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Relative hard Lefschetz for Soergel bimodules

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Abstract. We prove the relative hard Lefschetz theorem for Soergel bimodules. It follows that the structure constants of the Kazhdan–Lusztig basis are unimodal. We explain why the relative hard Lefschetz theorem implies that the tensor category associated by Lusztig to any two-sided cell in a Coxeter group is rigid and pivotal.

Keywords. Hard Lefschetz, Kazhdan–Lusztig theory, Soergel bimodules

Contents

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1. Introduction

Let (W, S) denote a Coxeter system and $\mathcal{H}$ its Hecke algebra. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with standard basis $\{H_x \mid x \in W\}$ and Kazhdan–Lusztig basis $\{\underline{H}_x \mid x \in W\}$. The Kazhdan–Lusztig positivity conjectures are the statements:

(1) ("positivity of Kazhdan–Lusztig polynomials") if we write $\underline{H}_x = \sum h_{y,x} H_y$, then $h_{y,x} \in \mathbb{Z}_{\geq 0}[v]$;

(2) ("positivity of structure constants") if we write $\underline{H}_x \underline{H}_y = \sum \mu_{x,y}^z \underline{H}_z$ then $\mu_{x,y}^z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

These conjectures have been known since the 1980s for Weyl groups of Kac–Moody groups [KL80, Spr82], using sophisticated geometric technology. More recently in [EW14] the authors proved these conjectures algebraically for arbitrary Coxeter systems by establishing Soergel's conjecture.

Let us briefly recall the setting of Soergel's conjecture. For a certain reflection representation $\mathfrak{h}$ of (W, S) over the real numbers, Soergel constructed a category $\mathcal{B}$ of Soergel bimodules, which is a full subcategory of the category of graded R-bimodules, where R denotes the polynomial functions on $\mathfrak{h}$. The category $\mathcal{B}$ of Soergel bimodules is monoidal under tensor product of bimodules, and is closed under grading shift. Soergel showed that one has a canonical isomorphism

$$\mathrm{ch} : [\mathcal{B}] \xrightarrow{\sim} \mathcal{H}$$

of $\mathbb{Z}[v^{\pm 1}]$-algebras between the split Grothendieck group of Soergel bimodules and the Hecke algebra. (The split Grothendieck group $[\mathcal{B}]$ is an algebra via $[B][B'] := [B \otimes_R B']$ and is a $\mathbb{Z}[v^{\pm 1}]$-algebra via $v[B] := [B(1)]$, where (1) denotes a grading shift.) In proving this isomorphism, Soergel constructed certain bimodules B_x for each $x \in W$ which give representatives for all indecomposable Soergel bimodules (up to isomorphism and grading shift). Soergel's conjecture is the statement that $\mathrm{ch}([B_x]) = \underline{H}_x$, which immediately implies the Kazhdan–Lusztig positivity conjectures. (Property (1) follows because the coefficient of H_y in $\mathrm{ch}([B])$ is given by the graded dimension of a certain hom space. Property (2) follows because $\mu_{x,y}^z$ gives the graded multiplicity of B_z as a summand in $B_x \otimes_R B_y$.)

The geometric techniques used to understand the Kazhdan–Lusztig basis yield another remarkable property of the structure constants $\mu_{x,y}^z$. Using duality, one can show that $\mu_{x,y}^z$ is preserved under swapping v and v^{-1}. The quantum numbers

$$[m] := \frac{v^m - v^{-m}}{v - v^{-1}} = v^{-m+1} + v^{-m+3} + \cdots + v^{m-3} + v^{m-1} \in \mathbb{Z}[v^{\pm 1}]$$

for $m \geq 1$ give a $\mathbb{Z}$-basis for those elements of $\mathbb{Z}[v^{\pm 1}]$ preserved under swapping v and v^{-1}. A folklore conjecture states:[1]

[1] Unimodality is stated as a question in [dC06, §5.1], but experts assure the authors that the conjecture is much older. In [dC06] positivity properties (2) and (3) are checked for W a finite reflection group of H_4 by computer (almost three trillion polynomials $\mu_{x,y}^z$ need to be computed!). For H_4, property (1) had already been checked by Alvis [Alv87] in 1987. In [dC06, §5.2] it is incorrectly stated that the unimodality conjecture is open for Weyl groups.

(3) ("unimodality of structure constants") if we write $\mu_{x,y}^z = \sum_{m \geq 1} a_m [m]$, then $a_m \geq 0$ for all m.

(In other words, each $\mu_{x,y}^z$ is the character of a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-representation.)

In geometric settings unimodality follows from the *relative hard Lefschetz theorem* of [BBD82]. Recall that the relative hard Lefschetz theorem states that if $f : X \to Y$ is a projective morphism of complex algebraic varieties and if η is a relatively ample line bundle on X then for all $i \geq 0$, η induces an isomorphism:

$$\eta^i : {}^p\mathcal{H}^{-i}(Rf_* IC_X) \xrightarrow{\sim} {}^p\mathcal{H}^i(Rf_* IC_X).$$

(Here IC_X denotes the intersection cohomology complex on X and ${}^p\mathcal{H}^i$ denotes perverse cohomology.) In this paper we prove unimodality for all Coxeter groups, by adapting the relative hard Lefschetz theorem to the context of Soergel bimodules.

Inside the category of Soergel bimodules we consider the full subcategory ${}^p\mathcal{B}$ consisting of direct sums of the indecomposable self-dual bimodules B_x without shifts. We call ${}^p\mathcal{B}$ the subcategory of *perverse Soergel bimodules*. Soergel's conjecture implies that each $B \in {}^p\mathcal{B}$ admits a canonical isotypic decomposition

$$B \cong \bigoplus_{x \in W} V_x \otimes_{\mathbb{R}} B_x$$

for certain real vector spaces V_x (in degree 0). If a Soergel bimodule is not perverse, its decomposition into indecomposable summands of the form $B_x(i)$ is not canonical. However, there is a canonical filtration on any Soergel bimodule called the *perverse filtration*, whose i-th subquotient has indecomposable summands of the form $B_x(-i)$ for some $x \in W$. Taking the subquotients of this filtration and shifting them appropriately, one obtains for each i the perverse cohomology functor

$$H^i : \mathcal{B} \to {}^p\mathcal{B}.$$

Any degree d map $B \to B'(d)$ induces a map $H^i(B) \to H^{i+d}(B')$ on perverse cohomology.

Remark 1.1. The category $\mathcal{B}$ is an analogue of semisimple complexes, ${}^p\mathcal{B}$ is an analogue of the category of semisimple perverse sheaves and H^i is an analogue of the perverse cohomology functor.

This main result of this paper is the following:

Theorem 1.2 (Relative hard Lefschetz for Soergel bimodules). *Let $x, y \in W$ be arbitrary and fix $\rho \in \mathfrak{h}^*$ dominant regular (i.e. $\langle \rho, \alpha_s^\vee \rangle > 0$ for all $s \in S$). The map*

$$\eta : B_x \otimes_R B_y \to B_x \otimes_R B_y(2), \qquad b \otimes b' \mapsto b \otimes \rho b' = b\rho \otimes b',$$

induces an isomorphism (for all $i \geq 0$)

$$\eta^i : H^{-i}(B_x \otimes_R B_y) \xrightarrow{\sim} H^i(B_x \otimes_R B_y).$$

Remark 1.3. A stronger version of the above theorem, involving iterated tensor products of indecomposable Soergel bimodules of arbitrary length is still open (see Conjecture 3.5). It is amusing that establishing Conjecture 3.5 for Bott–Samelson bimodules (i.e., when all $x_i \in S$, in the notation of Conjecture 3.5) was the authors' original plan of attack to settle Soergel's conjecture. This remains a very interesting Hodge-theoretic statement that we *cannot* prove!

Remark 1.4. Let us give some geometric context for Theorem 1.2 from the relative hard Lefschetz theorem, when W is the Weyl group of a complex reductive group G, with Borel subgroup $B \subset G$. In this case the indecomposable Soergel bimodule B_x arises as the $B \times B$-equivariant intersection cohomology of a $B \times B$-orbit closure $\overline{BxB} \subset G$ (the inverse image of a Schubert variety in G/B). Here the intersection cohomology of the Schubert variety $\overline{BxB}/B$ is given by

$$B_x \otimes_R \mathbb{R}$$

and the hard Lefschetz theorem and Hodge–Riemann predict remarkable structure on this R-module. Establishing these ("global") properties algebraically is the subject of [EW14].

Now let us turn to the relative setting. The morphism

$$m : G \times_B G \to G$$

induced by multiplication is smooth and proper, with fibre the flag variety G/B. The tensor product $B_x \otimes_R B_y$ is realized as the equivariant cohomology of the derived push-forward via m of the intersection cohomology complex of

$$\overline{BxB} \times_B \overline{ByB} \subset G \times_B G.$$

The relative hard Lefschetz theorem applied to this direct image is equivalent to the statement of Theorem 1.2. (The choice of relatively ample line bundle corresponds to the choice of ρ in Theorem 1.2.)

As was true in our previous work on hard Lefschetz type theorems for Soergel bimodules [EW14, Wil16], the inductive proof we use to establish our main theorem actually requires proving a stronger statement, analogous to the relative Hodge–Riemann bilinear relations [dCM05]. That is, we must calculate the signatures of certain forms on the multiplicity spaces of $H^{-i}(B_x \otimes_R B_y)$ (see Theorem 3.4).

Let us spell out an important combinatorial consequence of Theorem 1.2. As we explained above, for all $x, y \in W$ we have a (non-canonical) isomorphism

$$B_x \otimes B_y \cong \bigoplus V_z \otimes_{\mathbb{R}} B_z$$

for certain finite-dimensional graded vector spaces V_z. It is a consequence of Soergel's categorification theorem that the graded dimension of V_z agrees with the structure constant $\mu_{x,y}^z$ for multiplication in the Kazhdan–Lusztig basis. Theorem 1.2 implies that

η induces[2] a degree 2 operator on V_z which satisfies the hard Lefschetz theorem. In this way we are able to solve the folklore conjecture (3) above on the unimodality of structure constants.

Corollary 1.5. *The structure constants* $\mu_{x,y}^z$ *of multiplication in the Kazhdan–Lusztig basis are unimodal.*

Remark 1.6. In general, the "global" theory (developed in [EW14]) is quite different to the "relative" theory (developed here). However, let us briefly point out one amusing connection. Suppose that W is finite with longest element w_0. One can prove that

$$B_{w_0} \otimes_R B_x \cong \overline{B}_x \otimes_{\mathbb{R}} B_{w_0}.$$

Here $\overline{B}_x = B_x \otimes_R \mathbb{R}$ is the underlying "Soergel module", whose Hodge theory is considered in detail in [EW14]. In this case relative hard Lefschetz (resp. Hodge–Riemann) for $B_{w_0} \otimes_R B_x$ is equivalent to the global hard Lefschetz (resp. Hodge–Riemann) for $\overline{B}_x$. Thus, at least for finite W, the results of the current work are strictly stronger than those of [EW14].

Relative hard Lefschetz for Soergel bimodules also has important consequences for certain tensor categories associated to cells in Coxeter groups. Recall that to any two-sided cell $\mathbf{c} \subset W$ in a finite or affine Weyl group Lusztig has associated a tensor category, which categorifies the J-ring of $\mathbf{c}$. These categories (for finite Weyl groups) are fundamental for the representation theory of finite reductive groups of Lie type: by results of Bezrukavnikov, Finkelberg and Ostrik [BFO12] and Lusztig [Lus15], their (Drinfeld) centres are equivalent to the braided monoidal category of unipotent character sheaves corresponding to $\mathbf{c}$.

Given any two-sided cell $\mathbf{c} \subset W$ in an arbitrary Coxeter group Lusztig has generalized his construction to yield a monoidal category $\mathcal{J}$. (Note that $\mathcal{J}$ is only "locally unital" unless $\mathbf{c}$ contains finitely many left cells, and the existence of a unit relies on a conjecture in general, see Remark 5.2.) In the last section of this paper we explain why Theorem 1.2 implies that $\mathcal{J}$ is rigid and pivotal (see Theorem 5.3). (The rigidity was conjectured by Lusztig [Lus15, §10] when W is finite. Bezrukavnikov, Finkelberg and Ostrik were able to show rigidity for finite and affine Weyl groups [BFO12] via a very different method, which uses the geometric Satake isomorphism and the affine theory in a crucial way. See Remark 5.4 for additional comments.) Establishing the rigidity of $\mathcal{J}$ is an important step towards the study of "unipotent character sheaves" associated to any Coxeter system.

By a theorem of [Müg03, ENO05], rigidity of $\mathcal{J}$ implies that the (Drinfeld) center of $\mathcal{J}$ is a modular tensor category. We expect cells in non-crystallographic Coxeter groups to provide many new examples of modular tensor categories (see [Ost14, 5.4]).

[2] For the purposes of the introduction we are not quite precise here. Really we mean "after passing to the associated graded for the perverse filtration."

2. Background

2.1. Equality and isomorphism

Given objects B and B' we write $B \cong B'$ to mean that B and B' are isomorphic, often without a given isomorphism. We write $B = B'$ to indicate that B and B' are canonically isomorphic.

2.2. Soergel bimodules and duality

Let $\mathfrak{h}$ be an $\mathbb{R}$-linear realization of the Coxeter system (W, S), as in [Soe07, §2]. Thus $\mathfrak{h}$ is a finite-dimensional $\mathbb{R}$-vector space, equipped with linearly independent subsets of *roots* $\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^*$ and *coroots* $\{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$ such that

$$\langle \alpha_s, \alpha_t^\vee \rangle = -2\cos(\pi/m_{st}),$$

where m_{st} denotes the order (possibly ∞) of $st \in W$.[3] We have an action of W on $\mathfrak{h}$ given by the formula

$$s(v) = v - \langle \alpha_s, v \rangle \alpha_s^\vee$$

for all $s \in S$ and $v \in \mathfrak{h}$. The contragredient action of W on $\mathfrak{h}^*$ is defined by an analogous formula,

$$s(f) = f - \langle f, \alpha_s^\vee \rangle \alpha_s$$

for all $s \in S$ and $f \in \mathfrak{h}^*$.

Let R be the ring of polynomial functions on $\mathfrak{h}$, graded so that the linear terms $\mathfrak{h}^*$ have degree 2. Throughout this paper we work in the category of graded R-bimodules, with degree 0 morphisms. Given two such bimodules B and B' we denote by $\mathrm{Hom}(B, B')$ the morphisms in this category.

Our ring R comes equipped with an action of W. Define a graded R-bimodule

$$B_s := R \otimes_{R^s} R(1)$$

for each $s \in S$, where R^s denotes the s-invariant polynomial subring. We use the standard convention for grading shifts, so that the (1) above indicates that the minimal degree element $1 \otimes 1$ lives in degree -1. Given two graded R-bimodules B, B' their tensor product over R is denoted $BB' := B \otimes_R B'$. For a sequence $\underline{w} = (s_1, \ldots, s_d)$ with $s_i \in S$, the tensor product

$$BS(\underline{w}) := B_{s_1} \ldots B_{s_d}$$

is called a *Bott–Samelson bimodule*.

[3] The choice of roots and coroots plays a significant role in this paper, but only up to positive rescaling; what is important (in order that we may cite certain results from [Soe07] and [EW14]) is that our representation is reflection faithful [Soe07] and that there be a well-defined notion of positive roots. If the reader prefers, they may also take the representation given by a realization of a generalized Cartan matrix.

Soergel [Soe07] proved that when $\underline{x}$ is a reduced expression for an element $x \in W$, there is a unique indecomposable direct summand $B_x \overset{\oplus}{\subseteq} BS(\underline{x})$ which is not isomorphic to a summand of a shift of any Bott–Samelson bimodule corresponding to a shorter reduced expression. Moreover, this summand does not depend on the reduced expression of x, up to non-canonical isomorphism. (Using the main theorem of [EW14] one can make this isomorphism canonical.) Note that the two notations for B_s agree.

Let $\mathcal{B}$ denote the full subcategory of graded R-bimodules whose objects are finite direct sums of grading shifts of summands of Bott–Samelson bimodules. The objects in this category $\mathcal{B}$ are known as *Soergel bimodules*, and the bimodules $\{B_x\}_{x \in W}$ give a complete list of non-isomorphic indecomposable objects up to grading shift. Because Bott–Samelson bimodules are closed under tensor product, $\mathcal{B}$ is as well, and inherits its monoidal structure from R-bimodules.

If B is a Soergel bimodule we will often use the symbol B to denote the identity morphism on B. For example, if $f : B' \to B''$ is a morphism then $Bf : BB' \to BB''$ denotes the tensor product of the identity on B with f. Similarly, given $r \in R$ of degree m, rB (resp. Br) denotes the morphism $B \to B(m)$ given by left (resp. right) multiplication by r.

For two Soergel bimodules B and B', recall that $\mathrm{Hom}(B, B')$ denotes the degree 0 homomorphisms of R-bimodules. Write

$$\mathrm{Hom}^\bullet(B, B') := \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(B, B'(m))$$

for the graded vector space of bimodule homomorphisms of all degrees. A morphism $f \in \mathrm{Hom}(B, B'(m))$ is said to be a *degree* m morphism from B to B'. By a theorem of Soergel [Soe07, Theorem 5.15], $\mathrm{Hom}^\bullet(B, B')$ is free of finite rank as a left or right R-module.

Given a Soergel bimodule $B \in \mathcal{B}$ its dual is

$$\mathbb{D}B := \mathrm{Hom}^\bullet_{-R}(B, R)$$

where $\mathrm{Hom}^\bullet_{-R}$ denotes the graded vector space of right R-module homomorphisms of all degrees. We make $\mathbb{D}B$ into an R-bimodule via $r_1 f r_2(b) = f(r_1 b r_2)$. Because $\mathbb{D}BS(\underline{w}) \cong BS(\underline{w})$,[4] the functor $\mathbb{D}$ descends to a contravariant equivalence of $\mathcal{B}$. By the defining property of the indecomposable bimodule B_x, we must also have $\mathbb{D}B_x \cong B_x$. As usual, we have $\mathbb{D}\mathbb{D}B = B$ canonically, for any Soergel bimodule B.

A *pairing* on two Soergel bimodules B, B' is a homogeneous $\mathbb{Z}$-bilinear form

$$\langle -, - \rangle : B \times B' \to R$$

such that $\langle rb, b' \rangle = \langle b, rb' \rangle$ and $\langle br, b' \rangle = \langle b, b'r \rangle = \langle b, b' \rangle r$ for all $b \in B, b' \in B$ and $r \in R$. (Note the asymmetry in the conditions on the left and right R-actions.[5])

4 We even have a canonical isomorphism, as we will discuss later. However, this is not important at this stage.

5 This is the convention used in [EW14]. The opposite convention is used in [Wil16].

The homogeneous condition states that $\deg b + \deg b' = \deg\langle b, b'\rangle$. A pairing induces bimodule morphisms $B \to \mathbb{D}B'$ and $B' \to \mathbb{D}B$. We say that a pairing is *non-degenerate* if one (or equivalently both) of these morphisms is an isomorphism.[6]

A (*non-degenerate*) *form* on a Soergel bimodule is a (non-degenerate) pairing

$$\langle -, - \rangle : B \times B \to R$$

which is in addition symmetric: $\langle b, b'\rangle = \langle b', b\rangle$ for all $b, b' \in B$. A *polarized Soergel bimodule* is a pair $(B, \langle -, -\rangle_B)$ where $B \in \mathcal{B}$ is a Soergel bimodule and $\langle -, -\rangle_B$ is a non-degenerate form, in which case $\langle -, -\rangle_B$ is the *polarization*.

Given a map $f : B \to B'(m)$ between polarized Soergel bimodules its *adjoint* is the unique map $f^* : B' \to B(m)$ such that

$$\langle f(b), b'\rangle_B = \langle b, f^*(b')\rangle_{B'} \quad \text{for all } b \in B,\ b' \in B'.$$

Equivalently $f^* = \mathbb{D}f : \mathbb{D}(B'(m)) \to \mathbb{D}B$, where we use the polarizations to identify $B = \mathbb{D}B$, $B'(-m) = \mathbb{D}(B'(m))$.

2.3. *Perverse cohomology and graded multiplicity spaces*

All morphisms between indecomposable self-dual Soergel bimodules are of non-negative degree, and those of degree 0 are isomorphisms. That is,

$$\mathrm{Hom}(B_x, B_y) = \begin{cases} \mathbb{R} & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

$$\mathrm{Hom}(B_x, B_y(m)) = 0 \quad \text{for } x, y \in W \text{ and } m < 0. \tag{2.2}$$

These fundamental Hom-vanishing statements are equivalent to Soergel's conjecture (see [EW14, the paragraph following Theorem 3.6]).

A Soergel bimodule B is *perverse* if it is isomorphic to a direct sum of indecomposable bimodules B_x without shifts. We denote by ${}^p\mathcal{B}$ the full subcategory of perverse Soergel bimodules. As a consequence of (2.2), any perverse Soergel bimodule admits a canonical decomposition

$$B = \bigoplus_{x \in W} V_x \otimes_{\mathbb{R}} B_x \tag{2.3}$$

for some finite-dimensional real vector spaces V_x. (Concretely, $V_x = \mathrm{Hom}(B_x, B)$.) The rest of this section is dedicated to understanding what replaces this multiplicity space V_x in case the bimodule B in question is not perverse.

By the classification of indecomposable bimodules, every Soergel bimodule splits into a direct sum of shifts of perverse bimodules, but this splitting is not canonical. However, it is a consequence of (2.1) and (2.2) that B admits a unique functorial (non-canonically split) filtration, the *perverse filtration*, whose subquotients are isomorphic to a shift of a perverse Soergel bimodule. Before discussing the details, it is worth illustrating this subtle point in examples.

[6] Warning: this is stronger than the condition $\langle b, B\rangle = 0 \Rightarrow b = 0$.

Example 2.1. The Bott–Samelson bimodule $B_s B_s$ is isomorphic to $B_s(+1) \oplus B_s(-1)$. The degree -1 projection map, that is, the map $B_s B_s \to B_s(-1)$, is canonical up to a scalar. After all, it is easy to confirm from (2.1) and (2.2) that $\mathrm{Hom}^\bullet(B_s B_s, B_s) \cong \mathrm{Hom}^\bullet(B_s(+1) \oplus B_s(-1), B_s)$ is zero in degrees ≤ -2, and is one-dimensional in degree -1. The same can be said about the degree -1 inclusion map, that is, the map $B_s(+1) \to B_s B_s$. However, the degree $+1$ projection map $B_s B_s \to B_s(+1)$ (resp. the degree $+1$ inclusion map $B_s(-1) \to B_s B_s$) is not canonical; adding to it an R-multiple of the degree -1 projection map will give another valid projection map. Said another way, $B_s(+1)$ is a canonical submodule, and $B_s(-1)$ a canonical quotient, and this filtration of $B_s B_s$ splits, but not canonically.

Example 2.2. Suppose that W is of type A_2 with simple reflections $\{s, t\}$. Then $BS(stst)$ is isomorphic to $B_{sts}(-1) \oplus B_{sts}(+1) \oplus B_{st}$. The degree -1 projection map to $B_{sts}(-1)$ is canonical. However, the degree 0 projection map to B_{st} is not canonical! The morphism space $\mathrm{Hom}(BS(stst), B_{st})$ is two-dimensional; it has a one-dimensional subspace arising as the composition of the canonical projection to $B_{sts}(-1)$ followed by a non-split map $B_{sts}(-1) \to B_{st}$, and any morphism not in this one-dimensional subspace will serve as a projection map to B_{st}. This example is meant to loudly proclaim that even what appears to be an "isotypic component," such as the summand B_{st} which is the only one of its kind, is not canonically a direct summand, owing to the presence of other summands with lower degree shifts.

For any $i \in \mathbb{Z}$, define $\mathcal{B}^{\leq i}$ (resp. $\mathcal{B}^{>i}$) to be the full additive subcategory of $\mathcal{B}$ consisting of bimodules which are isomorphic to direct sums of $B_x(m)$ with $m \geq -i$ (resp. $m < -i$). In formulas,

$$\mathcal{B}^{\leq i} := \langle B_x(m) \mid x \in W,\ m \geq -i \rangle_{\oplus,\cong},$$
$$\mathcal{B}^{>i} := \langle B_x(m) \mid x \in W,\ m < -i \rangle_{\oplus,\cong}.$$

Similarly we define $\mathcal{B}^{<i}$ and $\mathcal{B}^{\geq i}$. We have ${}^p\mathcal{B} = \mathcal{B}^{\geq 0} \cap \mathcal{B}^{\leq 0}$. We can rephrase (2.2) as the statement

$$\mathrm{Hom}(\mathcal{B}^{\leq i}, \mathcal{B}^{>i}) = 0. \tag{2.4}$$

We now recall the construction of the perverse filtration, following[7] [EW14, §6.2]. For any Soergel bimodule B we can choose a decomposition

$$B \cong \bigoplus B_x^{\oplus m_{x,i}}(i)$$

into indecomposable bimodules. We define

$$\tau_{\leq j} B := \bigoplus_{i \geq -j} B_x^{\oplus m_{x,i}}(i),$$

[7] In [EW14, §6.2] the perverse filtration is defined on any p-split Soergel bimodule. The main result of [EW14] is that every Soergel bimodule is p-split, so the reader trying to follow along in [EW14, §6.2] can ignore this technicality.

which is a summand of B. Using (2.2) we can deduce that the submodule $\tau_{\leq j} B$ does not depend on the choice of decomposition, because it agrees with the smallest submodule containing the image of any map $B_x(i) \to B$ for $i \geq -j$. Note, however, that its complement

$$\bigoplus_{i<-j} B_x^{\oplus m_{x,i}}(i)$$

depends in general on the choice of decomposition.

We obtain in this way the *perverse filtration* on B

$$\cdots \subset \tau_{\leq i} B \subset \tau_{\leq i+1} B \subset \cdots$$

such that $\tau_{\leq i} B \subset \mathcal{B}^{\leq i}$ and $B/\tau_{\leq i} B \in \mathcal{B}^{>i}$. This filtration always splits, but the splitting is not canonical.

If $f : B \to B'$ is a morphism then $f(\tau_{\leq i} B) \subset \tau_{\leq i} B'$. We have

$$\tau_{\leq i}(B(m)) = (\tau_{\leq i+m} B)(m). \tag{2.5}$$

Dually, we set

$$\tau_{\geq j} B := B/\tau_{\leq j-1}.$$

Every Soergel bimodule has a unique *perverse cofiltration*

$$\cdots \twoheadrightarrow \tau_{\geq i} B \twoheadrightarrow \tau_{\geq i+1} B \twoheadrightarrow \cdots$$

where every arrow is a split surjection, each $\tau_{\geq i} B$ is in $\mathcal{B}^{\geq i}$ and the kernel of $B \twoheadrightarrow \tau_{\geq i} B$ belongs to $B^{<i}$. We have

$$\mathbb{D}(\tau_{\geq i} B) = \tau_{\leq -i}(\mathbb{D} B) \tag{2.6}$$

canonically.

Remark 2.3. Note that $\tau_{\geq j} B$ is canonically a quotient of B, not a submodule. It is not "the same as" the (non-canonical) complement of $\tau_{\leq j-1} B$, though any such complement will map isomorphically to $\tau_{\geq j} B$.

The *perverse cohomology* of a Soergel bimodule B is

$$H^i(B) := (\tau_{\leq i} B/\tau_{\leq i-1} B)(i).$$

(The shift (i) is included so that $H^i(B)$ is perverse.) By the third isomorphism theorem, $H^i(B)(-i)$ is also the kernel of the map $\tau_{\geq i} B \to \tau_{\geq i+1} B$. From this we conclude that

$$\mathbb{D}(H^i(B)) = H^{-i}(\mathbb{D}(B)). \tag{2.7}$$

Applying (2.3) we obtain canonical isotypic decompositions

$$H^i(B) = \bigoplus_{z \in W} H_z^i(B) \otimes_{\mathbb{R}} B_z$$

for certain finite-dimensional vector spaces $H_z^i(B)$. We have a non-canonical isomorphism

$$B \cong \mathrm{gr}\, B := \bigoplus_{i \in \mathbb{Z}} H^i(B)(-i)$$

and canonical isomorphisms

$$\mathrm{gr}\, B = \bigoplus_{i,z} H_z^i(B) \otimes B_z(-i) = \bigoplus_{z} H_z^\bullet(B) \otimes_{\mathbb{R}} B_z$$

where $H_z^\bullet(B)$ denotes the graded vector space $\bigoplus H_z^i(B)(-i)$. (That is, to form $H_z^\bullet(B)$ we place each $H_z^i(B)$ in degree i and take the direct sum over all i.) Below we call the graded vector spaces $H_z^\bullet(B)$ *multiplicity spaces*.

Remark 2.4. To reiterate the point made in Example 2.2: in general, it is not possible to produce separate multiplicity spaces $H_z^\bullet(B)$ for different $z \in W$, without first passing to the associated graded of the perverse filtration.

Let B, B' be Soergel bimodules and $f : B \to B'(m)$ a morphism. Then by (2.4) and (2.5) we have

$$f(\tau_{\le i} B) \subset \tau_{\le i}(B'(m)) = (\tau_{\le i+m} B')(m).$$

Thus f induces a map

$$f : H^i(B) \to H^{i+m}(B')$$

of Soergel bimodules, and hence a degree m map $\mathrm{gr}\, f$ from $\mathrm{gr}\, B$ to $\mathrm{gr}\, B'$. For any $z \in W$ this induces a map

$$\mathrm{gr}_z\, f : H_z^\bullet(B) \to H_z^{\bullet+m}(B')$$

of graded vector spaces. To simplify notation, we use f to denote all these maps: f, $\mathrm{gr}\, f$, $\mathrm{gr}_z\, f$ for all $z \in W$. We refer to the maps $\mathrm{gr}\, f$ and $\mathrm{gr}_z\, f$ as the *maps induced on perverse cohomology*.

The following triviality is important later:

Lemma 2.5. *If $f : B \to B'(m)$ is a map such that, for all $i \in \mathbb{Z}$,*

$$f(\tau_{\le i} B) \subset \tau_{\le i-1}(B'(m))$$

then f induces the zero map on perverse cohomology. In particular, this applies to the map given by left or right multiplication by any positive-degree polynomial in R on a Soergel bimodule B.

Proof. Only the second sentence requires proof. The perverse filtration is a filtration by R-bimodules, so left (resp. right) multiplication by an element of R preserves the submodule $\tau_{\le i} B$ for any i. (An arbitrary endomorphism might not.) If $r \in R$ is homogeneous of degree $d > 0$ then multiplication by r on the left (resp. right) induces a map (see (2.5))

$$\tau_{\le i} B \to (\tau_{\le i} B)(d) = \tau_{\le i-d}(B(d)).$$

Therefore, the hypothesis of the lemma applies to multiplication by r. $\square$

2.4. Polarizations of Soergel bimodules

In [EW14, §3.4, see also Corollary 3.9], the Bott–Samelson bimodule $BS(\underline{w})$ was equipped with a non-degenerate form called the *intersection form*. By restriction, one obtains a form on any summand of a Bott–Samelson bimodule. By [EW14, Lemma 3.7], there is, up to an invertible scalar, a unique non-zero form on an indecomposable Soergel bimodule B_x (this statement is equivalent to Soergel's conjecture), and it is non-degenerate. Letting $\underline{x}$ be any reduced expression for x, the restriction of the intersection form to $B_x \overset{\oplus}{\subseteq} BS(\underline{x})$ is non-zero.[8] Thus this restricted form is non-degenerate, and is a polarization of B_x. For all $x \in W$ we fix a reduced expression $\underline{x}$ of x and an embedding $B_x \subset BS(\underline{x})$, and hence a polarization $\langle -, - \rangle_{B_x}$ on B_x. We refer to $\langle -, - \rangle$ as the *intersection form* on B_x.

The intersection form has the following important positivity property:

Lemma 2.6 ([EW14, Lemma 3.10]). *If $\rho \in \mathfrak{h}^*$ is dominant regular (i.e. $\langle \rho, \alpha_s^\vee \rangle > 0$ for all $s \in S$) and $b \in B_x$ is any non-zero element of degree $-\ell(x)$ then*

$$\langle b, \rho^{\ell(x)} b \rangle > 0.$$[9]

Remark 2.7. This lemma and the discussion of the previous paragraph imply that the intersection form on B_x does not depend on the choice of reduced expression $\underline{x}$ or the choice of embedding $B_x \subset BS(\underline{x})$, up to multiplication by a *positive* scalar.

Given any polarized Soergel bimodule B, it is explained in [EW14, §3.6] how to produce a polarization on BB_s, called the *induced form*. Moreover, if $B = B_x$ is given its intersection form (i.e. the form restricted from our fixed inclusion $B_x \overset{\oplus}{\subseteq} BS(\underline{x})$ for a reduced expression) then the induced form on $B_x B_s$ agrees with the form restricted from the inclusion $B_x B_s \overset{\oplus}{\subseteq} BS(\underline{x}s)$. This is because the intersection form on any Bott–Samelson bimodule $BS(\underline{w})$ is constructed by being repeatedly induced from the canonical form on $BS(\emptyset) = R$. Let us generalize this notion of induced forms.

If B and B' are two polarized Soergel bimodules, we define an *induced form* on BB' by the formula

$$\langle b \otimes b', c \otimes c' \rangle_{BB'} := \langle (\langle b, c \rangle_B) \cdot b', c' \rangle_{B'} = \langle b', (\langle b, c \rangle_B) \cdot c' \rangle_{B'}. \tag{2.8}$$

Lemma 2.8 ([Wil16, §6.4]). *The induced form on BB' is non-degenerate, and thus is a polarization of BB'.*

[8] There are two ways to see this. The first is to note that the intersection form can be reinterpreted as an isomorphism $BS(\underline{x}) \to \mathbb{D}BS(\underline{x})$. By the Krull–Schmidt property, this isomorphism must restrict to an isomorphism $B_x \to \mathbb{D}B_x$, which is another way of saying that the restriction of the form to B_x is non-degenerate. Alternatively, one can use Lemma 2.6 below to see that the restriction to B_x is non-zero, hence non-degenerate.

[9] The reader should not forget that $\langle b, b' \rangle$ is, in general, an element of the ring R. Here, for degree reasons, one obtains a degree 0 element of R, hence an element of $\mathbb{R}$.

The induction of forms is associative: given polarized Soergel bimodules B, B', B'', the induced form on $BB'B''$ is well-defined. Note that the tensor product of two Bott–Samelson bimodules $BS(\underline{w}) \otimes BS(\underline{x})$ is a Bott–Samelson bimodule for the concatenation $BS(\underline{wx})$. If one induces the intersection forms on $BS(\underline{w})$ and $BS(\underline{x})$, one again obtains the intersection form on $BS(\underline{wx})$. This follows from associativity, once one confirms that the earlier definition of the induced form on BB_s matches with (2.8). Consequently, if $B \overset{\oplus}{\subseteq} BS(\underline{w})$ and $B' \overset{\oplus}{\subseteq} BS(\underline{x})$, then the induced form on BB' agrees with the restriction of the intersection form on $BS(\underline{wx})$. All tensor products of the form $B_{x_1} \dots B_{x_m}$ are always assumed to be polarized with respect to their intersection form.

Remark 2.9. Here is some additional motivation for (2.8), omitting many details which can be found in [EW14, §3.4]. The intersection form on a Bott–Samelson bimodule B can be defined in terms of a trace map $\mathrm{Tr}_B : B \to R$. There is a particular homogeneous basis of B as a right R-module, with a special basis element c_{top} of top degree, and the trace of an element $b \in B$ is the coefficient of c_{top} in the expansion of b. The basis of BB' is the tensor product of the respective bases, with top degree element $c_{\mathrm{top}} \otimes c_{\mathrm{top}}$. However, the right action of R on B does not match the right action of R on BB'; instead it is the "middle action" of R, or the left action of R on B'. If $\mathrm{Tr}_B(b) = r$, then the trace of $b \otimes b' \in BB'$ agrees with the trace of $c_{\mathrm{top}} \cdot r \otimes b' = c_{\mathrm{top}} \otimes r \cdot b'$, which agrees with the trace of $r \cdot b' \in B'$. In other words,

$$\mathrm{Tr}_{BB'}(b \otimes b') = \mathrm{Tr}_{B'}(\mathrm{Tr}_B(b) \cdot b').$$

This formula directly implies (2.8).

Let $(B, \langle -, - \rangle_B)$ be a polarized Soergel bimodule. If B is also perverse then by considering the isotypic decomposition (see (2.3))

$$B = \bigoplus_{x \in W} V_x \otimes_{\mathbb{R}} B_x$$

and the associated map $B \to \mathbb{D}B$, we see that $\langle -, - \rangle_B$ is orthogonal for this decomposition. Moreover, $\langle -, - \rangle_B$ is determined by symmetric forms $\langle -, - \rangle_{V_x}$ on each vector space V_x (i.e. if $v, v' \in V_x$ and $b, b' \in B_x$ then $\langle v \otimes b, v' \otimes b' \rangle = \langle v, v' \rangle_{V_x} \langle b, b' \rangle_{B_x}$). We say that B is *positively polarized* if $B = 0$ or the following conditions are satisfied:

(1) B is perverse and vanishes in even or odd degree (because B_x is non-zero in degree $-\ell(x)$, the second condition is equivalent to the existence of $q \in \{0, 1\}$ such that $V_x = 0$ for all x with $\ell(x)$ of the same parity as q).
(2) Let $z \in W$ denote an element of maximal length in W such that $V_z \neq 0$. If $V_y \neq 0$ then $\langle -, - \rangle_{V_y}$ is $(-1)^{(\ell(z)-\ell(y))/2}$ times a positive definite form, for all $y \in W$.

The canonical example of a positively polarized Soergel bimodule is given by the following lemma:

Lemma 2.10 ([EW14, Proposition 6.12]). *Suppose that $y \in W$ and $s \in S$ with $ys > y$ (resp. $sy > y$). Then $B_y B_s$ (resp. $B_s B_y$), equipped with its interesection form, is positively polarized.*

2.5. *Forms on multiplicity spaces*

Assume that $(B, \langle -, - \rangle)$ is a polarized Soergel bimodule. If we interpret $\langle -, - \rangle$ instead as an isomorphism

$$f : B \xrightarrow{\sim} \mathbb{D}(B)$$

then we deduce from the functoriality of the perverse filtration that

$$f(\tau_{\leq i} B) \subset \tau_{\leq i}(\mathbb{D}B) \overset{(2.6)}{=} \mathbb{D}(\tau_{\geq -i} B), \tag{2.9}$$

$$f \text{ induces an isomorphism } H^i(B) \xrightarrow{\sim} H^i(\mathbb{D}B) \overset{(2.7)}{=} \mathbb{D}H^{-i}(B). \tag{2.10}$$

Since $\tau_{\geq -i} B$ is the quotient of B by $\tau_{<-i} B$, the usual duality between subspaces and quotients implies that $\mathbb{D}(\tau_{\geq -i} B)$ is the space orthogonal to $\tau_{<-i} B$. Thus statement (2.9) is equivalent to saying that $\langle \tau_{\leq i} B, \tau_{<-i} B \rangle = 0$, which implies that $\langle -, - \rangle$ induces a pairing of Soergel bimodules

$$\langle -, - \rangle : H^i(B) \times H^{-i}(B) \to R. \tag{2.11}$$

Statement (2.10) tells us that this pairing is non-degenerate. By (2.1) the canonical decompositions

$$H^i(B) = \bigoplus_{z \in W} H^i_z(B) \otimes_{\mathbb{R}} B_z \quad \text{and} \quad H^{-i}(B) = \bigoplus_{z \in W} H^{-i}_z(B) \otimes_{\mathbb{R}} B_z$$

are orthogonal with respect to $\langle -, - \rangle$ (i.e. $\langle \gamma \otimes b, \gamma' \otimes b' \rangle = 0$ for $\gamma \otimes b \in H^i_z(B) \otimes_{\mathbb{R}} B_z$ and $\gamma' \otimes b' \in H^{-i}_{z'}(B) \otimes_{\mathbb{R}} B_{z'}$ if $z \neq z'$). Applying (2.1) again we conclude that (2.11) is completely determined by the non-degenerate bilinear pairing on the vector spaces

$$H^i_z(B) \times H^{-i}_z(B) \to \mathbb{R} \tag{2.12}$$

for all $z \in W$. To be precise, given $v \in H^i_z(B)$ and $v' \in H^{-i}_z(B)$, this pairing (2.12) is defined so that, for all $b, b' \in B_z$, one has

$$\langle v \otimes b, v' \otimes b' \rangle = \langle v, v' \rangle \langle b, b' \rangle. \tag{2.13}$$

The left hand side is (a summand of) the pairing in (2.11) between $H^i(B)$ and $H^{-i}(B)$, and the right hand side is the pairing in (2.12) multiplied by the intersection form on B_z.

Reassembling this data, we conclude that $\langle -, - \rangle$ descends to a symmetric non-degenerate form

$$\langle -, - \rangle : \operatorname{gr} B \times \operatorname{gr} B \to R.$$

and that this form is determined by the symmetric non-degenerate graded bilinear forms

$$\langle -, - \rangle : H^\bullet_z(B) \times H^\bullet_z(B) \to \mathbb{R}$$

on multiplicity spaces for all $z \in W$.

Here is another important triviality:

Lemma 2.11. *Let $B = BS(\underline{x})$ be a Bott–Samelson bimodule associated to a reduced expression $\underline{x}$ for an element $x \in W$, polarized with respect to its intersecton form. The summand B_x appears with multiplicity* 1 *having no grading shift, so that $H_x^\bullet(B) = \mathbb{R}$ in degree* 0*. Up to a positive scalar, the form $H_x^0(B) \times H_x^0(B) \to \mathbb{R}$ is just the standard form, with $\langle 1, 1\rangle = 1$.*

Proof. This follows immediately from (2.13), because the intersection form on $BS(\underline{x})$ restricts to a positive multiple of the intersection form on B_x (see Remark 2.7). □

3. Relative hard Lefschetz and Hodge–Riemann

3.1. Statement

We fix once and for all a dominant regular $\rho \in \mathfrak{h}^*$, that is, an element such that $\langle \rho, \alpha_s^\vee\rangle \geq 0$ for all $s \in S$.

Let $\mathbf{x} := (x_1, \dots, x_m)$ be a sequence of elements in W, and fix scalars $\mathbf{a} := (a_1, \dots, a_{m-1}) \in \mathbb{R}^{m-1}$. Consider the operator

$$L_{\mathbf{a}} : B_{x_1} B_{x_2} \dots B_{x_m} \to B_{x_1} B_{x_2} \dots B_{x_m}(2),$$
$$L_{\mathbf{a}} = a_1 B_{x_1} \rho B_{x_2} \dots B_{x_m} + a_2 B_{x_1} B_{x_2} \rho \dots B_{x_m} + \dots + a_{m-1} B_{x_1} B_{x_2} \dots \rho B_{x_m}.$$

In words, $L_{\mathbf{a}}$ is the sum of the operators of multiplication by $a_i \rho$ in the gap between B_{x_i} and $B_{x_{i+1}}$.

We have explained that to any $z \in W$ we may associate a graded vector space

$$V^\bullet := H_z^\bullet(B_{x_1} \dots B_{x_m})$$

equipped with

(1) a symmetric graded non-degenerate form $\langle -, -\rangle_{V^\bullet}$ obtained from the intersection form on $B_{x_1} \dots B_{x_m}$;
(2) a degree 2 Lefschetz operator $L_{\mathbf{a}} : V^\bullet \to V^{\bullet+2}$ obtained by taking perverse cohomology of $L_{\mathbf{a}}$.

Remark 3.1. The operator $L_{\mathbf{a}}$ involves only internal multiplication by polynomials. One could also consider the Lefschetz operator $L_{\mathbf{a}} + a_0\rho \cdot (-) + a_m(-) \cdot \rho$ which includes multiplication on the left and right. However, as observed in Lemma 2.5, left and right multiplication by polynomials act trivially on perverse cohomology, so this does not affect the degree 2 operator on $V^\bullet$.

We say that $L_{\mathbf{a}}$ *satisfies relative hard Lefschetz* if for any $d \geq 0$, $L_{\mathbf{a}}$ induces an isomorphism

$$L_{\mathbf{a}}^d : V^{-d} \xrightarrow{\sim} V^d.$$

We say that $L_{\mathbf{a}}$ *satisfies relative Hodge–Riemann* if $L_{\mathbf{a}}$ satisfies relative hard Lefschetz and the restriction of the Lefschetz form $(v, v') := \langle v, L_{\mathbf{a}}^d v'\rangle_{V^\bullet}$ on V^{-d} to

$$P^{-d} := \ker L_{\mathbf{a}}^{d+1} : V^{-d} \to V^{d+2}$$

is $(-1)^{\varepsilon(\mathbf{x},z,d)}$-definite for all $d \geq 0$, where

$$\varepsilon(\mathbf{x}, z, d) := \frac{1}{2}\Big(\sum_{i=1}^{m} \ell(x_i) - \ell(z) - d\Big).$$

Note that relative hard Lefschetz and relative Hodge–Riemann are both statements about $H_z^\bullet$ which are required to hold for all $z \in W$.

Remark 3.2. The sign $(-1)^{\varepsilon(\mathbf{x},z,d)}$ might appear mysterious. The following is a useful mnemonic. Set $B := B_{x_1} \dots B_{x_m}$ and consider the finite-dimensional graded vector space

$$\overline{B} := B \otimes_R \mathbb{R}.$$

We have a non-canonical isomomorphism

$$\overline{B} \cong \bigoplus_{z \in W} H_z^\bullet(B) \otimes \overline{B}_z.$$

Now $\varepsilon(\mathbf{x}, z, d)$ has the following meaning: it is half the difference between the smallest non-zero degree in $H_z^{-d}(B) \otimes_{\mathbb{R}} \overline{B}_z^{-\ell(z)}$ on the right hand side (i.e. $-\ell(z) - d$) and the smallest non-zero degree in $\overline{B}$ (i.e. $-\sum \ell(x_i)$). In this way one may see that the above definition is compatible with the signs predicted by Hodge theory in the geometric setting (see [dCM05, Theorem 2.1.8] and [Wil17, Theorem 3.12], where the signs are made explicit).

Remark 3.3. One important reason to fix the signs globally is that it makes the statement of relative Hodge–Riemann compatible under direct sums. Let $\mathbf{x} := (x_1, \dots, x_m)$ and $\mathbf{x}' := (x_1', \dots, x_m')$ be two sequences in W of the same length m, and let $B = B_{x_1} \dots B_{x_m}$ and $B' = B_{x_1'} \dots B_{x_m'}$. Define $V^\bullet$ and $(V')^\bullet$ as above, for some $z \in W$, and equip them with their respective Lefschetz operators $L_{\mathbf{a}}$ for the same sequence $\mathbf{a}$. Clearly then $L_{\mathbf{a}}$ acting on the direct sum $(V \oplus V')^\bullet$ will satisfy hard Lefschetz. As long as $\sum \ell(x_i)$ and $\sum \ell(x_i')$ have the same parity, we have $\varepsilon(\mathbf{x}, z, d) = \varepsilon(\mathbf{x}', z, d)$ for all $d \geq 0$, and the signs on the primitive subspaces of $V^\bullet$ and $(V')^\bullet$ will agree. Thus $L_{\mathbf{a}}$ acting on $(V \oplus V')^\bullet$ will satisfy the Hodge–Riemann bilinear relations, i.e. the Lefschetz form on primitives is definite with the expected sign.

For $x_1, \dots, x_m \in W$ as above we introduce the following abbreviations:

RHL$(x_1, \dots, x_m)$: $L_{\mathbf{a}}$ satisfies relative hard Lefschetz
for all $\mathbf{a} := (a_1, \dots, a_{m-1}) \in \mathbb{R}_{>0}^{m-1}$;

RHR$(x_1, \dots, x_m)$: $L_{\mathbf{a}}$ satisfies relative Hodge–Riemann
for all $\mathbf{a} := (a_1, \dots, a_{m-1}) \in \mathbb{R}_{>0}^{m-1}$.

As always, it is implicitly assumed in these statements that all tensor products of the form $B_{x_1} \dots B_{x_m}$ are equipped with their intersection form.

The main theorem of this paper is:

Theorem 3.4. *For any $x, y \in W$, RHR(x, y) holds.*

3.2. A conjecture

Conjecture 3.5. *For any $x_1, \dots, x_m \in W$, $RHR(x_1, \dots, x_m)$ holds.*

More generally, relative Hodge–Riemann should hold for any operator of the form

$$B_{x_1}\rho_1 B_{x_2} \dots B_{x_m} + B_{x_2} B_{x_2}\rho_2 \dots B_{x_m} + \dots + B_{x_1} B_{x_2} \dots \rho_{m-1} B_{x_m},$$

where $\rho_1, \dots, \rho_{m-1}$ is any sequence of dominant regular elements. (Such elements span the cone of relatively ample classes in the Weyl group case.) For the conjecture above, one sets $\rho_i = a_i \rho$.

3.3. Base cases

Lemma 3.6. *$RHL(x)$ and $RHR(x)$ hold, for any $x \in W$.*

Proof. The only non-vanishing $H_z^\bullet(B_x)$ occurs when $z = x$, and this multiplicity space is concentrated in degree 0. Thus $RHL(x)$ is trivial, and $RHR(x)$ is equivalent to the statement that the form $H_x^0(B_x) \times H_x^0(B_x) \to \mathbb{R}$ is positive definite, which holds by Lemma 2.11. □

Lemma 3.7. *If $RHL(x_1, \dots, x_m)$ holds, then so does $RHL(x_1, \dots, x_m, \mathrm{id})$ and $RHL(\mathrm{id}, x_1, \dots, x_m)$. The same statement can be made for RHR.*

Proof. Let us compare $RHL(x_1, \dots, x_m)$ and $RHL(x_1, \dots, x_m, \mathrm{id})$. Because $B = B_{x_1} \dots B_{x_m}$ and $B' = B_{x_1} \dots B_{x_m} B_{\mathrm{id}}$ are canonically isomorphic, the multiplicity spaces $H_z^\bullet(B) \cong H_z^\bullet(B')$ being studied are the same. If $\mathbf{a} = (a_1, \dots, a_{m-1})$ and $\mathbf{a}' = (a_1, \dots, a_{m-1}, a_m)$, then under the isomorphism $B \cong B'$ we have $L_{\mathbf{a}'} = L_{\mathbf{a}} + a_m B\rho$. That is, the difference between the Lefschetz operators in question is right multiplication by $a_m \rho$, which acts trivially on perverse cohomology (see Lemma 2.5 and Remark 3.1). Thus the induced Lefschetz operators on $H_z^\bullet(B)$ and $H_z^\bullet(B')$ are the same. □

To warm up, we consider the first interesting case: $x_1 = x$ and $x_2 = s$ for $s \in S$. This splits into two subcases: $xs < x$ and $xs > x$. Suppose that $xs > x$. Then $B_x B_s$ is perverse, and so each $H_z^\bullet(B_x B_s)$ is concentrated in degree 0 and $RHL(x, s)$ holds automatically. In this case $RHR(x, s)$ is equivalent to Lemma 2.10.

Suppose now that $xs < x$. Then $B_x B_s \cong B_x(+1) \oplus B_x(-1)$. The action of $B_x \rho B_s$ on the multiplicity spaces $H_x^\bullet(B_x B_s)$ is independent of x (see Lemma 4.15 below), and can be computed when $x = s$, where it is a simple exercise. (We have been brief here because this computation, expanded upon and in further generality, comprises the bulk of §4.2.)

3.4. Structure of the proof

Let us outline the major steps in the proof of Theorem 3.4, which will be carried out in the rest of this paper. The proof is by induction on $\ell(x) + \ell(y)$ and then on $\ell(y)$. More precisely, for integers M and N, consider the statements:

$X_{M,N}$: $RHR(x', y')$ holds whenever either
(1) $\ell(x') + \ell(y') < M$, or
(2) $\ell(x') + \ell(y') = M$ and $\ell(y') \leq N$;

$Y_{M,N}$: $RHR(x', s, y')$ holds, for all $s \in S$, whenever either
(1) $\ell(x') + \ell(y') + 1 < M$, or
(2) $\ell(x') + \ell(y') + 1 = M$ and $\ell(y') \leq N$.

(So M always bounds the length of the sequence, and N bounds the length of the final factor.) Let us write $X_{M,-}$ for the statement that $X_{M,N}$ holds for all $N \geq 0$, and similarly for $Y_{M,-}$.

Certain implications are obvious. For example, $X_{M,M}$ implies $X_{M,-}$. Also, $X_{M,0}$ is equivalent to $X_{M-1,-}$, because the only element with $\ell(y') = 0$ is $y' = \mathrm{id}$ (see Lemmas 3.6 and 3.7). Similarly, $Y_{M,M-1}$ implies $Y_{M,-}$, and $Y_{M,0}$ is equivalent to $Y_{M-1,-}$ together with $X_{M,1}$.

Suppose we knew $X_{M-1,-}$ and $Y_{M-1,-}$. Then we can deduce $X_{M,0}$. The warm-up case sketched above, where $y = s$, will prove that $X_{M,0}$ implies $X_{M,1}$. Then $X_{M,1}$ together with $Y_{M-1,-}$ imply $Y_{M,0}$. To continue the induction, let us fix $M > N \geq 1$. Our goal is to show that $X_{M,N}$ and $Y_{M,N-1}$ together imply $X_{M,N+1}$ and $Y_{M,N}$. From this we inductively deduce $X_{M,M}$ and $Y_{M,M-1}$, which yields $X_{M,-}$ and $Y_{M,-}$.

To reiterate, going forth let us fix $M > N \geq 1$. We now outline why $X_{M,N}$ and $Y_{M,N-1}$ together imply $X_{M,N+1}$ and $Y_{M,N}$.

Let $x, y \in W$ be such that $\ell(x) + \ell(y) = M$ and $\ell(y) = N + 1$. By a weak Lefschetz style argument (Proposition 4.7)

$$RHR({<}x, y) + RHR(x, {<}y) \Rightarrow RHL(x, y). \tag{3.1}$$

Let us fix $s \in S$ with $sy < y$ and set $\dot{y} := sy$. Again weak Lefschetz style arguments yield (Proposition 4.9)

$$RHR({<}x, s, \dot{y}) + RHR(x, s, {<}\dot{y}) \Rightarrow RHL(x, s, \dot{y}). \tag{3.2}$$

We now distinguish two cases. If $xs > x$ then an easy limit argument (Proposition 4.11) gives

$$RHR({\leq}xs, \dot{y}) + RHL(x, s, \dot{y}) \Rightarrow RHR(x, s, \dot{y}). \tag{3.3}$$

If $xs < x$ then a more complicated limit argument (Proposition 4.13) allows us to reach essentially the same conclusion:

$$RHR(x, \dot{y}) + RHL(x, s, \dot{y}) \Rightarrow RHR(x, s, \dot{y}). \tag{3.4}$$

Another limit argument (Proposition 4.12) yields

$$RHR(x, s, \dot{y}) + RHL(x, {\leq}y) \Rightarrow RHR(x, y). \tag{3.5}$$

Thus assuming $X_{M,N}$ and $Y_{M,N-1}$ we have concluded that $X_{M,N+1}$ holds.

Finally, if $x, y \in W$ and $t \in S$ is such that $\ell(x) + \ell(y) + 1 = M$ and $\ell(y) = N$ then as in (3.2) we deduce

$$RHR({<}x, t, y) + RHR(x, t, {<}y) \Rightarrow RHL(x, t, y). \tag{3.6}$$

If $xt < x$ then we have

$$RHR(x, y) + RHL(x, t, y) \Rightarrow RHR(x, t, y). \tag{3.7}$$

If $xt > x$ then

$$RHR({\le}xt, y) + RHL(x, t, y) \Rightarrow HR(x, t, y).$$

Thus assuming $X_{M,N}$ and $Y_{M,N-1}$ we have deduced that $Y_{M,N}$ holds.

Putting these two steps together we deduce

$$X_{M,N} + Y_{M,N-1} \Rightarrow X_{M,N+1} + Y_{M,N}.$$

We conclude by induction that $X_{M,-}$, $Y_{M,-}$ hold for all M. This reduces the proof of the theorem to the propositions listed above.

4. The proof

4.1. Hodge–Riemann implies hard Lefschetz

In [EW14] it was observed that homological algebra in the homotopy category of Soergel bimodules can be used to imitate the weak Lefschetz theorem. This is the key step to deduce the hard Lefschetz theorem by induction. In this section we show that the same idea is useful for studying relative hard Lefschetz.

Recall that $\mathcal{B}$ denotes the category of Soergel bimodules. Let

$$K := K^b(\mathcal{B})$$

denote its bounded homotopy category. As in [EW14, §6.1] we denote the cohomological degree of an object by an upper left index, so as not to get confused with the grading. Thus, an object in K is a complex

$$\cdots \to {}^iF \to {}^{i+1}F \to \cdots$$

with each ${}^iF$ in $\mathcal{B}$. We denote by $(K^{\le 0}, K^{\ge 0})$ the perverse t-structure on K (see [EW14, §6.3]).

Lemma 4.1. *Let $F = (0 \to {}^0F \stackrel{d_0}{\to} {}^1F \to \cdots)$ be a complex supported in non-negative homological degrees, and suppose that $F \in K^{\ge 0}$. Then the induced map*

$$d_0 : H^i({}^0F) \to H^i({}^1F)$$

is split injective for all $i < 0$.

Proof. Because $F \in K^{\geq 0}$, by definition we can find an isomorphism of complexes

$$F \cong F_p \oplus F_c$$

with F_c contractible and F_p such that $H^i({}^j F_p) = 0$ if $i < -j$. Only the summand F_c contributes to $H^i({}^0 F)$ for $i < 0$, but the first differential in a contractible complex is a split injection. □

Given any $x \in W$ we denote by

$$F_x = (\cdots \to {}^{-1}F_x = 0 \to {}^0 F_x = B_x \to {}^1 F_x \to \cdots)$$

a fixed choice of minimal complex for the Rouquier complex (unique up to isomorphism; see [EW14, §6.4]). The following lemma shows that tensor product with F_x is left t-exact.

Lemma 4.2. *For any $x \in W$, $(K^{\geq 0})F_x \subset K^{\geq 0}$ and $F_x(K^{\geq 0}) \subset K^{\geq 0}$.*

Proof. Because F_x is a tensor product of various F_s, $s \in S$, it is enough to prove the lemma for $x = s$. That $(-) \otimes F_s$ preserves $K^{\geq 0}$ is proven in [EW14, Lemma 6.6]; the proof deduces the general statement from [EW14, Lemma 6.5], which states that $B_x F_s \in K^{\geq 0}$ for all $x \in W$ and $s \in S$. The same proof shows that $F_s B_x \in K^{\geq 0}$, and consequently that $F_s \otimes (-)$ preserves $K^{\geq 0}$. □

The following proposition is fundamental for what follows. (In rough form it appears first in [EW14] as Theorem 6.9, Lemma 6.15 and Theorem 6.21.)

Proposition 4.3. *For any x there exists a map $d_x : B_x \to F(1)$ between positively polarized Soergel bimodules such that*

(1) *all summands of F are isomorphic to B_z with $z < x$, such that $\ell(z)$ and $\ell(x) - 1$ have the same parity;*
(2) *d_x is isomorphic to the first differential on a Rouquier complex;*
(3) *if $d_x^* : F \to B_x(1)$ denotes the adjoint of d, then*

$$d_x^* \circ d_x = B_x\rho - (x\rho)B_x.$$

Proof. Except for part (2) and the parity assumption in part (1), this proposition is [Wil16, Proposition 7.14]. However, the reader may easily check that the inductive proof of [Wil16, Proposition 7.14] goes through if one adds these assumptions to the induction. (Indeed, the proof mimics tensoring with a complex isomorphic to the Rouquier complex F_s to carry out the induction.) □

Exchanging left and right actions gives

Proposition 4.4. *For any y there exists a map $d_y : B_y \to G(1)$ between positively polarized Soergel bimodules such that*

(1) *all summands of G are isomorphic to B_z with $z < y$ such that $\ell(z)$ and $\ell(y) - 1$ have the same parity;*
(2) *d_y is isomorphic to the first differential on a Rouquier complex;*
(3) *if $d_y^* : G \to B_y(1)$ denotes the adjoint of d, then*

$$d_y^* \circ d_y = \rho B_y - B_y(y^{-1}\rho).$$

Putting these three statements together gives

Proposition 4.5. *Consider the map*

$$f := \begin{pmatrix} d_x B_y \\ B_x d_y \end{pmatrix} : B_x B_y \to E(1) := F B_y(1) \oplus B_x G(1).$$

Here, d_x and F are as in Proposition 4.3, and d_y and G are as in Proposition 4.4. Then

(1) *the induced map $f : H^i(B_x B_y) \to H^{i+1}(E)$ is split injective for $i < 0$;*
(2) *if $f^* : E \to B_x B_y(1)$ denotes the adjoint of f then*

$$f^* \circ f = B_x(2\rho)B_y - x(\rho)B_x B_y - B_x B_y(y^{-1}\rho).$$

Proof. The first claim follows by noticing that f is isomorphic to the first differential on a Rouquier complex representing

$$F_x F_y \cong (B_x \to F(1) \to \cdots)(B_y \to G(1) \to \cdots).$$

Because $F_x F_y \in K^{\geq 0}$ by Lemma 4.2, the first claim follows from Lemma 4.1.

The adjoint of f is given by the matrix $(d_x^* B_y \; B_x d_y^*)$ and hence

$$f^* \circ f = (d_x^* \circ d_x)B_y + B_x(d_y^* \circ d_y) = B_x(2\rho)B_y - x(\rho)B_x B_y - B_x B_y(y^{-1}\rho),$$

which is the second claim in the lemma. □

Similarly we have

Proposition 4.6. *Fix $a, b > 0$ and consider the map*

$$g_{a,b} := \begin{pmatrix} \sqrt{a} \cdot d_x B_s B_y \\ \sqrt{b} \cdot B_x B_s d_y \end{pmatrix} : B_x B_s B_y \to E(1) := F B_s B_y(1) \oplus B_x B_s G(1).$$

Then

(1) *the induced map $g_{a,b} : H^i(B_x B_s B_y) \to H^{i+1}(E)$ is split injective for $i < 0$;*
(2) *if $g_{a,b}^* : E \to B_x B_y(1)$ denotes the adjoint of $g_{a,b}$ then*

$$g_{a,b}^* \circ g_{a,b} = aB_x(\rho)B_s B_y + bB_x B_s(\rho)B_y - a(x\rho)B_x B_x B_y - bB_x B_y(y^{-1}\rho).$$

Proof. The argument for (2) is the same as for the previous proposition.

It remains to show (1). Note that $g_{a,b}$ is the first differential on a complex representing

$$F_x B_s F_y \cong (B_x \to E(1) \to \cdots)B_s(B_y \to F(1) \to \cdots),$$

and so $F_x B_s F_y \in K^{\geq 0}$ by Lemma 4.2. Now (1) follows from Lemma 4.1. □

The following two propositions explain the title of this section.

Proposition 4.7. *Fix $x, y \in W$ and suppose $RHR(x', y)$ and $RHR(x, y')$ hold for all $x' < x$, $y' < y$. Then $RHL(x, y)$ holds.*

Remark 4.8. This proposition is an instance of the philosophy that HR in dimension $\leq n-1$ implies HL in dimension n.

Proof. Let us keep the notation in the statement of Proposition 4.5. We assume that $B_x B_y$ is standardly polarized (see Lemma 2.8 and following) and E is polarized with the induced form. Fix $z \in W$ and consider the graded vector spaces

$$V := H_z^\bullet(B_x B_y) \quad \text{and} \quad U := H_z^\bullet(E).$$

These have operators $L : V^\bullet \to V^{\bullet+2}$ and $L : U^\bullet \to U^{\bullet+2}$ obtained by applying $H_z^\bullet(-)$ to the "middle multiplication" maps

$$B_x B_y \to B_x B_y(2) : b_1 b_2 \mapsto b_1 \rho b_2,$$
$$E \to E(2) : (b_1 b_2, b_3 b_4) \mapsto (b_1 \rho b_2, b_3 \rho b_4).$$

Also, the maps f, f^* of Proposition 4.5 induce maps (again by taking perverse cohomology)

$$U^\bullet \xrightarrow{f^*} V^{\bullet+1} \xrightarrow{f} U^{\bullet+2}.$$

These maps are morphisms of graded $\mathbb{R}[L]$-modules. We have:

(1) f is injective in degrees < 0, by Proposition 4.5(1).
(2) $\langle f(v), f(v') \rangle = \langle v, f^*(f(v')) \rangle = \langle v, 2Lv' \rangle$ for all $v, v' \in V^\bullet$. The first equality holds because f^* is the adjoint of f. The second equality holds by Proposition 4.5(2), and by Lemma 2.5.
(3) U satisfies the Hodge–Riemann bilinear relations, which we now justify. Recall that $E = FB_y \oplus B_x G$. Every direct summand of F has the form $B_{x'}$ for $x' < x$ with $\ell(x')$ having the same parity as $\ell(x) - 1$. We have assumed $RHR(x', y)$, which applies to every direct summand of FB_y (since E is given its standard polarization). Thus $H_z^\bullet(B_{x'} B_y)$ satisfies the Hodge–Riemann bilinear relations for each of these summands. Moreover, their direct sum $H_z^\bullet(FB_y)$ also satisfies the Hodge–Riemann bilinear relations by Remark 3.3, since the parity of each tensor product agrees with $\ell(x) + \ell(y) - 1$. The same argument applies to $B_x G$, with the same parity $\ell(x) + \ell(y) - 1$, and thus it also applies to the direct sum $H_z^\bullet(E)$.

Now we may repeat the proof of [EW14, Lemma 2.3] to deduce that $L_V^i : V^{-i} \to V^i$ is injective and hence is an isomorphism by comparison of dimension. The property $RHL(x, y)$ follows. □

Proposition 4.9. *Fix $x, y \in W$ and $s \in S$ and suppose $RHR(x', s, y)$ and $RHR(x, s, y')$ hold for all $x' < x$, $y' < y$. Then $RHL(x, s, y)$ holds.*

Proof. The proof is the same as that of the previous proposition, replacing Proposition 4.5 with Proposition 4.6. □

4.2. Signs via limit arguments

In this section we will repeatedly appeal to the *principle of conservation of signs*, which states that a continuous family of non-degenerate symmetric forms on a real vector space has constant signature. The following lemma, which was one of the key techniques used by de Cataldo and Migliorini in their proof of the Hodge–Riemann bilinear relations in geometry [dCM02], is an immediate consequence.

Lemma 4.10. *Consider a polarized graded vector space and a continuous family of operators L_t parametrized by a connected set. Assume all the operators in the family satisfy hard Lefschetz. If any member of the family satisfies the Hodge–Riemann bilinear relations, then they all do.*

To spell out this general argument in slightly more detail: one is given a finite-dimensional graded vector space $V^\bullet$ equipped with a non-degenerate graded symmetric form

$$\langle -, - \rangle : V^\bullet \times V^\bullet \to \mathbb{R}.$$

A degree 2 Lefschetz operator induces a symmetric form on each V^{-i} for $i \in \mathbb{Z}_{\geq 0}$ via $(v, w) := \langle v, L^i w \rangle$, which collectively are non-degenerate if and only if L satisfies hard Lefschetz. If L does satisfy hard Lefschetz, then L satisfies the Hodge–Riemann bilinear relations (in the sense of [EW14, §2]) if and only if the signature of the Lefschetz form on each V^{-i} agrees with a certain formula, which depends only on the graded dimension of V. From the principle of conservation of signs, one deduces the lemma above. The applications will become clear immediately.

Proposition 4.11. *Suppose $x, y \in W$, $s \in S$ and $xs > x$. Assume $RHL(x, s, y)$ and $RHR(\leq xs, y)$. Then $RHR(x, s, y)$ holds.*

Proof. For $a, b \in \mathbb{R}$, consider the Lefschetz operator

$$L_{a,b} := B_x(a\rho)B_sB_y + B_xB_s(b\rho)B_y : B_xB_sB_y \to B_xB_sB_y(2).$$

Recall that $HR(x, s, y)$ means that $L_{a,b}$ induces an operator on $H^\bullet_z(B_xB_sB_y)$ which satisfies hard Lefschetz and Hodge–Riemann, for any $a > 0, b > 0$.

However B_xB_s is perverse, and by $RHR(x, s)$ (see Lemma 2.10 above) the restriction of the intersection form on B_xB_s to each summand $B_z \overset{\oplus}{\subseteq} B_xB_s$ is a multiple of the intersection form on B_z with sign $(-1)^{(\ell(x)+1-\ell(z))/2}$. By $RHR(\leq xs, y)$, $L_{0,b}$ satisfies relative Hodge–Riemann on $B_xB_sB_y$ for any $b > 0$ (it is an exercise to confirm that the signs are correct). Thus $L_{a,b}$ satisfies relative hard Lefschetz for all $a \geq 0$ and $b > 0$ and satisfies relative Hodge–Riemann for $a = 0$, $b > 0$. We can now appeal to the principle of conservation of signs to conclude that relative Hodge–Riemann is satisfied for all $a \geq 0, b > 0$. Thus $RHR(x, s, y)$ holds. □

The previous proof uses the special case $a = 0, b > 0$ to deduce the general case $a > 0$, $b > 0$. Here we go the other way:

Proposition 4.12. *Suppose $x, y \in W$, $s \in S$ and that $sy > y$. Assume $RHR(x, s, y)$ and $RHL(x, \leq sy)$. Then $RHR(x, sy)$ holds.*

Proof. Let $L_{a,b}$ denote the Lefschetz operator considered in the previous proof. By our assumptions, $L_{a,b}$ satisfies Hodge–Riemann for $a > 0$, $b > 0$ and hard Lefschetz for $a > 0$, $b = 0$. By the principle of conservation of signs, Hodge–Riemann is also satisfied for $a > 0, b = 0$. Now $B_x B_{sy}$ is a summand of $B_x B_s B_y$ and the intersection form on $B_x B_s B_y$ restricts to a positive multiple of the intersection form on $B_x B_{sy}$. We conclude[10] that $L_{a,0}$ satisfies Hodge–Riemann on $B_x B_{sy}$, which is what we wanted. □

Proposition 4.13. *Let $x, y \in W$ and $s \in S$ be such that $xs < x$. Assume $HL(x, s, y)$ and $HR(x, y)$. Then $HR(x, s, y)$ holds.*

The proof of Proposition 4.13 is more complicated than that of Proposition 4.11, and will occupy the rest of this section. Here is a sketch of our approach. We fix a decomposition $B_x B_s = B_x(1) \oplus B_x(-1)$ and explicitly calculate the Lefschetz operator and forms in the decomposition

$$B_x B_s B_y = B_x B_y(1) \oplus B_x B_y(-1)$$

in terms of the corresponding operators on $B_x B_y$. Appealing to $RHR(x, y)$ we will see that the signs are correct for $b \gg a > 0$. By the principle of conservation of signs (which is applicable by our $RHL(x, s, y)$ assumption) we deduce that $RHR(x, s, y)$ holds, which is what we wanted to show.

For simplicity we assume $\rho(\alpha_s^\vee) = 1$ for all $s \in S$.

Lemma 4.14. *The map $r \mapsto (\partial_s(-rs(\rho)), \rho\partial_s(r))$ gives an isomorphism*

$$R = R^s \oplus \rho R^s \tag{4.1}$$

of R^s-bimodules.

Proof. R is free as an R^s-module with basis $\{1, \gamma\}$ where $\gamma \in R^2$ is any degree 2 element which is not s-invariant. In particular we can take $\gamma = \rho$. Under the map as in the statement of the lemma we have

$$1 \mapsto (\partial_s(-s\rho), \rho\partial_s(1)) = (1, 0),$$
$$\rho \mapsto (\partial_s(-\rho s(\rho)), \rho\partial_s(\rho) = (0, \rho),$$

and so our map sends a basis to a basis, and the lemma follows. □

By [Wil11, Proposition 7.4.3] there exists an (R, R^s)-bimodule B_x^s (a "singular Soergel bimodule") and a canonical isomorphism

$$B_x^s \otimes_{R^s} R = B_x. \tag{4.2}$$

Our choice of isomorphism (4.1) yields a decomposition

$$B_x B_s = B_x^s \otimes_{R^s} R \otimes_{R^s} R(1) = B_x(1) \oplus B_x(-1). \tag{4.3}$$

Now consider the endomorphism $B_x \rho B_s : B_x B_s \to B_x B_s(2)$.

[10] We are using the fact that relative Hodge–Riemann is preserved under taking polarized direct summands. See [Wil16, Lemma 4.5] for a related situation.

Lemma 4.15. *With respect to the decomposition* (4.3) *the degree* 2 *endomorphism* $B_x \rho B_s$ *is given by the matrix*

$$\begin{pmatrix} 0 & B_x(-\rho(s\rho)) \\ B_x & B_x(\rho + s\rho) \end{pmatrix} : B_x(1) \oplus B_x(-1) \to B_x(3) \oplus B_x(1). \tag{4.4}$$

Proof. We identify B_x with $B_x^s \otimes_{R^s} R$, and write an element of it as $b \otimes f$ for $b \in B_x^s$ and $f \in R$. Similarly, we identify $B_x B_s$ with $B_x^s \otimes_{R^s} R \otimes_{R^s} R(1)$.

Consider an element of the form $b \otimes 1 \in B_x$. We calculate the action of $B_x \rho B_s$ on the summand $B_x(1)$:

$$\begin{aligned} &B_x(1) \xhookrightarrow{(4.2)} B_x B_s \xrightarrow{B_x \rho B_s} B_x B_s \xrightarrow{(4.2)} B_x(1) \oplus B_x(-1), \\ &b \otimes 1 \mapsto b \otimes 1 \otimes 1 \mapsto b \otimes \rho \otimes 1. \end{aligned}$$

Similarly we calculate the action on the summand $B_x(-1)$:

$$\begin{aligned} &B_x(-1) \xhookrightarrow{(4.3)} B_x B_s \xrightarrow{B_x \rho B_s} B_x B_s \xrightarrow{(4.3)} B_x(1) \oplus B_x(-1), \\ &b \otimes 1 \mapsto b \otimes \rho \otimes 1 \mapsto b \otimes \rho^2 \otimes 1 \mapsto (b \otimes (-\rho s(\rho)), b \otimes (\rho + s\rho)). \end{aligned}$$

The lemma follows. □

Lemma 4.16. *The singular Soergel bimodule* B_x^s *admits a unique invariant form*

$$\langle -, - \rangle_{B_x^s} : B_x^s \times B_x^s \to R^s$$

such that $\langle -, - \rangle \otimes_{R^s} R$ *agrees with the intersection form under the identification* (4.2).

Here and in the following proof, an invariant form on an (R, R^s)-bimodule means a graded bilinear form $\langle -, - \rangle : B_x^s \times B_x^s \to R^s$ which satisfies $\langle rb, b' \rangle = \langle b, rb' \rangle$ and $\langle br', b' \rangle = \langle b, b'r' \rangle = \langle b, b' \rangle r'$ for all $b, b' \in B_x^s$, $r \in R$, $r' \in R^s$.

Proof of Lemma 4.16. Let ${}^s B_{x^{-1}}$ denote the (R^s, R)-bimodule obtained from B_x^s by interchanging left and right actions. Then ${}^s B_{x^{-1}}$ agrees with the indecomposable singular Soergel bimodule parametrized by the coset of x^{-1} in $\langle s \rangle \setminus W$, as described in [Wil11, Theorem 7.4.2]. Soergel's conjecture and [Wil11, Theorem 7.4.1] imply that $\mathrm{Hom}({}^s B_{x^{-1}}, \mathbb{D}({}^s B_{x^{-1}}))$ is one-dimensional. (We denote by $\mathbb{D}$ the duality functor on singular Soergel bimodules defined in [Wil11, §6.3].) We can regard elements in this Hom space as maps $B_x^s \to \mathrm{Hom}_{-R^s}(B_x^s, R^s)$ and hence as invariant forms

$$\langle -, - \rangle : B_x^s \times B_x^s \to R^s.$$

We conclude that B_x^s admits an invariant form which is unique up to scalar. Given any such form $\langle -, - \rangle$, $\langle -, - \rangle \otimes_{R^s} R$ is a non-degenerate form on B_x, and hence agrees with the intersection form on B_x up to scalar. The lemma follows. □

Our fixed decomposition (4.3) gives the basic identification

$$B_x B_s B_y = B_x B_y(1) \oplus B_x B_y(-1). \tag{4.5}$$

The following is immediate from the definitions:

Lemma 4.17. *Under* (4.5) *the invariant form is given by*

$$\langle (b_1, b_2), (b_1', b_2') \rangle = \langle b_1, b_2' \rangle + \langle b_2, b_1' \rangle + \langle \rho b_2, b_2' \rangle.$$

We now put the above calculations together. Until the end of the section let us in addition fix $z \in W$ and set

$$V^\bullet := H_z^\bullet(B_x B_y).$$

Then $V^\bullet$ is equipped with a symmetric form $\langle -, - \rangle_{V^\bullet}$ and a Lefschetz operator $L : V^\bullet \to V^{\bullet+2}$. This data satisfies Hodge–Riemann, by our assumption $HR(x, y)$. Our identification (4.5) fixes an isomorphism

$$H_z^\bullet(B_x B_s B_y) = V^\bullet(1) \oplus V^\bullet(-1). \tag{4.6}$$

Proposition 4.18. *Under the identification* (4.6)*:*

(1) *The invariant form is given by*

$$\langle (v_1, v_2), (v_1', v_2') \rangle = \langle v_1, v_2' \rangle + \langle v_2, v_1' \rangle + \langle L v_2, v_2' \rangle \tag{4.7}$$

for $v_1, v_1' \in V^\bullet(1)$ *and* $v_2, v_2' \in V^\bullet(-1)$.

(2) *The operator induced by* $L_{a,b} := B_x(a\rho)B_s B_y + B_x B_s(b\rho)B_y$ *is given by*

$$a\begin{pmatrix} 0 & X \\ \mathrm{id} & Y \end{pmatrix} + b\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \tag{4.8}$$

for certain (unspecified) maps $X : V(-1) \to V(1)$ *and* $Y : V(-1) \to V(-1)$.

Proof. (1) (resp. (2)) is an immediate consequence of Lemma 4.17 (resp. Lemma 4.15). □

Proposition 4.19. *Assume* $HR(x, y)$. *Then for* $b \gg a > 0$ *the operator* $L_{a,b}$ *satisfies HR on* $V^\bullet(1) \oplus V^\bullet(-1)$.

Proof. We roll up our sleeves and calculate everything in a basis.

Fix a degree $-d \leq 0$. By [EW14, Lemma 5.2] it is enough to show that for $b \gg a > 0$ the signature of the Lefschetz form on the degree $-d$ piece of $V^\bullet(1) \oplus V^\bullet(-1)$ is equal to the signature of the Lefschetz form on the primitive subspace

$$P^{-d+1} := \ker L^d : V^{-d+1} \to V^{d+1}.$$

To this end let us fix bases

$$x_1, \ldots, x_m \quad \text{for } V^{-d-1}, \qquad p_1, \ldots, p_n \quad \text{for } P^{-d+1}.$$

Because L satisfies hard Lefschetz on V we deduce that

$$Lx_1, \ldots, Lx_m, p_1, \ldots, p_n \quad \text{is a basis for } V^{-d+1}.$$

Thus a basis for $(V^\bullet(1) \oplus V^\bullet(-1))^d = V^{d+1} \oplus V^{d-1}$ is given by

$$(0, x_1), \ldots, (0, x_m), (Lx_1, 0), \ldots, (Lx_m, 0), (p_1, 0), \ldots, (p_n, 0).$$

Let us write

$$L_{a,b} = aA + bB$$

where

$$A = \begin{pmatrix} 0 & X \\ \mathrm{id} & Y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$$

are the matrices appearing in Proposition 4.8. We calculate the leading terms of the Lefschetz form $(v, w) \mapsto \langle v, L_{a,b}^d w\rangle$ in the above basis with respect to the parameter b. We have

$$\langle (0, x_i), L_{a,b}^d(0, x_j)\rangle = b^d\langle Lx_i, L^d x_j\rangle_{V^\bullet} + O(b^{d-1}) = b^d\langle x_i, L^{d+1}x_j\rangle_{V^\bullet} + O(b^{d-1}),$$
$$\langle (Lx_i, 0), L_{a,b}^d(0, x_j)\rangle = b^d\langle Lx_i, L^d x_j\rangle_{V^\bullet} + O(b^{d-1}) = b^d\langle x_i, L^{d+1}x_j\rangle_{V^\bullet} + O(b^{d-1}),$$
$$\langle (Lx_i, 0), L_{a,b}^d(Lx_i, 0)\rangle = b^d\langle (Lx_i, 0), (L^{d+1}x_i, 0)\rangle + O(b^{d-1}) = O(b^{d-1})$$

where $O(b^k)$ denotes a polynomial in b and a in which all powers of b are bounded by k. Using $L^d p_i = 0$ we have

$$\langle (0, x_i), L_{a,b}^d(p_i, 0)\rangle = dab^{d-1}\langle Lx_i, L^{d-1}p_i\rangle + O(b^{d-2}) = O(b^{d-2}),$$
$$\langle (Lx_i, 0), L_{a,b}^d(p_i, 0)\rangle = dab^{d-1}\langle Lx_i, L^{d-1}p_i\rangle + O(b^{d-2}) = O(b^{d-2}),$$
$$\langle (p_i, 0), L_{a,b}^d(p_i, 0)\rangle = dab^{d-1}\langle p_i, L^{d-1}p_i\rangle + O(b^{d-2}).$$

Thus if we define matrices

$$R := (\langle x_i, L^d x_j\rangle)_{1\le i,j\le m} \quad \text{and} \quad Q := (\langle p_i, L^d p_j\rangle)_{1\le i,j\le n}$$

then we can write the Gram matrix of the Lefschetz form $(v, w) \mapsto \langle v, L_{a,b}^d w\rangle$ as a block matrix with entries

$$\begin{pmatrix} b^d R + O(b^{d-1}) & b^d R + O(b^{d-1}) & O(b^{d-2}) \\ b^d R + O(b^{d-1}) & O(b^{d-1}) & O(b^{d-2}) \\ O(b^{d-2}) & O(b^{d-2}) & dab^{d-1}Q + O(b^{d-2}) \end{pmatrix}.$$

For $b \gg a > 0$ this matrix has the same signature as the matrix

$$\begin{pmatrix} R & R & 0 \\ R & 0 & 0 \\ 0 & 0 & Q \end{pmatrix}.$$

Now the submatrix $\left(\begin{smallmatrix} R & R \\ R & 0 \end{smallmatrix}\right)$ is easily seen to be non-degenerate with signature 0. Thus for $b \gg a > 0$ our matrix has the same signature as Q. We have already remarked that by [EW14, Lemma 5.2] this is what we wanted to know. □

Thus Proposition 4.13 holds (see the remarks immediately after the statement of the proposition).

5. Ridigity

We briefly recall Lusztig's notion of two-sided cells and the a-function, in the language of Soergel bimodules; for more details see [Lus14]. There is a preorder $\le_{LR}$ on W, where $z \le_{LR} x$ if and only if there exist some $y, y' \in W$ for which some shift of B_z is a direct summand of $B_y B_x B_{y'}$. The equivalence classes under this preorder are called two-sided cells. For $z \in W$ one defines $a(z)$ to be the maximal integer k (or equivalently, $-a(z)$ is the minimal integer k) such that $B_z(k)$ is a direct summand of $B_x B_y$ for any $x, y \in W$. It is a non-trivial fact (a consequence of Soergel's conjecture) that the a-function is well defined for any Coxeter group [Lus15, §10.1]. A theorem of Lusztig guarantees that the a-function is constant on two-sided cells.

Fix a two-sided cell $\mathbf{c} \subset W$ and let a be its a-value and $J = \bigoplus_{x \in \mathbf{c}} \mathbb{Z} j_x$ the J-ring associated to $\mathbf{c}$ (J is denoted $J^{\mathbf{c}}$ in [Lus14, §18.3]). Following Lusztig [Lus15, §10], we define a semisimple monoidal category $\mathcal{J}$ ($\mathcal{J}$ is denoted $C^{\mathbf{c}}$ in [Lus14, §18.5]), which categorifies J.

We first consider the full subcategory $\mathcal{B}_{\leq \mathbf{c}} \subset \mathcal{B}$, consisting of all direct sums of shifts of B_z with $z \leq_{LR} \mathbf{c}$. By definition of $\leq_{LR}$, $\mathcal{B}_{\leq \mathbf{c}}$ is closed under tensor products with arbitrary objects of $\mathcal{B}$, and thus inherits a monoidal structure (without unit). One can define $\mathcal{B}_{< \mathbf{c}}$ similarly.

Let $I_{\mathbf{c}}$ denote the tensor ideal in $\mathcal{B}$ consisting of all morphisms which factor through objects in $\mathcal{B}_{<\mathbf{c}}$. Thus the quotient of additive categories $\mathcal{B}'_{\mathbf{c}} := \mathcal{B}/I_{\mathbf{c}}$ inherits the structure of a graded additive monoidal category. We denote the image of B_x in $\mathcal{B}'_{\mathbf{c}}$ by $B^{\mathbf{c}}_x$. We set $\mathcal{B}_{\mathbf{c}}$ to be the full graded additive subcategory generated by $B^{\mathbf{c}}_x$ with $x \in \mathbf{c}$; in other words, it is the image of $\mathcal{B}_{\leq \mathbf{c}}$ inside $\mathcal{B}/I_{\mathbf{c}}$. The objects $B^{\mathbf{c}}_x(m)$ with $x \in W$ and $x \not< \mathbf{c}$ (resp. $x \in \mathbf{c}$) give representatives for the isomorphism classes of indecomposable objects in $\mathcal{B}'_{\mathbf{c}}$ (resp. $\mathcal{B}_{\mathbf{c}}$). Moreover $\mathcal{B}_{\mathbf{c}}$ is a graded additive monoidal category (without unit unless $\mathbf{c} = \{\mathrm{id}\}$).

The (obvious analogues of the) crucial vanishing statements (2.1) and (2.2) still hold in $\mathcal{B}'_{\mathbf{c}}$ and $\mathcal{B}_{\mathbf{c}}$, and hence the perverse filtration and perverse cohomology functors descend to $\mathcal{B}'_{\mathbf{c}}$ and $\mathcal{B}_{\mathbf{c}}$. We denote them by the same symbols. It is immediate from the definition of the a-function that, for all $x, y \in \mathbf{c}$,

$$H^i(B^{\mathbf{c}}_x B^{\mathbf{c}}_y) = 0 \quad \text{if } |i| > a. \tag{5.1}$$

Remark 5.1. Note that the tensor product $B_x B_y$ in $\mathcal{B}$ might have objects $B_z(k)$ with $|k| > a$ as direct summands, but only for $z < \mathbf{c}$, not for $z \in \mathbf{c}$ (as follows from the definition of a). Thus (5.1) only holds in $\mathcal{B}_{\mathbf{c}}$, not in $\mathcal{B}$. Similarly, (2.2) implies that

$$\mathrm{Hom}_{\mathcal{B}_{\mathbf{c}}}(B^{\mathbf{c}}_x, H^{-a}(B^{\mathbf{c}}_y B^{\mathbf{c}}_z)) \cong \mathrm{Hom}_{\mathcal{B}_{\mathbf{c}}}(B^{\mathbf{c}}_x, B^{\mathbf{c}}_y B^{\mathbf{c}}_z(-a)) \tag{5.2}$$

canonically, for any $x, y, z \in \mathbf{c}$. The analogous statement in $\mathcal{B}$ is false.

We now come to the definition of $\mathcal{J}$. It is a full subcategory of $\mathcal{B}_{\mathbf{c}}$, although with a different monoidal structure. The objects of $\mathcal{J}$ are given by direct sums (without shifts) of $B^{\mathbf{c}}_x$ with $x \in \mathbf{c}$, and thus by (2.1) the category is semisimple. The monoidal product is given by

$$B * B' := H^{-a}(BB') \in \mathcal{B}_{\mathbf{c}}$$

(the lowest potentially non-zero degree, by (5.1)). Lusztig proves that $\mathcal{J}$ is a semisimple monoidal category (this result relies in an essential way on [EW14]), and that the map $j_x \to [B^{\mathbf{c}}_x]$ induces an isomorphism $J \xrightarrow{\sim} [\mathcal{J}]$, where $[\mathcal{J}]$ denotes the Grothendieck group of $\mathcal{J}$.

Remark 5.2. The reader is warned that in general $\mathcal{J}$ is a "monoidal category without unit", i.e. it has an associator but no unit. In general, Lusztig conjectures [Lus14, §13.4] that the a-function is bounded (i.e. $a(z) \leq N$ for all $z \in W$ and some fixed constant N, which he describes explicitly). This boundedness is known to hold for finite and affine Weyl groups. Under the assumption of this conjecture, it turns out that $\mathcal{J}$ has a unit if and only if $\mathbf{c}$ contains finitely many left cells (as is always the case in finite and affine type). In this case Lusztig proves [Lus14, §18.5] that the object $\bigoplus_{x \in \mathcal{D} \cap \mathbf{c}} B_x^{\mathbf{c}}$ is a unit for $\mathcal{J}$ (here $\mathcal{D} \subset W$ denotes the set of distinguished involutions). When $\mathbf{c}$ contains infinitely many left cells, $\mathcal{J}$ is still "locally unital" (under the same boundedness assumption). For any given object $B \in \mathcal{J}$, only finitely many $B_x^{\mathbf{c}}$ with $x \in \mathcal{D} \cap \mathbf{c}$ satisfy $B_x^{\mathbf{c}} * B \neq 0$. The formal direct sum $\bigoplus_{x \in \mathcal{D} \cap \mathbf{c}} B_x^{\mathbf{c}}$, while not an object in $\mathcal{J}$ when $\mathcal{D} \cap \mathbf{c}$ is infinite, acts on any object, and it will act as a monoidal identity would.

Our aim in this section is to show that the relative hard Lefschetz theorem for Soergel bimodules implies

Theorem 5.3. *$\mathcal{J}$ is a rigid, pivotal monoidal category.*

Remark 5.4. For finite and affine Weyl groups the rigidity of $\mathcal{J}$ has been proved by Bezrukavnikov, Finkelberg and Ostrik [BFO09, §4.3] (using the geometric Satake equivalence). Lusztig has also proven rigidity for Weyl groups (see [Lus15, §9.3] and [Lus14, §18.19]). His techniques probably extend to crystallographic Coxeter groups. Lusztig also conjectured the rigidity to hold for any finite Coxeter group [Lus15, §10], in which case he expects the Drinfeld center $Z(\mathcal{J})$ to be related to the "unipotent characters" of W. Ostrik has informed us that for the interesting case of the two-sided cell in H_4 with a-value 6, he has been able to verify the rigidity of $\mathcal{J}$ by other means.

Remark 5.5. As we will see, the pivotal structure on $\mathcal{J}$ will depend on our fixed choice of regular dominant element $\rho \in \mathfrak{h}^*$. We do not know if the structure varies in an interesting way with ρ. It is possible that the Hodge–Riemann relations might allow one to show that $\mathcal{J}$ is unitary, and hope to address this question in future work.

Because $\mathcal{J}$ does not have a unit in general the standard definition of rigidity does not make sense. We will prove the following (which is equivalent to the usual notion of rigidity if $\mathcal{J}$ has a unit, see Remark 5.8 below):

Proposition 5.6. *There exists a contravariant functor $B \mapsto B^\vee$ on $\mathcal{J}$ which satisfies the following properties:*

(1) *For $B, X, Y \in \mathcal{J}$ we have canonical isomorphisms*

$$\mathrm{Hom}_{\mathcal{J}}(X, B * Y) \xrightarrow{\phi_{X,Y}} \mathrm{Hom}_{\mathcal{J}}(B^\vee * X, Y),$$
$$\mathrm{Hom}_{\mathcal{J}}(X, Y * B) \xrightarrow{\chi_{X,Y}} \mathrm{Hom}_{\mathcal{J}}(X * B^\vee, Y)$$

functorial in X and Y.

(2) *For $B, X, Y, Z \in \mathcal{J}$ the following diagrams commute:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{J}}(X, B * Y) & \xrightarrow{(-)*Z} & \mathrm{Hom}_{\mathcal{J}}(X * Z, B * Y * Z) \\ \downarrow \phi_{X,Y} & & \downarrow \phi_{X*Z,Y*Z} \\ \mathrm{Hom}_{\mathcal{J}}(B^\vee * X, Y) & \xrightarrow{(-)*Z} & \mathrm{Hom}_{\mathcal{J}}(B^\vee * X * Z, Y * Z) \end{array} \tag{5.3}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{J}}(X, Y * B) & \xrightarrow{Z*(-)} & \mathrm{Hom}_{\mathcal{J}}(Z * X, Z * Y * B) \\ \downarrow \chi_{X,Y} & & \downarrow \chi_{Z*X,Z*Y} \\ \mathrm{Hom}_{\mathcal{J}}(X * B^\vee, Y) & \xrightarrow{Z*(-)} & \mathrm{Hom}_{\mathcal{J}}(Z * X * B^\vee, Z * Y) \end{array} \tag{5.4}$$

(3) *We have a canonical isomorphism $B \xrightarrow{\sim} (B^\vee)^\vee$.*

We make some remarks before turning to the proof. It is easy to see that $B_s \in \mathcal{B}$ is self-dual (this is immediate in the language of [EW16], where the cup and cap maps provide the unit and counit). Note also that if M and N are rigid (i.e. if $M^\vee$ and $N^\vee$ exist) then one has $(MN)^\vee = N^\vee M^\vee$. It follows that any Bott–Samelson module is rigid. Hence $\mathcal{B}$ is rigid (taking the Karoubi envelope preserves rigidity). Let us denote by $B \mapsto B^\vee$ the duality on $\mathcal{B}$. It is easy to see that this structure is even pivotal (i.e. we have a canonical isomorphism $B \xrightarrow{\sim} (B^\vee)^\vee$). Note also that $B_x^\vee \cong B_{x^{-1}}$ canonically, and thus $(-)^\vee$ preserves two-sided cells, and $\mathcal{B}_{\leq \mathbf{c}}$ is also a rigid, pivotal monoidal category.

As quotients of a rigid, pivotal monoidal category, the monoidal categories $\mathcal{B}_{\mathbf{c}}$ and $\mathcal{B}'_{\mathbf{c}}$ are rigid and pivotal. We abuse notation and also denote the duality on $\mathcal{B}_{\mathbf{c}}$ by $B \mapsto B^\vee$.

Proof of Proposition 5.6. Note that $(B_x^{\mathbf{c}})^\vee \cong B_{x^{-1}}^{\mathbf{c}}$, so the objects of $\mathcal{J}$ are closed under the duality of $\mathcal{B}_{\mathbf{c}}$. Let us show that this definition of $(-)^\vee$ satisfies the desired conditions: (3) is immediate, it remains to check (1) and (2).

We first establish (1). We will construct the isomorphism $\phi_{X,Y}$, as the construction of $\chi_{X,Y}$ is similar. Let $X, Y, B \in \mathcal{J}$. We have canonical identifications (by definition and the analogue for $\mathcal{B}_{\mathbf{c}}$ of (2.4))

$$\begin{aligned} \mathrm{Hom}_{\mathcal{J}}(X, B * Y) &= \mathrm{Hom}_{\mathcal{B}_{\mathbf{c}}}(X, H^{-a}(BY)) = \mathrm{Hom}_{\mathcal{B}_{\mathbf{c}}}(X, BY(-a)) \\ &= \mathrm{Hom}_{\mathcal{B}_{\mathbf{c}}}(B^\vee X, Y(-a)) = \mathrm{Hom}_{\mathcal{B}_{\mathbf{c}}}(H^{a}(B^\vee X), Y). \end{aligned}$$

Precomposing with the isomorphism $H^{-a}(B^\vee X) \xrightarrow{\sim} H^{a}(B^\vee X)$ given by relative hard Lefschetz gives an isomorphism

$$\mathrm{Hom}_{\mathcal{B}_{\mathbf{c}}}(H^{a}(B^\vee X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}_{\mathbf{c}}}(H^{-a}(B^\vee X), Y) = \mathrm{Hom}_{\mathcal{J}}(B^\vee * X, Y).$$

The composition of these isomorphisms defines our isomorphism $\phi_{X,Y}$. It is immediate to check that this isomorphism is natural in X and Y.

We now turn to (2). As before we only establish the commutativity of (5.3), with (5.4) being similar. Choose $f \in \mathrm{Hom}_{\mathcal{J}}(X, B * Y)$ and let f_{NE} (resp. f_{SW}) denote the image of f in $\mathrm{Hom}_{\mathcal{J}}(B^\vee * X * Z, Y * Z)$ obtained by passing through the north-east (resp. south-west) corner of (5.3). We must prove that $f_{SW} = f_{NE}$.

Via $\mathrm{Hom}_{\mathcal{J}}(X, B * Y) = \mathrm{Hom}_{\mathcal{B}_{\mathbf{c}}}(X, BY(-a))$ we may regard f as a map

$$f : X \to BY(-a).$$

From f we obtain the following maps in $\mathcal{B}_{\mathbf{c}}$:

$$\begin{aligned} f' : B^\vee X &\to Y(-a), & \varphi &:= H^a(f') : H^a(B^\vee X) \to Y, \\ g := fZ : XZ &\to BYZ(-a), & & \\ g' := f'Z : B^\vee XZ &\to YZ(-a), & \gamma &:= H^{2a}(g') : H^{2a}(B^\vee XZ) \to H^a(YZ), \\ h : H^{-a}(XZ) &\to BH^{-a}(YZ)(-a), & h' &: B^\vee H^{-a}(XZ) \to H^{-a}(YZ)(-a). \end{aligned}$$

Here f' (resp. g', h') are obtained from f (resp. g, h) using the dual pair $(B, B^\vee)$ in $\mathcal{B}_{\mathbf{c}}$, and h is uniquely determined by $H^0(h) = H^{-a}(g)$ (note that $H^{-a}(BYZ(-a)) = H^0(BH^{-a}(YZ)(-a))$).

Consider the diagram given in Figure 1. The maps which have not been defined above are given as follows:

(1) All maps labelled $\sim$ are relative hard Lefschetz isomorphisms (given by our fixed choice of $\rho \in \mathfrak{h}^*$). At the top and bottom of the middle square we use the canonical identifications

$$\begin{aligned} H^{2a}(B^\vee XZ) &= H^a(H^a(B^\vee X)Z) = H^a(B^\vee H^a(XZ)), \\ H^{-2a}(B^\vee XZ) &= H^{-a}(H^{-a}(B^\vee X)Z) = H^{-a}(B^\vee H^{-a}(XZ)). \end{aligned}$$

(2) We set $l := H^{-a}(\varphi Z)$ and $r := H^a(h')$.

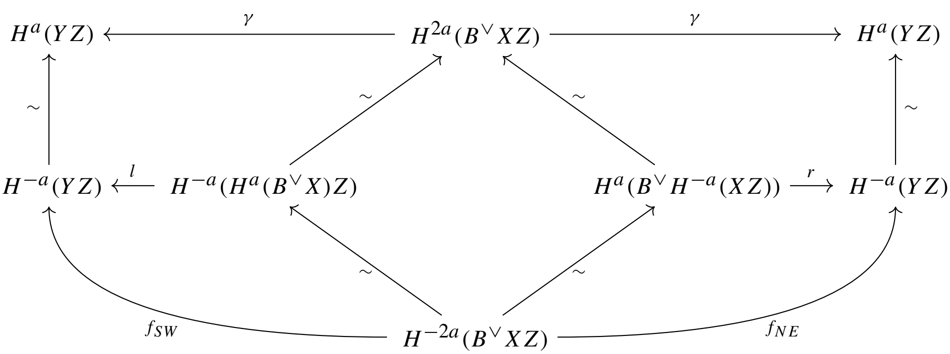

Fig. 1. Diagram for the proof of Proposition 5.6(2).

It is straightforward but tedious to check that all squares and triangles in Figure 1 commute (see also Remark 5.7 below). If q denotes the relative hard Lefschetz isomorphism $q : H^{-a}(YZ) \to H^a(YZ)$ we deduce from the commutativity of the diagram that $q \circ f_{NE} = q \circ f_{SW}$, and hence $f_{NE} = f_{SW}$, which is what we wanted to show. □

Remark 5.7. All the squares and triangles in Figure 1 commute for roughly the same reason, which we now explain. Relative hard Lefschetz provides us with a family of isomorphisms

$$\theta_{MN}\colon H^{-a}(MN) \xrightarrow{\sim} H^{a}(MN)$$

for all $M, N \in \mathcal{B}_{\mathbf{c}}$, which were used to produce the maps labelled $\sim$. However, these isomorphisms have the special property that they were induced by "middle multiplication" by a polynomial (independent of the choice of M and N) on the bimodule MN. For a triple tensor product MNP, this implies (very loosely stated) that $\theta_{MN}P$ should commute with $M\theta_{NP}$: this is effectively why the square in the middle of Figure 1 commutes. More generally, θ_{NP} will commute with any bimodule morphism involving the bimodule MN, and vice versa, which explains the other parts of this commutative diagram. In other words, middle multiplication on MN will commute with all bimodule operations involving $N \otimes (-)$ or $(-) \otimes M$, and this property is at the heart of the proof.

The reader will notice that we do not use the full strength of relative hard Lefschetz to establish Theorem 5.3. "All" that is needed is a family of isomorphisms θ_{MN} which satisfy a host of commutativity properties. That said, it is hard to imagine isomorphisms θ_{MN} which satisfy these properties but do not come from middle multiplication. This is a very strong motivation for using hard Lefschetz to attack the question of rigidity.

Remark 5.8. Suppose that $\mathbf{c}$ contains finitely many left cells. Then $\mathcal{J}$ has a unit (see Remark 5.2), which we denote by $\mathbb{1}$. Applying the isomorphisms of Proposition 5.6 to the identity maps in $\mathrm{Hom}_{\mathcal{J}}(B, B)$ and $\mathrm{Hom}_{\mathcal{J}}(B^\vee, B^\vee)$, we obtain morphisms $\varepsilon : \mathbb{1} \to B * B^\vee$ and $\mu : B^\vee * B \to \mathbb{1}$. For $f : X \to B * Y$, $\phi_{X,Y}(f)$ is given by the composition

$$B^\vee * X \xrightarrow{B^\vee * f} B^\vee * B * Y \xrightarrow{\mu * Y} \mathbb{1} * Y = Y.$$

To see this, use Proposition 5.6(2) to show that $\phi_{B*Y,Y}(\mathrm{id}_{B*Y}) = \mu * Y$. Then, by naturality of Proposition 5.6(1) under precomposition with f, one obtains the desired equality. Similarly, the inverse of $\phi_{X,Y}$ sends $g : B^\vee * X \to Y$ to

$$X = \mathbb{1} * X \xrightarrow{\varepsilon * X} B * B^\vee * X \xrightarrow{B * g} B * Y.$$

From this (replacing f by ε and g by μ) one easily deduces that $B^\vee$ (and ε, μ) is left dual to B. Similarly, one deduces that $B^\vee$ is right dual to B. Hence $\mathcal{J}$ is rigid in the usual sense. Finally, the canonical isomorphism $B \xrightarrow{\sim} (B^\vee)^\vee$ shows that $\mathcal{J}$ is pivotal.

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