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Categorical Plücker formula and homological projective duality

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Abstract. Kuznetsov's homological projective duality (HPD) theory [K4] is one of the most active and powerful recent developments in the homological study of algebraic geometry. The fundamental theorem of HPD systematically compares derived categories of dual linear sections of a pair of HP-dual varieties $(X, X^{\natural})$. In this paper we generalize the fundamental theorem of HPD beyond linear sections. More precisely, we show that for any two pairs of HP-duals $(X, X^{\natural})$ and $(T, T^{\natural})$ which intersect properly, there exist semiorthogonal decompositions of the derived categories $D(X \cap T)$ and $D(X^{\natural} \cap T^{\natural})$ into primitive and ambient parts, and that there is an equivalence of primitive parts ${}^{\mathrm{prim}}D(X \cap T) \simeq D(X^{\natural} \cap T^{\natural})^{\mathrm{prim}}$.

Keywords. Categorification, Plücker formula, homological projective duality

1. Introduction

Homological projective duality (HPD), as a homological generalization of classical projective duality, is one of the most powerful theories in the homological study of algebraic geometry. Since its introduction by Kuznetsov [K4], HPD theory has become the primary method to produce semiorthogonal decompositions of derived categories of coherent sheaves on smooth projective varieties, as well as relate derived categories of different varieties [K1, K3, K6, K8, IK, ABB, K10, ADS, T2, HT17, CT, BBF, HT16].

HPD theory is very closely related to equivalences and dualities in physical theories of gauged linear sigma models [HHP, HT, DSh, CDH+, S, Hor, HK]. Mathematically, this is related to the mathematical formulation of Landau–Ginzburg models [Is, Shi, ADS] and the theory of derived categories of geometric invariant theory (GIT) quotients [Kaw, Vd, Se, DS, HW, BFK, HL]. This thread of ideas has provided a powerful approach to construct examples of HPD as well as to find applications of HPD, see [BDF+, ADS, ST18, Re, RS].

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The *fundamental theorem of HPD* of Kuznetsov [K4, Thm. 6.3] is as follows. Suppose that $(X \to \mathbb{P}^n, X^\natural \to \check{\mathbb{P}}^n)$ is a pair of *homological projective dual* (*HP-dual*) varieties (which are by definition smooth and projective) with semiorthogonal decompositions (of Lefschetz and dual Lefschetz types, see §2.3)

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{i-1}(i-1)\rangle, \qquad D(X^\natural) = \langle \mathcal{A}^1(1-n), \ldots, \mathcal{A}^{n-1}(-1), \mathcal{A}^n\rangle,$$

where the building blocks $\mathcal{A}_\bullet$ and $\mathcal{A}^\bullet$ are "complementary" to each other, illustrated in the first diagram of Fig. 1 (§3.1; see also §2.4). Then first, there are *primitive decompositions* of the derived categories of all complete linear sections $X \times_{\mathbb{P}^n} L$ and $X^\natural \times_{\check{\mathbb{P}}^n} L^\perp$, where $L \subset \mathbb{P}^n$ is a linear subspace of codimension ℓ, and $L^\perp \subset \check{\mathbb{P}}^n$ is the orthogonal (also called the dual linear subspace) of L. More precisely, we have a semiorthogonal decomposition

$$D(X \times_{\mathbb{P}^n} L) = \langle {}^{\mathrm{prim}}D(X \times_{\mathbb{P}^n} L^\perp), \mathcal{A}_{\ell-1}(\ell-1), \ldots, \mathcal{A}_{i-1}(i-1)\rangle,$$

where ${}^{\mathrm{prim}}D(X \times_{\mathbb{P}^n} L)$ is the (left) primitive part, also called the "interesting" or "non-trivial" part of $D(X \times_{\mathbb{P}^n} L)$, and the remaining $\mathcal{A}_k(k)$-terms are ambient or "trivial" pieces coming from the ambient space. There is also a similar primitive decomposition of $D(X^\natural \times_{\check{\mathbb{P}}^n} L^\perp)$. Second, the primitive parts of the two categories are equivalent:

$${}^{\mathrm{prim}}D(X \times_{\mathbb{P}^n} L) \simeq D(X^\natural \times_{\check{\mathbb{P}}^n} L^\perp)^{\mathrm{prim}}$$

(see Thm. 2.18 for details). As the dual linear space $L^\perp$ is also the HP-dual of L (see Ex. 2.17), it is natural to ask: can we generalize the fundamental theorem of HPD beyond linear sections, by replacing $(L, L^\perp)$ by other nonlinear HP-dual spaces $(T, T^\natural)$?

Our main result answers this affirmatively: we can replace $(L, L^\perp)$ by *any* other pair of HP-dual spaces $(T, T^\natural)$, provided they intersect properly. Denote the (Lefschetz and dual Lefschetz) decompositions for the other HP-dual pair $(T, T^\natural)$ by

$$D(T) = \langle \mathcal{C}^1(1-n), \ldots, \mathcal{C}^{n-1}(-1), \mathcal{C}^n\rangle, \quad D(T^\natural) = \langle \mathcal{C}_0, \mathcal{C}_1(1), \ldots, \mathcal{C}_{\ell-1}(\ell-1)\rangle,$$

Main Theorem 1.1 (Thm. 3.1). *Assume that the pairs $(X, X^\natural)$ and $(T, T^\natural)$ intersect properly (i.e. they are* admissible*, see Def. 3.9). Then there are* primitive decompositions *of the derived categories of their fiber products into primitive and ambient parts:*

$$D(X \times_{\mathbb{P}^n} T) = \big\langle {}^{\mathrm{prim}}D(X \times_{\mathbb{P}^n} T), \langle (\mathcal{A}_k \boxtimes \mathcal{C}^k) \otimes \mathscr{O}(k)\rangle_{k\in\mathbb{Z}}\big\rangle,$$
$$D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural) = \big\langle \langle (\mathcal{A}^k \boxtimes \mathcal{C}_k) \otimes \mathscr{O}(1-\ell+k)\rangle_{k\in\mathbb{Z}}, D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)^{\mathrm{prim}}\big\rangle.$$

Moreover, there is an equivalence of the primitive components:

$${}^{\mathrm{prim}}D(X \times_{\mathbb{P}^n} T) \simeq D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)^{\mathrm{prim}}.$$

The categorical primitive decompositions of this main theorem can be visualized as in Fig. 1 (§3.1), and regarded as obeying a simple "Poincaré pairing rule" (Rmk. 3.4). The theorem shows that categorical primitive decompositions hold for intersections in great generality, a phenomenon which does not seem to occur at the level of classical cohomology (except for the Plücker formula for the Euler characteristic, to be discussed below).

The theorem also provides a systematic way of giving exciting and subtle relationships between different varieties that may not be easily seen from the classical perspective. Since the release of the preprint of this paper, our theorem has been used by Ottem–Rennemo [OR] and Borisov–Căldăraru–Perry [BCP] to provide counterexamples to the Birational Torelli Conjecture for Calabi–Yau threefolds, and by Manivel [M] to provide examples of the same phenomenon for Calabi–Yau manifolds of dimension 5.

If we take $(T, T^{\natural}) = (L, L^{\perp})$, then the theorem reduces to the fundamental theorem of HPD. In fact, our approach generalizes Richard Thomas' methods [T2], and solves the problem posed by Kuznetsov [K4, p. 182] (see Rmk. 3.2) in a very general form, providing a "more direct proof" of the original fundamental theorem of HPD.

1.1. Plücker formula

As Kuznetsov's HPD theory is the homological counterpart of the classical *Lefschetz theory* for cohomology of linear sections of projective varieties, our investigation is motivated by the topological *Plücker formula*. The Plücker formula of the second named author [L1] is a higher-dimensional generalization of the classical Plücker formula for dual curves in $\mathbb{P}^2$. It states that for any two subvarieties $X, T \subset \mathbb{P}^n$ which intersect transversely, and whose projective duals $X^\vee, T^\vee \subset \check{\mathbb{P}}^n$ also intersect transversely, one has

$$\chi(X \cap T) - \frac{\tilde{\chi}(X) \cdot \tilde{\chi}(T)}{n+1} = \pm\left(\chi(X^\vee \cap T^\vee) - \frac{\tilde{\chi}(X^\vee) \cdot \tilde{\chi}(T^\vee)}{n+1}\right) \tag{1.1}$$

(see [L1, CL]; the sign is determined by $\pm = (-1)^*$, $* = \dim X + \dim T + \dim X^\vee + \dim T^\vee$). Here $\chi(-)$ is the usual Euler characteristic and $\tilde{\chi}(X) = \chi(X, \mathrm{Eu}[X])$ is the weighted Euler characteristic for a (possibly singular) scheme X with respect to MacPherson's local Euler obstruction function $\mathrm{Eu}(X)$ [Be]. If X is smooth, then $\tilde{\chi}(X) = (-1)^{\dim X}\chi(X)$.

On the other hand, if we "de-categorify" twice (i.e. take the Hochschild homology and then the Euler characteristic of) our main theorem, we obtain

Corollary 1.2 (Cor. 3.5). *Assume $X \times_{\mathbb{P}^n} T$ and $X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural}$ are smooth. Then*

$$\chi(X \times_{\mathbb{P}^n} T) - \frac{\chi(X) \cdot \chi(T)}{n+1} = \chi(X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural}) - \frac{\chi(X^{\natural}) \cdot \chi(T^{\natural})}{n+1}.$$

This has the same form as the topological Plücker formula, with the classical projective duals $X^\vee, T^\vee$ replaced by the HP-duals $X^{\natural}, T^{\natural}$. Therefore our main theorem can be viewed as a "2-level categorification" of the Plücker formula. The Plücker formula is closely related to the theory of Lagrangian intersections inside hyperkähler manifolds under Mukai flops [L1, L2]. It would be very interesting to further explore the relationships between our main theorem and the categorification of Lagrangian intersections.

1.2. The "chess game" approach and strategy of proof

The "chess game" method, introduced by Richard Thomas [T2] and Kuznetsov, and further developed in this paper, is a systematic method of comparing two subcategories $\mathscr{D}_1$

and $\mathscr{D}_2$, through analyzing them on a two-dimensional diagram called "chessboard" (see Fig. 2–4).

Our strategy of proving the main theorem is as follows. First, we use base change of categories to put $D(X \times_{\mathbb{P}^n} T)$ and $D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)$ into a common ambient category $D(\mathcal{H}_{X,T^\natural}) = D(\mathcal{H}_{T^\natural,X})$ (see §3.2). Then we can play the "chess game" to compare these two categories. The "chess game" can be further divided into two steps: the proof of fully-faithfulness and of generation. The "fully faithful" part of the main theorem follows from analyzing the patterns of mutation functors on the chessboard (see §3.4). The most subtle part is the proof of generation, which is done through a specific "zig-zag" scheme in §3.5.

This "chess game" approach has the benefits that it is independent of the original linear fundamental theorem of HPD, and provides a more direct proof of it; the Fourier–Mukai functor for the equivalence of primitive parts is very explicit (see Thm. 3.1 or Thm. 3.12); the primitive decompositions obey a simple "Poincaré pairing rule" (see Rmk. 3.4). Moreover, our general result on the chess game, Thm. 3.12, can be applied to many other situations, including various situations of flops [JL2, JLX]. The chess game method can also be applied to study autoequivalences and is closely related to spherical functors [ST01].

Related work. While preparing this paper, we learned that Alexander Kuznetsov and Alex Perry also claimed similar results using very different methods [KP19].

Conventions. Our argument in this paper can be applied to noncommutative settings as well (as we did in the arXiv version of this paper). For simplicity and concreteness, we will stick to commutative world of algebraic varieties, explicitly present the Fourier–Mukai kernels involved, and verify the Tor-independent conditions in detail (§3.2).

All schemes are assumed to be embeddable k-varieties, i.e. k-varieties admitting finite surjections onto smooth k-varieties, where k is an algebraically closed field of characteristic zero. All categories are assumed to be k-linear. We use $D(X) := D^b(\mathrm{coh}(X))$ to denote the *bounded* derived category of coherent sheaves on an algebraic variety X. For a morphism $f : X \to Y$ we denote by $f_* : D(X) \to D(Y)$ and $f^* : D(Y) \to D(X)$ the *derived* pushforward and derived pullback functors. We use $\otimes$ for the derived tensor product. For a k-linear category $\mathcal{C}$, we denote by $\mathrm{Hom}_{\mathcal{C}}$ the k-linear hom spaces inside the category, and by $R\,\mathrm{Hom}_{\mathcal{C}}$ the derived Hom functor. When X is a variety, we denote by Hom_X the k-linear hom spaces inside $D(X)$, by $\mathscr{H}om_X$ the local sheaf hom functor, and by $R\,\mathrm{Hom}_X$, resp. $R\,\mathscr{H}om_X$ the corresponding derived functors. For objects $A, B \in \mathcal{C}$, we denote by $0 \in \mathrm{Hom}(A, B)$ the zero element of the k-vector space $\mathrm{Hom}(A, B)$. We also use $0 \in \mathcal{C}$ to denote the zero object of the k-linear category $\mathcal{C}$. The notation $\mathbb{P}^n = \mathrm{Proj}\,\mathrm{k}[x_0, \ldots, x_n]$ stands for the projective space of dimension n over k, and $\check{\mathbb{P}}^n$ denotes the dual projective space of $\mathbb{P}^n$, i.e. $\check{\mathbb{P}}^n$ parametrizes hyperplanes H of $\mathbb{P}^n$. For a variety with a morphism $X \to \mathbb{P}^n$, we use $X^\vee \subset \check{\mathbb{P}}^n$ to denote the (*classical*) *projective dual*, and $X^\natural$ for the *HP-dual* of X (see §2.4 for the definitions). We will use the "homological convention" $\mathcal{A}_k$ for Lefschetz decompositions and the "cohomological convention" $\mathcal{B}^j$ for dual Lefschetz ones (see §2.3).

2. Preliminaries

The bounded derived category of coherent sheaves, $D(X) = D^b(\mathrm{coh}\, X)$, on an algebraic variety X was introduced by Verdier [Ve] in the 1950's. Through the works of Beilinson [Bei], Mukai [Mu], Bondal [B], Orlov [BO], Kapranov [BK, Kap] and others from the 1980's, $D(X)$ has become a central object of investigation. The derived categories of coherent sheaves also play important roles in Kontsevich's homological mirror symmetry conjecture [Kon] and its applications (see for example [AAE$^+$, FLTZ, CPU, CHL, SS]), Bridgeland's stability conditions [Br2] and Donaldson–Thomas theory [T]. See [BO, H, Ca1] for more about $D(X)$. In the following we focus on semiorthogonal decompositions.

2.1. Semiorthogonal decompositions and mutations

References for this section are [B, BK, H, K4, K7]. A *semiorthogonal decomposition* of a triangulated category $\mathcal{T}$, written as

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle, \tag{2.1}$$

is formed by a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ of $\mathcal{T}$ such that

(1) $\mathrm{Hom}_{\mathcal{T}}(a_k, a_\ell) = 0$ for all $a_k \in \mathcal{A}_k$ and $a_\ell \in \mathcal{A}_\ell$, if $k > \ell$,
(2) for any object $a \in \mathcal{T}$, there is a sequence of objects t_k and morphisms

$$0 = t_n \to t_{n-1} \to \cdots \to t_1 \to t_0 = a$$

such that each cone $a_k = \mathrm{cone}(t_k \to t_{k-1}) \in \mathcal{A}_k$, $k = 1, \dots, n$.

The subcategories $\mathcal{A}_k$ are called the *components* of $\mathcal{T}$ with respect to (2.1). The first condition implies the objects $t_k \in \mathcal{T}$ and $a_k \in \mathcal{A}_k$ are uniquely determined by (and functorial in) a [K7, Lem. 2.4]. The functors $\mathcal{T} \to \mathcal{A}_k$, $a \mapsto a_k$, are called the *projection functors*, and a_k is called the *component* of a in $\mathcal{A}_k$ with respect to the decomposition (2.1). A sequence $\mathcal{A}_1, \dots, \mathcal{A}_n$ satisfying (1) is called *semiorthogonal*, and in this case for any subset $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$, we denote by $\langle \mathcal{A}_{j_1}, \dots, \mathcal{A}_{j_m} \rangle$ the smallest triangulated subcategory of $\mathcal{T}$ generated by $\mathcal{A}_{j_k}$'s for $k = 1, \dots, m$. The semiorthogonal sequence $\mathcal{A}_1, \dots, \mathcal{A}_n$ satisfies (2) if and only if $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$, justifying the notation (2.1).

Example 2.1. For a projective space $\mathbb{P}^{\ell-1}$ with $\ell \geq 1$, we have *Beilinson's decomposition* [Bei]

$$D(\mathbb{P}^{\ell-1}) = \langle \mathscr{O}_{\mathbb{P}^{\ell-1}}, \mathscr{O}_{\mathbb{P}^{\ell-1}}(1), \dots, \mathscr{O}_{\mathbb{P}^{\ell-1}}(\ell-1) \rangle, \tag{2.2}$$

by which we mean $\langle \mathscr{O}_{\mathbb{P}^{\ell-1}}(k) \rangle \simeq D(\mathrm{Vect}_{\Bbbk})$ (where $\mathrm{Vect}_{\Bbbk}$ is the abelian category of finite-dimensional k-vector spaces) for $k = 0, 1, \dots, \ell-1$ and the sequence of subcategories $\langle \mathscr{O}_{\mathbb{P}^{\ell-1}} \rangle, \dots, \langle \mathscr{O}_{\mathbb{P}^{\ell-1}}(\ell-1) \rangle$ gives rise to a semiorthogonal decomposition of $D(\mathbb{P}^{\ell-1})$.

Suppose $\mathcal{A}$ is a full triangulated subcategory of a triangulated category $\mathcal{T}$. Then denote by

$$\mathcal{A}^{\perp} := \{T \in \mathcal{T} \mid \mathrm{Hom}(\mathcal{A}, T) = 0\}, \qquad {}^{\perp}\mathcal{A} := \{T \in \mathcal{T} \mid \mathrm{Hom}(T, \mathcal{A}) = 0\}$$

the *right* and respectively *left orthogonal* of $\mathcal{A}$ inside $\mathcal{T}$. The subcategory $\mathcal{A}$ is called a *left* (resp. *right*) *admissible* subcategory of $\mathcal{T}$ if the inclusion functor $i = i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{T}$ has a left adjoint $i^* : \mathcal{T} \to \mathcal{A}$ (resp. right adjoint $i^! : \mathcal{T} \to \mathcal{A}$). $\mathcal{A}$ is called *admissible* if it is both left and right admissible. If $\mathcal{A} \subset \mathcal{T}$ is admissible, then $\mathcal{A}^\perp$ is left admissible and ${}^\perp\mathcal{A}$ is right admissible, and we have the semiorthogonal decompositions $\mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle$.

For simplicity we will only consider semiorthogonal decompositions (2.1) with admissible components. This condition is automatically satisfied for any semiorthogonal decompositions (2.1) if $\mathcal{T} = D(X)$ and X is a smooth projective variety; in other cases we will give the arguments why the components considered are admissible.

Lemma 2.2. *Assume $\mathcal{T}$ admits a semiorthogonal decomposition (2.1), and let $\mathcal{A}$ be a triangulated subcategory of $\mathcal{T}$. Let $a \in \mathcal{T}$, and a_k be its component in $\mathcal{A}_k$. If $a \in \mathcal{A}^\perp$ (resp. $a \in {}^\perp\mathcal{A}$), and $a_k \in \mathcal{A}^\perp$ (resp. $a_k \in {}^\perp\mathcal{A}$) for all $k \neq \ell$, then $a_\ell \in \mathcal{A}^\perp$ (resp. $a_\ell \in {}^\perp\mathcal{A}$).*

Proof. The lemma follows directly from the fact that $\mathcal{A}^\perp$ (resp. ${}^\perp\mathcal{A}$) is a triangulated subcategory, i.e. is closed under shifts and taking cones. □

Starting with a semiorthogonal decomposition, one can obtain a whole collection of new decompositions by *mutations*. Let $\mathcal{A}$ be an admissible subcategory of a triangulated category $\mathcal{T}$. Then the functor $\mathbb{L}_{\mathcal{A}} := i_{\mathcal{A}^\perp} i^*_{\mathcal{A}^\perp} : \mathcal{A} \to \mathcal{A}$ (resp. $\mathbb{R}_{\mathcal{A}} := i_{{}^\perp\mathcal{A}} i^!_{{}^\perp\mathcal{A}} : \mathcal{A} \to \mathcal{A}$) is called the *left* (resp. *right*) *mutation through* $\mathcal{A}$. For simplicity we will mainly focus on left mutation functors in this paper, and the statements on right mutations are exactly similar. The following results are standard [B, BK, K4].

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{A}_1, \dots, \mathcal{A}_n$ be admissible subcategories of a triangulated category $\mathcal{T}$ where $n \geq 2$. Let k be an integer, $2 \leq k \leq n$.*

(1) *For any $b \in \mathcal{T}$, there are distinguished triangles*

$$i_{\mathcal{A}} i^!_{\mathcal{A}}(b) \to b \to \mathbb{L}_{\mathcal{A}}\, b \xrightarrow{[1]}, \qquad \mathbb{R}_{\mathcal{A}}\, b \to b \to i_{\mathcal{A}} i^*_{\mathcal{A}}(b) \xrightarrow{[1]}.$$

(2) *$(\mathbb{L}_{\mathcal{A}})|_{\mathcal{A}} = 0$ and $(\mathbb{R}_{\mathcal{A}})|_{\mathcal{A}} = 0$ are the zero functors, and $(\mathbb{L}_{\mathcal{A}})|_{\mathcal{A}^\perp} = \mathrm{Id}_{\mathcal{A}^\perp} : \mathcal{A}^\perp \to \mathcal{A}^\perp$, $(\mathbb{R}_{\mathcal{A}})|_{{}^\perp\mathcal{A}} = \mathrm{Id}_{{}^\perp\mathcal{A}} : {}^\perp\mathcal{A} \to {}^\perp\mathcal{A}$ are the identity functors. Furthermore $(\mathbb{L}_{\mathcal{A}})|_{{}^\perp\mathcal{A}} : {}^\perp\mathcal{A} \to \mathcal{A}^\perp$ and $(\mathbb{R}_{\mathcal{A}})|_{\mathcal{A}^\perp} : \mathcal{A}^\perp \to {}^\perp\mathcal{A}$ are mutually inverse equivalences.*

(3) *If $\mathcal{A}_1, \dots, \mathcal{A}_n$ is a semiorthogonal sequence, then*

$$\mathbb{L}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle} = \mathbb{L}_{\mathcal{A}_1} \circ \cdots \circ \mathbb{L}_{\mathcal{A}_n}.$$

(4) *If $\mathcal{A}_1, \dots, \mathcal{A}_{k-1}, \mathcal{A}_k, \mathcal{A}_{k+1}, \dots, \mathcal{A}_n$ is a semiorthogonal sequence inside $\mathcal{T}$, then*

$$\mathcal{A}_1, \dots, \mathcal{A}_{k-2}, \mathbb{L}_{\mathcal{A}_{k-1}}(\mathcal{A}_k), \mathcal{A}_{k-1}, \mathcal{A}_{k+1}, \dots, \mathcal{A}_n$$

is also a semiorthogonal sequence, and it generates the same subcategory:

$$\langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, \mathcal{A}_k, \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle = \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-2}, \mathbb{L}_{\mathcal{A}_{k-1}}(\mathcal{A}_k), \mathcal{A}_{k-1}, \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle.$$

(5) *Let $F : \mathcal{T} \to \mathcal{T}$ be any autoequivalence. Then $F \circ \mathbb{L}_{\mathcal{A}} \simeq \mathbb{L}_{F(\mathcal{A})} \circ F$.*

For a triangulated category $\mathcal{T}$, a *Serre functor* is a (covariant) autoequivalence $S : \mathcal{T} \to \mathcal{T}$ such that for any two objects $F, G \in \mathcal{T}$, there is a bi-functorial isomorphism

$$\mathrm{Hom}(F, G) = \mathrm{Hom}(G, S(F))^{\vee},$$

where $(-)^{\vee}$ denotes the dual vector space over k. If a Serre functor exists, then it is unique up to canonical isomorphisms. If X is a smooth projective variety of dimension n, then $D(X)$ has a Serre functor given by $S_X(-) = - \otimes \omega_X[n]$, where ω_X is the dualizing sheaf on X. For example, when $X = \mathbb{P}^{\ell-1}$ is the projective space, then $S_X = \otimes \mathscr{O}(-\ell)[\ell - 1]$. The Serre functor $S_{\mathcal{T}}$ commutes with any k-linear autoequivalence of $\mathcal{T}$.

Lemma 2.4 ([B, BK]). *Let $\mathcal{T}$ be a triangulated subcategory with a Serre functor S, and $\mathcal{A} \subset \mathcal{T}$ be an admissible subcategory. Then*

(1) *$S({}^{\perp}\mathcal{A}) = \mathcal{A}^{\perp}$ and $S^{-1}(\mathcal{A}^{\perp}) = {}^{\perp}\mathcal{A}$. In particular, if $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$, then $\mathcal{B} = {}^{\perp}\mathcal{A}$, $\mathcal{A} = \mathcal{B}^{\perp}$, and $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle = \langle S(\mathcal{B}), \mathcal{A} \rangle = \langle \mathcal{B}, S^{-1}(\mathcal{A}) \rangle$.*
(2) *$\mathcal{A}$ also admits a Serre functor given by $S_{\mathcal{A}} = i_{\mathcal{A}}^{!} \circ S \circ i_{\mathcal{A}}$.*

Let X, Y be smooth varieties. A *Fourier–Mukai functor* is a functor of the form

$$\Phi_{\mathcal{P}}(-) := \pi_{Y*}(\pi_X^*(-) \otimes \mathcal{P}) : D(X) \to D(Y)$$

for some $\mathcal{P} \in D(X \times Y)$, where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the natural projections. The object $\mathcal{P}$ is called the *kernel* of the functor. The readers are referred to [H, BKR, Br1] for properties and applications of Fourier–Mukai functors.

2.2. *Derived categories over a base*

References for this section are [K2, K4, K9]. Assume S is a smooth k-scheme, and let $f : X \to S$ be a map between smooth varieties. An admissible subcategory $\mathcal{A} \subset D(X)$ is called *S-linear* if for any $a \in \mathcal{A}$ and $F \in D(S)$, we have $a \otimes f^*F \in \mathcal{A}$. We may regard an S-linear triangulated subcategory as a family of categories over S. Under certain conditions we can pull back the family through base change $\phi : T \to S$ to get a family of categories over T with desired properties.

Definition 2.5. A base change $\phi : T \to S$ is called *faithful* with respect to a morphism $f : X \to S$ if the cartesian square

$$\begin{tikzcd} X_T \arrow[r, "\phi_T"] \arrow[d, "f_T"'] & X \arrow[d, "f"] \\ T \arrow[r, "\phi"] & S \end{tikzcd} \tag{2.3}$$

is *exact cartesian*, i.e., the natural transformation $f^* \circ \phi_* \to \phi_{T*} \circ f_T^* : D(T) \to D(X)$ is an isomorphism. A base change $\phi : T \to S$ is called *faithful with respect to a pair* (X, Y) if ϕ is faithful with respect to morphisms $f : X \to S$, $g : Y \to S$, and $f \times_S g : X \times_S Y \to S$.

A base change $\phi : T \to S$ for $X \to S$ is faithful if and only if the square (2.3) is *Tor-independent*, i.e. for all $t \in T$, $x \in X$, and $s \in S$ with $\phi(t) = s = f(x)$,

$\mathrm{Tor}_i^{\mathscr{O}_{S,s}}(\mathscr{O}_{T,t}, \mathscr{O}_{X,x}) = 0$ for all $i > 0$ (see [LM, Thm. 3.10.3], [K2]). The following are certain typical situations when this condition is satisfied.

Lemma 2.6 ([K2, Cor. 2.23, 2.27], [K4, Lem. 2.31, 2.35]). *Let $f : X \to S$ be a morphism and $\phi : T \to S$ be a base change.*

(1) *If ϕ is flat, then it is faithful.*
(2) *If T and X are smooth and X_T has expected dimension $\dim X_T = \dim X + \dim T - \dim S$, then $\phi : T \to S$ is faithful with respect to the morphism $f : X \to S$.*
(3) *If $\phi : T \to S$ is a closed embedding and $T \subset S$ is a locally complete intersection, and both S and X are Cohen–Macaulay, and X_T has expected dimension, i.e. $\dim X_T = \dim X + \dim T - \dim S$, then $\phi : T \to S$ is faithful with respect to the morphism $f : X \to S$.*

The following will be useful later:

Lemma 2.7 ([K2, Lem. 2.25]). *Consider the following commutative diagram of cartesian squares of varieties:*

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

If the right square is exact cartesian, then the ambient square is exact cartesian if and only if the left square is.

The power of faithful base change is that it preserves S-linear fully faithful Fourier–Mukai transforms and semiorthogonal decompositions.

Proposition 2.8 ([K4, Prop. 2.38]). *Suppose $\phi : T \to S$ is a faithful base change for a pair (X, Y) where $f : X \to S$ and $g : Y \to S$, the varieties X and Y are projective over P and smooth, and $K \in D(X \times_S Y)$ is a kernel such that $\Phi_K : D(X) \to D(Y)$ is fully faithful. Then $\phi_{K_T} : D(X_T) \to D(Y_T)$ is fully faithful, where the Fourier–Mukai kernel is $K_T := \phi_T^* K$.*[1]

Proposition 2.9 ([K9, Thm. 5.6]). *Suppose $f : X \to S$ is a map between smooth varieties, and $D(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition by admissible S-linear subcategories. Let $\phi : T \to S$ be a faithful base change for f. Then we have a T-linear semiorthogonal decomposition*

$$D(X_T) = \langle \mathcal{A}_{1T}, \ldots, \mathcal{A}_{nT} \rangle$$

where $\mathcal{A}_{kT}$ is the base change category of $\mathcal{A}_k$ to T, which depends only on $\mathcal{A}_k$, i.e. is independent of the embedding $\mathcal{A}_k \subset D(X)$, and satisfies $\phi_T^*(a) \in \mathcal{A}_{kT}$ for any $a \in \mathcal{A}_k$, and $\phi_{T*}(b) \in \mathcal{A}_k$ for $b \in \mathcal{A}_k$ with proper support over X.

---

[1] K_T a priori only belongs to the bounded above derived category $D^-(X_T \times_T Y_T)$ of quasi-coherent complexes with coherent cohomology. [K2, Lem. 2.4] guarantees that Φ_{K_T} preserves boundedness.

Suppose X, Y are smooth varieties with semiorthogonal decompositions

$$D(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle \quad \text{and} \quad D(Y) = \langle \mathcal{B}_1, \ldots, \mathcal{B}_n \rangle.$$

For $i = 1, \ldots, m$, define the exterior product $\mathcal{A}_i \boxtimes D(Y) \subset D(X \times Y)$ to be the base change category $\mathcal{A}_{iY} \subset D(X \times Y)$ of $\mathcal{A}_i \subset D(X)$ from Prop. 2.9 applied to the base change $Y \to \operatorname{Spec} k$. We define $D(X) \boxtimes \mathcal{B}_j$ similarly. Then the results of [K9, §5.6] can be summarized:

Proposition-Definition 2.10. *In the above situation, there are Y-linear and resp. X-linear semiorthogonal decompositions*

$$D(X \times Y) = \langle \mathcal{A}_i \boxtimes D(Y) \rangle_{1 \le i \le m} \quad \textit{and} \quad D(X \times Y) = \langle D(X) \boxtimes \mathcal{B}_j \rangle_{1 \le j \le n}.$$

Furthermore,

$$D(X \times Y) = \langle \mathcal{A}_i \boxtimes \mathcal{B}_j \rangle_{1 \le i \le m, 1 \le j \le n},$$

where the exterior product $\mathcal{A}_i \boxtimes \mathcal{B}_j$ *is defined by*

$$\mathcal{A}_i \boxtimes \mathcal{B}_j := \mathcal{A}_i \boxtimes D(Y) \cap D(X) \boxtimes \mathcal{B}_j \subset D(X \times Y).$$

Then $D(X) \boxtimes D(Y) = D(X \times Y)$, and we have semiorthogonal decompositions

$$\mathcal{A}_i \boxtimes D(Y) = \langle \mathcal{A}_i \boxtimes \mathcal{B}_1, \ldots, \mathcal{A}_i \boxtimes \mathcal{B}_n \rangle \quad \textit{and} \quad D(X) \boxtimes \mathcal{B}_j = \langle \mathcal{A}_1 \boxtimes \mathcal{B}_j, \ldots, \mathcal{A}_m \boxtimes \mathcal{B}_j \rangle.$$

By the proof of [K9, Thm. 5.8], the exterior product could also be characterized by

$$\mathcal{A}_i \boxtimes \mathcal{B}_j = \langle \pi_X^* \mathcal{A}_i \otimes \pi_Y^* \mathcal{B}_j \rangle \subset D(X \times Y),$$

by which we mean $\mathcal{A}_i \boxtimes \mathcal{B}_j$ is the minimal triangulated subcategory of $D(X \times Y)$ which is closed under taking summands and contains all objects of the form $\pi_X^* a \otimes \pi_Y^* b$ for $a \in \mathcal{A}_i$ and $b \in \mathcal{B}_j$, where π_X, π_Y are the natural projections from $X \times Y$ to X and Y.

2.3. *Lefschetz decompositions*

Lefschetz decomposition, introduced by Kuznetsov [K4], is a type of semiorthogonal decomposition behaving well with respect to the autoequivalence $\otimes \mathscr{O}(1)$ for a line bundle $\mathscr{O}(1)$.

Definition 2.11. Let X be a variety and $\mathscr{O}(1)$ be a line bundle on X. A (*right*) *Lefschetz decomposition* of $D(X)$ is a semiorthogonal decomposition of the form

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{i-1}(i-1) \rangle \tag{2.4}$$

with $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{i-1} \supset 0$ a descending sequence of admissible subcategories, where $\mathcal{A}(k)$ denotes the image of $\mathcal{A} \subset D(X)$ under the autoequivalence $\otimes \mathscr{O}(k)$.

The number $i = \min\{k \mid \mathcal{A}_k = 0\} \in \mathbb{Z}_{\geq 0}$ is called the *length* of the Lefschetz decomposition. The Lefschetz decomposition (2.4) is called *rectangular* if $\mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_{i-1}$. A Lefschetz decomposition is determined by its first component $\mathcal{A}_0$: all other $\mathcal{A}_k$'s can be reconstructed from $\mathcal{A}_0$ via $\mathcal{A}_k = {}^{\perp}\mathcal{A}_0(-k) \cap \mathcal{A}_{k-1}$ [K5, Lem. 2.18].

Following Kuznetsov [K4], we denote by $\mathfrak{a}_k$ the right orthogonal of $\mathcal{A}_{k+1}$ inside $\mathcal{A}_k$ for $0 \leq k \leq i-1$. Then there are semiorthogonal decompositions

$$\mathcal{A}_k = \langle \mathfrak{a}_k, \mathfrak{a}_{k+1}, \dots, \mathfrak{a}_{i-1} \rangle.$$

Hence we could also extend the definition of $\mathcal{A}_k$ to all $k \in \mathbb{Z}$ by setting $\mathcal{A}_k = \mathcal{A}_0$ if $k \leq 0$ and $\mathcal{A}_k = 0$ if $k \geq i$.

If X is a smooth projective variety, then $\mathfrak{a}_k$'s are admissible subcategories of $\mathcal{A}_k$. Let $\alpha_0 : \mathcal{A}_0 \hookrightarrow D(X)$ be the inclusion functor and $\alpha_0^* : D(X) \to \mathcal{A}_0$ be its left adjoint.

Lemma 2.12. *In the above situation, let $k \in \{1, \dots, i\}$. Then $\alpha_0^*(\mathfrak{a}_0(1)), \dots, \alpha_0^*(\mathfrak{a}_{k-1}(k))$ is a semiorthogonal sequence, and if we denote $\mathcal{A}_i = 0$, then*

$$\langle \alpha_0^*(\mathfrak{a}_0(1)), \dots, \alpha_0^*(\mathfrak{a}_{k-1}(k)), \mathcal{A}_1(1), \dots, \mathcal{A}_k(k) \rangle = \langle \mathcal{A}_0(1), \dots, \mathcal{A}_{k-1}(k) \rangle.$$

In particular, if $k = i$, then $\mathcal{A}_0 = \langle \alpha_0^*(\mathfrak{a}_0(1)), \alpha_0^*(\mathfrak{a}_1(2)), \dots, \alpha_0^*(\mathfrak{a}_{i-1}(i)) \rangle$. This gives another semiorthogonal decomposition of $\mathcal{A}_0$.

Proof. See [K4, Lem. 4.3]. We give another proof based on properties of mutations without using Serre functors. By properties (3) and (4) of Lem. 2.3, we have

$$\begin{aligned}
&\langle \mathcal{A}_0(1), \dots, \mathcal{A}_{k-1}(k) \rangle = \langle \mathfrak{a}_0(1), \mathcal{A}_1(1), \mathfrak{a}_1(2), \mathcal{A}_2(2), \dots, \mathfrak{a}_{k-1}(k), \mathcal{A}_k(k) \rangle \\
&\quad = \langle \mathfrak{a}_0(1), \mathbb{L}_{\mathcal{A}_1(1)} \mathfrak{a}_1(2), \mathcal{A}_1(1), \mathcal{A}_2(2), \mathfrak{a}_2(3), \dots, \mathfrak{a}_{k-1}(k), \mathcal{A}_k(k) \rangle \\
&\quad = \langle \mathfrak{a}_0(1), \mathbb{L}_{\mathcal{A}_1(1)} \mathfrak{a}_1(2), \mathbb{L}_{\langle \mathcal{A}_1(1), \mathcal{A}_2(2) \rangle} \mathfrak{a}_2(3), \mathcal{A}_1(1), \mathcal{A}_2(2), \mathcal{A}_3(3) \dots, \mathfrak{a}_{k-1}(k), \mathcal{A}_k(k) \rangle \\
&\quad = \langle \mathfrak{a}_0(1), \mathbb{L}_{\mathcal{A}_1(1)} \mathfrak{a}_1(2), \dots, \mathbb{L}_{\langle \mathcal{A}_1(2), \dots, \mathcal{A}_{k-1}(k-1) \rangle} \mathfrak{a}_{k-1}(k), \mathcal{A}_1(1), \mathcal{A}_2(2), \dots, \mathcal{A}_k(k) \rangle.
\end{aligned}$$

Since $\alpha_0^* = \mathbb{L}_{\langle \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1) \rangle}$, and we have the semiorthogonal decomposition

$$D(X) = D(X)(1) = \langle \mathfrak{a}_0(1), \mathcal{A}_1(1), \mathfrak{a}_1(2), \mathcal{A}_2(2), \dots, \mathfrak{a}_{i-1}(i) \rangle,$$

it follows that we have $\mathfrak{a}_0(1) = \alpha_0^*(\mathfrak{a}_0(1))$, $\mathbb{L}_{\mathcal{A}_1(1)} \mathfrak{a}_1(2) = \alpha_0^*(\mathfrak{a}_1(2)), \dots,$ and $\mathbb{L}_{\langle \mathcal{A}_1(2), \dots, \mathcal{A}_{k-1}(k-1) \rangle} \mathfrak{a}_{k-1}(k) = \alpha_0^*(\mathfrak{a}_{k-1}(k))$. Hence the lemma follows. □

From this lemma, the new decomposition of $\mathcal{A}_0$ can be used to build another series of ascending subcategories

$$\mathcal{A}^k := \langle \alpha_0^*(\mathfrak{a}_0(1)), \dots, \alpha_0^*(\mathfrak{a}_{k-1}(k)) \rangle \subset \mathcal{A}_0 \quad \text{for } k = 1, \dots, n. \tag{2.5}$$

Then $0 = \mathcal{A}^0 \subset \mathcal{A}^1 \subset \cdots \subset \mathcal{A}^i = \mathcal{A}^{i+1} = \cdots = \mathcal{A}^n = \mathcal{A}_0$. Notice the components of $\mathcal{A}^k$ are each equivalent to $\mathfrak{a}_0, \mathfrak{a}_1, \dots, \mathfrak{a}_{k-1}$, hence one could intuitively regard $\mathcal{A}^k$ as the "complement" of $\mathcal{A}_k$ inside $\mathcal{A}_0$. The new $\mathcal{A}^k$'s will be the building blocks of another space, the HP-dual $X^\natural$ of X, to be defined in the next section.

One can similarly define the concept of *dual* (or *left*) *Lefschetz decomposition*, which is by definition a semiorthogonal decomposition of the form

$$D(X) = \langle \mathcal{B}^0, \mathcal{B}^1(1), \dots, \mathcal{B}^{j-1}(j-1) \rangle, \tag{2.6}$$

with $0 \neq \mathcal{B}^0 \subset \mathcal{B}^1 \subset \cdots \subset \mathcal{B}^{j-1}$ an ascending sequence of admissible subcategories.

A dual Lefschetz decomposition is determined by its last component $\mathcal{B}^{j-1}$. The number j is called the length of the dual Lefschetz decomposition. Given a Lefschetz decomposition (2.4), there is a unique dual Lefschetz decomposition of the same length $j = i$ with $\mathcal{B}^{j-1} = \mathcal{A}_0$, and vice versa [K5, Lem. 2.19]. It is also common to twist (2.6) by the autoequivalence $\otimes \mathscr{O}(1-j) = (\mathscr{O}(1)^\vee)^{\otimes j-1}$, and write equivalently

$$D(X) = \langle \mathcal{B}^0(1-j), \mathcal{B}^1(2-j), \dots, \mathcal{B}^{j-1} \rangle.$$

2.4. *Homological projective duality*

Let X be a *smooth* projective variety, with a morphism $f : X \to \mathbb{P}^n$, and suppose X has a Lefschetz decomposition of the form (2.4) with respect to $\mathscr{O}_X(1) := f^*\mathscr{O}_{\mathbb{P}^n}(1)$. Following Kuznetsov [K4], we make the following assumption throughout all considerations of HPD theory in this paper:

Assumption (†)**.** X is a smooth projective variety with a map $f : X \to \mathbb{P}^n$ and a Lefschetz decomposition (2.4) of length less than $n+1$.

Definition 2.13. The *universal hyperplane section* $\mathcal{H}_X$ is defined to be the subscheme $X \times_{\mathbb{P}^n} Q \subset X \times \check{\mathbb{P}}^n$, where $Q \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ is the incidence quadric $Q = \{(x, H) \mid x \in H\}$.

It is easy to see that $\mathcal{H}_X \subset X \times \check{\mathbb{P}}^n$ is a divisor of the line bundle $\mathscr{O}(1,1)$, and $\mathcal{H}_X$ is a projective bundle with fiber $\mathbb{P}^{n-1}$ over X, hence a smooth projective variety of dimension $\dim X + n - 1$.

A map $f : X \to \mathbb{P}^n$ is called *non-degenerate* if the image $f(X)$ is not contained in any proper linear subspace of $\mathbb{P}^n$. This is equivalent to the natural map $H^0(\mathbb{P}^n, \mathscr{O}(1)) \to H^0(X, \mathscr{O}_X(1))$ being an injection. The following observation will be useful:

Lemma 2.14. *Suppose X is integral and $f : X \to \mathbb{P}^n$ is non-degenerate. Then the universal hyperplane section $\mathcal{H}_X$ is* flat *over the dual projective space $\check{\mathbb{P}}^n$.*

Proof. Note that $\mathcal{H}_X \subset X \times \check{\mathbb{P}}^n$ is an effective Cartier divisor, and $X \times \check{\mathbb{P}}^n$ is flat over $\check{\mathbb{P}}^n$. The non-degeneracy condition for f exactly says that for any $s \in \check{\mathbb{P}}^n$, the corresponding section $s \in \Gamma(X, \mathscr{O}_X(1))$ is non-zero. Since X is integral, this implies that the divisor $\mathcal{H}_X \subset X \times \check{\mathbb{P}}^n$ is cut out locally at any point $x \in \mathcal{H}_X \times_{\check{\mathbb{P}}^n} \{s\}$ of the fiber $X_s := X \times \{s\}$ by a non-zerodivisor $s_x \in \mathscr{O}_{X_s,x} = \mathscr{O}_{X \times \check{\mathbb{P}}^n} \otimes_{\mathscr{O}_{\check{\mathbb{P}}^n,s}} k(s)$. By [Ko, Def.-Prop. 1.11] or [FAG, Lem. 9.3.4], $\mathcal{H}_X$ is a relative effective Cartier divisor over $\check{\mathbb{P}}^n$, i.e. it is flat over $\check{\mathbb{P}}^n$. □

From Prop. 2.10, we have a $\check{\mathbb{P}}^n$-linear semiorthogonal decomposition for $X \times \check{\mathbb{P}}^n$:

$$D(X \times \check{\mathbb{P}}^n) = \langle \mathcal{A}_0 \boxtimes D(\check{\mathbb{P}}^n), \mathcal{A}_1(1) \boxtimes D(\check{\mathbb{P}}^n), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes D(\check{\mathbb{P}}^n) \rangle.$$

Denote by $i_{\mathcal{H}_X} : \mathcal{H}_X \hookrightarrow X \times \check{\mathbb{P}}^n$ the inclusion, and by $i_{\mathcal{H}_X}^*$ the derived pullback functor.

Lemma-Definition 2.15. *The functor $i^*_{\mathcal{H}_X}$ is fully faithful on the subcategories $\mathcal{A}_1(1) \boxtimes D(\check{\mathbb{P}}^n), \ldots, \mathcal{A}_{i-1}(i-1) \boxtimes D(\check{\mathbb{P}}^n)$, and induces a $\check{\mathbb{P}}^n$-linear semiorthogonal decomposition*

$$D(\mathcal{H}_X) = \langle \mathscr{C}, (\mathcal{A}_1(1) \boxtimes D(\check{\mathbb{P}}^n))|_{\mathcal{H}}, \ldots, (\mathcal{A}_{i-1}(i-1) \boxtimes D(\check{\mathbb{P}}^n))|_{\mathcal{H}} \rangle. \tag{2.7}$$

The $\check{\mathbb{P}}^n$-linear subcategory $\mathscr{C} \subset D(\mathcal{H}_X)$ defined by the above formula is called the HP-dual category *of $X \to \mathbb{P}^n$ with respect to* (2.4).

Here the subscript $(-)|_{\mathcal{H}}$ denotes the image of a category under $i^*_{\mathcal{H}_X}$, and we will sometimes omit it if no confusion can arise. Notice that as $\mathcal{H}_X$ is a smooth projective variety, all components of (2.7), including the HP-dual category $\mathscr{C}$, are admissible subcategories.

Proof. This is [K4, Lem. 5.3]. The result follows directly from computing $R\operatorname{Hom}_{\mathcal{H}_X}$ in terms of $R\operatorname{Hom}_{X \times \check{\mathbb{P}}^n}$; see the more general cone lemma 3.13 if we take $T^\natural = \check{\mathbb{P}}^n$. □

Definition 2.16. A projective variety $X^\natural$ with a morphism $X^\natural \to \check{\mathbb{P}}^n$ is called *homological projective dual* (*HP-dual*) to $X \to \mathbb{P}^n$ with respect to the Lefschetz decomposition (2.4) if there is a Fourier–Mukai functor $\Phi_{\mathcal{E}_X} : D(X^\natural) \to D(\mathcal{H}_X)$ with kernel $\mathcal{E}_X \in D(\mathcal{H}_X \times_{\check{\mathbb{P}}^n} X^\natural)$ which induces an equivalence of categories $D(X^\natural) \simeq \mathscr{C}$, where $\mathscr{C}$ is the HP-dual category in (2.7).

Example 2.17 (Linear duality). Let $L \subset \mathbb{P}^n$ be a projective linear subspace. Then the *dual linear subspace* $L^\perp := \{s \in \check{\mathbb{P}}^n \mid s(x) = 0,\ \forall x \in L\}$ is *HP-dual* to L with respect to Beilinson decomposition (2.2) of L. See [K4, Thm. 8.2] or [T2, Prop. 3.6].

There exists a dual Lefschetz decomposition for $D(X^\natural)$:

$$D(X^\natural) = \langle \mathcal{A}^1(1-n), \ldots, \mathcal{A}^{n-1}(-1), \mathcal{A}^n \rangle, \quad \mathcal{A}^1 \subset \cdots \subset \mathcal{A}^n, \tag{2.8}$$

where $\mathcal{A}^k$ is the complementary block of $\mathcal{A}_k$ inside $\mathcal{A}_0$ defined by (2.5) (see [K4, Thm. 6.3]). Notice that by [K5, Lem. 2.19] there is also a unique Lefschetz decomposition for $D(X^\natural)$ determined by (2.8). The relations between $\mathcal{A}_k$ and $\mathcal{A}^k$ are illustrated in Figure 1. The HP-dual variety $X^\natural$ is always smooth. The classical *projective dual* variety

$$X^\vee := \{H \in \check{\mathbb{P}}^n \mid X \cap H \text{ non-transversely}\} \subset \check{\mathbb{P}}^n$$

to $X \to \mathbb{P}^n$ can be reconstructed from the HP-dual $X^\natural$ by using the fact that $X^\vee$ is the set of critical values of the morphism $X^\natural \to \check{\mathbb{P}}^n$ [K4, Thm.7.9].

HP-duality is a duality relation: $(X^\natural)^\natural \simeq X$. More precisely, if $X^\natural$ is HP-dual to X as above, then X is also HP-dual to $X^\natural$ [K4, Thm. 7.3]. Therefore one can say X and $X^\natural$ are HP-dual to each other, or $(X, X^\natural)$ is a pair of HP-duals.

The fundamental result of HPD theory is the following: Suppose $X^\natural \to \check{\mathbb{P}}^n$ is HP-dual to $X \to \mathbb{P}^n$ with respect to the Lefschetz decomposition (2.4) as before. Further assume the morphism $X \to \mathbb{P}^n$ is non-degenerate. Then we have

Theorem 2.18 (Fundamental theorem of HPD, Kuznetsov [K4, Thm. 6.3]). *For any projective linear subspace $L \subset \mathbb{P}^n$ of codimension ℓ, $0 \le \ell \le n$, and $L^\perp \subset \check{\mathbb{P}}^n$ the dual linear space, if the fiber products $X \times_{\mathbb{P}^n} L$ and $X^\natural \times_{\check{\mathbb{P}}^n} L^\perp$ are of expected dimensions:*

$$\dim X \times_{\mathbb{P}^n} L = \dim X - \ell, \quad \dim X^\natural \times_{\check{\mathbb{P}}^n} L^\perp = \dim X^\natural + \ell - n - 1,$$

then there are semiorthogonal decompositions (*i.e. primitive decompositions*)

$$D(X \times_{\mathbb{P}^n} L) = \langle {}^{\mathrm{prim}}D(X \times_{\mathbb{P}^n} L), \mathcal{A}_\ell(\ell), \ldots, \mathcal{A}_{i-1}(i-1)\rangle,$$
$$D(X^\natural \times_{\check{\mathbb{P}}^n} L^\perp) = \langle \mathcal{A}^1(2-\ell), \ldots, \mathcal{A}^{\ell-2}(-1), \mathcal{A}^{\ell-1}, D(X^\natural \times_{\check{\mathbb{P}}^n} L^\perp)^{\mathrm{prim}}\rangle,$$

and an equivalence of categories

$${}^{\mathrm{prim}}D(X \times_{\mathbb{P}^n} L) \simeq D(X^\natural \times_{\check{\mathbb{P}}^n} L^\perp)^{\mathrm{prim}}.$$

The theorem produces interesting semiorthogonal decompositions of all (complete) linear sections of algebraic varieties. Almost all known examples of semiorthogonal decompositions of algebraic varieties fit into the framework of HP-duality or its variants (see [K10] for a survey). Secondly it gives striking relations between derived categories of different varieties. For various applications of the fundamental theorem of HPD, the readers are referred to [K1, K3, K6, K8, IK, ABB, K10, ADS, T2, HT17, CT, BBF, HT16] and [BDF+, HL, ADS, ST18, Re, RS].

Convention. Note we use $X^\natural$ instead of Y for the HP-dual of X, and $\mathcal{A}^k$ instead of $\mathcal{B}_k$ for the building blocks of $X^\natural$, to reduce the burden of notations. Our "cohomological" convention $\mathcal{A}^k$ and Kuznetsov's "homological" convention $\mathcal{B}_k$ in [K4] are related by

$$\mathcal{A}^k = \mathcal{B}_{n-k}.$$

Note that the starting few terms of the ascending chain $\mathcal{A}^\bullet$ may be zero. In fact, if we set $m := \max\{k \mid \mathcal{A}_k = \mathcal{A}_0\}$, which is an integer between 0 and $i-1$, then $\mathcal{A}^0 = \mathcal{A}^1 = \cdots = \mathcal{A}^m = 0$, and the non-zero terms of the chain $\mathcal{A}^\bullet$ are $0 \neq \mathcal{A}^{m+1} \subset \mathcal{A}^{m+2} \subset \cdots \subset \mathcal{A}^i = \cdots = \mathcal{A}^n = \mathcal{A}_0$, of length $n - m$.

Notations for HP-duals. For the HP-dual pair $(X, X^\natural)$, we will always use the same notations as in this subsection and the preceding one (§2.3, 2.4). In the next section we will consider another HP-dual pair $(T \to \mathbb{P}^n, T^\natural \to \check{\mathbb{P}}^n)$, and the notations are given similarly as follows. We assume the defining decomposition (2.7) for $(T, T^\natural)$ is given by

$$D(\mathcal{H}_{T^\natural}) = \langle \Phi_{\mathcal{E}_{T^\natural}}(D(T)), (D(\mathbb{P}^n) \boxtimes \mathcal{C}_1(1))|_{\mathcal{H}_{T^\natural}}, \ldots, (D(\mathbb{P}^n) \boxtimes \mathcal{C}_{\ell-1}(\ell-1))|_{\mathcal{H}_{T^\natural}}\rangle, \tag{2.9}$$

where $\Phi_{\mathcal{E}_{T^\natural}} : D(T) \hookrightarrow D(\mathcal{H}_{T^\natural})$ is the Fourier–Mukai functor in Definition 2.16 of HP-dual, and the (dual) Lefschetz decompositions are denoted by

$$D(T) = \langle \mathcal{C}^1(1-n), \ldots, \mathcal{C}^{n-1}(-1), \mathcal{C}^n\rangle, \quad D(T^\natural) = \langle \mathcal{C}_0, \mathcal{C}_1(1), \ldots, \mathcal{C}_{\ell-1}(\ell-1)\rangle$$

with $\mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_{\ell-1}$ $(1 \le \ell \le n+1)$. Define $\mathfrak{c}_k$ to be the right orthogonal of $\mathcal{C}_{k+1}$ inside $\mathcal{C}_k$ for $k = 0, 1, \dots, \ell - 1$; then the $\mathcal{C}^k$'s are defined as in (2.5) by

$$\mathcal{C}^k = \langle \gamma_0^*(\mathfrak{c}_0(1)), \dots, \gamma_0^*(\mathfrak{c}_{k-1}(k)) \rangle \subset \mathcal{C}_0 \quad \text{for } k = 1, 2, \dots, \tag{2.10}$$

where $\gamma_0 : \mathcal{C}_0 \hookrightarrow D(T^\natural)$ denotes the inclusion functor and $\gamma_0^* : D(T^\natural) \to \mathcal{C}_0$ is its left adjoint. These notations for $(X, X^\natural)$ and $(T^\natural, T)$ are illustrated in Figure 1. We also extend the definition of $\mathcal{C}^k$ to all $k \in \mathbb{Z}$ by setting $\mathcal{C}^k = 0$ if $k \le 0$.

Now we introduce further notations:

$$^L\mathcal{C}_k := S_{D(T^\natural)}(\mathcal{C}_k) \otimes \mathscr{O}_{T^\natural}(\ell) \quad \text{for } k = 1, \dots, \ell - 1,$$

where $S_{D(T^\natural)}$ is the Serre functor. Then $^L\mathcal{C}_k \simeq \mathcal{C}_k$ and $^L\mathcal{C}_0 \supset {}^L\mathcal{C}_1 \supset \cdots \supset {}^L\mathcal{C}_{\ell-1}$. Notice that if the decomposition for $D(T^\natural)$ is rectangular, i.e. $\mathcal{C}_0 = \mathcal{C}_1 = \cdots = \mathcal{C}_{\ell-1}$, then $^L\mathcal{C}_k = \mathcal{C}_k$ (see for example [T2, Rmk. 4.10]).

The $^L\mathcal{C}_k$'s allow us to extend the decomposition (2.9) to negative degrees:

$$\begin{aligned} D(\mathcal{H}_{T^\natural}) = \langle (D(\mathbb{P}^n) \boxtimes {}^L\mathcal{C}_{k+1}(k+2-\ell))|_{\mathcal{H}_{T^\natural}}, \dots, (D(\mathbb{P}^n) \boxtimes {}^L\mathcal{C}_{\ell-1})|_{\mathcal{H}_{T^\natural}}, \\ \Phi_{\mathcal{E}_{T^\natural}}(D(T)), (D(\mathbb{P}^n) \boxtimes \mathcal{C}_1(1))|_{\mathcal{H}_{T^\natural}}, \dots, (D(\mathbb{P}^n) \boxtimes \mathcal{C}_k(k))|_{\mathcal{H}_{T^\natural}} \rangle \end{aligned} \tag{2.11}$$

for all $k = 0, 1, \dots, \ell - 1$.[2] In fact, since $\omega_{\mathcal{H}_{T^\natural}} = (\omega_{\mathbb{P}^n} \boxtimes \omega_{T^\natural})(1, 1)|_{\mathcal{H}_{T^\natural}}$, if we apply the Serre functor $S_{D(\mathcal{H}_{T^\natural})} = \otimes\, \omega_{\mathcal{H}_{T^\natural}}[\dim \mathcal{H}_{T^\natural}]$ to (2.9), we obtain (2.11) by Lemma 2.4.

Remark 2.19. Notice that we use the dual convention for the pair $(T, T^\natural)$, and the reasons are (i) we treat X and $T^\natural$ symmetrically, and this point of view allows us to apply base change in §3.2 to compare the derived categories of intersections and of the intersections of the dual; (ii) the "homological convention" of $\mathcal{A}_k$'s for X and the "cohomological convention" of $\mathcal{C}^k$'s for T allow us to express Thms. 2.18 and 3.1 in nice forms (see "Poincaré pairing rule", Rmk. 3.4).

3. Categorical Plücker formula

3.1. Main results

Our main result is the generalization of the fundament theorem of HPD beyond linear sections. Assume $(X \to \mathbb{P}^n, X^\natural \to \check{\mathbb{P}}^n)$ and $(T \to \mathbb{P}^n, T^\natural \to \check{\mathbb{P}}^n)$ are two HP-dual pairs, with HP-dual relations given by Fourier–Mukai functors $\Phi_{\mathcal{E}_X} : D(X^\natural) \to D(\mathcal{H}_X)$ and $\Phi_{\mathcal{E}_{T^\natural}} : D(T) \to D(\mathcal{H}_{T^\natural})$ as in Def. 2.16 and respectively (2.9), and with Lefschetz and dual Lefschetz decompositions

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1) \rangle, \qquad D(X^\natural) = \langle \mathcal{A}^1(1-n), \dots, \mathcal{A}^{n-1}(-1), \mathcal{A}^n \rangle,$$
$$D(T) = \langle \mathcal{C}^1(1-n), \dots, \mathcal{C}^{n-1}(-1), \mathcal{C}^n \rangle, \quad D(T^\natural) = \langle \mathcal{C}_0, \mathcal{C}_1(1), \dots, \mathcal{C}_{\ell-1}(\ell-1) \rangle,$$

[2] If $k = 0$ (resp. $k = \ell - 1$), then it is understood that there are no terms on the right (resp. left) of $\Phi_{\mathcal{E}_{T^\natural}}(D(T))$ in the above decomposition.

where $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{i-1}$ and $\mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_{\ell-1}$, $1 \le i, \ell \le n+1$, are chains of admissible subcategories, and $\mathcal{A}^k$ (resp. $\mathcal{C}^k$) is the "complement" of $\mathcal{A}_k$ (resp. $\mathcal{C}_k$) inside $\mathcal{A}_0$ (resp. $\mathcal{C}_0$) defined by formula (2.5) (resp. 2.10). The relations are illustrated in Figure 1.

Theorem 3.1 (Categorical Plücker formula). *If the two HP-dual pairs* $(X, X^\natural)$ *and* $(T, T^\natural)$ *are admissible* (*Def.* 3.9), *then there are* primitive decompositions[3]

$$D(X \times_{\mathbb{P}^n} T) = \big\langle {}^{\mathrm{prim}}D(X \times_{\mathbb{P}^n} T), \langle (\mathcal{A}_k \boxtimes \mathcal{C}^k) \otimes \mathscr{O}(k)\rangle_{k\in\mathbb{Z}}\big\rangle,$$
$$D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural) = \big\langle \langle (\mathcal{A}^k \boxtimes \mathcal{C}_k) \otimes \mathscr{O}(1-\ell+k)\rangle_{k\in\mathbb{Z}}, D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)^{\mathrm{prim}}\big\rangle,$$

and an equivalence of the primitive parts

$${}^{\mathrm{prim}}D(X \times_{\mathbb{P}^n} T) \simeq D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)^{\mathrm{prim}}.$$

The above equivalence is given by restriction of the functor

$$(\Phi_{\mathcal{E}_X|T^\natural})^* \circ \Phi_{\mathcal{E}_{T^\natural}|X} \colon D(X \times_{\mathbb{P}^n} T) \to D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural),$$

where $\Phi_{\mathcal{E}_X|T^\natural}$ *(resp.* $\Phi_{\mathcal{E}_{T^\natural}|X}$*) is the base change of* $\Phi_{\mathcal{E}_X}$ *(resp.* $\Phi_{\mathcal{E}_{T^\natural}}$*) along* $T^\natural \to \check{\mathbb{P}}^n$ *(resp.* $X \to \mathbb{P}^n$*), and* $(\Phi_{\mathcal{E}_X|T^\natural})^*$ *is a left adjoint functor of* $\Phi_{\mathcal{E}_X|T^\natural}$.[4]

The admissibility condition to be defined in §3.2 is a technical way to say the two pairs *intersect properly*, and holds for almost all known examples of HP-duals (for example, linear sections of HPD varieties of expected dimensions, quadric sections of HPD varieties of expected dimensions, intersections of quadrics, intersections of Grassmannians, etc). Symmetrically, we also have ${}^{\mathrm{prim}}D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural) \simeq D(X \times_{\mathbb{P}^n} T)^{\mathrm{prim}}$.

The proof of this main theorem will be given in the remaining sections, and examples are given at the end of this section.

Remark 3.2. In the degenerate case[5] $(T, T^\natural) = (\emptyset, \check{\mathbb{P}}^n)$, our argument can be used to imply that if $D(X)$ has Lefschetz decomposition (2.4), then its HP-dual category $\mathscr{C} = D(X^\natural)$ has dual Lefschetz decomposition (2.8). Thus our approach solves the problem posed by Kuznetsov [K4, p. 182] of "finding a more direct proof" of the decomposition (2.8) for the HP-dual category.

If we apply the main theorem to the case of dual linear sections $(T, T^\natural) = (L, L^\perp)$, then the admissibility condition of the main theorem is equivalent to the expected dimension conditions of Thm. 2.18, and the theorem implies Thm. 2.18. Hence our approach also provides a "more direct proof" of the fundamental theorem of HPD of Kuznetsov [K4].

[3] Notice there are only finitely many components $\mathcal{A}_k \boxtimes \mathcal{C}^k$ and $\mathcal{A}^k \boxtimes \mathcal{C}_k$ indexed by $k \in \mathbb{Z}$, since $\mathcal{A}_k = 0$ and $\mathcal{C}_k = 0$ for $k \gg 0$ and $\mathcal{C}^k = \mathcal{A}^k = 0$ for $k < 0$.

[4] The existence of the left adjoint functor is established in Prop. 3.11.

[5] Although the case $(T, T^\natural)$ does not satisfy the conventional definition of HP-dual pairs, it is easy to see that our argument, especially Thm. 3.12, still works.

Remark 3.3. Notice that in the main theorem we do not require the fiber products to be smooth. If we consider Orlov's singularity categories $D_{\mathrm{sg}}(Z) := D(Z)/\mathrm{Perf}(Z)$ for schemes and $\mathcal{A}_{\mathrm{sg}}$ for an admissible subcategory $\mathcal{A}$ in [O], then from the fact that $\mathcal{A} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ implies $\mathcal{A}_{\mathrm{sg}} = \langle (\mathcal{A}_1)_{\mathrm{sg}}, \dots, (\mathcal{A}_n)_{\mathrm{sg}} \rangle$ ([O]), our theorem yields $D_{\mathrm{sg}}(X \times_{\mathbb{P}^n} T) \simeq D_{\mathrm{sg}}(X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural})$. In particular, $X \times_{\mathbb{P}^n} T$ is smooth if and only if $X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural}$ is.

Remark 3.4 ("Poincaré pairing rule"). If we regard the Lefschetz decomposition of $D(X)$ as $\sum \mathcal{A}_k$, and the dual Lefschetz decomposition of $D(T)$ similarly but in a cohomological convention as $\sum \mathcal{C}^j$, then our theorem implies that the ambient component of $D(X \times_{\mathbb{P}^n} T)$ follows a "Poicaré pairing" rule: the ambient part is given by $\sum_k \mathcal{A}_k \boxtimes \mathcal{C}^k$, the summations of all "pairings" between $\{\mathcal{A}_a\}$ and $\{\mathcal{C}^b\}$ with the same indices $a = b = k$. Similarly for $D(X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural})$. This can be visualized as in Fig. 1.

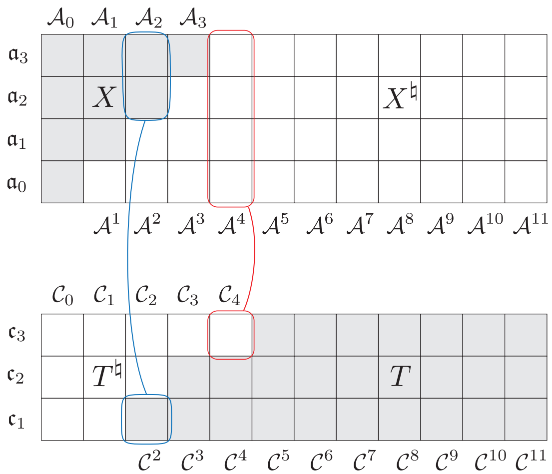

Fig. 1. Lefschetz and dual Lefschetz decompositions for HP-dual pairs $(X, X^{\natural})$ and $(T, T^{\natural})$. The vertical lines indicate the "pairing rule" for the ambient component of $D(X \times_{\mathbb{P}^n} T)$ and $D(X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural})$ in our main theorem.

If we "de-categorify" twice (i.e. take the Hochschild homology and then the Euler characteristic of) our main theorem, we obtain:

Corollary 3.5 (Plücker formula for HP-duals). *In the same situation as in Thm. 3.1, further assume that $X \times_{\mathbb{P}^n} T$ and $X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural}$ are smooth. Then*

$$\chi(X \times_{\mathbb{P}^n} T) - \frac{\chi(X) \cdot \chi(T)}{n+1} = \chi(X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural}) - \frac{\chi(X^{\natural}) \cdot \chi(T^{\natural})}{n+1},$$

where $\chi(-)$ is the ordinary topological Euler characteristic for topological spaces.

Note that the above formula for HP-duals has the same form as the topological Plücker formula (1.1). Therefore our main theorem can be viewed as a 2-level categorification of the Plücker formula. To obtain the corollary from the main theorem, we now briefly review the theory of Hochschild homology for admissible subcategories.

Recall that the Hochschild homology of a smooth projective variety X can be defined to be $\mathrm{HH}_\bullet(X) := \mathrm{Ext}^\bullet_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\omega_X)$, where $\Delta : X \to X \times X$ is the diagonal embedding, and ω_X is the dualizing complex [Ke1, Ke2, Ca2]. For an admissible subcategory $\mathcal{A} \subset D(X)$, the Hochschild homology $\mathrm{HH}_*(\mathcal{A})$ can be defined using Fourier–Mukai kernels in $D(X \times X)$, and this definition agrees with the definition using dg-enhancements [K7]. Suppose that $D(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n\rangle$ is a semiorthogonal decomposition into admissible subcategories. Then $\mathrm{HH}_\bullet(X) = \mathrm{HH}_\bullet(\mathcal{A}_1) \oplus \cdots \oplus \mathrm{HH}_\bullet(\mathcal{A}_n)$ [K7, Thm. 7.3]. Notice that the Künneth formula $\mathrm{HH}_\bullet(\mathcal{A} \boxtimes \mathcal{B}) = \mathrm{HH}_\bullet(\mathcal{A}) \otimes \mathrm{HH}_\bullet(\mathcal{B})$ holds (see Lem. A.1).

For an admissible subcategory $\mathcal{A} \subset D(X)$, we define its *homological Euler characteristic* to be $\chi^H(\mathcal{A}) := \sum_k (-1)^k \operatorname{rank} \mathrm{HH}_k(\mathcal{A}) \in \mathbb{Z}$. Notice that for a smooth projective variety X, from the Hochschild–Kostant–Rosenberg (HKR) isomorphism (see [Ca2]) $\mathrm{HH}_i(X) = \bigoplus_{p-q=i} H^q(X, \Omega^p)$, we have $\chi^H(X) = \chi(X)$, where $\chi(X)$ is the topological Euler characteristic. From the above properties of HH_*, we infer that the homological Euler characteristic is additive for semiorthogonal decompositions, i.e. $\chi(X) = \sum_k \chi^H(\mathcal{A}_k)$ if $D(X) = \langle \mathcal{A}_0, \dots, \mathcal{A}_n\rangle$. Moreover the homological Euler characteristic is multiplicative for exterior products, i.e. $\chi^H(\mathcal{A} \boxtimes \mathcal{B}) = \chi^H(\mathcal{A}) \cdot \chi^H(\mathcal{B})$, by the Künneth formula.

Proof of Corollary 3.5. By applying the above properties of χ^H to our main theorem, we obtain

$$\chi(X \times_{\mathbb{P}^n} T) - \sum_{k=1}^{i-1} \chi^H(\mathcal{A}_k) \cdot \chi^H(\mathcal{C}^k) = \chi(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural) - \sum_{k=1}^{\ell-1} \chi^H(\mathcal{A}^k) \cdot \chi^H(\mathcal{C}_k).$$

Adding the term $\frac{1}{n+1}\chi(X) \cdot \chi(T^\natural) - \sum_{k=1}^{\min\{i-1,\ell-1\}} \chi^H(\mathcal{A}_k) \cdot \chi^H(\mathcal{C}_k)$ on both sides, and using the equalities

$$\chi(X) + \chi(X^\natural) = (n+1) \cdot \chi^H(\mathcal{A}_0), \qquad \chi(T) + \chi(T^\natural) = (n+1) \cdot \chi^H(\mathcal{C}_0)$$

(which follow from the semiorthogonal decompositions of $\mathcal{A}_\bullet$, $\mathcal{A}^\bullet$, $\mathcal{C}_\bullet$, $\mathcal{C}^\bullet$ into $\mathfrak{a}_k$'s and $\mathfrak{c}_k$'s and additivity of $\chi^H(-)$; cf. also Figure 1), we obtain the desired formula. □

Example 3.6 (Intersections of quadrics). Consider two odd-dimensional nondegenerate quadrics $Q_1, Q_2 \subset \mathbb{P}^{2m}$, $m \geq 2$, which intersect transversely, with Kapranov's decomposition [Kap]. Our theorem implies that there is a primitive decomposition

$$D(Q_1 \cap Q_2) = \langle {}^{\mathrm{prim}}D(Q_1 \cap Q_2), \mathscr{O}(1), \dots, \mathscr{O}(2m-3)\rangle.$$

According to [K6, Cor. 5.7], ${}^{\mathrm{prim}}D(Q_1 \cap Q_2) \simeq D(C)$, where C is an orbifold $\mathbb{P}^1$ with $\mathbb{Z}/2\mathbb{Z}$-stack structure over $2m+1$ points. The HP-dual $Q_j^\natural$ is a $2m$-dimensional quadric with a ramified double covering map onto the dual projective space $\check{\mathbb{P}}^{2m}$, ramified over the dual quadric $\check{Q}_i \subset \check{\mathbb{P}}^{2m}$ of Q_i, $i = 1, 2$. Then $Q_1^\natural \times_{\check{\mathbb{P}}^{2m}} Q_2^\natural$ is a smooth $2m$-dimensional manifold which admits a degree 4 finite surjection onto $\check{\mathbb{P}}^{2m}$. Then the second decomposition of our theorem implies there is a decomposition

$$D(Q_1^\natural \times_{\check{\mathbb{P}}^{2m}} Q_2^\natural) = \langle \langle S_2, \mathscr{O}\rangle(2-2m), \mathscr{O}(1-2m), \dots, \mathscr{O}(-2), \mathscr{O}(-1), \langle S_1, \mathscr{O}\rangle, D(C)\rangle,$$

where $S_i = (S_{Q_i^\natural} \boxtimes \mathscr{O}_{Q_{i\pm1}^\natural})|_{Q_1^\natural \times_{\check{\mathbb{P}}^{2m}} Q_2^\natural}$, and $S_{Q_i^\natural}$ is one of the spinor bundles on $Q_i^\natural$, $i = 1, 2$.

If $Q_i \subset \mathbb{P}^{2m+1}$, $m \geq 1$, $i = 1, 2$, are two even-dimensional quadrics which intersect transversely, and $Q_i^\natural = Q_i^\vee \subset \check{\mathbb{P}}^{2m}$ are the dual quadrics, then our theorem implies

$$D(Q_1 \cap Q_2) = \langle {}^{\mathrm{prim}}D(Q_1 \cap Q_2), \mathscr{O}(1), \ldots, \mathscr{O}(2m-2)\rangle,$$

and a similar decomposition for $Q_1^\vee \cap Q_2^\vee$. By Bondal–Orlov [BO] we know that ${}^{\mathrm{prim}}D(Q_1 \cap Q_2) \simeq D(C)$ and $D(Q_1^\vee \cap Q_2^\vee)^{\mathrm{prim}} \simeq D(C')$, where C, C' are two hyperelliptic curves. Our theorem then further implies that $D(C) \simeq D(C')$, hence $C \simeq C'$ (see [H]); but this is not surprising since one can show directly $Q_1 \cap Q_2 \simeq Q_1^\vee \cap Q_2^\vee$ using [R].

Example 3.7 (Intersections of Grassmannians). Following [OR, BCP], consider $X = \mathrm{Gr}(2,5) \subset \mathbb{P}^9$ via the Plücker embedding and $T = g \cdot \mathrm{Gr}(2,5) \subset \mathbb{P}^9$ for a generic $g \in \mathrm{PGL}(10, \mathbb{C})$. Then according to Kuznetsov [K2, §6.1], the HP-duals $X^\natural = X^\vee \subset \mathbb{P}^{9*}$ and $T^\natural = (T)^\vee \subset \mathbb{P}^{9*}$ coincide with their projective duals with natural Lefschetz decompositions. Since the Lefschetz decompositions are rectangular in this case, it follows directly that the ambient parts for the two decompositions vanish. Therefore our theorem implies (see [OR, BCP])

$$D(X \cap T) \simeq D(X^\natural \cap T^\natural).$$

For a generic g, the intersections $X \cap T$ and $X^\natural \cap T^\natural$ are smooth Calabi–Yau 3-folds that are deformation equivalent. By a result of Addington–Căldăraru (see [ADM, footnote p. 857], also [OR, Prop. 2.1]) the above derived equivalence implies that $X \cap T$ and $X^\natural \cap T^\natural$ have the same polarized Hodge structure. Moreover, it is shown independently by [OR, BCP] that $X \cap T$ and $X^\natural \cap T^\natural$ are non-birational for generic g, and therefore this example provides a *counterexample* to the birational Torelli conjecture for Calabi–Yau threefolds.

Example 3.8 (Intersection of spinor varieties). Following [M], let $X = \mathcal{S} \subset \mathbb{P}(\Delta)$ be the spinor variety for $\mathrm{Spin}_{10}(\mathbb{C})$, and $T = g \cdot \mathcal{S} \subset \mathbb{P}(\Delta)$ the translation by a generic $g \in \mathrm{PGL}(\Delta)$, where Δ is one of the half-spin representations of $\mathrm{Spin}_{10}(\mathbb{C})$. Then $X^\natural = \mathcal{S}^\vee \subset \mathbb{P}(\Delta)^*$ and $T^\natural = (T)^\vee \subset \mathbb{P}(\Delta)^*$, i.e. the HP-duals agree with the projective duals of X and T (see [K2, §6.2]). Our theorem implies that (see [M])

$$D(X \cap T) \simeq D(X^\natural \cap T^\natural).$$

Here $X \cap T$ and $X^\natural \cap T^\natural$ are Calabi–Yau 5-folds that are deformation-equivalent. Similarly, the derived equivalence implies the two Calabi–Yau manifolds have the same polarized Hodge structure. Furthermore [M] proved that $X \cap T$ and $X^\natural \cap T^\natural$ are non-birational for generic g, therefore this provides a counterexample to the birational Torelli conjecture for Calabi–Yau manifolds of dimension 5.

3.2. Base change and admissibility

In order to compare the derived categories of $X \times_{\mathbb{P}^n} T$ and $X^\natural \times_{\check{\mathbb{P}}^n} T^\natural$, notice that from (2.9) and Def. 2.16 we have fully faithful embeddings $D(T) \hookrightarrow D(\mathcal{H}_{T^\natural})$ and $D(X^\natural) \hookrightarrow D(\mathcal{H}_X)$, so if we perform faithful base change of the two embeddings along $X \to \mathbb{P}^n$ and respectively $T^{\natural} \to \check{\mathbb{P}}^n$, we will have embeddings $D(X \times_{\mathbb{P}^n} T) \hookrightarrow D(X \times_{\mathbb{P}^n} \mathcal{H}_{T^{\natural}})$ and $D(X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural}) \hookrightarrow D(\mathcal{H}_X \times_{\check{\mathbb{P}}^n} T^{\natural})$. Notice that

$$\mathcal{H}_X \times_{\check{\mathbb{P}}^n} T^{\natural} = X \times_{\mathbb{P}^n} \mathcal{H}_{T^{\natural}} =: \mathcal{H},$$

which is nothing but the universal quadric of $X \times T^{\natural}$, i.e. defined by the incidence relation $\mathcal{H} = \{(x, s) \mid s(x) = 0\} \subset X \times T^{\natural}$. Thus we will put the two categories of interest into a common ambient category $D(\mathcal{H})$. The condition allowing the above base change procedures to work is called *admissibility*. More precisely, we introduce

Definition 3.9. Two pairs $(X \to \mathbb{P}^n, X^{\natural} \to \check{\mathbb{P}}^n)$ and $(T \to \mathbb{P}^n, T^{\natural} \to \check{\mathbb{P}}^n)$ are called *admissible* if the morphisms $X \to \mathbb{P}^n$ and $T^{\natural} \to \check{\mathbb{P}}^n$ considered as base changes are *faithful* (see Def. 2.5) with respect to $(\mathcal{H}_{T^{\natural}}, T)$ and respectively $(\mathcal{H}_X, T^{\natural})$.

The admissibility condition is a technical way to say the two pairs *intersect properly*, and this condition automatically holds if we are in the proper context of dg-categories or stable-∞ categories of derived intersections. For commutative varieties, this condition holds for intersections of varieties in "generic position", and holds for almost all known examples of HP-duals. The criterion will be given below.

Note that if the above two pairs are admissible, then $\mathcal{H} \subsetneq X \times T^{\natural}$ is a Cartier divisor. In fact, consider the following commutative diagram of cartesian squares of varieties:

$$\begin{array}{ccccc}
\mathcal{H} & \longrightarrow & X \times T^{\natural} & \longrightarrow & T^{\natural} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{H}_X & \longrightarrow & X \times \check{\mathbb{P}}^n & \longrightarrow & \check{\mathbb{P}}^n
\end{array}$$

The right product square is always exact cartesian. Admissibility implies the ambient square is exact cartesian, hence from Lem. 2.7, the left square is exact cartesian. Notice $\mathcal{H}_X \subsetneq X \times \check{\mathbb{P}}^n$ is a proper divisor, so $\mathcal{H} \subsetneq X \times T$ is a divisor.

To give a criterion for admissibility, we introduce the following notations. For two morphisms $X \to \mathbb{P}^n$ and $S \to \check{\mathbb{P}}^n$, we denote the incidence quadric by

$$Q(X, S) := X \times_{\mathbb{P}^n} Q \times_{\check{\mathbb{P}}^n} S \subset X \times S,$$

where $Q \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ is the universal quadric. More informally, $Q(X, S) = \{(x, s) \mid \langle x, s\rangle = 0\} \subset X \times S$, where $\langle -, - \rangle$ is the pairing between $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$. Note $Q(X, S) \subset X \times S$ is cut out by a natural section of the line bundle $\mathscr{O}(1, 1) := \mathscr{O}_X(1) \boxtimes \mathscr{O}_S(1)$, hence it is a divisor of $X \times S$ if and only if the section is non-zero on each component of $X \times S$.

Lemma 3.10 (Criterion for admissibility). *Let $(f : X \to \mathbb{P}^n, g : X^{\natural} \to \check{\mathbb{P}}^n)$ and $(p : T \to \mathbb{P}^n, q : T^{\natural} \to \check{\mathbb{P}}^n)$ be two pairs of morphisms. Assume one of the following holds:*

(1) *Both $f : X \to \mathbb{P}^n$ and $q : T^{\natural} \to \check{\mathbb{P}}^n$ are non-degenerate.*
(2) *$f : X \to \mathbb{P}^n$ is non-degenerate, and either*
   (i) *$Q(T, T^{\natural}) = T \times T^{\natural}$ (e.g. $(T, T^{\natural}) = (L, L^{\perp})$ dual linear subspaces), or*
   (ii) *$Q(T, T^{\natural}) \neq T \times T^{\natural}$, and $Q(X \times_{\mathbb{P}^n} T, T^{\natural})$ is a divisor inside $X \times_{\mathbb{P}^n} T \times T^{\natural}$.*

(3) *$q : T^{\natural} \to \check{\mathbb{P}}^n$ is non-degenerate, and either*
   (i) *$Q(X, X^{\natural}) = X \times X^{\natural}$, or*
   (ii) *$Q(X, X^{\natural}) \neq X \times X^{\natural}$, and $Q(X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural}, X)$ is a divisor inside $X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural} \times X$.*

(4) *Both $f : X \to \mathbb{P}^n$ and $q : T^{\natural} \to \check{\mathbb{P}}^n$ are degenerate, $Q(X, T^{\natural}) \neq X \times T^{\natural}$, one of* (2i), (2ii) *holds and one of* (3i), (3ii) *holds.*

Then the two pairs are admissible if $X \times_{\mathbb{P}^n} T$ and $X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural}$ are of expected dimensions:

$$\dim X \times_{\mathbb{P}^n} T = \dim X + \dim T - n, \qquad \dim X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural} = \dim X^{\natural} + \dim T^{\natural} - n.$$

Proof. The key is to check that six squares are exact cartesian. The first three squares are

$$\begin{tikzcd} X \times_{\mathbb{P}^n} T \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r, "f"] & \mathbb{P}^n \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d] & \mathcal{H}_{T^{\natural}} \arrow[d] \\ X \arrow[r, "f"] & \mathbb{P}^n \end{tikzcd} \qquad \begin{tikzcd} Q(X \times_{\mathbb{P}^n} T, T^{\natural}) \arrow[r] \arrow[d] & Q(T, T^{\natural}) \arrow[d] \\ X \arrow[r, "f"] & \mathbb{P}^n \end{tikzcd}$$

The first square is exact cartesian if $X \times_{\mathbb{P}^n} T$ is of expected dimension by Lem. 2.6(2). The second is exact cartesian if $\mathcal{H} \equiv Q(X, T^{\natural}) \neq X \times T^{\natural}$. For the last square, if $Q(T, T^{\natural}) = T \times T^{\natural}$, then $Q(X \times_{\mathbb{P}^n} T, T^{\natural}) = X \times_{\mathbb{P}^n} T \times T^{\natural}$, and the square is exact cartesian if $X \times_{\mathbb{P}^n} T$ are of expected dimension. If $q : T^{\natural} \to \check{\mathbb{P}}^n$ is non-degenerate, then $\mathcal{H}_{T^{\natural}} \to \mathbb{P}^n$ is a flat family, and so is $Q(T, T^{\natural}) \to T$. Since we have

$$\begin{tikzcd} Q(X \times_{\mathbb{P}^n} T, T^{\natural}) \arrow[r] \arrow[d, "f_Q"'] & X \times_{\mathbb{P}^n} T \arrow[r] \arrow[d, "f_T"] & X \arrow[d, "f"] \\ Q(T, T^{\natural}) \arrow[r] & T \arrow[r] & \mathbb{P}^n \end{tikzcd}$$

the ambient square is exact cartesian by Lem. 2.7. If $Q(T, T^{\natural})$ is a divisor, then consider the squares

$$\begin{tikzcd} Q(X \times_{\mathbb{P}^n} T, T^{\natural}) \arrow[r, hook] \arrow[d] & X \times Q(T, T^{\natural}) \arrow[r] \arrow[d] & Q(T, T^{\natural}) \arrow[d] \\ X \arrow[r, hook, "\Gamma_f"] & X \times \mathbb{P}^n \arrow[r] & \mathbb{P}^n \end{tikzcd}$$

where Γ_f is the graph embedding of X. The right square is exact cartesian since the projection $X \times \mathbb{P}^n \to \mathbb{P}^n$ is a smooth map. For the left square, the condition $Q(X \times_{\mathbb{P}^n} T, T^{\natural})$ is a divisor guarantees it is of expected dimension, and since Γ_f is a locally complete intersection embedding, $X \times Q(T, T^{\natural})$ is Cohen–Macaulay, the left square is also exact cartesian by Lem. 2.6(3). From Lem. 2.7, the ambient square is exact cartesian. The argument is the same for the other three squares:

$$\begin{tikzcd} X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural} \arrow[r] \arrow[d] & Y \arrow[d] \\ T^{\natural} \arrow[r, "q"] & \check{\mathbb{P}}^n \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d] & \mathcal{H}_X \arrow[d] \\ T^{\natural} \arrow[r, "q"] & \check{\mathbb{P}}^n \end{tikzcd} \qquad \begin{tikzcd} Q(X, X^{\natural} \times_{\check{\mathbb{P}}^n} T^{\natural}) \arrow[r] \arrow[d] & Q(X, X^{\natural}) \arrow[d] \\ T^{\natural} \arrow[r, "q"] & \check{\mathbb{P}}^n \end{tikzcd}$$

Hence the lemma follows. □

Proposition 3.11. *If the two HP-dual pairs* $(X, X^\natural)$ *and* $(T, T^\natural)$ *are* admissible*, then* $\Phi_{\mathcal{E}_{T^\natural}|X} : D(X \times_{\mathbb{P}^n} T) \to D(\mathcal{H})$ *and the compositions* $D(X) \boxtimes \mathcal{C}_r(r) \hookrightarrow D(X \times T^\natural) \xrightarrow{i^*_{\mathcal{H}}} D(\mathcal{H})$ *and* $D(X) \boxtimes {}^L\mathcal{C}_r(r) \hookrightarrow D(X \times T^\natural) \xrightarrow{i^*_{\mathcal{H}}} D(\mathcal{H})$*, where* $1 \le r \le \ell - 1$*, are all fully faithful, and for any* $k = 0, 1, \dots, \ell - 1$ *there is a semiorthogonal decomposition*

$$D(\mathcal{H}) = \langle D(X) \boxtimes {}^L\mathcal{C}_{k+1}(k + 2 - \ell)|_{\mathcal{H}}, \dots, D(X) \boxtimes {}^L\mathcal{C}_{\ell-1}|_{\mathcal{H}}, \\ \mathscr{D}_1 , D(X) \boxtimes \mathcal{C}_1(1)|_{\mathcal{H}}, \dots , D(X) \boxtimes \mathcal{C}_k(k)|_{\mathcal{H}}\rangle. \tag{3.1}$$

where $\mathscr{D}_1 \simeq D(X \times_{\mathbb{P}^n} T)$ *denotes the fully faithful image of* $\Phi_{\mathcal{E}_{T^\natural}|X}$*. Similarly, the functors* $\Phi_{\mathcal{E}_X|T^\natural} : D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural) \to D(\mathcal{H})$ *and* $\mathcal{A}_r(r) \boxtimes D(T^\natural) \hookrightarrow D(X \times T^\natural) \xrightarrow{i^*_{\mathcal{H}}} D(\mathcal{H})$ $(1 \le r \le i - 1)$ *are all fully faithful, and give rise to a semiorthogonal decomposition*

$$D(\mathcal{H}) = \langle \mathscr{D}_2, \mathcal{A}_1(1) \boxtimes D(T^\natural)|_{\mathcal{H}}, \dots , \mathcal{A}_{i-1}(i - 1) \boxtimes D(T^\natural)|_{\mathcal{H}}\rangle.$$

where $\mathscr{D}_2 \simeq D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)$ *denotes the fully faithful image of* $\Phi_{\mathcal{E}_X|T^\natural}$*. All components of the above semiorthogonal decompositions are admissible subcategories of* $D(\mathcal{H})$*.*

Here $\Phi_{\mathcal{E}_X|T^\natural}$ (resp. $\Phi_{\mathcal{E}_{T^\natural}|X}$) is the base change of $\Phi_{\mathcal{E}_X}$ (resp. $\Phi_{\mathcal{E}_{T^\natural}}$) along $T^\natural \to \check{\mathbb{P}}^n$ (resp. $X \to \mathbb{P}^n$), and by Prop. 2.8 the Fourier–Mukai kernels are given by restrictions.

Proof. Consider the $\check{\mathbb{P}}^n$-linear decomposition (2.7). Since the base change $T^\natural \to \check{\mathbb{P}}^n$ is faithful for $\mathcal{H}_X \to \check{\mathbb{P}}^n$, by Prop. 2.9 it induces a $T^\natural$-linear semiorthogonal decomposition of $D(\mathcal{H}_X \times_{\check{\mathbb{P}}^n} T^\natural) = D(\mathcal{H})$. It is clear that under base change the fully faithful $\check{\mathbb{P}}^n$-linear functor $\mathcal{A}_r(r) \boxtimes D(\check{\mathbb{P}}^n) \subset D(X \times \check{\mathbb{P}}^n) \xrightarrow{i^*_{\mathcal{H}_X}} D(\mathcal{H}_X)$ induces a fully faithful $T^\natural$-linear functor $\mathcal{A}_r(r) \boxtimes D(T^\natural) \subset D(X \times T^\natural) \xrightarrow{i^*_{\mathcal{H}}} D(\mathcal{H})$, where $1 \le r \le i - 1$, and the images of these coincide with the components obtained from base change in Prop. 2.9, i.e. $[i^*_{\mathcal{H}_X}(\mathcal{A}_r(r) \boxtimes D(\check{\mathbb{P}}^n))]_{T^\natural} = i^*_{\mathcal{H}}(\mathcal{A}_r(r) \boxtimes D(T^\natural))$. For the first component, since $T^\natural \to \check{\mathbb{P}}^n$ is faithful for the pair $(\mathcal{H}_X, T^\natural)$, the fully faithful embedding $\Phi_{\mathcal{E}_X} : D(X^\natural) \to D(\mathcal{H}_X)$ induces a fully faithful embedding $\Phi_{\mathcal{E}_X|T^\natural} : D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural) \to D(\mathcal{H})$ by Prop. 2.8, where the Fourier–Mukai kernel is given by $\mathcal{E}_X|T^\natural := \phi^*\mathcal{E}_X$, where ϕ is the natural map $X^\natural \times_{\check{\mathbb{P}}^n} T^\natural \times_{T^\natural} \mathcal{H} \to X^\natural \times_{\check{\mathbb{P}}^n} \mathcal{H}_X$. It is not hard to see that the image of the Fourier–Mukai transform coincides with the first component of the induced decomposition from (2.7) [K9, Thm. 6.4]. Similarly, the other case follows from the same argument applied to the $\mathbb{P}^n$-linear decomposition (2.11). Since an admissible subcategory remains admissible after applying a faithful quasi-projective base change [K9, Cor. 5.7], all the above components are admissible. □

3.3. *Chess game*

If two HP-dual pairs $(X \to \mathbb{P}^n, X^\natural \to \check{\mathbb{P}}^n)$ and $(T \to \mathbb{P}^n, T^\natural \to \check{\mathbb{P}}^n)$ are admissible, then in the preceding subsection we have put $D(X \times_{\mathbb{P}^n} T)$ and $D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)$ into a common category $D(\mathcal{H})$, with images $\mathscr{D}_1 \simeq D(X \times_{\mathbb{P}^n} T)$ and $\mathscr{D}_2 \simeq D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)$.

Then we are in a situation of the following *chess game*. For simplicity of notations, we introduce

$$\mathcal{E}(\alpha,\beta) := (\mathcal{A}_\alpha(\alpha) \boxtimes \mathcal{C}_\beta(\beta))|_{\mathcal{H}} \subset D(\mathcal{H}), \qquad \alpha \in [0, i-1],\ \beta \in [0, \ell-1],$$
$$^L\mathcal{E}(\alpha,\beta) := (\mathcal{A}_\alpha(\alpha) \boxtimes {}^L\mathcal{C}_{\beta+\ell-1}(\beta))|_{\mathcal{H}} \subset D(\mathcal{H}), \qquad \alpha \in [0, i-1],\ \beta \in [2-\ell, 0].$$

Note that in general $\mathcal{E}(\alpha,\beta)$ are all of different sizes and $\mathcal{E}(\alpha,\beta) \otimes \mathscr{O}(-1,-1) \subset \mathcal{E}(\alpha-1,\beta-1)$ will be strict inclusion for $\alpha,\beta \geq 1$. But if the decompositions for $D(X)$ and $D(T^\natural)$ are rectangular, then all $\mathcal{E}(\alpha,\beta)$ are of the same size, and $\mathcal{E}(\alpha,\beta) = \mathcal{E}(0,0) \otimes \mathscr{O}(\alpha,\beta)$ for all $\beta \geq 0$ and α; and $^L\mathcal{E}(\alpha,\beta) = \mathcal{E}(0,0) \otimes \mathscr{O}(\alpha,\beta)$ for all $\beta \leq 0$ and α, therefore all $\mathcal{E}(-,-)$'s and $^L\mathcal{E}(-,-)$'s are twistings of the same box $\mathcal{E}(0,0)$ by a line bundle.

We further introduce

$$\mathcal{E}(*,\beta) := \langle \mathcal{E}(k,\beta) \mid k \in [0,i-1] \rangle, \qquad \mathcal{E}(\alpha,*) := \langle \mathcal{E}(\alpha,k) \mid k \in [0,\ell-1] \rangle,$$
$$^L\mathcal{E}(*,\beta) := \langle {}^L\mathcal{E}(k,\beta) \mid k \in [0,i-1] \rangle, \qquad {}^L\mathcal{E}(\alpha,*) := \langle {}^L\mathcal{E}(\alpha,k) \mid k \in [2-\ell,0] \rangle.$$

For further simplicity of notations, we will sometimes omit the symbol $(-)|_{\mathcal{H}}$ for fully faithful images of subcategories of $D(X \times T^\natural)$ in $D(\mathcal{H})$. Notice that

$$\mathcal{E}(\alpha,*) = {}^L\mathcal{E}(\alpha,*) = \mathcal{A}_\alpha(\alpha) \boxtimes D(T^\natural), \qquad \alpha \in [0,i-1],$$
$$\mathcal{E}(*,\beta) = D(X) \boxtimes \mathcal{C}_\beta(\beta), \qquad \beta \in [0,\ell-1],$$
$$^L\mathcal{E}(*,\beta) = D(X) \boxtimes {}^L\mathcal{C}_\beta(\beta), \qquad \beta \in [2-\ell,0].$$

Therefore the main result Prop. 3.11 of the preceding subsection now becomes

$$D(\mathcal{H}) = \langle \mathscr{D}_1, \mathcal{E}(*,1), \ldots, \mathcal{E}(*,\ell-1) \rangle = \langle \mathscr{D}_2, \mathcal{E}(1,*), \ldots, \mathcal{E}(i-1,*) \rangle, \tag{3.2}$$

Furthermore, (3.1) allows us to extend the decompositions to negative values of β:

$$D(\mathcal{H}) = \langle {}^L\mathcal{E}(*,k+1-\ell), \ldots, {}^L\mathcal{E}(*,0), \mathscr{D}_1, \mathcal{E}(*,1), \ldots, \mathcal{E}(*,k-1) \rangle \tag{3.3}$$

for all $k = 1, \ldots, \ell-1, \ell$. One may remember this suggestively as[6]

$$D(\mathcal{H}) = \langle {}^L\mathcal{E}(*, I_{\leq 0}), \mathscr{D}_1, \mathcal{E}(*, I_{>0}) \rangle,$$

for all open interval $I \subset [1-\ell, \ell]$ of length $|I| = \ell$, where $I_{\leq 0} = I \cap \mathbb{Z}_{\leq 0}$, $I_{>0} = I \cap \mathbb{Z}_{>0}$.

Then the way how $\mathscr{D}_1 \simeq D(X \times_{\mathbb{P}^n} T)$ and $\mathscr{D}_2 \simeq D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)$ sit inside $D(\mathcal{H})$ can be illustrated via a diagram known as a *chessboard* as in Figs. 2–4. Each category $\mathcal{E}(\alpha,\beta)$ or $^L\mathcal{E}(\alpha,\beta)$ corresponds to a box in these figures, located by the values of α and β. The boxes $\mathcal{E}(-,-)$ and $^L\mathcal{E}(-,-)$ correspond to the "linear" part of the decompositions of $D(\mathcal{H})$.

Denote the inclusion functors by $I_j : \mathscr{D}_j \hookrightarrow D(\mathcal{H})$, $j = 1, 2$, and the left (resp. right) adjoint functors of I_j by I_j^* (resp. $I_j^!$). Then our main Theorem 3.1 is equivalent to the following:

[6] Similarly one extends to negative values of α, $D(\mathcal{H}) = \langle {}^L\mathcal{E}(I_{\leq 0}, *), \mathscr{D}_2, \mathcal{E}(I_{>0}, *) \rangle$, which will not be used.

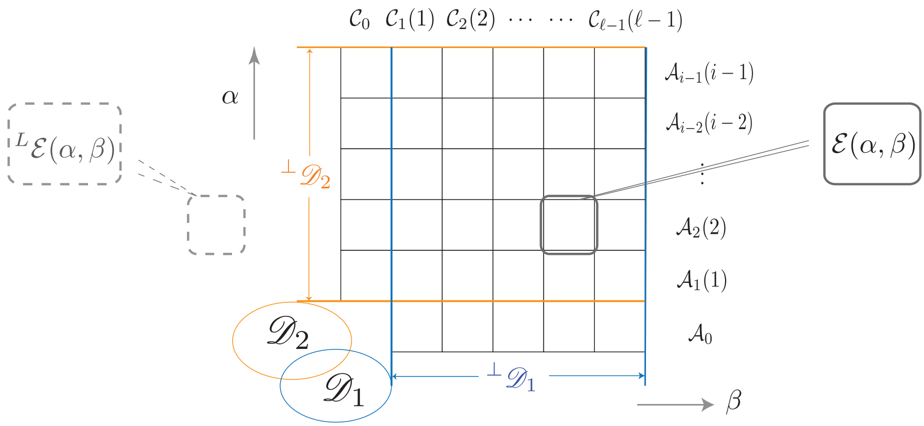

Fig. 2. The chessboard describing the decompositions $D(\mathcal{H}) = \langle \mathscr{D}_1, {}^\perp\mathscr{D}_1\rangle = \langle \mathscr{D}_2, {}^\perp\mathscr{D}_2\rangle = \langle \mathscr{D}_1^\perp, \mathscr{D}_1\rangle$, where $\mathscr{D}_1 \simeq D(X \times_{\mathbb{P}^n} T)$ and $\mathscr{D}_2 \simeq D(X^\natural \times_{\check{\mathbb{P}}^n} T^\natural)$. The horizontal (resp. vertical) direction corresponds to the $\mathscr{O}_{T^\natural}(\beta)$- (resp. $\mathscr{O}_X(\alpha)$-) direction.

Theorem 3.12. (1) $I_j^* : D(\mathcal{H}) \to \mathscr{D}_j$, $j = 1, 2$, *are fully faithful on the subcategories* $\mathcal{A}_k \boxtimes \mathcal{C}^k$ *respectively* $\mathcal{A}^k \boxtimes {}^L\mathcal{C}_k$, $k \in \mathbb{Z}$, *and induce semiorthogonal decompositions*

$$\mathscr{D}_1 = \langle {}^{\mathrm{prim}}\mathscr{D}_1, I_1^* i_{\mathcal{H}}^*(\mathcal{A}_1(1) \boxtimes \mathcal{C}^1), \ldots, I_1^* i_{\mathcal{H}}^*(\mathcal{A}_{i-1}(i-1) \boxtimes \mathcal{C}^{i-1})\rangle,$$
$$\mathscr{D}_2 = \langle I_2^* i_{\mathcal{H}}^*(\mathcal{A}^1 \boxtimes {}^L\mathcal{C}_1(2-\ell)), \ldots, I_2^* i_{\mathcal{H}}^*(\mathcal{A}^{\ell-1} \boxtimes {}^L\mathcal{C}_{\ell-1}), \mathscr{D}_2^{\mathrm{prim}}\rangle.$$

(2) *The functor* $I_2^* I_1$ *induces an equivalence of categories* ${}^{\mathrm{prim}}\mathscr{D}_1 \simeq \mathscr{D}_2^{\mathrm{prim}}$.

We will prove this by playing the "chess game" on the "chessboard" of Fig. 2–4 in the following two subsections. Notice $i_{\mathcal{H}*}$ fits into an exact triangle of functors[7]

$$\otimes\, \mathscr{O}_{X\times T^\natural}(-1,-1) \to \mathrm{Id} \to i_{\mathcal{H}*}\, i_{\mathcal{H}}^* \xrightarrow{[1]} .$$

This enables us to compute the Hom space of the "linear" part $\mathcal{E}(*,*)$ or ${}^L\mathcal{E}(*,*)$ of $D(\mathcal{H})$ in terms of the Hom space on the product space $X \times T^\natural$.

Lemma 3.13 (Cone lemma). *For any* $F_1, F_2 \in D(X)$, $G_1, G_2 \in D(T^\natural)$, *we have*

$$\begin{aligned} R\,\mathrm{Hom}_{\mathcal{H}}((F_1 \boxtimes G_1)|_{\mathcal{H}}, (F_2 \boxtimes G_2)|_{\mathcal{H}}) = \mathrm{cone}\big(&R\,\mathrm{Hom}_X(F_1, F_2(-1)) \\ &\otimes R\,\mathrm{Hom}_{T^\natural}(G_1, G_2(-1)) \to R\,\mathrm{Hom}_X(F_1, F_2) \otimes R\,\mathrm{Hom}_{T^\natural}(G_1, G_2)\big). \end{aligned} \tag{3.4}$$

Proof. Since $i_{\mathcal{H}*} i_{\mathcal{H}}^* = \mathrm{cone}(\mathscr{O}(-1,-1) \to \mathrm{Id})$, letting $E_1 = F_1 \boxtimes G_1$, $E_2 = F_2 \boxtimes G_2$, we find that $R\,\mathrm{Hom}_{\mathcal{H}}(i_{\mathcal{H}}^*(E_1), i_{\mathcal{H}}^*(E_2)) = R\,\mathrm{Hom}_{X\times T^\natural}(E_1, i_{\mathcal{H}*}\, i_{\mathcal{H}}^*(E_2))$ is the cone of

$$R\,\mathrm{Hom}_{X\times T^\natural}(E_1, E_2(-1,-1)) \to R\,\mathrm{Hom}_{X\times T^\natural}(E_1, E_2).$$

Now by the Künneth formula, we are done. ∎

[7] Or equivalently, the cotwist functor of $i_{\mathcal{H}}^*$ is isomorphic to $\mathscr{O}_{X\times T^\natural}(-1,-1)$. The sequence can be regarded as an exact triangle in the categories of Fourier–Mukai functors on $D(X \times T)$ [CW].

Remark 3.14. From the lemma, suppose we want to show the vanishing

$$R\operatorname{Hom}_{\mathcal{H}}(\mathcal{E}(\alpha_1,\beta_1),\mathcal{E}(\alpha_2,\beta_2))=0. \tag{3.5}$$

This is equivalent to showing that for all $F_j\in\mathcal{A}_{\alpha_j}(\alpha_j)$, $G_j\in\mathcal{C}_{\beta_j}(\beta_j)$, $j=1,2$,

$$R\operatorname{Hom}_{\mathcal{H}}(F_1\boxtimes G_1,F_2\boxtimes G_2)=0.$$

Then from equality (3.4) it suffices to show that both the following terms inside the cone are zero:

- the *twisted term* $R\operatorname{Hom}_X(F_1,F_2(-1))\otimes R\operatorname{Hom}_{T^\natural}(G_1,G_2(-1))$, and
- the *untwisted term* $R\operatorname{Hom}_X(F_1,F_2)\otimes R\operatorname{Hom}_{T^\natural}(G_1,G_2)$.

We will refer to the vanishing of the terms caused by the vanishing of the $R\operatorname{Hom}_X$-factor (resp. the $R\operatorname{Hom}_{T^\natural}$-factor) as caused by *$\alpha$-vanishing* (resp. *$\beta$-vanishing*). Then both the twisted and untwisted terms are zero, hence (3.5) holds if

- (α-vanishing) both $R\operatorname{Hom}_X$-factors are zero, e.g. $i-1\geq\alpha_1>\alpha_2\geq 1$, or
- (β-vanishing) both $R\operatorname{Hom}_{T^\natural}$-factors are zero, e.g. $\ell-1\geq\beta_1>\beta_2\geq 1$, or
- (mixed type) one $R\operatorname{Hom}_X$-factor is zero, the other $R\operatorname{Hom}_{T^\natural}$-factor is zero, e.g. $\alpha_1=\alpha_2\geq 1$, $\beta_1>\beta_2=0$ or $\alpha_1>\alpha_2=0$, $\beta_1=\beta_2\geq 1$.

One also has a similar criterion for vanishing on the ${}^L\mathcal{E}(\alpha,\beta)$-part.

3.4. Fully-faithfulness

We first prove the statements about fully-faithfulness. The strategy is as follows: for $j\in\{1,2\}$, suppose we want to show that $I_j^*:D(\mathcal{H})\to\mathscr{D}_j$ is fully faithful when restricted to a certain subcategory $\mathcal{C}\subset D(\mathcal{H})$, i.e. to show that for $a,b\in\mathcal{C}$,

$$R\operatorname{Hom}_{\mathcal{H}}(a,b)=R\operatorname{Hom}_{\mathscr{D}_j}(I_j^*a,I_j^*b)=R\operatorname{Hom}_{\mathcal{H}}(a,I_jI_j^*b).$$

Then it is equivalent to showing $R\operatorname{Hom}_{\mathcal{H}}(a,\operatorname{cone}(b\to I_jI_j^*(b))=0$, where $b\to I_jI_j^*(b)$ is the natural unit map by the adjunction $I_j^*\dashv I_j$. Since I_j^* are mutation functors passing through a certain region of type $\mathcal{E}(-,-)$ or ${}^L\mathcal{E}(-,-)$, we can analyze $\operatorname{cone}(b\to I_jI_j^*(b))$ on the chessboard of Figs. 3 and 4, and show it belongs to a region receiving no (non-zero) Homs from $\mathcal{C}$. For simplicity of notations, we will write b for I_jb for an element $b\in\mathscr{D}_j\subset D(\mathcal{H})$.

Lemma 3.15. *Let* $b\in(\mathcal{A}_k(k)\boxtimes\mathcal{C}^k)|_{\mathcal{H}}\subset\mathcal{E}(k,0)$*,* $k=1,\dots,i-1$*. If* $k\in[1,\ell]$*, then* $\operatorname{cone}(b\to I_1^*(b))$ *belongs to the subcategory*[8]

$$\left\langle\begin{matrix}\mathcal{E}(k-1,1), & \mathcal{E}(k-1,2), & \dots, & \mathcal{E}(k-1,k-2), & \mathcal{E}(k-1,k-1)\\ \mathcal{E}(k-2,1), & \mathcal{E}(k-2,2), & \dots, & \mathcal{E}(k-2,k-2) & \\ \vdots & & & & \\ \mathcal{E}(1,1) & & & & \end{matrix}\right\rangle.$$

[8] The order of the semiorthogonal sequence is from bottom to top, and from left to right.

If $k \in [\ell, i-1]$*, then* $\mathcal{C}^k = \mathcal{C}_0$*, and* $\mathrm{cone}(b \to I_1^*(b))$ *belongs to the subcategory*

$$\left\langle \begin{array}{l} \mathcal{E}(k-1,1),\ \mathcal{E}(k-1,2),\ \ldots,\ \mathcal{E}(k-1,\ell-2),\ \mathcal{E}(k-1,\ell-1) \\ \mathcal{E}(k-2,1),\ \mathcal{E}(k-2,2),\ \ldots,\ \mathcal{E}(k-2,\ell-2) \\ \qquad\vdots \\ \mathcal{E}(k-\ell+1,1) \end{array} \right\rangle.$$

The result is illustrated in Figure 3. Similar patterns also appear in [K4, Lem. 5.6] and [T2, proof of Thm. 4.7].

From this lemma, the image of $\mathcal{S} = \langle (\mathcal{A}_1(1) \boxtimes \mathcal{C}^1)|_{\mathcal{H}}, \ldots, (\mathcal{A}_{i-1}(i-1) \boxtimes \mathcal{C}^{i-1})|_{\mathcal{H}} \rangle$ under I_1^* is contained in the subcategory generated by $\mathcal{S}$ itself and

$$\mathcal{E}(\alpha, \beta) \quad \text{for } \beta \in [1, \ell-1],\ \alpha \in [\beta, i-2],$$

thus in particular contained in ${}^{\perp}\mathcal{D}_2 = \langle \mathcal{E}(1,*), \ldots, \mathcal{E}(i-1,*) \rangle$. This will be useful later.

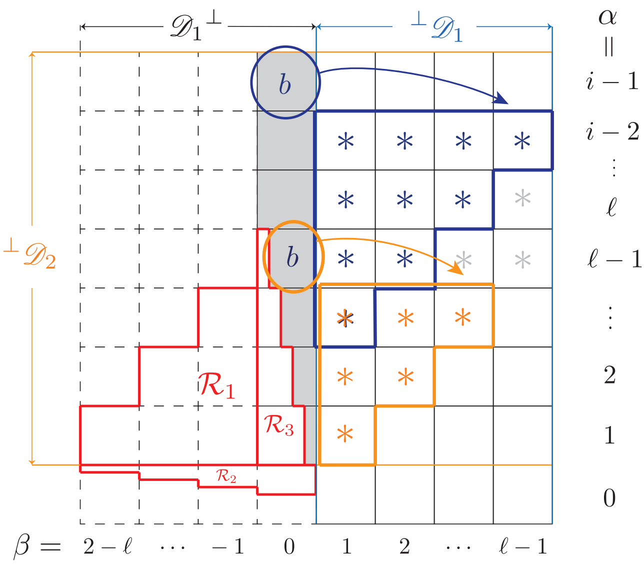

Fig. 3. The chessboard indicating blocks $\mathcal{E}(-,-)$ and ${}^{L}\mathcal{E}(-,-)$ in the case $i \geq \ell$. The shaded region indicates the ambient components $\mathcal{A}^k(k) \boxtimes \mathcal{C}^k \subset \mathcal{E}(k,0)$ to be embedded into $\mathcal{D}_1$. For b in such a component (e.g. the circled ones), Lem. 3.15 implies $\mathrm{cone}(b \to I_1^* b)$ belongs to the pointed staircase region marked with "$*$".

Proof of Lemma 3.15. We show the case $1 \leq k \leq \ell$; the argument for $k \geq \ell$ is similar. Note I_1^* is the left mutation functor passing through the semiorthogonal sequence

$$\langle \mathcal{E}(*,1), \ldots, \mathcal{E}(*,k-1), \mathcal{E}(*,k) \ldots, \mathcal{E}(*,\ell-1) \rangle.$$

We first observe that the subcategories $\mathcal{E}(*,k) \ldots, \mathcal{E}(*,\ell-1)$ have no (non-zero derived) Homs to b. This is from what we call "β-vanishings" in Rmk. 3.14. In fact, let

$\beta \in [k, \ell - 1]$. To show $R\operatorname{Hom}(\mathcal{E}(\alpha,\beta), \mathcal{A}_k(k) \boxtimes \mathcal{C}^k) = 0$, as in Rmk. 3.14 observe that the untwisted term is zero by $R\operatorname{Hom}_{T^\natural}(\mathcal{C}_\beta(\beta), \mathcal{C}^k) = 0$ since $\mathcal{C}^k \subset \mathcal{C}_0$, and the twisted term is zero since

$$\begin{aligned} R\operatorname{Hom}_{T^\natural}(\mathcal{C}_\beta(\beta), \mathcal{C}^k(-1)) &= R\operatorname{Hom}_{T^\natural}(\mathcal{C}_\beta(\beta+1), \mathcal{C}^k) \\ &= R\operatorname{Hom}_{\mathcal{C}_0}((\gamma_0^*(\mathcal{C}_\beta(\beta+1)), \mathcal{C}^k) = R\operatorname{Hom}_{\mathcal{C}_0}(\gamma_0^*(\mathfrak{c}_\beta(\beta+1)), \mathcal{C}^k) = 0 \end{aligned}$$

exactly by the way we define $\mathcal{C}^k \subset \mathcal{C}_0$ in (2.10) and Lem. 2.12, where $\gamma_0 \colon \mathcal{C}_0 \to D(T^\natural)$ denotes the inclusion. Therefore by Lem. 2.3, the left mutations passing through the last $\ell - k$ terms are the identity functors on b, and we have

$$I_1^* b = \mathbb{L}_{\mathcal{E}(*,1)} \circ \cdots \circ \mathbb{L}_{\mathcal{E}(*,k-1)} b.$$

Let $b^{(0)} = b$ and $b^{(\gamma)} = \mathbb{L}_{\mathcal{E}(*,k-\gamma)} b^{(\gamma-1)}$ for $\gamma = 1, \ldots, k-1$. Then $b^{(k-1)} = I_1^*(b)$. Notice that if $k = 1$, then $I_1^*(b) = b$, hence we are already done; therefore we may assume $k \geq 2$.

To prove the lemma, we show by induction on γ that, for each step, $\operatorname{cone}(b \to b^{(\gamma)})$ belongs to a similar staircase subregion (of size γ, on the top right inside the desired region), and then the case $\gamma = k-1$ implies the lemma. More precisely, we show that for every $\gamma = 1, \ldots, k-1$, $\operatorname{cone}(b \to b^{(\gamma)})$ belongs to the subcategory generated by

$$\mathcal{E}(\alpha,\beta) = \mathcal{A}_\alpha(\alpha) \boxtimes \mathcal{C}_\beta(\beta) \quad \text{for } \beta \in [k-\gamma, k-1],\ \alpha \in [\beta, k-1]. \tag{3.6}$$

Base case. For $\gamma = 1$, to compute $b^{(1)} = \mathbb{L}_{\mathcal{E}(*,k-1)} b$, where $\mathcal{E}(*, k-1) = D(X) \boxtimes \mathcal{C}_{k-1}(k-1)$, we use the decomposition[9]

$$D(X) = \langle \mathcal{A}_{k-1}(k-1), {}^\perp(\mathcal{A}_{k-1}(k-1)) \rangle.$$

Note that of all components of $\mathcal{E}(*, k-1) = D(X) \boxtimes \mathcal{C}_{k-1}(k-1)$ induced by the above decomposition, only $\mathcal{E}(k-1, k-1) = \mathcal{A}_{k-1}(k-1) \boxtimes \mathcal{C}_{k-1}(k-1)$ has Homs to b. This is a vanishing of "mixed type": in fact, to compute such Homs using cone lemma 3.13 and Rmk. 3.14, the untwisted term vanishes since it has factor $R\operatorname{Hom}_S(\mathcal{C}_{k-1}(k-1), \mathcal{C}^k) = 0$, by $k \geq 2$, and the twisted term vanishes since it has the $R\operatorname{Hom}_X$-factor

$$R\operatorname{Hom}_X({}^\perp(\mathcal{A}_{k-1}(k-1))(1), \mathcal{A}_k(k)) = R\operatorname{Hom}_X({}^\perp(\mathcal{A}_{k-1}(k-1)), \mathcal{A}_k(k-1)) = 0$$

because $\mathcal{A}_k(k-1) \subset \mathcal{A}_{k-1}(k-1)$. Thus $\operatorname{cone}(b \to b^{(1)}) \in \mathcal{E}(k-1, k-1)$ as claimed.

Induction step. Next, suppose the claim holds for γ, i.e. $\operatorname{cone}(b \to b^{(\gamma)})$ belongs to the region (3.6). Then $b^{(\gamma)}$ belongs the subcategory generated by $\mathcal{A}_k(k) \boxtimes \mathcal{C}^k$ and (3.6). To analyze $b^{(\gamma+1)} = \mathbb{L}_{\mathcal{E}(*,k-\gamma-1)} b^{(\gamma)}$, we use the decomposition

$$D(X) = \langle \mathcal{A}_{k-\gamma-1}(k-\gamma-1), \ldots, \mathcal{A}_{k-1}(k-1), {}^\perp\langle \mathcal{A}_{k-\gamma-1}(k-\gamma-1), \ldots, \mathcal{A}_{k-1}(k-1) \rangle \rangle.$$

[9] We know explicitly that the second component is

$${}^\perp(\mathcal{A}_{k-1}(k-1)) = \mathcal{A}_k(k), \ldots, \mathcal{A}_{i-1}(i-1), S_X^{-1}(\mathcal{A}_0), \ldots, S_X^{-1}(\mathcal{A}_{k-1}(k-1))$$

but this will be irrelevant for our computation.

We claim that of all components of $\mathcal{E}(*, k-\gamma-1) = D(X) \boxtimes \mathcal{C}_{k-\gamma-1}(k-\gamma-1)$ induced by the above decomposition, only the first $\gamma + 1$ terms

$$\mathcal{E}(k-\gamma-1, k-\gamma-1), \ldots, \mathcal{E}(k-1, k-\gamma-1) \tag{3.7}$$

have Homs to $b^{(\gamma)}$. To prove the claim, first notice that the remainder of the above inside $\mathcal{E}(*, k-\gamma-1)$,

$$^{\perp}\langle \mathcal{A}_{k-\gamma-1}(k-\gamma-1), \ldots, \mathcal{A}_{k-1}(k-1)\rangle \boxtimes \mathcal{C}_{k-\gamma-1}(k-\gamma-1), \tag{3.8}$$

has no Homs to $\mathcal{A}_k(k) \boxtimes \mathcal{C}^k$. The reason is similar to the base case: the untwisted term is zero by β-vanishing, and the twisted term is zero by α-vanishing since $\mathcal{A}_k(k-1) \subset \mathcal{A}_{k-1}(k-1)$. It remains to show there are no Homs from (3.8) to (3.6), and this now follows from α-vanishing: since the $\mathcal{A}_\alpha(\alpha)$-factor which appears in (3.6) has range $k-\gamma \leq \alpha \leq k-1$, it follows that $\mathcal{A}_\alpha(\alpha)$ and $\mathcal{A}_\alpha(\alpha-1)$ are both contained in $\langle \mathcal{A}_{k-\gamma-1}(k-\gamma-1), \ldots, \mathcal{A}_{k-1}(k-1)\rangle$, therefore the $R\operatorname{Hom}_X$-factors of both twisted and untwisted terms are zero from the decomposition of $D(X)$. Therefore the claim is proved.

From the claim, $\operatorname{cone}(b^{(\gamma)} \to b^{(\gamma+1)})$ belongs to the subcategory generated by (3.7). Then from the distinguished triangle from the octahedral axiom of triangulated categories:

$$\operatorname{cone}(b \to b^{(\gamma)}) \to \operatorname{cone}(b \to b^{(\gamma+1)}) \to \operatorname{cone}(b^{(\gamma)} \to b^{(\gamma+1)}) \xrightarrow{[1]},$$

$\operatorname{cone}(b \to b^{(\gamma+1)})$ belongs to the union of (3.6) and the "new column" (3.7), so it belongs to the desired region of the form (3.6) with γ replaced by $\gamma+1$. Then by induction we are done. □

From the above lemma, we can directly prove

Proposition-Definition 3.16. *The functor $I_1^* : D(\mathcal{H}) \to \mathscr{D}_1$ is fully faithful on the subcategories $(\mathcal{A}_1(1) \boxtimes \mathcal{C}^1)|_{\mathcal{H}}, \ldots, (\mathcal{A}_{i-1}(i-1) \boxtimes \mathcal{C}^{i-1})|_{\mathcal{H}} \subset D(\mathcal{H})$, and induces a semiorthogonal decomposition, called the* (left) primitive decomposition *of $\mathscr{D}_1$:*

$$\mathscr{D}_1 = \langle {}^{\mathrm{prim}}\mathscr{D}_1, I_1^*(\mathcal{A}_1(1) \boxtimes \mathcal{C}^1)|_{\mathcal{H}}, \ldots, I_1^*(\mathcal{A}_{i-1}(i-1) \boxtimes \mathcal{C}^{i-1})|_{\mathcal{H}}\rangle$$

where the left admissible subcategory ${}^{\mathrm{prim}}\mathscr{D}_1$ is called the left primitive component *of $\mathscr{D}_1$.*

Proof. For any $a \in i_{\mathcal{H}}^*(\mathcal{A}_m(m) \boxtimes \mathcal{C}^m)$ and $b \in i_{\mathcal{H}}^*(\mathcal{A}_k(k) \boxtimes \mathcal{C}^k)$ with $1 \leq k \leq m \leq i-1$, from Lem. 3.15, $\operatorname{cone}(b \to I_1^* b)$ belongs to the subcategory $\langle \mathcal{E}(1,*), \ldots, \mathcal{E}(k-1,*)$, but $a \in \mathcal{E}(m,*)$ and $k-1 < m \leq i-1$, hence $R\operatorname{Hom}(a, \operatorname{cone}(b \to I_1^* b)) = 0$. Therefore

$$R\operatorname{Hom}_{\mathscr{D}_1}(I_1^*(a), I_1^*(b)) = R\operatorname{Hom}_{\mathcal{H}}(a, I_1 I_1^*(b)) = R\operatorname{Hom}_{\mathcal{H}}(a, b),$$

which gives the "fully faithful" statements. Notice the subcategories $I_1^*(\mathcal{A}_k(k) \boxtimes \mathcal{C}^k)|_{\mathcal{H}} \simeq \mathcal{A}_k(k) \boxtimes \mathcal{C}^k$ are saturated in the sense of [B, BV], since they are equivalent to the admissible subcategories $\mathcal{A}_k(k) \boxtimes \mathcal{C}^k \subset D(X \times T)$ for the smooth projective variety $X \times T$ (see [BV]), therefore by [B] (see also [K4, Lem. 2.11]) they are also admissible subcategories of $\mathscr{D}_1$, and thus ${}^{\mathrm{prim}}\mathscr{D}_1$ is well-defined and left admissible by [K4, Lem. 2.4]. □

By the same argument, we show that the functor $I_2^* : D(\mathcal{H}) \to \mathscr{D}_2$ is fully faithful on $\langle (\mathcal{A}^1 \boxtimes \mathcal{C}_1(1))|_{\mathcal{H}}, \ldots, (\mathcal{A}^{\ell-1} \boxtimes \mathcal{C}_{\ell-1}(\ell-1))|_{\mathcal{H}} \rangle$, and induces a primitive decomposition

$$\mathscr{D}_2 = \langle {}^{\mathrm{prim}}\mathscr{D}_2, I_2^*(\mathcal{A}^1 \boxtimes \mathcal{C}_1(1))|_{\mathcal{H}}, \ldots, I_2^*(\mathcal{A}^{\ell-1} \boxtimes \mathcal{C}_{\ell-1}(\ell-1))|_{\mathcal{H}} \rangle.$$

But this is *not* the decomposition we need: it will be the *right* primitive component $\mathscr{D}_2^{\mathrm{prim}}$ which compares nicely with ${}^{\mathrm{prim}}\mathscr{D}_1$ under the functor $I_2^* I_1 : \mathscr{D}_1 \to \mathscr{D}_2$.

Lemma 3.17. (1) *Let* $b \in (\mathcal{A}^k \boxtimes {}^L\mathcal{C}_k(k+1-\ell))|_{\mathcal{H}} \subset {}^L\mathcal{E}(0, k+1-\ell)$, $k \in [1, \ell-1]$. *If* $k \in [1, i]$, *then* $\mathrm{cone}(b \to I_2^*(b))$ *belongs to*

$$\left\langle \begin{array}{r} {}^L\mathcal{E}(k-1, k-\ell) \\ \vdots \\ {}^L\mathcal{E}(2, 3-\ell),\ \ldots,\ {}^L\mathcal{E}(2, k-\ell-1),\ {}^L\mathcal{E}(2, k-\ell) \\ {}^L\mathcal{E}(1, 2-\ell),\ {}^L\mathcal{E}(1, 3-\ell),\ \ldots,\ {}^L\mathcal{E}(1, k-\ell-1),\ {}^L\mathcal{E}(1, k-\ell) \end{array} \right\rangle.$$

If $k \in [i, \ell-1]$, *then* $\mathcal{A}^k = \mathcal{A}_0$, *and* $\mathrm{cone}(b \to I_2^*(b))$ *belongs to*

$$\left\langle \begin{array}{r} {}^L\mathcal{E}(i-1, k-\ell) \\ \vdots \\ {}^L\mathcal{E}(2, k-i+3-\ell),\ \ldots,\ {}^L\mathcal{E}(2, k-\ell) \\ {}^L\mathcal{E}(1, k-i+2-\ell),\ {}^L\mathcal{E}(1, k-i+3-\ell),\ \ldots,\ {}^L\mathcal{E}(1, k-\ell) \end{array} \right\rangle.$$

Therefore the image of $\mathcal{S}' = \langle (\mathcal{A}^1 \boxtimes {}^L\mathcal{C}_1(2-\ell)))|_{\mathcal{H}}, \ldots, (\mathcal{A}^{l-1} \boxtimes {}^L\mathcal{C}_{l-1}))|_{\mathcal{H}} \rangle$ *under* I_2^* *is contained in the subcategory generated by* $\mathcal{S}'$ *and*

$${}^L\mathcal{E}(\alpha, \beta) \quad \textit{for } \alpha \in [1, i-1],\ \beta \in [\alpha - \ell + 1, -1].$$

(2) *Let* $b \in {}^{\mathrm{prim}}\mathscr{D}_1$. *Then* $\mathrm{cone}(b \to I_2^*(b))$ *belongs to the subcategory generated by*

$${}^L\mathcal{E}(\alpha, \beta) \quad \textit{for } \alpha \in [1, i-1],\ \beta \in [\alpha - \ell + 1, 0].$$

In particular, the images of both $\langle i_{\mathcal{H}}^*(\mathcal{A}^1 \boxtimes {}^L\mathcal{C}_1(2-\ell)), \ldots, i_{\mathcal{H}}^*(\mathcal{A}^{l-1} \boxtimes {}^L\mathcal{C}_{l-1}) \rangle$ and ${}^{\mathrm{prim}}\mathscr{D}_1$ under I_2^* are contained in $\mathscr{D}_1^{\perp} = \langle {}^L\mathcal{E}(*, 2-\ell), \ldots, {}^L\mathcal{E}(*, 0) \rangle$.

For statement (1), the ambient component (to be embedded into $\mathscr{D}_2$) corresponds to the region "$\mathcal{R}_2$" in Figs. 3 and 4, and the readers are encouraged to mark the corresponding region for $\mathrm{cone}(b \to I_2^* b)$ (a staircase-shape subregion on the bottom left inside the region "$\mathcal{R}_1$") as we did in Figure 3 for Lem. 3.15.

Proof of Lemma 3.17. The proof is similar to that of Lem. 3.15. For the proof of (1), assume without loss of generality $k \in [1, i]$; the case $k \in [i, l-1]$ is similar. Note I_2^* is the left mutation through

$${}^L\mathcal{E}(1, *), \ldots, {}^L\mathcal{E}(k-1, *), {}^L\mathcal{E}(k, *), \ldots, {}^L\mathcal{E}(i-1, *).$$

(Recall ${}^L\mathcal{E}(\alpha, *) = \mathcal{E}(\alpha, *) = \mathcal{A}_\alpha(\alpha) \boxtimes D(T^{\natural})$.) As before, b receives no Homs from

${}^L\mathcal{E}(k,*), \dots, {}^L\mathcal{E}(i-1,*)$ by α-vanishing, which is a consequence of the way we define $\mathcal{A}^k$ in (2.10). Therefore

$$I_2^* b = \mathbb{L}_{{}^L\mathcal{E}(1,*)} \circ \cdots \circ \mathbb{L}_{{}^L\mathcal{E}(k-1,*)} b.$$

Let

$$b^{(0)} = b \quad \text{and} \quad b^{(\gamma)} = \mathbb{L}_{{}^L\mathcal{E}(k-\gamma,*)} b^{(\gamma-1)} \quad \text{for } \gamma = 1, \dots, k-1.$$

Then $b^{(k-1)} = I_2^* b$. Again we can show by induction that for each $\gamma = 1, 2, \dots, k-1$, $\mathrm{cone}(b \to b^{(\gamma)})$ belongs to a similar subregion (on the top right, of size γ) inside the desired region, i.e. generated by

$${}^L\mathcal{E}(\alpha,\beta) \quad \text{for } \beta \in [k-\gamma+1-\ell, k-\ell],\ \alpha \in [k-\gamma, \beta+\ell-1].$$

Then the case $\gamma = k-1$ gives the result of the lemma. For the induction step, to analyze $b^{(\gamma+1)} = \mathbb{L}_{{}^L\mathcal{E}(k-\gamma-1,*)} b^{(\gamma)}$, we consider the decomposition

$$D(T^\natural) \\ = \big\langle \langle {}^L\mathcal{C}_{k-\gamma-1}(k-\gamma-\ell), \dots, {}^L\mathcal{C}_{k-1}(k-\ell) \rangle, {}^\perp\langle {}^L\mathcal{C}_{k-\gamma-1}(k-\gamma-\ell), \dots, {}^L\mathcal{C}_{k-1}(k-\ell) \rangle \big\rangle.$$

As before, for the decomposition of ${}^L\mathcal{E}(k-\gamma-1,*) = \mathcal{A}_{k-\gamma-1}(k-\gamma-1) \boxtimes D(T^\natural)$ induced by the above decomposition of $D(T^\natural)$, only the "new row"

$$\langle {}^L\mathcal{E}(k-\gamma-1, k-\gamma-\ell), \dots, {}^L\mathcal{E}(k-\gamma-1, k-\ell) \rangle$$

has Homs to $b^{(\gamma)}$, and this completes the induction step and the proof of (1).

The proof of (2) is similar; the only thing we need to check is the vanishing for $b \in {}^{\mathrm{prim}}\mathscr{D}_1$ which does not come from the cone lemma. Assume without loss of generality $\ell \le i$. Again we set

$$b^{(0)} = b \quad \text{and} \quad b^{(\gamma)} = \mathbb{L}_{\mathcal{A}_{\ell-\gamma}(\ell-\gamma)\boxtimes D(T^\natural)} b^{(\gamma-1)} \quad \text{for } \gamma = 1, \dots, \ell-1.$$

and show by induction on $\gamma \in [1, \ell-1]$ that each $\mathrm{cone}(b \to b^{(\gamma)})$ belongs to a similar subregion (of size γ on the top-right inside the desired region), i.e. the subcategory generated by

$${}^L\mathcal{E}(\alpha,\beta) \quad \text{for } \beta \in [1-\gamma, \ell-1],\ \alpha \in [\ell-\gamma, \beta+\ell-1],$$

and the case $\gamma = \ell-1$ implies the lemma.

For the induction step, as before, we only need to show that for the decomposition of ${}^L\mathcal{E}(\ell-\gamma-1,*) = \mathcal{A}_{\ell-\gamma-1}(\ell-\gamma-1) \boxtimes D(T^\natural)$ induced by the decomposition

$$D(T^\natural) = \langle {}^L\mathcal{C}_{\ell-\gamma-1}(-\gamma), {}^L\mathcal{C}_{\ell-\gamma}(1-\gamma), \dots, {}^L\mathcal{C}_{l-1}, \mathcal{C}_0(1), \dots, \mathcal{C}_{\ell-\gamma-2}(\ell-\gamma-1) \rangle,$$

only the first $\gamma+1$ terms (i.e. the "new row")

$$\langle {}^L\mathcal{E}(\ell-\gamma-1, -\gamma), \dots, {}^L\mathcal{E}(\ell-\gamma-1, 0) \rangle$$

have Homs to $b^{(\gamma)}$; or equivalently $b^{(\gamma)}$ receives no Homs from

$$\mathcal{A}_{\ell-\gamma-1}(\ell-\gamma-1) \boxtimes \langle \mathcal{C}_0(1), \ldots, \mathcal{C}_{\ell-\gamma-2}(\ell-\gamma-1)\rangle. \tag{3.9}$$

As before, cone$(b \to b^{(\gamma)})$ receives no Homs from the above by the induction hypothesis and the cone lemma. The only thing to show is that $b \in {}^{\mathrm{prim}}\mathscr{D}_1$ also receives no Homs from (3.9). This can be shown as follows. Since $b \in \mathscr{D}_1$ is in the right orthogonal of all $\mathcal{E}(-,k)$ for $k \in [1, i-1]$, it receives no Homs from

$$\mathcal{A}_{\ell-\gamma-1}(\ell-\gamma-1) \boxtimes \langle \mathcal{C}_1(1), \ldots, \mathcal{C}_{\ell-\gamma-1}(\ell-\gamma-1).$$

Also from the definition of ${}^{\mathrm{prim}}\mathscr{D}_1$ in Prop.-Def. 3.16, $b \in {}^{\mathrm{prim}}\mathscr{D}_1$ receives no Homs from

$$\mathcal{A}_{\ell-\gamma-1}(\ell-\gamma-1) \boxtimes \mathcal{C}^{\ell-\gamma-1}.$$

But recall $\mathcal{C}^{\ell-\gamma-1} = \langle \gamma_0^*(\mathfrak{c}_0(1)), \ldots, \gamma_0^*(\mathfrak{c}_{\ell-\gamma-2}(\ell-\gamma-1))\rangle$, and from Lem. 2.12 for $D(T^{\natural})$ we have

$$\langle \mathcal{C}_0(1), \ldots, \mathcal{C}_{\ell-\gamma-2}(\ell-\gamma-1)\rangle = \langle \mathcal{C}^{\ell-\gamma-1}, \mathcal{C}_1(1), \ldots, \mathcal{C}_{\ell-\gamma-1}(\ell-\gamma-1)\rangle$$

for all $\gamma = 0, 1, \ldots, \ell-2$. Hence the above two vanishings imply b itself receives no Homs from (3.9). Hence we finish the induction step as before. □

From this lemma, we directly have

Proposition-Definition 3.18. (1) *The functor $I_2^* : D(\mathcal{H}) \to \mathscr{D}_2$ is fully faithful on $(\mathcal{A}^1 \boxtimes {}^{L}\mathcal{C}_1(2-\ell))|_{\mathcal{H}}, \ldots, (\mathcal{A}^{\ell-1} \boxtimes {}^{L}\mathcal{C}_{\ell-1})|_{\mathcal{H}}$, and induces a primitive decomposition*

$$\mathscr{D}_2 = \langle (\mathcal{A}^1 \boxtimes {}^{L}\mathcal{C}_1(2-\ell))|_{\mathcal{H}}, \ldots, (\mathcal{A}^{l-1} \boxtimes {}^{L}\mathcal{C}_{l-1})|_{\mathcal{H}}, \mathscr{D}_2^{\mathrm{prim}}\rangle;$$

the right admissible subcategory $\mathscr{D}_2^{\mathrm{prim}}$ is called the right primitive component *of $\mathscr{D}_2$.*
(2) *The functor $I_2^* I_1$ is fully faithful on ${}^{\mathrm{prim}}\mathscr{D}_1$, and induces ${}^{\mathrm{prim}}\mathscr{D}_1 \hookrightarrow \mathscr{D}_2^{\mathrm{prim}}$.*

Proof. To prove the proposition, we show

$$R\,\mathrm{Hom}(a, \mathrm{cone}(b \to I_2^* b)) = 0$$

for any a, b in one of the three cases:

(1) $a \in (\mathcal{A}^m \boxtimes {}^{L}\mathcal{C}_m(m+1-\ell))|_{\mathcal{H}}, b \in (\mathcal{A}^k \boxtimes {}^{L}\mathcal{C}_k(k+1-\ell))|_{\mathcal{H}}, 1 \le k \le m \le \ell-1$;
(2) $a \in {}^{\mathrm{prim}}\mathscr{D}_1, b \in (\mathcal{A}^k \boxtimes {}^{L}\mathcal{C}_k(k+1-\ell))|_{\mathcal{H}}, 1 \le k \le \ell-1$;
(3) $a \in {}^{\mathrm{prim}}\mathscr{D}_1, b \in {}^{\mathrm{prim}}\mathscr{D}_1$.

For cases (1) and (2), from Lem. 3.17, cone$(b \to I_2^* b)$ belongs to the subcategory generated by ${}^{L}\mathcal{E}(*, 2-\ell), \ldots, {}^{L}\mathcal{E}(*, k-\ell)$, hence receives no Homs from ${}^{L}\mathcal{E}(*, m+1-\ell)$ for $k \le m \le l-1$ or ${}^{\mathrm{prim}}\mathscr{D}_1 \subset \mathscr{D}_1$, by semiorthogonality of (3.3). For case (3), Lem. 3.17 and (3.3) imply that cone$(b \to I_2^* b)$ belongs to $\mathscr{D}_1^{\perp}$, hence receives no Homs from $a \in {}^{\mathrm{prim}}\mathscr{D}_1 \subset \mathscr{D}_1$. □

3.5. Generation

Now we have finished the proof of the first part of Thm. 3.12, and also the part ${}^{\mathrm{prim}}\mathscr{D}_1 \hookrightarrow \mathscr{D}_2^{\mathrm{prim}}$. To complete the proof of $I_2^* I_1 : {}^{\mathrm{prim}}\mathscr{D}_1 \simeq \mathscr{D}_2^{\mathrm{prim}}$, hence our main Theorem 3.1, it remains to show that the image of ${}^{\mathrm{prim}}\mathscr{D}_1$ generates $\mathscr{D}_2^{\mathrm{prim}}$. This is equivalent to showing that the images of $(\mathcal{A}^1 \boxtimes {}^L\mathcal{C}_1(2-\ell))|_{\mathcal{H}}, \ldots, (\mathcal{A}^{\ell-1} \boxtimes {}^L\mathcal{C}_{\ell-1}))|_{\mathcal{H}}$ and ${}^{\mathrm{prim}}\mathscr{D}_1$ under I_2^* generate $\mathscr{D}_2$. But from the comment after Lem. 3.15, the (left) orthogonal of ${}^{\mathrm{prim}}\mathscr{D}_1$ inside $\mathscr{D}_1$ (i.e. the ambient part) is contained in ${}^{\perp}\mathscr{D}_2$, hence ${}^{\mathrm{prim}}\mathscr{D}_1$ and $\mathscr{D}_1$ have the same image under $I_2^* = \mathbb{L}_{{}^{\perp}\mathscr{D}_2}$, since $\mathbb{L}_{{}^{\perp}\mathscr{D}_2}|_{{}^{\perp}\mathscr{D}_2} = 0$ (see Lem. 2.3). Therefore we only need to prove a slightly weaker statement: the images of

$$(\mathcal{A}^1 \boxtimes {}^L\mathcal{C}_1(2-\ell)))|_{\mathcal{H}}, \ldots, (\mathcal{A}^{\ell-1} \boxtimes {}^L\mathcal{C}_{\ell-1}))|_{\mathcal{H}} \quad \text{and} \quad \mathscr{D}_1$$

under I_2^* generate $\mathscr{D}_2$. We will omit the subscript $|_{\mathcal{H}}$ as before. Our strategy is to show that the *right orthogonal* of these images inside $\mathscr{D}_2$ is zero, i.e. if $b \in \mathscr{D}_2$ is such that

$$R\operatorname{Hom}_{\mathcal{H}}(a, I_2\, b) = 0 \quad \forall a \in \mathscr{D}_1, \tag{3.10}$$

$$R\operatorname{Hom}_{\mathcal{H}}(a, I_2\, b) = 0 \quad \forall a \in \mathcal{A}^1 \boxtimes {}^L\mathcal{C}_1(2-l), \ldots, \mathcal{A}^{l-1} \boxtimes {}^L\mathcal{C}_{l-1}, \tag{3.11}$$

then $b = 0$. Here we use the adjunction $R\operatorname{Hom}_{\mathcal{H}}(a, I_2\, b) = R\operatorname{Hom}_{\mathscr{D}_2}(I_2^*\, a, b)$. From now on we will simply write b also for the image $I_2\, b$. Note that the first condition (3.10) is equivalent to $b \in \mathscr{D}_1^{\perp} = \langle {}^L\mathcal{E}(\alpha, \beta) \mid \alpha \in [0, i-1], \beta \in [2-\ell, 0] \rangle$, therefore b is built from the components

$$b_\beta^\alpha \in {}^L\mathcal{E}(\alpha, \beta - \ell + 1) = \mathcal{A}_\alpha(\alpha) \boxtimes {}^L\mathcal{C}_\beta(\beta - \ell + 1),$$

where $\alpha \in [0, i-1]$, $\beta \in [1, \ell-1]$. We further put $b_\beta^0 \in {}^L\mathcal{E}(0, \beta)$ into a distinguished triangle $b_\beta^R \to b_\beta^0 \to b_\beta^L$ using the decomposition $\mathcal{A}_0 = \langle (\mathcal{A}^\beta)^{\perp}, \mathcal{A}^\beta \rangle$, $\beta \in [1, \ell-1]$, with

$$b_\beta^L \in ((\mathcal{A}^\beta)^{\perp} \boxtimes {}^L\mathcal{C}_\beta(\beta + 1 - \ell))|_{\mathcal{H}}, \quad b_\beta^R \in (\mathcal{A}^\beta \boxtimes {}^L\mathcal{C}_\beta(\beta + 1 - \ell))|_{\mathcal{H}}. \tag{3.12}$$

Now it remains to show that the condition (3.11) and $b \in \mathscr{D}_1$ force all the above components b_β^α, b_β^L, b_β^R to be zero. This will be done by using a specific zig-zag induction scheme. The region indicated by "$\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$" in Figs. 3 and 4 will be helpful to visualize the proof.

Step 1. *The components of b which are "above the staircase region $\mathcal{R}_1$" are zero, i.e.*

$$b_\beta^\alpha = 0 \quad \textit{for } \beta \in [1, \ell-1],\ \alpha \in [\beta+1, i-1]. \tag{3.13}$$

Proof. We show this by induction, with a zig-zag induction routine (see Fig. 4):

$$(b_1^{i-1} = 0) \implies (b_1^{i-2} = 0) \implies \cdots \implies (b_1^2 = 0) \implies (b_2^{i-1} = 0)$$
$$\implies \cdots \implies (b_{i-1}^{i-2} = 0).$$

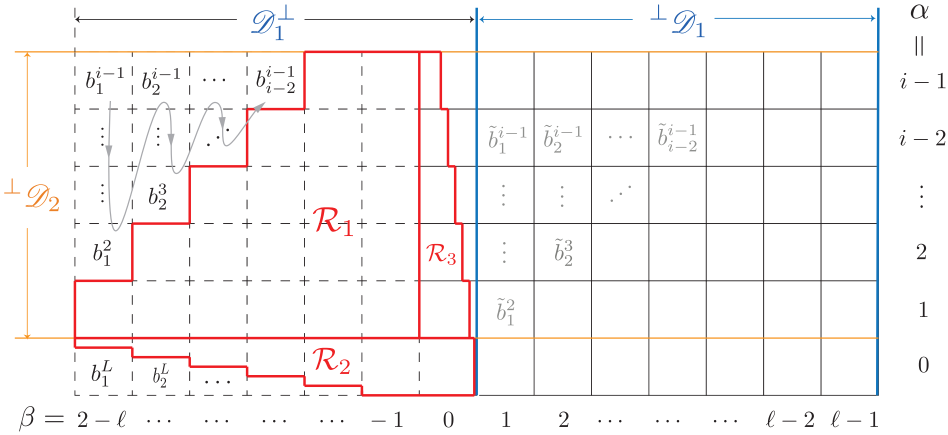

Fig. 4. The chessboard indicating blocks $\mathcal{E}(-,-)$ and ${}^{L}\mathcal{E}(-,-)$ in the case $l \geq i$. The grey line with arrows indicates the "zig-zag" induction routine ($b_1^{i-1} \to \cdots \to b_1^2 \to b_2^{i-1} \to \cdots \to b_2^3 \to \cdots \to b_{i-1}^{i-1}$) in Step 1 in the proof of generation, and the grey "$\tilde{b}_\beta^\alpha$" on the right indicates the key block for detecting the vanishing of b_β^α.

Assume we have proved (3.13) for those b_β^α's in the region with smaller β or with the same β but larger α.[10] To show $b_\beta^\alpha = 0$, we look at Homs from

$$\mathcal{E}(*, \beta) = D(X) \boxtimes \mathcal{C}_\beta(\beta) = D(X)(-1) \boxtimes \mathcal{C}_\beta(\beta). \tag{3.14}$$

Note that $b_\beta^\alpha \in {}^{L}\mathcal{E}(*, \beta + 1 - \ell)$ and by (3.3) for $k = \beta$, there are no Homs from

$$\langle {}^{L}\mathcal{E}(*, \beta + 2 - \ell), \ldots, {}^{L}\mathcal{E}(*, 0), \mathscr{D}_1, \mathcal{E}(*, 1), \ldots, \mathcal{E}(*, \beta - 1)\rangle.$$

If there are no Homs from $\mathcal{E}(*, \beta)$ to b_β^α, then by (3.3) again for $k = \beta + 1$, there are no Homs from the whole $\mathcal{H}$ to b_β^α. Hence this forces $b_\beta^\alpha = 0$ and we are done.

To show there are no Homs from (3.14) to b_β^α, we first observe that the non-zero Homs from (3.14) to b_β^α can only come from the "key subcategory"

$$\mathcal{A}_\alpha(\alpha - 1) \boxtimes \mathcal{C}_\beta(\beta) \subset \mathcal{E}(\alpha - 1, \beta) \tag{3.15}$$

(marked by gray "$\tilde{b}_\beta^\alpha$" in Fig. 4). The reason is that if we consider $D(X) = \langle \mathcal{A}_\alpha(\alpha), {}^{\perp}(\mathcal{A}_\alpha(\alpha))\rangle$,[11] then $D(X)(-1) \boxtimes \mathcal{C}_\beta(\beta) = \langle \mathcal{A}_\alpha(\alpha-1), {}^{\perp}(\mathcal{A}_\alpha(\alpha))(-1)\rangle \boxtimes \mathcal{C}_\beta(\beta)$. The Hom from the latter component to $b_\beta^\alpha \in \mathcal{A}_\alpha(\alpha) \boxtimes {}^{L}\mathcal{C}_\beta(\beta + 1 - l)$ is a cone of the form (3.4), where the untwisted term is zero by β-vanishing,[12] and the twisted term is zero by

$$R\operatorname{Hom}_X({}^{\perp}(\mathcal{A}_\alpha(\alpha))(-1), \mathcal{A}_\alpha(\alpha)(-1)) = R\operatorname{Hom}_X({}^{\perp}(\mathcal{A}_\alpha(\alpha)), \mathcal{A}_\alpha(\alpha)) = 0.$$

10 For the base case $(\alpha, \beta) = (i - 1, 1)$, this assumption is empty.

11 Explicitly, ${}^{\perp}(\mathcal{A}_\alpha(\alpha)) = \langle \mathcal{A}_{\alpha+1}(\alpha+1), \ldots, \mathcal{A}_{i-1}(i-1), S_X^{-1}(\mathcal{A}_0), \ldots, S_X^{-1}(\mathcal{A}_{\alpha-1}(\alpha-1))\rangle$.

12 More precisely, $\operatorname{Hom}_S(\mathcal{C}_{\beta-1}(\beta), {}^{L}\mathcal{C}_\beta(\beta + 1 - l)) = 0$ and $\mathcal{C}_\beta(\beta) \subset \mathcal{C}_{\beta-1}(\beta)$.

Now it remains to show the vanishing from the key box (3.15) to b^α_β, and this is where the "zig-zag" induction routine comes in. We show there are no Homs from (3.15) to any other surviving component of b other than b^α_β. This together with the vanishing for b implies the desired vanishing for b^α_β. These other surviving components fall into three categories:

- The components with larger β, i.e. components $b^{\alpha'}_{\beta'}$ with $\beta' > \beta$; there are no Homs from (3.15) to these simply by β-vanishing.
- The components with the same β but smaller α, i.e. the components in

$$\langle \mathcal{A}_0, \mathcal{A}_1(1), \dots \mathcal{A}_{\alpha-1}(\alpha-1)\rangle \boxtimes {}^L\mathcal{C}_\beta(\beta+1-l);$$

  Homs from (3.15) to these are zero since the untwisted terms are zero by β-vanishing, and the twisted terms contain zero factors

$$R\operatorname{Hom}_X(\mathcal{A}_\alpha(\alpha), \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots \mathcal{A}_{\alpha-1}(\alpha-1)\rangle) = 0.$$

- The components with smaller β; by induction assumption, these are components of the form $b^{\alpha'}_{\beta'}$ with $0 \le \alpha' \le \beta'$, $\beta' < \beta$. Since $\alpha' \le \beta - 1 \le \alpha - 2$, Homs from (3.15) to these components are zero by α-vanishing,

Note that $b \in \mathscr{D}_2$ itself receives no Homs from (3.15), so by Lem. 2.2, b^α_β also receives no Homs from (3.15), hence from (3.14). Hence $b^\alpha_\beta = 0$, and by induction we are done. □

Step 2. *The components of b which are "below the staircase region $\mathcal{R}_2$" are zero:*

$$b^L_\beta = 0 \quad \textit{for } \beta = 1, \dots, l-1.$$

Proof. We show this by induction on $\beta \in [1, \ell - 1]$. Assume it is true for components with smaller β (for the base case $\beta = 1$ the assumption is empty). Then to show $b^L_\beta = 0$ we again only need to look at Homs from $\mathcal{E}(*, \beta) = D(X) \boxtimes \mathcal{C}_\beta(\beta)$. This time we consider the decomposition

$$D(X) = \langle \mathfrak{a}_0, \mathcal{A}_1, \mathfrak{a}_1(1), \mathcal{A}_2(1), \dots, \mathfrak{a}_{i-2}(i-2), \mathcal{A}_{i-1}(i-2), \mathfrak{a}_{i-1}(i-1)\rangle. \tag{3.16}$$

First, there are no Homs from $\mathcal{A}_1 \boxtimes \mathcal{C}_\beta(\beta), \mathcal{A}_2(1) \boxtimes \mathcal{C}_\beta(\beta), \dots, \mathcal{A}_{i-1}(i-2) \boxtimes \mathcal{C}_\beta(\beta)$ to b^L_β, since using the cone lemma for these Homs, the untwisted terms are zero by β-vanishing, and the twisted terms are zero by α-vanishing,

$$R\operatorname{Hom}_X(\langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}(i-2)\rangle(1), \mathcal{A}_0) = 0.$$

Second, there are also no Homs from $\langle \mathfrak{a}_0, \mathfrak{a}_1(1), \dots, \mathfrak{a}_{\beta-1}(\beta-1)\rangle \boxtimes \mathcal{C}_\beta(\beta)$ to b^L_β by the definition of b^L_β. More precisely, Homs from these all come from the twisted terms, which are zero exactly by adjoint pairs (α_0^*, α_0) and the defining equation (3.12) of b^L_β:

$$R\operatorname{Hom}_X(\langle \mathfrak{a}_0, \mathfrak{a}_1(1), \dots, \mathfrak{a}_{\beta-1}(\beta-1)\rangle(1), (\mathcal{A}^\beta)^\perp) = R\operatorname{Hom}_{\mathcal{A}_0}(\mathcal{A}^\beta, (\mathcal{A}^\beta)^\perp) = 0.$$

Hence it remains to show there are no Homs from the following subcategories to b^L_β:

$$\mathfrak{a}_\beta(\beta) \boxtimes \mathcal{C}_\beta(\beta), \dots, \mathfrak{a}_{i-1}(i-1) \boxtimes \mathcal{C}_\beta(\beta). \tag{3.17}$$

Now the "staircase shape" of the region and the induction hypothesis come into play. We show there are no Homs from (3.17) to all other non-zero components other than b^L_β. These components fall into three different categories:

- The components with larger β's, i.e. ones in ${}^L\mathcal{E}(*, \beta+2-\ell), \ldots, {}^L\mathcal{E}(*, 0)$; there are no Homs from (3.17) to these simply by β-vanishing.
- The surviving components from the staircase region of Step 1 with the same β or smaller β, i.e. the components $b^{\alpha'}_{\beta'}$ with $1 \leq \beta' \leq \beta$, and $1 \leq \alpha' \leq \beta'$. Homs from (3.17) to these components are zero by α-vanishing. More precisely, for components with $\beta' = \beta$, the untwisted terms are zero by $R\operatorname{Hom}_X(\langle \mathfrak{a}_\beta(\beta), \ldots \mathfrak{a}_{i-1}(i-1)\rangle, \mathcal{A}_0) = 0$, and the twisted terms also have vanishing factors:
$$R\operatorname{Hom}_X(\langle \mathfrak{a}_\beta(\beta), \ldots, \mathfrak{a}_{i-1}(i-1)\rangle, \mathcal{A}_{\alpha'}(\alpha'-1)) = 0$$
since $\mathcal{A}_{\alpha'}(\alpha'-1) \subset \mathcal{A}_{\alpha'-1}(\alpha'-1)$ and $1 \leq \alpha' \leq \beta$. For components with $\beta' \leq \beta-1$, apart from the twisted terms being zero as above, we also have for the untwisted terms
$$R\operatorname{Hom}_X(\langle \mathfrak{a}_\beta(\beta), \ldots, \mathfrak{a}_{i-1}(i-1)\rangle, \mathcal{A}_{\alpha'}(\alpha')) = 0$$
since $\alpha' \leq \beta' \leq \beta - 1$.
- The surviving $b^R_{\beta'}$ with $1 \leq \beta' \leq \beta$ from our induction; for Homs from (3.17) to these components, the untwisted terms are always zero since $R\operatorname{Hom}_X(\mathfrak{a}_\beta(\beta), \mathcal{A}^{\beta'}) = 0$ since $\beta \geq 1$ and $\mathcal{A}^{\beta'} \subset \mathcal{A}_0$. For the twisted terms,
$$\begin{aligned} &R\operatorname{Hom}_X(\langle \mathfrak{a}_\beta(\beta), \ldots, \mathfrak{a}_{i-1}(i-1)\rangle(1), \mathcal{A}^{\beta'}) \\ &\qquad = R\operatorname{Hom}_{\mathcal{A}_0}(\langle \alpha_0^*(\mathfrak{a}_\beta(\beta+1)), \ldots, \alpha_0^*(\mathfrak{a}_{i-1}(i)), \mathcal{A}^{\beta'}) = 0 \end{aligned}$$
by $\mathcal{A}^{\beta'} \subset \mathcal{A}^\beta$ for $\beta' \leq \beta$, Lem. 2.12, and the definition (2.5) of $\mathcal{A}^\beta$.

Finally b itself receives no Homs from (3.17) where $\beta \geq 1$ by Lem. 2.3, and by Lem. 2.2, b^L_β receives no Homs from (3.17). Altogether, $R\operatorname{Hom}(\mathcal{E}(*, \beta), b^L_\beta) = 0$, which again forces $b^L_\beta = 0$ as in Step 1. This completes the proof of Step 2 by induction. □

Final step: "checkmate". By the previous two steps, the only surviving components of b belong to the subcategory generated by
$$\mathcal{E}(\alpha, \beta+1-\ell) \quad \text{for } \alpha \in [1, \min\{\beta, i-1\}],\ \beta \in [1, \ell-1],$$
and
$$\mathcal{A}^1 \boxtimes {}^L\mathcal{C}_1(2-\ell), \ldots, \mathcal{A}^{\ell-1} \boxtimes {}^L\mathcal{C}_{\ell-1}.$$
See the region "$\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$" in Figs. 3 and 4. But $b \in \mathscr{D}_2$ together with (3.11) implies b receives no Homs from these categories as well. Therefore $R\operatorname{Hom}_{\mathcal{H}}(b, b) = 0$, and hence $b = 0$. This completes the proof of the generation part, and hence the proof of our main theorem. □

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Appendix. Künneth formula for exterior product

The following is known to experts; we include a detailed proof for the lack of direct references.

Lemma A.1 (Künneth formula). *For any admissible subcategories $\mathcal{A} \subset D(X)$ and $\mathcal{B} \subset D(Y)$, where X and Y are smooth projective varieties, the following Künneth formula holds:*

$$\mathrm{HH}_\bullet(\mathcal{A} \boxtimes \mathcal{B}) = \mathrm{HH}_\bullet(\mathcal{A}) \otimes \mathrm{HH}_\bullet(\mathcal{B}),$$

where the right hand side is regarded as the tensor product of graded vector spaces.

Proof. By [K9, Thm. 7.1] there are kernels $P \in D(X \times X)$ and $Q \in D(Y \times Y)$ such that the projection functors $i^*_{\mathcal{A}} = \Phi_P : D(X) \to \mathcal{A}$ and $i^*_{\mathcal{B}} = \Phi_Q : D(Y) \to \mathcal{B}$ are the corresponding Fourier–Mukai functors. Then the projection functor $D(X \times Y) \to \mathcal{A} \boxtimes \mathcal{B}$ is given by $i^*_{\mathcal{A}\boxtimes\mathcal{B}} = \Phi_{P\boxtimes Q}$ for $P \boxtimes Q \in D(X \times X \times Y \times Y)$. (This follows from the fact that $D(X \times Y)$ is split generated by elements of the form $a \boxtimes b \in D(X \times Y)$ for $a \in D(X)$, $b \in D(Y)$, and $\Phi_{P\boxtimes Q}(a \boxtimes b) = (i^*_{\mathcal{A}} a) \boxtimes (i^*_{\mathcal{B}} b) \in \mathcal{A} \boxtimes \mathcal{B}$ satisfies $\Phi_{P\boxtimes Q}(a \boxtimes b) = a \boxtimes b$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$, and that $\mathrm{cone}(\Phi_{P\boxtimes Q}(a \boxtimes b) \to (a \boxtimes b)) \in {}^\perp(\mathcal{A} \boxtimes \mathcal{B})$ for any $a \in D(X)$, $b \in D(Y)$; see [BK].)

(Proof via dg-categories). If E and F are strong compact generators for $D(X)$ and $D(Y)$, then by [K7, Lem. 4.3], $E_{\mathcal{A}} = i^*_{\mathcal{A}} E$ and $F_{\mathcal{B}} = i^*_{\mathcal{B}} F$ are the respective strong generators for $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{A} \simeq H^0(\mathrm{Perf}(A))$, $\mathcal{B} \simeq H^0(\mathrm{Perf}(B))$, where $A = R\,\mathrm{Hom}(E_{\mathcal{A}}, E_{\mathcal{A}})$, $B = R\,\mathrm{Hom}(F_{\mathcal{B}}, F_{\mathcal{B}})$ are the dg-algebras, and $\mathrm{Perf}(A)$ and $\mathrm{Perf}(B)$ are the dg-categories of perfect A- and resp. B-modules. By [BV, Lem. 3.4.1 & Thm. 3.1.4], $E \boxtimes F$ is a strong generator for $D(X \times Y)$, and by [K7, Lem. 4.3], $E_{\mathcal{A}} \boxtimes F_{\mathcal{B}}$ is a strong generator for $\mathcal{A} \boxtimes \mathcal{B}$, and $\mathcal{A} \boxtimes \mathcal{B} \simeq H^0(\mathrm{Perf}(A \otimes B))$, since $R\,\mathrm{Hom}_{X\times Y}(E_{\mathcal{A}} \boxtimes F_{\mathcal{B}}, E_{\mathcal{A}} \boxtimes F_{\mathcal{B}}) = A \otimes B$. By definition of [K7], $\mathrm{HH}_\bullet(\mathcal{A}) := \mathrm{HH}_\bullet(\mathrm{Perf}(A)) = \mathrm{HH}_\bullet(A)$, and similarly for $\mathcal{B}$ and $\mathcal{A} \boxtimes \mathcal{B}$. Therefore the desired result follows from the fact that the composition of the natural maps of Hochschild chains

$$C_\bullet(\mathrm{Perf}(A)) \otimes C_\bullet(\mathrm{Perf}(B)) \to C_\bullet(\mathrm{Perf}(A) \otimes \mathrm{Perf}(B)) \to C_\bullet(\mathrm{Perf}(A \otimes B))$$

is a quasi-isomorphism [Sh, Prop. 2.9, §2.4]. The formula $\mathrm{HH}_\bullet(A \otimes B) = \mathrm{HH}_\bullet(A) \otimes \mathrm{HH}_\bullet(B)$ is known as Künneth for dg-algebras (see [PV, Prop. 1.1.4] and [Sh, §2.4] for more details).

(Proof via kernel functors). If we denote by $P^T \in D(X \times X)$ and $Q^T \in D(Y \times Y)$ the transposes of the kernels, then $(P \boxtimes Q)^T = P^T \boxtimes Q^T \in D(X \times X \times Y \times Y)$. Therefore by

[K7, Thm. 4.5]), $\mathrm{HH}_\bullet(\mathcal{A}) = H^\bullet(X \times X, P \otimes P^T)$ and $\mathrm{HH}_\bullet(\mathcal{B}) = H^\bullet(Y \times Y, Q \otimes Q^T)$, hence

$$\begin{aligned}\mathrm{HH}_\bullet(\mathcal{A} \boxtimes \mathcal{B}) &= H^\bullet(X \times X \times Y \times Y, (P \otimes P^T) \boxtimes (Q \otimes Q^T)) \\ &= H^\bullet(X \times X, P \otimes P^T) \otimes H^\bullet(Y \times Y, Q \otimes Q^T) \\ &= \mathrm{HH}_\bullet(\mathcal{A}) \otimes \mathrm{HH}_\bullet(\mathcal{B}). \qquad \square\end{aligned}$$

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