 and W as in the statement of the lemma.

The cone W_0 being given, we can choose W to be a thickening of W_0, i.e.

$$W = \left\{\eta \in \mathbb{R}^n \setminus \{0\} : \exists \xi \in W_0, \left|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\right| \leq \delta\right\}$$

for some fixed positive δ. This means that small angular perturbations of covectors in W_0 will lie in the neighborhood W. Choose some smooth compactly supported function φ which equals 1 on the support of u, hence we have the identity $\widehat{u} = \widehat{u\varphi}$. We compute the Fourier transform of the product:

$$|\widehat{u\varphi}(\xi)| \leq \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi-\eta)\widehat{u}(\eta)|\, d\eta.$$

We decompose

$$\int_{\mathbb{R}^n} |\widehat{\varphi}(\xi-\eta)\widehat{u}(\eta)|\, d\eta = \underbrace{\int_{|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}|\leq\delta} |\widehat{\varphi}(\xi-\eta)\widehat{u}(\eta)|\, d\eta}_{I_1(\xi)} + \underbrace{\int_{|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}|\geq\delta} |\widehat{\varphi}(\xi-\eta)\widehat{u}(\eta)|\, d\eta}_{I_2(\xi)},$$

and we will estimate the two terms $I_1(\xi)$, $I_2(\xi)$ separately.

Start with $I_1(\xi)$. If $\xi \in W_0$ then, by definition of W, η belongs to W. Hence, using the Cauchy–Schwarz inequality, this yields the estimate

$$\begin{aligned}
I_1(\xi) &= \int_{|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}|\leq\delta} |\widehat{\varphi}(\xi-\eta)\widehat{u}(\eta)|\, d\eta \\
&= (1+|\xi|)^{-N} \\
&\quad \times \int_{|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}|\leq\delta} \big|\widehat{\varphi}(\xi-\eta)(1+|\xi-\eta|)^N \widehat{u}(\eta)(1+|\eta|)^N\big| \frac{(1+|\xi|)^N}{(1+|\eta|)^N(1+|\xi-\eta|)^N}\, d\eta \\
&\leq (1+|\xi|)^{-N} \sup_{\xi,\eta} \frac{(1+|\xi|)^N}{(1+|\eta|)^N(1+|\xi-\eta|)^N} \|\varphi\|_{H^N} \|(1+|\xi|)^N \widehat{u}(\xi)\|_{L^2(W)} \\
&\leq C_{\varphi,N}(1+|\xi|)^{-N} \|(1+|\xi|)^N \widehat{u}(\xi)\|_{L^2(W)},
\end{aligned}$$

where we use the triangle inequality $|\xi| \leq |\xi-\eta|+|\eta|$ in order to bound $\frac{(1+|\xi|)^N}{(1+|\eta|)^N(1+|\xi-\eta|)^N}$ by some constant C uniformly in ξ and in η.

To estimate the second term $I_2(\xi)$, we shall use the fact that the integral is over η such that $\left|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\right| \geq \delta$. This implies that the angle between ξ and η is bounded from below by some $\alpha \in (0, \pi/2)$ which depends only on the aperture δ. We now observe that

$$a^2 + b^2 - 2ab\cos c = (a - b\cos c)^2 + b^2 \sin^2 c \geq b^2 \sin^2 c,$$

and we apply this lower bound to $a = |\xi|$, $b = |\eta|$ and c the angle between ξ and η. Thus,

$$\forall (\xi, \eta) \in V \times^c W, \qquad |(\sin\alpha)\eta| \leq |\xi - \eta|, |(\sin\alpha)\xi| \leq |\xi - \eta|.$$

Then, for such ξ and η, there exists some constant C (depending only on N, $\tilde{N}$ and δ) such that

$$(1+|\xi-\eta|)^{-N-\tilde{N}} \leq (1+|(\sin\alpha)\eta|)^{-\tilde{N}}(1+|(\sin\alpha)\xi|)^{-N} \leq C(1+|\eta|)^{-\tilde{N}}(1+|\xi|)^{-N}.$$

Thus, up to increasing the value of C and by applying the Cauchy–Schwarz inequality, we find

$$\int_{\left|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}\right|\geq\delta} |\widehat{\varphi}(\xi-\eta)\widehat{u}(\eta)|\,d\eta \leq C\|\varphi\|_{H^{N+\tilde{N}}}(1+|\xi|)^{-N}\left(\int_{\mathbb{R}^n}(1+|\eta|)^{-2\tilde{N}}|\widehat{u}(\eta)|^2\,d\eta\right)^{1/2}.$$

Gathering the two estimates yields the final result.

Acknowledgments. When we gave talks on our previous work [17], a recurrent question was about the relation between our results and the Witten Laplacian. We thank the many people who asked us this question as it motivated us to seriously explore this problem. We also warmly thank J. F. Bony, A. Drouot, C. Guillarmou, F. Hérau, F. Laudenbach and B. Merlet for useful discussions related to this work. Finally, we warmly thank the two anonymous referees for their careful and detailed reading: their suggestions allowed us to improve the presentation of this work. The second author is partially supported by the Agence Nationale de la Recherche through the Labex CEMPI (ANR-11-LABX-0007-01) and the ANR project GERASIC (ANR-13-BS01-0007-01).

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