Yukinobu Toda

# Semiorthogonal decompositions of stable pair moduli spaces via d-critical flips

**Abstract.** We show the existence of semiorthogonal decompositions (SOD) of Pandharipande–Thomas (PT) stable pair moduli spaces on Calabi–Yau 3-folds with irreducible curve classes, assuming relevant moduli spaces are non-singular. The above result is motivated by categorifications of the wall-crossing formula for PT invariants in the derived category, and also by a d-critical analogue of Bondal–Orlov's and Kawamata's D/K equivalence conjecture.

We also give SOD of stable pair moduli spaces on K3 surfaces, which categorifies Kawai–Yoshioka's formula proving Katz–Klemm–Vafa's formula for PT invariants on K3 surfaces with irreducible curve classes.

**Keywords.** Derived categories of coherent sheaves, Pandharipande–Thomas invariants

## Contents

## 1. Introduction

The purpose of this paper is to give applications of d-critical birational geometry proposed in [Tod] to the study of derived categories of coherent sheaves on moduli spaces of stable objects on Calabi–Yau (CY for short) 3-folds. The main result is that Pandharipande–Thomas (PT for short) stable pair moduli spaces [PT09] on CY 3-folds with irreducible curve classes admit certain semiorthogonal decompositions (SOD for short), assuming relevant moduli spaces are non-singular. Our results are motivated by categorifications of the wall-crossing formula for Donaldson–Thomas invariants on CY 3-folds [JS12, KS] in

Y. Toda: Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo (WPI), 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan; e-mail: yukinobu.toda@ipmu.jp

the derived category, and also by a d-critical analogue of the D/K equivalence conjecture of Bondal–Orlov and Kawamata [BO, Kaw02].

### 1.1. SOD of stable pair moduli spaces

Let $X$ be a smooth projective CY 3-fold over $\mathbb{C}$. By definition, a *stable pair* on $X$ consists of data [PT09]

$$(F, s), \quad s \colon \mathcal{O}_X \to F, \tag{1.1}$$

where $F$ is a pure one-dimensional coherent sheaf on $X$ and $s$ is surjective in dimension 1. For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we denote by

$$P_n(X, \beta) \tag{1.2}$$

the moduli space of stable pairs (1.1) such that $[F] = \beta$ and $\chi(F) = n$, where $[F]$ is the homology class of the fundamental one-cycle of $F$. The moduli space (1.2) is a projective scheme with a symmetric perfect obstruction theory. The integration of its zero-dimensional virtual class defines the *PT invariant*

$$P_{n,\beta} := \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

The study of PT invariants is one of the central topics in curve counting on CY 3-folds (see [PT14]).

Suppose that $n \geq 0$ and $\beta$ is an irreducible curve class, i.e. $\beta$ cannot be written as $\beta_1 + \beta_2$ for effective curve classes $\beta_i$. Then we have the following diagram:

$$
\begin{array}{ccc}
P_n(X, \beta) & & P_{-n}(X, \beta) \\
& \searrow{\scriptstyle \pi^+} \quad \swarrow{\scriptstyle \pi^-} & \\
& U_n(X, \beta) &
\end{array}
\tag{1.3}
$$

Here $U_n(X, \beta)$ is the moduli space of one-dimensional Gieseker stable sheaves $F$ on $X$ with $[F] = \beta$ and $\chi(F) = n$. The maps $\pi^{\pm}$ are defined by

$$\pi^+(F, s) := F, \quad \pi^-(F', s') := \mathcal{E}xt_X^2(F', \mathcal{O}_X).$$

For a variety $Y$, we denote by $D^b(Y)$ the bounded derived category of coherent sheaves on $Y$. The following is the main result in this paper:

**Theorem 1.1** (Theorem 5.7). *Suppose that $U_n(X, \beta)$ is fine and non-singular. Then $P_{\pm n}(X, \beta)$ are also non-singular, and we have the following:*

(i) *The Fourier–Mukai functor*

$$\Phi_P \colon D^b(P_{-n}(X, \beta)) \to D^b(P_n(X, \beta))$$

*with kernel the structure sheaf of the fiber product of* (1.3) *is fully faithful.*

(ii) *There is a $\pi^+$-ample line bundle $\mathcal{O}_P(1)$ on $P_n(X, \beta)$ such that if $n \geq 1$ then the functor*

$$\Upsilon^i_P \colon D^b(U_n(X, \beta)) \to D^b(P_n(X, \beta))$$

*defined by $\mathbf{L}\pi^{+*}(-) \otimes \mathcal{O}_P(i)$ is fully faithful.*

(iii) *We have the SOD*

$$D^b(P_n(X, \beta)) = \langle \operatorname{Im} \Upsilon^{-n+1}_P, \ldots, \operatorname{Im} \Upsilon^0_P, \operatorname{Im} \Phi_P \rangle. \tag{1.4}$$

Here $U_n(X, \beta)$ is called *fine* if it admits a universal sheaf on $X \times U_n(X, \beta)$, which is guaranteed if $(D \cdot \beta, n)$ is coprime for some divisor $D$. The result of Theorem 1.1 will also be applied to some non-compact CY 3-folds. We apply Theorem 1.1 in the case of

$$X = \operatorname{Tot}_S(K_S), \qquad H^i(\mathcal{O}_S) = 0, \quad i = 1, 2,$$

where $S$ is a smooth projective surface. The assumption of Theorem 1.1 is satisfied when $-K_S \cdot \beta > 0$, and we obtain the SOD of derived categories of relative Hilbert schemes of points on the universal curve over a complete linear system on $S$ (see Corollary 5.10).

We also apply Theorem 1.1 in the case of

$$X = \operatorname{Tot}_C(L_1 \oplus L_2), \qquad L_i \in \mathcal{P}ic(C), \qquad L_1 \otimes L_2 \cong \omega_C,$$

where $C$ is a smooth projective curve. For a generic choice of $L_i$, the diagram (1.3) is a classical diagram of symmetric products of $C$ and their Abel–Jacobi maps. Then Theorem 1.1 implies the SOD of derived categories of coherent sheaves on symmetric products of $C$ (see Corollary 5.12):

$$D^b(C^{[n+g-1]}) = \langle \overbrace{D^b(J_C), \ldots, D^b(J_C)}^{n}, D^b(C^{[-n+g-1]}) \rangle.$$

Here $n \in \mathbb{Z}_{\geq 0}$, $C^{[k]}$ is the $k$-th symmetric product of $C$, $g$ is the genus of $C$, and $J_C$ is the Jacobian of $C$. The above SOD seems to give a new result on the properties of symmetric products of curves and the associated Abel–Jacobi maps.

## 1.2. Motivations behind Theorem 1.1

We have two motivations behind the result of Theorem 1.1. The first one is to give a categorification of the formula (see [PT10, Tod12a])

$$P_{n,\beta} - P_{-n,\beta} = (-1)^{n-1} n N_{n,\beta}. \tag{1.5}$$

Here $N_{n,\beta} \in \mathbb{Z}$ is obtained by the integration of the virtual class on $U_n(X, \beta)$. The identity (1.5) is the key ingredient to show the rationality of the generating series of PT invariants

$$P_\beta(X) = \sum_{n \in \mathbb{Z}} P_{n,\beta} q^n$$

when $\beta$ is irreducible (see [PT10]). As observed in [Tod12a], the diagram (1.3) is a wall-crossing diagram in $D^b(X)$, and (1.5) is the associated wall-crossing formula. Under the assumption of Theorem 5.7, the invariants in (1.5) are given by

$$P_{\pm n, \beta} = (-1)^{n+d-1} e(P_{\pm n}(X, \beta)), \quad N_{n, \beta} = (-1)^d e(U_n(X, \beta))$$

where $d$ is the dimension of $U_n(X, \beta)$. Therefore the SOD in (1.4) categorifies the formula (1.5), as it recovers the formula (1.5) by taking the Euler characteristics of the Hochschild homology of both sides of (1.4).

The second motivation is to give an evidence for a d-critical analogue of Bondal–Orlov's and Kawamata's D/K equivalence conjecture [BO, Kaw02]. The original D/K equivalence conjecture asserts that for a flip of smooth varieties $Y^+ \dashrightarrow Y^-$ there exists a fully faithful functor

$$D^b(Y^-) \hookrightarrow D^b(Y^+).$$

On the other hand, the diagram (1.3) is an example of a *d-critical flip* introduced in [Tod]. Therefore Theorem 1.1(i) gives evidence for a d-critical analogue of the D/K equivalence conjecture. We will come back to this point of view in Subsection 1.4.

### 1.3. Categorification of Kawai–Yoshioka's formula

We will apply the argument as in the proof of Theorem 1.1 to show the existence of SOD on relative Hilbert schemes of points associated with linear systems on K3 surfaces. Let $S$ be a smooth projective K3 surface such that $\mathcal{P}ic(S)$ is generated by $\mathcal{O}_S(H)$ for an ample divisor $H$ with $H^2 = 2g - 2$. Let

$$\pi : \mathcal{C} \to |H| = \mathbb{P}^g \tag{1.6}$$

be the universal curve. Below we fix $n > 0$, and define

$$\mathcal{C}^{[n+g-1]} \to \mathbb{P}^g \tag{1.7}$$

to be the $\pi$-relative Hilbert scheme of $n + g - 1$ points. The moduli space (1.7) is known to be isomorphic to the moduli space of PT stable pairs $P_n(S, [H])$ on $S$.

For each $k \geq 0$, let $U_k$ be the moduli space of $H$-Gieseker stable sheaves $E$ on $S$ such that

$$v(E) = (k, H, k + n) \in H^{2*}(S, \mathbb{Z})$$

where $v(-)$ is the Mukai vector. The moduli space $U_k$ is an irreducible holomorphic symplectic manifold. We assume that $U_k$ is a fine moduli space for all $k \geq 0$, e.g. it is satisfied if $(2g - 2, n)$ is coprime. Let $N \geq 0$ be defined to be the largest $k \geq 0$ such that $U_k \neq \emptyset$. In this situation, we have the following:

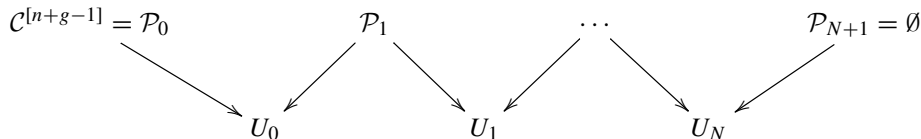**Theorem 1.2** (Corollary 6.3). *We have the following SOD:*

$$D^b(\mathcal{C}^{[n+g-1]}) = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle$$

*where each $\mathcal{A}_k$ has SOD*

$$\mathcal{A}_k = \langle \mathcal{A}_k^{(1)}, \mathcal{A}_k^{(2)}, \ldots, \mathcal{A}_k^{(n+2k)} \rangle$$

*such that each $\mathcal{A}_k^{(i)}$ is equivalent to $D^b(U_k)$.*

Theorem 1.2 is proved by using the zigzag diagram



constructed by Kawai–Yoshioka [KY00]. We show that each step of the above diagram is described in terms of a d-critical simple flip, by investigating wall-crossing diagrams on a CY 3-fold $X = S \times C$ for an elliptic curve $C$. Then Theorem 1.2 is proved by applying the argument used for Theorem 1.1 to each step of the diagram.

The SOD in Theorem 1.2 is interpreted as a categorification of Kawai–Yoshioka's formula [KY00] for PT invariants on K3 surfaces with irreducible curve classes, defined by $P_{n,g} := (-1)^{n-1} e(\mathcal{C}^{[n+g-1]})$. Indeed, the following formula is proved in [KY00]:

$$e(\mathcal{C}^{[n+g-1]}) = \sum_{k=0}^{N} (n + 2k) e(U_k). \tag{1.8}$$

The SOD in Theorem 1.2 recovers the formula (1.8) by taking the Euler characteristics of Hochschild homologies of both sides of (1.2). In [KY00], the formula (1.8) led to the Katz–Klemm–Vafa (KKV) formula for PT invariants with irreducible curve classes (see Remark 6.4).

### 1.4. D-critical analogue of the D/K equivalence conjecture

Here we explain the notion of d-critical flips for Joyce's d-critical loci [Joy15], and an analogue of the D/K equivalence conjecture mentioned earlier. By definition, a *d-critical locus* consists of data

$$(M, s), \quad s \in \Gamma(M, \mathcal{S}_M^0),$$

where $M$ is a $\mathbb{C}$-scheme or an analytic space and $\mathcal{S}_M^0$ is a certain sheaf of $\mathbb{C}$-vector spaces on $M$. The section $s$ is called a *d-critical structure* of $M$. Roughly speaking, if $M$ admits a d-critical structure $s$, this means that $M$ is locally written as a critical locus of some function on a smooth space, and the section $s$ remembers how $M$ is locally written as a

critical locus. If $M$ is a truncation of a derived scheme with a $(-1)$-shifted symplectic structure [PT$^+$13], then it has a canonical d-critical structure [BB$^+$15].

Let $(M^{\pm}, s^{\pm})$ be two d-critical loci and consider a diagram of morphisms of schemes or analytic spaces

$$
\begin{array}{ccc}
M^+ & & M^- \\
& \searrow \quad \swarrow & \\
& U &
\end{array}
\tag{1.9}
$$

The above diagram is called a *d-critical flip* if for any $p \in U$, there is a commutative diagram

$$
\begin{array}{ccc}
Y^+ & \overset{\phi}{\dashrightarrow} & Y^- \\
& Z & \\
w^+ & \downarrow w & w^- \\
& \mathbb{C} &
\end{array}
\tag{1.10}
$$

where $\phi \colon Y^+ \dashrightarrow Y^-$ is a flip of smooth varieties (or complex manifolds) such that locally near $p \in U$ there exist isomorphisms of $M^{\pm}$ and $\{dw^{\pm} = 0\}$ as d-critical loci (see [Tod14b, Definition 3.7] for details). A d-critical flip is called *simple* if $\phi \colon Y^+ \dashrightarrow Y^-$ is a simple toric flip [Rei92].

We expect that an analogue of the D/K equivalence conjecture may hold for d-critical loci. Namely for a d-critical locus $(M, s)$ [1] there may exist a certain triangulated category $\mathcal{D}(M, s)$ such that if the diagram (1.9) is a d-critical flip, we have a fully faithful functor

$$
\mathcal{D}(M^-, s^-) \hookrightarrow \mathcal{D}(M^+, s^+).
\tag{1.11}
$$

The category $\mathcal{D}(M^-, s^-)$ may be constructed as a gluing of $\mathbb{Z}/2\mathbb{Z}$-periodic triangulated categories of matrix factorizations defined locally on each d-critical chart, though its construction seems to be a hard problem at this moment (see [Joy, (J)], [Toë14, Section 6.1]).

For a flip $Y^+ \dashrightarrow Y^-$ in the diagram (1.10), suppose that the D/K equivalence conjecture holds, i.e. we have a fully faithful functor

$$
D^b(Y^-) \hookrightarrow D^b(Y^+).
$$

Then it induces the fully faithful functor (see Theorem 2.1)

$$
D(Y^-, w^-) \hookrightarrow D(Y^+, w^+)
\tag{1.12}
$$

where $D(Y^{\pm}, w^{\pm})$ are the derived factorization categories associated with pairs $(Y^{\pm}, w^{\pm})$. If the desired categories $\mathcal{D}(M^{\pm}, s^{\pm})$ are gluings of $D(Y^{\pm}, w^{\pm})$ defined locally

---

[1] Probably we need to assume that $(M, s)$ is induced by a $(-1)$-shifted symplectic derived scheme, equipped with some additional data (orientation data, or something more).

on $U$, then we may try to globalize the functor (1.12) to give a fully faithful functor (1.11). If this is possible, then the numerical realization of a semiorthogonal complement of the embedding (1.11) may recover the wall-crossing formula for DT invariants [JS12, KS].[2]

For a d-critical flip (1.9), suppose that $M^{\pm}$ are smooth, so in particular $s^{\pm} = 0$. In this case, we can use usual derived categories of coherent sheaves to ask an analogue of the above question. Namely for a d-critical flip (1.9) with $M^{\pm}$ smooth, we can ask whether we have a fully faithful functor

$$D^b(M^-) \hookrightarrow D^b(M^+).$$

The results of Theorems 1.1 and 1.2 are proved by establishing such a result in the case of d-critical simple flips (see Theorem 4.5).

### 1.5. Relation to other works

There exist some recent works studying wall-crossing behavior of derived categories of moduli spaces of stable objects on algebraic surfaces. In [Bal17], Ballard showed the existence of SOD under wall-crossing of Gieseker moduli spaces of stable sheaves on rational surfaces. Also Halpern-Leistner [HL] announces that, under wall-crossing of Bridgeland moduli spaces of stable objects on K3 surfaces, their derived categories are equivalent. The results in our paper can be regarded as a CY 3-fold version of these works. One of the crucial differences is that, although the moduli spaces considered in [Bal17, HL] are birational under wall-crossing, the moduli spaces in our paper are not necessarily birational under wall-crossing. For example the moduli spaces $P_{\pm n}(X, \beta)$ in the diagram (1.3) have different dimensions if $n > 0$. Instead, the fact that they are birational in d-critical birational geometry plays an important role for the existence of SOD in Theorems 1.1 and 1.2.

### 1.6. Outline of the paper

The outline of this paper is as follows. In Section 2, we review basics on derived factorization categories which we will use in later sections. In Section 3, we show the existence of SOD of gauged LG models on simple flips over a complete local base, and describe the relevant kernel objects. In Section 4, we globalize the result in Section 3 and show the SOD for formal d-critical simple flips. In Section 5, we use the result in Section 4 to show Theorem 1.1. In Section 6, we prove Theorem 1.2.

### 1.7. Notation and conventions

In this paper, all the varieties and schemes are defined over $\mathbb{C}$. For $\mathbb{C}$-schemes $U$, $S$, $T$ and a morphism $f \colon S \to T$, we set

$$S_U := S \times U, \quad f_U := f \times \mathrm{id}_U \colon S_U \to T_U.$$

---

[2] See [Efi17] for the relation between cyclic homology of the categories of matrix factorizations and hypercohomology of perverse sheaves of vanishing cycles.

For a variety $Y$, we denote by $D^b(Y)$ the bounded derived category of coherent sheaves on $Y$. If an algebraic group $G$ acts on $Y$, we also denote by $D^b_G(Y)$ the bounded derived category of $G$-equivariant coherent sheaves on $Y$. For smooth varieties $Y_1, Y_2$ with projective morphisms $Y_i \to T$, and an object $\mathcal{P} \in D^b(Y_1 \times Y_2)$ supported on $Y_1 \times_T Y_2$, we denote by $\Phi^{\mathcal{P}}$ the *Fourier–Mukai functor*

$$\Phi^{\mathcal{P}}(-) := \mathbf{R}p_{2*}(p_1^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{P}) \colon D^b(Y_1) \to D^b(Y_2).$$

Here $p_i \colon Y_1 \times Y_2 \to Y_i$ are the projections. The object $\mathcal{P}$ is called the *kernel* of the functor $\Phi^{\mathcal{P}}$.

Recall that a *semiorthogonal decomposition* of a triangulated category $\mathcal{D}$ is a collection $\mathcal{C}_1, \ldots, \mathcal{C}_n$ of full triangulated subcategories such that $\mathrm{Hom}(\mathcal{C}_i, \mathcal{C}_j) = 0$ for all $i > j$ and the smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{C}_1, \ldots, \mathcal{C}_n$ coincides with $\mathcal{D}$. In this case, we write $\mathcal{D} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_n \rangle$. If each $\mathcal{C}_i$ is equivalent to $D^b(M_i)$ for a variety $M_i$, we also write $\mathcal{D} = \langle D^b(M_1), \ldots, D^b(M_n) \rangle$ for simplicity.

## 2. Review of derived factorization categories

In this section, we recall the notion of gauged Landau–Ginzburg (LG) models, and the associated derived factorization categories introduced by Positselski. We refer to the articles [EP15] for basics on these notions.

### 2.1. Definitions of derived factorization categories

Let us consider data (called *gauged LG model*)

$$(Y, G, \chi, w) \tag{2.1}$$

where $Y$ is a $\mathbb{C}$-scheme, $G$ is a reductive algebraic group which acts on $Y$, $\chi \colon G \to \mathbb{C}^*$ is a character and $w \in \Gamma(\mathcal{O}_Y)$ satisfies $g^*w = \chi(g)w$ for any $g \in G$. Given data as above, the *derived factorization category*

$$D_G(Y, \chi, w) \tag{2.2}$$

is defined as a triangulated category whose objects consist of *factorizations of* $w$, i.e. sequences of $G$-equivariant morphisms of $G$-equivariant coherent sheaves $\mathcal{F}_0, \mathcal{F}_1$ on $Y$

$$\mathcal{F}_0 \overset{\alpha}{\to} \mathcal{F}_1 \overset{\beta}{\to} \mathcal{F}_0(\chi) \tag{2.3}$$

satisfying

$$\alpha \circ \beta = \cdot w, \quad \beta \circ \alpha = \cdot w.$$

The category (2.2) is defined to be the localization of the homotopy category of the factorizations (2.3) by its subcategory of acyclic factorizations. When $Y$ is a smooth affine

scheme and $G = \{1\}$, the category (2.2) is equivalent to the triangulated category of matrix factorizations of $w$ (see [Orl09]). In the case of $G = \mathbb{C}^*$ and $\chi = \mathrm{id}$, we simply write

$$D_{\mathbb{C}^*}(Y, w) := D_{\mathbb{C}^*}(Y, \chi = \mathrm{id}, w).$$

For a character $\chi : G \to \mathbb{C}^*$, let $\widetilde{\chi}$ be defined by

$$\widetilde{\chi} : G \times \mathbb{C}^* \to \mathbb{C}^*, \quad (g, t) \mapsto \chi(g)t.$$

We have the functor

$$\Xi : D_G^b(Y) \to D_{G \times \mathbb{C}^*}(Y, \widetilde{\chi}, w = 0) \tag{2.4}$$

where $\mathbb{C}^*$ acts on $Y$ trivially, sending $(\mathcal{F}^\bullet, d) \in D_G^b(Y)$ to

$$\left( \bigoplus_{i \in \mathbb{Z}} \mathcal{F}^{2i}(-i\widetilde{\chi}) \right) \xrightarrow{d} \left( \bigoplus_{i \in \mathbb{Z}} \mathcal{F}^{2i+1}(-i\widetilde{\chi}) \right) \xrightarrow{d} \left( \bigoplus_{i \in \mathbb{Z}} \mathcal{F}^{2i}(-i\widetilde{\chi}) \right)(\widetilde{\chi}).$$

When $G = \{1\}$, the functor (2.4) gives the equivalence (see [Isi13, Shi12, Hir17b])

$$\Xi : D^b(Y) \xrightarrow{\sim} D_{\mathbb{C}^*}(Y, 0). \tag{2.5}$$

### 2.2. Derived functors between derived factorization categories

Let $(Y, G, \chi, w)$ be a gauged LG model (2.1), and $W$ be another variety with a $G$-action. For a $G$-equivariant projective morphism $f : W \to Y$, we have another gauged LG model

$$(W, G, \chi, f^*w).$$

Similarly to the usual derived functors between derived categories, if $Y$ is smooth we have the derived functors

$$\begin{aligned} \mathbf{R}f_* &: D_G(W, \chi, f^*w) \to D_G(Y, \chi, w), \\ \mathbf{L}f^* &: D_G(Y, \chi, w) \to D_G(W, \chi, f^*w). \end{aligned}$$

Also for another object $\mathcal{P} \in D_G(Y, \chi, w')$, we have the derived tensor product

$$\overset{\mathbf{L}}{\otimes} \mathcal{P} : D_G(Y, \chi, w) \to D_G(Y, \chi, w + w').$$

Below we omit the subscripts $\mathbf{R}$, $\mathbf{L}$ when the relevant functors are exact functors of coherent sheaves, e.g. write $\mathbf{L}f^*$ as $f^*$ when $f$ is flat.

Let $Y_1, Y_2$ be regular $\mathbb{C}$-schemes with $G$-actions. Let $T$ be a $\mathbb{C}$-scheme with a $G$-action and consider $G$-equivariant projective morphisms $Y_i \to T$. Let us take $w \in \Gamma(\mathcal{O}_T)$ and

a character $\chi \colon G \to \mathbb{C}^*$ satisfying $g^* w = \chi(g) w$ for any $g \in G$. We consider the commutative diagram
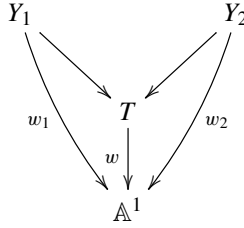


Let $p_i \colon Y_1 \times Y_2 \to Y_i$ be the projections, and suppose $G$ acts on $Y_1 \times Y_2$ diagonally. For any object

$$\mathcal{P} \in D_G(Y_1 \times Y_2, \chi, -p_1^* w_1 + p_2^* w_2)$$

we have the Fourier–Mukai type functor

$$\Psi^{\mathcal{P}} := \mathbf{R} p_{2*}(p_1^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{P}) \colon D_G(Y_1, \chi, w_1) \to D_G(Y_2, \chi, w_2).$$

Let $i \colon Y_1 \times_T Y_2 \hookrightarrow Y_1 \times Y_2$ be the closed embedding. We have the diagram

$$
\begin{array}{ccccc}
D_G^b(Y_1 \times_T Y_2) & \overset{\Xi}{\to} & D_{G \times \mathbb{C}^*}(Y_1 \times_T Y_2, \widetilde{\chi}, 0) & \overset{\mathrm{forg}^{\mathbb{C}^*}}{\longrightarrow} & D_G(Y_1 \times_T Y_2, \chi, 0) \\
\downarrow{\scriptstyle \mathrm{forg}^G} & & \downarrow{\scriptstyle \mathrm{forg}^G} & & \downarrow{\scriptstyle i_*} \\
D^b(Y_1 \times_T Y_2) & \overset{\Xi}{\longrightarrow} & D_{\mathbb{C}^*}(Y_1 \times_T Y_2, 0) & & D_G(Y_1 \times Y_2, \chi, -p_1^* w_1 + p_2^* w_2)
\end{array}
$$

Here $\Xi$ is given in (2.4) and $\mathrm{forg}^G$, $\mathrm{forg}^{\mathbb{C}^*}$ are forgetting the $G$-action and the $\mathbb{C}^*$-action respectively. For $\mathcal{P} \in D^b(Y_1 \times_T Y_2)$ and $\Xi(\mathcal{P}) \in D_{\mathbb{C}^*}(Y_1 \times_T Y_2, 0)$, the following diagram commutes:

$$
\begin{array}{ccc}
D^b(Y_1) & \overset{\sim}{\underset{\Xi}{\to}} & D_{\mathbb{C}^*}(Y_1, 0) \\
\downarrow{\scriptstyle \Phi^{\mathcal{P}}} & & \downarrow{\scriptstyle \Psi^{\Xi(\mathcal{P})}} \\
D^b(Y_2) & \overset{\sim}{\underset{\Xi}{\to}} & D_{\mathbb{C}^*}(Y_2, 0)
\end{array}
\qquad (2.6)
$$

Moreover we have the following:

**Theorem 2.1** ([BP10, Hir17b]). *For $\mathcal{Q} \in D_G^b(Y_1 \times_T Y_2)$, suppose that the functor*

$$\Phi^{\mathrm{forg}^G(\mathcal{Q})} \colon D^b(Y_1) \to D^b(Y_2)$$

*is fully faithful* (*resp. an equivalence*). *Then for the object*

$$\widetilde{\mathcal{Q}} := i_* \circ \mathrm{forg}^{\mathbb{C}^*} \circ \Xi(\mathcal{Q}) \in D_G(Y_1 \times Y_2, \chi, -p_1^* w_1 + p_2^* w_2)$$

*the functor*

$$\Psi^{\widetilde{\mathcal{Q}}} \colon D_G(Y_1, \chi, w_1) \to D_G(Y_2, \chi, w_2)$$

*is fully faithful* (*resp. an equivalence*).

### 2.3. Knörrer periodicity

Let $\mathcal{E} \to Y$ be an algebraic vector bundle on a regular $\mathbb{C}$-scheme $Y$, and $s : Y \to \mathcal{E}$ be a regular section of it, i.e. its zero locus

$$Z := (s = 0) \subset Y$$

has codimension equal to the rank of $\mathcal{E}$. The section $s$ naturally defines a morphism $Q_s : \mathcal{E}^\vee \to \mathbb{A}^1$ sending $(y, v)$ for $y \in Y$ and $v \in \mathcal{E}|_y^\vee$ to $\langle s(y), v \rangle$. We have the diagram

$$
\begin{array}{ccccc}
\mathcal{E}|_Z^\vee & \overset{i}{\hookrightarrow} & \mathcal{E}^\vee & \overset{Q_s}{\longrightarrow} & \mathbb{A}^1 \\
{\scriptstyle p}\downarrow & & \downarrow & & \\
Z & \hookrightarrow & Y & &
\end{array}
$$

Note that $Q_s = 0$ on $i(\mathcal{E}|_Z^\vee) \subset \mathcal{E}^\vee$. Let $\mathbb{C}^*$ act on $Z$ trivially, and on $\mathcal{E}^\vee$ with weight 1 on the fibers of the projection $\mathcal{E}^\vee \to Y$. The following is the version of Knörrer periodicity used in this paper:

**Theorem 2.2** ([Isi13, Shi12, Hir17a]). *The functor*

$$i_* \circ p^* : D_{\mathbb{C}^*}(Z, 0) \to D_{\mathbb{C}^*}(\mathcal{E}^\vee, Q_s)$$

*is an equivalence of triangulated categories. By composing it with the equivalence* (2.5), *we obtain the equivalence*

$$i_* \circ p^* \circ \Xi : D^b(Z) \overset{\sim}{\to} D_{\mathbb{C}^*}(\mathcal{E}^\vee, Q_s).$$

## 3. SOD via simple flips

Let $\widehat{U}$ be the formal completion of an affine space at the origin. In this section, we show that for a simple d-critical flip

$$\widehat{M}^+ \to \widehat{U} \leftarrow \widehat{M}^-$$

for smooth schemes $\widehat{M}^\pm$ satisfying some conditions, we have the SOD

$$D^b(\widehat{M}^+) = \langle \overbrace{D^b(\widehat{U}), \ldots, D^b(\widehat{U})}^{n}, D^b(\widehat{M}^-) \rangle. \tag{3.1}$$

The result is proved by combining derived factorization analogue of Bondal–Orlov's SOD associated with simple flips [BO] (see Theorem 3.3) with the Knörrer periodicity of derived factorization categories (see Theorem 2.2).

The main ingredient in this section is to show that the kernel object of the fully faithful functor $D^b(\widehat{M}^-) \hookrightarrow D^b(\widehat{M}^+)$ is given by the structure sheaf of the fiber product $\widehat{M}^+ \times_{\widehat{U}} \widehat{M}^-$. This explicit description of the kernel will be important in the next section to globalize the result of this section.

### 3.1. Simple toric flips

Let $V^+$, $V^-$ be $\mathbb{C}$-vector spaces of dimensions $a$, $b$ respectively. We assume that $a \geq b$, and set $n := a - b \geq 0$. Let $\mathbb{C}^*$ act on $V^+$, $V^-$ with weight $1$, $-1$ respectively. We fix bases of $V^\pm$ and denote the coordinates of $V^+$, $V^-$ by

$$\vec{x} = (x_1, \ldots, x_a), \quad \vec{y} = (y_1, \ldots, y_b)$$

respectively. Let $V^{\pm *} := V^\pm \setminus \{0\}$, and define

$$\begin{aligned}
Y^+ &:= [((V^{+*}) \times V^-)/\mathbb{C}^*] = \mathrm{Tot}_{\mathbb{P}(V^+)}(\mathcal{O}_{\mathbb{P}(V^+)}(-1) \otimes V^-), \\
Y^- &:= [(V^+ \times (V^{-*}))/\mathbb{C}^*] = \mathrm{Tot}_{\mathbb{P}(V^-)}(\mathcal{O}_{\mathbb{P}(V^-)}(-1) \otimes V^+), \\
Z &:= (V^+ \times V^-)/\!\!/\mathbb{C}^* = \mathrm{Spec}\,\mathbb{C}[x_i y_j : 1 \leq i \leq a, 1 \leq j \leq b].
\end{aligned} \tag{3.2}$$

We have a toric flip diagram, called a *simple flip* (see [Rei]):
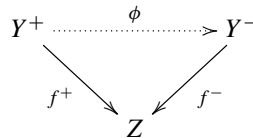


We also have the projections and closed embeddings

$$\mathrm{pr}^\pm \colon Y^\pm \to \mathbb{P}(V^\pm), \quad i^\pm \colon \mathbb{P}(V^\pm) \hookrightarrow Y^\pm \tag{3.3}$$

where $i^\pm$ are the zero sections of $\mathrm{pr}^\pm$.

By setting $W := Y^+ \times_Z Y^-$, we have the diagram



where $p^\pm$ are the projections. Note that $p^\pm$ are the blow-ups of $Y^\pm$ at the smooth loci $i^\pm(\mathbb{P}(V^\pm))$. The fiber product $W$ is also described as

$$W = \mathrm{Tot}_{\mathbb{P}(V^+) \times \mathbb{P}(V^-)}(\mathcal{O}_{\mathbb{P}(V^+) \times \mathbb{P}(V^-)}(-1, -1)) = [((V^{+*}) \times (V^{-*}) \times \mathbb{C})/(\mathbb{C}^*)^2]. \tag{3.5}$$

Here $(s_1, s_2) \in (\mathbb{C}^*)^2$ acts on $V^+ \times V^- \times \mathbb{C}$ by

$$(s_1, s_2) \cdot (\vec{x}, \vec{y}, t) = (s_1 \vec{x}, s_2^{-1} \vec{y}, s_1^{-1} s_2 t).$$

Under the description of $W$ in (3.5), the projections $p^\pm \colon W \to Y^\pm$ are induced by the maps

$$V^+ \times V^- \times \mathbb{C} \to V^+ \times V^-, \quad (\mathbb{C}^*)^2 \to \mathbb{C}^*$$

defined by

$$p^+ : (\vec{x}, \vec{y}, t) \mapsto (\vec{x}, t\vec{y}), \qquad (s_1, s_2) \mapsto s_1,$$
$$p^- : (\vec{x}, \vec{y}, t) \mapsto (t\vec{x}, \vec{y}), \qquad (s_1, s_2) \mapsto s_2,$$

respectively. Let $s \in \mathbb{C}^*$ act on $Y^+$, $Y^-$, $W$ by

$$s \cdot (\vec{x}, \vec{y}) = (\vec{x}, s\vec{y}), \qquad s \cdot (\vec{x}, \vec{y}) = (s\vec{x}, \vec{y}), \qquad s \cdot (\vec{x}, \vec{y}, t) = (\vec{x}, \vec{y}, st) \qquad (3.6)$$

respectively. Then the diagram (3.4) is equivariant with respect to the above $\mathbb{C}^*$-actions.
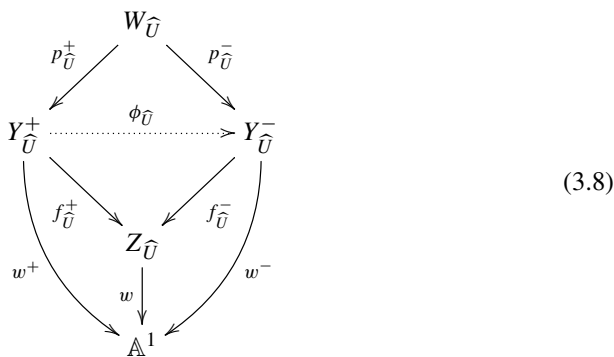
### 3.2. Critical loci

Let $\widehat{U}$ be a smooth $\mathbb{C}$-scheme of dimension $g$, given by

$$\widehat{U} := \mathrm{Spec}\, \mathbb{C}[[u_1, \ldots, u_g]].$$

Let us take an element $w \in \Gamma(\mathcal{O}_{Z_{\widehat{U}}})$ written as

$$w = \sum_{i=1}^{a} \sum_{j=1}^{b} x_i y_j w_{ij}(\vec{u}) \qquad (3.7)$$

for some $w_{ij}(\vec{u}) \in \Gamma(\mathcal{O}_{\widehat{U}})$. We consider the following commutative diagram:



$$\widetilde{w} := p_{\widehat{U}}^{+*} w^+ = p_{\widehat{U}}^{-*} w^- : W_{\widehat{U}} \to \mathbb{A}^1$$

can be written as

$$\widetilde{w} = t \sum_{i,j} x_i y_j w_{ij}(\vec{u}) \qquad (3.9)$$

in the description of $W$ by (3.5). We define $\widehat{M}^{\pm}$ to be

$$\widehat{M}^{\pm} := \{dw^{\pm} = 0\} \subset Y_{\widehat{U}}^{\pm}.$$

**Lemma 3.1.** *Suppose that* $\widehat{M}^{\pm}$ *are smooth and irreducible of dimension*

$$\dim \widehat{M}^{+} = n + g - 1, \quad \dim \widehat{M}^{-} = -n + g - 1 \tag{3.10}$$

*respectively. Then* $\widehat{M}^{\pm}$ *are contained in the images of* $i_{\widehat{U}}^{\pm} : \mathbb{P}(V^{\pm})_{\widehat{U}} \hookrightarrow Y_{\widehat{U}}^{\pm}$, *where* $i^{\pm}$ *are given in* (3.3). *Moreover,*

$$\widehat{M}^{+} = \left\{ (\vec{x}, \vec{u}) \in \mathbb{P}(V^{+})_{\widehat{U}} : \sum_{i=1}^{a} x_i w_{ij}(\vec{u}) = 0 \, \text{for all } 1 \leq j \leq b \right\},$$

$$\widehat{M}^{-} = \left\{ (\vec{y}, \vec{u}) \in \mathbb{P}(V^{-})_{\widehat{U}} : \sum_{j=1}^{b} y_j w_{ij}(\vec{u}) = 0 \, \text{for all } 1 \leq i \leq a \right\}. \tag{3.11}$$

*Proof.* Let $N^{+}$ be the scheme defined by the RHS of (3.11). Note that we obviously have the closed embedding

$$N^{+} \hookrightarrow \widehat{M}^{+}, \quad (\vec{x}, \vec{u}) \mapsto (\vec{x}, \vec{y} = 0, \vec{u}). \tag{3.12}$$

Since $N^{+}$ is defined by $b$ equations on the smooth scheme $\mathbb{P}(V^{+})_{\widehat{U}}$ of dimension $a + g - 1$, we have

$$\dim N^{+} \geq a + g - 1 - b = n + g - 1.$$

Therefore the assumption on $\widehat{M}^{+}$ implies that the embedding (3.12) is an isomorphism. The claim for $\widehat{M}^{-}$ is proved similarly. $\qquad \square$

**Remark 3.2.** The assumption of Lemma 3.1 is satisfied if $g = ab$ and $w_{ij}(\vec{u}) = u_{ij}$ where $\{u_{ij}\}_{1 \leq i \leq a, 1 \leq j \leq b}$ is a coordinate system of $\widehat{U}$. If the assumption of Lemma 3.1 is satisfied, then the projections $\widehat{M}^{\pm} \to \widehat{U}$ are well-presented families of projective spaces defined in [Kem73, Section 3].

Under the assumption of Lemma 3.1, we have $f_{\widehat{U}}^{\pm}(\widehat{M}^{\pm}) \subset \{0\} \times \widehat{U}$. Therefore $\pi^{\pm} := (f_{\widehat{U}}^{\pm})|_{\widehat{M}^{\pm}}$ induces the diagram

$$
\begin{array}{ccc}
\widehat{M}^{+} & & \widehat{M}^{-} \\
& \searrow \pi^{+} \quad \pi^{-} \swarrow & \\
& \widehat{U} &
\end{array}
\tag{3.13}
$$

Moreover for each $c \in \mathbb{Z}$, we have the line bundles

$$\mathcal{O}_{\widehat{M}^{\pm}}(c) := \mathcal{O}_{\mathbb{P}(V^{\pm})_{\widehat{U}}}(c)|_{\widehat{M}^{\pm}} \in \mathcal{P}ic(\widehat{M}^{\pm}).$$

### 3.3. SOD of derived factorization categories under simple flips

Let us consider the diagram (3.8). Since $w^{+}$, $w^{-}$ and $\widetilde{w}$ are of weight 1 with respect to the $\mathbb{C}^{*}$-actions (3.6), we have the associated derived factorization categories

$$D_{\mathbb{C}^{*}}(Y_{\widehat{U}}^{+}, w^{+}), \quad D_{\mathbb{C}^{*}}(Y_{\widehat{U}}^{-}, w^{-}), \quad D_{\mathbb{C}^{*}}(W_{\widehat{U}}, \widetilde{w})$$

respectively. Since the diagram (3.4) is $\mathbb{C}^*$-equivariant, we have the functors

$$\mathbf{L}p_{\widehat{U}}^{-*} \colon D_{\mathbb{C}^*}(Y_{\widehat{U}}^-, w^-) \to D_{\mathbb{C}^*}(W_{\widehat{U}}, \widetilde{w}), \quad \mathbf{R}p_{\widehat{U}*}^+ \colon D_{\mathbb{C}^*}(W_{\widehat{U}}, \widetilde{w}) \to D_{\mathbb{C}^*}(Y_{\widehat{U}}^+, w^+).$$

By composing them, we obtain the functor

$$\Psi_Y := \mathbf{R}p_{\widehat{U}*}^+ \circ \mathbf{L}p_{\widehat{U}}^{-*} \colon D_{\mathbb{C}^*}(Y_{\widehat{U}}^-, w^-) \to D_{\mathbb{C}^*}(Y_{\widehat{U}}^+, w^+). \tag{3.14}$$

Let

$$g \colon \mathbb{P}(V^+)_{\widehat{U}} \to \widehat{U}, \quad i_{\widehat{U}}^+ \colon \mathbb{P}(V^+)_{\widehat{U}} \hookrightarrow Y_{\widehat{U}}^+$$

be the projection and the inclusion into the zero section (3.3) respectively. We also have the functor

$$\Upsilon_Y := i_{\widehat{U}*}^+ \circ g^* \colon D_{\mathbb{C}^*}(\widehat{U}, 0) \to D_{\mathbb{C}^*}(Y_{\widehat{U}}^+, w^+). \tag{3.15}$$

Here $\mathbb{C}^*$ acts on $\widehat{U}, \mathbb{P}(V^+)_{\widehat{U}}$ trivially. The following result should be well-known, but we include a proof here as we cannot find a reference.

**Theorem 3.3.**   (i) *The functor $\Psi_Y$ in* (3.14) *is fully faithful.*
 (ii) *If $n \geq 1$, the functor $\Upsilon_Y$ in* (3.15) *is fully faithful.*
(iii) *By setting $\Upsilon_Y^i := \otimes \mathcal{O}_{Y_{\widehat{U}}^+}(i) \circ \Upsilon_Y$, we have the SOD*

$$D_{\mathbb{C}^*}(Y_{\widehat{U}}^+, w^+) = \langle \operatorname{Im} \Upsilon_Y^{-n}, \dots, \operatorname{Im} \Upsilon_Y^{-1}, \operatorname{Im} \Psi_Y \rangle.$$

*Here $\mathcal{O}_{Y_{\widehat{U}}^+}(i)$ is the pull-back of $\mathcal{O}_{\mathbb{P}(V^+)}(i)$ under the projection $Y_{\widehat{U}}^+ \to \mathbb{P}(V^+)$.*

*Proof.*   (i) The functor $\Psi_Y$ is written as $\Psi^{\widetilde{\mathcal{O}}_W}$ in the notation of Theorem 2.1. On the other hand, the functor

$$\Phi^{\mathcal{O}_W} \colon D^b(Y_{\widehat{U}}^-) \to D^b(Y_{\widehat{U}}^+) \tag{3.16}$$

is fully faithful by [BO]. Therefore (i) follows from Theorem 2.1. The proof of (ii) is similar.

   We prove (iii). Let us recall that we have a similar SOD using *windows* [HL15, BFK19]. Let $T_j = \mathbb{C}^*$ for $j = 1, 2$ act on $V^+ \times V^- \times \widehat{U}$ with weight $(1, -1, 0)$ for $j = 1$, and $(1, 0, 0)$ for $j = 2$. Then the open immersions

$$\eta^{\pm} \colon Y_{\widehat{U}}^{\pm} \hookrightarrow [(V^+ \times V^-)_{\widehat{U}}/T_1] \tag{3.17}$$

are $\mathbb{C}^*$-equivariant, where the $\mathbb{C}^*$-action on the LHS is given by (3.6) and that on the RHS is given by the above $T_2$-action. Let $\mathcal{O}_\bullet(i)$ be the $T_1$-equivariant line bundle on $\operatorname{Spec}\mathbb{C}$, given by a one-dimensional $T_1$-representation with weight $i$. We denote by $\mathcal{O}_{(V^+ \times V^-)_{\widehat{U}}}(i)$ the pull-back of $\mathcal{O}_\bullet(i)$ under the structure morphism

$$(V^+ \times V^-)_{\widehat{U}} \to \operatorname{Spec}\mathbb{C}.$$

Let $\chi : T_1 \times T_2 \to \mathbb{C}^*$ be the second projection, and take $w \in \Gamma(\mathcal{O}_{(V^+ \times V^-)_{\widehat{U}}})$ as in (3.7). For a subset $I \subset \mathbb{R}$, the window subcategory

$$\mathcal{W}_I \subset D_{T_1 \times T_2}((V^+ \times V^-)_{\widehat{U}}, \chi, w) \qquad (3.18)$$

is defined to be the thick triangulated subcategory generated by the factorizations (2.3) where $\mathcal{F}_0$, $\mathcal{F}_1$ are of the form

$$\mathcal{F}_j = \bigoplus_{-i \in I \cap \mathbb{Z}} \mathcal{O}_{(V^+ \times V^-)_{\widehat{U}}}(i)^{\oplus l_{i,j}}, \qquad j = 0, 1, \; l_{i,j} \in \mathbb{Z}_{\geq 0},$$

as $T_1$-equivariant sheaves. Here we regard the $(T_1 \times T_2)$-equivariant sheaves $\mathcal{F}_j$ as $T_1$-equivariant sheaves by the inclusion $T_1 \hookrightarrow T_1 \times T_2$, $t_1 \mapsto (t_1, 1)$. By [BFK19, Theorem 3.5.2], there exists a fully faithful functor

$$\Psi'_Y : D_{\mathbb{C}^*}(Y^-_{\widehat{U}}, w^-) \to D_{\mathbb{C}^*}(Y^+_{\widehat{U}}, w^+)$$

which fits into the commutative diagram

$$
\begin{array}{ccc}
\mathcal{W}_{(-b,0]} & \xrightarrow[\sim]{\eta^{-*}} & D_{\mathbb{C}^*}(Y^-_{\widehat{U}}, w^-) \\
\downarrow & & \downarrow{\scriptstyle \Psi'_Y} \\
\mathcal{W}_{(-b,a-b]} & \xrightarrow[\eta^{+*}]{\sim} & D_{\mathbb{C}^*}(Y^+_{\widehat{U}}, w^+)
\end{array}
\qquad (3.19)
$$

Here the horizontal arrows are equivalences of triangulated categories, defined by pullbacks via open immersions (3.17) restricted to $\mathcal{W}_I$, and the left vertical arrow is a natural inclusion. Moreover by *loc. cit.*, we have the SOD

$$D_{\mathbb{C}^*}(Y^+_{\widehat{U}}, w^+) = \langle \operatorname{Im} \Upsilon_Y^{-n}, \ldots, \operatorname{Im} \Upsilon_Y^{-1}, \operatorname{Im} \Psi'_Y \rangle.$$

It is enough to show that $\operatorname{Im} \Psi_Y = \operatorname{Im} \Psi'_Y$. Note that

$$\eta^{-*}(\mathcal{O}_{(V^+ \times V^-)_{\widehat{U}}}(i)) = \mathcal{O}_{Y^-_{\widehat{U}}}(-i), \qquad \eta^{+*}(\mathcal{O}_{(V^+ \times V^-)_{\widehat{U}}}(i)) = \mathcal{O}_{Y^+_{\widehat{U}}}(i).$$

By the diagram (3.19), it follows that $\operatorname{Im} \Psi'_Y$ is generated by factorizations (2.3) such that $\mathcal{F}_0$, $\mathcal{F}_1$ are of the form

$$\mathcal{F}_j = \bigoplus_{0 \leq i \leq b-1} \mathcal{O}_{Y^+_{\widehat{U}}}(-i)^{\oplus l_{i,j}}, \qquad j = 0, 1, \; l_{i,j} \in \mathbb{Z}_{\geq 0}.$$

On the other hand, an easy calculation shows that

$$\Phi^{\mathcal{O}_W}(\mathcal{O}_{Y^-_{\widehat{U}}}(i)) = \mathcal{O}_{Y^+_{\widehat{U}}}(-i), \qquad 0 \leq i \leq b-1, \qquad (3.20)$$

where $\Phi^{\mathcal{O}_W}$ is the functor (3.16). Together with the equivalence of the top horizontal arrow of (3.19), it follows that $\operatorname{Im} \Psi_Y$ is also generated by the objects of the form (3.20). Therefore $\operatorname{Im} \Psi_Y = \operatorname{Im} \Psi'_Y$.                                                $\square$

### 3.4. SOD in the complete local setting

We return to the situation of Subsection 3.2. Under the assumption of Lemma 3.1, we have the diagram

$$
\begin{array}{ccc}
A^{\pm} & \overset{j^{\pm}}{\hookrightarrow} Y^{\pm}_{\widehat{U}} & \overset{w^{\pm}}{\longrightarrow} \mathbb{A}^1 \\
{\scriptstyle q^{\pm}}\downarrow & \square & \downarrow{\scriptstyle \mathrm{pr}^{\pm}_{\widehat{U}}} \\
\widehat{M}^{\pm} & \hookrightarrow \mathbb{P}(V^{\pm})_{\widehat{U}} &
\end{array}
$$

Here $A^{\pm}$ are defined by the above Cartesian square. By Theorem 2.2, the above diagram induces the equivalence

$$
j^{\pm}_* \circ q^{\pm*} \colon D_{\mathbb{C}^*}(\widehat{M}^{\pm}, 0) \overset{\sim}{\to} D_{\mathbb{C}^*}(Y^{\pm}_{\widehat{U}}, w^{\pm}). \tag{3.21}
$$

Here $\mathbb{C}^*$ acts on $M^{\pm}$ trivially, and on $Y^{\pm}_{\widehat{U}}$ with weight 1 on fibers of $\mathrm{pr}^{\pm}_{\widehat{U}}$. Set

$$
B := \left\{ (\vec{x}, \vec{y}, \vec{u}) \in (\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}} : \sum_{i,j} x_i y_j w_{ij}(\vec{u}) = 0 \right\}.
$$

Similarly, we have the diagram

$$
\begin{array}{ccc}
E & \overset{\widetilde{j}}{\hookrightarrow} W_{\widehat{U}} & \overset{\widetilde{w}}{\longrightarrow} \mathbb{A}^1 \\
{\scriptstyle \widetilde{q}}\downarrow & \square & \downarrow{\scriptstyle \mathrm{pr}_{\widehat{U}}} \\
B & \hookrightarrow (\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}} &
\end{array}
$$

where the right vertical arrow is the projection and $E$ is defined by the above Cartesian square. Again by Theorem 2.2 and the description of $\widetilde{w}$ in (3.9), the above diagram induces the equivalence

$$
\widetilde{j}_* \circ \widetilde{q}^* \colon D_{\mathbb{C}^*}(B, 0) \overset{\sim}{\to} D_{\mathbb{C}^*}(W_{\widehat{U}}, \widetilde{w}). \tag{3.22}
$$

Here $\mathbb{C}^*$ acts on $B$ trivially, and on $W_{\widehat{U}}$ with weight 1 on fibers of $\mathrm{pr}_{\widehat{U}}$. Set

$$
F^+ := \left\{ (\vec{x}, \vec{y}, \vec{u}) \in (\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}} : \sum_{i=1}^{a} x_i w_{ij}(\vec{u}) = 0 \text{ for all } 1 \leq j \leq b \right\},
$$

$$
F^- := \left\{ (\vec{x}, \vec{y}, \vec{u}) \in (\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}} : \sum_{j=1}^{b} y_j w_{ij}(\vec{u}) = 0 \text{ for all } 1 \leq i \leq a \right\}.
$$

Note that $F^{\pm} \subset B$. Also the projections $(\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}} \to \mathbb{P}(V^{\pm})_{\widehat{U}}$ restricted to $F^{\pm}$ give morphisms $F^{\pm} \to \widehat{M}^{\pm}$, which are trivial $\mathbb{P}(V_{\mp})$-bundles. So we have the diagram

$$
\begin{array}{ccc}
F^{\pm} & \overset{k^{\pm}}{\hookrightarrow} & B \\
{\scriptstyle r^{\pm}}\downarrow & & \\
\widehat{M}^{\pm} & &
\end{array}
$$

Let $\Theta^{\pm}$ be the functor defined by

$$\Theta^{\pm} := k_*^{\pm} \circ r^{\pm *} \colon D_{\mathbb{C}^*}(\widehat{M}^{\pm}, 0) \to D_{\mathbb{C}^*}(B, 0).$$

**Lemma 3.4.** *The following diagram is commutative:*

$$
\begin{array}{ccc}
D_{\mathbb{C}^*}(\widehat{M}^{\pm}, 0) & \xrightarrow{\;\;\Theta^{\pm}\;\;} & D_{\mathbb{C}^*}(B, 0) \\
{\scriptstyle j_*^{\pm} \circ q^{\pm *}}\Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{j}_* \circ \widetilde{q}^*} \\
D_{\mathbb{C}^*}(Y_{\widehat{U}}^{\pm}, w^{\pm}) & \xrightarrow{\;\mathbf{L}p_{\widehat{U}}^{\pm *}\;} & D_{\mathbb{C}^*}(W_{\widehat{U}}, \widetilde{w})
\end{array}
\tag{3.23}
$$

*Here the vertical arrows are the equivalences* (3.21)*,* (3.22)*.*

*Proof.* Let $\widetilde{F}^{\pm} \subset W_{\widehat{U}}$ be defined by the Cartesian square

$$
\begin{array}{ccc}
\widetilde{F}^{\pm} & \xrightarrow{\;\;\widetilde{i}^{\pm}\;\;} & W_{\widehat{U}} \\
\Big\downarrow & \square & \Big\downarrow{\scriptstyle \mathrm{pr}_{\widehat{U}}} \\
F^{\pm} & \lhook\joinrel\longrightarrow & (\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}}
\end{array}
$$

Here the right vertical arrow is the projection. We have two diagrams



Since all Cartesians in the above diagrams are derived Cartesians, the base change shows that

$$\mathbf{L}p_{\widehat{U}}^{\pm *} \circ j_*^{\pm} \circ q^{\pm *} \cong \widetilde{i}_*^{\pm} \circ \widetilde{q}^{\pm *} \cong \widetilde{j}_* \circ \widetilde{q}^* \circ \Theta^{\pm}.$$

Therefore the lemma holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Lemma 3.5.** *The following diagram is commutative:*

$$
\begin{array}{ccc}
D_{\mathbb{C}^*}(\widehat{M}^{\pm}, 0) & \xleftarrow{\;\;\Theta_R^{\pm}\;\;} & D_{\mathbb{C}^*}(B, 0) \\
{\scriptstyle j_*^{\pm} \circ q^{\pm *}}\Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{j}_* \circ \widetilde{q}^*} \\
D_{\mathbb{C}^*}(Y_{\widehat{U}}^{\pm}, w^{\pm}) & \xleftarrow{\;\mathbf{R}p_{\widehat{U} *}^{\pm}\;} & D_{\mathbb{C}^*}(W_{\widehat{U}}, \widetilde{w})
\end{array}
\tag{3.24}
$$

Here $\Theta_R^{\pm}$ are the right adjoint functors of $\Theta$, i.e.

$$\Theta_R^{\pm} := \mathbf{R}r_*^{\pm} \circ (k^{\pm})^!,$$

where $(k^{\pm})^!$ are the right adjoint functors of $k_*^{\pm}$. They are written as

$$(k^+)^! = \otimes \mathcal{O}_{F^+}(b - 1, -1) \circ \mathbf{L}k^{+*}[1 - b], \tag{3.25}$$
$$(k^-)^! = \otimes \mathcal{O}_{F^-}(-1, a - 1) \circ \mathbf{L}k^{-*}[1 - a].$$

Here $\mathcal{O}_{F^{\pm}}(c, d) := \mathcal{O}_{(\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}}}(c, d)|_{F^{\pm}}$.

*Proof.* The commutativity of (3.24) follows from that of (3.23) together with the fact that $\mathbf{R}p_{\widehat{U}*}^{\pm}$, $\Theta_R^{\pm}$ are the right adjoint functors of $\mathbf{L}p_{\widehat{U}}^{\pm*}$, $\Theta^{\pm}$ respectively. As for the formula for $(k^{\pm})^!$, note that

$$(k^{\pm})^!(-) = \otimes \det N_{F^{\pm}/B} \circ \mathbf{L}k^{\pm*}(-) \circ [\dim F^{\pm} - \dim B].$$

By the exact sequences

$$0 \to N_{F^{\pm}/B} \to N_{F^{\pm}/(\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}}} \to N_{B/(\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}}} \to 0$$

we have

$$\det N_{F^+/B} = \mathcal{O}_{F^+}(b - 1, -1), \quad \det N_{F^-/B} = \mathcal{O}_{F^-}(a - 1, -1).$$

Together with the dimension computations

$$\dim F^+ = g + a - 2, \quad \dim F^- = g + b - 2, \quad \dim B = g + a + b - 3 \tag{3.26}$$

we obtain (3.25). $\qquad\square$

**Proposition 3.6.** *Suppose that*

$$\dim(\widehat{M}^+ \times_{\widehat{U}} \widehat{M}^-) \leq g - 1. \tag{3.27}$$

*Then the following diagram is commutative:*

$$
\begin{array}{ccc}
D_{\mathbb{C}^*}(\widehat{M}^-, 0) & \xrightarrow{\Theta_{\widehat{M}}} & D_{\mathbb{C}^*}(\widehat{M}^+, 0) \\
{\scriptstyle j_*^- \circ q^{-*}} \downarrow & & \downarrow {\scriptstyle j_*^+ \circ q^{+*}} \\
D_{\mathbb{C}^*}(Y_{\widehat{U}}^-, w^{\pm}) & \xrightarrow{\Psi_Y} & D_{\mathbb{C}^*}(Y_{\widehat{U}}^+, \widetilde{w})
\end{array}
$$

*Here the vertical arrows are the equivalences* (3.21), $\Psi_Y$ *is given by* (3.14) *and* $\Theta_{\widehat{M}}$ *is defined by*

$$\Theta_{\widehat{M}} := \otimes \mathcal{O}_{\widehat{M}^+}(b - 1) \circ \Psi^{\Xi(\mathcal{O}_{\widehat{M}^+ \times_{\widehat{U}} \widehat{M}^-})} \circ \otimes \mathcal{O}_{\widehat{M}^-}(-1)[1 - b]$$

*where* $\Xi$ *is the equivalence in* (2.5)*:*

$$\Xi : D^b(\widehat{M}^+ \times_{\widehat{U}} \widehat{M}^-) \xrightarrow{\sim} D_{\mathbb{C}^*}(\widehat{M}^+ \times_{\widehat{U}} \widehat{M}^-, 0).$$

*Proof.* By Lemmas 3.4 and 3.5, it is enough to check that $\Theta_R^+ \circ \Theta^-$ is isomorphic to $\Theta_{\widehat{M}}$. By setting $\mathcal{O}_B(c, d) := \mathcal{O}_{(\mathbb{P}(V^+) \times \mathbb{P}(V^-))_{\widehat{U}}}(c, d)|_B$, and using (3.25), we have

$$
\begin{aligned}
\Theta_R^+ \circ \Theta^-(-) &= \mathbf{R}r_*^+ \big(\mathbf{L}k^{+*}k_*^- r^{-*}(-) \otimes \mathcal{O}_{F^+}(b - 1, -1)\big)[1 - b] \\
&= \mathbf{R}r_*^+ \mathbf{L}k^{+*} \big(k_*^- r^{-*}(-) \otimes \mathcal{O}_B(0, -1)\big) \otimes \mathcal{O}_{\widehat{M}^+}(b - 1)[1 - b] \\
&= \mathbf{R}r_*^+ \mathbf{L}k^{+*} k_*^- r^{-*}(- \otimes \mathcal{O}_{\widehat{M}^-}(-1)) \otimes \mathcal{O}_{\widehat{M}^+}(b - 1)[1 - b].
\end{aligned}
$$

Let us consider the composition $\mathbf{L}k^{+*} \circ k_*^-$ in the above formula. We have the Cartesian diagram

$$
\begin{array}{ccc}
F^+ \cap F^- & \overset{\theta^+}{\hookrightarrow} & F^+ \\
{\scriptstyle\theta^-}\big\downarrow & \square & \big\downarrow{\scriptstyle k^+} \\
F^- & \underset{k^-}{\hookrightarrow} & B
\end{array}
\tag{3.28}
$$

By the definition of $F^\pm$, we have

$$
F^+ \cap F^- = \widehat{M}^+ \times_{\widehat{U}} \widehat{M}^-.
\tag{3.29}
$$

Since $F^+ \subset B$ is of codimension $b - 1$ and $F^- \subset B$ is of codimension $a - 1$, we have

$$
\dim(F^+ \cap F^-) \geq \dim B - (b - 1) - (a - 1) = g - 1.
$$

Then the assumption (3.27) on the dimension of the fiber product implies $\dim(F^+ \cap F^-) = g - 1$ and (3.28) is a derived Cartesian diagram. Therefore by base change,

$$
\mathbf{L}k^{+*} \circ k_*^- \cong \theta_*^+ \circ \mathbf{L}\theta^{-*}.
$$

By substituting into the above formula for $\Theta_R^+ \circ \Theta^-$, and again noting (3.29), we have

$$
\begin{aligned}
\Theta_R^+ \circ \Theta^-(-) &= \mathbf{R}r_*^+ \theta_*^+ \mathbf{L}\theta^{-*} r^{-*}(- \otimes \mathcal{O}_{\widehat{M}^-}(-1)) \otimes \mathcal{O}_{\widehat{M}^+}(b - 1)[1 - b] \\
&= \Psi^{\Xi(\mathcal{O}_{\widehat{M}^+ \times_{\widehat{U}} \widehat{M}^-})}(- \otimes \mathcal{O}_{\widehat{M}^-}(-1)) \otimes \mathcal{O}_{\widehat{M}^+}(b - 1)[1 - b] = \Theta_{\widehat{M}}(-).
\end{aligned}
$$

Therefore the proposition holds. $\qquad\qquad\square$

**Lemma 3.7.** *The following diagram is commutative:*

$$
\begin{array}{ccc}
D_{\mathbb{C}^*}(\widehat{U}, 0) & \overset{\overline{\Upsilon}_{\widehat{M}}^{i+b}[-b]}{\longrightarrow} & D_{\mathbb{C}^*}(\widehat{M}^+, 0) \\
\big\| & & \big\downarrow{\scriptstyle j_*^+ \circ q^{+*}} \\
D_{\mathbb{C}^*}(\widehat{U}, 0) & \underset{\Upsilon_Y^i}{\longrightarrow} & D_{\mathbb{C}^*}(Y_{\widehat{U}}^+, w^+)
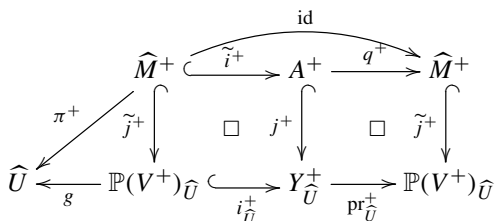\end{array}
\tag{3.30}
$$

*Here $\Upsilon_Y^i$ is defined in Theorem 3.3(iii), and $\overline{\Upsilon}_{\widehat{M}}^i$ is defined by*

$$
\overline{\Upsilon}_{\widehat{M}}^i := \otimes \mathcal{O}_{\widehat{M}^+}(i) \circ \mathbf{L}\pi^{+*} \colon D_{\mathbb{C}^*}(\widehat{U}, 0) \to D_{\mathbb{C}^*}(\widehat{M}^+, 0).
\tag{3.31}
$$

*Proof.* The inverse of the equivalence of the right vertical arrow in (3.30) is given by $\mathbf{R}q_*^+ \circ j^{+!}$. Therefore it is enough to check that

$$\mathbf{R}q_*^+ \circ j^{+!} \circ \otimes \mathcal{O}_{Y_{\widehat{U}}^+}(i) \circ i_{\widehat{U}*}^+ \circ g^* \cong \overline{\Upsilon}_{\widehat{M}}^{i+b}[-b]. \tag{3.32}$$

We use the commutative diagram



Since the left Cartesian diagram above is a derived Cartesian, by base change we have $\mathbf{L}j^{+*} \circ i_{\widehat{U}*}^+ \cong \widetilde{i}_*^+ \circ \mathbf{L}\widetilde{j}^{+*}$. Together with $\mathcal{O}_{Y_{\widehat{U}}^+}(1)|_{A^+} = q^{+*}\mathcal{O}_{\widehat{M}^+}(1)$, we have

$$\mathbf{R}q_*^+ \circ j^{+!} \circ \otimes \mathcal{O}_{Y_{\widehat{U}}^+}(i) \circ i_{\widehat{U}*}^+ \circ g^*$$

$$\cong \mathbf{R}q_*^+ \circ \otimes \mathcal{O}_{Y_{\widehat{U}}^+}(b)|_{A^+} \circ \mathbf{L}j^{+*} \circ \otimes \mathcal{O}_{Y_{\widehat{U}}^+}(i) \circ i_{\widehat{U}*}^+ \circ g^*[-b]$$

$$\cong \otimes \mathcal{O}_{\widehat{M}^+}(b+i) \circ \mathbf{R}q_*^+ \circ \mathbf{L}j^{+*} \circ i_{\widehat{U}*}^+ \circ g^*[-b]$$

$$\cong \otimes \mathcal{O}_{\widehat{M}^+}(b+i) \circ \mathbf{R}q_*^+ \circ \widetilde{i}_*^+ \circ \mathbf{L}\widetilde{j}^{+*} \circ g^*[-b] \cong \otimes \mathcal{O}_{\widehat{M}^+}(b+i) \circ \mathbf{L}\pi^{+*}[-b]$$

as expected. $\qquad \square$

By putting all the arguments in this subsection together, we have the following:

**Proposition 3.8.** *In the setting of Subsection* 3.2, *suppose that the assumptions of Lemma* 3.1 *and the dimension condition* (3.27) *hold. Then we have the following:*

(i) *The functor*

$$\Phi_{\widehat{M}} := \Phi^{\mathcal{O}_{\widehat{M}^- \times_{\widehat{U}} \widehat{M}^+}} : D^b(\widehat{M}^-) \to D^b(\widehat{M}^+)$$

*is fully faithful.*

(ii) *If $n \geq 1$, the functor*

$$\Upsilon_{\widehat{M}}^i := \otimes \mathcal{O}_{\widehat{M}^+}(i) \circ \mathbf{L}\pi^{+*} : D^b(\widehat{U}) \to D^b(\widehat{M}^+)$$

*is fully faithful.*

(iii) *We have the SOD*

$$D^b(\widehat{M}^+) = \langle \operatorname{Im} \Upsilon_{\widehat{M}}^{-n+1}, \ldots, \operatorname{Im} \Upsilon_{\widehat{M}}^0, \operatorname{Im} \Phi_{\widehat{M}} \rangle.$$

*Proof.* By Theorem 3.3(i) and Proposition 3.6, the functor

$$\Psi^{\Xi(\mathcal{O}_{\widehat{M}^- \times_{\widehat{U}} \widehat{M}^+})} : D_{\mathbb{C}^*}(\widehat{M}^-, 0) \to D_{\mathbb{C}^*}(\widehat{M}^+, 0)$$

is fully faithful. Therefore (i) follows by the commutative diagram (2.6). Similarly, (ii) follows from Theorem 3.3(ii), Lemma 3.7 and the commutative diagram (2.6). As for (iii), by Theorem 3.3(iii), Proposition 3.6 and Lemma 3.7 we have the SOD

$$D^b(\widehat{M}^+) = \langle \operatorname{Im} \Upsilon_{\widehat{M}}^{b-n}, \ldots, \operatorname{Im} \Upsilon_{\widehat{M}}^{b-1}, \otimes \mathcal{O}_{\widehat{M}^+}(b-1) \circ \operatorname{Im} \Phi_{\widehat{M}} \rangle.$$

By tensoring with $\mathcal{O}_{\widehat{M}^+}(1-b)$, we obtain the desired SOD. □

## 4. SOD via d-critical simple flips

In this section, we show that for a d-critical simple flip $M^+ \to U \leftarrow M^-$ satisfying some conditions, we have an associated SOD of $D^b(M^+)$. The SOD in this section is obtained by globalizing the SOD in Proposition 3.8.

### 4.1. D-critical simple flips

Let $U$ be a smooth variety with $g := \dim U$. Let $(M^{\pm}, s^{\pm})$ be two d-critical loci, and suppose that we have projective morphisms

$$
\begin{array}{ccc}
M^+ & & M^- \\
& \searrow_{\pi^+} \quad \swarrow_{\pi^-} & \\
& U &
\end{array}
\tag{4.1}
$$

For each $p \in U$, we set

$$\widehat{U}_p := \operatorname{Spec} \widehat{\mathcal{O}}_{U,p}, \qquad \widehat{M}_p^{\pm} := M^{\pm} \times_U \widehat{U}_p. \tag{4.2}$$

**Definition 4.1.** A diagram (4.1) is called a *formal d-critical simple flip* if for any $p \in U$, there exist finite-dimensional vector spaces $V^{\pm}$ with $\dim V^+ \geq \dim V^-$ such that, with $Y^{\pm}, Z$ defined as in (3.2), and

$$\widehat{Z}_U := \operatorname{Spec} \widehat{\mathcal{O}}_{Z_U,(0,p)}, \qquad \widehat{Y}_U^{\pm} := Y_U^{\pm} \times_{Z_U} \widehat{Z}_U, \tag{4.3}$$

there exist $\widehat{w} \in \mathcal{O}_{\widehat{Z}_U}$ and a commutative diagram

$$
\begin{array}{ccc}
\widehat{M}_p^{\pm} & \xhookrightarrow{\iota^{\pm}} & \widehat{Y}_U^{\pm} \\
{\scriptstyle \pi^{\pm}} \downarrow & {\scriptstyle \widehat{f}_U^{\pm}} \downarrow & \searrow {\scriptstyle \widehat{w}^{\pm}} \\
\widehat{U}_p & \xhookrightarrow{j} \widehat{Z}_U & \xrightarrow{\widehat{w}} \mathbb{A}^1
\end{array}
\tag{4.4}
$$

where the horizontal arrows are closed immersions, $\widehat{w}^{\pm}$ are defined by the above commutative diagram, $j$ sends $p$ to $(0, p)$ and $\iota^{\pm}$ induce the isomorphisms of d-critical loci

$$\iota^{\pm} \colon \widehat{M}_p^{\pm} \xrightarrow{\cong} \{d\widehat{w}^{\pm} = 0\} \subset \widehat{Y}_U^{\pm}. \tag{4.5}$$

For a formal d-critical simple flip (4.1) and $p \in U$, let $V^{\pm}$ be vector spaces as in Definition 4.1. Below we use the notation in Subsection 3.1, e.g. $a = \dim V^+$, $b = \dim V^-$, $n := a - b \geq 0$, the coordinates $\vec{x}, \vec{y}$ of $V^+, V^-$, etc. Note that $(a, b)$ may depend on the choice of $p \in U$. We assume the following on the diagram (4.1):

**Assumption 4.2.**   (i)  The diagram (4.1) is a formal d-critical simple flip.
 (ii)  For any $p \in U$, the formal function $\widehat{w}$ in (4.4) is of the form

$$\widehat{w} = \sum_{i,j} x_i y_j w_{ij}^{(1)}(\vec{u}) + \sum_{i,i',j,j'} x_i x_{i'} y_j y_{j'} w_{ii'jj'}^{(2)}(\vec{u}) + \cdots \qquad (4.6)$$

for some $w_*^{(k)}(\vec{u}) \in \widehat{\mathcal{O}}_{U,p}$, and $w_{ij}^{(1)}(\vec{u})$ is written as

$$w_{ij}^{(1)}(\vec{u}) = \sum_{k=1}^{g} a_{ijk} u_k + \text{(higher order terms in } \vec{u}) \qquad (4.7)$$

for some $a_{ijk} \in \mathbb{C}$. Moreover the bilinear map

$$\psi : \mathbb{C}^a \otimes \mathbb{C}^b \to \mathbb{C}^g, \quad \psi(\vec{\alpha}, \vec{\beta}) = \left( \sum_{i,j} a_{ijk} \alpha_i \beta_j \right)_{1 \leq k \leq g}, \qquad (4.8)$$

is injective on each factor, i.e. for any non-zero $\vec{\alpha} \in \mathbb{C}^a$ and $\vec{\beta} \in \mathbb{C}^b$, the maps $\psi(\vec{\alpha}, -)$, $\psi(-, \vec{\beta})$ are injective maps $\mathbb{C}^b \to \mathbb{C}^g$, $\mathbb{C}^a \to \mathbb{C}^g$.
 (iii)  There exists a $\pi^+$-ample line bundle $\mathcal{O}_{M^+}(1)$ on $M^+$ such that under the isomorphism (4.5), we have an isomorphism of line bundles

$$(\iota^+)^* \mathcal{O}_{\widehat{Y}_U^+}(1) \cong \mathcal{O}_{M^+}(1)|_{\widehat{M}_p^+}. \qquad (4.9)$$

**Lemma 4.3.** *Suppose that a diagram* (4.1) *satisfies Assumption* 4.2(i, ii). *Then for any* $p \in U$, *the critical loci* $\{d\widehat{w}^{\pm} = 0\} \subset \widehat{Y}_U^{\pm}$ *in the diagram* (4.4) *are written as*

$$\{d\widehat{w}^+ = 0\} = \left\{ (\vec{x}, \vec{u}) \in \mathbb{P}(V^+)_{\widehat{U}_p} : \sum_{i=1}^{a} x_i w_{ij}^{(1)}(\vec{u}) = 0 \text{ for all } 1 \leq j \leq b \right\},$$
$$\{d\widehat{w}^- = 0\} = \left\{ (\vec{y}, \vec{u}) \in \mathbb{P}(V^-)_{\widehat{U}_p} : \sum_{j=1}^{b} y_j w_{ij}^{(1)}(\vec{u}) = 0 \text{ for all } 1 \leq i \leq a \right\}. \qquad (4.10)$$

*Moreover* $n = a - b \geq 0$ *is independent of* $p \in U$, $M^{\pm}$ *are smooth and*

$$\dim M^{\pm} = \pm n + g - 1.$$

*Proof.* For $p \in U$, let us consider the diagram (4.4). The subscheme $\{d\widehat{w}^+ = 0\} \subset \widehat{Y}_U^+$ is contained in the closed subscheme of $\widehat{Y}_U^+$ defined by the equations

$$\frac{\partial \widehat{w}^+(\vec{u})}{\partial u_k} = \sum_{i,j} x_i y_j \frac{\partial w_{ij}^{(1)}(\vec{u})}{\partial u_k} + \sum_{i,i',j,j'} x_i x_{i'} y_j y_{j'} \frac{\partial w_{ii'jj'}^{(2)}(\vec{u})}{\partial u_k} + \cdots = 0 \qquad (4.11)$$

for all $1 \le k \le g$. Note that

$$\frac{\partial w_{ij}^{(1)}(\vec{u})}{\partial u_k} = a_{ijk} + O(\vec{u}).$$

Then by the assumption on the map (4.8), the subscheme

$$\left\{ \sum_{i,j} x_i y_j \frac{\partial w_{ij}^{(1)}(\vec{u})}{\partial u_k} = 0 : 1 \le k \le g \right\} \subset (V^{+*} \times V^-)_{\widehat{U}_p}$$

coincides with $V^{+*} \times \{0\} \times \widehat{U}_p$. Since the higher order terms in (4.11) have degrees greater than or equal to 2 in $\vec{y}$, by the Nakayama lemma the zero locus defined by (4.11) equals $\vec{y} = 0$ on $\widehat{Y}_U^+$, i.e. the zero section $\mathbb{P}(V^+)_{\widehat{U}_p} \subset \widehat{Y}_U^+$. Therefore $\{d\widehat{w}^+ = 0\} \subset \widehat{Y}_U^+$ is as in (4.10).

Let $g_j$ for $1 \le j \le b$ be the defining equations in the RHS of (4.10). Again the property of the map (4.8) implies that the Jacobian matrix

$$\left( \frac{\partial g_j}{\partial x_i}, \frac{\partial g_j}{\partial u_k} \right)_{1 \le i \le a, 1 \le j \le b, 1 \le k \le g}$$

is of maximal rank $b$ at any point in the RHS of (4.10). Therefore $\{d\widehat{w}^+ = 0\}$ is smooth of dimension $a - 1 + g - b = n + g - 1$. By the isomorphism (4.5), $\widehat{M}_p^+$ is smooth of dimension of $n + g - 1$ for any $p \in U$, hence $M^+$ is smooth of dimension $n + g - 1$. The claim for $M^-$ is similarly proved. $\qquad\square$

**Lemma 4.4.** *In the situation of Lemma* 4.3, *we have*

$$\dim(M^- \times_U M^+) \le g - 1. \tag{4.12}$$

*Proof.* Let us take $p \in U$, and vector spaces $V^\pm$ as in Definition 4.1 with $a = \dim V^+$, $b = \dim V^-$ as before. For each $k \ge 0$, let $U^{(k)}$ be the locally closed subset $U$ defined by

$$U^{(k)} := \{x \in U : \dim (\pi^-)^{-1}(x) = k - 1\}.$$

Then $p \in U^{(b)}$ as $(\pi^-)^{-1}(p) = \mathbb{P}(V^-)$, and the descriptions of $\{d\widehat{w}^\pm = 0\}$ in (4.10) and the isomorphisms (4.5) show that

$$U^{(b)} \cap \widehat{U}_p = \mathrm{Spec}\big(\widehat{\mathcal{O}}_{U,p}/(w_{ij}^{(1)}(\vec{u}) : 1 \le i \le a, \ 1 \le j \le b)\big).$$

It follows from the description of $w_{ij}^{(1)}(\vec{u})$ in (4.7) that the tangent space of $U^{(b)}$ at $p$ is

$$TU^{(b)}|_p = \left\{ (u_1, \ldots, u_g) \in \mathbb{C}^g : \sum_{k=1}^g a_{ijk} u_k = 0, \ 1 \le i \le a, 1 \le j \le b \right\}.$$

Therefore the dimension of $TU^{(b)}|_p$ is the dimension of the cokernel of $\psi$ in (4.8). By the assumption on the map (4.8), the Hopf lemma (see [Gin, Lemma 2]) implies that $\dim \mathrm{Cok}(\psi) \le g - a - b + 1$. Therefore

$$\dim U^{(b)} \le g - a - b + 1.$$

It follows that

$$\dim\left((\pi^+)^{-1}(U^{(b)}) \times_{U^{(b)}} (\pi^-)^{-1}(U^{(b)})\right)$$
$$\leq (a-1) + (b-1) + (g-a-b+1) = g-1.$$

By applying the above argument for all $p \in U$, we see that for any $k \geq 0$ we have

$$\dim\left((\pi^+)^{-1}(U^{(k)}) \times_{U^{(k)}} (\pi^-)^{-1}(U^{(k)})\right) \leq g-1.$$

Since $U$ is a disjoint union of strata $U^{(k)}$, the condition (4.12) holds.          □

### 4.2. SOD under d-critical simple flips

The following is the main result in this section.

**Theorem 4.5.** *Suppose that the diagram* (4.1) *satisfies Assumption* 4.2, *so that $M^\pm$ are smooth of dimension $\pm n + g - 1$ for some $n \in \mathbb{Z}_{\geq 0}$ by Lemma* 4.3. *Then:*

(i) *The functor*

$$\Phi_M := \Phi^{\mathcal{O}_{M^- \times_U M^+}} : D^b(M^-) \to D^b(M^+)$$

*is fully faithful.*
(ii) *If $n \geq 1$, the functor*

$$\Upsilon_M^i := \otimes \mathcal{O}_{M^+}(i) \circ \mathbf{L}\pi^{+*} : D^b(U) \to D^b(M^+)$$

*is fully faithful.*
(iii) *We have the SOD*

$$D^b(M^+) = \langle \operatorname{Im} \Upsilon_M^{-n+1}, \ldots, \operatorname{Im} \Upsilon_M^0, \operatorname{Im} \Phi_M \rangle. \tag{4.13}$$

The proof of Theorem 4.5 is in three steps.

**Step 1.** *For each $p \in U$, we may assume that the formal function* (4.6) *satisfies $w_*^{(k)}(\vec{u}) = 0$ for $k \geq 2$.*

*Proof.* In the notation of Assumption 4.2(ii), let $w \in \mathcal{O}_Z \otimes \widehat{\mathcal{O}}_{U,p}$ be defined by

$$w = \sum_{i,j} x_i y_j w_{ij}^{(1)}(\vec{u}).$$

We set $w^\pm \colon Y_{\widehat{U}_p}^\pm \to \mathbb{A}^1$ as in the diagram (3.8) for $\widehat{U} = \widehat{U}_p$. Then the argument used for Lemma 4.3 shows that $\{dw^\pm = 0\} \subset Y_{\widehat{U}_p}^\pm$ are described as in the RHS of (4.10) and the isomorphisms (4.5) give

$$\iota^\pm \colon \widehat{M}_p \xrightarrow{\ \widehat{\cong}\ } \{dw^\pm = 0\} \subset Y_{\widehat{U}_p}^\pm.$$

Therefore we may replace $\widehat{w}$ with $w$ and assume that $w_*^{(k)}(\vec{u}) = 0$ for $k \geq 2$.          □

**Step 2.** *Item* (i) *and* (ii) *of Theorem* 4.5 *hold.*

*Proof.* Let $\Phi_{M,R}$ be the right adjoint functor of $\Phi_M$, and let $\mathcal{P} \in D^b(M^- \times M^-)$ be the kernel object for the composition functor

$$\Phi_{M,R} \circ \Phi_M \colon D^b(M^-) \to D^b(M^+) \to D^b(M^-).$$

Then there is a canonical morphism

$$\mathcal{O}_{\Delta_{M^-}} \to \mathcal{P} \tag{4.14}$$

corresponding to the adjunction $\mathrm{id}_{M^-} \to \Phi_{M,R} \circ \Phi_M$. Let $\mathcal{Q}$ be the cone of the morphism (4.14). In order to show that $\Phi_M$ is fully faithful, it is enough to show that $\mathcal{Q} = 0$. Indeed, if this is the case, then the adjunction $\mathrm{id}_{M^-} \to \Phi_{M,R} \circ \Phi_M$ is an isomorphism hence $\Phi_M$ is fully faithful.

Note that $\mathcal{Q}$ is supported on the fiber product $M^- \times_U M^-$ by construction. Since $\widehat{U}_p \to U$ is faithfully-flat, the vanishing $\mathcal{Q} = 0$ is equivalent to

$$\mathcal{Q} \otimes_{\mathcal{O}_{M^- \times M^-}} \mathcal{O}_{\widehat{M}_p^- \times \widehat{M}_p^-} = 0 \tag{4.15}$$

for all $p \in U$.

Now by Step 1 and Lemmas 4.3 and 4.4, the diagram

$$\widehat{M}_p^+ \to \widehat{U}_p \leftarrow \widehat{M}_p^-$$

satisfies the assumptions in Proposition 3.8. Then Proposition 3.8(i) shows that the morphism (4.14) is an isomorphism after pulling it back by $\widehat{M}_p^- \times \widehat{M}_p^- \to M^- \times M^-$. Therefore (4.15) holds for any $p \in U$, and Theorem 4.5(i) is proved. The proof of (ii) is similar.                                                                                                    □

**Step 3.** *Theorem* 4.5(iii) *holds.*

*Proof.* We first show the semiorthogonality of the RHS of (4.13), i.e. the vanishings

$$\mathrm{Hom}(\mathrm{Im}\,\Phi_M, \mathrm{Im}\,\Upsilon_M^i) = 0, \quad \mathrm{Hom}(\mathrm{Im}\,\Upsilon_M^i, \mathrm{Im}\,\Upsilon_M^j) = 0,$$

for $i < j$. It is enough to check that

$$\Phi_{M,R} \circ \Upsilon_M^i = 0, \quad \Upsilon_{M,R}^i \circ \Upsilon_M^j = 0$$

where $\Phi_{M,R}$, $\Upsilon_{M,R}^i$ are the right adjoint functors of $\Phi_M$, $\Upsilon_M^i$ respectively. Again it is enough to check these vanishings formally locally at every $p \in U$, and Proposition 3.8(iii) implies that they indeed hold.

Let $E \in D^b(M^+)$ be an object in the right orthogonal complement of the RHS of (4.13). Then Proposition 3.8(iii) implies that $E = 0$ on $\widehat{M}_p^+$ for any $p \in U$. Therefore $E = 0$, and the RHS of (4.13) generates the LHS.                                                                               □

## 5. SOD for stable pair moduli spaces

In this section, we apply Theorem 4.5 to prove Theorem 1.1, i.e. the existence of certain SOD on moduli spaces of Pandharipande–Thomas stable pairs on CY 3-folds.

### 5.1. Stable pairs and stable sheaves

Let $X$ be a smooth quasi-projective variety. By definition, a *stable pair* in the sense of Pandharipande–Thomas [PT09] consists of data

$$(F, s), \quad s \colon \mathcal{O}_X \to F,$$

where $F$ is a pure one-dimensional coherent sheaf on $X$ with compact support, and $s$ is surjective in dimension 1. For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the moduli space of stable pairs $(F, s)$ satisfying the condition

$$[F] = \beta, \quad \chi(F) = n \tag{5.1}$$

is denoted by $P_n(X, \beta)$. Here $[F]$ is the homology class of the fundamental one-cycle associated with $F$. The moduli space $P_n(X, \beta)$ is a quasi-projective scheme (see [PT09]). We define the open subscheme

$$P_n^\circ(X, \beta) \subset P_n(X, \beta)$$

to consist of stable pairs $(F, s)$ such that the fundamental one-cycle associated with $F$ is irreducible.

We denote by $U_n(X, \beta)$ the moduli space of compactly supported one-dimensional Gieseker stable sheaves $F$ on $X$ with respect to a fixed polarization, satisfying the condition (5.1). The moduli space $U_n(X, \beta)$ is a quasi-projective scheme (see [HL97]). We define the open subscheme

$$U_n^\circ(X, \beta) \subset U_n(X, \beta)$$

consisting of one-dimensional stable sheaves whose fundamental one-cycles are irreducible. Note that $U_n^\circ(X, \beta)$ is the moduli space of pure one-dimensional sheaves $F$ with irreducible fundamental one-cycles satisfying (5.1). In particular, $U_n^\circ(X, \beta)$ is independent of the choice of a polarization. Below we always assume that $U_n^\circ(X, \beta)$ is a *fine* moduli space, i.e. it admits a universal sheaf

$$\mathcal{F} \in \mathrm{Coh}(X \times U_n^\circ(X, \beta)). \tag{5.2}$$

**Remark 5.1.** Alternatively, $U_n^\circ(X, \beta)$ parametrizes pairs $(C, F)$ where $C \subset X$ is an irreducible projective curve with $[F] = \beta$, and $F \in \mathrm{Coh}(C)$ is a rank-one torsion free sheaf satisfying $\chi(F) = n$.

**Remark 5.2.** The existence of the universal sheaf (5.2) is guaranteed if $(D \cdot \beta, n)$ is coprime for some divisor $D$ on $X$. See [HL97, Corollary 4.6.6].

### 5.2. Wall-crossing diagram of stable pair moduli spaces

Suppose that $X$ is a smooth projective CY 3-fold, i.e.

$$\dim X = 3, \quad K_X = 0.$$

Let us take $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}_{\geq 0}$. Then as in [PT10], we have the diagram

$$
\begin{array}{ccc}
P_n^\circ(X, \beta) & & P_{-n}^\circ(X, \beta) \\
& \searrow{\scriptstyle \pi^+} \quad {\scriptstyle \pi^-}\swarrow & \\
& U_n^\circ(X, \beta) &
\end{array}
\tag{5.3}
$$

Here $\pi^{\pm}$ are defined by

$$\pi^+(F, s) = F, \quad \pi^-(F', s') = \mathcal{E}xt_X^2(F', \mathcal{O}_X).$$

If furthermore $H^1(\mathcal{O}_X) = 0$, then the diagram (5.3) gives an example of an analytic (in particular formal) d-critical simple flip (see [Tod, Theorem 6.18]). Here we recall some more details.

Let us take a point $p \in U_n^\circ(X, \beta)$ corresponding to a pure one-dimensional sheaf $F$ on $X$. We write

$$\widehat{U}_n(X, \beta)_p := \operatorname{Spec} \widehat{\mathcal{O}}_{U_n(X, \beta), p}, \qquad \widehat{P}_n(X, \beta)_p := P_n^\circ(X, \beta) \times_{U_n^\circ(X, \beta)} \widehat{U}_n(X, \beta)_p.$$

We take a collection of objects in $D^b(X)$

$$E_\bullet = (E_1, E_2), \quad E_1 = \mathcal{O}_X, \quad E_2 = F[-1]. \tag{5.4}$$

We define vector spaces $V^+$, $V^-$ and $U$ as follows:

$$
\begin{aligned}
V^+ &:= \operatorname{Ext}_X^1(E_1, E_2) = H^0(F), \\
V^- &:= \operatorname{Ext}_X^1(E_2, E_1) = H^1(F)^\vee, \\
U &:= \operatorname{Ext}_X^1(E_2, E_2) = \operatorname{Ext}_X^1(F, F).
\end{aligned}
\tag{5.5}
$$

Below we use the notation and conventions of Subsections 3.1 and 3.2, e.g. $\mathbb{C}^*$-actions on $V^\pm$, the GIT quotients $Y^\pm$, $Z$, coordinates $\vec{x}$, $\vec{y}$, $\vec{u}$ on $V^\pm$, $U$, $a = \dim V^+$, $b = \dim V^-$, $g = \dim U$, etc. We also take the formal completion $\widehat{Z}_U$ of $Z_U$ at $(0, 0)$, and define $\widehat{f}_U^\pm \colon \widehat{Y}_U^\pm \to \widehat{Z}_U$ as in (4.3). The following result is obtained in [Tod]:

**Theorem 5.3** ([Tod, Theorem 6.18]). *In the above situation, there exist an element $\widehat{w} \in \widehat{\mathcal{O}}_{Z_U, (0,0)}$ and the commutative diagram*

$$
\begin{array}{ccccccc}
\widehat{P}_n(X, \beta)_p & \xrightarrow[\iota^\pm]{\cong} & \{d\widehat{w}^\pm = 0\} & \lhook\joinrel\longrightarrow & \widehat{Y}_U^\pm & & \\
{\scriptstyle \pi^+}\downarrow & & \downarrow & & {\scriptstyle \widehat{f}_U^\pm}\downarrow & \searrow{\scriptstyle \widehat{w}^\pm} & \\
\widehat{U}_n(X, \beta)_p & \xrightarrow{\cong} & \{dw^{(0)} = 0\} & \underset{j}{\lhook\joinrel\longrightarrow} & \widehat{Z}_U & \xrightarrow{\widehat{w}} & \mathbb{A}^1
\end{array}
\tag{5.6}
$$

*Here $\widehat{w}^\pm$ are defined by the above commutative diagram, the bottom left arrow sends $p$ to $(0, 0)$, and the map $j$ is the composition of the inclusion $\{dw^{(0)} = 0\} \subset \widehat{U}$ with the inclusion $\widehat{U} \hookrightarrow \widehat{Z}_U$ given by $u \mapsto (0, u)$.*

**Remark 5.4.** In [Tod14b, Theorem 6.18], it is stated that we can take $\widehat{w}$ as an analytic function on an analytic open neighborhood of $0 \in Z_U$, and the diagram (5.6) can be extended to analytic neighborhoods of $0 \in Z_U$ and $p \in U_n^\circ(X, \beta)$. The formal version in Theorem 5.3 is weaker than the analytic version in [Tod14b, Theorem 6.18], but enough for the purpose of this paper.

Let us write the formal function $\widehat{w}$ in Theorem 5.3 as

$$\widehat{w} = w^{(0)}(\vec{u}) + \sum_{i,j} x_i y_j w_{ij}^{(1)}(\vec{u}) + \sum_{i,i',j,j'} x_i x_{i'} y_j y_{j'} w_{ii'jj'}^{(2)}(\vec{u}) + \cdots \qquad (5.7)$$

for $w_*^{(k)}(\vec{u}) \in \widehat{\mathcal{O}}_{U,0}$. The function (5.7) is constructed using the minimal cyclic $A_\infty$-structure on the subcategory of $D^b(X)$ generated by $E_1$ and $E_2$ (see [Tod, Subsection 5.1]). In particular, the linear term of $w_{ij}^{(1)}(\vec{u})$ is given as follows. Let us consider the triple product

$$\operatorname{Ext}_X^1(E_2, E_2) \otimes \operatorname{Ext}_X^1(E_1, E_2) \otimes \operatorname{Ext}_X^1(E_2, E_1) \to \operatorname{Ext}_X^3(E_2, E_2) \cong \mathbb{C} \qquad (5.8)$$

given by composition, where the last isomorphism is given by the Serre duality. For $1 \le i \le a$, $1 \le j \le b$ and $1 \le k \le g$, let

$$x_i^\vee \in \operatorname{Ext}_X^1(E_1, E_2), \quad y_j^\vee \in \operatorname{Ext}_X^1(E_2, E_1), \quad u_k^\vee \in \operatorname{Ext}_X^1(E_2, E_2)$$

be the dual basis of $x_i$, $y_j$, $u_k$ respectively. Then using the triple product (5.8), we have

$$w_{ij}^{(1)}(\vec{u}) = \frac{1}{2} \sum_{k=1}^{g} (x_i^\vee \cdot y_j^\vee \cdot u_k^\vee) u_k + \text{(higher order terms in } \vec{u}). \qquad (5.9)$$

We show that $w_{ij}^{(1)}(\vec{u})$ satisfies Assumption 4.2(ii):

**Lemma 5.5.** *The map*

$$\operatorname{Ext}_X^1(E_1, E_2) \otimes \operatorname{Ext}_X^1(E_2, E_1) \to \operatorname{Ext}_X^2(E_2, E_2) \qquad (5.10)$$

*given by composition is injective on each factor.*

*Proof.* Recall that $E_1$, $E_2$ are as in (5.4), i.e. $E_1 = \mathcal{O}_X$ and $E_2 = F[-1]$ for a pure one-dimensional sheaf $F$ on $X$ with irreducible fundamental one-cycle. So $F$ can be written as $j_* E$ where $j \colon C \hookrightarrow X$ is an irreducible Cohen–Macaulay curve and $E$ is a rank-one torsion free sheaf on $C$. Therefore the map (5.10) is

$$H^0(C, E) \otimes \operatorname{Ext}_X^2(j_* E, \mathcal{O}_X) \to \operatorname{Ext}_X^2(j_* E, j_* E). \qquad (5.11)$$

Note that

$$\operatorname{Ext}_X^2(j_* E, \mathcal{O}_X) = \operatorname{Ext}_C^2(E, j^! \mathcal{O}_X) = \operatorname{Hom}(E, \omega_C)$$

where $\omega_C$ is the dualizing sheaf on $C$. Also $H^1(C, \mathcal{E}nd(E)) \subset \mathrm{Ext}^1_X(j_*E, j_*E)$, and the Serre duality gives the surjection

$$\mathrm{Ext}^2_X(j_*E, j_*E) \twoheadrightarrow \mathrm{Hom}(\mathcal{E}nd(E), \omega_C).$$

By composing it with (5.11) we obtain the map

$$H^0(C, E) \otimes \mathrm{Hom}(E, \omega_C) \to \mathrm{Hom}(\mathcal{E}nd(E), \omega_C). \tag{5.12}$$

The above bilinear map is given by the natural composition map. Since $E$ and $\mathcal{E}nd(E)$ are torsion free on $C$, and $\omega_C$ is also torsion free on $C$ as $C$ is Cohen–Macaulay, the bilinear map (5.12) is injective on each factor. Therefore the lemma holds.               $\square$

We also have the following lemma:

**Lemma 5.6.** *There is a $\pi^+$-ample line bundle $\mathcal{O}_P(1)$ on $P_n^\circ(X, \beta)$ such that for any $p \in U_n^\circ(X, \beta)$, the isomorphism $\iota^+$ in the diagram (5.3) satisfies*

$$(\iota^+)^*(\mathcal{O}_{\widehat{Y}_U^+}(1)|_{\{d\widehat{w}^+=0\}}) \cong \mathcal{O}_P(1)|_{\widehat{P}_n(X,\beta)_p}.$$

*Proof.* Let $H$ be a sufficiently ample divisor on $X$ such that for any $[F] \in U_n^\circ(X, \beta)$, the sheaf $F(H) := F \otimes \mathcal{O}_X(H)$ satisfies $H^1(X, F(H)) = 0$, and the natural map $F \to F(H)$ defined by taking the tensor product with $\mathcal{O}_X \subset \mathcal{O}_X(H)$ is injective. Such an ample divisor $H$ exists as $U_n^\circ(X, \beta)$ is of finite type. By setting $d = H \cdot \beta$, we have the commutative diagram

$$\begin{array}{ccc}
P_n^\circ(X, \beta) & \hookrightarrow & P_{n+d}^\circ(X, \beta) \\
{\scriptstyle \pi^+}\downarrow & & \downarrow{\scriptstyle \pi^+} \\
U_n^\circ(X, \beta) & \xrightarrow{\cong} & U_{n+d}^\circ(X, \beta)
\end{array} \tag{5.13}$$

Here the top arrow is given by

$$(\mathcal{O}_X \to F) \mapsto (\mathcal{O}_X \to F \hookrightarrow F(H))$$

and the bottom arrow sends a stable sheaf $F$ to $F(H)$. By the condition $H^1(X, F(H)) = 0$, the right arrow is a projective bundle with fiber $\mathbb{P}(H^0(X, F(H)))$. Indeed, using the universal sheaf (5.2), we have an isomorphism over $U_n^\circ(X, \beta)$

$$P_{n+d}^\circ(X, \beta) \cong \mathbb{P}(p_{U*}(\mathcal{F} \otimes p_X^* \mathcal{O}_X(H))).$$

Here $p_X$, $p_U$ are the projections from $X \times U_n^\circ(X, \beta)$ to $X$, $U_n^\circ(X, \beta)$ respectively. By restricting the tautological line bundle on $P_{n+d}^\circ(X, \beta)$ to $P_n^\circ(X, \beta)$ by the top arrow of (5.13), we obtain the desired $\mathcal{O}_P(1)$.               $\square$

### 5.3. SOD for stable pair moduli spaces

We keep the situation of the previous subsections. For the diagram (5.3), let $\mathcal{W}^\circ$ be the fiber product

$$\mathcal{W}^\circ := P_n^\circ(X, \beta) \times_{U_n^\circ(X,\beta)} P_{-n}^\circ(X, \beta).$$

The following is the main result in this section:

**Theorem 5.7.** *For $n \geq 0$ and $\beta \in H_2(X, \mathbb{Z})$, suppose that $U_n^\circ(X, \beta)$ is fine and non-singular of dimension $g$. Then $P_{\pm n}^\circ(X, \beta)$ are also non-singular of dimension $\pm n + g - 1$, and we have the following:*

(i) *The functor*

$$\Phi_P := \Phi^{\mathcal{O}_{\mathcal{W}^\circ}} : D^b(P_{-n}^\circ(X, \beta)) \to D^b(P_n^\circ(X, \beta))$$

*is fully faithful.*

(ii) *There is a $\pi^+$-ample line bundle $\mathcal{O}_P(1)$ on $P_n^\circ(X, \beta)$ such that if $n \geq 1$, the functor*

$$\Upsilon_P^i : D^b(U_n^\circ(X, \beta)) \to D^b(P_n^\circ(X, \beta))$$

*defined by $\mathbf{L}\pi^{+*}(-) \otimes \mathcal{O}_P(i)$ is fully faithful.*

(iii) *We have the semiorthogonal decomposition*

$$D^b(P_n^\circ(X, \beta)) = \langle \operatorname{Im} \Upsilon_P^{-n+1}, \ldots, \operatorname{Im} \Upsilon_P^0, \operatorname{Im} \Phi_P \rangle.$$

*Proof.* We show that the diagram (5.3) satisfies Assumption 4.2. Let us take $p \in U_n^\circ(X, \beta)$ corresponding to a pure one-dimensional sheaf $F$. The assumption that $U_n^\circ(X, \beta)$ is smooth and the bottom left isomorphism in the diagram (5.6) indicate that, for the formal function $\widehat{w}$ written as in (5.7), we may assume that $w^{(0)}(\vec{u}) = 0$. Then (i) of Assumption 4.2 follows from Theorem 5.3, (ii) follows from Lemma 5.5, and (iii) follows from Lemma 5.6. Therefore the theorem follows from Theorem 4.5. $\square$

**Remark 5.8.** When $X$ is a non-compact CY 3-fold, suppose that $X$ has a smooth compactification $\overline{X}$ such that $H^i(\mathcal{O}_{\overline{X}}) = 0$ for $i = 1, 2$. Then the result of Theorem 5.7 also holds without any modification with $X$ replaced by $\overline{X}$. This is because for $E_1 = \mathcal{O}_{\overline{X}}$ and $E_2 = F[-1]$ where the support of $F$ is contained in $X$, we have the perfect pairing

$$\operatorname{Ext}_X^1(E_i, E_j) \otimes \operatorname{Ext}_X^2(E_j, E_i) \to \mathbb{C}$$

by the CY3 condition for $X$ and since $H^i(\mathcal{O}_{\overline{X}}) = 0$ for $i = 1, 2$.

### 5.4. Stable pairs on local surfaces

We apply Theorem 5.7 to some local surfaces. Let $S$ be a smooth projective surface satisfying $H^i(\mathcal{O}_S) = 0$ for $i = 1, 2$. We consider the non-compact CY 3-fold

$$X = \operatorname{Tot}_S(K_S).$$

We will apply Theorem 5.7 to show the existence of SOD of relative Hilbert schemes of points on the universal curve over a complete linear system.
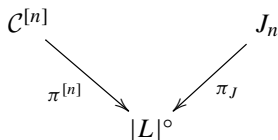
Let us take $\beta \in H_2(S, \mathbb{Z}) = H^2(S, \mathbb{Z})$ such that $-K_S \cdot \beta > 0$. By the assumption $H^i(\mathcal{O}_S) = 0$ for $i = 1, 2$, there is a unique $L \in \mathcal{P}ic(S)$ such that $c_1(L) = \beta$. Let $|L|^\circ \subset |L|$ be the open subset consisting of irreducible curves and

$$\pi : \mathcal{C} \to |L|^\circ$$

the universal curve. Note that any member $C \in |L|^\circ$ has arithmetic genus

$$g = 1 + \tfrac{1}{2}(\beta^2 + K_S \cdot \beta).$$

We have the diagram

$$
\begin{array}{ccc}
\mathcal{C}^{[n]} & & J_n \\
& \searrow{\scriptstyle \pi^{[n]}} \quad \swarrow{\scriptstyle \pi_J} & \\
& |L|^\circ &
\end{array}
$$

Here $\pi^{[n]}$ is the $\pi$-relative Hilbert scheme of $n$ points, and $\pi_J$ is the $\pi$-relative rank-one torsion free sheaf on the fibers of $\pi$ with Euler characteristic $n$. Let $i \colon S \hookrightarrow X$ be the zero section. We have the following lemma:

**Lemma 5.9.** (i) *We have isomorphisms*

$$\mathcal{C}^{[n+g-1]} \xrightarrow{\cong} P_n^\circ(S, \beta) \xrightarrow{\cong} P_n^\circ(X, i_*\beta)$$

*and they are non-singular.*

(ii) *We have isomorphisms*

$$J_n \xrightarrow{\cong} U_n^\circ(S, \beta) \xrightarrow{\cong} U_n^\circ(X, i_*\beta)$$

*and they are non-singular.*

*Proof.* As for (i), the isomorphism $\mathcal{C}^{[n+g-1]} \xrightarrow{\cong} P_n^\circ(S, \beta)$ and the smoothness of $P_n^\circ(S, \beta)$ follow by applying the arguments used for [PT10, Propositions B.8, C.2]. The assumption $-K_S \cdot \beta > 0$ implies that any compactly supported irreducible curve on $X$ with homology class $i_*\beta$ must lie on the zero section $S \subset X$. Therefore we have the set-theoretic bijection $P_n^\circ(S, \beta) \to P_n^\circ(X, i_*\beta)$, and these schemes have the same scheme structures by [KT14, Proposition 3.4].

As for (ii), the smoothness of $U_n^\circ(S, \beta)$ follows from

$$\mathrm{Ext}_S^2(F, F) = \mathrm{Hom}(F, F \otimes \mathcal{O}_S(K_S))^\vee = 0$$

for a sheaf $F$ corresponding to a point in $U_n(X, \beta)$, by the Serre duality and the assumption $-K_S \cdot \beta > 0$. The isomorphism $J_n \xrightarrow{\cong} U_n^\circ(S, \beta)$ follows from the argument in [MT18, Subsection 5.3], and $U_n^\circ(S, \beta) \xrightarrow{\cong} U_n^\circ(X, i_*\beta)$ follows similarly to (i). $\qquad\square$

By Lemma 5.9, the diagram (5.3) in this case is

$$
\begin{array}{ccc}
\mathcal{C}^{[n+g-1]} & & \mathcal{C}^{[-n+g-1]} \\
& \searrow{\scriptstyle \pi^+} \quad \swarrow{\scriptstyle \pi^-} & \\
& J_n &
\end{array}
\qquad (5.14)
$$

Again we assume that $n \geq 0$ and $U_n^\circ(S, \beta)$ is fine, which is guaranteed if $\gcd(\beta \cdot D, n) = 1$ for some divisor $D$ on $S$ (see Remark 5.2). Applying Theorem 5.7 to $X = \mathrm{Tot}_S(K_S)$ and noting Remark 5.8, we have the following:

**Corollary 5.10.** *In the above situation, we have the SOD*

$$D^b(\mathcal{C}^{[n+g-1]}) = \langle \overbrace{D^b(J_n), \ldots, D^b(J_n)}^{n}, D^b(\mathcal{C}^{[-n+g-1]}) \rangle.$$

### 5.5. SOD of symmetric product of curves

Let $C$ be a smooth projective curve over $\mathbb{C}$ of genus $g$. Its $k$-fold symmetric product $C^{[k]}$ is defined by

$$C^{[k]} := \overbrace{(C \times \cdots \times C)}^{k} / \mathfrak{S}_k$$

where the action of the symmetric group $\mathfrak{S}_k$ is by permutation. The variety $C^{[k]}$ is a smooth projective variety of dimension $k$, and identified with the Hilbert scheme of $k$-points on $C$.

Let $\mathcal{P}ic^k(C)$ be the moduli space of degree $k$ line bundles on $C$, which is a $g$-dimensional complex torus. Once we fix a point $c \in C$, we have the isomorphism

$$\mathcal{P}ic^k(C) \xrightarrow{\cong} J_C := \mathcal{P}ic^0(C) \tag{5.15}$$

which sends $[L] \in \mathcal{P}ic^k(C)$ to $[L(-kc)] \in J_C$. Below we fix the above isomorphisms for each $k \in \mathbb{Z}$. We also have the Abel–Jacobi map

$$\mathrm{AJ} \colon C^{[k]} \to \mathcal{P}ic^k(C) \tag{5.16}$$

which sends a length $k$ subscheme $Z \subset C$ to the line bundle $\mathcal{O}_C(Z)$.

**Remark 5.11.** For $k > 2g - 2$, the map (5.16) is a projective bundle. In general, the map (5.16) is a stratified projective bundle, where strata on $\mathcal{P}ic^k(C)$ are given by Brill–Noether loci. The geometry of Brill–Noether loci is complicated and depends on the complex structure of $C$, whose study is a classical subject in the study of symmetric products of curves (see [Fla, Section 5], [Kas13, Examples 1.0.7–1.0.10]).

For $n \geq 0$, we consider the diagram

$$\begin{array}{ccc} C^{[n+g-1]} & & C^{[-n+g-1]} \\ & \searrow{\scriptstyle \mathrm{AJ}} \quad {\scriptstyle \mathrm{AJ}^\vee}\swarrow & \\ & \mathcal{P}ic^{n+g-1}(C) & \end{array} \tag{5.17}$$

Here $\mathrm{AJ}^\vee$ sends $Z \subset C$ to $\omega_C(-Z)$. Applying Theorem 5.7 and using the isomorphism (5.15), we obtain the following corollary:

**Corollary 5.12.** *For each $n \geq 0$, we have the SOD*

$$D^b(C^{[n+g-1]}) = \langle \overbrace{D^b(J_C), \ldots, D^b(J_C)}^{n}, D^b(C^{[-n+g-1]}) \rangle.$$

*Proof.* Let $X$ be the non-compact CY 3-fold

$$X = \mathrm{Tot}_C(L_1 \oplus L_2)$$

where $L_1$, $L_2$ are general line bundles of degree $g-1$ satisfying $L_1 \otimes L_2 \cong \omega_C$. Then the diagram (5.3) in this case coincides with the diagram (5.17). As mentioned in [Tod, Example 9.22, Remark 9.23], Theorem 5.3 applies to the non-compact CY 3-fold $X$. Therefore the result follows by the same argument as for Theorem 5.7 and isomorphisms (5.15). □

For $n = 0$, the images of AJ and $\mathrm{AJ}^\vee$ coincide with the theta divisor

$$\Theta := \{[L] \in \mathcal{P}ic^{g-1}(C) : h^0(L) \neq 0\} \subset \mathcal{P}ic^{g-1}(C),$$

which is singular in general, but has only rational singularities [Kem73]. So we have the diagram
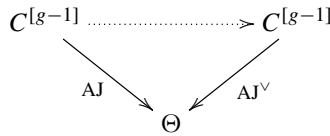


which gives a (possibly non-isomorphic) resolutions of $\Theta$. Let $\mathcal{W}$ be the fiber product of the above diagram. Applying Theorem 5.7 as in the proof of Corollary 5.12 for $n = 0$, we have the following:

**Corollary 5.13.** *We have the autoequivalence*

$$\Phi^{\mathcal{O}_\mathcal{W}} : D^b(C^{[g-1]}) \xrightarrow{\sim} D^b(C^{[g-1]}). \tag{5.18}$$

Below we give some examples related to Corollaries 5.12 and 5.13.

**Example 5.14.** Suppose that $n > g - 1$. Then $C^{[-n+g-1]} = \emptyset$ and

$$\mathrm{AJ} : C^{[n+g-1]} \to \mathcal{P}ic^{n+g-1}(C)$$

is a projective bundle whose fibers are $\mathbb{P}^{n-1}$. Then the SOD in Theorem 5.12 is

$$D^b(C^{[n+g-1]}) = \langle \overbrace{D^b(J_C), \ldots, D^b(J_C)}^{n} \rangle,$$

which is nothing other than Orlov's SOD for projective bundles [Orl92].

**Example 5.15.** Suppose that $n = g - 1$. Then $C^{[-n+g-1]} = \operatorname{Spec} \mathbb{C}$ and

$$\mathrm{AJ} \colon C^{[2g-2]} \to \mathcal{P}ic^{2g-2}(C)$$

is a projective bundle outside the point $[\omega_C] \in \mathcal{P}ic^{2g-2}(C)$. For the fiber $F = \mathrm{AJ}^{-1}([\omega_C])$, its structure sheaf $\mathcal{O}_F$ is exceptional, and the SOD in Theorem 5.12 is

$$D^b(C^{[2g-2]}) = \langle \overbrace{D^b(J_C), \ldots, D^b(J_C)}^{g-1}, \mathcal{O}_F \rangle.$$

**Example 5.16.** Suppose that $n = g - 2$. Then $C^{[-n+g-1]} = C$ and

$$\mathrm{AJ} \colon C^{[2g-3]} \to \mathcal{P}ic^{2g-3}(C)$$

is a projective bundle outside $\mathrm{AJ}^\vee(C) \subset \mathcal{P}ic^{2g-3}(C)$. In this case, the SOD in Theorem 5.12 is

$$D^b(C^{[2g-3]}) = \langle \overbrace{D^b(J_C), \ldots, D^b(J_C)}^{g-2}, D^b(C) \rangle.$$

**Example 5.17.** Suppose that $g = 3$ and $n = 0$. Then the birational map

$$\mathrm{AJ} \colon C^{[2]} \to \Theta$$

is not an isomorphism if and only if $C$ is a hyperelliptic curve (see [Kas13, Example 1.0.9]). In this case, the above map contracts a $(-2)$-curve on $C^{[2]}$ to a rational double point in $\Theta$. The equivalence (5.18) is the spherical twist along with the $(-2)$-curve.

**Example 5.18.** Suppose that $g = 4$ and $n = 1$. Then the birational map

$$\mathrm{AJ} \colon C^{[4]} \to \mathcal{P}ic^4(C)$$

contracts a divisor $E \subset C^{[4]}$ to the surface $\mathrm{AJ}^\vee(C^{[2]}) \subset \mathcal{P}ic^4(C)$ (see [Kas13, Example 1.0.10]). If $C$ is not hyperelliptic, then $E$ is a $\mathbb{P}^1$-bundle over $\mathrm{AJ}^\vee(C^{[2]}) \cong C^{[2]}$. The SOD in Theorem 5.12 becomes

$$D^b(C^{[4]}) = \langle D^b(J_C), D^b(C^{[2]}) \rangle.$$

If $C$ is not hyperelliptic, the above SOD seems to be the blow-up formula of derived categories obtained in [Orl92].

**Example 5.19.** Suppose that $g = 4$ and $n = 0$. Then the birational map

$$\mathrm{AJ} \colon C^{[3]} \to \Theta$$

is a crepant resolution of $\Theta$ which is a divisorial contraction if $C$ is hyperelliptic, and a small resolution which contracts one or two smooth rational curves if $C$ is not hyperelliptic (see [Kas13, Example 1.0.10]). In the latter case, the equivalence (5.18) seems to be the derived equivalence under flops [BO01, Bri02].

## 6. Categorification of Kawai–Yoshioka's formula

In this section, we prove Theorem 1.2 as another application of Theorem 4.5. We use Kawai–Yoshioka's diagram [KY00] relating moduli spaces of stable pairs on K3 surfaces to moduli spaces of stable sheaves on them. The key ingredient, which was essentially observed in [Tod12b], is to interpret Kawai–Yoshioka's diagram in terms of a wall-crossing diagram in a CY 3-fold defined by the product of the K3 surface and an elliptic curve.

### 6.1. SOD of relative Hilbert schemes of points

Let $S$ be a smooth projective K3 surface such that

$$\mathcal{P}ic(S) = \mathbb{Z}[\mathcal{O}_S(H)]$$

for an ample divisor $H$ on $S$. Let $g \in \mathbb{Z}$ be defined by $H^2 = 2g - 2$. We have the complete linear system $|H|$ and the universal curve

$$\pi : \mathcal{C} \to |H| = \mathbb{P}^g.$$

In what follows, we fix $n > 0$. Let

$$\mathcal{C}^{[n+g-1]} \to \mathbb{P}^g \qquad (6.1)$$

be the $\pi$-relative Hilbert scheme of $n + g - 1$ points on $\mathcal{C}$. As in Lemma 5.9, the $\pi$-relative Hilbert scheme (6.1) is isomorphic to the moduli space of Pandharipande–Thomas stable pair moduli space $P_n(S, [H])$ on $S$.

Let $\Gamma_S$ be the *Mukai lattice* of $S$,

$$\Gamma_S := H^0(S, \mathbb{Z}) \oplus \mathbb{Z}[H] \oplus H^4(S, \mathbb{Z}).$$

For $E \in D^b(S)$ its *Mukai vector* is defined by

$$v(E) := \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}_S} \in \Gamma_S.$$

For elements $(r_i, \beta_i, m_i) \in \Gamma_S$ with $i = 1, 2$, the *Mukai pairing* is defined by

$$((r_1, \beta_1, m_1), (r_2, \beta_2, m_2)) := \beta_1 \beta_2 - r_2 m_1 - r_1 m_2.$$

For each $k \in \mathbb{Z}_{\geq 0}$, we define $U_k$ to be the moduli space of $H$-Gieseker stable sheaves $E$ on $S$ satisfying

$$v(E) = \mathbf{v}_k := (k, [H], k + n).$$

Here we refer to [HL97] for basics on moduli spaces of stable sheaves and their properties. The moduli space $U_k$ is known to be a projective irreducible holomorphic symplectic manifold with

$$\dim U_k = 2 + (\mathbf{v}_k, \mathbf{v}_k) = 2(g - k^2 - kn). \qquad (6.2)$$

Below we assume that $U_k$ is a fine moduli space, i.e. there is a universal sheaf on $S \times U_k$.

**Remark 6.1.** By [HL97, Corollary 4.6.7], the moduli space $U_k$ is fine if $(k, 2g-2, 2k+n)$ is coprime. In particular if $(2g - 2, n)$ is coprime, then $U_k$ is fine for any $k \in \mathbb{Z}_{\geq 0}$.

Let $\mathcal{P}_k$ be the moduli space of pairs

$$(E, s), \quad s \colon \mathcal{O}_S \to E,$$

where $[E] \in U_k$ and $s$ is a non-zero morphism. By [KY00, Lemma 5.117], the moduli space $\mathcal{P}_k$ is a smooth projective variety with

$$\dim \mathcal{P}_k = 1 + \langle \mathbf{v}_{k-1}, \mathbf{v}_k \rangle = 2(g - k^2 - kn) + 2k + n - 1. \tag{6.3}$$

We have the diagram (see [KY00, Lemma 5.113])

$$
\begin{array}{ccc}
\mathcal{P}_k & & \mathcal{P}_{k+1} \\
& \searrow \quad \swarrow & \\
\pi_k^+ & & \pi_k^- \\
& U_k &
\end{array}
\tag{6.4}
$$

where $\pi_k^{\pm}$ are defined by

$$\pi_k^+(E, s) := E, \quad \pi_k^-(E', s') := \mathrm{Cok}(s').$$

As an application of Theorem 4.5, we have the following result whose proof will be given in Subsection 6.5:

**Theorem 6.2.** *For $k \geq 0$, we have the following SOD:*

$$D^b(\mathcal{P}_k) = \langle \overbrace{D^b(U_k), \ldots, D^b(U_k)}^{n+2k}, D^b(\mathcal{P}_{k+1}) \rangle.$$

Let

$$N := \max\{k \geq 0 : g - k^2 - kn \geq 0\}.$$

Applying the above theorem from $k = 0$ to $k = N$, and noting that

$$\mathcal{P}_0 = P_n(S, [H]) \cong \mathcal{C}^{[n+g-1]}, \quad \mathcal{P}_{N+1} = \emptyset,$$

where the latter is due to (6.2), we have the following result:

**Corollary 6.3.** *For $n > 0$, we have the SOD*

$$D^b(\mathcal{C}^{[n+g-1]}) = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle \tag{6.5}$$

*where each $\mathcal{A}_k$ has the SOD*

$$\mathcal{A}_k = \langle \overbrace{D^b(U_k), \ldots, D^b(U_k)}^{n+2k} \rangle.$$

**Remark 6.4.** As mentioned in Subsection 1.3, the SOD (6.5) recovers Kawai–Yoshioka's formula (1.8),

$$P_{n,g} = (-1)^{n-1} \sum_{k=0}^{N} (n + 2k) e(U_k).$$

In [KY00], the formula (1.8) is the key ingredient to prove the Katz–Klemm–Vafa (KKV) formula for PT invariants with irreducible curve classes. Together with the identities

$$e(U_k) = e(\text{Hilb}^{g-k(k+n)}(S)), \qquad \sum_{k \geq 0} e(\text{Hilb}^k(S)) = \prod_{k \geq 1} (1 - q^k)^{-24}$$

the formula (1.8) implies (see [KY00])

$$\sum_{g \geq 0} \sum_{n \in \mathbb{Z}} P_{n,g} z^n q^{g-1} = \left( \sqrt{z} - \frac{1}{\sqrt{z}} \right)^{-2} \frac{1}{\Delta(z, q)}. \tag{6.6}$$

Here

$$\Delta(z, q) := q \prod_{n \geq 1} (1 - q^n)^{20} (1 - zq^n)^2 (1 - z^{-1}q^n)^2.$$

The formula (6.6) is the KKV formula mentioned above.

### 6.2. Tilting on $S \times C$

Let $S$ be a K3 surface as in the previous subsection. We fix a smooth elliptic curve $C$ and consider a compact CY 3-fold $X := S \times C$ with projections $p_S$, $p_C$,

$$X = S \times C \xrightarrow{\;p_C\;} C$$
$$\left. p_S \right\downarrow$$
$$S$$

In what follows, we will interpret the diagram (6.4) in terms of wall-crossing diagrams in $D^b(X)$.

We define the triangulated subcategory

$$\mathcal{D}_0 \subset D^b(X)$$

consisting of objects whose cohomology is supported on fibers of $p_C$. The triangulated category $\mathcal{D}_0$ is the derived category of the abelian subcategory

$$\text{Coh}_0(X) \subset \text{Coh}(X)$$

consisting of sheaves supported on fibers of $p_C$. For $c \in C$, let

$$i_c \colon S \times \{c\} \hookrightarrow S \times C = X. \tag{6.7}$$

The category $\text{Coh}_0(X)$ is the extension closure of objects of the form $i_{c*}F$ for some $c \in C$ and $F \in \text{Coh}(S)$.

For $F \in \mathcal{D}_0$, we set $v(F) \in \Gamma_S$ to be

$$v(F) := v(p_{S*}F) = (v_0(F), v_1(F), v_2(F))$$

for $v_i(F) \in H^{2i}(S, \mathbb{Z})$. We define the following *slope function* on $\mathrm{Coh}_0(X)$:

$$\mu(F) := \frac{v_1(F) \cdot H}{v_0(F)} \in \mathbb{Q} \cup \{\infty\}.$$

Here $F \in \mathrm{Coh}_0(X)$ and we set $\mu(F) = \infty$ if $v_0(F) = 0$. The slope function defines $\mu$-stability on $\mathrm{Coh}_0(X)$ in the usual way: a non-zero object $F \in \mathrm{Coh}_0(X)$ is $\mu$-*(semi)stable* if for any non-zero subsheaf $F' \subsetneq F$,

$$\mu(F') < (\leq) \mu(F/F').$$

Let $\mathcal{T}, \mathcal{F}$ be the subcategories of $\mathrm{Coh}_0(X)$ defined by

$$\mathcal{T} := \langle F \in \mathrm{Coh}_0(X) : F \text{ is } \mu\text{-semistable with } \mu(F) > 0 \rangle_{\mathrm{ex}},$$
$$\mathcal{F} := \langle F \in \mathrm{Coh}_0(X) : F \text{ is } \mu\text{-semistable with } \mu(F) \leq 0 \rangle_{\mathrm{ex}}.$$

Here $\langle - \rangle_{\mathrm{ex}}$ means extension closure. The pair of subcategories $(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\mathrm{Coh}_0(X)$. We have the associated tilting

$$\mathcal{B} := \langle \mathcal{F}, \mathcal{T}[-1] \rangle_{\mathrm{ex}} \subset \mathcal{D}_0.$$

For $t \in \mathbb{R}_{>0}$, let

$$Z_t \colon K(\mathcal{D}_0) \to \mathbb{C} \qquad\qquad (6.8)$$

be the group homomorphism defined by

$$Z_t(F) := \int_S e^{-tH\sqrt{-1}} v(F) = v_2(F) + (1-g)t^2 v_0(F) - (tH \cdot v_1(F))\sqrt{-1}.$$

Then the pair

$$(Z_t, \mathcal{B}), \quad t \in \mathbb{R}_{>0},$$

is a Bridgeland stability condition on $\mathcal{D}_0$ (see [Tod12b, Lemma 3.3]). In particular it defines $Z_t$-(semi)stable objects: a non-zero object $E \in \mathcal{B}$ is $Z_t$-*(semi)stable* if for any non-zero subobject $0 \neq E' \subsetneq E$ in $\mathcal{B}$, we have the inequality in $(0, \pi]$:

$$\arg Z_t(E') < (\leq) \arg Z_t(E).$$

We have the following lemma:

**Lemma 6.5.** *An object $E \in \mathcal{B}$ with $v_1(E) = -[H]$, $v_0(E) \leq 0$ is $Z_t$-stable if and only if $E \cong i_{c*}F[-1]$ for some $c \in C$ and some $H$-Gieseker stable sheaf $F \in \mathrm{Coh}(S)$.*

*Proof.* The lemma is well-known (for example see [Bay18, proof of Lemma 6.1]). Let $\mathcal{C} \subset \mathcal{B}$ be the subcategory defined by

$$\mathcal{C} := \{F \in \mathcal{B} : \Im Z_t(F) = 0\}$$
$$= \langle U, \mathcal{O}_x[-1] : U \in \mathrm{Coh}_0(X) \text{ is } \mu\text{-stable with } \mu(U) = 0, \ x \in X \rangle_{\mathrm{ex}}.$$

Suppose that $E \in \mathcal{B}$ satisfies $v_1(E) = -[H]$ and $v_0(E) \leq 0$. Since $-v_1(-) \cdot H$ is non-negative on $\mathcal{B}$, and $-v_1(E) \cdot H = H^2$ is the smallest positive value of $-v_1(-) \cdot H$ on $\mathcal{B}$, the object $E$ is $Z_t$-stable if and only if $\mathrm{Hom}(\mathcal{C}, E) = 0$.

First suppose that $E$ is $Z_t$-stable, so $\mathrm{Hom}(\mathcal{C}, E) = 0$. Then $\mathcal{H}^0(E) = 0$, and $\mathcal{H}^1(E)$ is either a $\mu$-stable two-dimensional sheaf or a one-dimensional $H$-Gieseker stable sheaf. It follows that $E \cong i_{c*}F[-1]$ for some $c \in C$, where $F$ is a $\mu$-stable sheaf on $S$ or an $H$-Gieseker stable one-dimensional sheaf on $S$. In the former case, since the Mukai vector of $F$ is primitive, its $\mu$-stability is equivalent to its $H$-Gieseker stability. Conversely, if $E \cong i_{c*}F[-1]$ as in the statement, then it is obvious that $\mathrm{Hom}(\mathcal{C}, E) = 0$. Therefore the lemma is proved. $\qquad\square$

We define the following subcategory of $D^b(X)$:

$$\mathcal{A} := \langle p_C^* \mathcal{P}ic(C), \mathcal{B} \rangle_{\mathrm{ex}}. \tag{6.9}$$

The category $\mathcal{A}$ is the heart of a bounded t-structure on the triangulated subcategory of $D^b(X)$ generated by $p_C^* \mathcal{P}ic(C)$ and objects in $\mathcal{D}_0$ (see [Tod12b, Proposition 2.9]). In particular, $\mathcal{A}$ is an abelian category. Note that $E \in \mathcal{A}$ satisfies $\mathrm{rank}(E) = 0$ if and only if $E \in \mathcal{B}$. We will use the following property of $\mathcal{A}$:

**Lemma 6.6.** *For any object $E \in \mathcal{A}$, there is an exact sequence in $\mathcal{A}$*

$$0 \to E' \to E \to E'' \to 0 \tag{6.10}$$

*such that $E' \in \mathcal{B}$ and $E'' \in \langle p_C^* \mathcal{P}ic(C) \rangle_{\mathrm{ex}}$.*

*Proof.* This is proved in [Tod12b, Lemma 7.5] in the case of $S \times \mathbb{P}^1$, and the same argument works here. For simplicity, we prove the lemma when $E$ fits into a non-split extension in $\mathcal{A}$

$$0 \to p_C^* \mathcal{L} \to E \to i_{c*}F[-1] \to 0 \tag{6.11}$$

for $\mathcal{L} \in \mathcal{P}ic(C)$, $[F] \in U_k$, and $c \in C$. The full details are in [Tod12b, Lemma 7.5].

Let $\xi$ be the extension class of (6.11). Then since $i_c^! p_C^* \mathcal{L} = \mathcal{O}_S[-1]$, we have

$$\xi \in \mathrm{Ext}_X^2(i_{c*}F, p_C^* \mathcal{L}) = \mathrm{Ext}_S^1(F, \mathcal{O}_S).$$

Therefore $\xi$ gives rise to the non-trivial extension of sheaves on $S$

$$0 \to \mathcal{O}_S \to F' \to F \to 0.$$

It is easy to see that $F'$ is $H$-Gieseker stable so $[F'] \in U_{k+1}$. We have the commutative diagram

$$
\begin{array}{ccc}
E & & p_C^* \mathcal{L}(c) \\
\downarrow & & \downarrow \\
i_{c*}F'[-1] \longrightarrow i_{c*}F[-1] \longrightarrow i_{c*}\mathcal{O}_S \\
\quad\downarrow{\scriptstyle \xi} & & \downarrow \\
p_C^* \mathcal{L}[1] \overset{\mathrm{id}}{\longrightarrow} p_C^* \mathcal{L}[1]
\end{array}
$$

Here horizontal and vertical sequences are distinguished triangles. By the above diagram, we obtain the exact sequence in $\mathcal{A}$

$$0 \to i_{c*}F'[-1] \to E \to p_C^*\mathcal{L}(c) \to 0.$$

The above exact sequence is the desired sequence (6.10).                      □

### 6.3. Weak stability conditions on $\mathcal{A}$

Let $\mathcal{A}$ be the abelian category given in (6.9). For $t \in \mathbb{R}_{>0}$ and $E \in \mathcal{A}$, we define $\mu_t^\star(E) \in \mathbb{R} \cup \{\infty\}$ by

$$
\mu_t^\star(E) := \begin{cases} 0, & \mathrm{rank}(E) \neq 0, \\ -\frac{\Re Z_t(E)}{\Im Z_t(E)}, & \mathrm{rank}(E) = 0. \end{cases}
$$

Here if $\mathrm{rank}(E) = 0$, then $E \in \mathcal{B}$ and $Z_t(E) \in \mathbb{C}$ is given in (6.8). The following stability condition on $\mathcal{A}$ appeared in [Tod12b] in the framework of weak stability conditions:

**Definition 6.7.** A non-zero object $E \in \mathcal{A}$ is $\mu_t^\star$-(*semi*)*stable* if for any exact sequence $0 \to E' \to E \to E'' \to 0$ in $\mathcal{A}$ with non-zero $E'$, $E''$, we have

$$\mu_t^\star(E') < (\leq) \, \mu_t^\star(E'').$$

Below we fix $n \in \mathbb{Z}_{>0}$ and characterize $\mu_t^\star$-semistable objects $E \in \mathcal{A}$ satisfying

$$\mathrm{ch}(E) = (1, 0, -i_{c*}[H], -n) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X). \tag{6.12}$$

**Proposition 6.8.** *For* $k \in \mathbb{Z}_{>0}$, *suppose that* $t \in \mathbb{R}_{>0}$ *satisfies*

$$t_k < t < t_{k-1}, \quad t_k := \sqrt{\frac{n+k}{(g-1)k}}, \quad t_0 := \infty. \tag{6.13}$$

*Then an object* $E \in \mathcal{A}$ *satisfying* (6.12) *is* $\mu_t^\star$-*semistable if and only if* $E$ *is isomorphic to a two-term complex*

$$E \cong (p_C^*\mathcal{L} \overset{s}{\to} i_{c*}F) \tag{6.14}$$

*for some* $c \in C$, $[F] \in U_k$, $\mathcal{L} \in \mathcal{P}ic^k(C)$ *and* $s$ *is a non-zero morphism. Here* $\mathcal{P}ic^k(C) \subset \mathcal{P}ic(C)$ *is the subset of degree* $k$ *line bundles, and* $p_C^*\mathcal{L}$ *is located in degree zero. Moreover in this case,* $E$ *is* $\mu_t^\star$-*stable.*

*Proof.*

**Step 1.** *The 'only if' direction.* Let us take $t \in (t_k, t_{k-1})$ and a $\mu_t^\star$-semistable object $E \in \mathcal{A}$ satisfying (6.12). By [Tod12b, Lemma 7.5], there is an exact sequence in $\mathcal{A}$

$$0 \to A \to E \to p_C^* \mathcal{L} \to 0 \tag{6.15}$$

for some $A \in \mathcal{B}$, $\mathcal{L} \in \mathcal{P}ic^r(C)$ for some $r \in \mathbb{Z}$. The condition (6.12) imply $v(A) = -\mathbf{v}_r$. The above exact sequence and the $\mu_t^\star$-semistability of $E$ yield

$$\Re Z_t(A) = -r - n + r(g-1)t^2 \geq 0.$$

As $t < t_{k-1}$ and $n > 0$, the above inequality yields $r \geq k > 0$.

The $\mu_t^\star$-semistability of $E$ implies that $\mathrm{Hom}(\mathcal{C}, E) = 0$, where $\mathcal{C} \subset \mathcal{B}$ is defined in the proof of Lemma 6.5. By the exact sequence (6.15) we have $\mathrm{Hom}(\mathcal{C}, A) = 0$, and Lemma 6.5 shows that $A \cong i_{c*}F[-1]$ for some $c \in C$ and $[F] \in U_r$. Therefore $E$ is isomorphic to a two-term complex

$$E = (p_C^* \mathcal{L} \xrightarrow{s'} i_{c*}F)$$

where $s'$ must be non-zero due to the $\mu_t^\star$-semistability of $E$. Let us show that $r = k$. By taking the cohomology of $E$, we obtain the exact sequence in $\mathcal{A}$

$$0 \to p_C^* \mathcal{L}(-c) \to E \to G[-1] \to 0 \tag{6.16}$$

where $G$ is the cokernel of $s'$. Since $v(G) = \mathbf{v}_{r-1}$, the $\mu_t^\star$-semistability of $E$ yields

$$\Re Z_t(G[-1]) = -r + 1 - n + (r-1)(g-1)t^2 \leq 0. \tag{6.17}$$

As $t > t_k$, the above inequality implies that $k \geq r$. As we already proved $r \geq k$, it follows that $r = k$. Therefore we have proved the 'only if' direction of the proposition.

**Step 2.** *The 'if' direction.* Conversely, let us take an object $E \in \mathcal{A}$ of the form (6.14). We show that $E$ is $\mu_t^\star$-stable if $t \in (t_k, t_{k-1})$. Let us take an exact sequence in $\mathcal{A}$

$$0 \to A \to E \to B \to 0 \tag{6.18}$$

such that $A, B$ are non-zero. We will show that

$$\mu_t^\star(A) < \mu_t^\star(B). \tag{6.19}$$

Since $\mathrm{rank}(E) = 1$, we have $(\mathrm{rank}(A), \mathrm{rank}(B)) = (0, 1)$ or $(1, 0)$. We will show (6.19) in each case.

First suppose that $\mathrm{rank}(A) = 0$, i.e. $A \in \mathcal{B}$. By the exact sequence in $\mathcal{A}$

$$0 \to i_{c*}F[-1] \to E \to p_C^* \mathcal{L} \to 0 \tag{6.20}$$

we have $\mathrm{Hom}(\mathrm{Coh}_0(X), E) = 0$. Therefore $\mathcal{H}^0(A) = 0$ and $A \in \mathcal{T}[-1]$. Then $A$ is given as an iterative extension of objects of the form $i_{c*}T[-1]$ for some $c \in C$, where

$T \in \mathrm{Coh}(S)$ is either torsion or $\mu$-stable with $\mu(T) > 0$. By the Serre duality and the $\mu$-stability of $T$, we have

$$\mathrm{Hom}(i_{c*}T[-1], p_C^*\mathcal{L}) = \mathrm{Hom}(T, \mathcal{O}_S) = 0.$$

Therefore $\mathrm{Hom}(A, p_C^*\mathcal{L}) = 0$. By the exact sequences (6.18) and (6.20), we have an injection $A \hookrightarrow i_{c*}F[-1]$ in $\mathcal{B}$. By Lemma 6.5, the object $i_{c*}F[-1]$ is $Z_t$-stable in $\mathcal{B}$. Therefore

$$\mu_t^\star(A) \le \mu_t^\star(i_{c*}F[-1]) = \mu_t^\star(-\mathbf{v}_k) < 0 = \mu_t^\star(B)$$

where $\mu_t^\star(-\mathbf{v}_k) < 0$ is due to $t_k < t$. Hence (6.19) holds.

Next suppose that $\mathrm{rank}(A) = 1$, i.e. $B \in \mathcal{B}$. Let $T \subset \mathcal{H}^0(B)$ be the HN factor of $\mathcal{H}^0(B)$ in $\mu$-stability such that $\mu(-)$ is maximal. Note that $\mu(T) \le 0$ by the definition of $\mathcal{B}$. If $\mu(T) = 0$, then $\mu_t^\star(T) = \infty$ and we have

$$\mu_t^\star(B) \ge \mu_t^\star(B/T).$$

Therefore after replacing $B$ by $B/T$, we may assume that $\mu(T) < 0$. This implies

$$\mathrm{Hom}(p_C^*\mathcal{P}ic(C), B) = 0. \tag{6.21}$$

Similarly to (6.16), we have the exact sequence in $\mathcal{A}$

$$0 \to p_C^*\mathcal{L}(-c) \to E \to G[-1] \to 0 \tag{6.22}$$

where $G$ is the cokernel of $s$ in (6.14). By the exact sequences (6.18), (6.22), and the vanishing (6.21), there is a surjection $G[-1] \twoheadrightarrow B$ in $\mathcal{B}$. By Lemma 6.5, the object $G[-1] \in \mathcal{B}$ is $Z_t$-stable. Therefore

$$\mu_t^\star(B) \ge \mu_t^\star(G[-1]) = \mu_t^\star(-\mathbf{v}_{k-1}) > 0 = \mu_t^\star(A)$$

where $\mu_t^\star(-\mathbf{v}_{k-1}) > 0$ due to $t < t_{k-1}$. Hence (6.19) holds. $\qquad \square$

When $t$ lies on a wall, the $\mu_t^\star$-semistable objects are characterized by the following lemma.

**Lemma 6.9.** *An object $E \in \mathcal{A}$ satisfying (6.12) is $\mu_{t_k}^\star$-semistable if and only if $E$ is S-equivalent to a $\mu_{t_k}^\star$-polystable object of the form*

$$E_1 \oplus E_2, \quad E_1 = p_C^*\mathcal{L}, \quad E_2 = i_{c*}F[-1], \tag{6.23}$$

*for some $c \in C$, $[F] \in U_k$ and $\mathcal{L} \in \mathcal{P}ic^k(C)$.*

*Proof.* The 'if' direction is obvious as both $E_1$, $E_2$ are $\mu_{t_k}^\star$-semistable with $\mu_{t_k}^\star(E_1) = \mu_{t_k}^\star(E_2) = 0$. The 'only if' direction is proved similarly to Step 1 in the proof of Proposition 6.8. If we apply the proof above for $t = t_k$, the only point to notice is that, just after (6.17) we only have $k \ge r - 1$ as we take $t = t_k$. Therefore either $r = k$ or $r = k + 1$. In the latter case, the exact sequence (6.16) shows that $E$ is S-equivalent to an object of the form (6.23). $\qquad \square$

### 6.4. Moduli stacks of semistable objects

Let $\mathcal{M}$ be the 2-functor

$$\mathcal{M} \colon \mathcal{S}ch/\mathbb{C} \to \mathcal{G}roupoid$$

sending a $\mathbb{C}$-scheme $S$ to the groupoid of relatively perfect objects $\mathcal{E} \in D^b(X \times S)$ such that for each point $s \in S$, the object $\mathcal{E}_s := \mathbf{L}i_s^*\mathcal{E}$ for the inclusion $i_s \colon X \times \{s\} \hookrightarrow X \times S$ satisfies $\mathrm{Ext}^{<0}(\mathcal{E}_s, \mathcal{E}_s) = 0$. The stack $\mathcal{M}$ is known to be an Artin stack locally of finite type [Lie06]. For a fixed $n \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{R}_{>0}$, we define the substack

$$\mathcal{M}_t^\star \subset \mathcal{M} \tag{6.24}$$

to be the stack whose $S$-valued points consist of $\mathcal{E} \in \mathcal{M}(S)$ such that for each $s \in S$, the object $\mathcal{E}_s$ is a $\mu_t^\star$-semistable object in $\mathcal{A}$ satisfying (6.12). Using Proposition 6.8 and Lemma 6.9, we show the following:

**Proposition 6.10.** *The stack $\mathcal{M}_t^\star$ is an Artin stack of finite type such that* (6.24) *is an open immersion. Moreover if $t \in (t_k, t_{k-1})$, the stack $\mathcal{M}_t^\star$ is smooth.*

*Proof.* By [Tod12b, Lemma 4.13(ii)], $\mathcal{M}_t^\star \subset \mathcal{M}$ is constructible. Therefore for the first statement, it is enough to show that $\mathcal{M}_t^\star \subset \mathcal{M}$ is open in the analytic topology.

By Lemma 6.9, for $t = t_k$ an object corresponding to a $\mathbb{C}$-valued point of $\mathcal{M}_t^\star$ is a small deformation of an object of the form (6.23). Set

$$V^+ = \mathrm{Ext}_X^1(E_1, E_2), \quad V^- = \mathrm{Ext}_X^1(E_2, E_1), \quad U = \mathrm{Ext}_X^1(E_1, E_1) \oplus \mathrm{Ext}_X^1(E_2, E_2).$$

Then the analytic local deformation space of $E_1 \oplus E_2$ is given by the critical locus of some analytic function $w$ defined in an analytic neighborhood of $0 \in V^+ \times V^- \times U$. Similarly to the case of stable pairs in (5.7), the function $w$ is invariant under the conjugate $\mathrm{Aut}(E_1 \oplus E_2) = (\mathbb{C}^*)^2$-action on $V^+ \times V^- \times U$, so it is of the form

$$w = w^{(0)}(\vec{u}) + \sum_{i,j} x_i y_j w_{ij}^{(1)}(\vec{u}) + \sum_{i,i',j,j'} x_i x_{i'} y_j y_{j'} w_{ii'jj'}^{(2)}(\vec{u}) + \cdots$$

where $\vec{x}$, $\vec{y}$ and $\vec{u}$ are coordinates of $V^+$, $V^-$ and $U$ respectively. As in [Tod, Subsection 5.1], the function $w$ is constructed using the minimal $A_\infty$-structure on $D^b(X)$. By the construction in *loc. cit.*, the function $w^{(0)}(\vec{u})$ can be written as

$$w^{(0)}(\vec{u}) = w_1^{(0)}(\vec{u}_1) + w_2^{(0)}(\vec{u}_2), \quad \vec{u} = (\vec{u}_1, \vec{u}_2), \quad \vec{u}_i \in \mathrm{Ext}_X^1(E_i, E_i),$$

such that the critical locus of $w_i^{(0)}(\vec{u}_i)$ in $\mathrm{Ext}_X^1(E_i, E_i)$ gives the local deformation space of $E_i$. Since the deformation space of $E_i$ is smooth, we may assume that $w^{(0)}(\vec{u}) = 0$.

Similarly to Subsection 5.2, the function $w_{ij}^{(1)}(\vec{u})$ can be written as in (4.7) such that the coefficients of the linear terms $a_{ijk}$ are determined by the triple product

$$V^+ \times V^- \times U \to \mathbb{C}$$

given by composition and the Serre duality. Then by Lemma 6.11 below, the coefficients $a_{ijk}$ satisfy the condition in Assumption 4.2(ii). Therefore Lemma 4.3 shows that

$$\begin{aligned}
\{dw = 0\} \cap (V^{+*} \times V^- \times U) \subset (V^{+*} \times \{0\} \times U), \\
\{dw = 0\} \cap (V^+ \times V^{-*} \times U) \subset (\{0\} \times V^{-*} \times U).
\end{aligned} \tag{6.25}$$

This implies that any small deformation $E'$ of $E_1 \oplus E_2$ fits into one of the following exact sequences in $\mathcal{A}$:

$$\begin{aligned}
0 \to i_{c'*}F'[-1] \to E' \to p_C^* \mathcal{L}' \to 0, \\
0 \to p_C^* \mathcal{L}' \to E' \to i_{c*}F'[-1] \to 0
\end{aligned} \tag{6.26}$$

where $(F', \mathcal{L}', c')$ is a small deformation of $(F, \mathcal{L}, c)$, so that $[F'] \in U_k$ and $\mathcal{L}' \in \mathcal{P}ic^k(C)$. Therefore $E'$ is $\mu_{t_k}^\star$-semistable, and $\mathcal{M}_{t_k}^\star \subset \mathcal{M}$ is open.

Suppose that $t \in (t_k, t_{k-1})$, and take an object $E$ as in (6.14) which corresponds to a $\mathbb{C}$-valued point of $\mathcal{M}_t^\star$. Then $E$ is isomorphic to a small deformation of the object $E_1 \oplus E_2$ as above, which lies in the LHS of (6.25). Then any small deformation $E'$ of $E$ fits into a non-split sequence (6.26). Therefore $E'$ is again $\mu_t^\star$-semistable by Proposition 6.8, and $\mathcal{M}_t^\star \subset \mathcal{M}$ is open. Moreover the argument used for Lemma 4.3 implies that the LHS of (6.25) is smooth, hence $\mathcal{M}_t^\star$ is smooth.                      $\square$

We have used the following lemma, which is an analogue of Lemma 5.5.

**Lemma 6.11.** *For the objects $E_1$, $E_2$ in (6.23), the composition map*

$$\operatorname{Ext}_X^1(E_1, E_2) \otimes \operatorname{Ext}_X^1(E_2, E_1) \to \operatorname{Ext}_X^2(E_2, E_2) \tag{6.27}$$

*is injective.*

*Proof.* Note that

$$\operatorname{Ext}_X^1(E_1, E_2) = H^0(S, F), \quad \operatorname{Ext}_X^1(E_2, E_1) = \operatorname{Ext}_S^1(F, \mathcal{O}_S).$$

We also have the surjection

$$\operatorname{Ext}_X^2(E_2, E_2) = \operatorname{Ext}_X^2(i_{c*}F, i_{c*}F) \twoheadrightarrow \operatorname{Ext}_S^1(F, F)$$

which is Serre dual to the natural map $\operatorname{Ext}_S^1(F, F) \to \operatorname{Ext}_X^1(i_{c*}F, i_{c*}F)$. By composing it with (6.11), we obtain the composition map

$$H^0(S, F) \otimes \operatorname{Ext}_S^1(F, \mathcal{O}_S) \to \operatorname{Ext}_S^1(F, F). \tag{6.28}$$

It is enough to show that the map (6.28) is injective. Let us take the universal extension in $\operatorname{Coh}(S)$

$$0 \to \operatorname{Ext}_S^1(F, \mathcal{O}_S)^\vee \otimes \mathcal{O}_S \to \mathcal{U} \to F \to 0. \tag{6.29}$$

Then it is well-known that $\mathcal{U}$ is a $\mu$-stable sheaf (see [Yos99, Tod14a]). Applying $\operatorname{Hom}(-, F)$ to the above exact sequence, we obtain the exact sequence

$$0 \to \mathbb{C} \to \operatorname{Hom}(\mathcal{U}, F) \to H^0(S, F) \otimes \operatorname{Ext}_S^1(F, \mathcal{O}_S) \to \operatorname{Ext}_S^1(F, F). \tag{6.30}$$

Since (6.29) is the universal extension, applying $\mathrm{Hom}(-, \mathcal{O}_S)$ to (6.29) we obtain $\mathrm{Ext}^1_S(\mathcal{U}, \mathcal{O}_S) = 0$. Then applying $\mathrm{Hom}(\mathcal{U}, -)$ to (6.29) and using the stability of $\mathcal{U}$, we get

$$\mathrm{Hom}(\mathcal{U}, F) = \mathrm{Hom}(\mathcal{U}, \mathcal{U}) = \mathbb{C}.$$

Therefore by the exact sequence (6.30), we see that (6.28) is injective.                    □

For $t \in \mathbb{R}_{>0}$, let

$$\mathcal{M}^\star_t \to M^\star_t \tag{6.31}$$

be the good moduli space for the stack $\mathcal{M}^\star_t$, which exists by [AHLH]. The good moduli space $M^\star_t$ is an algebraic space of finite type which parametrizes $\mu^\star_t$-polystable objects in $\mathcal{A}$ satisfying (6.12), i.e. direct sums of $\mu^\star_t$-stable objects with $\mu^\star_t(-) = 0$. By Proposition 6.8, the moduli space $M^\star_t$ is constant if $t \in (t_k, t_{k-1})$ for some $k$. So we can write

$$\mathbf{M}^\star_k := M^\star_t, \quad t \in (t_k, t_{k-1}).$$

By Proposition 6.8, $\mathbf{M}^\star_k$ consists of $\mu^\star_t$-stable objects for $t \in (t_k, t_{k-1})$ and is also smooth by Proposition 6.10.

Recall that $J_C := \mathcal{P}ic^0(C)$ is defined to be the moduli space of degree zero line bundles on $C$, which is isomorphic to $C$ itself as $C$ is an elliptic curve. In the $k = 1$ case, we can describe $\mathbf{M}^\star_1$ by the stable pair moduli space:

**Lemma 6.12.** *For $\beta = i_{c*}[H]$, we have the isomorphism*

$$P_{-n}(X, \beta) \times J_C \xrightarrow{\cong} \mathbf{M}^\star_1 \tag{6.32}$$

*given by*

$$((\mathcal{O}_X \to i_{c*}F'), L) \mapsto p^*_C L \otimes \mathbb{D}(\mathcal{O}_X \to i_{c*}F'). \tag{6.33}$$

*Here $\mathbb{D} := \mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)$ is the derived dual.*

*Proof.* First we need to show that the map (6.33) is well-defined, i.e. the object

$$p^*_C L \otimes \mathbb{D}(\mathcal{O}_X \to i_{c*}F') \in \mathcal{A} \tag{6.34}$$

on the RHS of (6.33) corresponds to a point in $\mathbf{M}^\star_1$. By [Tod, Remark 9.8], an object in $E \in \mathcal{A}$ is of the form (6.34) if and only if $E$ fits into an exact sequence in $\mathcal{A}$

$$0 \to p^*_C L \to E \to i_{c*}F''[-1] \to 0 \tag{6.35}$$

where $F''$ is a pure one-dimensional sheaf on $S$ such that $\mathrm{Hom}(T[-1], E) = 0$ for any one-dimensional sheaf $T$ on $X$. Moreover in this case we have $i_{c*}F'' = \mathcal{E}xt^2_X(i_{c*}F', \mathcal{O}_X)$. The proof of Lemma 6.6 shows that $E$ fits into an exact sequence

$$0 \to i_{c*}F'''[-1] \to E \to p^*_C L(c) \to 0 \tag{6.36}$$

for $[F'''] \in U_1$. Therefore $E$ is isomorphic to $(p^*_C L(c) \xrightarrow{s} i_{c*}F''')$ for a non-zero $s$, hence gives a point in $\mathbf{M}^\star_1$ by Proposition 6.8.

Conversely, by Proposition 6.8, any object $[E] \in \mathbf{M}_1^\star$ fits into an exact sequence of the form (6.36). By taking the cohomology of $E$, it also fits into a non-split exact sequence of the form (6.35). On the other hand, by the exact sequence (6.36) we see that $\mathrm{Hom}(T[-1], E) = 0$ for any one-dimensional sheaf $T$ on $X$. Therefore $E$ is of the form (6.34), and the map (6.33) is bijective on closed points. Since both sides of (6.32) are smooth, it is an isomorphism. □

In general for $k > 0$, we can describe $\mathbf{M}_k^\star$ in terms of pair moduli spaces $\mathcal{P}_k$ on $S$:

**Lemma 6.13.** *For $k > 0$, we have an isomorphism*

$$\mathcal{P}_k \times (C \times J_C) \xrightarrow{\cong} \mathbf{M}_k^\star \tag{6.37}$$

*given by*

$$((\mathcal{O}_S \to F), c, L) \mapsto (p_C^*(\mathcal{O}_C(k[c]) \otimes L) \to i_{c*}F). \tag{6.38}$$

*Proof.* The map (6.38) is a morphism of smooth algebraic spaces which is bijective on closed points by Proposition 6.8. Hence it is an isomorphism. □

We also set

$$U_k^\star := (M_{t_k}^\star)^{\mathrm{red}}, \quad k \in \mathbb{Z}_{>0}.$$

By the open immersions $\mathcal{M}_{t_k+\varepsilon}^\star \subset \mathcal{M}_{t_k}^\star \supset \mathcal{M}_{t_k-\varepsilon}^\star$ for $0 < \varepsilon \ll 1$, and noting that $\mathbf{M}_k^\star$ is smooth, we have the induced morphisms

$$
\begin{array}{ccc}
\mathbf{M}_k^\star & & \mathbf{M}_{k+1}^\star \\
& \searrow \quad \swarrow & \\
\pi_k^{\star+} & U_k^\star & \pi_k^{\star-}
\end{array}
\tag{6.39}
$$

**Lemma 6.14.** (i) *We have an isomorphism*

$$U_k \times (C \times J_C) \xrightarrow{\cong} U_k^\star. \tag{6.40}$$

(ii) *Under the isomorphisms (6.37), (6.40), the diagram (6.39) is identified with the diagram (6.4) $\times \mathrm{id}_{C \times J_C}$.*

*Proof.* (i) By Lemma 6.9, a point in $M_{t_k}^\star$ corresponds to a $\mu_{t_k}^\star$-polystable object of the form (6.23). Therefore we have the morphism

$$U_k \times (C \times J_C) \to M_{t_k}^\star$$

defined by

$$(F, c, L) \mapsto p_C^*(\mathcal{O}_C(k[c]) \otimes L) \oplus i_{c*}F[-1]. \tag{6.41}$$

The morphism (6.41) is a bijection on closed points. Moreover the proof of [Tod, Lemma 9.21] shows that (6.41) is a closed immersion. Therefore we have the isomorphism (6.40) by taking the reduced parts of (6.41).

(ii) The statement $\pi_k^{\star+} = \pi_k \times \mathrm{id}_{C \times J_C}$ is obvious from the descriptions of the maps (6.38), (6.41). As for $\pi_k^{\star-}$, let us take a point

$$((\mathcal{O}_S \xrightarrow{s'} F'), c, L') \in \mathcal{P}_{k+1} \times (C \times J_C).$$

Under the map (6.37), it corresponds to a point in $\mathbf{M}_{k+1}^\star$ of the form

$$E' = (p_C^* \mathcal{L}' \to i_{c*} F') \in \mathcal{A}, \quad \mathcal{L}' = \mathcal{O}_C((k+1)[c]) \otimes L' \in \mathcal{P}ic^{k+1}(C).$$

By taking the cohomology of $E'$, we have an exact sequence in $\mathcal{A}$

$$0 \to p_C^* \mathcal{L}'(-c) \to E' \to G[-1] \to 0$$

where $G$ is the cokernel of $s'$. Then the map $\pi_k^{\star-}$ is given by

$$\pi_k^{\star-}(E') = p_C^* \mathcal{L}'(-c) \oplus G[-1].$$

As $\mathcal{L}'(-c) = \mathcal{O}_C(k[c]) \otimes L'$, it comes from $(G, c, L') \in U_k \times (C \times J_C)$ under the map (6.41). Therefore the identity $\pi_k^{\star-} = \pi_k^- \times \mathrm{id}_{C \times J_C}$ also holds.                    □

**Proposition 6.15.** *The diagram* (6.39) *satisfies Assumption* 4.2 *by setting*

$$M^+ = \mathbf{M}_k^\star, \quad M^- = \mathbf{M}_{k+1}^\star, \quad U = U_k^\star, \quad \pi^\pm = \pi_k^{\star\pm}.$$

*Proof.* Note that the diagram (6.39) is a wall-crossing diagram in the CY 3-fold $X$. Together with the fact that a point in $U_k^\star$ corresponds to a $\mu_t^\star$-polystable object (6.23), it is a d-critical simple flip by [Tod, Example 6.3] (see also [Tod, proof of Theorem 9.22]). Therefore Assumption 4.2(i) holds. By Lemma 6.11 Assumption 4.2(ii) also holds, and Assumption 4.2(iii) holds by the same argument as used for Lemma 5.6.                    □

### 6.5. Proof of Theorem 6.2

We first prove Theorem 6.2 for $k > 0$. Let $\mathcal{W}_k$ be the fiber product of the diagram (6.39), and $\mathcal{O}_{\mathbf{M}_k^\star}(1)$ be a $\pi_k^{\star+}$-ample line bundle on $\mathbf{M}_k^\star$ satisfying Assumption 4.2(iii) for the diagram (6.39). By Theorem 4.5 and Proposition 6.15, we have the fully faithful functors

$$\Phi^{\mathcal{O}_{\mathcal{W}_k}} : D^b(\mathbf{M}_{k+1}^\star) \hookrightarrow D^b(\mathbf{M}_k^\star), \quad \Upsilon_k^i : D^b(U_k^\star) \hookrightarrow D^b(\mathbf{M}_k^\star).$$

Here $\Upsilon_k^i$ is given by $\mathbf{L}(\pi_k^{\star+})^*(-) \otimes \mathcal{O}_{\mathbf{M}_k^\star}(i)$. Moreover we have the SOD

$$D^b(\mathbf{M}_k^\star) = \langle \mathrm{Im}\, \Upsilon_k^{-2k-n+1}, \dots, \mathrm{Im}\, \Upsilon_k^0, \mathrm{Im}\, \Phi^{\mathcal{O}_{\mathcal{W}_k}} \rangle. \tag{6.42}$$

Then by Lemma 6.14(ii), the functors (6.42) are linear over $C \times J_C$ under the isomorphisms (6.37), (6.40), so Theorem 6.2 for $k > 0$ follows by restricting the SOD (6.42) to $U_k \times \{(0, 0)\} \subset U_k^\star$ (see [Kuz11, Proposition 5.1, Theorem 6.4]).

Finally, we prove Theorem 6.2 for $k = 0$. By setting $\beta = i_{c*}[H]$ we define

$$\mathbf{M}_0^\star := P_n(X, \beta) \times J_C, \quad U_0^\star := U_n(X, \beta) \times J_C.$$

Then we have the diagram

$$
\begin{array}{ccc}
\mathbf{M}_0^\star & & \mathbf{M}_1^\star \\
& & \\
\searrow \pi_0^{\star+} & & \pi_0^{\star-} \swarrow \\
& U_0^\star &
\end{array}
\tag{6.43}
$$

by taking the product of the diagram (5.3) with $J_C$ via the isomorphism (6.32). The diagram (6.43) satisfies Assumption 4.2 as in the proof of Theorem 5.7. On the other hand, similarly to Lemma 6.13 and Lemma 6.14, we have isomorphisms

$$
\mathcal{P}_0 \times (C \times J_C) \xrightarrow{\cong} \mathbf{M}_0^\star, \quad ((\mathcal{O}_S \xrightarrow{s} F), c, L) \to ((\mathcal{O}_X \xrightarrow{s} i_{c*}F), L),
$$
$$
U_0 \times (C \times J_C) \xrightarrow{\cong} U_0^\star, \quad (F, c, L) \mapsto (i_{c*}F, L).
$$

Under the above isomorphisms, the diagram (6.43) is identified with the diagram (6.4) $\times$ $\mathrm{id}_{C \times J_C}$ for $k = 0$. Therefore the argument for $k > 0$ also implies Theorem 6.2 for $k = 0$.

$\square$

# References

[AHLH]    Alper, J., Halpern-Leistner, D., Heinloth, J.: Existence of moduli spaces for algebraic stacks. arXiv:1812.01128 (2018)

[Bal17]   Ballard, M. R.: Wall crossing for derived categories of moduli spaces of sheaves on rational surfaces. Algebr. Geom. **4**, 263–280 (2017)   Zbl 1405.14110   MR 3652079

[BFK19]   Ballard, M., Favero, D., Katzarkov, L.: Variation of geometric invariant theory quotients and derived categories. J. Reine Angew. Math. **746**, 235–303 (2019)   Zbl 1432.14015   MR 3895631

[BP10]    Baranovsky, V., Pecharich, J.: On equivalences of derived and singular categories. Cent. Eur. J. Math. **8**, 1–14 (2010)   Zbl 1191.14004   MR 2593258

[Bay18]   Bayer, A.: Wall-crossing implies Brill–Noether: applications of stability conditions on surfaces. In: Algebraic Geometry (Salt Lake City, 2015), Proc. Sympos. Pure Math. 97, Amer. Math. Soc., Providence, RI, 3–27 (2018)   Zbl 07272583   MR 3821144

[BB⁺15]   Ben-Bassat, O., Brav, C., Bussi, V., Joyce, D.: A 'Darboux Theorem' for shifted symplectic structures on derived Artin stacks, with applications. Geom. Topol. **19**, 1287–1359 (2015)   Zbl 1349.14003   MR 3352237

[BO]      Bondal, A., Orlov, D.: Semiorthogonal decomposition for algebraic varieties. arXiv:alg-geom/9506012 (1995)

[BO01]    Bondal, A., Orlov, D.: Reconstruction of a variety from the derived category and groups of autoequivalences. Compos. Math. **125**, 327–344 (2001)   Zbl 0994.18007   MR 1818984

[Bri02]    Bridgeland, T.: Flops and derived categories. Invent. Math. **147**, 613–632 (2002)
           Zbl 1085.14017  MR 1893007

[Efi17]    Efimov, A. I.: Cyclic homology of categories of matrix factorizations. Int. Math. Res.
           Notices **2018**, 3834–3869 (2018)  Zbl 1435.18013  MR 3815168

[EP15]     Efimov, A. I., Positselski, L.: Coherent analogues of matrix factorizations and
           relative singularity categories. Algebra Number Theory **9**, 1159–1292 (2015)
           Zbl 1333.14018  MR 3366002

[Fla]      Flamini, F.: Lectures on Brill–Noether theory. Lecture notes at Workshop on Curves
           and Jacobians (Yangyang, 2010), 19 pp. (online)

[Gin]      Ginensky, A.: A shorter proof of Marten's theorem. arXiv:1606.03652 (2016)

[HL]       Halpern-Leistner, D.: The D-equivalence conjecture for moduli spaces of sheaves on
           a K3 surface. http://www.math.columbia.edu/~danhl/

[HL15]     Halpern-Leistner, D.: The derived category of a GIT quotient. J. Amer. Math. Soc. **28**,
           871–912 (2015)  Zbl 1354.14029  MR 3327537

[Hir17a]   Hirano, Y.: Derived Knörrer periodicity and Orlov's theorem for gauged Landau–
           Ginzburg models. Compos. Math. **153**, 973–1007 (2017)        Zbl 1370.14019
           MR 3631231

[Hir17b]   Hirano, Y.: Equivalences of derived factorization categories of gauged Landau–
           Ginzburg models. Adv. Math. **306**, 200–278 (2017)  Zbl 1386.14075  MR 3581302

[HL97]     Huybrechts, D., Lehn, M.: The Geometry of Moduli Spaces of Sheaves. Aspects Math.
           E31, Vieweg, Braunschweig (1997)  Zbl 0872.14002  MR 1450870

[Isi13]    Isik, M. U.: Equivalence of the derived category of a variety with a singularity cate-
           gory. Int. Math. Res. Notices **2013**, 2787–2808  Zbl 1312.14052  MR 3071664

[Joy]      Joyce, D.: Shifted symplectic geometry, Calabi–Yau moduli spaces, and generaliza-
           tions of Donaldson–Thomas theory: our current and future research. Talks given at
           Oxford, October 2013, at a workshop for EPSRC Programme Grant research group;
           https://people.maths.ox.ac.uk/joyce/PGhandout.pdf

[Joy15]    Joyce, D.: A classical model for derived critical loci. J. Differential Geom. **101**, 289–
           367 (2015)  Zbl 1368.14027  MR 3399099

[JS12]     Joyce, D., Song, Y.: A theory of generalized Donaldson–Thomas invariants. Mem.
           Amer. Math. Soc. **217**, no. 1020, iv+199 pp. (2012)  Zbl 1259.14054  MR 2951762

[Kas13]    Kass, J. L.: Singular curves and their compactified Jacobians. In: A Celebration of
           Algebraic Geometry, Clay Math. Proc. 18, Amer. Math. Soc., Providence, RI, 391–
           427 (2013)  Zbl 1317.14068  MR 3114949

[KY00]     Kawai, T., Yoshioka, K.: String partition functions and infinite products. Adv. Theor.
           Math. Phys. **4**, 397–485 (2000)  Zbl 1013.81043  MR 1838446

[Kaw02]    Kawamata, Y.: $D$-equivalence and $K$-equivalence. J. Differential Geom. **61**, 147–171
           (2002)  Zbl 1056.14021  MR 1949787

[Kem73]    Kempf, G.: On the geometry of a theorem of Riemann. Ann. of Math. (2) **98**, 178–185
           (1973)  Zbl 0275.14023  MR 0349687

[KS]       Kontsevich, M., Soibelman, Y.: Stability structures, motivic Donaldson–Thomas in-
           variants and cluster transformations. arXiv:0811.2435 (2008)

[KT14]     Kool, M., Thomas, R.: Reduced classes and curve counting on surfaces I: theory. Al-
           gebr. Geom. **1**, 334–383 (2014)  Zbl 1322.14085  MR 3238154

[Kuz11]    Kuznetsov, A.: Base change for semiorthogonal decompositions. Compos. Math. **147**,
           852–876 (2011)  Zbl 1218.18009  MR 2801403

[Lie06]    Lieblich, M.: Moduli of complexes on a proper morphism. J. Algebraic Geom. **15**,
           175–206 (2006)  Zbl 1085.14015  MR 2177199

[MT18]      Maulik, D., Toda, Y.: Gopakumar–Vafa invariants via vanishing cycles. Invent. Math.
            **213**, 1017–1097 (2018)  Zbl 1400.14141  MR 3842061

[Orl92]     Orlov, D. O.: Projective bundles, monoidal transformations, and derived categories of
            coherent sheaves. Izv. Ross. Akad. Nauk Ser. Mat. **56**, 852–862 (1992) (in Russian)
            Zbl 0798.14007  MR 1208153

[Orl09]     Orlov, D.: Derived categories of coherent sheaves and triangulated categories of sin-
            gularities. In: Algebra, Arithmetic, and Geometry: in Honor of Yu. I. Manin, Progr.
            Math. 270, Birkhäuser, 503–531 (2009)  Zbl 1200.18007  MR 2641200

[PT09]      Pandharipande, R., Thomas, R. P.: Curve counting via stable pairs in the derived cate-
            gory. Invent. Math. **178**, 407–447 (2009)  Zbl 1204.14026  MR 2545686

[PT10]      Pandharipande, R., Thomas, R. P.: Stable pairs and BPS invariants. J. Amer. Math.
            Soc. **23**, 267–297 (2010)  Zbl 1250.14035  MR 2552254

[PT14]      Pandharipande, R., Thomas, R. P.: 13/2 ways of counting curves. In: Moduli Spaces,
            London Math. Soc. Lecture Note Ser. 411, Cambridge Univ. Press, Cambridge, 282–
            333 (2014)  Zbl 1310.14031  MR 3221298

[PT$^+$13]  Pantev, T., Toën, B., Vaquie, M., Vezzosi, G.: Shifted symplectic structures. Publ.
            Math. IHES **117**, 271–328 (2013)  Zbl 1328.14027  MR 3090262

[Rei]       Reid, M.: Minimal models of canonical 3-folds. In: Algebraic Varieties and Analytic
            Varieties, S. Iitaka (ed.), Adv. Stud. Pure Math. 1, Kinokuniya, Tokyo, and North-
            Holland, Amsterdam, 131–180 (1983)  Zbl 0558.14028  MR 0715649

[Rei92]     Reid, M.: What is a flip. Colloquium talk, Univ. of Utah

[Shi12]     Shipman, I.: A geometric approach to Orlov's theorem. Compos. Math. **148**, 1365–
            1389 (2012)  Zbl 1253.14019  MR 2982435

[Tod]       Toda, Y.: Birational geometry for d-critical loci and wall-crossing in Calabi–Yau
            3-folds. arXiv:1805.00182 (2018)

[Tod12a]    Toda, Y.: Stability conditions and curve counting invariants on Calabi–Yau 3-folds.
            Kyoto J. Math. **52**, 1–50 (2012)  Zbl 1244.14047  MR 2892766

[Tod12b]    Toda, Y.: Stable pairs on local K3 surfaces. J. Differential. Geom. **92**, 285–370 (2012)
            Zbl 1260.14045  MR 2998674

[Tod14a]    Toda, Y.: A note on Bogomolov–Gieseker type inequality for Calabi–Yau 3-folds.
            Proc. Amer. Math. Soc. **142**, 3387–3394 (2014)  Zbl 1337.14019  MR 3238415

[Tod14b]    Toda, Y.: Stability conditions and birational geometry of projective surfaces. Compos.
            Math. **150**, 1755–1788 (2014)  Zbl 1329.14032  MR 3269467

[Toë14]     Toën, B.: Derived algebraic geometry and deformation quantization. In: Proc. Int.
            Congress of Mathematicians (Seoul, 2014), Vol. II, Kyung Moon Sa, Seoul, 769–792
            (2014)  Zbl 1373.14003  MR 3728637

[Yos99]     Yoshioka, K.: Some examples of Mukai's reflections on K3 surfaces. J. Reine Angew.
            Math. **515**, 97–123 (1999)  Zbl 0940.14026  MR 1717621