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A set of positive Gaussian measure with uniformly zero density everywhere

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Abstract. Existing negative results on invalidity of analogues of classical Density and Differentiation Theorems in infinite-dimensional spaces are considerably strengthened by a construction of a Gaussian measure γ on a separable Hilbert space H for which the Density Theorem fails uniformly, i.e., there is a set $M \subset H$ of positive γ-measure such that

$$\lim_{r \searrow 0} \sup_{x \in H} \frac{\gamma(B(x,r) \cap M)}{\gamma B(x,r)} = 0.$$

Keywords. Gaussian measures on Hilbert spaces, Density Theorem

1. Introduction

Our aim here is to show that already for Gaussian measures on separable Hilbert spaces the classical Density Theorem may fail in a very strong, and perhaps surprising, way. Recall that for a given locally finite Borel measure μ on a metric space X the validity of this theorem means that for every Borel set $M \subset X$,

$$\lim_{r \searrow 0} \frac{\mu(B(x,r) \cap M)}{\mu B(x,r)} = \mathbf{1}_M(x) \quad \text{for } \mu\text{-almost every } x \in X. \tag{1.1}$$

This was first proved by Lebesgue for the Lebesgue measure on the real line. Nowadays there are a number of different short arguments showing this result of Lebesgue, for example [18] and [4], but most textbook proofs have as their main step the Vitali Covering Theorem. These proofs or their simple modifications can show the Density Theorem for measures absolutely continuous with respect to the Lebesgue measure on any finite-dimensional Banach space. However, the Density Theorem is known to hold for every locally finite Borel measure on every finite-dimensional Banach space, which is usually

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proved using much stronger covering result than Vitali's, namely the Besicovitch–Morse covering theorem. (See, for example, [7].)

When one abandons the assumption of finite-dimensionality, the situation becomes quite different. Since it is not difficult to see that in every infinite-dimensional Banach spaces there are measures for which the Density Theorem fails, the main question is whether it or similar results hold for measures that in some respects act as a suitable infinite-dimensional replacement for the Lebesgue measure. Gaussian measures are the most natural candidate, both because of their importance in mathematics (for which see, for example, [2, Chapter 7]) and because of their known use in geometric problems of nature similar to the Density Theorem. For example, an analogue of Rademacher's Theorem on almost everywhere differentiability of real-valued (and even some vector-valued) Lipschitz functions holds (with Gateaux derivatives) in every separable Banach space for every non-degenerate Gaussian measure. (See [10] or [1] for further results in this direction.) Nearer to our theme, [16] shows that some Gaussian measures are so well approximated by finite-dimensional ones that it is possible to use the dimension independent estimate of the Hardy–Littlewood maximal operator from [15] to show the following theorem giving a class of infinite-dimensional Gaussian measures on a Hilbert space for which the Differentiation Theorem holds for all L^p functions with $p > 1$. The quality of approximation of a given Gaussian measure by finite-dimensional ones may be measured, for example, by the speed of decrease of the eigenvalues of its covariance operator. (We will actually not use the covariance operator but a representation of Gaussian measures on Hilbert spaces in which these eigenvalues are directly related to the norm; see Section 2.)

Theorem T (Tišer 1988). *Suppose the eigenvalues λ_k of the covariance operator of a non-degenerate Gaussian measure γ on a separable Hilbert space H satisfy*

$$\lim_{k\to\infty} k^s \frac{\lambda_{k+1}}{\lambda_k} = 0$$

for some $s > 5/2$. Then for every $f \in L^p(\gamma)$ where $p > 1$,

$$\lim_{r\searrow 0} \frac{1}{\gamma B(x,r)} \int_{B(x,r)} f\, d\gamma = f(x) \tag{1.2}$$

for γ-almost every $x \in H$.

The first negative result related to our problem was a simple observation made in [11] that the Vitali Covering Theorem need not hold for Gaussian measures on infinite-dimensional separable Hilbert spaces. This result was strengthened in [17] by showing that this theorem fails for every infinite-dimensional Gaussian measure on a separable Hilbert space and in [12] by showing that even the Density Theorem may fail for Gaussian measures on Hilbert spaces: there are a Gaussian measure γ on a separable Hilbert space and a Borel set M with $\gamma M > 0$ such that the limit in (1.1) is equal to zero γ-almost everywhere. Since the set

$$\left\{ x \in M \;\middle|\; \lim_{r\searrow 0} \frac{\gamma(B(x,r)\cap M)}{\gamma B(x,r)} = 0 \right\}$$

is γ-measurable and has strictly positive γ-measure, it contains a compact set of strictly positive γ-measure, and we easily see that for this set the limit in (1.1) is zero everywhere.

The above negative results left open the possibility that at least a (very) weak version of the Density Theorem holds for any Gaussian measure γ on a separable Hilbert space H, namely, that for any Borel set $M \subset H$ with $\gamma M > 0$, and any $\eta > 0$, there are arbitrarily small balls $B(x, r)$ such that

$$\frac{\gamma(B(x,r) \cap M)}{\gamma B(x,r)} > 1 - \eta. \tag{1.3}$$

An analogous question for the Differentiation Theorem was answered in a surprising way in [13] by providing a rather artificial example of a Gaussian measure γ on a separable Hilbert space H together with an integrable function $f \in L^1(\gamma)$ such that

$$\liminf_{r \searrow 0} \left\{ \frac{1}{\gamma B(x,r)} \int_{B(x,r)} f \, d\gamma \;\middle|\; x \in H \right\} = \infty. \tag{1.4}$$

In other words, the averages of an integrable function over balls may, instead of converging to the function almost everywhere as in (1.2), tend to infinity uniformly over points of the space.

Here we refute even the above very weak version of the Density Theorem in perhaps the strongest possible way: not only is the ratio on the left of (1.3) not bigger than $1-\eta$ for small balls, but, as r tends to zero, it converges to zero uniformly over points of H. Rather naturally, based on Theorem T, one expects that this may hold for those Gaussian measures that are badly approximated by finite-dimensional ones. The following main result of this note shows that this is indeed the case. Moreover, the Gaussian measures for which we show that the Density Theorem (and, as we will see shortly, also the Differentiation Theorem) fails in so strong way are no longer artificial: they include the Gaussian measures for which the eigenvalues of the covariance operator are k^{-s} where $1 < s < 6/5$.

Theorem 1. *Suppose the eigenvalues λ_k of the covariance operator of a non-degenerate Gaussian measure γ on a separable Hilbert space H form a non-increasing sequence satisfying*

$$\limsup_{k \to \infty} k \left(\frac{\lambda_k}{\lambda_{k+1}} - 1 \right) < \frac{6}{5}.$$

Then for every $\varepsilon > 0$ there is a Borel set $M \subset H$ such that $\gamma(H \setminus M) < \varepsilon$ and

$$\limsup_{r \searrow 0} \left\{ \frac{\gamma(B(x,r) \cap M)}{\gamma B(x,r)} \;\middle|\; x \in H \right\} = 0.$$

This theorem will be proved at the end of this note as a consequence of the rather technical Proposition 14 which can be used to provide a host of other examples of Gaussian measures with the same property. (See the remark following this proposition.) Nevertheless, our methods do not allow us to decide what happens when the eigenvalues are k^{-s} with $s \geq 6/5$.

As it is customary, and often more convenient, to state the Density Theorem in an equivalent form for the complement $Q := H \setminus M$, we also state the corresponding result for Q.

Corollary 2. *Suppose γ satisfies the assumptions of Theorem 1. Then for every $\varepsilon > 0$ there is a Borel set $Q \subset H$ such that $\gamma Q < \varepsilon$ and*

$$\lim_{r \searrow 0} \inf \left\{ \frac{\gamma(B(x,r) \cap Q)}{\gamma B(x,r)} \;\middle|\; x \in H \right\} = 1.$$

A simple consequence of our main result is that a function satisfying (1.4) can actually belong to all $L^p(\gamma)$ for $1 \le p < \infty$. To see this, it suffices to use the following corollary with $\varphi(x) := e^x$.

Corollary 3. *Suppose γ satisfies the assumptions of Theorem 1 and $\varphi \colon [0,\infty) \to [0,\infty)$ is non-decreasing. Then there is a function $f \in L^1(\gamma)$ such that $\int \varphi(|f|)\,d\gamma < \infty$ and*

$$\lim_{r \searrow 0} \inf \left\{ \frac{1}{\gamma B(x,r)} \int_{B(x,r)} f\,d\gamma \;\middle|\; x \in H \right\} = \infty.$$

Proof. Let $\psi(x) := x + \varphi(x)$ and choose numbers $\varepsilon_1 \ge \varepsilon_2 \ge \cdots > 0$ such that $\sum_{k=1}^{\infty} \varepsilon_k \psi(k^2) < \infty$. By Corollary 2 we choose sets Q_n with $\gamma Q_n < 2^{-n}\varepsilon_n$ satisfying

$$\lim_{r \searrow 0} \inf \left\{ \frac{\gamma(B(x,r) \cap Q_n)}{\gamma B(x,r)} \;\middle|\; x \in H \right\} = 1.$$

Put $A_k := \bigcup_{n=k}^{\infty} Q_n$ and $f := \sum_{k=1}^{\infty} k\mathbf{1}_{A_k}$. Then $\gamma A_k \le \varepsilon_k$ and $f(x) \le k^2$ for $x \in A_k \setminus A_{k+1}$, hence

$$\int \psi(f)\,d\gamma \le \sum_{k=1}^{\infty} \psi(k^2)\gamma(A_k \setminus A_{k+1}) \le \sum_{k=1}^{\infty} \varepsilon_k \psi(k^2) < \infty,$$

which shows that $f \in L^1(\gamma)$ and that $\int \varphi(|f|)\,d\gamma < \infty$.

To prove the remaining statement, fix any n and choose s_n small enough to satisfy

$$\inf \left\{ \frac{\gamma(B(x,r) \cap Q_n)}{\gamma B(x,r)} \;\middle|\; x \in H \right\} \ge \tfrac{1}{2}$$

for all $0 < r < s_n$. Then, since $f \ge n\mathbf{1}_{Q_n}$, for any $x \in H$ and $0 < r < s_n$ we have

$$\frac{1}{\gamma B(x,r)} \int_{B(x,r)} f\,d\gamma \ge n \frac{\gamma(B(x,r) \cap Q_n)}{\gamma B(x,r)} \ge \tfrac{1}{2} n. \qquad \square$$

2. Gaussian measures, other notions and notation

We collect some of the notions and results used throughout the paper. We will use two notations for norms in vector spaces, $|\cdot|$ and $\|\cdot\|$. Both will be induced by a scalar product, the one giving $|\cdot|$ will be denoted by $\langle \cdot, \cdot \rangle$ but we will not need any notation for the one inducing $\|\cdot\|$. For the Euclidean norm in $\mathbb{R}^n$ we will always use the symbol $|\cdot|$. If U is a closed linear subspace of a Hilbert space H, then the same symbol U will denote the orthogonal projection from H onto U. In particular, Ux is the orthogonal projection of an element $x \in H$ onto U. We denote by $B(x, r)$ the closed ball centred at x with radius $r > 0$. We may use the same symbol for balls in different spaces (or different norms); when it is not clear from the context which space is intended, we will specify it.

It will be convenient to use, in any finite-dimensional Hilbert space $(H, |\cdot|)$, notions that we introduce only in Euclidean spaces. All that we need may be obtained by choosing an orthonormal basis of H and identifying H with $\mathbb{R}^n$ in the usual way (the result will not depend on the choice of the basis). In particular, the Lebesgue measure $\mathcal{L}^n$ on H may be defined in this way; or it may be defined as the Hausdorff measure $\mathcal{H}^n$ of dimension $n = \dim H$.

We will often use the special case of the coarea formula, or of the polar coordinates, saying that for every non-negative Borel function f on an $(n+1)$-dimensional Hilbert space H,

$$\int_H f(x)\, d\mathcal{L}^{n+1}(x) = \int_W \int_0^\infty f(sw) s^n\, ds\, d\mathcal{H}^n(w) \tag{2.1}$$

where $W := \{w \in H \mid |w| = 1\}$. In particular, for every Borel set $E \subset \mathbb{R}$,

$$\int_{\{x\in H \mid |x|\in E\}} e^{-c|x|^2}\, d\mathcal{L}^{n+1}(x) = \omega_n \int_E e^{-cs^2} s^n\, ds \tag{2.2}$$

where $\omega_n := \mathcal{H}^n\{w \in \mathbb{R}^{n+1} \mid |w| = 1\}$.

The term “measure” will be used only for locally finite Borel measures on separable Banach spaces; such measures are often called Radon measures. (In fact, with the exception of the Lebesgue and Hausdorff measures all our measures will be finite.) The support of a measure μ is defined as the set of x such that $\mu B(x, r) > 0$ for every $r > 0$.

An important role in our arguments is played by log-concave measures and functions. Of a number of equivalent definitions of log-concavity of measures we choose, as in [6], the one that is easiest to apply, namely the requirement that it satisfies the Prékopa–Leindler inequality. So a measure on a separable Banach space X will be called *log-concave* provided that

$$\int_X f\, d\mu \geq \left(\int_X g\, d\mu\right)^s \left(\int_X h\, d\mu\right)^t \tag{2.3}$$

whenever $0 < s, t < 1$, $s+t = 1$ and f, g, h are non-negative Borel measurable functions satisfying

$$f(sx + ty) \geq g(x)^s h(y)^t \tag{2.4}$$

for every $x, y \in X$. Notice that the usual statement of the Prékopa–Leindler inequality says that $\mathcal{L}^n$ is a log-concave measure on $\mathbb{R}^n$. (See [9, 14].)

A Borel measurable function $f : X \to [0, \infty)$ is said to be *log-concave* if the function $x \mapsto -\log f(x)$ is convex. Here we let $\log t = -\infty$ for $t \leq 0$ and allow a convex function to attain also the value $+\infty$.

The following properties of log-concave measures and functions, which we will freely use, follow immediately from the definition.

- If μ is a log-concave measure on X and f a μ-integrable log-concave function on X, then the measure $\mu E := \int_E f \, d\mu$ is log-concave.
- If μ is a log-concave measure on Y and f a log-concave function on $X \times Y$ such that $y \mapsto f(x, y)$ is μ-integrable for all $x \in X$, the function $g(x) := \int_Y f(x, y) \, d\mu(x)$ is log-concave.

We will make a deep use of some basic instances of the concentration of measure phenomenon. (See, for example, [8] for the basic techniques and uses of this important concept.) At this moment it suffices to say that what we need originates in the special case of the concentration phenomenon according to which in high-dimensional spaces for any given point x log-concave measures and so (integrals of) log-concave functions tend to be concentrated close to some sphere $\{y \mid |y - x| = c\}$. Unusually, it will be important for us to relate the values of c for two concentration problems (in spaces of different dimension). For that, the main tool will be concentration of log-concave functions close to their maximum.

To gain information on the position of the point at which given log-concave functions attain their maximum, the subdifferential criterion for a convex function to attain its minimum will be useful. Recall that when f is a convex function on $\mathbb{R}^n$ and $f(x) < \infty$, the *subdifferential* of f at x is defined as the set

$$\partial f(x) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x \rangle \leq f(y) - f(x) \text{ for all } y \in \mathbb{R}^n\}.$$

Clearly, f attains its minimum at x if and only if zero belongs to $\partial f(x)$.

There is a vast literature on Gaussian measures, both on finite-dimensional spaces and on Banach spaces; see, for example, [2] and references there. We will only recall the notions that we need.

Definition 4. The *standard Gaussian measure* on $\mathbb{R}^n$ is defined by

$$\gamma F = \frac{1}{(2\pi)^{n/2}} \int_F \exp(-|x|^2/2) \, dx$$

for Borel sets $F \subset \mathbb{R}^n$. The standard Gaussian measure on $\mathbb{R}^{\mathbb{N}}$ is defined as the countable product of the one-dimensional standard Gaussian measures.

Gaussian measures in infinite dimensions will be seriously used only in Section 6, but we introduce them already now in order to enable an informal presentation of the thinking behind the proof of Theorem 1 in the next section. Up to an isomorphism, non-degenerate Gaussian measures on infinite-dimensional separable Hilbert spaces are fully described in the following way.

Gaussian measures on Hilbert spaces. The measure γ is the restriction of the standard Gaussian measure from $\mathbb{R}^{\mathbb{N}}$ to

$$H := \left\{ x \in \mathbb{R}^{\mathbb{N}} \,\middle|\, \sum_{i=1}^{\infty} \lambda_i x_i^2 < \infty \right\}$$

equipped with the norm

$$\|x\| := \Big(\sum_{i=1}^{\infty} \lambda_i x_i^2\Big)^{1/2},$$

where $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ satisfy $\sum_{i=1}^{\infty} \lambda_i < \infty$.

The summability condition on the λ_i is sufficient (as well as necessary) for γ to be a Borel measure on H. In this representation, the λ_i are precisely the eigenvalues of the covariance operator of γ that have been used in the statements of Theorem T and Theorem 1. The corresponding eigenvectors are $u_i := (\delta_{i,j})_{j\in\mathbb{N}} \in H$, where $\delta_{i,j} = 1$ when $i = j$ and $\delta_{i,j} = 0$ otherwise. We will also denote

$$|x| = \Big(\sum_{i=1}^{\infty} x_i^2\Big)^{1/2} \quad \text{and} \quad \langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i \quad \text{provided that } |x|, |y| < \infty$$

and point out that, perhaps somewhat illogically,

$$B(x, r) = \{y \in H \mid \|y - x\| \leq r\}$$

denotes a ball in the norm $\|\cdot\|$.

Finally, we recall that Gaussian measures are log-concave. Indeed, on each of the spaces $H_n := \operatorname{span}\{u_1, \ldots, u_n\}$ the function $x \mapsto e^{-|x|^2/2}$ is log-concave and integrable with respect to the log-concave measure $\mathcal{L}^n$. Hence the standard Gaussian measure on H_n is log-concave, and by [6, Corollary 5] so is their weak limit γ.

3. Sketch of main arguments

Although some of our arguments may seem to be quite technical, the basic idea behind them is rather simple. We choose mutually orthogonal finite-dimensional subspaces H_i of H, each spanned by a subset of the u_j, with $\dim H_i = n_i \nearrow \infty$, and for suitable $\tau_i > 0$ define

$$M := \bigcap_{i=1}^{\infty} M_i \quad \text{where} \quad M_i := \Big\{y \in H \;\Big|\; \big||H_i y| - \sqrt{n_i}\big| \leq \tau_i\Big\}.$$

By the well known result on concentration of norm for the standard Gaussian measure (which will also be given in Corollary 7) the set M has positive γ-measure for τ_i much smaller than $\sqrt{n_i}$, for example for $\tau_i = 2\sqrt{\log n_i}$. Given any $z \in H$, $r > 0$ and $\varepsilon > 0$, the concentration phenomenon should also provide constants c_i such that the restriction of γ to the ball $B(z, r)$ is concentrated close to $\{y \in B(z, r) \mid |H_i y| = c_i\}$. In other words there are (small) $\sigma_i > 0$ such that

$$\gamma\Big\{y \in B(z, r) \;\Big|\; \big||H_i y| - c_i\big| > \sigma_i\Big\} \leq \varepsilon \gamma B(z, r).$$

Provided that $|\sqrt{n_i} - c_i| > \tau_i + \sigma_i$ for some i, we get $\gamma(M \cap B(z, r)) \leq \varepsilon\gamma B(z, r)$, and if for all small $r > 0$ this can be done for all z, we are done.

The main source of the technicalities in our arguments is that the c_i are not easy to estimate. We therefore explain the reasoning that lead us to the conclusion that with suitable choices of λ_i and H_i the above approach may go through. For simplicity we will assume that, when restricted to H_i, $\|\cdot\|$ is a constant multiple of $|\cdot|$; this corresponds to what was used in [12] and [13] and what we called artificial examples.

We first look at what happens when we fix some i and use Fubini's Theorem to calculate the measures $\gamma B(z,r)$ and $\gamma(M_i \cap B(z,r))$. After letting $n = n_i$, $x = H_i z$ and $T := \{y \in H_i \mid \big||y| - \sqrt{n}\big| \le \tau_i\}$, this gives

$$\gamma B(z,r) = \int_{H_i} \Phi(y) e^{-|y|^2/2}\, d\mathcal{L}^n(y) \quad \text{and} \quad \gamma(M_i \cap B(z,r)) = \int_T \Phi(y) e^{-|y|^2/2}\, d\mathcal{L}^n(y)$$

where $\Phi\colon H_i \to [0,\infty)$ and $\Phi(y)$ depends only on the distance from y to x in the norm $\|\cdot\|$ and so also in the norm $|\cdot|$. (Without loss of any significant details we may assume that Φ is the indicator of some ball about x.) Moreover, Φ is log-concave and so the concentration phenomenon implies that for some $\rho = \rho_i(z,r) > 0$ the integral of Φ is concentrated close to the sphere $\{y \in H_i \mid |y - x| = \rho\}$. The situation is illustrated in Figures 1 and 2 where we ignore the widths of concentration (the constants τ_i and σ_i) as they are much smaller than the radii of concentration ($\sqrt{n_i}$ and ρ).

We will distinguish several cases, one "good" (yielding the failure of the Density Theorem) in the sense that $\gamma(M_i \cap B(z,r)) \le \varepsilon\gamma B(z,r)$, and three "bad" when these inequalities do not hold. For that, we will denote by $S(y,t)$ the sphere in H_i centred at y with radius t and use two basic instances of the concentration phenomenon: in any cone C in H_i with vertex at the origin, the standard Gaussian measure is concentrated close to $C \cap S(0,\sqrt{n})$ and the $(n-1)$-dimensional measure of a spherical cap is concentrated close to its boundary. So, for example, if C has spherical base, we may approximate γC by $\kappa_n e^{-n/2} t^{n-2}$ where t is the radius of the sphere $S(0,\sqrt{n}) \cap \partial C$ and κ_n is a constant depending on n only.

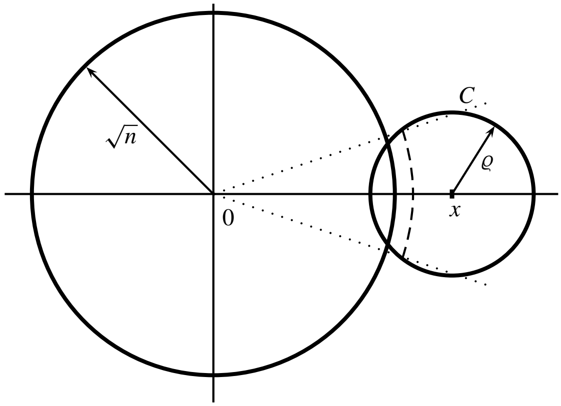

Fig. 1. Concentration outside $S(0,\sqrt{n})$.

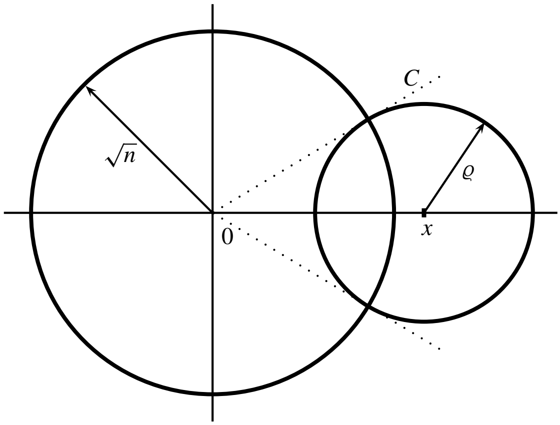

Fig. 2. Concentration on $S(0, \sqrt{n})$.

Figures 1 and 2 indicate a "good" and a "bad" case, respectively. We explain them in (a) and (b) below, and add two additional simple "bad" cases.

(a) Figure 1 gives an example of a "good" case. To explain it, we pick a suitable $s > 0$ whose choice will be indicated shortly, and let $U := S(0, \sqrt{n} + s) \cap S(x, \rho)$. (In Figure 1, U is the boundary of the dashed spherical cap, or equivalently its intersection with $S(x, \rho)$.) Also let

$$W := S(0, \sqrt{n}) \cap S(x, \rho) \quad \text{and} \quad V := S(0, \sqrt{n}) \cap \partial C.$$

We notice that U, V, W are $n-2$-dimensional spheres and denote their radii u, v, w, respectively. The way in which $S(0, \sqrt{n})$ and $S(x, \rho)$ intersect (as opposed to the way in which they intersect in Figure 2) shows that $v \geq w + cs$ where $c > 0$ is a (small) constant independent of n. Hence

$$u = (1 + s/\sqrt{n})v \geq (1 + s/\sqrt{n} + cs/w)w.$$

Since Φ is constant on $S(x, \rho)$, the concentration arguments indicated above show that

$$\gamma(M \cap B(z, r)) \leq \kappa_n e^{-n/2} w^{n-2} \quad \text{and} \quad \gamma B(z, r) \geq \kappa_n e^{-(\sqrt{n}+s)^2/2} u^{n-2}.$$

Hence

$$\gamma(M \cap B(z, r)) \leq e^{s\sqrt{n}+s^2/2}(1 + s/\sqrt{n} + cs/w)^{-n+2} \gamma B(z, r).$$

For suitable s (and under reasonable assumptions on the sizes involved in Figure 1), expansion of the coefficient of $\gamma B(z, r)$ leads to the main term $e^{-ncs/w}$, which is a small number.

(b) As stated above, inside the whole cone C the standard Gaussian measure is concentrated close to the sphere $S(0, \sqrt{n}) \cap \partial C$. But $S(0, \sqrt{n}) \cap \partial C$ is contained in $S(x, \rho)$ which is contained in C. Since Φ is constant on $S(x, \rho)$, the integral of $\Phi(y)e^{-|y|^2/2}$ is also concentrated close to $S(0, \sqrt{n}) \cap \partial C$.
(c) Another "bad" case occurs when x is close to $S(0, \sqrt{n})$ and ρ is small, for example because then the support of Φ may be contained in T.
(d) A final "bad" case occurs when the set $S(0, \sqrt{n}) \cap x^{\perp}$ (close to which the Gaussian measure on the whole of H_i is concentrated) is (almost) contained in $S(x, \rho)$.

Understanding these cases was enough to show that the almost everywhere version of the Density Theorem fails for some Gaussian measures. Since the centre belongs to M, the situation from (b) cannot occur for any i (or, more precisely, it is subsumed in (c)), and since r is small, (c) occurs for small i. Hence the only way in which the density ratio may be close to 1 is that (c) occurs for some i while (d) occurs for $i + 1$. Roughly, this would mean that $\rho_i(x, r)$ should be about τ_i and $\rho_{i+1}(x, r)$ about $\sqrt{2n_{i+1}}$. But this is impossible when the dimensions of H_i and H_{i+1} as well as the ratios of $\|\cdot\|$ and $|\cdot|$ on these spaces are not too far from each other. The reason for this is best seen by noticing that if both these dimensions and ratios were the same, symmetry would show that $\rho_i(x, r)$ is very close to $\rho_{i+1}(x, r)$.

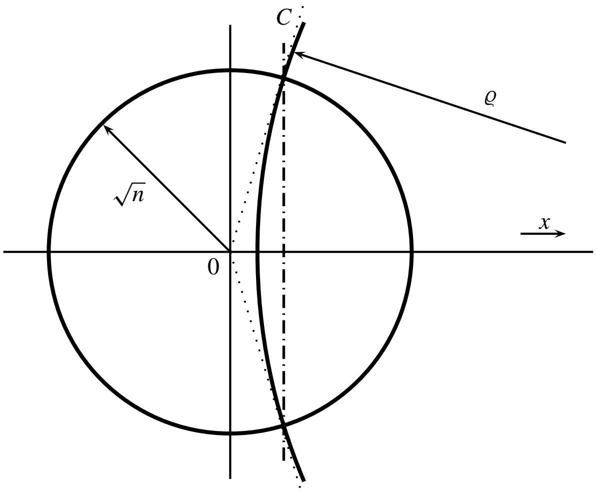

Fig. 3. Concentration far from $S(0, \sqrt{n}) \cap x^{\perp}$.

The above programme was realized in [12], and in [13] it was refined to get a function satisfying (1.4). Nevertheless, these ideas were too weak to show Theorem 1 till the second named author made several key observations that we summarize in the following two points.

(R_1) The possibility (b) for the choice of “bad” centre and radius is also far from (d). As illustrated in Figure 3, when we are in the situation from (b) then even if ρ is quite close to $|x|^2 + n$ (which means that $S(0, \sqrt{n}) \cap S(x, \rho)$ is close to $S(0, \sqrt{n}) \cap x^{\perp}$), the integral of $\Phi(y)e^{-|y|^2/2}$ is concentrated close to the hyperplane indicated by the dash line which is far from $x^{\perp}$. The only way in which this discrepancy may disappear is to have ρ very big, but this should imply that $|x|$ is big. As we are treating only small values of r, this should mean that $B(x, r)$ is far from $S(0, \sqrt{n})$, implying that $\gamma(M \cap B(x, r)) = 0$. However, the sizes of both x and r are measured in the norm $\|\cdot\|$ (which is a small multiple of $|\cdot|$), so this argument needs refining. For this, consider the dash line in Figure 3 as representing the hyperplane $\{y \in H_i \mid \langle y, x\rangle = \alpha\}$ close to which the restriction of the function $\langle \cdot, x\rangle$ to the ball $B(x, r)$ is concentrated. In Lemma 12(iii) we not only find that $|x|$ is big, but obtain a lower bound $\alpha \geq n/4$. (Incidentally, α has the same lower bound also in the case (c), since then $|x| = \sqrt{n}$ and $B(x, \rho)$ is a small ball around x. So the two “bad” cases, (b) and (c), may be treated as one, which will allow us to reduce the number of cases in Lemmas 12 and 13 from four to three.)

(R_2) For one index i, (R_1) does not give a strong enough estimate. However, assuming the case (a) never occurs, there is a chain of indices $k, k+1, \dots, l$ starting at (b), ending at (d) and such that for every $i = k, \dots, l-1$ either (b) or (c) occurs. A strengthened discrepancy argument (see below) shows that this chain is long, and (R_1) provides a lower estimate of $\|H_i z\|$ for mutually orthogonal vectors $H_i z$, $i = k, \dots, l-1$. Under conditions that are reflected in our assumptions on the eigenvalues of the covariance operator of the Gaussian measure γ, this finally implies that $\|z\|$ is big, and since r has an upper bound, $B(z, r)$ does not meet M and so $\gamma(M \cap B(z, r)) = 0$.

As pointed out, in (R_2) we need a strengthened discrepancy argument of [12]. More precisely, we need to understand, given z, r, what happens in H_{i+1} provided that (b) occurred for H_i. Calculating $\gamma B(z, r)$ and $\gamma(M_i \cap M_{i+1} \cap B(z, r))$ using Fubini’s Theorem, we are faced with two concentration problems for a function, say, Ψ in the space $H_i \oplus H_{i+1}$, namely with the problem of relation of the concentration constant of $|H_i y|$ to the concentration constant of $|H_{i+1} y|$. As we need rather sharp estimates, we use the fact that $\Psi(y)$ depends only on four variables: $\langle y, H_i z\rangle$, $\langle y, H_{i+1} z\rangle$, the shortest distances of $H_i y$ to a multiple of $H_i z$ and the shortest distances of $H_{i+1} y$ to a multiple of $H_{i+1} z$. This allows us to transform the problem to a four-dimensional one for a new function that happens to be logarithmically concave. For this function we use the fact that its integral is concentrated close to its maximum, which relatively easily allows comparison of concentration constants for different functions.

The above discussion assumed a simplifying condition that on each H_i the norms $\|\cdot\|$ and $|\cdot|$ are multiple of each other. As this cannot be the case for the most interesting choices of the eigenvalues in Theorem 1 including $\lambda_k = k^{-s}$ for $1 < s < 6/5$, the estimates we need are more technical than needed for the simplified case. We therefore devote the next section to showing basic results on concentration of integrals of log-concave functions close to their maximum. One of the standard results on concentration of Gaussian measures is given in Corollary 7 as an immediate consequence. Section 5 contains the main technical estimates needed to prove Theorem 1. There we introduce a class of log-concave functions that can appear by an application of Fubini's Theorem alluded to above. This allows us to reduce the dimension of the spaces in which the concentration is needed to either two or four. In order to estimate concentration of these functions, we obtain an equation for their maximum in Lemma 11, and prove the corresponding variant of the discrepancy between the "bad" cases in Lemma 12, where (i) corresponds to (a), (ii) to (d) and (iii) to joined (b) and (c). The main Lemma 13 of this section concerns a longer sum of the spaces H_i to realize the idea of (R_2).

Finally, Proposition 14 of Section 6 does in a more general form what was indicated here. It gives a rather technical criterion for uniform failure of the Density Theorem that involves the behaviour of the eigenvalues λ_k. Theorem 1 as well as other results indicated in the final Remark easily follow.

4. Concentration around maximum

Results on concentration of log-concave functions are nowadays standard (see, for example, [3]). Most often, they treat concentration about the mean value or median, while to prove our main concentration estimates in Section 5, concentration about the maximum is considerably more convenient. To make our proof complete, we therefore provide the full basic argument.

Lemma 5. *Let $g\colon \mathbb{R} \to [0, \infty)$ be log-concave, $a \in \mathbb{R}$ and $g(a) > 0$. Then for $t \geq 0$,*

$$\int_{a+t}^{\infty} g(s)\, ds \leq \frac{g(a+t)}{g(a)} \int_{a}^{\infty} g(s)\, ds,$$

and for $t \leq 0$,

$$\int_{-\infty}^{a+t} g(s)\, ds \leq \frac{g(a+t)}{g(a)} \int_{-\infty}^{a} g(s)\, ds.$$

Proof. We show only the first statement, the second being analogous. The case $g(a+t) \geq g(a)$ is obvious, since the integral on the left is clearly bounded by the integral on the right, and so is the case $g(a+t) = 0$ since then $g(s) = 0$ for all $s \geq a+t$. So assume $0 < g(a+t) < g(a)$ and let

$$h(s) := -\alpha s + \beta$$

be an affine function passing through the points $(a, \log g(a))$ and $(a+t, \log g(a+t))$. Since $\log g(a+t) < \log g(a)$, we have $\alpha > 0$. Further, by concavity of $\log g$ we have $\log g(s) \geq h(s)$ for $s \in (a, a+t)$ and $\log g(s) \leq h(s)$ for $s \in (a+t, \infty)$. Equivalently,

$$g(s) \geq e^{-\alpha s+\beta} \quad \text{and} \quad g(s) \leq e^{-\alpha s+\beta}$$

for s belonging to $(a, a+t)$ and $(a+t, \infty)$, respectively. Denote $A = 1/\alpha$. Integrating the first inequality over $(a, a+t)$ and the second over $(a+t, \infty)$, we get

$$\int_a^{a+t} g(s)\, ds \geq A(g(a) - g(a+t))$$

and

$$\int_{a+t}^{\infty} g(s)\, ds \leq Ag(a+t).$$

Combining these two estimates we obtain

$$\int_a^{a+t} g(s)\, ds \geq \frac{g(a) - g(a+t)}{g(a+t)} \int_{a+t}^{\infty} g(s)\, ds = \frac{g(a)}{g(a+t)} \int_{a+t}^{\infty} g(s)\, ds - \int_{a+t}^{\infty} g(s)\, ds,$$

and the statement follows by adding $\int_{a+t}^{\infty} g(s)\, ds$. □

Lemma 6. *Suppose* $\varphi \colon [0,\infty) \to [0,\infty)$ *is log-concave and* $c, r \geq 0$.

(i) *If* $t \mapsto \varphi(t)e^{ct^2}$ *is non-increasing on* $[r, \infty)$, *then for every* $s \geq r$ *and* $0 \leq k \leq 2cr^2$,

$$\int_s^{\infty} \varphi(t)t^k\, dt \leq e^{-c(s-r)^2} \int_r^{\infty} \varphi(t)t^k\, dt.$$

(ii) *If* $t \mapsto \varphi(t)e^{ct^2}$ *is non-decreasing on* $(0, r]$, *then for every* $0 < s \leq r$ *and* $k \geq 2cr^2$,

$$\int_0^{s} \varphi(t)t^k\, dt \leq e^{-c(s-r)^2} \int_0^{r} \varphi(t)t^k\, dt.$$

Proof. First notice that the case $c = 0$ is trivial and the statement (ii) is vacuously true when $r = 0$. Also, under the assumption of (i),

$$\varphi(t)e^{ct^2} \geq \varphi(t+s)e^{c(t+s)^2} \geq \varphi(t+s)e^{ct^2+cs^2}$$

for every $t, s \geq 0$. Thus $\varphi(t+s) \leq \varphi(t)e^{-cs^2}$ and integrating over $t \in (0,\infty)$ shows that the inequality (i) holds with $r = 0$. Hence we may assume that $c, r > 0$.

Notice that the integrand $\varphi(t)t^k$ is log-concave. We multiply the inequality $\varphi(s)e^{cs^2} \leq \varphi(r)e^{cr^2}$, which holds in both cases, by e^{-cs^2} to get

$$\varphi(s) \leq e^{cr^2 - cs^2}\varphi(r).$$

Also,

$$k \log s = k \log r + k \log(1 + (s-r)/r) \leq k \log r + k(s-r)/r, \text{ i.e. } s^k \leq r^k e^{k(s-r)/r}.$$

Combining the last two inequalities and using the fact that our assumptions imply $k(s-r)/r \leq 2c(s-r)r$, we get

$$\varphi(s)s^k \leq e^{cr^2-cs^2}\varphi(r)r^k e^{k(s-r)/r} \leq e^{cr^2-cs^2+2c(s-r)r}\varphi(r)r^k = e^{-c(s-r)^2}\varphi(r)r^k.$$

By the first statement of Lemma 5 with $a = r$ and $a + t = s$,

$$\int_s^\infty \varphi(t)t^k\,dt \le \frac{\varphi(s)s^k}{\varphi(r)r^k}\int_r^\infty \varphi(t)t^k\,dt \le e^{-c(s-r)^2}\int_r^\infty \varphi(t)t^k\,dt,$$

which is (i). The second statement of Lemma 5 with the same choice establishes (ii). □

Corollary 7. *Let $n \in \mathbb{N}$, $c > 0$ and $r = \sqrt{(n-1)/(2c)}$. Then for every $\lambda > 0$,*

$$\int_{\{x\in\mathbb{R}^n \mid\, ||x|-r|>\lambda\}} e^{-c|x|^2}\,d\mathcal{L}^n(x) \le e^{-c\lambda^2}\int_{\mathbb{R}^n} e^{-c|x|^2}\,d\mathcal{L}^n(x).$$

Proof. Using (2.2) we get

$$\int_{\{x\in\mathbb{R}^n \mid\, ||x|-r|>\lambda\}} e^{-c|x|^2}\,d\mathcal{L}^n(x) = \omega_{n-1}\int_0^{r-\lambda} e^{-ct^2}t^{n-1}\,dt + \omega_{n-1}\int_{r+\lambda}^\infty e^{-ct^2}t^{n-1}\,dt \tag{4.1}$$

where $\int_0^{r-\lambda} e^{-ct^2}t^{n-1}\,dt$ is set equal to zero if $\lambda \ge r$. We apply Lemma 6 to $\varphi(t) = e^{-ct^2}$ and $k = 2cr^2 = n-1$. Observe that φ satisfies the assumptions of both (i) and (ii). Hence the estimate (4.1) may be continued by

$$\le \omega_{n-1}e^{-c\lambda^2}\int_0^r e^{-ct^2}t^{n-1}\,dt + \omega_{n-1}e^{-c\lambda^2}\int_r^\infty e^{-ct^2}t^{n-1}\,dt = e^{-c\lambda^2}\int_{\mathbb{R}^n} e^{-c|x|^2}\,d\mathcal{L}^n(x).$$

□

Lemma 8. *Let $\varphi\colon \mathbb{R}^n \to [0,\infty)$ attain its maximum at $p \in \mathbb{R}^n$. Assume further that a positive semi-definite quadratic form Q on $\mathbb{R}^n$ is such that the function $x \mapsto \varphi(x)e^{Q(x)}$ is log-concave. Then $\psi(x) := \varphi(x)e^{Q(x-p)}$ is log-concave and attains its maximum at p.*

Proof. The function ψ is log-concave since $\psi(x) = \varphi(x)e^{Q(x)}e^{h(x)}$ where the function $h(x) := Q(x-p) - Q(x)$ is affine. Hence $g(x) := -\log\psi(x) = -\log\varphi(x) - Q(x-p)$ is a convex function. Assuming, as we may, that $\varphi(p) > 0$, we see that $-\log\varphi$ attains its minimum at p. Using this, we infer

$$\begin{aligned}\liminf_{t\to 0}\frac{g(p+tx)-g(p)}{t} &= \liminf_{t\to 0}\frac{-\log\varphi(p+tx)+\log\varphi(p)-Q(tx)}{t}\\ &\ge \liminf_{t\to 0}\frac{-t^2Q(x)}{t} = 0.\end{aligned}$$

This estimate means that zero belongs to the subdifferential of g at p. Hence g attains its minimum at p, and consequently $\psi = e^{-g}$ attains its maximum at p. □

Lemma 9. *Suppose $\varphi\colon \mathbb{R}^n \to [0,\infty)$ attains its maximum at $p \in \mathbb{R}^n$ and a positive semi-definite quadratic form Q on $\mathbb{R}^n$ is such that the function $x \mapsto \varphi(x)e^{Q(x)}$ is log-concave. Then for every $\tau \ge (n-1)/2$,*

$$\int_{\{x\in\mathbb{R}^n \mid\, Q(x-p)\ge\tau\}} \varphi\,d\mathcal{L}^n \le e^{-\sigma}\int_{\mathbb{R}^n}\varphi\,d\mathcal{L}^n, \tag{4.2}$$

where $\sigma := \left(\sqrt{\tau} - \sqrt{(n-1)/2}\right)^2$.

Proof. Clearly, only the situation when $\varphi(p) > 0$ and the integral on the right of (4.2) is finite needs treatment. By Lemma 8, $\psi(x) := \varphi(x)e^{Q(x-p)}$ is log-concave and attains its maximum at p.

Let $S = \{u \in \mathbb{R}^n \mid |u| = 1,\ Q(u) > 0\}$. For $u \in S$ define $\varphi_u\colon \mathbb{R} \to [0,\infty)$ by $\varphi_u(t) := \varphi(p+tu)$. Given $\tau \ge (n-1)/2$, we let for $u \in S$,

$$r_u := \left(\frac{\tau}{Q(u)}\right)^{1/2} - \left(\frac{\sigma}{Q(u)}\right)^{1/2} \quad \text{and} \quad s_u := \left(\frac{\tau}{Q(u)}\right)^{1/2}.$$

Then by (2.1),

$$\int_{\{x\in\mathbb{R}^n \mid Q(x-p)\ge\tau\}} \varphi\, d\mathcal{L}^n = \int_S \int_{s_u}^{\infty} \varphi_u(t)\, t^{n-1}\, dt\, d\mathcal{H}^{n-1}(u). \tag{4.3}$$

We estimate the inner integral on the right side of this inequality by an application of Lemma 6(i) with $\varphi_u(t) = \psi(p+tu)e^{-Q(tu)}$, $c = Q(u)$, $r = r_u$, $s = s_u$ and $k = n-1$. To see that its assumptions hold is straightforward: since ψ is log-concave, $\varphi_u(t)$ and $\varphi_u(t)e^{ct^2}$ are also log-concave. Together with the assumption that $\varphi_u(t)e^{Q(tu)}$ attains its maximum at $t = 0$ this shows that $\varphi_u(t)e^{ct^2}$ is non-increasing on $[0,\infty)$. Finally, by assumption, $2cr^2 = 2(\sqrt{\tau} - \sqrt{\sigma})^2 = n-1 = k$ and clearly $0 \le r \le s$. Hence, using first Lemma 6(i) and then (2.1) we can finish the estimates started at (4.3),

$$\begin{aligned}
&\le \int_S e^{-Q(u)(s_u-r_u)^2} \int_{r_u}^{\infty} \varphi_u(t)\, t^{n-1}\, dt\, d\mathcal{H}^{n-1}(u)\\
&= e^{-\sigma} \int_S \int_{r_u}^{\infty} \varphi_u(t)\, t^{n-1}\, dt\, d\mathcal{H}^{n-1}(u) \le e^{-\sigma} \int_{\mathbb{R}^n} \varphi\, d\mathcal{L}^n. \qquad \square
\end{aligned}$$

5. Main concentration estimates

We recall from Section 2 that both $|\cdot|$ and $\|\cdot\|$ are used to denote a norm induced by a scalar product on a vector space H and that $\langle\cdot,\cdot\rangle$ denotes the scalar product inducing $|\cdot|$. Additionally, it will be convenient to let $\langle u, v\rangle_+ := \max\{0, \langle u, v\rangle\}$. To indicate the reason for distinguishing the two norms, we notice that $|\cdot|$ is used for the norm related to the standard Gaussian measure γ (so it is the Euclidean norm in $\mathbb{R}^n$ or the usual norm in ℓ_2 in the infinite-dimensional situation) while $\|\cdot\|$ is used for a norm in which γ is σ-additive or for its approximation in the finite-dimensional case. In statements in which only one norm is used, and so this distinction is immaterial, we try to use the notation that corresponds best to later usage.

When H is equipped with $\|\cdot\|$ and $x \in H$, we denote by $\mathcal{F}(H, \|\cdot\|, x)$ the collection of bounded log-concave $\|\cdot\|$-upper semicontinuous functions $\Psi\colon H \to [0,\infty)$ such that $\Psi(u)$ depends only on $\|u - x\|$ and $\Psi(u) > 0$ for all u from a $\|\cdot\|$-neighbourhood of x. In the case when $\|u\| = \langle u, A(u)\rangle^{1/2}$ where A is a bounded positive definite symmetric linear operator on $(H, |\cdot|)$, we will write $\|\cdot\|_A$ and $\mathcal{F}(H, A, x)$ instead of $\|\cdot\|$ and $\mathcal{F}(H, \|\cdot\|, x)$, respectively.

Lemma 10. *Let V be a closed subspace of the Hilbert space $(H, \|\cdot\|)$, Z its orthogonal complement, and ν a finite log-concave Borel measure on Z. Suppose further that $x \in H$ is such that Zx belongs to the support of ν and $\Psi \in \mathcal{F}(H, \|\cdot\|, x)$. Then the function $\Phi(v) := \int_Z \Psi(v+z)\, d\nu(z)$ belongs to $\mathcal{F}(V, \|\cdot\|, Vx)$.*

Proof. As noticed in Section 2, Φ is log-concave. Clearly, it is bounded, and using the fact that Ψ is bounded and upper semicontinuous, we infer from Fatou's Lemma that Φ is upper semicontinuous. Since $\Psi(y) = f(\|y-x\|^2)$ for some f, we have

$$\Phi(v) = \int_Z f(\|v+z-x\|^2)\, d\nu(z) = \int_Z f(\|v-Vx\|^2 + \|z-Zx\|^2)\, d\nu(z).$$

Hence $\Phi(v)$ depends on the value of $\|v - Vx\|$ only.

Finally, to show that $\Phi > 0$ on a neighbourhood of Vx, let $r > 0$ be such that $\Psi > 0$ on $B(x, r)$. Then $\Psi(v+z) > 0$ whenever $v \in VB\big(x, \frac{1}{2}r\big)$ and $z \in ZB\big(x, \frac{1}{2}r\big)$. Since $\nu\, ZB\big(x, \frac{1}{2}r\big) > 0$, we see that $\Phi(v) > 0$ for $v \in VB\big(x, \frac{1}{2}r\big)$. □

Lemma 11. *Suppose that A is a positive definite symmetric linear operator on a finite-dimensional Hilbert space $(H, |\cdot|)$, $x \in H$, and $\Psi \in \mathcal{F}(H, A, x)$. Suppose further that I is a finite index set, and for each $i \in I$ we are given $n_i \in \mathbb{N}$ and $w_i \in H$ such that $\langle x, w_i\rangle \le 0$ and*

$$\{u \in H \mid \Psi(u) > 0\} \cap \bigcap_{i\in I} \{u \in H \mid \langle u, w_i\rangle > 0\} \neq \emptyset. \tag{5.1}$$

Then there is a unique point $p \in H$ at which the log-concave function

$$f(u) := e^{-|u|^2/2}\Psi(u) \prod_{i\in I} \langle u, w_i\rangle_+^{n_i}$$

attains its maximum. Moreover, $\langle p, w_i\rangle > 0$ for each $i \in I$ and there is $\lambda \ge 0$ such that

$$p + \lambda A(p-x) - \sum_{i\in I} \frac{n_i w_i}{\langle p, w_i\rangle} = 0.$$

Proof. Recall that $h(u) := -\log \Psi(u)$ is a convex function depending only on $\|u-x\|_A$. Since h is even with respect to x (i.e. $h(u) = h(2x-u)$), it attains its minimum at x. Consider any point $p \in H$ such that $p \neq x$ and $h(p) < \infty$. We show that any $y \in \partial h(p)$ is a non-negative multiple of $A(p-x)$. For this, it suffices to show that $\langle y, u\rangle \le 0$ whenever $\langle u, A(p-x)\rangle < 0$. So assume that $u \in H$ is such that $\langle u, A(p-x)\rangle < 0$. Then for small $t > 0$,

$$\|tu + p - x\|_A^2 = t^2\|u\|_A^2 + 2t\langle u, A(p-x)\rangle + \|p-x\|_A^2 \le \|p-x\|_A^2.$$

It follows that $h(tu+p) \le h(p)$ and so

$$\langle tu, y\rangle \le h(tu+p) - h(p) \le 0.$$

So indeed $\langle y, u\rangle \le 0$ whenever $\langle u, A(p-x)\rangle < 0$, and we infer that $y = \lambda A(p-x)$ for some $\lambda \ge 0$.

To finish the proof, we introduce the function $g := -\log f$, i.e.

$$g(u) = |u|^2/2 + h(u) - \sum_{i \in I} n_i \log \langle u, w_i \rangle_+.$$

The function g is convex, lower semicontinuous, and not identically $+\infty$ due to the condition (5.1). Moreover $g(u)$ tends to infinity when $|u| \to \infty$ and hence g attains its minimum at some point $p \in H$. From (5.1) we see that $g(p) < \infty$, and since g is strictly convex on the set $\{u \in H \mid g(u) < \infty\}$, the point p is unique. Since $g(p) < \infty$, we have $\langle p, w_i \rangle > 0$ for all $i \in I$ and so $p \neq x$. Further, zero belongs to the subdifferential of g at p. Since $|u|^2/2 - \sum_{i \in I} n_i \log \langle u, w_i \rangle$ is smooth on a neighbourhood of p, the latter condition implies that

$$0 = p + y - \sum_{i \in I} \frac{n_i w_i}{\langle p, w_i \rangle}$$

for some $y \in \partial h(p)$. Recalling that $y = \lambda A(p - x)$ we obtain

$$p + \lambda A(p - x) - \sum_{i \in I} \frac{n_i w_i}{\langle p, w_i \rangle} = 0. \qquad \square$$

Lemma 12. *Let $\{v, w\}$ be an orthonormal basis of a 2-dimensional Hilbert space $(U, |\cdot|)$ and let A be a symmetric linear operator on U with eigenvalues $\alpha \geq \beta \geq 8\alpha/9 > 0$. Suppose further that $n \in \mathbb{N}$, $0 < \tau \leq 2^{-6}\sqrt{n}$, and x is a multiple of v satisfying $|x| \leq 2^{-6} n/\tau$ and $(\alpha - \beta)|x| \leq \alpha\sqrt{n}/18$. Finally, let $\lambda \geq 0$ and $p \in U$ satisfy $\langle p, w \rangle > 0$ and*

$$p + \lambda A(p - x) - \frac{nw}{\langle p, w \rangle} = 0. \tag{5.2}$$

Then at least one of the following statements holds:

(i) $\big||p| - \sqrt{n}\big| > 2\tau$;
(ii) $\lambda\alpha \leq 2^3\tau/\sqrt{n}$ *and* $|p - \sqrt{n}\, w| \leq 2^5\tau(1 + |x|/\sqrt{n})$;
(iii) $\lambda\alpha \geq 2^{-3} n/(\sqrt{n} + |x|)^2$ *and* $|\langle p, x \rangle| \geq n/4$.

Proof. If $\lambda = 0$ we get $p = \sqrt{n}\, w$ and (ii) holds. Hence we may assume that $\lambda > 0$ and, replacing A by λA, that $\lambda = 1$. Observe that

$$|\langle z, Au \rangle| \leq \alpha |z|\, |u|, \qquad \langle u, Au \rangle \geq \beta |u|^2, \qquad |\langle u, Az \rangle - \alpha \langle u, z \rangle| \leq (\alpha - \beta)|u|\, |z|.$$

These inequalities will be used without a reference.

Suppose that (i) fails. Then, letting $\kappa := 1 + 2^{-5}$, we have

$$|p| \leq \sqrt{n} + 2\tau \leq \kappa\sqrt{n} \tag{5.3}$$

and

$$\big||p|^2 - n\big| = \big||p| - \sqrt{n}\big|(|p| + \sqrt{n}) \leq 2\tau(\kappa\sqrt{n} + \sqrt{n}) \leq 4\kappa\tau\sqrt{n}. \tag{5.4}$$

Multiplying (5.2) by v we obtain

$$\begin{aligned}|\langle p, v\rangle| = |\langle v, A(p-x)\rangle| &\le \alpha|p-x| \le \alpha(|p|+|x|)\\ &\le \alpha(\kappa\sqrt{n}+|x|) \le 2\alpha(\sqrt{n}+|x|).\end{aligned} \tag{5.5}$$

If $\alpha \le 2^3\tau/\sqrt{n}$, (5.5) shows

$$|\langle p, v\rangle| \le 2^4\tau(1+|x|/\sqrt{n}).$$

By the assumptions $0 < \tau \le 2^{-6}\sqrt{n}$ and $|x| \le 2^{-6}n/\tau$ this implies $|\langle p, v\rangle| \le \sqrt{n}/2$. So $\langle p, v\rangle^2 \le |\langle p, v\rangle|\sqrt{n}/2$ and, using also (5.4) and (5.5), we obtain the second inequality in (ii) by estimating

$$\begin{aligned}|p-\sqrt{n}\,w| &\le |\sqrt{n}-\sqrt{|p|^2-\langle p, v\rangle^2}| + |\langle p, v\rangle| \le \frac{|n-|p|^2+\langle p, v\rangle^2|}{\sqrt{n}} + |\langle p, v\rangle|\\ &\le 4\kappa\tau + \tfrac{3}{2}|\langle p, v\rangle| \le 4\kappa\tau + 24\tau(1+|x|/\sqrt{n}) \le 2^5\tau(1+|x|/\sqrt{n}).\end{aligned}$$

It remains to assume $\alpha > 2^3\tau/\sqrt{n}$ and show (iii). Multiplying (5.2) by p and using (5.4) we get

$$\begin{aligned}\langle p, Ax\rangle &= \langle p, Ap\rangle + |p|^2 - n \ge \beta|p|^2 + |p|^2 - n \ge \frac{8\alpha}{9}\big(n-4\kappa\tau\sqrt{n}\big) - 4\kappa\tau\sqrt{n}\\ &\ge \frac{8\alpha n}{9}(1-2^{-4}\kappa) - \frac{\alpha n}{2}\kappa = \frac{(8-5\kappa)\alpha n}{9},\end{aligned}$$

where we have estimated the first occurrence of τ by $\tau \le 2^{-6}\sqrt{n}$ and the second by $\tau \le 2^{-3}\alpha\sqrt{n}$. Using $(\alpha-\beta)|x| \le \alpha\sqrt{n}/18$, (5.3) and $\kappa \le 23/22$, we get

$$\begin{aligned}\alpha|\langle p, x\rangle| &\ge |\langle p, Ax\rangle| - (\alpha-\beta)|p|\,|x| \ge \frac{(8-5\kappa)\alpha n}{9} - \frac{\alpha\sqrt{n}}{18}\kappa\sqrt{n}\\ &= \frac{(16-11\kappa)\alpha n}{18} \ge \frac{\alpha n}{4}.\end{aligned}$$

Since $\alpha > 0$, this gives the second inequality in (iii). Using also (5.5) and the assumption that x is a multiple of v, we get the first inequality of (iii) by estimating

$$2\alpha(\sqrt{n}+|x|)^2 \ge |x|\,|\langle p, v\rangle| = |\langle p, x\rangle| \ge n/4. \qquad \square$$

Lemma 13. *Let $I = \{k, k+1, \dots, l\}$ where $k, l \in \mathbb{N}$, $k \le l$. For each $i \in I$, let $\{v_i, w_i\}$ be an orthonormal basis of a 2-dimensional Hilbert space $(U_i, |\cdot|)$ and A_i a symmetric linear operator on U_i with eigenvalues $\alpha_i \ge \beta_i \ge 8\alpha_i/9 > 0$. Suppose further that $n_i \in \mathbb{N}$, $\tau_i \ge 4$, x_i is a multiple of v_i satisfying $(\alpha_i - \beta_i)|x_i| \le \alpha_i\sqrt{n_i}/18$ and*

$$165\tau_j(1+|x_i|/\sqrt{n_i})^2 \le \sqrt{n_j}\,\alpha_j/\alpha_i \quad \text{when } k \le i \le j \le \min\{i+1, l\}. \tag{5.6}$$

Let U denote the orthogonal direct sum of the U_i, $A = \sum_{i\in I} A_i \circ U_i$, $x = \sum_{i\in I} x_i$, $\Psi \in \mathcal{F}(U, A, x)$ and μ be the Borel measure on U defined by

$$\mu E := \int_E e^{-|u|^2/2}\Psi(u)\prod_{i\in I}\langle u, w_i\rangle_+^{n_i}\, d\mathcal{L}^{2s}(u), \quad \text{where } s = l-k+1. \tag{5.7}$$

Then at least one of the following statements holds:

(i) $\mu\{u \in U \mid |U_k u - \sqrt{n_k} w_k| \geq \sqrt{n_k}/2\} \leq e^{-\tau_k^2/4}\mu U$.
(ii) $\mu\{u \in U \mid \big||U_i u| - \sqrt{n_i}\big| \leq \tau_i\} \leq e^{-\tau_i^2/4}\mu U$ *for some* $i \in I$.
(iii) $\mu\{u \in U \mid |\langle U_i u, x_i\rangle| \leq n_i/5\} \leq e^{-\tau_i^2/4}\mu U$ *for each* $i \in I$.

Proof. The case $\mu \equiv 0$ being trivial, we assume $\mu \not\equiv 0$. In particular,

$$\{u \in U \mid \Psi(u) > 0\} \cap \bigcap_{i \in I} \{u \in U \mid \langle u, w_i\rangle > 0\} \neq \emptyset.$$

We will also assume that (i) and (ii) fail, i.e.

$$\mu\{u \in U \mid |U_k u - \sqrt{n_k}\, w_k| \geq \sqrt{n_k}/2\} > e^{-\tau_k^2/4}\mu U \tag{5.8}$$

and for every $i \in I$,

$$\mu\Big\{u \in U \;\Big|\; \big||U_i u| - \sqrt{n_i}\big| \leq \tau_i\Big\} > e^{-\tau_i^2/4}\mu U. \tag{5.9}$$

The proof will have five steps in which we will consider the validity of the inequalities

$$\mu\{u \in U \mid |U_i u - \sqrt{n_i}\, w_i| \geq 33\tau_i(1 + |x_i|/\sqrt{n_i})\} \leq e^{-\tau_i^2/4}\mu U \tag{5.10}$$

and (iii), i.e.

$$\mu\{u \in U \mid |\langle U_i u, x_i\rangle| \leq n_i/5\} \leq e^{-\tau_i^2/4}\mu U. \tag{5.11}$$

First we make a simple observation about the incompatibility of (5.10) and (5.11). Then we show that for each i at least one of these two inequalities holds, and observe that (5.10) fails for $i = k$. The last observation is then extended to all i, which combined with the incompatibility of (5.10) and (5.11) easily finishes the proof.

Step 1. *The inequalities* (5.10) *and* (5.11) *cannot both hold for the same* i. Indeed, the inequality (5.6) with $j = i$ implies

$$33\tau_i|x_i|(1 + |x_i|/\sqrt{n_i}) \leq \frac{|x_i|\sqrt{n_i}}{5(1 + |x_i|/\sqrt{n_i})} \leq \frac{n_i}{5},$$

and so for each $u \in U$, either

$$|U_i u - \sqrt{n_i}\, w_i| \geq 33\tau_i(1 + |x_i|/\sqrt{n_i})$$

or

$$|\langle U_i u, x_i\rangle| = |\langle U_i u - \sqrt{n_i}\, w_i, x_i\rangle| \leq 33\tau_i|x_i|(1 + |x_i|/\sqrt{n_i}) \leq n_i/5.$$

Hence the validity of both (5.10) and (5.11) would give a contradiction:

$$\begin{aligned}\mu U &\leq \mu\{u \in U \mid |U_i u - \sqrt{n_i} w_i| \geq 33\tau_i(1 + |x_i|/\sqrt{n_i})\}\\ &\quad + \mu\{u \in U \mid |\langle U_i u, x_i\rangle| \leq n_i/5\}\\ &\leq 2e^{-\tau_i^2/4}\mu U < \mu U.\end{aligned}$$

∎

Step 2. *For each $i \in I$, either (5.10) or (5.11) holds.* Fix $i \in I$. We apply Lemma 10 with $H = U$, $V = U_i$, $Z = \bigoplus_{m \in I \setminus \{i\}} U_m$, x and Ψ given in the assumptions, and the Borel measure ν on Z defined by

$$\nu F = \int_F e^{-|z|^2/2} \prod_{j \in I \setminus \{i\}} \langle z, w_j \rangle_+^{n_j} \, d\mathcal{L}^{2s-2}(z).$$

Hence $\Phi(u) = \int_Z \Psi(u+z)\, d\nu(z)$ belongs to $\mathcal{F}(U_i, A_i, x_i)$, and by Fubini's Theorem for every Borel set $E \subset U_i$,

$$\mu(U_i^{-1} E) = \int_E e^{-|u|^2/2} \Phi(u) \langle u, w_i \rangle_+^{n_i} \, d\mathcal{L}^2(u). \tag{5.12}$$

By Lemma 11 the integrand of (5.12) attains its maximum at a point $p \in U_i$ such that $\langle p, w_i \rangle > 0$ and for some $\lambda \geq 0$,

$$p + \lambda A_i(p - x_i) - \frac{n_i w_i}{\langle p, w_i \rangle} = 0. \tag{5.13}$$

Since the integrand multiplied by $e^{|u|^2/2}$ is log-concave, we may use Lemma 9 in $\mathbb{R}^2$ with $Q(u) = |u|^2/2$ and $\tau = \tau_i^2/2$. Since $\tau_i \geq 4$, we get

$$\sigma = (\tau_i/\sqrt{2} - 1/\sqrt{2})^2 \geq \tfrac{1}{2}\tau_i^2\big(1 - \tfrac{1}{4}\big)^2 \geq \tau_i^2/4,$$

and so

$$\mu\{u \in U \mid |U_i u - p| \geq \tau_i\} \leq e^{-\sigma} \mu U \leq e^{-\tau_i^2/4} \mu U. \tag{5.14}$$

By Lemma 12 with the choice $n = n_i$ and $\tau = \tau_i$ at least one of the following statements holds:

(a) $\big||p| - \sqrt{n_i}\big| > 2\tau_i$;
(b) $|p - \sqrt{n_i}\, w_i| \leq 2^5 \tau_i (1 + |x_i|/\sqrt{n_i})$;
(c) $|\langle p, x_i \rangle| \geq n_i/4$.

If (a) holds, then $\big\{u \in U \;\big|\; \big||U_i u| - \sqrt{n_i}\big| \leq \tau_i\big\} \subset \{u \in U \mid |U_i u - p| \geq \tau_i\}$ and hence (5.14) implies

$$\mu\big\{u \in U \;\big|\; \big||U_i u| - \sqrt{n_i}\big| \leq \tau_i\big\} \leq \mu\{u \in U \mid |U_i u - p| \geq \tau_i\} \leq e^{-\tau_i^2/4} \mu U.$$

Since this contradicts (5.9), we infer that (a) fails.

When (b) holds,

$$\{u \in U \mid |U_i u - \sqrt{n_i}\, w_i| \geq 33\tau_i(1 + |x_i|/\sqrt{n_i})\} \subset \{u \in U \mid |U_i u - p| \geq \tau_i\},$$

and we get (5.10) by inferring from (5.14) that

$$\begin{aligned} \mu\{u \in U \mid |U_i u - \sqrt{n_i}\, w_i| \geq 33\tau_i(1 + |x_i|/\sqrt{n_i})\} &\leq \mu\{u \in U \mid |U_i u - p| \geq \tau_i\} \\ &\leq e^{-\tau_i^2/4} \mu U. \end{aligned}$$

Finally, when (c) holds, we observe that $x_i \neq 0$ and use (5.6) to infer that $\tau_i \leq n_i/(20|x_i|)$. Hence $|\langle U_i u, x_i\rangle| \leq n_i/5$ implies

$$|U_i u - p| \geq (|\langle p, x_i\rangle| - |\langle U_i u, x_i\rangle|)/|x_i| \geq n_i/(20|x_i|) \geq \tau_i$$

and we obtain (5.11) by estimating

$$\mu\{u \in U \mid |\langle U_i u, x_i\rangle| \leq n_i/5\} \leq \mu\{u \in U \mid |U_i u - p| \geq \tau_i\} \leq e^{-\tau_i^2/4}\mu U. \qquad \blacksquare$$

Step 3. *The inequality* (5.10) *fails for* $i = k$. This follows from (5.8) since (5.6) implies $33\tau_k(1 + |x_k|/\sqrt{n_k}) < \sqrt{n_k}/2$. ■

Step 4. *The inequality* (5.10) *fails for each* $k \leq i \leq l$. By Step 3, if this is not the case, there is $k \leq i < l$ such that (5.10) fails for i but holds for $j := i + 1$, and by Step 1 this implies that (5.11) fails for j.

Let $U_{i,j} := U_i \oplus U_j$, $x_{i,j} := x_i + x_j$ and $A_{i,j} := A_i \circ U_i + A_j \circ U_j$. Similarly to the proof of Step 2, we use Lemma 10 with $H = U$, $V = U_{i,j}$, $Z = \bigoplus_{m \in I\setminus\{i,j\}} U_m$, the given x and Ψ, and the Borel measure ν on Z defined by

$$\nu F = \int_F e^{-|z|^2/2} \prod_{m \in I\setminus\{i,j\}} \langle z, w_m\rangle_+^{n_m}\, d\mathcal{L}^{2s-4}(z).$$

Hence $\Phi(u) = \int_Z \Psi(u+z)\, d\nu(z)$ belongs to $\mathcal{F}(U_{i,j}, A_{i,j}, x_{i,j})$, and by Fubini's Theorem for every Borel set $E \subset U_{i,j}$,

$$\mu(U_{i,j}^{-1}E) = \int_E e^{-|u|^2/2}\Phi(u)\langle u, w_i\rangle_+^{n_i}\langle u, w_j\rangle_+^{n_j}\, d\mathcal{L}^4(u). \tag{5.15}$$

Again, similarly to Step 2 it suffices to make appropriate estimates of the integral in (5.15).

By Lemma 11 the integrand of (5.15) attains its maximum at a point $p = p_i + p_j$, where $p_i \in U_i$ and $p_j \in U_j$, such that $\langle p_i, w_i\rangle > 0$, $\langle p_j, w_j\rangle > 0$ and for some $\lambda \geq 0$,

$$p + \lambda A_{i,j}(p - x_{i,j}) - \frac{n_i w_i}{\langle p, w_i\rangle} - \frac{n_j w_j}{\langle p, w_j\rangle} = 0. \tag{5.16}$$

Notice that (5.16) holds coordinatewise, i.e. for each $\iota = i, j$,

$$p_\iota + \lambda A_\iota(p_\iota - x_\iota) - \frac{n_\iota w_\iota}{\langle p_\iota, w_\iota\rangle} = 0.$$

Hence by Lemma 12 for each $\iota = i, j$ at least one of the following statements holds:

(a) $\big||p_\iota| - \sqrt{n_\iota}\big| > 2\tau_\iota$;
(b) $\lambda\alpha_\iota \leq 2^3\tau_\iota/\sqrt{n_\iota}$ and $|p_\iota - \sqrt{n_\iota}\, w_\iota| \leq 2^5\tau_\iota(1 + |x_\iota|/\sqrt{n_\iota})$;
(c) $\lambda\alpha_\iota \geq 2^{-3}n_\iota/(\sqrt{n_\iota} + |x_\iota|)^2$ and $|\langle p_\iota, x_\iota\rangle| \geq n_\iota/4$.

In a way completely similar to the end of the proof of Step 2 we see that for each $\iota = i, j$, (a) fails, the condition (b) implies (5.10), and (c) implies (5.11). Since (5.10) fails for i, we see that (b) fails and hence (c) holds for $\iota = i$. Since (5.11) fails for j, (c) fails and hence (b) holds for $\iota = j$. Summarizing, (b) holds for $\iota = j$ and (c) for $\iota = i$. Moreover, the validity of (c) for $\iota = i$ implies that $\lambda > 0$, and we get the final contradiction by using (5.6) to estimate

$$\frac{\alpha_j}{\alpha_i} \le \frac{2^3\tau_j}{\sqrt{n_j}} \frac{2^3(\sqrt{n_i} + |x_i|)^2}{n_i} = \frac{64\tau_j}{\sqrt{n_j}}\left(1 + \frac{|x_i|}{\sqrt{n_i}}\right)^2 < \frac{\alpha_j}{\alpha_i}. \qquad ■$$

Step 5. *End of proof.* By Steps 4 and 2, (5.11) holds for each $k \le i \le l$, which is exactly the statement of (iii). □

6. Invalid density theorems

Proposition 14. *Suppose that $\lambda_j > 0$, $j \in \mathbb{N}$, are such that for some $n_i, m_i \in \mathbb{N}$ satisfying $m_{i+1} > m_i + n_i + 1$ and some $\sigma_i \ge 1$ and $\xi_i, \tau_i > 0$,*

(i) $\xi_i \le \lambda_j \le \xi_i\sigma_i$ *whenever* $m_i \le j \le m_i + n_i + 1$;
(ii) $\sigma_i = 1 + O(\sqrt{\xi_i n_i})$;
(iii) $\tau_i = O(\xi_i\sqrt{n_i}\min\{n_i, n_{i-1}\})$;
(iv) $\sum_{i=1}^{\infty} e^{-\tau_i^2} < \infty$.

Then for every Gaussian measure γ in a separable Hilbert space H whose covariance operator has eigenvalues λ_j and for every $\varepsilon > 0$ there is a Borel set $M \subset H$ with $\gamma M > 1 - \varepsilon$ such that

$$\lim_{r\to 0}\sup_{x\in H}\frac{\gamma(M \cap B(x,r))}{\gamma B(x,r)} = 0. \tag{6.1}$$

Proof. Let $C \in (0,\infty)$ be such that $\sigma_i \le 1 + C\sqrt{\xi_i n_i}$ and $\tau_i \le \frac{1}{4}C\xi_i\sqrt{n_i}\min\{n_i, n_{i-1}\}$, and choose $\eta \in (0,1)$ such that

$$18C\eta \le 1 \quad \text{and} \quad 8\cdot 165C\eta^2 \le 1.$$

Recalling that existence of γ implies $\sum_{j=1}^{\infty}\lambda_j < \infty$ and that $\lim_{i\to\infty}\tau_i = \infty$ because of (iv), we find $i_0 \in \mathbb{N}$ such that $m_{i_0} > 1$, $\sum_{j=m_{i_0}}^{\infty}\lambda_j < \eta^2$ and $\tau_i \ge 1$ for $i \ge i_0$. Observing that then $\xi_i n_i < \eta^2$ for $i \ge i_0$ by (i), we shift the parameter i by redefining

$$(n_i, m_i, \sigma_i, \xi_i, \tau_i) \quad \text{as} \quad (n_{i_0+i}, m_{i_0+i}, \sigma_{i_0+i}, \xi_{i_0+i}, 4\tau_{i_0+i}),$$

respectively, to achieve, in addition to (i), also the validity of the following inequalities for each i:

(v) $\xi_i n_i \le \eta^2/16$;
(vi) $\sigma_i \le 1 + C\sqrt{\xi_i n_i}$, so in particular $\sigma_i \le 1 + C\eta$ and $\sigma_i \le 9/8$;
(vii) $\tau_i \le C\xi_i\sqrt{n_i}\min\{n_i, n_{i-1}\}$ if $i \ge 2$, in particular $\tau_i \le \sqrt{n_i}$;
(viii) $\tau_i \ge 4$ and $\sum_{i=1}^{\infty} e^{-\tau_i^2/16} < \infty$.

We recall the notation introduced in Section 2: $H := \{x \in \mathbb{R}^{\mathbb{N}} \mid \sum_{i=1}^{\infty} \lambda_i x_i^2 < \infty\}$ equipped with the norm $\|x\| = (\sum_{i=1}^{\infty} \lambda_i x_i^2)^{1/2}$, γ is the restriction of the countable product of the one-dimensional standard Gaussian measures to H, $|x| = (\sum_{i=1}^{\infty} x_i^2)^{1/2}$, $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ when $|x|, |y| < \infty$, $B(x,r) = \{y \in H \mid \|y - x\| \le r\}$ and $u_j \in H$ are defined by $u_j := (\delta_{i,j})_{i \in \mathbb{N}}$, where $\delta_{i,j} = 1$ when $i = j$ and $\delta_{i,j} = 0$ otherwise. Additionally, for $i \in \mathbb{N}$ we let

$$H_i := \operatorname{span}\{u_j \in H \mid m_i \le j \le m_i + n_i + 1\}.$$

The rest of the proof consists of three steps. In the first step we define the desired set M, in the second we introduce measures μ_w, and finally in the third step we apply Lemma 13 to μ_w for a suitably chosen parameter w.

Step 1. Our plan is to find for each $\varepsilon > 0$ a Borel set $L = L_\varepsilon \subset H$ and $r_0 > 0$ such that $\gamma L > 1 - \varepsilon$ and $\gamma(L \cap B(x,r)) \le \varepsilon \gamma B(x,r)$ for every $x \in H$ and $0 < r < r_0$. Clearly, the set M required in the proposition can then be obtained as $M = \bigcap_{i=1}^{\infty} L_{\varepsilon/2^i}$.

For the rest of the proof we fix $\varepsilon > 0$ and find $k \in \mathbb{N}$, $k \ge 2$, large enough that $\sum_{i=k}^{\infty} e^{-\tau_i^2/8} < \varepsilon$. We let

$$L := \bigcap_{i=k}^{\infty} M_i,$$

where $M_i := \left\{x \in H \;\middle|\; \left||H_i x| - \sqrt{n_i}\right| \le \tau_i\right\}$. Since $\tau_i \ge 1 \ge 1/\sqrt{n_i}$, we see that

$$\tau_i - (\sqrt{n_i + 1} - \sqrt{n_i}) \ge \tau_i - 1/(2\sqrt{n_i}) \ge \tau_i/2.$$

Hence $M_i \supset \left\{x \in H \;\middle|\; \left||H_i x| - \sqrt{n_i + 1}\right| \le \tau_i/2\right\}$. Noticing that $\dim H_i = n_i + 2$, we infer from Corollary 7 with $c = 1/2$, $n = n_i + 2$ and $\lambda = \tau_i/2$ that $\gamma M_i \ge 1 - e^{-\tau_i^2/8}$ and so

$$\gamma L \ge 1 - \sum_{i=k}^{\infty} e^{-\tau_i^2/8} > 1 - \varepsilon.$$

Let $r_0 = \sqrt{\xi_k n_k}/2$ and suppose, for a contradiction, that $\gamma(L \cap B(x,r)) > \varepsilon \gamma B(x,r)$ for some $x \in H$ and $0 < r < r_0$. Fix such x and r and find $\rho > 0$ such that

$$\gamma(L \cap B(x,r)) > \varepsilon \gamma B(x,r) + \rho. \tag{6.2}$$

Choose $l > k$ such that $e^{-\tau_l^2/4}/(1 - e^{-\tau_l^2/4}) < \rho$ and put $I = \{k, k+1, \dots, l\}$ and $J = \mathbb{N} \setminus I$. Let U be the linear span of $\bigcup_{i \in I} H_i$ and Z the $\|\cdot\|$-closed linear span of $\bigcup_{j \in J} H_j$. Also denote $n := \dim U = \sum_{i \in I}(n_i + 2)$, $s := \#I = l - k + 1$ and $q := \sum_{i \in I} n_i$. Lemma 10 applied with $V = U$, x chosen above, $\Psi = \mathbf{1}_{B(x,r)}$, and the standard Gaussian measure ν on Z shows that the function $\Phi(u) = \int_Z \Psi(u + z)\, d\nu(z)$ belongs to $\mathcal{F}(U, \|\cdot\|, Ux)$. Clearly, $\Phi \le 1$, $\Phi(u) = 0$ for $\|u - Ux\| > r$ and by Fubini's Theorem for every Borel set $E \subset U$,

$$\gamma\{y \in B(x,r) \mid Uy \in E\} = (2\pi)^{-n/2} \int_E e^{-|u|^2/2} \Phi(u)\, d\mathcal{L}^n(u). \tag{6.3}$$

Step 2. For $i \in I$ let $x_i := H_i x$. Notice that $M_i \cap B(x_i, r) \neq \emptyset$ since $M_i \cap B(x_i, r) = \emptyset$ would imply $L \cap B(x, r) = \emptyset$, which contradicts (6.2). Choosing $u \in M_i \cap B(x_i, r)$ and using the fact that $\tau_i \le \sqrt{n_i}$ by (vii) and $|u - x_i| \le \|u - x_i\|/\sqrt{\xi_i}$ by (i), we use (v) and $r \le \eta/2$ to get

$$|x_i| \le |u| + |u - x_i| \le \sqrt{n_i} + \tau_i + r/\sqrt{\xi_i} \le 2\sqrt{n_i} + r/\sqrt{\xi_i} \le \eta/\sqrt{\xi_i}. \tag{6.4}$$

Choose now $v_i \in H_i$ with $|v_i| = 1$ such that $x_i = |x_i| v_i$ and put

$$W = \{w \in U \mid \langle w, v_i\rangle = 0 \text{ and } |H_i w| = 1 \text{ for every } i \in I\}.$$

As pointed out by one of the referees, it may help to notice that the set W, being a product of mutually orthogonal spheres $S_i = \{u \in H_i \mid \langle u, v_i\rangle = 0,\ |u| = 1\},\ i \in I$, has a torus structure.

Let $w \in W$, $w = (w_i)_{i \in I}$, $w_i \in H_i$. We denote $U_w := \operatorname{span}\{v_j, w_j \mid j \in I\}$ and define a Borel measure μ_w on U_w by

$$\mu_w F = (2\pi)^{-n/2} \int_F e^{-|u|^2/2} \Phi(u) \prod_{j \in I} \langle u, w_j\rangle_+^{n_j} \, d\mathcal{L}^{2s}(u). \tag{6.5}$$

By (6.3) and iterated application of cylindrical coordinates we obtain, for every Borel set $E \subset U$,

$$\gamma\{y \in B(x, r) \mid Uy \in E\} = \int_W \mu_w(E \cap U_w)\, d\mathcal{H}^q. \tag{6.6}$$

Using this with the orthogonal projection of L on U, so with the set

$$E_1 := UL = \bigcap_{i \in I} \{x \in H \mid \big||H_i x| - \sqrt{n_i}\big| \le \tau_i\},$$

and recalling (6.2), we get

$$\int_W \mu_w(E_1 \cap U_w)\, d\mathcal{H}^q = \gamma\{y \in B(x, r) \mid Uy \in E_1\} \ge \gamma(L \cap B(x, r)) > \varepsilon \gamma B(x, r) + \rho.$$

Since (6.6) with $E_2 := U$ gives

$$\gamma B(x, r) = \int_W \mu_w U_w \, d\mathcal{H}^q,$$

we conclude that

$$\int_W \mu_w(E_1 \cap U_w)\, d\mathcal{H}^q \ge \gamma(L \cap B(x, r)) > \int_W \left(\varepsilon \mu_w U_w + \rho\, (\mathcal{H}^q W)^{-1}\right) d\mathcal{H}^q.$$

So there is $w \in W$ such that

$$\mu_w(E_1 \cap U_w) \ge \varepsilon \mu_w U_w + \rho\, (\mathcal{H}^q W)^{-1}. \tag{6.7}$$

Step 3. We fix such a w and use Lemma 13 to show that this leads to a contradiction. First we define the remaining parameters needed for an application of that lemma. For $i \in I$ we let $(U_i, |\cdot|)$ be the span of $\{v_i, w_i\}$ (so U_w is the space denoted by U in Lemma 13) and choose a positive definite symmetric linear operator A_i on U_i such that $\|u\|^2 = \langle u, A_i u\rangle$ for $u \in U_i$. Using the last inequality from (vi), we see that the eigenvalues $\alpha_i \geq \beta_i$ of A_i satisfy

$$\xi_i\sigma_i \geq \alpha_i \geq \beta_i \geq \xi_i \geq 8\xi_i\sigma_i/9 \geq 8\alpha_i/9 > 0.$$

Since n_i, τ_i and x_i have already been defined and x_i is a multiple of v_i, we just need to verify the remaining inequalities required in the assumptions of Lemma 13. The inequality $\tau_i \geq 4$ is in (viii). Using the estimate of $|x_i|$ from (6.4) and (vi), we get

$$18(\alpha_i - \beta_i)|x_i| \leq 18(\sigma_i - 1)\xi_i\eta/\sqrt{\xi_i} \leq 18C\eta\xi_i\sqrt{n_i} \leq \alpha_i\sqrt{n_i}.$$

Whenever $i \geq 1$ and $j = i$ or $j = i + 1$ we use (6.4), $n_i \leq \eta^2/\xi_i$ (see (v)) and

$$4 \cdot 165C\sigma_i\eta^2 \leq 8 \cdot 165C\eta^2 \leq 1$$

to estimate

$$\begin{aligned}
165\tau_j(1 + |x_i|/\sqrt{n_i})^2 &\leq 2 \cdot 165C\xi_j\sqrt{n_j}\, n_i(1 + \eta^2/(\xi_i n_i))\\
&\leq 2 \cdot 165C\xi_j\sqrt{n_j}\,\eta^2/\xi_i + 2 \cdot 165C\xi_j\sqrt{n_j}\,\eta^2/\xi_i\\
&= 4 \cdot 165C\sigma_i\xi_j\sqrt{n_j}\,\eta^2/(\sigma_i\xi_i) \leq \sqrt{n_j}\,\alpha_j/\alpha_i.
\end{aligned}$$

Hence Lemma 13 may be applied and consequently at least one of the following statements holds:

(a) $\mu_w\{u \in U_w \mid |U_k u - \sqrt{n_k}\, w_k| \geq \sqrt{n_k}/2\} \leq e^{-\tau_k^2/4}\mu_w U_w$.
(b) $\mu_w\big\{u \in U_w \;\big|\; \big||U_i u| - \sqrt{n_i}\big| \leq \tau_i\big\} \leq e^{-\tau_i^2/4}\mu_w U_w$ for some $i \in I$.
(c) $\mu_w\{u \in U_w \mid |\langle U_i u, x_i\rangle| \leq n_i/5\} \leq e^{-\tau_i^2/4}\mu_w U_w$ for each $i \in I$.

We show that each of these possibilities leads to a contradiction.

Using the fact that x_k is a multiple of v_k to infer that $|x_k - \sqrt{n_k}\, w_k| \geq \sqrt{n_k}$, and recalling that $\Phi(u) = 0$ when $\|u - U_w x\| > r$, we see that the support of μ_w is contained in

$$\begin{aligned}
\{u \in U_w \mid \|u - U_w x\| \leq r\} &\subset \{u \in U_w \mid \|U_k u - x_k\| \leq r\}\\
&\subset \{u \in U_w \mid |U_k u - x_k| \leq r/\sqrt{\xi_k}\}\\
&\subset \{u \in U_w \mid |U_k u - x_k| < \sqrt{n_k}/2\}\\
&\subset \{u \in U_w \mid |U_k u - \sqrt{n_k}\, w_k| \geq \sqrt{n_k}/2\},
\end{aligned}$$

which clearly contradicts (a).

If (b) were true, then

$$\mu_w(E_1 \cap U_w) \leq \mu_w\big\{u \in U_w \;\big|\; \big||U_i u| - \sqrt{n_i}\big| \leq \tau_i\big\} \leq e^{-\tau_i^2/4}\mu_w U_w < \varepsilon\mu_w U_w,$$

which contradicts (6.7).

Finally, we show that (c) fails as well. To this end we observe that for any $i \in I$ the standard formulas for the Γ-function and area of the sphere (see, e.g., [5, pp. 250–251]) give

$$\int_{U_i} e^{-|u|^2/2} \langle u, w_i\rangle_+^{n_i}\, d\mathcal{L}^2(u) = \int_{\mathbb{R}} e^{-t^2/2} dt \cdot \int_0^\infty e^{-t^2/2} t^{n_i}\, dt = (2\pi)^{(n_i+2)/2} (\mathcal{H}^{n_i} W_i)^{-1},$$

where $W_i := \{u \in H_i \mid |u| = 1, \langle u, v_i\rangle = 0\}$. We look at $i \neq l$ and use the above equality together with the facts that $\Phi \le 1$ and $n = \sum_{i\in I}(n_i + 2)$ to get

$$\begin{aligned}
&\mu_w\{u \in U_w \mid |\langle U_l u, v_l\rangle| \ge \tau_l\} \\
&\quad\le (2\pi)^{-n/2} \int_{\{u\in U_w \mid |\langle u, v_l\rangle| \ge \tau_l\}} e^{-|u|^2/2} \prod_{i\in I} \langle u, w_i\rangle_+^{n_i}\, d\mathcal{L}^{2s}(u) \\
&\quad= \Big((2\pi)^{(n_l+2)/2} \prod_{i=k}^{l-1} \mathcal{H}^{n_i} W_i\Big)^{-1} \int_{\{u\in U_l \mid |\langle u, v_l\rangle| \ge \tau_l\}} e^{-|u|^2/2} \langle u, w_l\rangle_+^{n_l}\, d\mathcal{L}^2(u).
\end{aligned}$$

The last integrand attains its maximum at $\sqrt{n_l}\, w_l$ and its multiple by $e^{|u|^2/2}$ is log-concave. Hence, Lemma 9 with $n = 2$, $\varphi(u) = e^{-|u|^2/2}\langle u, w_l\rangle_+^{n_l}$, $Q(u) = |u|^2/2$, and $\tau = \tau_l^2/2$ gives

$$\begin{aligned}
&\mu_w\{u \in U_w \mid |\langle U_l u, v_l\rangle| \ge \tau_l\} \\
&\le \Big((2\pi)^{(n_l+2)/2} \prod_{i=k}^{l-1} \mathcal{H}^{n_i} W_i\Big)^{-1} \int_{\{u\in U_l \mid |\langle u, v_l\rangle| \ge \tau_l\}} e^{-|u|^2/2} \langle u, w_l\rangle_+^{n_l}\, d\mathcal{L}^2(u) \\
&\le \Big((2\pi)^{(n_l+2)/2} \prod_{i=k}^{l-1} \mathcal{H}^{n_i} W_i\Big)^{-1} \int_{\{u\in U_l \mid |u - \sqrt{n_l} w_l| \ge \tau_l\}} e^{-|u|^2/2} \langle u, w_l\rangle_+^{n_l}\, d\mathcal{L}^2(u) \\
&\le \Big((2\pi)^{(n_l+2)/2} \prod_{i=k}^{l-1} \mathcal{H}^{n_i} W_i\Big)^{-1} e^{-\tau_l^2/4} \int_{U_l} e^{-|u|^2/2} \langle u, w_l\rangle_+^{n_l}\, d\mathcal{L}^2(u) = e^{-\tau_l^2/4} (\mathcal{H}^q W)^{-1}.
\end{aligned}$$

Since (6.4), (vii) and (v) imply $\tau_l |x_l| \le \eta\tau_l/\sqrt{\xi_l} \le C\eta\sqrt{\xi_l n_l}\, n_l \le C\eta^2 n_l \le n_l/5$, we have

$$\{u \in U_w \mid |\langle U_l u, v_l\rangle| \le \tau_l\} \subset \{u \in U_w \mid |\langle U_l u, x_l\rangle| \le n_l/5\}.$$

Hence, assuming (c) holds,

$$\begin{aligned}
\mu_w U_w &\le \mu_w\{u \in U_w \mid |\langle u, v_l\rangle| \le \tau_l\} + \mu_w\{u \in U_w \mid |\langle u, v_l\rangle| \ge \tau_l\} \\
&\le e^{-\tau_l^2/4} \mu_w U_w + e^{-\tau_l^2/4} (\mathcal{H}^q W)^{-1}.
\end{aligned}$$

Consequently,

$$\mu_w U_w \le \frac{e^{-\tau_l^2/4}}{1 - e^{-\tau_l^2/4}} (\mathcal{H}^q W)^{-1}.$$

Recalling that l satisfies $e^{-\tau_l^2/4}/(1 - e^{-\tau_l^2/4}) < \rho$, this yields $\mu_w U_w \le \rho\, (\mathcal{H}^q W)^{-1}$, which contradicts (6.7) and so finishes the proof. □

7. Proof of Theorem 1

Choose $m \in \mathbb{N}$ and $1 < p < 6/5$ such that $k(\lambda_k/\lambda_{k+1} - 1) < p$ for $k \geq m$. Then for every $k \geq m$,

$$\frac{k^p \lambda_k}{(k+1)^p \lambda_{k+1}} < \frac{1 + p/k}{(1 + 1/k)^p} < 1,$$

and hence the sequence $k^p \lambda_k$ is increasing for $k \geq m$. If necessary, we increase m so that $m \geq 2$ and $(1 + (m^{2-p} + 2)/m)^p < 2$.

We show that the assumptions of Proposition 14 hold with

$$\begin{aligned}
m_i &:= m^i, \\
n_i &:= \lceil m_i^{2-p} \rceil, \text{ i.e., } n_i \in \mathbb{N} \text{ and } m_i^{2-p} \leq n_i < m_i^{2-p} + 1, \\
\sigma_i &:= (1 + (n_i + 1)/m_i)^p, \\
\xi_i &:= \lambda_{m_i}/\sigma_i, \\
\tau_i &:= m_i^{3-5p/2}.
\end{aligned}$$

For that, we observe that our assumptions on m imply $\sigma_i \leq 2$, and we make the following estimates:

- For $m_i \leq j \leq m_i + n_i + 1$, $\lambda_j \leq \lambda_{m_i} = \xi_i \sigma_i$ and $\lambda_j \geq (m_i/j)^p \lambda_{m_i} \geq \xi_i$; hence $\xi_i \leq \lambda_j \leq \xi_i \sigma_i$.
- Clearly, $\sigma_i - 1 = O(n_i/m_i) = O(m_i^{1-p})$. On the other hand recalling that $k^p \lambda_k$ is increasing we obtain $m_i^p \lambda_{m_i} \geq m^p \lambda_m$, hence $\lambda_{m_i} \geq \lambda_m m^p m_i^{-p}$ and

$$\xi_i n_i \geq \frac{\lambda_{m_i}}{2} m_i^{2-p} \geq \frac{\lambda_m m^p}{2} m_i^{2-2p}.$$

  Hence $\sigma_i = 1 + O(\sqrt{\xi_i n_i})$.
- If $i > m$ and $j = i$ or $j = i - 1$, then

$$\xi_i \sqrt{n_i}\, n_j \geq \tfrac{1}{2} \lambda_{m_i} m_i^{1-p/2} m_j^{2-p} \geq \tfrac{1}{2} m^{2p-2} \lambda_m m_i^{3-5p/2},$$

  and hence $\tau_i = O(\xi_i \sqrt{n_i} \min\{n_i, n_{i-1}\})$.
- $\sum_{i=1}^{\infty} e^{-\tau_i^2} < \infty$ since $p < 6/5$.

Hence the statement follows from Proposition 14.

Remark. In addition to those given in Theorem 1, there are many other choices of λ_j satisfying the conditions of Proposition 14. In the introduction we indicated perhaps the simplest way of choosing them which may be realized, for example, by letting $\lambda_j = 32^{-i}$ for $16^{i-1} \leq j \leq 16^i$, with the remaining parameters required by Proposition 14 given by $n_i = m_i = 16^i$, $\sigma_i = 1$, $\xi_i = 32^{-i}$, and $\tau_i = 2^i$.

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