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Lower bounds for Dirichlet Laplacians and uncertainty principles

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Abstract. We prove lower bounds for the Dirichlet Laplacian on possibly unbounded domains in terms of natural geometric conditions. This is used to derive uncertainty principles for low energy functions of general elliptic second order divergence form operators with not necessarily continuous main part.

Keywords. Uncertainty principle, Dirichlet Laplacians, unique continuation

1. Introduction

Generalized eigensolutions to energies near the bottom of the spectrum of infinite volume Laplacians should be well spread out in configuration space. This can be seen as a version of the uncertainty principle: Low (and thus well determined) kinetic energy of a quantum particle cannot occur simultaneously with a sharp concentration of the position of the particle. Mathematically, this is usually associated with quantitative forms of unique continuation for solutions of second order linear differential equations. Starting with the groundbreaking work of Carleman [13] we just mention [2, 3, 4, 20, 21] as a small list of references, as well as [26], which contains a good overview of the literature up to about 2006.

While this is a classical topic, it has found renewed interest in recent years in connection with describing the fluctuation boundary regime of localization in Anderson-type models with a random potential which only partially covers configuration space (also referred to as "trimmed" Anderson models by some authors, e.g. [17, 38]). Eigenvectors or generalized eigenvectors of the unperturbed Hamiltonian have to feel the random perturbation in order to see a Lifshitz tail regime and lead to an associated Wegner estimate. The starting point of this development was the celebrated paper [8] by Bourgain and Kenig, who were the first who could treat the Bernoulli–Anderson model and used uncertainty principles in their analysis. For the subsequent development in this direction see [7, 9, 19,

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24, 25, 39, 35] and the references therein. Recently, in [16, 30], the authors were able to treat the discrete Bernoulli–Anderson model in 2 and 3 dimensions, respectively, again with an uncertainty principle as a main ingredient in the proof.

This has provided ample motivation for more thorough studies of the geometric properties required for subsets of configuration space to guarantee that these subsets carry a "substantial" part of the mass of low energy states of the Laplacian, both in the continuous setting and for discrete Laplacians on graphs. Our goal here is to establish a result in the continuous case, similar to work in the discrete setting in [29], and we refer to the literature cited in that paper. We mention that from a harmonic analysis point of view, our results are close in spirit to Logvinenko–Sereda theorems (see [27]), with the important difference that we have to restrict ourselves to spectral projectors with energy intervals close to the ground state energy.

We should stress that the kind of uncertainty principle we aim at is complementary to the classical unique continuation results referred to above: on the one hand, it does not imply global vanishing of eigensolutions that vanish on a ball or vanish at some point to infinite order. On the other hand, it can be established in situations where such classical unique continuation results are known not to hold: for certain graphs and elliptic divergence operators with discontinuous main part, which is the focus of the present paper. Let us moreover point out that an important issue is the uniformity of estimates with respect to the coefficients and with respect to the underlying domain.

This will allow us to prove localization at band edges for new classes of random models, an application we will not work out here (compare e.g. [39] for the use of results on quantitative unique continuation in localization proofs).

Here is the set-up for our main result:

Let $d \geq 3$ and H^G (1/2 times) the Neumann Laplacian, characterized by the quadratic form

$$\mathcal{E}[u] := \frac{1}{2} \int_G |\nabla u(x)|^2 \, dx \quad \text{ on } W^{1,2}(G), \tag{1.1}$$

on an open and convex, not necessarily bounded, domain G in $\mathbb{R}^d$. The reason for including the factor $1/2$ here and in the following is that we will study $\mathcal{E}$ through its associated Markov process and we want to get the usual Brownian motion for $\Omega = \mathbb{R}^d$. We denote by $P_I(H^G)$ the spectral projection for H^G onto an interval I.

The *inradius* of G is

$$R_G := \sup\{r \mid \exists x \in G : B_r(x) \subset G\} \in (0, \infty]. \tag{1.2}$$

Let $R \geq \delta > 0$. A closed subset $B \subset G$ is said to be (R, δ)-*relatively dense* in G with covering radius R and thickness δ provided

$$\forall x \in G \; \exists y \in B : \quad B_R(x) \cap B \supset B_\delta(y). \tag{1.3}$$

Note that this trivially implies that $\delta \leq R_G$.

In this language a set is relatively dense (in the classical sense) if it is $(R, 0)$-relatively dense for some $R > 0$. Typical (R, δ)-relatively dense sets are given by fattened relatively dense sets, i.e., their δ-neighborhoods.

The main result of our work is the following quantitative unique continuation bound for low energy states of H^G and, more generally, elliptic second order divergence form operators of the type $-\nabla \mathbf{a} \nabla$ with $\mathbf{a} \in L^\infty$, $\mathbf{a} \geq \eta_0$ that we introduce now:

Assume that $\mathbf{a}(x)$, $x \in G$, is a symmetric $d \times d$-matrix whose entries are bounded measurable functions of x such that

$$\mathbf{a}(x) \geq \eta_0 > 0. \tag{1.4}$$

Denote by $H^G_{\mathbf{a}}$ the unique selfadjoint operator defined by the form

$$\mathcal{E}_{\mathbf{a}}[u] := \frac{1}{2} \int_G (\mathbf{a}(x) \nabla u(x) \mid \nabla u(x))\, dx \quad \text{on } W^{1,2}(G), \tag{1.5}$$

where we use $(\cdot \mid \cdot)$ for the inner product in $\mathbb{R}^d$.

Theorem 1.1. *Let $d \geq 3$. Then there exist constants $a, b, C, c > 0$, only depending on d, such that for every open and convex $G \subset \mathbb{R}^d$, any (R, δ)-relatively dense subset B in G, and every elliptic $\mathbf{a}$ as in (1.4) above,*

$$\|f 1_B\|^2 \geq \eta_0 \kappa \|f\|^2 \tag{1.6}$$

for all f in the range of $P_I(H^G_{\mathbf{a}})$, where

$$I = \left[0, C\eta_0 \frac{\delta^{d-2}}{R^d}\right] \quad \text{and} \quad \kappa = c\left(\frac{\delta}{R}\right)^d \left[\frac{b}{(R \wedge R_G)^2} + \left|\log \frac{a\delta^{d-2}}{R^d}\right|\right]^{-2}. \tag{1.7}$$

While our method of proof allows estimates only for low energies, the bound in (1.6) is quite satisfactory. It only differs from the optimal estimate $(\delta/R)^d$ (attained for constant functions) by a logarithmic correction term and is much better than what appears in the literature so far: see [35] for a comparison.

Maybe more importantly, it is the first uncertainty principle in $d \geq 3$ that holds without any continuity or smoothness assumption on the coefficient matrix $\mathbf{a}$. Usual PDE–techniques are known to break down beyond Lipschitz continuity of the main coefficient, as can be seen from the examples in [32, 34].

A nice feature of our method of proof is that we can mainly concentrate on the easier case of the Laplacian H^G. The uncertainty principle then easily extends to any operator bounded below by a positive multiple of H^G which covers the above case of elliptic second order operators in divergence form. We could as well add positive potentials and consider other boundary conditions, as long as a lower bound is available. For a more complete discussion and possible applications we refer to Section 4 below.

Our proof of Theorem 1.1 consists of three parts, covered in the remaining three sections of this paper. The same general strategy has been used in [29] to prove corresponding results for Laplacians on graphs. The continuous setting considered here leads to some additional complications.

The overall idea, presented in the conclusion of the proof of our main theorem in Section 4, is to reduce the uncertainty principle (1.6) to showing that the bottom of the spectrum of $H^G + \beta 1_B$ rises above the energy interval I in the large coupling limit $\beta \to \infty$

(where 1_B is the characteristic function of B). This approach to uncertainty principles was introduced in [10]. It provides explicit lower bounds on κ which will yield (1.7).

So we have to understand the large coupling limit of $H^G + \beta 1_B$, which we start in Section 2 by studying the case of infinite coupling. This means we will find a lower bound for $H^{G,S}$, the Laplacian on $\Omega := G \setminus S$ with Neumann condition on the boundary of G and an additional Dirichlet condition on the boundary of a set S (whose relation to B will be explained below). It is here that we encounter one of the main differences between the discrete and continuous cases: Points in $\mathbb{R}^d$, for $d \geq 2$, are not massive in the sense of 1-capacities. We further illustrate this in the Appendix by providing a simple (and certainly not new) example of a set with finite inradius whose Dirichlet Laplacian has spectrum $[0, \infty)$. The key insight in this part of our proof is that we can quantify how lower bounds of Dirichlet Laplacians with δ-fat and relatively dense complement depend on δ. The crucial geometric quantity we identify in Theorem 2.5 can be interpreted as the capacity per unit volume of the set S of obstacles, reminiscent of the "crushed ice problem" (see Section 2).

As a last part of the strategy we need to be able to relate the lower bounds for finite and infinite coupling, respectively. Here it is crucial for our proof that the set S is chosen as a slightly smaller ("semi-fat") version of B. The space created between the boundaries of B and S will allow us to compare the spectral minima of $H^{G,S}$ and $H^G + \beta 1_B$ via a norm bound on the difference of the corresponding heat semigroups. The latter bound will be proven via the Feynman–Kac formula in Section 3. In particular, this will use a "hit and run" lemma which bounds the probability that a Brownian path can hit the center of a fat set and then leave the set (by crossing the space between B and S) within a short time.

In addition to our main result, some of the auxiliary results obtained in Sections 2 and 3 should be of independent interest. The lower bounds on Dirichlet Laplacians of sets with (R, ρ)-relatively dense complement shown in Theorem 2.5 improve on a classical result in [15] in their dependence on the ratio ρ/R (and allow for an additional Neumann part of the boundary); see the comments at the end of Section 2. Also, while the "hit and run" Lemma 3.1 has been used in spectral theory before (e.g. [33]), we feel that this tool deserves additional advertising. Moreover, as we point our here, it also holds for reflected Brownian motion, i.e., in the study of the heat semigroup of Neumann Laplacians.

2. Lower bounds for the Dirichlet Laplacian on unbounded domains with uniform relatively dense complement

The first ingredient in our strategy of proof is provided by quantitative lower bounds for Dirichlet Laplacians $-\Delta_\Omega$ on sets Ω with "fat" relatively dense complement in the sense of (1.3). More generally, we consider a set-up where this is done relative to a convex open subset G of $\mathbb{R}^d$, on whose boundary we will place a Neumann condition. The assumption on G could be weakened in some ways, but we make it for clarity and because it provides a convenient class of sets which have all the properties required for our proofs.

In particular, we will use the fact that convex sets are star shaped and that intersections of convex sets are convex. Also, convex sets satisfy the segment property and thus, by [1, Theorem 3.22], we have the first claim in

$$\{u \in C^1(G) \cap C_c(\overline{G}) \mid \|u\|_{1,2} < \infty\} \text{ is dense in } (W^{1,2}(G), \|\cdot\|_{1,2}) \text{ and in } (C_c(\overline{G}), \|\cdot\|_\infty). \tag{2.1}$$

Here $\overline{G}$ denotes the closure of G and

$$\|u\|_{1,2} = \left(\int_G (|\nabla u(x)|^2 + |u(x)|^2)\, dx\right)^{1/2} \tag{2.2}$$

is the Sobolev norm. The second claim in (2.1) can be seen from the Stone–Weierstrass Theorem: For $f \in C_c(\overline{G})$ let $K := \operatorname{supp} f$ and choose an open ball U and a closed ball B in $\mathbb{R}^d$ such that $K \subset B \subset U$. Stone–Weierstrass shows that $\{\varphi|_B \mid \varphi \in C_c^\infty(U)\}$ is dense in $C(B)$, i.e., there exist $\varphi_n \in C_c^\infty(U)$ such that $\sup_{x\in B} |\varphi_n(x) - f(x)| \to 0$. Finally, choose $\chi \in C_c^\infty(U)$ such that $\chi|_K = 1$. Then $\chi\varphi_n \in C^1(G) \cap C_c(\overline{G})$ with $\sup_{x\in\overline{G}} |(\chi\varphi_n)(x) - f(x)| \to 0$.

Note that $C_c(\overline{G})$-functions are not supposed to vanish at the boundary of G. Therefore the form (1.1) can be regarded as a *regular Dirchlet form* on the locally compact space $\overline{G}$ (see [18] for the basics on Dirichlet forms and potential theory). In particular, there is a process associated with $\mathcal{E}$, via reflected Brownian motion, a fact that will be of primary importance in what follows.

Let $H = H^G$ (mostly, we omit the superscript) be the associated Laplacian, which is $-\frac{1}{2}\Delta$ in $L^2(G)$ with Neumann boundary conditions. The Dirichlet Laplacians referred to in the title are given by an additional Dirichlet boundary condition on a closed set S, which is defined via forms again through $\Omega := \overline{G} \setminus S$ and

$$\mathcal{E}^{G,S} = \mathcal{E} \quad \text{on } \operatorname{dom}(\mathcal{E}^{G,S}) = \overline{\{u \in C^1(G) \cap C_c(\Omega) \mid \|u\|_{1,2} < \infty\}}^{W^{1,2}}. \tag{2.3}$$

As will be discussed in Section 3 below, this form is associated with a process that is related to the one of H by killing paths once they hit S. Note that $\mathcal{E}^{G,S}$ is closed and densely defined in $L^2(\Omega)$ and denote the associated mixed Neumann–Dirichlet Laplacian on $L^2(\Omega)$ by $H^{G,S}$.

The main result of this section is a lower bound for

$$\lambda^{G,S} := \inf \sigma(H^{G,S}) \tag{2.4}$$

whenever $\Omega := \overline{G} \setminus S$ for an (R, ρ)-relatively dense closed subset S of G. Note that, by (2.3) and the variational principle,

$$\lambda^{G,S} = \inf\left\{\frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx \;\middle|\; u \in C^1(G) \cap C_c(\Omega),\ \int_\Omega |u(x)|^2\, dx = 1\right\}. \tag{2.5}$$

We start with a finite volume estimate: Here we let G be open and convex and assume in addition that

$$B_\rho(0) \subset G \subset B_R(0) \quad \text{for } 0 < \rho < R < \infty. \tag{2.6}$$

Denote $\mathbb{S}^{d-1} = \{u \in \mathbb{R}^d \mid |u| = 1\}$ and let du be the surface measure on $\mathbb{S}^{d-1}$ induced by Lebesgue measure on $\mathbb{R}^d$. Let ω_d be the volume of the d-dimensional unit ball. Let $R : \mathbb{S}^{d-1} \to (\rho, R]$ be the "radius function" of G, i.e., $R_u := \sup\{t \mid tu \in G\}$ for $u \in \mathbb{S}^{d-1}$. Note that R is lower semicontinuous and hence measurable.

Proposition 2.1. *Let $d \geq 3$, G open and convex satisfying (2.6), $S := B_\rho(0)$ and $\Omega := \overline{G} \setminus S$. Then for $H^{G,S}$ defined as above, we have*

$$d(d-2)\frac{\rho^{d-2}}{R^d} \leq \lambda^{G,S}. \tag{2.7}$$

If furthermore $B_{2\rho}(0) \subset G$, then

$$\lambda^{G,S} \leq 2^d \omega_d \frac{\rho^{d-2}}{\mathrm{vol}(G) - \omega_d (2\rho)^d}. \tag{2.8}$$

Proof. For the lower bound, by (2.5), it suffices to consider $f \in C^1(G)$, $f = 0$ on $B_\rho(0)$, and prove an estimate for $\|f\|_2^2$ in terms of $\|\nabla f\|_2^2$. So let $u \in \mathbb{S}^{d-1}$ and $r \in [\rho, R_u)$. We have $f(ru) = \int_\rho^r \partial_u f(tu)\, dt$ and thus

$$\begin{aligned}|f(ru)|^2 &\leq \int_\rho^r |\partial_u f(tu)|^2 t^{d-1}\, dt \cdot \int_\rho^r t^{1-d}\, dt \\ &\leq \int_\rho^r |\partial_u f(tu)|^2 t^{d-1}\, dt \cdot \frac{1}{d-2}\rho^{2-d}.\end{aligned} \tag{2.9}$$

Integrating with respect to surfaces we get

$$\begin{aligned}\|f\|^2 &= \int_G |f(x)|^2\, dx = \int_{\mathbb{S}^{d-1}} \int_0^{R_u} |f(ru)|^2\, dr\, du \\ &\leq \int_{\mathbb{S}^{d-1}} \int_\rho^{R_u} r^{d-1} \int_\rho^r |\nabla f(tu)|^2 t^{d-1}\, dt\, \frac{1}{d-2}\, \frac{1}{\rho^{d-2}}\, dr\, du \\ &\leq \frac{1}{(d-2)\rho^{d-2}} \int_\rho^R r^{d-1}\, dr \int_{\mathbb{S}^{d-1}} \int_\rho^{R_u} |\nabla f(tu)|^2 t^{d-1}\, dt\, du \\ &\leq \frac{1}{d(d-2)}\, \frac{R^d}{\rho^{d-2}} \|\nabla f\|_2^2,\end{aligned} \tag{2.10}$$

which gives the asserted lower bound.

The upper bound can be shown by a test function of the following form: $f(x) = \varphi(|x|)$ where $\varphi(s) = 0$ for $0 \leq s \leq \rho$, $\varphi(s) = (s-\rho)/\rho$ for $\rho < s \leq 2\rho$ and $\varphi(s) = 1$ for $s > 2\rho$. It follows that $|\nabla f(x)| = \rho^{-1} \cdot 1_{B_{2\rho}(0)\setminus B_\rho(0)}$ and therefore

$$\|\nabla f\|_2^2 \leq \rho^{-2}\, \mathrm{vol}(B_{2\rho}(0)) = 2^d \omega_d \rho^{d-2}. \tag{2.11}$$

The assertion now follows from $\|f\|_2^2 \geq \mathrm{vol}(G) - \mathrm{vol}(B_{2\rho}(0))$. □

Remark 2.2. (i) We think of ρ as small compared to R, and G a set of almost the size of B_R. In such a case the upper and lower bounds in the preceding proposition match up to constants and are both of the form

$$\frac{\rho^{d-2}}{R^d}. \tag{2.12}$$

(ii) One can modify the above calculations to get bounds for $d = 2$, but due to the logarithmic terms appearing we do not easily see a two-sided bound comparable to (2.12) in this case. This is the main reason why, here and in the following, we limit our discussion to $d \geq 3$.

As a first special case of the main result of this section (Theorem 2.5 below), we apply the above local result to a standard geometric situation considered in recent unique continuation results, e.g. [7, 39]. For obvious reasons it is called a "ball pool" by some experts in the field. The lower bound we present is a first step towards a quantitative unique continuation estimate which is very explicit as far as constants are concerned.

Consider $\rho > 0$ and $\ell > 0$ with $\rho < \ell/2$ and a sequence of balls $B_\rho(y_k) \subset k + (0, \ell)^d$ for $k \in (\ell\mathbb{Z})^d$. Let $\Gamma \subset (\rho\mathbb{Z})^d$ be an arbitrary subset of lattice points and

$$S := \bigcup_{k\in\Gamma} B_\rho(y_k). \tag{2.13}$$

Then S is contained in the interior

$$G = \Big(\bigcup_{k\in\Gamma} (k + [0, \ell]^d)\Big)^\circ \tag{2.14}$$

of the corresponding union of closed cubes. Clearly, this gives an example of a set S which is (R, ρ)-relatively dense in G for $R = \sqrt{d}\,\ell$.

Corollary 2.3. *Let S and G be given by (2.13) and (2.14). Consider $H^{G,S}$ as defined above with $\Omega := \overline{G} \setminus S$. Then*

$$\lambda^{G,S} \geq (d-2)\frac{(\rho/\sqrt{d})^{d-2}}{\ell^d}. \tag{2.15}$$

Proof. By (2.5), it suffices to bound $\|f\|_2^2$ in terms of $\|\nabla f\|_2^2$ for any $f \in C^1(G) \cap C_c(\Omega)$. This follows easily from (2.10), applied to each of the sets $\Omega_k := (k + (0, \ell)^d) \setminus B_\rho(y_k)$, since $B_\rho(y_k) \subset k + (0, \ell)^d \subset B_{\ell\sqrt{d}}(y_k)$:

$$\begin{aligned}\|f\|_2^2 = \sum_{k\in\Gamma} \|f 1_{k+(0,\ell)^d}\|^2 &\leq \frac{1}{d(d-2)} \frac{(\ell\sqrt{d})^d}{\rho^{d-2}} \sum_{k\in\Gamma} \|\nabla f \cdot 1_{k+(0,\ell)^d}\|_2^2 \\ &= \frac{1}{d-2} \frac{\ell^d}{(\rho/\sqrt{d})^{d-2}} \|\nabla f\|_2^2. \end{aligned}$$

□

The main result of this section, Theorem 2.5 below, extends this to general (R, ρ)-relatively dense subsets S of open and convex sets G, without requiring the specific geometry used in Corollary 2.3. We start with a preliminary geometrical result that will be helpful later.

Proposition 2.4. *Let $d \geq 3$, let $G \subset \mathbb{R}^d$ be open and convex, and $S \subset G$ be (R, ρ)-relatively dense in G. Then there is a $\Sigma \subset S$ with the following properties:*

(a) *$B_\rho(\Sigma) := \bigcup_{p\in\Sigma} B_\rho(p)$ is $(3R, \rho)$-relatively dense in G and $B_\rho(\Sigma) \subset S$.*
(b) *$\bigcup_{p\in\Sigma} B_{3R}(p) \supset \overline{G}$.*
(c) *If $p \in \Sigma$ and $\Sigma \setminus \{p\} \neq \emptyset$, then*

$$R \leq \operatorname{dist}(p, \Sigma \setminus \{p\}) \leq 6R; \tag{2.16}$$

in particular, Σ is uniformly discrete and $B_\rho(\Sigma \setminus \{p\})$ is $(6R, \rho)$-relatively dense in G.

We call such a set Σ a *skeleton* of S.

Proof. (R, ρ)-relative denseness of S ensures that we can find a subset $D \subset S$ such that

$$\bigcup_{p\in D} B_R(p) \supset G \tag{2.17}$$

and $B_\rho(p) \subset S$ for any $p \in D$. We may pick a subset $\tilde{D} \subset D$ such that

$$p, q \in \tilde{D},\ p \neq q \Longrightarrow |p - q| \geq R, \tag{2.18}$$

i.e., $\tilde{D}$ is uniformly discrete, which is nothing but the lower bound appearing in (c). By Zorn's lemma, there exists a maximal subset $\Sigma \subset D$ with this property. Then Σ satisfies (a)–(c):

By construction, Σ satisfies $B_\rho(\Sigma) \subset S$ and the lower bound in (c), i.e., uniform discreteness.

To show (b), assume that there is $x \in G$ such that $\operatorname{dist}(x, \Sigma) \geq 3R$. By (2.17) the ball $B_R(x)$ contains at least one $p_0 \in D$. The triangle inequality implies that $\{p_0\} \cup \Sigma$ still satisfies (2.18), contradicting the assumed maximality of Σ. This shows $\bigcup_{p\in\Sigma} B_{3R}(p) \supset G$. The union on the left is closed (by uniform discreteness), so that (b) follows. This readily implies that $B_\rho(\Sigma)$ is $(3R, \rho)$-relatively dense in G, completing the verification of (a).

It remains to prove the upper bound in (c) under the assumptions that $p \in \Sigma$ and $\Sigma \setminus \{p\} \neq \emptyset$. So let $R' := \operatorname{dist}(p, \Sigma \setminus \{p\}) = |p - q|$ for $q \in \Sigma \setminus \{p\}$. By uniform discreteness of Σ, we can find such a q. The midpoint s of the line segment $[p, q]$ belongs to G by convexity and so there is an $s' \in \Sigma$ such that $|s - s'| \leq 3R$. The minimality of $|p - q|$ gives $|s - q| \leq 3R$ as well, implying that $|p - q| = 2|s - q| \leq 6R$. □

Theorem 2.5. *Let $d \geq 3$, let $G \subset \mathbb{R}^d$ be open and convex, and let $S \subset G$ be (R, ρ)-relatively dense in G. Then, for $\Omega := \overline{G} \setminus S$, we have*

$$\lambda^{G,S} \geq \frac{d(d-2)}{3^d} \frac{\rho^{d-2}}{R^d}. \tag{2.19}$$

Proof. First, by monotonicity it suffices to prove a bound for any subset $\tilde{S} \subset S$.

We pick $\tilde{S} = B_\rho(\Sigma)$, where Σ is a skeleton of S, the existence of which is granted by Proposition 2.4 above. Define the corresponding Voronoï decomposition of $\overline{G}$ by

$$G_p := \{x \in \overline{G} \mid |x - p| \leq |x - q| \text{ for all } q \in \Sigma\}, \quad p \in \Sigma. \tag{2.20}$$

By construction we see that

(i) $\bigcup_{p \in \Sigma} G_p = \overline{G}$,
(ii) $\mathring{G}_p \cap \mathring{G}_q = \emptyset$ for $p, q \in \Sigma$, $p \neq q$,
(iii) $B_\rho(p) \subset G_p \subset B_{3R}(p)$

and G_p is the intersection of $\overline{G}$ and a finite number of half-spaces. In particular, all the sets G_p as well as their interiors are convex.

To prove the assertion, it suffices to bound $\|f\|_2^2$ appropriately in terms of $\|\nabla f\|_2^2$ for given $f \in C^1(G) \cap C_c(\Omega)$. Note that (iii) above allows us to apply Proposition 2.1 to G_p with R replaced by $3R$. Therefore we get, also using (i) and (ii),

$$\begin{aligned}\|f\|_2^2 = \sum_{p \in \Sigma} \|f 1_{G_p}\|_2^2 &\leq \frac{1}{d(d-2)} \frac{(3R)^d}{\rho^{d-2}} \sum_{p \in \Sigma} \|(\nabla f) 1_{G_p}\|_2^2 \\ &= \frac{3^d}{d(d-2)} \frac{R^d}{\rho^{d-2}} \|\nabla f\|_2^2. \end{aligned}$$

□

We remark that wanting to work with a Voronoï decomposition required to choose a uniformly discrete skeleton Σ of D in the above proof. That is why the constants in (2.19) and the special case (2.15), where the Voronoï cells are given a priori, differ by a factor 3^d.

In case $G = \mathbb{R}^d$, we could employ [15, Theorem 1.5.3] which gives a lower bound on $H^{\mathbb{R}^d, S}$, the Dirichlet Laplacian on $\Omega = \mathbb{R}^d \setminus S$, in terms of

$$d_u(x) := \min\{|t| \mid x + tu \in S\}, \quad u \in \mathbb{S}^{d-1}, \tag{2.21}$$

$$\frac{1}{m(x)^2} := \int_{\mathbb{S}^{d-1}} \frac{du}{d_u(x)^2}. \tag{2.22}$$

More precisely,

$$H^{G,S} \geq \frac{d}{8m^2} \tag{2.23}$$

in the sense of quadratic forms. In the case at hand and in the regime $0 < \rho \ll R$ we could bound $d_u(x)$ by R on a set of unit vectors of size ρ^{d-1}/R^{d-1}, so that we would get a lower bound on λ_Ω of the form

$$\text{const} \, \frac{\rho^{d-1}}{R^{d+1}}, \tag{2.24}$$

which is worse (by a factor of ρ/R) than what we have proven above. More importantly, it is not clear how to adapt Davies' method of proof to the case of the Neumann Laplacian on subdomains.

It is well known that the capacity of a ball of radius r in $\mathbb{R}^d$ behaves like r^{d-2} for $d \geq 3$ and small $r \geq 0$ (see the discussion in the Appendix). For well-spaced S this means that the crucial geometric characteristic of S that determines the lower bound in (2.19) can be regarded as capacity per unit volume.

This is well in accordance with the results for the "crushed ice problem" in the celebrated article [36] by Rauch and Taylor.

We will now discuss some consequences of Theorem 2.5 for related situations that shed some light on "singular homogenization" in the following sense.

Fix $G \subset \mathbb{R}^d$ for $d \geq 3$ and consider a sequence S_n of sets that are (R_n, ρ_n)-relatively dense. We think of each S_n as a union of ρ_n-balls with $\rho_n \to 0$ as $n \to \infty$. If we increase the number of balls so that

$$\inf_{n\in\mathbb{N}} \frac{\rho_n^{d-2}}{R_n^d} > 0, \tag{2.25}$$

the presence of the tiny obstacles will be felt in the limit, since there is a uniform lower bound for the operators H^{G,S_n} by (2.19) above.

If

$$\frac{\rho_n^{d-2}}{R_n^d} \to \infty \quad \text{as } n \to \infty, \tag{2.26}$$

the operators H^{G,S_n} "diverge to ∞" in the sense that

$$\|(H^{G,S_n} + 1)^{-1}\| \to 0 \quad \text{as } n \to \infty,$$

again by (2.19) above. To relate this behavior to the set-up in [36], let us specialize to the case where G is bounded and S_n consists of n balls of radius ρ_n (called r_n in the above paper). There it is shown that for $n\rho_n^{d-2} \to 0$, the effect of the small holes vanishes in the limit, the obstacles are fading. This is a consequence of the fact that the capacity of S_n tends to 0 in this case. Actually, using [40, Theorem 1], it follows that the semigroup of H^{G,S_n} converges to the semigroup of H^G in the Hilbert–Schmidt norm, which gives a quite strong convergence result. A volume counting argument shows that

$$n \sim R_n^{-d},$$

so that we recover the different phases identified in [36], where the limit of the operators is studied while we restrict ourselves to the analysis of lower bounds. However, the estimates in (2.25) and (2.26) give information for fixed configurations, in contrast to what is found in [36].

3. A norm estimate for the heat semigroup at large coupling

In comparison with the discrete case, [29], this is probably the most tricky part of the present analysis.

We fix an open and convex set G and a closed (R, ρ)-relatively dense subset S of G, and set

$$B := B_\rho(S). \tag{3.1}$$

To get a lower bound for eigenfunctions of $H = H^G$ we will use a lower bound on

$$\lambda_\beta := \inf \sigma(H_\beta), \tag{3.2}$$

where $H_\beta := H + \beta 1_B$. To this end, we will introduce an additional Dirichlet boundary condition on S and compare, in this section, e^{-H_β} and $e^{-H_\beta^{G,S}}$ in the operator norm. Here $\Omega := \overline{G} \setminus S$ and

$$H_\beta^{G,S} = H^{G,S} + \beta 1_{B \setminus S} \tag{3.3}$$

on $L^2(\Omega)$ and, as usual, $e^{-H_\beta^{G,S}}$ is interpreted as an operator on $L^2(G)$ by setting it to be 0 on $L^2(S)$.

The main idea is that this additional Dirichlet boundary condition at S does not matter too much for large β, since the potential barrier given by $\beta 1_{B \setminus S}$ is almost impenetrable from within Ω. To formalize and quantify this heuristic we use the probabilistic representation of the semigroup, the Feynman–Kac formula, which shows how the potential and the Dirichlet boundary condition enter the probabilistic formulae and, most importantly, the “hit and run” lemma, which shows that, with an overwhelming probability, each Brownian path that hits S stays around at least for some time in the ρ-neighborhood B of S.

This additional twist is necessary, since there are no quantitative results that allow one to control the convergence of λ_β as $\beta \to \infty$ directly. We refer to [6, 11] and the results cited there for partial results.

First note that since, by assumption, H corresponds to a regular Dirichlet form, by [18, Thm. 6.2.1, p. 184] there is a process $(\Omega, (\mathbb{P}_x)_{x \in \overline{G}}, (X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0})$ which is associated with H in the sense that for any $t \geq 0$ and $f \in L^p(G)$ $(1 \leq p \leq \infty)$,

$$\mathbb{E}_x(f \circ X_t) = e^{-tH} f(x) \tag{3.4}$$

almost everywhere. Here $\mathbb{E}_x$ is expectation with respect to $\mathbb{P}_x$.

By [18, p. 89f.] we know that this process has the strong Markov property, and since the form is strongly local, the paths are continuous; see [18, Thm. 6.2.2, p. 184]. In the case at hand, $(X_t)_{t \geq 0}$ is the *reflected Brownian motion* RBM, which coincides with the usual Brownian motion on $\mathbb{R}^d$, denoted $(W_t)_{t \geq 0}$, as long as particles do not hit the boundary of G. The exact meaning of this will be elaborated in our arguments below.

From the general theory we infer the Feynman–Kac formula [18]

$$e^{-tH_\beta} f(x) = \mathbb{E}_x\left[f \circ X_t \cdot \exp\left(- \int_0^t \beta 1_B \circ X_s \, dx \right) \right] \tag{3.5}$$

and denote the appearing *occupation time* for $t = 1$ by

$$T = T(\omega) = \int_0^1 1_B \circ X_s \, dx = \text{meas}\,\{s \in [0, 1] \mid X_s \in B\}. \tag{3.6}$$

Moreover, we denote the *first hitting time* of S by

$$\sigma := \sigma_S := \inf\{s \geq 0 \mid X_s \in S\} \tag{3.7}$$

and infer from [18] that the additional Dirchlet boundary condition kills the Brownian motion, i.e.,

$$e^{-tH^{G,S}} f(x) = \mathbb{E}_x[f \circ X_t \cdot 1_{\sigma > t}] \tag{3.8}$$

as well as

$$e^{-tH^{G,S}_\beta} = \mathbb{E}_x\left[f \circ X_t \cdot \exp\left(-\int_0^t \beta 1_B \circ X_s \, ds\right) 1_{\sigma > t}\right]. \tag{3.9}$$

We specialize to $t = 1$, where the r.h.s. of (3.9) becomes $\mathbb{E}_x[f \circ X_1 \cdot e^{-\beta T} 1_{\sigma > 1}]$.

Lemma 3.1 ("Hit and Run" Lemma). *In the situation above, for $x \in G$,*

$$\mathbb{P}_x\{\sigma \le 1, \ T \le \alpha\} \le 2^{d/2+2} \exp\left(-\frac{\rho^2}{16\alpha}\right). \tag{3.10}$$

Let us mention the very convincing intuitive meaning of (3.10), at least at a qualitative level: A Brownian path belonging to the event in question has to do a full crossing of a wall of thickness ρ in time at most α, i.e., "hit" S and then quickly "run" away from it again. Clearly, the probability for this to happen should be quite small if ρ is large and α small.

In the case of $G = \mathbb{R}^d$ the "hit and run" lemma was already used for spectral-theoretic purposes in [33, Lemma 3] (see also [42] for related techniques). Let us briefly explain why reflected Brownian motion agrees with the usual one up to the hitting time of the boundary. For bounded regions, much more precise statements are known: see [14] and [12], where a calculation quite like the one we use below is presented. Since we allow unbounded regions, however, these references do not settle the case, although it is quite obvious that boundedness should not matter. Our argument goes as follows: the process $(X_t)_{t\ge 0}$ in question is, as we saw above, associated with the regular Dirichlet form of $H = H^G$; adding a killing or Dirichlet b.c. at ∂G results in the same form that one obtains when adding a Dirichlet condition on G^c for the usual Laplacian on $\mathbb{R}^d$, for which we get the usual Brownian motion $(W_t)_{t\ge 0}$, killed at ∂G. Since processes are essentially uniquely determined by the form (see [18, Theorem 4.2.8]), this means that $(X_t)_{t\ge 0}$ and $(W_t)_{t\ge 0}$ agree up to the time when they hit ∂G.

Proof of Lemma 3.1. We introduce the following auxiliary set and stopping time:

$$B' := B_{\rho/2}(S) \subset B, \tag{3.11}$$

$$\tau := \inf\{s > 0 \mid X_s \in B'\}, \tag{3.12}$$

as well as the event

$$E := \{\omega \in \Omega \mid X_0(\omega) \in B' \text{ and } |X_s(\omega) - X_0(\omega)| \ge \rho/2 \text{ for some } s \le \alpha\}. \tag{3.13}$$

Since $B_{\rho/2}(y) \subset B$ for $y \in B'$, X_s agrees with the classical Brownian motion up to the exit time $\tau^W_{\rho/2}$ for the Wiener process,

$$\mathbb{P}_x(E) = \mathbb{P}_0[\tau^W_{\rho/2} \le \alpha]. \tag{3.14}$$

By the reflection principle,

$$\mathbb{P}_0[\tau^W_{\rho/2} \leq \alpha] \leq 2\mathbb{P}_0[|W_\alpha| \geq \rho/2\}. \tag{3.15}$$

From the explicit formula for the latter we get

$$\begin{aligned}\mathbb{P}_0[|W_\alpha| \geq \rho/2] &= (2\pi\alpha)^{-d/2} \int_{|y|\geq\rho/2} \exp\Big(-\frac{|y|^2}{2\alpha}\Big)\,dy \\ &\leq (2\pi\alpha)^{-d/2} \exp\Big(-\frac{\rho^2}{16\alpha}\Big) \int_{|y|\geq\rho/2} \exp\Big(-\frac{|y|^2}{4\alpha}\Big)\,dy \\ &\leq 2^{d/2} \exp\Big(-\frac{\rho^2}{16\alpha}\Big)(4\pi\alpha)^{-d/2} \int_{\mathbb{R}^d} \exp\Big(-\frac{|y|^2}{4\alpha}\Big)\,dy \\ &= 2^{d/2} \exp\Big(-\frac{\rho^2}{16\alpha}\Big). \end{aligned} \tag{3.16}$$

We conclude that

$$\mathbb{P}_x(E) \leq 2^{d/2+1} \exp\Big(-\frac{\rho^2}{16\alpha}\Big). \tag{3.17}$$

We go on to estimate the probability in question by

$$\mathbb{P}_x\{\sigma \leq 1,\ T \leq \alpha\} \leq \mathbb{P}_x(\Omega_1) + \mathbb{P}_x(\Omega_2) \tag{3.18}$$

for the events $\Omega_1 := \{\sigma \leq 1,\ T \leq \alpha,\ \tau \leq 1-\alpha\}$ and $\Omega_2 := \{\sigma \leq 1,\ T \leq \alpha,\ \tau > 1-\alpha\}$.

First consider Ω_1 and $x \notin B'$. In this case, as $X_0(\omega) = x$ for $\mathbb{P}_x$-a.e. $\omega \in \Omega_1$, we know by continuity of sample paths that $\tau(\omega) \leq \sigma(\omega)$ and $X_{\tau(\omega)}(\omega) \in \partial B'$. From $T \leq \alpha$ we conclude that ω must leave B before $\tau + \alpha$ (≤ 1). In particular, ω must leave

$$B_{\rho/2}(X_{\tau(\omega)}(\omega)) \subset B, \tag{3.19}$$

and therefore

$$(X_{\tau+s}(\omega))_{s\geq 0} \in E. \tag{3.20}$$

Denoting conditional expectation (in $L^\infty(\Omega)$) by $\mathbb{E}_\bullet$, this can be put together as

$$\begin{aligned}\mathbb{P}_x(\Omega_1) &= \mathbb{E}_x(\mathbb{E}_\bullet(\Omega_1 \mid \mathcal{F}_\tau)) \\ &\leq \mathbb{E}_x(\mathbb{E}_\bullet((X_{\tau+s})_{s\geq 0} \in E \mid \mathcal{F}_\tau)) = \mathbb{E}_x(\mathbb{P}_{X_{\tau(\omega)}}(E)) \end{aligned} \tag{3.21}$$

by the strong Markov property. Finally, by (3.17),

$$\mathbb{P}_x(\Omega_1) \leq 2^{d/2+1} \exp\Big(-\frac{\rho^2}{16\alpha}\Big). \tag{3.22}$$

For $x \in B'$ it is clear that $\tau(\omega) = 0$ for $\mathbb{P}_x$-a.e. $\omega \in \Omega_1$ and, by the reasoning above, (3.22) holds in this case as well.

Concerning the second term in (3.18), it is clear that $\mathbb{P}_x(\Omega_2) = 0$ for $x \in B'$, so we can stick to the case $x \notin B'$. For $\mathbb{P}_x$-a.e. $\omega \in \Omega_2$ we know that $\tau \leq \sigma$ and $X_{\tau(\omega)}(\omega) \in \partial B'$

by continuity of sample paths. Since $\tau < 1-\alpha$ and $\sigma \le 1$, any $\omega \in \Omega_2$ must get from $\partial B'$ to S. Therefore, as above, $(X_{\tau+s}(\omega))_{s\ge 0} \in E$, so that

$$\mathbb{P}_x(\Omega_2) \le 2^{d/2+1} \exp\left(-\frac{\rho^2}{16\alpha}\right). \tag{3.23}$$

Putting the above together, we get the assertion. □

Our main result in this section is

Proposition 3.2. *In the situation above, for $\beta > 0$,*

$$\|e^{-H_\beta} - e^{-H_{\Omega,\beta}}\| \le \sqrt{1+4\cdot 2^{d/2}} \exp\left(-\frac{\rho\sqrt{\beta}}{4\sqrt{2}}\right). \tag{3.24}$$

Proof. By the above probabilistic interpretation we get, for $f \in L^2$, $\|f\|_2 \le 1$ and $x \in \overline{G}$,

$$\begin{aligned} |e^{-H_\beta} f(x) - e^{-H_{\Omega,\beta}} f(x)| &= |\mathbb{E}_x[f\circ X_1 \cdot \exp(-\beta T) - f\circ X_1 \cdot \exp(-\beta T)\cdot 1_{\{\sigma>1\}}]| \\ &= |\mathbb{E}_x[f\circ X_1 \cdot \exp(-\beta T)\cdot 1_{\{\sigma\le 1\}}]|. \end{aligned} \tag{3.25}$$

Therefore, by Cauchy–Schwarz,

$$|(e^{-H_\beta} - e^{-H_{\Omega,\beta}}) f(x)|^2 \le \mathbb{E}_x[|f|^2 \circ X_1]\cdot c(x,\rho,\beta), \tag{3.26}$$

where we have set $c(x,\rho,\beta) := \mathbb{E}_x[\exp(-2\beta T)\cdot 1_{\{\sigma\le 1\}}]$. Note that $|f|^2 \in L^1$ with $\||f|^2\|_1 = \|f\|_2^2 \le 1$ and that

$$\mathbb{E}_x[|f|^2\circ X_1] = e^{-H}(|f|^2)(x). \tag{3.27}$$

Integrating over G gives

$$\|(e^{-H_\beta} - e^{-H-\Omega,\beta}) f\|_2 \le 1\cdot \sqrt{\sup_x c(x,\rho,\beta)} \tag{3.28}$$

since $\|e^{-H} : L^1 \to L^1\| \le 1$ as H generates a Dirichlet form.

We are left with estimating $c(x,\rho,\beta)$ appropriately. To this end we fix $\alpha \in (0,1)$, to be specified later, and write

$$\begin{aligned} c(x,\rho,\beta) &= \mathbb{E}_x[\ldots \mid T \ge \alpha] + \mathbb{E}_x[\ldots \mid T < \alpha] \\ &\le \exp(-2\beta\alpha) + \mathbb{P}_x[\sigma \le 1,\ T \le \alpha]. \end{aligned} \tag{3.29}$$

The second term was estimated in the hit-and-run lemma by

$$\mathbb{P}_x[\sigma \le 1, T \le \alpha] \le 4\cdot 2^{d/2}\cdot \exp\left(-\frac{\rho^2}{16\alpha}\right). \tag{3.30}$$

To get the desired bound on $c(x, \rho, \beta)$ we pick α so as to equate exponents in (3.29) above, i.e.,

$$\frac{\rho^2}{16\alpha} = 2\beta\alpha, \quad \text{so} \quad \alpha = \frac{\rho}{4\sqrt{2}\sqrt{\beta}}. \tag{3.31}$$

Plugged back into (3.29) this gives

$$c(x, \rho, \beta) = (1 + 4 \cdot 2^{d/2}) \exp\Big(-\frac{\rho\sqrt{\beta}}{2\sqrt{2}}\Big), \tag{3.32}$$

as was to be shown. □

4. The uncertainty principle: Proof of Theorem 1.1

In this section we will combine Theorem 2.5 and Proposition 3.2 with a spectral-theoretic uncertainty principle from [10] to derive our main result, Theorem 1.1, a quantitative unique continuation bound for low energy states of Neumann Laplacians on arbitrary convex, not necessarily bounded, subsets G of $\mathbb{R}^d$. Actually, we will deduce a slightly stronger, more abstract version in Theorem 4.1 below, which relates directly to the spectral uncertainty principle we recall next.

Theorem 1.1 from [10] refers to a bounded non-negative perturbation W of a semi-bounded selfadjoint operator H in any Hilbert space. If I is an interval and $P_I = P_I(H)$ the corresponding spectral projection of H, then the theorem says that

$$P_I W P_I \geq \kappa P_I \tag{4.1}$$

as long as there is a $\beta > 0$ with

$$\max I < \min \sigma(H + \beta W) =: \lambda_\beta. \tag{4.2}$$

A lower bound for κ is given by

$$\kappa \geq \sup_{\beta > 0} \frac{\lambda_\beta - \max I}{\beta}, \tag{4.3}$$

meaning, in fact, that (4.1) holds with κ replaced by $(\lambda_\beta - \max I)/\beta$ for every $\beta > 0$ which satisfies (4.2).

In our application, $H = H^G$ will be the Neumann Laplacian, characterized by the quadratic form (1.1), on an open and convex domain G in $\mathbb{R}^d$. We choose $W = 1_B$, the indicator function of a set B which arises as a “fattened” relatively dense subset of G.

To determine the maximal energy interval I of applicability of (4.1)–(4.3) in this case, we will need to find (at least a lower bound) for

$$\lim_{\beta \to \infty} \lambda_\beta = \lim_{\beta \to \infty} \min \sigma(H_\beta) \tag{4.4}$$

with $H_\beta := H^G + \beta 1_B$. This will be done in two steps, using our results from Sections 2 and 3: Theorem 2.5 will provide a lower bound on the lowest eigenvalue of a mixed

Neumann–Dirchlet Laplacian, with Neumann condition on ∂G and Dirichlet condition on a “semi-fat” subset S of B. Then the norm bound on the difference of semigroups found in Proposition 3.2 will show that this eigenvalue is sufficiently close to λ_β, giving the desired lower bound for the latter.

In the proof of Theorem 4.1 we will frequently use the fact that the first Dirichlet eigenvalue λ^R of a ball of radius R in $\mathbb{R}^d$ is given by

$$j_d R^{-2}, \tag{4.5}$$

where j_d is the first positive zero of the Bessel function $J_{d/2-1}$. We refrain from telling the whole history and refer to the survey article [5] instead.

Theorem 4.1. *Let $d \geq 3$. Then there exist constants $a, b, C, c > 0$, only depending on d, such that for every open and convex $G \subset \mathbb{R}^d$, any (R, δ)-relatively dense subset B in G, and $\lambda_\beta := \min \sigma(H^G + \beta 1_B)$ as above,*

$$\sup_{\beta>0} \frac{\lambda_\beta - E}{\beta} \geq \kappa(R, \delta), \tag{4.6}$$

where

$$E = C\frac{\delta^{d-2}}{R^d} \quad \text{and} \quad \kappa(R,\delta) = c\left(\frac{\delta}{R}\right)^d \left[\frac{b}{(R \wedge R_G)^2} + \left|\log \frac{a\delta^{d-2}}{R^d}\right|\right]^{-2}. \tag{4.7}$$

Proof. First note that by monotonicity we can replace B by any subset. Thus, without restriction, we modify the set-up slightly, choosing a skeleton $\Sigma \subset B$ for B (see Proposition 2.4). We replace B by $B_\delta(\Sigma)$ and keep the name so that B is now $(3R, \delta)$-dense. Moreover, we set $\rho := \frac{1}{2}\delta$ and $S := B_\rho(\Sigma)$, so that S is $(3R, \rho)$-dense (a “semi-fat” subset of B).

We may assume further that $\Omega = G \setminus B \neq \emptyset$ (as our result would be trivial otherwise), giving that

$$\lambda_\Omega := \inf \sigma(H^{G,B}) < \infty. \tag{4.8}$$

Now we proceed in two steps. First, we will prove the theorem with an expression for κ where the term $b/(R \wedge R_G)^2$ in (1.7) will be replaced by λ_Ω. Then we will use some additional geometric considerations to get the more explicit final form of (1.7).

First step: By estimate (2.19) from Theorem 2.5 we find that $\mu_0 := \lambda^{G,S}$ satisfies

$$\mu_0 \geq c\frac{\delta^{d-2}}{R^d}, \tag{4.9}$$

where we have set $c := d(d-2)/18^d$ (which is not the final value of c in the theorem). Our aim is a lower bound for

$$\lambda_\beta = \min \sigma(H^G + \beta 1_B), \tag{4.10}$$

which we achieve by comparing it to

$$\mu_\beta := \inf \sigma(H^{G,S} + \beta 1_{B\setminus S}) \geq \mu_0, \tag{4.11}$$

noting that $\lambda_\beta \le \mu_\beta \le \lambda_\Omega$. In fact, the difference of the corresponding semigroups is estimated in norm by

$$\|e^{-(H^G+\beta 1_B)} - e^{-(H^{G,S}+\beta 1_{B\setminus S})}\| \le (1+2^{d/2+2})^{1/2} \exp\left(-\frac{\rho\sqrt{\beta}}{4\sqrt{2}}\right) \tag{4.12}$$

by Proposition 3.2. Finally, we pick $t \in (0,1)$ and $E_0 := t\mu_0 < \mu_0$, so that, by monotonicity,

$$\mu_\beta - E_0 \ge (1-t)\mu_0. \tag{4.13}$$

If

$$\mu_\beta - \lambda_\beta \le \tfrac{1}{2}(1-t)\mu_0, \tag{4.14}$$

we get

$$\lambda_\beta - E_0 \ge \tfrac{1}{2}(1-t)\mu_0 > 0, \tag{4.15}$$

giving a desired lower bound 4.7 with $\kappa(R,\delta)$ determined by the corresponding β.

Towards (4.14), we observe that (4.12) gives

$$e^{-\lambda_\beta} - e^{-\mu_\beta} \le A\exp(-a\rho\sqrt{\beta}) \tag{4.16}$$

with the obvious (not final) choice of the explicit constants a, A. The mean value theorem implies that there is $\xi \in [\lambda_\beta, \mu_\beta]$ with

$$\mu_\beta - \lambda_\beta = e^{\xi}(e^{-\lambda_\beta} - e^{-\mu_\beta}) \le e^{\lambda_\Omega}(e^{-\lambda_\beta} - e^{-\mu_\beta}). \tag{4.17}$$

Combining (4.16) and (4.17) we must determine β in such a way that

$$A\exp(-a\rho\sqrt{\beta})e^{\lambda_\Omega} \le \tfrac{1}{2}(1-t)\mu_0. \tag{4.18}$$

Solving for β in the previous formula gives

$$\beta_0 = (a\rho)^{-2}\left[\lambda_\Omega - \log\left(\frac{(1-t)\mu_0}{2A}\right)\right]^2. \tag{4.19}$$

We plug this value into the right hand side of (4.3), using (4.15), and obtain

$$\kappa \ge \tfrac{1}{2}(1-t)\mu_0(a\rho)^2\left[\lambda_\Omega - \log\left(\frac{(1-t)\mu_0}{2A}\right)\right]^{-2}, \tag{4.20}$$

which gives, by (4.9),

$$\begin{aligned}\kappa &\ge (1-t)\tfrac{1}{2}(a\rho)^2 c\frac{\delta^{d-2}}{R^d}\left[\lambda_\Omega - \log\left(\frac{(1-t)c}{2A}\frac{\delta^{d-2}}{R^d}\right)\right]^{-2}\\ &= (1-t)c'\frac{\delta^d}{R^d}\left[\lambda_\Omega - \log\left((1-t)a'\frac{\delta^{d-2}}{R^d}\right)\right]^{-2},\end{aligned} \tag{4.21}$$

with constants only depending on d,

$$c' = \tfrac{1}{8}a^2 c = 2^{-8}\frac{d(d-2)}{18^d}, \tag{4.22}$$

$$a' = \tfrac{1}{2}cA^{-1} = \frac{d(d-2)}{2\cdot 18^d A}, \tag{4.23}$$

$$A = (1+2^{d/2+2})^{1/2}. \tag{4.24}$$

We thus get an uncertainty estimate with

$$\sup_{\beta>0}\frac{\lambda_\beta - E_t}{\beta} \geq \kappa_t := (1-t)c'\frac{\delta^d}{R^d}\left[\lambda_\Omega - \log\left((1-t)a'\frac{\delta^{d-2}}{R^d}\right)\right]^{-2} \tag{4.25}$$

valid in the energy range up to

$$E_t := tc\frac{\delta^{d-2}}{R^d}. \tag{4.26}$$

On the one hand, this is more general than what we asserted (which we get for $t = 1/2$), but not yet the bound we strive for: the dependence of κ_t on λ_Ω might be unpleasant if Ω is small. On the other hand, this would imply that B is large, a situation which clearly is in favor of our overall result and provides the reason behind the following modifications.

Second step: We now modify B (and Ω) so as to get an upper bound on λ_Ω. This will require some geometrical considerations, partly based on Proposition 2.4 above.

Fix R_0 so that

$$\begin{cases} \frac{1}{4}R_G \leq R_0 < \frac{1}{2}R_G & \text{if } R_G < \infty, \\ 4R \leq R_0 & \text{if } R_G = \infty. \end{cases} \tag{4.27}$$

By definition of R_G, in both cases there is $x_0 \in G$ such that

$$B_{2R_0}(x_0) \subset G. \tag{4.28}$$

We first consider

Case 1: $4R \leq R_0$, including the possibility that $R_G < \infty$. Clearly, in this case the skeleton Σ introduced at the beginning of the proof must contain at least two elements.

Case 1.1: $\Sigma \cap B_{R_0}(x_0) = \emptyset$. Since $\delta \leq R$ by definition (and we have set $B = B_\delta(\Sigma)$ as before), it follows that

$$\operatorname{dist}(x_0, B) \geq 4R - \delta \geq 3R, \tag{4.29}$$

therefore the open ball $U_R(x_0)$ is contained in $G \setminus B$ and so

$$\lambda_\Omega \leq j_d R^{-2} \tag{4.30}$$

by identity (4.5) above. Plugging this bound into estimate (4.25) above, we get the assertion of the theorem with a suitable b since $R^{-2} \leq (R \wedge R_G)^{-2}$.

Case 1.2: $\Sigma \cap B_{R_0}(x_0) \neq \emptyset$. Choose $s_0 \in \Sigma \cap B_{R_0}(x_0)$ and denote $\Sigma_0 := \Sigma \setminus \{s_0\}$ and $B_0 := B_{\delta/2}(\Sigma_0)$. Note that, by Proposition 2.4, $\mathrm{dist}(s_0, \Sigma_0) \geq R$ and B_0 is $\left(6R, \frac{1}{2}\delta\right)$-dense. Carrying out the above calculations with this smaller subset of B, rather than the set $B_\delta(\Sigma)$ used before, we arrive at the estimate (4.25) with λ_Ω replaced by λ_{Ω_0} and suitably modified d-dependent constants.

We obtain

$$\mathrm{dist}(s_0, B_0) \geq R - \delta/2 \geq \tfrac{1}{2}R, \tag{4.31}$$

so that

$$U_{\frac{1}{2}R}(s_0) \subset G \setminus B_0 =: \Omega_0, \tag{4.32}$$

giving $\lambda_{\Omega_0} \leq bR^{-2}$ and thus the assertion.

Case 2: $R_0 < 4R$. Consequently, $R_G < \infty$, so that R_0 and R_G are comparable by the definition of R_0 above.

Case 2.1: $\Sigma \cap B_{R_0}(x_0) = \emptyset$. This is treated much like Case 1.1 above. In fact, by definition, $\delta \leq R_G \leq 4R_0$. Replacing $B = B_\delta(\Sigma)$ by $B = B_{\delta/8}(\Sigma)$, i.e., $\Omega = G \setminus B_{\delta/8}(\Sigma)$, we obtain

$$U_{\frac{1}{4}R_0}(x_0) \subset G \setminus B, \tag{4.33}$$

and therefore the assertion follows with $\lambda_\Omega \leq bR_G^{-2}$ for suitable b.

Case 2.2: $\Sigma \cap B_{R_0}(x_0) \neq \emptyset$ and Σ contains at least two elements. Then we proceed as in Case 1.2 above, this time getting a bound of the form bR_G^{-2}. Since no new ideas are involved, we skip the details.

Case 2.3: $\Sigma = \{s_0\} \subset B_{R_0}(x_0)$. Again replacing δ by $\frac{1}{8}\delta$, we see that $B_{2R_0}(x_0) \setminus B_{R_0}(x_0)$ contains a ball of radius R_0 that does not intersect B, once more giving a bound of the form bR_G^{-2} for the corresponding λ_Ω.

This completes the proof of Theorem 4.1. □

Combining the previous estimate with the spectral uncertainty principle, Theorem 1.1 from [10], as explained above, we immediately get:

Corollary 4.2. *Let $d \geq 3$. Then there exist constants $a, b, C, c > 0$, only depending on d, such that for every open and convex $G \subset \mathbb{R}^d$, any (R, δ)-relatively dense B in G, and every selfadjoint operator $H^\sharp$ satisfying*

$$H^\sharp \geq \eta_0 H^G \quad \textit{for some } \eta_0 > 0,$$

we have

$$\|f1_B\|^2 \geq \eta_0\kappa\|f\|^2 \tag{4.34}$$

for all f in the range of $P_I(H^\sharp)$, where

$$I = \left[0, C\eta_0\frac{\delta^{d-2}}{R^d}\right] \quad \textit{and} \quad \kappa = c\left(\frac{\delta}{R}\right)^d\left[\frac{b}{(R \wedge R_G)^2} + \left|\log\frac{a\delta^{d-2}}{R^d}\right|\right]^{-2}. \tag{4.35}$$

As a special case we obtain our main Theorem 1.1 stated in the introduction. Note that

(i) While lower bounds of the form (4.34) and (4.35) have important applications also for bounded sets G (for example for large cubes, where we get volume independent bounds), the result is already new and well illustrated in the case $G = \mathbb{R}^d$ or other sets with infinite inradius. In this case it gives the following small-δ and large-R asymptotics:

For fixed $R = R_0$ we have $\kappa \sim \delta^d/|\log \delta|$ on $I = [0, C\delta^{d-2}]$ as $\delta \to 0$.
For fixed $\delta = \delta_0$ we have $\kappa \sim \frac{1}{R^d(\log R)^2}$ on $I = [0, CR^{-d}]$ as $R \to \infty$.

(ii) In principle, our methods could also be used to get bounds for $d = 2$, but the constants would look less satisfactory (and contain more logarithms).
(iii) Totally different methods are available for $d = 1$; see [23].

We refrain from spelling out more consequences in the form of corollaries and instead list a few more possibilities of exploiting the flexibility of the preceding corollary.

- We can regard different b.c., in particular periodic b.c. when G is a cube and obtain the same estimates as above for the corresponding operator $H^G_{\text{b.c.}}$.
- We can add a non-negative potential V and get the same estimates as above for the corresponding operator $H^G_{\text{b.c.}} + V$.
- More generally, not necessarily positive lower order terms that are controlled by H^G can be added, i.e., we can treat $H^G + B$ as long as $B \geq -\gamma H^G$ for some $\gamma < 1$.

We end our discussion by mentioning that the above results can be used to prove *localization* (see [22, 41] for the general phenomenon of bound states for random models) for new classes of random models. As remarked in the introduction, uncertainty principles are used to derive Wegner and Lifshitz tail estimates when the random perturbation obeys no covering condition. With the uniform estimates above, one could treat models with a random second order main term plus a random potential.

Appendix. Capacities of balls in $\mathbb{R}^d$

As compared to the discrete case of graphs, euclidean space is more complicated in many ways. One important difference that matters for our analysis is that points are not massive at all, at least in dimension $d \geq 2$. This is why a finite inradius of an open set $\Omega \subset \mathbb{R}^d$ does not imply that $\inf \sigma(-\Delta_\Omega) > 0$ for the *Dirichlet Laplacian* $-\Delta_\Omega$, defined via forms as the Friedrichs extension of $-\Delta$ on $C_c^\infty(\Omega)$, or, equivalently, as the selfadjoint operator associated with the form

$$\mathcal{E}[u] := \int_\Omega |\nabla u(x)|^2 \, dx \quad \text{on } W_0^{1,2}(\Omega). \tag{A.1}$$

Example A.1. In $\mathbb{R}^d$ for $d \geq 2$ consider $D = \mathbb{Z}^d$ and the union of closed balls $S := \bigcup_{k \in D} B_{r_k}(k)$ with $0 < r_k < 1/2$ for $k \in D$. For

$$\Omega := \mathbb{R}^d \setminus S \tag{A.2}$$

we see that the inradius $R_\Omega = \sup\{s > 0 \mid \exists x \in \Omega : B_s(x) \subset \Omega\}$ is bounded above by $\sqrt{d}/2$. However, as we will see below,

$$\operatorname{cap}(B_r(x)) = \operatorname{cap}(B_r(0)) \to 0 \quad \text{as } r \to 0, \tag{A.3}$$

so that we can pick r_k such that

$$\operatorname{cap}(S) \le \sum_k \operatorname{cap}(B_{r_k}(k)) < \infty. \tag{A.4}$$

In that case, by [40, Theorem 1], we find that $e^{-\Delta} - e^{-\Delta_\Omega}$ is Hilbert–Schmidt and therefore $\sigma_{\text{ess}}(-\Delta_\Omega) = \sigma(-\Delta_\Omega) = [0, \infty)$.

As different notions of capacity are around, let us briefly settle the case of (A.3) above:

In the above result, capacity refers to the 1-capacity, often used in potential theory for Dirichlet forms and defined by the following variational principle:

$$\operatorname{cap}(B_r(0)) := \inf\{\|\nabla f\|^2 + \|f\|^2 \mid f \in C_c^1(\mathbb{R}^d),\ f \ge 1_{B_r(0)}\}. \tag{A.5}$$

Set $\phi(x) = 1$ if $0 \le x \le 1$, $\phi(x) = 2 - x$ if $1 \le x \le 2$ and $\phi(x) = 0$ for $x > 2$ and define $f_r(x) = \phi(|x|/r)$ on $\mathbb{R}^d$. Then $\operatorname{cap}(B_r(0)) \le \|\nabla f_r\|^2 + \|f_r\|^2 \le C_d r^{d-2}$, which gives the claim for $d \ge 3$. In $d = 2$ this only gives boundedness, but can be combined with $\|f_r\|^2 \to 0$, weak compactness of the unit ball in $W^{1,2}$ and Hahn–Banach to give a sequence r_n with $\operatorname{cap}(B_{r_n}(0)) \to 0$, proving (A.3) by monotonicity of the capacity.

We go on to show that for $d \ge 3$,

$$\operatorname{cap}(B_r(0)) \sim r^{d-2} \quad \text{for } r \le 1. \tag{A.6}$$

This is most easily seen by using the slightly smaller Newtonian capacity

$$\operatorname{cap}_N(B_r(0)) := \inf\{\|\nabla f\|^2 \mid f \in C_c^1(\mathbb{R}^d),\ f \ge 1_{B_r(0)}\}. \tag{A.7}$$

The above scaling shows immediately that $\operatorname{cap}_N(B_r(0)) \sim r^{d-2}$, so that (A.6) follows, since $\operatorname{cap}_N(B_r(0)) \le \operatorname{cap}(B_r(0))$. We cannot resist to mention two classical papers on capacities, [37, 43]. For a thorough discussion, we refer to [31, Section 11.15], as well as to classical textbooks like [28].

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