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Bounds for twists of GL(3) L-functions

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Abstract. Let π be a fixed Hecke–Maass cusp form for $\mathrm{SL}(3,\mathbb{Z})$ and χ be a primitive Dirichlet character modulo M, which we assume to be a prime. Let $L(s,\pi\otimes\chi)$ be the L-function associated to $\pi\otimes\chi$. For any given $\varepsilon>0$, we establish a subconvex bound $L(1/2+it,\pi\otimes\chi)\ll_{\pi,\varepsilon}(M(|t|+1))^{3/4-1/36+\varepsilon}$, uniformly in both the M- and t-aspects.

Keywords. L-functions, subconvexity, Hecke–Maass cusp forms

1. Introduction and statement of results

The subconvexity problem, which asks for an estimate of an automorphic L-function on the critical line $s=1/2+it$ that is better by a power saving than the bound implied by the functional equation and the Phragmén–Lindelöf principle, is one of the central problems in analytic number theory. Many cases have been treated in the past; see [21] for results with full generality on GL(1) and GL(2). It has only been recently that people have started making progress on GL(3) with the introduction of new techniques.

In this paper, we are interested in certain degree 3 L-functions. Let π be a fixed Hecke–Maass cusp form of type (ν_1,ν_2) for $\mathrm{SL}(3,\mathbb{Z})$ with normalized Fourier coefficients $(\lambda(m,n))_{m,n\geq 0}$. Let χ be a primitive Dirichlet character modulo M. Let

$$L(s,\pi)=\sum_{n=1}^{\infty}\frac{\lambda(1,n)}{n^s}\quad\text{and}\quad L(s,\pi\otimes\chi)=\sum_{n=1}^{\infty}\frac{\lambda(1,n)\chi(n)}{n^s}$$

be the L-series associated with π and $\pi\otimes\chi$; these series can be analytically continued to entire functions of $s\in\mathbb{C}$ with functional equations. The Phragmén–Lindelöf principle implies the convexity bound $L(1/2+it,\pi\otimes\chi)\ll_{\pi,\varepsilon}(M(|t|+1))^{3/4+\varepsilon}$, which one aims to improve upon.

For the L-functions $L(s,\pi)$, the first breakthrough was made by Li [17] who resolved the subconvexity problem in the t-aspect for fixed self-dual cusp forms π. Using a first moment method, partially inspired by the approach in [6], Li showed that $L(1/2+it,\pi)\ll_\pi(|t|+1)^{3/4-\delta+\varepsilon}$ with $\delta=1/16$. Li's approach also implies a subcon-

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Mathematics Subject Classification (2020): Primary 11F66; Secondary 11F55, 11M41 vexity bound in the GL(2) spectral aspect for certain GL(3) × GL(2) L-functions. The method depends on the nonnegativity of central values of certain L-functions, which requires the self-duality assumption on the cusp forms π. Li's exponent of saving $\delta = 1/16$ was later improved to $\delta = 1/12$ by Mckee, Sun, and Ye [20], and to $\delta = 1/8$ by Nunes [32].

For the case where M, the conductor of the Dirichlet characters χ, is varying, in the special case that the π's are self-dual and χ's are quadratic, a subconvex bound was obtained by Blomer [2]. He showed that $L(1/2, \pi \otimes \chi) \ll_\pi M^{5/8+\varepsilon}$ by using a first moment method as in Li's work, where M is assumed to be prime. Later Huang [9], with input from [41], managed to extend the results of Li and Blomer to the hybrid setting $L(1/2 + it, \pi \otimes \chi) \ll_\pi (M(|t| + 1))^{3/4-\delta}$, for some $\delta > 0$, under the same self-duality assumptions on π and χ.

From a theorem of Miller [22], self-dual cusp forms f_j on $\mathrm{SL}(3, \mathbb{Z})\backslash\mathfrak{h}^3$ are sparse in the sense that

$$\lim_{T\to\infty} \frac{\#\{\lambda_j \le T \mid \Delta f_j = \lambda_j f_j,\ f_j \text{ self-dual}\}}{\#\{\lambda_j \le T\}} = 0.$$

It is therefore desirable to remove the self-duality assumptions in the previous works.

In a series of papers [26–30], Munshi proposed a new approach to the subconvexity problem. This method does not need the nonnegativity of central values of certain L-functions, which enabled Munshi to deal with more general cusp forms than just the self-dual subclass.

In the t-aspect setting, by adopting Kloosterman's refinement of the circle method and enhanced by a "conductor lowering" mechanism, Munshi [28] obtained the bound $L(1/2 + it, \pi) \ll_\pi (|t| + 1)^{3/4-1/16+\varepsilon}$, thus extending Li's result [17] to arbitrary fixed cusp forms π.

In the Dirichlet character twist case, by using a variant of the δ-symbol method of Duke, Friedlander, and Iwaniec [7], a GL(2) Petersson δ-symbol method, Munshi established $L\,(1/2, \pi \otimes \chi) \ll_\pi M^{3/4-1/1612+\varepsilon}$ [29], under the Ramanujan conjecture for π. In a follow-up preprint [30], with a much cleaner treatment, he removed that assumption and improved the exponent of saving to $\delta = 1/308$. Again, this approach does not require nonnegativity of central values of certain L-functions, thereby removing the self-duality assumptions on the cusp forms π and characters χ in Blomer's work [2].

More recently Holowinsky and Nelson [8] discovered that there is a hidden identity within the proof of [30], which allowed them to produce a method that removes the use of the Petersson δ-symbol method and also improves the exponent of saving. They obtained a stronger subconvex exponent, $L\,(1/2, \pi \otimes \chi) \ll_\pi M^{3/4-1/36+\varepsilon}$.

It is then desirable to ask, "Can one prove a subconvex bound for the L-functions $L\,(s, \pi \otimes \chi)$, simultaneously in both the M- and t-aspects, for general $\mathrm{SL}(3, \mathbb{Z})$ Hecke–Maass cusp forms and primitive Dirichlet characters?" Our main result answers this affirmatively.

Theorem 1.1. *Let π be a Hecke–Maass cusp form for* $\mathrm{SL}(3, \mathbb{Z})$ *and χ be a primitive Dirichlet character modulo M, which we assume to be prime. Given any $\varepsilon > 0$, we have*

$$L(1/2 + it, \pi \otimes \chi) \ll_{\pi,\varepsilon} (M(|t| + 1))^{3/4-1/36+\varepsilon}. \tag{1}$$

Remark 1.2. Below we will carry out the proof under the assumption $|t| > M^{\varepsilon}$ for any $\varepsilon > 0$. We make such an assumption so as to effectively control the error terms coming from the stationary phase analysis in our approach. For the case where $|t| < M^{\varepsilon}$, the bound (1) would follow from the work [8], since there the bound $L(1/2 + it, \pi \otimes \chi) \ll_{t,\pi} M^{3/4-1/36+\varepsilon}$ is of polynomial dependence in t.

For subconvexity bounds on GL(3) in other aspects or over more general number fields, see [3, 4, 13, 34, 35, 39, 40].

Recently, Schumacher [36] has been able to provide another interpretation of the methods that we follow, at least in the t-aspect case, from the perspective of integral representations under the framework of Michel–Venkatesh [21], and he produces the same bound (1). Aggarwal [1], who revisited Munshi's work in [28] by removing the "conductor lowering" trick, was able to improve the exponent of saving in the t-aspect case to $3/40$. The exponent of saving in the M-aspect was recently improved to $1/32$ by Sharma [37]. Following Li's work in [17], there have been recent developments in the subconvexity problem on GL(3) $\times$ GL(2) in different aspects, and the reader is referred to [15, 16, 18, 19, 31, 35, 37, 38].

Our approach is a variant of the methods introduced in the works [30] and [8]. In Section 2, we will give a brief outline of our approach to guide the readers through.

Notation

We use $e(x)$ to denote $\exp(2\pi i x)$. We denote by ε an arbitrary small positive constant, which might change from line to line. In this paper the notation $A \asymp B$ (sometimes even $A \approx B$) means that $B/(M|t|)^{\varepsilon} \ll |A| \ll B(M|t|)^{\varepsilon}$. We reserve the letters p and ℓ to denote primes. The notations $p \sim P$ and $\ell \sim L$ denote primes in the dyadic segments $[P, 2P]$ and $[L, 2L]$ respectively.

2. An outline of the proof

Our approach is inspired by the work [30] and makes use of an observation due to Holowinsky and Nelson [8]. We now give a brief introduction to the approach in [30].

Let p be a prime number, and let $k \equiv 3 \bmod 4$ be a positive integer. Let ψ be a character of $\mathbb{F}_p^{\times}$ satisfying $\psi(-1) = -1 = (-1)^k$. One can view ψ as a character modulo pM. Let $H_k(pM, \psi)$ be an orthogonal Hecke basis of the space $S_k(pM, \psi)$ of cusp forms of level pM, nebentypus ψ and weight k. For $f \in H_k(pM, \psi)$, let $(\lambda_f(n))_{n\ge 1}$ denote its Fourier coefficients. Denote $P^{\star} = \sum_{P<p<2P} \sum_{\psi \bmod p} (1 - \psi(-1))$. Then we have the following averaged version of the Petersson formula:

$$\delta(r,n) = \frac{1}{P^{\star}} \sum_{p\sim P} \sum_{\psi \bmod p} (1-\psi(-1)) \sum_{f \in H_k(pM,\psi)} \omega_f^{-1} \overline{\lambda_f(r)} \lambda_f(n)$$
$$- \frac{2\pi i}{P^{\star}} \sum_{p\sim P} \sum_{\psi \bmod p} (1-\psi(-1)) \sum_{c=1}^{\infty} \frac{S_{\psi}(r,n;cpM)}{cpM} J_{k-1}\left(\frac{4\pi\sqrt{rn}}{cpM}\right), \qquad (2)$$

where $\delta(r,n)$ denotes the Kronecker symbol, $\omega_f^{-1}=\frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|^2}$ is the spectral weight, and $S_\psi(r,n;c)=\sum^*_{\alpha \bmod c}\psi(\alpha)e\big(\frac{r\alpha+n\bar{\alpha}}{c}\big)$ is the generalized Kloosterman sum.

Let $\mathcal{L}$ be the set of primes in the interval $[L,2L]$ and let $L^\star=|\mathcal{L}|$ denote the cardinality of $\mathcal{L}$. By writing his main sum of interest $\sum\sum_{m,n=1}^\infty\lambda(m,n)\chi(n)W\big(\frac{nm^2}{N}\big)V\big(\frac{n}{N}\big)$ as

$$\frac{1}{L^\star}\sum_{\ell\in\mathcal{L}}\bar{\chi}(\ell)\sum\sum_{m,n=1}^{\infty}\lambda(m,n)W\left(\frac{nm^2}{N}\right)\sum_{r=1}^{\infty}\chi(r)V\left(\frac{r}{N\ell}\right)\delta(r,n\ell),$$

and then substituting the formula (2) for $\delta(r,n\ell)$ inside, Munshi breaks the main sum into two pieces, $\mathcal{F}^\star+\mathcal{O}^\star$, with $\mathcal{F}^\star$ and $\mathcal{O}^\star$ appropriately defined. Here the introduction of the extra summation over ℓ serves the role of an amplification technique. Successfully bounding $\mathcal{F}^\star$ and $\mathcal{O}^\star$ simultaneously with suitable choices of P and L to balance the contribution enables Munshi to get his main result $L(1/2,\pi\otimes\chi)\ll_\pi M^{3/4-1/308+\varepsilon}$.

Now we turn to our case. From Lemma 3.3, it suffices to improve the trivial bound $O(N)$ for the smooth sum

$$S(N):=\sum_{n\geq 1}\lambda(1,n)\chi(n)n^{-it}w(n/N)$$

for $(Mt)^{3/2-\delta}<N<(Mt)^{3/2+\varepsilon}$, where $w(x)$ is some smooth function with compact support contained in $\mathbb{R}_{>0}$ satisfying $w^{(j)}(x)\ll_j 1$ for all $j\geq 0$.

Let P and L be two large parameters to be specified later. In our case, instead of using the Petersson δ-symbol method (2), we use a "key identity" (13),

$$\begin{aligned}&\chi(n)n^{-it}V_A\left(\frac{n}{N}\right)\\&=\left(\frac{2\pi}{Mt}\right)^{it}e\left(\frac{t}{2\pi}\right)\frac{M^2t^{3/2}\ell}{Npg_{\bar{\chi}}}\sum_{r=1}^{\infty}\chi(r\ell\bar{p})\left(\frac{r\ell}{p}\right)^{-it}e\left(-\frac{np\bar{M}}{\ell r}\right)V\left(\frac{r}{Np/M\ell t}\right)\\&\quad-\left(\frac{2\pi}{N}\right)^{it}e\left(\frac{t}{2\pi}\right)\frac{t^{1/2}}{g_{\bar{\chi}}}\sum_{r\neq 0}S_{\bar{\chi}}(n,rp\bar{\ell};M)\mathcal{J}_{it}(n,rp/\ell;M)+O(t^{1/2-A}),\end{aligned}$$

where g_χ denotes the Gauss sum associated with χ and

$$\mathcal{J}_{it}(n,rp/\ell;M)=\int_{\mathbb{R}}V(x)x^{-it}e\left(-\frac{nt}{Nx}\right)e\left(-\frac{rNpx}{M^2\ell t}\right)\,\mathrm{d}x.$$

Here $V(x)$ is a smooth compactly supported function satisfying $V^{(j)}(x)\ll_j 1$ for all $j\geq 0$.

Thus we can write, for an arbitrarily large $A\geq 1$, that

$$S(N)\asymp|\mathcal{F}|+|\mathcal{O}|+O(Nt^{-A}),$$

where

$$\begin{aligned}\mathcal{F}&=\frac{M^{3/2}t^{3/2}}{NP^2}\\&\quad\times\sum_{p\sim P}\sum_{\ell\sim L}\sum_{r\sim NP/(MLt)}\chi(r\ell\bar{p})\left(\frac{r\ell}{p}\right)^{-it}\sum_{n=1}^{\infty}\lambda(1,n)e\left(-\frac{np\bar{M}}{\ell r}\right)w\left(\frac{n}{N}\right),\end{aligned}\tag{3}$$

and

$$\mathcal{O} = \frac{t^{1/2}}{PLM^{1/2}} \sum_{n=1}^{\infty} \lambda(1,n) w\left(\frac{n}{N}\right) \sum_{p\sim P} \sum_{\ell \sim L} \sum_{0\neq |r| \ll \frac{M^2t^2L}{NP}} S_{\bar{\chi}}(n, rp\bar{\ell}; M) \mathcal{J}_{it}(n, rp/\ell; M).$$

Now our task is to beat the bound $O(N)$ for $\mathcal{F}$ and $\mathcal{O}$ simultaneously. We estimate the term $\mathcal{O}$ first. The integral $\mathcal{J}_{it}(n, rp/\ell; M)$ restricts the length of the r-sum to $0 \neq |r| \ll \frac{M^2t^2L}{NP}$. For this sketch we pretend that $r \sim \frac{M^2t^2L}{NP}$.

From the second derivative test we have $\mathcal{J}_{it}(n, rp/\ell; M) \ll t^{-1/2}$. Using this, along with the Weil bound for Kloosterman sums, and estimating directly, we find that

$$\mathcal{O} \ll \frac{t^{1/2}}{PLM^{1/2}} NPL \frac{M^2t^2L}{NP} M^{1/2} t^{-1/2} \ll N \frac{M^2t^2L}{NP}.$$

So we need to save more than $\frac{M^2t^2L}{NP}$ for $\mathcal{O}$.

We apply the Cauchy–Schwarz inequality to reduce the task to saving the same amount from

$$\frac{N^{1/2}t^{1/2}}{PLM^{1/2}} \left(\sum_{n\sim N} \left| \sum_{p\sim P} \sum_{\ell\sim L} \sum_{r\sim \frac{M^2t^2L}{NP}} S_{\bar{\chi}}(n, rp\bar{\ell}; M) \mathcal{J}_{it}(n, rp/\ell; M) \right|^2 \right)^{1/2},$$

or equivalently, saving $\frac{M^4t^4L^2}{N^2P^2}$ from the sum

$$\sum_{n\sim N} \left| \sum_{p\sim P} \sum_{\ell\sim L} \sum_{r\sim \frac{M^2t^2L}{NP}} S_{\bar{\chi}}(n, rp\bar{\ell}; M) \mathcal{J}_{it}(n, rp/\ell; M) \right|^2.$$

For the diagonal term $(p_1, \ell_1, r_1) = (p_2, \ell_2, r_2)$, we save $PL\frac{M^2t^2L}{NP} = \frac{M^2t^2L^2}{N}$, which is satisfactory as long as $\frac{M^2t^2L^2}{N} > \frac{M^4t^4L^2}{N^2P^2}$, i.e., $P \gg \frac{Mt}{N^{1/2}}$.

Opening the square above and applying Poisson summation to the n-sum, only the zero frequency contributes. For the off-diagonal $(p_1, \ell_1, r_1) \neq (p_2, \ell_2, r_2)$, applying Poisson summation in the n-sum we save M from evaluating

$$\sum_{a\,(M)} S_{\bar{\chi}}(a, r_1p_1\bar{\ell}_1; M) \overline{S_{\bar{\chi}}(a, r_2p_2\bar{\ell}_2; M)},$$

and save t from estimating the integral

$$\mathfrak{J} = \int_{\mathbb{R}} \mathcal{J}_{it}(Ny, r_1p_1/\ell_1; M) \overline{\mathcal{J}_{it}(Ny, r_2p_2/\ell_2; M)}\, w(y)\, \mathrm{d}y$$

upon using the first derivative test for oscillatory integral (which is the content of Lemma 5.1). So the estimates for the off-diagonal are satisfactory as long as $Mt \gg \frac{M^4t^4L^2}{N^2P^2}$, i.e., $P > \frac{M^{3/2}t^{3/2}L}{N}$. Hence $\mathcal{O}$ is fine for our purpose if $P > \max\left\{\frac{Mt}{N^{1/2}}, \frac{M^{3/2}t^{3/2}L}{N}\right\}$.

Next, we try to bound the $\mathcal{F}$ term in (3). Estimating trivially, we see that

$$\mathcal{F} \ll \frac{M^{3/2}t^{3/2}}{NP^2} PL \frac{NP}{MLt} N \ll N(Mt)^{1/2}.$$

So our job is to save more than $(Mt)^{1/2}$.

We apply Voronoi summation (Lemma 3.5) to the n-sum to get

$$\mathcal{F} \asymp \frac{M^{3/2}t^{3/2}N^{1/2}}{NP^2} \times \left| \sum_{p\sim P} \sum_{\ell\sim L} \sum_{r\sim NP/(MLt)} \chi(r\ell\bar{p}) \left(\frac{r\ell}{p}\right)^{-it} \sum_{n\sim(\ell r)^3/N} \frac{\lambda(n,1)}{\sqrt{n}} \frac{S(\bar{p}M, n; r\ell)}{\sqrt{r\ell}} \right|.$$

Using the Weil bound and estimating directly we get $\mathcal{F} \ll (NP)^{3/2}/(Mt)$. We save $\frac{M^{3/2}t^{3/2}}{N^{1/2}P^{3/2}}$ from this process, compared to the original trivial bound $O(N(Mt)^{1/2})$, and we still need to save a little more than $\frac{N^{1/2}P^{3/2}}{Mt}$ from the new sum. Pulling the r, n-sums outside, and applying the Cauchy–Schwarz inequality, our job is to save $\frac{NP^3}{M^2t^2}$ from the sum

$$\sum_{r\sim\frac{NP}{MLt}} \sum_{n\sim\frac{N^2P^3}{M^3t^3}} \left| \sum_{p\sim P} \sum_{\ell\sim L} \chi(\ell\bar{p})(\ell/p)^{-it} S(\bar{p}M, n; r\ell) \right|^2.$$

We can save PL from the diagonal, which is satisfactory if $PL > \frac{NP^3}{M^2t^2}$, that is, $L > \frac{NP^2}{M^2t^2}$. Our final step involves opening the square and applying Poisson summation to the n-sum to gain saving for the off-diagonal terms $(p_1, \ell_1) \neq (p_2, \ell_2)$. The zero frequency (which vanishes unless $\ell_1 = \ell_2$) makes a contribution that is dominated by the diagonal $(p_1, \ell_1) = (p_2, \ell_2)$ contribution. The original n-sum can be estimated by

$$\sum_{n\sim\frac{N^2P^3}{M^3t^3}} S(\bar{p}_1M, n; r\ell_1) S(\bar{p}_2M, n; r\ell_2) \ll \frac{N^2P^3}{M^3t^3} \sqrt{r\ell_1 \cdot r\ell_2}.$$

After applying Poisson summation in the n-sum, we gain square-root cancellation for the character sum

$$\sum_{a(r\ell_1\ell_2)} S(\bar{p}_1M, a; r\ell_1) S(\bar{p}_2M, a; r\ell_2) e\left(\frac{an}{r\ell_1\ell_2}\right)$$

in "generic" cases, so that the dualized n-sum is dominated by $r^{3/2}\ell_1\ell_2$. We save $\frac{N^{3/2}P^{5/2}}{M^{5/2}t^{5/2}L^{1/2}}$, which is more than $\frac{NP^3}{M^2t^2}$ if $N/(Mt) > PL$. Hence $\mathcal{F}$ is fine if $\frac{NP^2}{M^2t^2} < L < \frac{N}{PMt}$.

Now it turns out that we can optimally choose $P = (Mt)^{5/18}$ and $L = (Mt)^{1/9}$ to simultaneously beat the bound $O(N)$ for $\mathcal{F}$ and $\mathcal{O}$, which in turn implies a nontrivial bound for $S(N)$, for $(Mt)^{3/2-1/18} < N < (Mt)^{3/2+\varepsilon}$. This yields a subconvexity bound $L(1/2+it, \pi\otimes\chi) \ll_\pi (M(|t|+1))^{3/4-1/36+\varepsilon}$.

3. Some lemmas

In this section, we collect some lemmas that we may use in our proof.

Let $(\lambda(m,n))_{m,n\neq 0}$ be the Fourier–Whittaker coefficients of the $\mathrm{SL}(3,\mathbb{Z})$ Hecke–Maass cusp form π.

First we have the following Rankin–Selberg estimate (see for example [25]).

Lemma 3.1. *One has*

$$\sum\sum_{m^2 n\leq X} |\lambda(m,n)|^2 \ll_\pi X.$$

From the lemma, we readily have the similar estimate

$$\sum_{n\leq X} |\lambda(1,n)| \ll_\pi X, \tag{4}$$

by the Cauchy–Schwarz inequality.

Following [41] and [14], we make the following definition.

Definition 3.2. We say a smooth function $f(x_1,\dots,x_n)$ on $\mathbb{R}^n$ is *inert* if

$$x_1^{j_1}\cdots x_n^{j_n} f^{(j_1,\dots,j_n)}(x_1,\dots,x_n) \ll_{j_1,\dots,j_n} 1, \tag{5}$$

for all nonnegative integers $j_1,\dots,j_n$. Here the superscript denotes partial differentiation.

For any $N\geq 1$, let

$$S(N)=\sum_{n\geq 1}\lambda(1,n)\chi(n)n^{-it}\varpi(n/N), \tag{6}$$

where $\varpi(x)$ is an *inert* function on $\mathbb{R}$ with compact support contained in $\mathbb{R}_{>0}$.

By symmetry, we assume $t>2$ from now on. Using a standard approximate functional equation argument [11, Theorem 5.3] and the estimate (4), one can derive the following.

Lemma 3.3. *For any $\delta>0$ and $\varepsilon>0$, we have*

$$L(1/2+it,\pi\otimes\chi)\ll (Mt)^{\varepsilon}\sup_N \frac{|S(N)|}{\sqrt{N}}+(Mt)^{3/4-\delta/2+\varepsilon},$$

where the supremum is taken over N in the range $(Mt)^{3/2-\delta}<N<(Mt)^{3/2+\varepsilon}$.

From Lemma 3.3, it suffices to beat the convexity bound $O(N)$ for $S(N)$ for N in the range $(Mt)^{3/2-\delta}<N<(Mt)^{3/2+\varepsilon}$, which we henceforth assume. Here $0<\delta<1/2$ is a small constant to be optimized later. We observe for later convenience that

$$(Mt)^{1+\varepsilon}<N. \tag{7}$$

Let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$ be the Langlands parameters associated to the Maass cusp form π, with

$$\alpha_1+\alpha_2+\alpha_3=0 \quad\text{and}\quad |\Re\alpha_i|\leq\theta.$$

We recall the Ramanujan–Selberg conjecture predicts $\theta = 0$, while from a result [12] of Jacquet and Shalika one at least knows that $|\Re\alpha_i| < 1/2$.

Let

$$G_\delta(s) := \begin{cases} 2(2\pi)^{-s}\Gamma(s)\cos(\pi s/2) & \text{if } \delta = 0, \\ 2i(2\pi)^{-s}\Gamma(s)\sin(\pi s/2) & \text{if } \delta = 1, \end{cases}$$

and let

$$G_{(\boldsymbol{\alpha},\boldsymbol{\delta})}(s) = \prod_{j=1}^{3} G_{\delta_j}(s+\alpha_j), \qquad \text{where } \boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3).$$

Define

$$j_{(\boldsymbol{\alpha},\boldsymbol{\delta})}(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} G_{(\boldsymbol{\alpha},\boldsymbol{\delta})}(s) x^{-s}\, \mathrm{d}s, \qquad x > 0,$$

where $\mathcal{C}$ is a curved contour such that all the singularities of $G_{(\boldsymbol{\alpha},\boldsymbol{\delta})}(s)$ are to the left of $\mathcal{C}$, defined as in [33, Definition 3.2].

Let

$$J_{\pi,\pm}(x) := J_{(\boldsymbol{\alpha},\boldsymbol{\delta})}(\pm x) = \tfrac{1}{2}\big(j_{(\boldsymbol{\alpha},\boldsymbol{\delta})}(x) \pm j_{(\boldsymbol{\alpha},\boldsymbol{\delta}+\boldsymbol{e})}(x)\big),$$

where $\boldsymbol{e} = (1, 1, 1)$, and $\boldsymbol{\delta} + \boldsymbol{e}$ is taken modulo 2. The Bessel function $J_{\pi,\pm}(x)$ satisfies the following properties.

Lemma 3.4. (1) *Let* $\rho > \max\{-\Re\alpha_1, -\Re\alpha_2, -\Re\alpha_3\}$. *For* $x \ll 1$, *we have*

$$x^j J_{\pi,\pm}^{(j)}(x) \ll_{\alpha_1,\alpha_2,\alpha_3,\rho,j} x^{-\rho}.$$

(2) *Let* $K \geq 0$ *be a fixed nonnegative integer. For* $x > 0$, *we may write*

$$J_{\pi,\pm}(x^3) = \frac{e(\pm 3x)}{x} W_\pi^{\pm}(x) + E_\pi^{\pm}(x),$$

where $W_\pi^{\pm}(x)$ *and* $E_\pi^{\pm}(x)$ *are real-analytic functions on* $(0, \infty)$ *satisfying*

$$W_\pi^{\pm}(x) = \sum_{m=0}^{K-1} B_m^{\pm}(\pi) x^{-m} + O_{K,\alpha_1,\alpha_2,\alpha_3}(x^{-K})$$

and

$$E_\pi^{\pm,(j)}(x) \ll_{\alpha_1,\alpha_2,\alpha_3,j} \frac{\exp(-3\sqrt{3}\,\pi x)}{x},$$

for $x \gg_{\alpha_1,\alpha_2,\alpha_3} 1$, *where* $B_m^{\pm}(\pi)$ *are constants depending on* α_1, α_2 *and* α_3.

Proof. See [33, Theorem 14.1]; note that our $J_{\pi,\pm}(x)$ is the $J_{(\lambda,\delta)}(x)$ in the notation of [33]. □

Now we recall the Voronoi formula for GL(3), in which the Bessel function $J_{\pi,\pm}(x)$ appears naturally.

Lemma 3.5 ([24]). *For* $(a, c) = 1$, $\bar{a}a \equiv 1 \pmod c$, *we have*

$$\sum_{n=1}^{\infty} \lambda(m,n) e\left(-\frac{na}{c}\right) w(n) = c \sum_{\pm} \sum_{m'|mc} \sum_{n=1}^{\infty} \frac{\lambda(n,m')}{m'n} S(\bar{a}m, \pm n; mc/m') \frac{m'^2 n}{mc^3} W^{\pm}\left(\frac{m'^2 n}{mc^3}\right), \tag{8}$$

where

$$W^{\pm}(x) = \int_0^{\infty} w(y) J_{\pi,\mp}(xy)\, dy.$$

In particular, replacing $w(n)$ by $w(n/N)$ gives

$$\sum_{n=1}^{\infty} \lambda(m,n) e\left(-\frac{na}{c}\right) w\left(\frac{n}{N}\right) = c \sum_{\pm} \sum_{m'|mc} \sum_{n=1}^{\infty} \frac{\lambda(n,m')}{m'n} S(\bar{a}m, \pm n; mc/m') \frac{Nm'^2 n}{mc^3} W^{\pm}\left(\frac{Nm'^2 n}{mc^3}\right).$$

If $w^{(j)}(y) \ll 1$, then from the oscillation of $J_{\pi,\pm}(x)$ when $|x| > N^{\varepsilon}$, $W^{\pm}(\frac{Nm'^2 n}{mc^3})$ is negligibly small as long as $m'^2 n$ is such that $\frac{Nm'^2 n}{mc^3} \gg N^{\varepsilon}$.

If we write

$$\mathcal{U}^{\pm}(x) = x W^{\pm}(x),$$

then (8) becomes

$$\sum_{n=1}^{\infty} \lambda(m,n) e\left(-\frac{na}{c}\right) w(n) = c \sum_{\pm} \sum_{m'|mc} \sum_{n=1}^{\infty} \frac{\lambda(n,m')}{m'n} S(\bar{a}m, \pm n; mc/m') \mathcal{U}^{\pm}\left(\frac{m'^2 n}{mc^3}\right), \tag{9}$$

which is the usual version of Voronoi's formula given in [24] and in other works.

Remark 3.6. Here the normalization of (8) is different from the usual version (9). With this normalization, the weight function on the right is the Hankel transform of the original Schwartz class function, matching the rank one and rank two cases. We thank Zhi Qi for making us aware of this.

Lemma 3.7 (Miller's bound, [23]). *Uniformly in* $\alpha \in \mathbb{R}$, *we have*

$$\sum_{n \le X} \lambda(1,n) e(\alpha n) \ll_{\pi,\varepsilon} X^{3/4+\varepsilon}. \tag{10}$$

Lemma 3.8 ([8, Lemma 2]). *Let s_1 and s_2 be natural numbers. Let t_1, t_2, and n be integers. Set*

$$\mathcal{C} := \sum_{x([s_1,s_2])} S(t_1x, 1; s_1)S(t_2x, 1; s_2)e\left(\frac{nx}{[s_1, s_2]}\right).$$

Write $s_i = w_i(s_1, s_2)$, $i = 1, 2$, and set $\Delta = w_2^2 t_1 - w_1^2 t_2$. Then

$$|\mathcal{C}| \leq 2^{O(\omega([s_1,s_2]))}(s_1s_2[s_1, s_2])^{1/2}\frac{(\Delta, n, s_1, s_2)}{(n, s_1, s_2)^{1/2}},$$

where $\omega(s)$ denotes the number of distinct prime factors of s, and the implied constant in the O-symbol is absolute.

Lemma 3.9. *Let V be a smooth function with compact support in $\mathbb{R}_{>0}$, satisfying $V^{(j)}(x) \ll_j 1$ for all $j \geq 0$. Assume $(M, r) = 1$ and $n \asymp N$. For any integer $A \geq 1$, there exists an inert function $V_A(x)$ compactly supported in $\mathbb{R}_{>0}$ such that*

$$\begin{aligned}
&\sum_{r=1}^{\infty} \chi(r)r^{-it}e\left(-\frac{n\bar{M}}{r}\right)V\left(\frac{r}{N/(Mt)}\right)\\
&\quad = \frac{N}{M^{3/2}t^{3/2}}\frac{g_{\bar{\chi}}}{\sqrt{M}}\left(\frac{2\pi}{Mt}\right)^{-it}e\left(-\frac{t}{2\pi}\right)\chi(n)n^{-it}V_A\left(\frac{2\pi n}{N}\right) + O\left(\frac{N}{M^{3/2}t^{1+A}}\right)\\
&\qquad + \frac{1}{M}\left(\frac{N}{Mt}\right)^{1-it}\sum_{\tilde{r}\neq 0} S_{\bar{\chi}}(n, \tilde{r}; M)\int_{\mathbb{R}} x^{-it}e\left(-\frac{nt}{Nx}\right)V(x)e\left(-\frac{\tilde{r}N}{M^2t}x\right)\mathrm{d}x, \qquad (11)
\end{aligned}$$

where $S_{\bar{\chi}}(n, \tilde{r}; M) = \sum_{a\,(M)} \bar{\chi}(a)e\left(\frac{na+\tilde{r}\bar{a}}{M}\right)$ is the generalized Kloosterman sum and $g_{\chi} = \sum_{a\,(M)} \chi(a)e\left(\frac{a}{M}\right)$ denotes the Gauss sum.

Proof. Writing

$$e\left(-\frac{n\bar{M}}{r}\right) = e\left(\frac{n\bar{r}}{M}\right)e\left(-\frac{n}{Mr}\right),$$

which follows from reciprocity, and applying Poisson summation, the r-sum becomes

$$\begin{aligned}
&\sum_{r=1}^{\infty} \chi(r)e\left(\frac{n\bar{r}}{M}\right)r^{-it}e\left(-\frac{n}{Mr}\right)V\left(\frac{r}{N/(Mt)}\right)\\
&\quad = \frac{N}{M^2t}\sum_{\tilde{r}\in\mathbb{Z}}\sum_{a\,(M)} \chi(a)e\left(\frac{n\bar{a}}{M}\right)e\left(\frac{a\tilde{r}}{M}\right)\int_{\mathbb{R}}\left(\frac{N}{Mt}x\right)^{-it}e\left(-\frac{nt}{Nx}\right)V(x)e\left(-\frac{\tilde{r}N}{M^2t}x\right)\mathrm{d}x.
\end{aligned}$$

In particular, the zero frequency $\tilde{r} = 0$ contribution is

$$\frac{1}{M}\left(\frac{N}{Mt}\right)^{1-it} g_{\bar{\chi}}\chi(n)\int_{\mathbb{R}} x^{-it}e\left(-\frac{nt}{Nx}\right)V(x)\,\mathrm{d}x.$$

Considering the integral, by [14, Main Theorem], there is an inert function V_A supported on $x_0 \asymp 1$ such that

$$\int_{\mathbb{R}} x^{-it} e\Big(-\frac{nt}{Nx}\Big) V(x)\,\mathrm{d}x = \int_{\mathbb{R}} e\Big(-\frac{t\log x}{2\pi} - \frac{nt}{Nx}\Big) V(x)\,\mathrm{d}x$$
$$= \frac{e(f(x_0))}{\sqrt{t}} V_A(x_0) + O_A(t^{-A}),$$

where $f(x) = -\frac{t\log x}{2\pi} - \frac{nt}{Nx}$, and $x_0 = \frac{2\pi n}{N}$ is the unique solution for $f'(x) = 0$, and $A \ge 1$ is any arbitrarily large constant. Therefore,

$$\int_{\mathbb{R}} x^{-it} e\Big(-\frac{nt}{Nx}\Big) V(x)\,\mathrm{d}x = \Big(\frac{2\pi}{N}\Big)^{-it} \frac{e(-t/(2\pi))}{\sqrt{t}} n^{-it} V_A\Big(\frac{2\pi n}{N}\Big) + O(t^{-A}).$$

Hence

$$\sum_{r=1}^{\infty} \chi(r) e\Big(\frac{n\bar{r}}{M}\Big) r^{-it} e\Big(-\frac{n}{Mr}\Big) V\Big(\frac{r}{N/(Mt)}\Big)$$
$$= \frac{1}{M}\Big(\frac{N}{Mt}\Big)^{1-it} g_{\bar{\chi}}\, \chi(n) \Big(\frac{2\pi}{N}\Big)^{-it} \frac{e(-t/(2\pi))}{\sqrt{t}} n^{-it} V_A\Big(\frac{2\pi n}{N}\Big) + O\Big(\frac{N}{M^{3/2} t^{1+A}}\Big)$$
$$+ \frac{1}{M}\Big(\frac{N}{Mt}\Big)^{1-it} \sum_{\tilde{r}\neq 0} S_{\bar{\chi}}(n,\tilde{r};M) \int_{\mathbb{R}} x^{-it} e\Big(-\frac{nt}{Nx}\Big) V(x) e\Big(-\frac{\tilde{r}N}{M^2 t} x\Big)\,\mathrm{d}x,$$

and (11) follows. □

Remark 3.10. The identity (11),

$$\chi(n) n^{-it} V_A\Big(\frac{n}{N}\Big)$$
$$= \Big(\frac{2\pi}{Mt}\Big)^{it} e\Big(\frac{t}{2\pi}\Big) \frac{M^2 t^{3/2}}{N g_{\bar{\chi}}} \sum_{r=1}^{\infty} \chi(r) e\Big(\frac{n\bar{r}}{M}\Big) r^{-it} e\Big(-\frac{n}{Mr}\Big) V\Big(\frac{r}{N/(Mt)}\Big)$$
$$- \Big(\frac{2\pi}{N}\Big)^{it} e\Big(\frac{t}{2\pi}\Big) \frac{t^{1/2}}{g_{\bar{\chi}}} \sum_{\tilde{r}\neq 0} S_{\bar{\chi}}(n,\tilde{r};M) \int_{\mathbb{R}} V(x) x^{-it} e\Big(-\frac{nt}{Nx}\Big) e\Big(-\frac{\tilde{r}Nx}{M^2 t}\Big)\,\mathrm{d}x$$
$$+ O(t^{1/2-A}),$$

is a variant of the following key identity in [8, (3.6)]:

$$\chi(n)\hat{V}(0) = \frac{M}{R g_{\bar{\chi}}} \sum_{r\in\mathbb{Z}} \chi(r) e\Big(\frac{n\bar{r}}{M}\Big) V\Big(\frac{r}{R}\Big) - \frac{1}{g_{\bar{\chi}}} \sum_{\tilde{r}\neq 0} S_{\bar{\chi}}(n,\tilde{r};M) \hat{V}\Big(\frac{\tilde{r}}{M/R}\Big),$$

where $R > 0$ is a parameter and $\hat{V}$ denotes the Fourier transform of the Schwartz function V, which is compactly supported in $\mathbb{R}_{>0}$. Inserting the identity, with an amplification technique, one can express the smoothed sum $\sum_{n\ge 1} \lambda(1,n)\chi(n) w(n/N)$ as $\mathcal{F} + \mathcal{O}$. Balancing the contribution of $\mathcal{F}$ and $\mathcal{O}$ properly, the authors of [8] obtained $L(1/2, \pi\otimes\chi) \ll M^{3/4-1/36+\varepsilon}$.

From Lemma 3.9, assuming $(M, \ell r) = 1$ and $n \asymp N$, one has

$$\sum_{r=1}^{\infty} \chi(r) r^{-it} e\left(-\frac{np\bar{M}}{\ell r}\right) V\left(\frac{r}{Np/(M\ell t)}\right)$$
$$= \frac{Np}{M^{3/2}t^{3/2}\ell} \frac{g_{\bar{\chi}}}{\sqrt{M}} \left(\frac{2\pi p}{M\ell t}\right)^{-it} e\left(-\frac{t}{2\pi}\right) \chi(p\bar{\ell}) \chi(n) n^{-it} V_A\left(\frac{2\pi n}{N}\right)$$
$$+ \frac{1}{M}\left(\frac{Np}{M\ell t}\right)^{1-it} \chi(p\bar{\ell}) \sum_{\tilde{r}\neq 0} S_{\bar{\chi}}(n, \tilde{r} p\bar{\ell}; M) \mathcal{J}_{it}(n, \tilde{r} p/\ell; M) + O\left(\frac{Np}{M^{3/2}\ell t^{1+A}}\right), \tag{12}$$

or, in another form,

$$\chi(n) n^{-it} V_A\left(\frac{2\pi n}{N}\right)$$
$$= \left(\frac{2\pi}{Mt}\right)^{it} e\left(\frac{t}{2\pi}\right) \frac{M^2 t^{3/2}\ell}{Npg_{\bar{\chi}}} \sum_{r=1}^{\infty} \chi(r\ell\bar{p}) \left(\frac{r\ell}{p}\right)^{-it} e\left(-\frac{np\bar{M}}{\ell r}\right) V\left(\frac{r}{Np/(M\ell t)}\right)$$
$$- \left(\frac{2\pi}{N}\right)^{it} e\left(\frac{t}{2\pi}\right) \frac{t^{1/2}}{g_{\bar{\chi}}} \sum_{\tilde{r}\neq 0} S_{\bar{\chi}}(n, \tilde{r} p\bar{\ell}; M) \mathcal{J}_{it}(n, \tilde{r} p/\ell; M) + O(t^{1/2-A}), \tag{13}$$

where

$$\mathcal{J}_{it}(n, \tilde{r} p/\ell; M) := \int_{\mathbb{R}} x^{-it} e\left(-\frac{nt}{Nx}\right) V(x) e\left(-\frac{\tilde{r} Np}{M^2 \ell t} x\right) \mathrm{d}x. \tag{14}$$

We shall use (12) as a "key identity" in our proof; see Section 4.

Lemma 3.11. *For any $\varepsilon > 0$, one has*

$$\sum_{p_1\sim P} \sum_{p_2\sim P} \sum_{\ell_1\sim L} \sum_{\ell_2\sim L} \sum_{r_1\sim R} \sum_{\substack{r_2\sim R \\ \ell_1 r_2 p_2 \neq \ell_2 r_1 p_1}} \frac{1}{|\ell_1 r_2 p_2 - \ell_2 r_1 p_1|} \ll (LPR)^{1+\varepsilon}$$

and

$$\sum_{p_1\sim P} \sum_{p_2\sim P} \sum_{\ell_1\sim L} \sum_{\ell_2\sim L} \sum_{r_1\sim R} \sum_{\substack{r_2\sim R \\ \ell_1 r_2 p_2 \neq \ell_2 r_1 p_1 \\ \ell_1 r_2 p_2 \equiv \ell_2 r_1 p_1\ (M)}} \frac{1}{|r_1 p_1 \ell_2 - r_2 p_2 \ell_1|} \ll \frac{(LPR)^{1+\varepsilon}}{M}.$$

Proof. The first sum is bounded by

$$\sum_{m_1\sim LPR} \sum_{\substack{m_2\sim LPR \\ m_2\neq m_1}} \frac{\tau_3(m_1)\tau_3(m_2)}{|m_1 - m_2|} \leq (LPR)^{\varepsilon} \sum_{m\sim LPR} \sum_{1\leq h\ll LPR} \frac{1}{h} \ll (LPR)^{1+\varepsilon}.$$

Hence the first inequality follows. Here $\tau_3(m) := \sum_{abc=m} 1$.

The second inequality can be proven using a similar argument. □

4. Reducing $S(N)$ to $\mathcal{F}_1$ and $\mathcal{O}$

Our basic strategy is to introduce more "points" of summation to mimic the smoothed sum $S(N)$ in (6), which is our main object of study. Throughout the paper we assume that $|t| > M^{\varepsilon}$ for any $\varepsilon > 0$.

Let P and L be two large parameters. We begin by introducing the following sum:

$$\mathcal{F}_1 = \frac{M^{3/2}t^{3/2}}{NP^2} \sum_{p\sim P} \bar{\chi}(p)p^{it} \sum_{\ell\sim L} \chi(\ell)\ell^{-it} \sum_{r=1}^{\infty} \chi(r)r^{-it} V\left(\frac{r}{Np/(M\ell t)}\right) \times \sum_{n=1}^{\infty} \lambda(1,n) e\left(-\frac{np\bar{M}}{\ell r}\right) w\left(\frac{n}{N}\right), \tag{15}$$

where $p \sim P$ and $\ell \sim L$ denote primes in the dyadic segments $[P, 2P]$ and $[L, 2L]$, respectively; w and V are smooth functions with compact supports in $\mathbb{R}_{>0}$ satisfying $w^{(j)}(x), V^{(j)}(x) \ll_j 1$ for all $j \geq 0$.

We shall see that if one applies Poisson summation to the r-sum (which is the content of Lemma 3.9), then the contribution of the zero frequency $\tilde{r} = 0$ ($\tilde{r}$ the variable dual to r) will give rise to the sum $S(N)$ that we are initially interested in. In order to bound $S(N)$, it suffices to bound $\mathcal{F}_1$ and the sum arising from the nonzero frequencies $\tilde{r} \neq 0$ of the dual sum, which we denote by $\mathcal{O}$. This observation is initially due to Holowinsky and Nelson [8, B.4], in their work on the Dirichlet character twist case.

Plugging the identity (12) in, we get

$$\begin{aligned}\mathcal{F}_1 = &\left(\frac{2\pi}{Mt}\right)^{-it} e\left(-\frac{t}{2\pi}\right) \frac{g_{\bar{\chi}}}{M^{1/2}} \\ &\times \sum_{p\sim P} p/P^2 \sum_{\ell\sim L} \ell^{-1} \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it} w\left(\frac{n}{N}\right) V_A\left(\frac{2\pi n}{N}\right) \\ &+ \left(\frac{N}{Mt}\right)^{-it} \frac{t^{1/2}}{P^2M^{1/2}} \sum_{n=1}^{\infty} \lambda(1,n) w\left(\frac{n}{N}\right) \sum_{p\sim P} p \sum_{\ell\sim L} \ell^{-1} \\ &\times \sum_{\tilde{r}\neq 0} S_{\bar{\chi}}(n, \tilde{r}p\bar{\ell}; M) \mathcal{J}_{it}(n, \tilde{r}p/\ell; M) + O(Nt^{1/2-A}),\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{\log P \log L} &\sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it} w\left(\frac{n}{N}\right) V_A\left(\frac{2\pi n}{N}\right) \asymp |\mathcal{F}_1| + O(Nt^{1/2-A}) \\ &+ \frac{t^{1/2}}{M^{1/2}PL} \left|\sum_{n=1}^{\infty} \lambda(1,n) w\left(\frac{n}{N}\right) \sum_{p\sim P} \sum_{\ell\sim L} \sum_{\tilde{r}\neq 0} S_{\bar{\chi}}(n, \tilde{r}p\bar{\ell}; M) \mathcal{J}_{it}(n, \tilde{r}p/\ell; M)\right|.\end{aligned}$$

We have shown the following.

Lemma 4.1. *For any positive integer $A \geq 1$, there exists an inert function $V_A(x)$ with compact support in $\mathbb{R}_{>0}$ such that asymptotically, one has*

$$\frac{1}{\log P \log L} \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it} w\left(\frac{n}{N}\right) V_A\left(\frac{2\pi n}{N}\right) \asymp |\mathcal{F}_1| + |\mathcal{O}| + O(Nt^{1/2-A}) \tag{16}$$

with

$$\mathcal{F}_1 = \frac{M^{3/2}t^{3/2}}{NP^2} \sum_{p\sim P} \bar{\chi}(p)p^{it} \sum_{\ell\sim L} \chi(\ell)\ell^{-it} \sum_{r=1}^{\infty} \chi(r)r^{-it} V\left(\frac{r}{Np/(M\ell t)}\right) \times \sum_{n=1}^{\infty} \lambda(1,n) e\left(-\frac{np\bar{M}}{\ell r}\right) w\left(\frac{n}{N}\right)$$

and

$$\mathcal{O} = \frac{t^{1/2}}{M^{1/2}PL} \sum_{n=1}^{\infty} \lambda(1,n) w\left(\frac{n}{N}\right) \sum_{p\sim P} \sum_{\ell\sim L} \sum_{\tilde{r}\neq 0} S_{\bar{\chi}}(n, \tilde{r}p\bar{\ell}; M) \mathcal{J}_{it}(n, \tilde{r}p/\ell; M), \tag{17}$$

where $\mathcal{J}_{it}(n, \tilde{r}p/\ell; M)$ is given by (14).

For any given $\varepsilon > 0$, we can make the error term $O(Nt^{1/2-A})$ negligibly small by assuming $|t| > M^{\varepsilon}$ and taking A to be sufficiently large. It is easily seen that the function $\varpi(x) := w(x)V_A(2\pi x)$ is an *inert* function (under Definition 3.2); see for instance [14, Example 4]. From the lemma, to bound

$$\sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it} w\left(\frac{n}{N}\right) V_A\left(\frac{2\pi n}{N}\right) = \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it} \varpi\left(\frac{n}{N}\right),$$

which is our original object of study (6), it suffices to bound the terms $\mathcal{F}_1$ and $\mathcal{O}$. Note that a priori $w(x)V_A(2\pi x)$ need not be an arbitrary bump function, but this can be done by adjusting the weight function $w(x)$ in our initial definition of $\mathcal{F}_1$ in (15) appropriately (e.g. replacing the original $w(x)$ by $w(x)/V_A(2\pi x)$).

5. Treatment of $\mathcal{O}$

This section is devoted to giving a nontrivial bound for the sum

$$\mathcal{O} = \frac{t^{1/2}}{M^{1/2}PL} \sum_{n=1}^{\infty} \lambda(1,n) w\left(\frac{n}{N}\right) \sum_{p\sim P} \sum_{\ell\sim L} \sum_{r\neq 0} S_{\bar{\chi}}(n, rp\bar{\ell}; M) \mathcal{J}_{it}(n, rp/\ell; M),$$

introduced in (17). Here

$$\mathcal{J}_{it}(n, rp/\ell; M) = \int_{\mathbb{R}} V(x)x^{-it} e\left(-\frac{nt}{Nx}\right) e\left(-\frac{rNpx}{M^2\ell t}\right) \mathrm{d}x,$$

defined in (14).

Our goal is to improve the bound $\mathcal{O} = O(N)$.

For $r \neq 0$, integrating by parts implies that the integral $\mathcal{J}_{it}(n, rp/\ell; M)$ is negligibly small, unless $0 \neq |r| \leq N^{\varepsilon} \frac{M^2t^2L}{NP}$ (by [5, Lemma 8.1]). Moreover, using the second derivative test [10, Lemma 5.1.3] we find that $\mathcal{J}_{it}(n, rp/\ell; M) \ll t^{-1/2+\varepsilon}$.

To estimate $\mathcal{O}$, by a dyadic subdivision, it suffices to bound the sum

$$\mathcal{O}(R) := \frac{t^{1/2}}{M^{1/2}PL} \sum_{n=1}^{\infty} \lambda(1,n) w\left(\frac{n}{N}\right) \sum_{p\sim P} \sum_{\ell\sim L} \sum_{r\sim R} S_{\bar{\chi}}(n, rp\bar{\ell}; M) \mathcal{J}_{it}(n, rp/\ell; M)$$

for all R that satisfy

$$1 \ll R \ll N^{\varepsilon} \frac{M^2t^2L}{NP}.$$

By the Cauchy–Schwarz inequality and Lemma 3.1,

$$\begin{aligned}
\mathcal{O}(R) &\ll \frac{N^{1/2+\varepsilon}t^{1/2}}{M^{1/2}PL} \left(\sum_{n=1}^{\infty} \left| \sum_{p\sim P} \sum_{\ell\sim L} \sum_{r\sim R} S_{\bar{\chi}}(n, rp\bar{\ell}; M) \mathcal{J}_{it}(n, rp/\ell; M) \right|^2 w\left(\frac{n}{N}\right) \right)^{1/2} \\
&= \frac{N^{1/2+\varepsilon}t^{1/2}}{M^{1/2}PL} \Bigg(\sum_{p_1\sim P} \sum_{p_2\sim P} \sum_{\ell_1\sim L} \sum_{\ell_2\sim L} \sum_{r_1\sim R} \sum_{r_2\sim R} \sum_{n=1}^{\infty} S_{\bar{\chi}}(n, r_1p_1\bar{\ell}_1; M) \\
&\qquad \times \overline{S_{\bar{\chi}}(n, r_2p_2\bar{\ell}_2; M)} \mathcal{J}_{it}(n, r_1p_1/\ell_1; M) \overline{\mathcal{J}_{it}(n, r_2p_2/\ell_2; M)}\, w\left(\frac{n}{N}\right) \Bigg)^{1/2}. \qquad (18)
\end{aligned}$$

Next, we apply Poisson summation to the n-sum, yielding

$$\begin{aligned}
&\sum_{n=1}^{\infty} S_{\bar{\chi}}(n, r_1p_1\bar{\ell}_1; M) \overline{S_{\bar{\chi}}(n, r_2p_2\bar{\ell}_2; M)} \mathcal{J}_{it}(n, r_1p_1/\ell_1; M) \overline{\mathcal{J}_{it}(n, r_2p_2/\ell_2; M)}\, w\left(\frac{n}{N}\right) \\
&= \frac{N}{M} \sum_{n\in\mathbb{Z}} \sum_{a\,(M)} S_{\bar{\chi}}(a, r_1p_1\bar{\ell}_1; M) \overline{S_{\bar{\chi}}(a, r_2p_2\bar{\ell}_2; M)}\, e\left(\frac{an}{M}\right) \\
&\qquad \times \int_{\mathbb{R}} \mathcal{J}_{it}(Ny, r_1p_1/\ell_1; M) \overline{\mathcal{J}_{it}(Ny, r_2p_2/\ell_2; M)}\, w(y) e\left(-\frac{nN}{M}y\right) \mathrm{d}y.
\end{aligned}$$

Taking into account the oscillations of $\mathcal{J}_{it}(Ny, r_1p_1/\ell_1; M)$ and $\overline{\mathcal{J}_{it}(Ny, r_2p_2/\ell_2; M)}$, the integral is arbitrarily small for $n \neq 0$ (since $N \gg (Mt)^{1+\varepsilon}$; see (7)). Hence only the zero frequency contributes significantly to the dual sum:

$$\begin{aligned}
&\sum_{n=1}^{\infty} S_{\bar{\chi}}(n, r_1p_1\bar{\ell}_1; M) \overline{S_{\bar{\chi}}(n, r_2p_2\bar{\ell}_2; M)} \mathcal{J}_{it}(n, r_1p_1/\ell_1; M) \overline{\mathcal{J}_{it}(n, r_2p_2/\ell_2; M)}\, w\left(\frac{n}{N}\right) \\
&\qquad = \frac{N}{M} \mathfrak{C}\mathfrak{J} + O(N^{-2018}), \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{C} = \sum_{a\,(M)} S_{\bar{\chi}}(a, r_1p_1\bar{\ell}_1; M) \overline{S_{\bar{\chi}}(a, r_2p_2\bar{\ell}_2; M)} &= M \sideset{}{^\star}\sum_{\beta\,(M)} e\left(\frac{(r_1p_1\bar{\ell}_1 - r_2p_2\bar{\ell}_2)\beta}{M}\right) \\
&= M[M\delta_{\ell_2r_1p_1\equiv\ell_1r_2p_2\,(M)} - 1]
\end{aligned}$$

and

$$\mathfrak{J}=\int_{\mathbb{R}}\mathcal{J}_{it}(Ny,r_1p_1/\ell_1;M)\overline{\mathcal{J}_{it}(Ny,r_2p_2/\ell_2;M)}\,w(y)\,\mathrm{d}y. \tag{20}$$

One readily sees that

$$\mathfrak{C}=\begin{cases}O(M^2), & \ell_1r_2p_2\equiv\ell_2r_1p_1\ (M),\\ O(M), & \text{otherwise}.\end{cases} \tag{21}$$

For the integral $\mathfrak{J}$, if we use the previously mentioned second derivative bound $\mathcal{J}_{it}(n,rp/\ell;M)\ll t^{-1/2+\varepsilon}$ we get $\mathfrak{J}\ll t^{-1+\varepsilon}$. However, there are more cancellations beyond $O(t^{-1+\varepsilon})$, as long as the parameters (r_i,p_i,ℓ_i) satisfy $r_1p_1\ell_2\neq r_2p_2\ell_1$. Indeed, we have the following precise estimate in terms of (r_i,p_i,ℓ_i).

Lemma 5.1. *For $\mathfrak{J}$ defined as in (20), we have*

$$\mathfrak{J}\ll t^{\varepsilon}(\max\{t,|X|\})^{-1},\quad \textit{where}\quad X:=\frac{N(\ell_2r_1p_1-\ell_1r_2p_2)}{M^2\ell_1\ell_2}.$$

This can be proven by using stationary phase expansion for each of the $\mathcal{J}_{it}(Ny,rp/\ell;M)$ in the definition of $\mathfrak{J}$ and then applying the first derivative test. The following simpler proof was kindly suggested to us by the referee.

Proof. By definition (14) of $\mathcal{J}_{it}(Ny,rp/\ell;M)$, we express $\mathfrak{J}$ as

$$\mathfrak{J}=\int V(u)V(v)w(y)e(f(u)-f(v))\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}u.$$

Observe that

$$f(u)-f(v)=\frac{t}{uv}(u-v)y+\left(-\frac{t\log u}{2\pi}+\frac{t\log v}{2\pi}+\frac{Nr_2p_2}{M^2t\ell_2}v-\frac{Nr_1p_1}{M^2t\ell_1}u\right)$$

and the integral $\int w(y)e(ty(u-v)/(uv))\,\mathrm{d}y$ is negligible if $|u-v|\gg t^{\varepsilon-1}$. Let q be a non-negative smooth function supported on $[-1,1]$ which takes the value 1 on $[-1/2,1/2]$. For an arbitrarily large constant A, set

$$Q(u-v):=q\left(\frac{u-v}{t^{\varepsilon-1}}\right)\left(1-q\left(\frac{u-v}{t^{-A}}\right)\right),$$

supported in $t^{-A}\ll|u-v|\ll t^{\varepsilon-1}$. Then

$$\mathfrak{J}=\int Q(u-v)V(u)V(v)w(y)e(f(u)-f(v))\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}u+O_{A,\varepsilon}(t^{-A}).$$

Integrating by parts shows that the integral is

$$\frac{1}{2\pi i\cdot t}\int Q(u-v)\frac{uV(u)vV(v)}{u-v}w'(y)e(f(u)-f(v))\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}u. \tag{22}$$

If $|X| \ll t^{1+2\varepsilon}$, then we are done, for the integral in (22) is trivially $\ll \log t$. Otherwise, the change of variable $h = u - v$ transforms the integral to

$$\int \frac{Q(h)}{h} w'(y)(v+h)vV(v+h)V(v)e(f(v+h)-f(v))\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}y.$$

Now

$$f(v+h)-f(v) = \frac{X}{t}v - \frac{t}{2\pi}\log\left(1+\frac{h}{v}\right) + th\frac{y}{v(v+h)} - \frac{Nr_1p_1}{M^2t\ell_1}h,$$

hence

$$\left|\frac{d}{dv}(f(v+h)-f(v))\right| \gg \frac{|X|}{t} - O(th) \gg \frac{|X|}{t} - O(t^{\varepsilon}) \gg \frac{|X|}{t}.$$

It follows by the first derivative test that the expression in (22) is $\ll |X|^{-1}\log t$. □

Remark 5.2. For $\ell_1 r_2 p_2 \neq \ell_2 r_1 p_1$ and $r_i \sim \frac{M^2t^2L}{NP}$, typically $|X|^{-1} \asymp t^{-2}$, so that the second bound of the lemma shows that we save an extra t over the "trivial bound" $O(t^{-1+\varepsilon})$. The estimation of this lemma is an analytic analogue of the bound (21).

Now we return to the estimate of $\mathcal{O}(R)$ in (18). Plugging the n-sum (19) into $\mathcal{O}(R)$, up to a negligible error, we have

$$\begin{aligned}\mathcal{O}(R) &\ll \frac{N^{1/2+\varepsilon}t^{1/2}}{M^{1/2}PL}\Bigg(\sum_{p_1\sim P}\sum_{p_2\sim P}\sum_{\ell_1\sim L}\sum_{\ell_2\sim L}\sum_{r_1\sim R}\sum_{r_2\sim R}\frac{N}{M}|\mathfrak{C}|\,|\mathfrak{J}|\Bigg)^{1/2}\\ &\ll \frac{N^{1+\varepsilon}t^{1/2}}{MPL}\Bigg(\sum_{p_1\sim P}\sum_{p_2\sim P}\sum_{\ell_1\sim L}\sum_{\ell_2\sim L}\sum_{r_1\sim R}\sum_{\substack{r_2\sim R\\ \ell_1r_2p_2\equiv\ell_2r_1p_1\,(M)}} M^2|\mathfrak{J}|\\ &\quad+\sum_{p_1\sim P}\sum_{p_2\sim P}\sum_{\ell_1\sim L}\sum_{\ell_2\sim L}\sum_{r_1\sim R}\sum_{\substack{r_2\sim R\\ \ell_1r_2p_2\not\equiv\ell_2r_1p_1\,(M)}} M\frac{M^2\ell_1\ell_2}{N\,|\ell_1r_2p_2-\ell_2r_1p_1|}\Bigg)^{1/2},\end{aligned}\tag{23}$$

by using (21) and Lemma 5.1. We remind the reader that $1 \le R \ll N^{\varepsilon}\frac{M^2t^2L}{NP}$.

Using Lemma 5.1 again, we see that the first term inside the parentheses is bounded by

$$M^2t^{\varepsilon}\sum_{p_1\sim P}\sum_{p_2\sim P}\sum_{\ell_1\sim L}\sum_{\ell_2\sim L}\sum_{r_1\sim R}\sum_{\substack{r_2\sim R\\ \ell_1r_2p_2=\ell_2r_1p_1}} t^{-1}$$
$$+\frac{M^4L^2t^{\varepsilon}}{N}\sum_{p_1\sim P}\sum_{p_2\sim P}\sum_{\ell_1\sim L}\sum_{\ell_2\sim L}\sum_{r_1\sim R}\sum_{\substack{r_2\sim R\\ \ell_1r_2p_2\neq\ell_2r_1p_1\\ \ell_1r_2p_2\equiv\ell_2r_1p_1\,(M)}}\frac{1}{|\ell_1r_2p_2-\ell_2r_1p_1|},$$

which is further dominated by

$$\ll N^{\varepsilon} M^2 t^{-1} PLR + N^{\varepsilon} \frac{M^4 L^2}{N} \cdot \frac{LPR}{M} \ll N^{\varepsilon} \frac{M^4 t L^2}{N} + N^{\varepsilon} \frac{M^5 t^2 L^4}{N^2},$$

by using Lemma 3.11 and by noting that $R \ll N^{\varepsilon} \frac{M^2 t^2 L}{NP}$.

Similarly, the second term inside the parentheses of (23) is bounded by

$$\frac{M^3 L^2}{N} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\ell_1 \sim L} \sum_{\substack{\ell_2 \sim L \\ \ell_1 r_2 p_2 \neq \ell_2 r_1 p_1}} \sum_{r_1 \sim R} \sum_{r_2 \sim R} \frac{1}{|\ell_1 r_2 p_2 - \ell_2 r_1 p_1|} \ll N^{\varepsilon} \frac{M^3 L^2}{N} \cdot PLR \ll N^{\varepsilon} \frac{M^5 t^2 L^4}{N^2},$$

upon using Lemma 3.11.

Returning to the estimate of $\mathcal{O}(R)$, we have shown that

$$\mathcal{O}(R) \ll \frac{N^{1+\varepsilon} t^{1/2}}{MPL} \left(\frac{M^4 t L^2}{N} + \frac{M^5 t^2 L^4}{N^2} \right)^{1/2} \ll \frac{N^{1/2+\varepsilon} M t}{P} + N^{\varepsilon} \frac{M^{3/2} t^{3/2} L}{P},$$

which holds for any $1 \leq R \ll N^{\varepsilon} \frac{M^2 t^2 L}{NP}$,

We summarize the main result of this section.

Proposition 5.3. *For any $\varepsilon > 0$, we have the bound*

$$\mathcal{O} \ll \frac{N^{1/2+\varepsilon} M t}{P} + N^{\varepsilon} \frac{M^{3/2} t^{3/2} L}{P},$$

for $\mathcal{O}$ defined as in (17).

Remark 5.4. If we only use the "trivial" bound $\mathfrak{J} \ll t^{-1+\varepsilon}$ for the estimate of the integral $\mathfrak{J}$, then one will see that for the second term we get $O(N^{\varepsilon} M^{3/2} t^2 L/P)$ instead. It is thus crucial to use Lemma 5.1 to get an extra $t^{1/2}$ saving in order to beat the convexity bound for $L(1/2 + it, \pi \otimes \chi)$ in the t-aspect.

6. Treatment of $\mathcal{F}_1$

The purpose of this section is to give a nontrivial bound for

$$\mathcal{F}_1 = \frac{M^{3/2} t^{3/2}}{NP^2} \sum_{p \sim P} \bar{\chi}(p) p^{it} \sum_{\ell \sim L} \chi(\ell) \ell^{-it} \sum_{r=1}^{\infty} \chi(r) r^{-it} V\left(\frac{r}{Np/(M\ell t)} \right) \times \sum_{n=1}^{\infty} \lambda(1, n) e\left(-\frac{np\bar{M}}{\ell r} \right) w\left(\frac{n}{N} \right),$$

defined in (15), where w and V are smooth compactly supported functions with bounded derivatives.

Our goal is to improve the bound $\mathcal{F}_1 = O(N)$.

Bounding the sum directly with Miller's bound (10), we have $\mathcal{F}_1 \ll N^{3/4+\varepsilon} (Mt)^{1/2}$, which is not satisfactory yet for our purpose.

We shall apply the Voronoi summation to the n-sum. To this end, one may assume $(p,r)=1$ in $\mathcal{F}_1$, as the contribution from the terms $(p,r)>1$ is negligible, compared to the generic terms $(p,r)=1$. We briefly justify this. Denote the terms with $p\,|\,r$ in $\mathcal{F}_1$ by $\mathcal{F}_1^\sharp$. Then

$$\mathcal{F}_1^\sharp = \frac{M^{3/2}t^{3/2}}{NP^2}\sum_{p\sim P}\sum_{\ell\sim L}\chi(\ell)\ell^{-it}\sum_{r=1}^{\infty}\chi(r)r^{-it}V\left(\frac{r}{N/(M\ell t)}\right) \times\sum_{n=1}^{\infty}\lambda(1,n)e\left(-\frac{n\bar{M}}{\ell r}\right)w\left(\frac{n}{N}\right).$$

An application of Voronoi summation (9) takes the n-sum to the following dual sum:

$$\ell r\sum_{\pm}\sum_{m|\ell r}\sum_{n=1}^{\infty}\frac{\lambda(n,m)}{mn}S(M,\pm n;\ell r/m)\,\mathcal{U}^{\pm}\left(\frac{m^2n}{(\ell r)^3/N}\right),$$

where the new length can be truncated at $m^2n<N^{\varepsilon}(\ell r)^3/N$, at the cost of a negligible error.

Hence we can estimate $\mathcal{F}_1^\sharp$ as follows:

$$\begin{aligned}\mathcal{F}_1^\sharp &\asymp \frac{M^{3/2}t^{3/2}}{NP\log P}\left|\sum_{\ell\sim L}\chi(\ell)\ell^{-it}\sum_{r=1}^{\infty}\chi(r)r^{-it}V\left(\frac{r}{N/(M\ell t)}\right)\right.\\ &\qquad\left.\times\,\ell r\sum_{m|\ell r}\sum_{n=1}^{\infty}\frac{\lambda(m,n)}{mn}S(M,n;\ell r/m)\,\mathcal{U}^{+}\left(\frac{m^2n}{(\ell r)^3/N}\right)\right|\\ &\ll N^{\varepsilon}\frac{M^{3/2}t^{3/2}}{NP}\sum_{\ell\sim L}\sum_{r\sim N/(MLt)}\ell r\sum\sum_{m^2n<(\ell r)^3/N}\frac{|\lambda(m,n)|}{mn}\left(\frac{\ell r}{m}\right)^{1/2}\\ &\ll N^{\varepsilon}\frac{N^{3/2}}{PMt},\end{aligned}$$

upon using Weil's bound. This bound turns out to be satisfactory for our purpose.

From now on we assume that $(p,\ell r)=1$. Then an application of Voronoi summation (9) to the n-sum yields

$$\sum_{n=1}^{\infty}\lambda(1,n)e\left(-\frac{np\bar{M}}{\ell r}\right)w\left(\frac{n}{N}\right) = \ell r\sum_{\pm}\sum_{m|\ell r}\sum_{n=1}^{\infty}\frac{\lambda(n,m)}{mn}S(\bar{p}M,\pm n;\ell r/m)\,\mathcal{U}^{\pm}\left(\frac{m^2nN}{(\ell r)^3}\right).$$

Here the contribution from the terms with $m^2n\gg N^{\varepsilon}(\ell r)^3/N$ is negligibly small, and we can truncate the (m,n)-sum at $m^2n\ll N^{2+\varepsilon}P^3/(M^3t^3)$, at the cost of a negligible error.

For those $m^2n \ll N^{2+\varepsilon}P^3/(M^3t^3)$, the result $|\Re\alpha_i| < 1/2$ for the Langlands parameter $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ of Jacquet and Shalika gives us the bound

$$\mathcal{U}^{\pm}\left(\frac{m^2nN}{\ell^3r^3}\right) \ll \sqrt{\frac{m^2nN}{\ell^3r^3}},$$

while in general we have $y^j\mathcal{U}^{\pm,(j)}(y) \ll \sqrt{y}$.

After applying Voronoi summation, we have

$$\begin{aligned}\mathcal{F}_1 = \frac{M^{3/2}t^{3/2}}{NP^2}\sum_{\pm}\sum_{p\sim P}\bar{\chi}(p)p^{it}\sum_{\ell\sim L}\chi(\ell)\ell^{-it}\sum_{r=1}^{\infty}\chi(r)r^{-it}V\left(\frac{r}{Np/(M\ell t)}\right)\\ \times \ell r \sum_{\substack{m|\ell r \\ m^2n\ll N^{2+\varepsilon}P^3/M^3t^3}}\sum_{n\geq 1}\frac{\lambda(n,m)}{mn}S(\bar{p}M, \pm n; \ell r/m)\mathcal{U}^{\pm}\left(\frac{m^2nN}{\ell^3r^3}\right) + O\left(\frac{N^{3/2+\varepsilon}}{PMt}\right)\\ := \sum_{\pm}\mathcal{F}_1^{\pm} + O\left(\frac{N^{3/2+\varepsilon}}{PMt}\right).\end{aligned}$$

Consider for example one of the two sums, $\mathcal{F}_1^+$. Pulling the ℓ-sum inside the (m,n)-sum and applying the Cauchy–Schwarz inequality to $\mathcal{F}_1^+$, we obtain

$$\begin{aligned}\mathcal{F}_1^+ \asymp \frac{(Mt)^{1/2}}{P}\Bigg|\sum_{r=1}^{\infty}\chi(r)r^{-it}V\left(\frac{r}{NP/(MLt)}\right)\\ \times \sum_{\substack{m,n\geq 1 \\ m^2n\ll N^{2+\varepsilon}P^3/(M^3t^3)}}\frac{\lambda(n,m)}{mn}\sum_{\substack{\ell\sim L \\ m|\ell r}}\chi(\ell)\ell^{-it}\sum_{p\sim P}\bar{\chi}(p)p^{it}S(\bar{p}M, n; \ell r/m)\mathcal{U}^{+}\left(\frac{m^2nN}{\ell^3r^3}\right)\Bigg|\\ \ll \frac{N^{1/2+\varepsilon}}{P^{1/2}L^{1/2}}\,\Sigma^{1/2},\end{aligned}$$

where

$$\begin{aligned}\Sigma := \sum_{\substack{r\sim NP/(MLt) \\ (r,M)=1}}\sum_{\substack{m,n\geq 1 \\ m^2n\ll N^2P^3/(M^3t^3)}}\frac{1}{mn}\\ \times\left|\sum_{\substack{\ell\sim L \\ m|\ell r}}\chi(\ell)\ell^{-it}\sum_{p\sim P}\bar{\chi}(p)p^{it}S(\bar{p}M, n; \ell r/m)\mathcal{U}^{+}\left(\frac{m^2nN}{\ell^3r^3}\right)\right|^2, \qquad (24)\end{aligned}$$

by noting that

$$\sum_{\substack{m,n\geq 1 \\ m^2n\ll N^2P^3/(M^3t^3)}}\frac{|\lambda(n,m)|^2}{mn} \ll N^{\varepsilon}.$$

Now it remains to treat the sum Σ. Opening the square and interchanging the order of summations, we find

$$\Sigma \leq \sum_{r\sim NP/(MLt)} \sum_{m<NP^{3/2}/(M^{3/2}t^{3/2})} \frac{1}{m} \sum_{\substack{\ell_1\sim L\\ m|\ell_1 r}} \sum_{\substack{\ell_2\sim L\\ m|\ell_2 r}} \sum_{p_1\sim P} \sum_{p_2\sim P}$$
$$\left| \sum_{n\ll N^2P^3/(m^2M^3t^3)} \frac{1}{n} S(\bar{p}_1 M, n; \ell_1 r/m) S(\bar{p}_2 M, n; \ell_2 r/m)\, \mathcal{U}^+\!\left(\frac{m^2nN}{\ell_1^3 r^3}\right) \overline{\mathcal{U}^+\!\left(\frac{m^2nN}{\ell_2^3 r^3}\right)} \right|,$$

where $\bar{p}_1$ and $\bar{p}_2$ denote the multiplicative inverses of p_1 and p_2 modulo $\ell_1 r$ and $\ell_2 r$, respectively.

Our next step is to apply Poisson summation to the n-sum. To this end, by a smooth dyadic partition of unity, one can insert into the n-sum a nonnegative smooth function $F(x)$ which is supported on, say $[1/2, 3]$, and constantly 1 on $[1, 2]$.

Now for any

$$1 \ll \mathcal{N}_m \ll N^2P^3/(m^2M^3t^3), \tag{25}$$

an application of Poisson summation with modulus $[\ell_1 r/m, \ell_2 r/m]$ gives

$$\sum_{n\geq 1} \frac{1}{n} S(\bar{p}_1 M, n; \ell_1 r/m) S(\bar{p}_2 M, n; \ell_2 r/m)\, \mathcal{U}^+\!\left(\frac{m^2nN}{\ell_1^3 r^3}\right) \overline{\mathcal{U}^+\!\left(\frac{m^2nN}{\ell_2^3 r^3}\right)} F\!\left(\frac{n}{\mathcal{N}_m}\right)$$
$$= \frac{1}{[\ell_1,\ell_2]r/m} \sum_{n\in\mathbb{Z}} \mathcal{C}_{\ell_1,\ell_2}(n) \mathcal{T}(n,\ell_1,\ell_2),$$

where

$$\mathcal{C}_{\ell_1,\ell_2}(n) = \sum_{a([\ell_1,\ell_2]r/m)} S(\bar{p}_1 M, a; \ell_1 r/m) S(\bar{p}_2 M, a; \ell_2 r/m)\, e\!\left(\frac{an}{[\ell_1,\ell_2]r/m}\right) \tag{26}$$

and

$$\mathcal{T}(n,\ell_1,\ell_2) = \int_{\mathbb{R}} F(x)\, \mathcal{U}^+\!\left(\frac{m^2\mathcal{N}_m Nx}{\ell_1^3 r^3}\right) \overline{\mathcal{U}^+\!\left(\frac{m^2\mathcal{N}_m Nx}{\ell_2^3 r^3}\right)} e\!\left(-\frac{n\mathcal{N}_m}{[\ell_1,\ell_2]r/m}x\right) \frac{\mathrm{d}x}{x}.$$

By integrating by parts repeatedly, we see that the integral $\mathcal{T}(n,\ell_1,\ell_2)$ is negligibly small if $|n| \geq N^\varepsilon \frac{[\ell_1,\ell_2]r}{m\mathcal{N}_m}$. Therefore we can truncate the dual n-sum at $N^\varepsilon \frac{[\ell_1,\ell_2]r}{m\mathcal{N}_m}$, at the cost of a negligible error. Meanwhile in the range $|n| \ll N^\varepsilon \frac{[\ell_1,\ell_2]r}{m\mathcal{N}_m}$, we use the bounds $y^j \mathcal{U}^{+,(j)}(y) \ll \sqrt{y}$ to obtain

$$\mathcal{T}(n,\ell_1,\ell_2) \ll \frac{m^2\mathcal{N}_m N}{(\ell_1\ell_2)^{3/2} r^3} \ll \frac{m^2\mathcal{N}_m N}{(NP/(Mt))^3}. \tag{27}$$

Let us also observe in particular that

$$\mathcal{T}(n,\ell_1,\ell_2) \ll 1 \quad \text{and} \quad \mathcal{N}_1 \ll N^2P^3/(M^3t^3)$$

for later convenience.

We arrive at

$$\mathcal{F}_1 \ll \frac{N^{1/2+\varepsilon}}{P^{1/2}L^{1/2}}\,\Omega^{1/2} + \frac{N^{3/2+\varepsilon}}{PMt}, \tag{28}$$

where

$$\Omega = \sum_{r\sim NP/(MLt)} \sum_{m<NP^{3/2}/(M^{3/2}t^{3/2})} \sum_{\substack{\ell_1\sim L\\ m|\ell_1 r}} \sum_{\substack{\ell_2\sim L\\ m|\ell_2 r}} \sum_{p_1\sim P} \sum_{p_2\sim P} \sup_{\mathcal{N}_m \ll \frac{N^2P^3}{m^2M^3t^3}} \sum_{|n|<[\ell_1,\ell_2]r/(m\mathcal{N}_m)} \frac{1}{[\ell_1,\ell_2]r} |\mathcal{C}_{\ell_1,\ell_2}(n)\mathcal{T}(n,\ell_1,\ell_2)|.$$

We have essentially square-root cancellation for the character sum $\mathcal{C}_{\ell_1,\ell_2}(n)$, defined in (26). The details of this calculation were carried out in [8]. We have collected their results relevant to our present setting in Lemma 3.8.

Bounding our sum (26) using Lemma 3.8, we get

$$|\mathcal{C}_{\ell_1,\ell_2}(n)| \le 2^{O(\omega(r))}\left(\frac{r}{m}\right)^{3/2} \frac{\ell_1\ell_2}{(\ell_1,\ell_2)^{1/2}} \frac{(\Delta, n, \ell_1 r/m, \ell_2 r/m)}{(n, \ell_1 r/m, \ell_2 r/m)^{1/2}}, \tag{29}$$

where

$$\Delta := \frac{\bar{p}_1\ell_2^2 - \bar{p}_2\ell_1^2}{(\ell_1,\ell_2)^2} M,$$

and $\bar{p}_1$ and $\bar{p}_2$ denote the multiplicative inverses of p_1 and p_2 modulo $\ell_1 r/m$ and $\ell_2 r/m$, respectively, and $\omega(r)$ denotes the number of distinct prime factors of r.

We write

$$\Omega = \Omega_0 + \Omega_1,$$

where Ω_0 denotes the contribution from the terms $n = \Delta = 0$, and Ω_1 denotes the complement.

Remark 6.1. In fact, Ω_0 is the diagonal contribution $(\ell_1, p_1) = (\ell_2, p_2)$ to the sum (24), and Ω_1 is the off-diagonal $(\ell_1, p_1) \ne (\ell_2, p_2)$ contribution.

If $\Delta = 0$, then $\bar{p}_1\ell_2^2 - \bar{p}_2\ell_1^2 = 0$. Necessarily, $\ell_1 = \ell_2 := \ell$ and $p_1 = p_2 := p$. Under this condition,

$$|C_{\ell,\ell}(n)| \le 2^{O(\omega(r))}\left(\frac{\ell r}{m}\right)^{3/2}\left(n, \frac{\ell r}{m}\right)^{1/2}.$$

In particular, $|C_{\ell,\ell}(0)| \le 2^{O(\omega(r))}(\ell r/m)^2$. Therefore,

$$\Omega_0 \ll \sum_{r\sim NP/(MLt)} \sum_{\ell\sim L} \sum_{m|\ell r} \sum_{p\sim P} 2^{\omega(r)} \frac{1}{\ell r}\left(\frac{\ell r}{m}\right)^2 \sup_{\mathcal{N}_m \ll \frac{N^2P^3}{m^2M^3t^3}} |\mathcal{T}(0,\ell,\ell)| \ll \frac{N^{2+\varepsilon}P^3}{M^2t^2}. \tag{30}$$

Here we have used the fact that $\omega(r) \ll \frac{\log r}{\log\log r}$.

Meanwhile for Ω_1, we further write

$$\Omega_1 = \Omega_{1a} + \Omega_{1b},$$

where Ω_{1a} denotes the contribution coming from the $n \neq 0$ terms, and Ω_{1b} denotes the contribution of the zero frequency: $n = 0,\ \Delta \neq 0$. Plugging the bounds (27) and (29) in, we see that

$$\Omega_{1a} \ll \sum_{r\sim NP/(MLt)} \sum_{\ell_1\sim L} \sum_{\ell_2\sim L} \sum_{m|(\ell_1,\ell_2)r} \sum_{p_1\sim P} \sum_{p_2\sim P} \sup_{\mathcal{N}_m \ll \frac{N^2P^3}{m^2M^3t^3}} \sum_{0\neq|n|<[\ell_1,\ell_2]r/(m\mathcal{N}_m)} \frac{|\mathcal{C}_{\ell_1,\ell_2}(n)|}{[\ell_1,\ell_2]r} |\mathcal{T}(n,\ell_1,\ell_2)|$$

$$\ll N^{\varepsilon} \sum_{r\sim NP/(MLt)} \sum_{\substack{\ell_1\sim L\\ \ell_1\neq\ell_2}} \sum_{\ell_2\sim L} \sum_{m|r} \sum_{p_1\sim P} \sum_{p_2\sim P} \sup_{\mathcal{N}_m \ll \frac{N^2P^3}{m^2M^3t^3}} \sum_{0\neq|n|<\ell_1\ell_2 r/(m\mathcal{N}_m)} \frac{r^{1/2}}{m^{3/2}} \frac{(\Delta,n,r/m)}{(n,r/m)^{1/2}} \frac{m^2\mathcal{N}_m N}{(NP/(Mt))^3}$$

$$\ll N^{\varepsilon}\frac{NP}{MLt}L^2P^2 \sup_{\mathcal{N}_1 \ll \frac{N^2P^3}{M^3t^3}} \frac{NPL}{\mathcal{N}_1 Mt}\left(\frac{NP}{MLt}\right)^{1/2} \frac{\mathcal{N}_1 N}{(NP/(Mt))^3} \ll N^{\varepsilon}(NMt)^{1/2}(PL)^{3/2}.$$

Here in the second inequality above we have used the fact that the contribution from the case $\ell_1 = \ell_2$ is comparably smaller.

Now we treat the case of Ω_{1b}, which by our definition is

$$\Omega_{1b} = \sum_{r\sim NP/(MLt)} \sum_{m<NP^{3/2}/(M^{3/2}t^{3/2})} \sum_{\substack{\ell_1\sim L\\ m|\ell_1 r}} \sum_{\substack{\ell_2\sim L\\ m|\ell_2 r}} \sum_{p_1\sim P} \sum_{p_2\sim P} \frac{1_{\Delta\neq 0}}{[\ell_1,\ell_2]r} \sup_{\mathcal{N}_m \ll \frac{N^2P^3}{m^2M^3t^3}} |\mathcal{C}_{\ell_1,\ell_2}(0)\mathcal{T}(0,\ell_1,\ell_2)|.$$

A direct evaluation of $\mathcal{C}_{\ell_1,\ell_2}(0)$ from the definition (26) shows that it vanishes unless $\ell_1 = \ell_2 =: \ell$. In the latter case we have

$$\mathcal{C}_{\ell,\ell}(0) = \frac{\ell r}{m} \sideset{}{^\star}\sum_{\beta(\ell r/m)} e\left(\frac{p_2-p_1}{\ell r/m}\beta\right) \ll \frac{\ell r}{m} \sum_{h|(\ell r/m,\, p_1-p_2)} h.$$

Recall for $\ell_1 = \ell_2 = \ell$, we have $\Delta = (\bar{p}_1 - \bar{p}_2)M$, where $\bar{p}_1$ and $\bar{p}_2$ are the multiplicative inverses of p_1 and p_2 modulo $\ell r/m$, respectively. As $\Delta \neq 0$, we have $p_1 \neq p_2$.

We thus have

$$\Omega_{1b} \ll N^{\varepsilon} \sum_{r\sim NP/(MLt)} \sum_{\ell\sim L} \sum_{m|\ell r} \sum_{h|\ell r/m} \frac{h}{m} \sum_{\substack{p_1\neq p_2\sim P\\ p_1\equiv p_2\,(h)}} 1 \ll \frac{N^{1+\varepsilon}P^3}{Mt}.$$

This is dominated by the diagonal contribution Ω_0 (30), since $Mt < N$.

Hence we obtain the bound

$$\Omega = \Omega_0 + \Omega_{1a} + \Omega_{1b} \ll \frac{N^{2+\varepsilon} P^3}{M^2 t^2} + N^{\varepsilon} (NMt)^{1/2} (PL)^{3/2}. \tag{31}$$

Combining (28) and (31), we retrieve the bound on $\mathcal{F}_1$ in the following.

Proposition 6.2. *For any given $\varepsilon > 0$,*

$$\mathcal{F}_1 \ll \frac{N^{3/2+\varepsilon} P}{MtL^{1/2}} + N^{3/4+\varepsilon} (MtPL)^{1/4}.$$

Remark 6.3. We will assume $L < P$, so that the term $O\left(\frac{N^{3/2+\varepsilon}}{PMt}\right)$ in (28) is negligible.

7. The choices of the parameters P and L

Recall from Proposition 6.2 that

$$\mathcal{F}_1 \ll \frac{N^{3/2+\varepsilon} P}{MtL^{1/2}} + N^{3/4+\varepsilon} (MtPL)^{1/4},$$

while Proposition 5.3 gives

$$\mathcal{O} \ll \frac{N^{1/2+\varepsilon} Mt}{P} + N^{\varepsilon} \frac{M^{3/2} t^{3/2} L}{P}.$$

Plugging these bounds into (16) yields

$$S(N) \ll \frac{N^{3/2+\varepsilon} P}{MtL^{1/2}} + N^{3/4+\varepsilon} (MtPL)^{1/4} + \frac{N^{1/2+\varepsilon} Mt}{P} + N^{\varepsilon} \frac{M^{3/2} t^{3/2} L}{P}.$$

Substituting this into Lemma 3.3 and noting that $(Mt)^{3/2-\delta} < N < (Mt)^{3/2+\varepsilon}$, one gets

$$\begin{aligned} &L(1/2 + it, \pi \otimes \chi) \\ &\ll \frac{(Mt)^{1/2+\varepsilon} P}{L^{1/2}} + (Mt)^{5/8+\varepsilon} (PL)^{1/4} + \frac{(Mt)^{1+\varepsilon}}{P} + \frac{(Mt)^{3/4+\delta/2+\varepsilon} L}{P} + (Mt)^{3/4-\delta/2+\varepsilon} \\ &= \frac{(Mt)^{5/8+\varepsilon} P^{1/4}}{L^{1/2}} \left(\frac{P^{3/4}}{(Mt)^{1/8}} + L^{3/4} \right) + (Mt)^{\varepsilon} \left(\frac{Mt}{P} + (Mt)^{3/4-\delta/2} \right), \end{aligned}$$

upon assuming $L < (Mt)^{1/4-\delta/2}$.

Equate the first two terms by letting $L = P(Mt)^{-1/6}$ to get

$$L(1/2 + it, \pi \otimes \chi) \ll (Mt)^{7/12+\varepsilon} P^{1/2} + (Mt)^{1+\varepsilon}/P + (Mt)^{3/4-\delta/2+\varepsilon}.$$

Letting $P = (Mt)^{5/18}$ gives

$$L(1/2 + it, \pi \otimes \chi) \ll (Mt)^{13/18+\varepsilon} + (Mt)^{3/4-\delta/2+\varepsilon}. \tag{32}$$

Finally, by choosing $\delta = 1/18$, (32) implies that

$$L(1/2 + it, \pi \otimes \chi) \ll (Mt)^{3/4 - 1/36 + \varepsilon}.$$

Note that with such choices, $L = (Mt)^{1/9}$ satisfies the assumption $L < (Mt)^{1/4-\delta/2} = (Mt)^{2/9}$. Theorem 1.1 follows.

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