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Volume preserving flow and Alexandrov–Fenchel type inequalities in hyperbolic space

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Abstract. In this paper, we study flows of hypersurfaces in hyperbolic space, and apply them to prove geometric inequalities. In the first part of the paper, we consider volume preserving flows by a family of curvature functions including positive powers of k-th mean curvatures with $k = 1, \dots, n$, and positive powers of p-th power sums S_p with $p > 0$. We prove that if the initial hypersurface M_0 is smooth and closed and has positive sectional curvatures, then the solution M_t of the flow has positive sectional curvature for any time $t > 0$, exists for all time and converges to a geodesic sphere exponentially in the smooth topology. The convergence result can be used to show that certain Alexandrov–Fenchel quermassintegral inequalities, known previously for horospherically convex hypersurfaces, also hold under the weaker condition of positive sectional curvature.

In the second part of this paper, we study curvature flows for strictly horospherically convex hypersurfaces in hyperbolic space with speed given by a smooth, symmetric, increasing and degree one homogeneous function f of the shifted principal curvatures $\lambda_i = \kappa_i - 1$, plus a global term chosen to impose a constraint on the quermassintegrals of the enclosed domain, where f is assumed to satisfy a certain condition on the second derivatives. We prove that if the initial hypersurface is smooth, closed and strictly horospherically convex, then the solution of the flow exists for all time and converges to a geodesic sphere exponentially in the smooth topology. As applications of the convergence result, we prove a new rigidity theorem on smooth closed Weingarten hypersurfaces in hyperbolic space, and a new class of Alexandrov–Fenchel type inequalities for smooth horospherically convex hypersurfaces in hyperbolic space.

Keywords. Volume preserving flow, Alexandrov–Fenchel inequalities, hyperbolic space, horospherically convex hypersurfaces

Contents

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1. Introduction

Let $X_0 : M^n \to \mathbb{H}^{n+1}$ be a smooth embedding such that $M_0 = X_0(M)$ is a closed smooth hypersurface in the hyperbolic space $\mathbb{H}^{n+1}$. We consider a smooth family of immersions $X : M^n \times [0, T) \to \mathbb{H}^{n+1}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} X(x,t) = (\phi(t) - \Psi(x,t))\nu(x,t), \\ X(\cdot, 0) = X_0(\cdot), \end{cases} \tag{1.1}$$

where $\nu(x,t)$ is the unit outward normal of $M_t = X(M,t)$, Ψ is a smooth curvature function evaluated at the point (x,t) of M_t, and the global term $\phi(t)$ is chosen to impose a constraint on the enclosed volume or quermassintegrals of M_t.

The volume preserving mean curvature flow in hyperbolic space was first studied by Cabezas-Rivas and Miquel [12] in 2007. By imposing horospherical convexity (the condition that all principal curvatures exceed 1, which will also be called h-convexity) on the initial hypersurface, they proved that the solution exists for all time and converges smoothly to a geodesic sphere. Some other mixed volume preserving flows were considered in [23, 32] with speed given by degree one homogeneous functions of the principal curvatures. Recently Bertini and Pipoli [11] succeeded in treating flows by more general functions of mean curvature, including in particular any positive power of mean curvature. In a recent paper [10], the first and third authors proved the smooth convergence of quermassintegral preserving flows with speed given by any positive power of a degree one homogeneous function f of the principal curvatures for which the dual function $f_*(x_1, \ldots, x_n) = (f(x_1^{-1}, \ldots, x_n^{-1}))^{-1}$ is concave and approaches zero on the boundary of the positive cone. This includes in particular the volume preserving flow by positive powers of k-th mean curvature for h-convex hypersurfaces. Note that in all the above mentioned work, the initial hypersurface is assumed to be h-convex.

One reason to consider constrained flows of the kind considered here is to prove geometric inequalities: In particular, the convergence of the volume preserving mean curvature flow to a sphere implies that the area of the initial hypersurface is no less than that of a geodesic sphere with the same enclosed volume, since the area is non-increasing while the volume remains constant under the flow. The same motivation lies behind [32], where inequalities between quermassintegrals were deduced from the convergence of certain flows.

In this paper, we make the following contributions:

(1) In the first part of the paper, we weaken the horospherical convexity condition, allowing instead hypersurfaces for which the intrinsic sectional curvatures are positive. We consider the flow (1.1) for hypersurfaces with positive sectional curvature and with

speed Ψ given by any positive power of a smooth, symmetric, strictly increasing and degree one homogeneous function of the Weingarten matrix $\mathcal{W}$ of M_t. Here we say a hypersurface M in hyperbolic space has *positive sectional curvature* if its sectional curvatures satisfy $R^M_{ijij} > 0$ for any $1 \leq i < j \leq n$, which by the Gauss equation is equivalent to the principal curvatures of M satisfying $\kappa_i \kappa_j > 1$ for $1 \leq i \neq j \leq n$. This is a weaker condition than h-convexity. As a consequence we deduce inequalities between volume and other quermassintegrals for hypersurfaces with positive sectional curvature, extending inequalities previously known only for horospherically convex hypersurfaces.

(2) In the second part of this paper, we consider flows (1.1) for strictly h-convex hypersurfaces in which the speed Ψ is homogeneous as a function of the shifted Weingarten matrix $\mathcal{W} - \mathrm{I}$ of M_t, rather than the Weingarten matrix itself. Using these flows we are able to prove a new class of integral inequalities for horospherically convex hypersurfaces.

(3) In order to understand these new functionals we introduce some new machinery for horospherically convex regions, including a horospherical Gauss map and a horospherical support function. We also develop an interesting connection (closely related to the results of [15]) between flows of h-convex hypersurfaces in hyperbolic space by functions of principal curvatures, and conformal flows of conformally flat metrics on S^n by functions of the eigenvalues of the Schouten tensor. This allows us to translate our results to convergence theorems for metric flows, and our isoperimetric inequalities to corresponding results for conformally flat metrics. We expect that these will prove useful in future work.

We will describe our results in more detail in the rest of this section.

1.1. Volume preserving flow with positive sectional curvature

Suppose that the initial hypersurface M_0 has positive sectional curvature. We consider a smooth family of immersions $X : M^n \times [0, T) \to \mathbb{H}^{n+1}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} X(x,t) = (\phi(t) - F^\alpha(\mathcal{W}))\nu(x,t), \\ X(\cdot, 0) = X_0(\cdot), \end{cases} \tag{1.2}$$

where $\alpha > 0$, $\nu(x,t)$ is the unit outward normal of $M_t = X(M,t)$, F is a smooth, symmetric, strictly increasing and degree one homogeneous function of the Weingarten matrix $\mathcal{W}$ of M_t. The global term $\phi(t)$ in (1.2) is defined by

$$\phi(t) = \frac{1}{|M_t|} \int_{M_t} F^\alpha \, d\mu_t \tag{1.3}$$

such that the volume of Ω_t remains constant along the flow (1.2), where $d\mu_t$ is the area measure on M_t with respect to the induced metric.

Since $F(\mathcal{W})$ is symmetric with respect to the components of $\mathcal{W}$, by a theorem of Schwarz [27] we can write $F(\mathcal{W}) = f(\kappa)$ as a symmetric function of the eigenvalues of $\mathcal{W}$. We assume that f satisfies the following assumption:

Assumption 1.1. *Suppose f is a smooth symmetric function defined on the positive cone $\Gamma_+ := \{\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n : \kappa_i > 0,\ \forall i = 1, \ldots, n\}$, and satisfies*

(i) *f is positive, strictly increasing, homogeneous of degree one and is normalized such that $f(1, \ldots, 1) = 1$;*
(ii) *for any $i \neq j$,*

$$\left(\frac{\partial f}{\partial \kappa_i}\kappa_i - \frac{\partial f}{\partial \kappa_j}\kappa_j\right)(\kappa_i - \kappa_j) \geq 0; \tag{1.4}$$

(iii) *for all $(y_1, \ldots, y_n) \in \mathbb{R}^n$,*

$$\sum_{i,j} \frac{\partial^2 \log f}{\partial \kappa_i \partial \kappa_j} y_i y_j + \sum_{i=1}^n \frac{1}{\kappa_i} \frac{\partial \log f}{\partial \kappa_i} y_i^2 \geq 0. \tag{1.5}$$

Examples satisfying Assumption 1.1 include $f = n^{-1/k} S_k^{1/k}$ $(k > 0)$ and $f = E_k^{1/k}$ (see, e.g., [16, 18]), where

$$E_k = \binom{n}{k}^{-1} \sigma_k(\kappa) = \binom{n}{k}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}, \qquad k = 1, \ldots, n,$$

is the (normalized) k-th mean curvature of M_t and $S_k(\kappa) = \sum_{i=1}^n \kappa_i^k$ is the k-th power sum of κ for $k > 0$. The inequalities (1.4) and (1.5) are equivalent to the statement that $\log F$ is a convex function of the components of $\log \mathcal{W}$, which is the map with the same eigenvectors as $\mathcal{W}$ and eigenvalues $\log \kappa_i$. In particular, if f_1 and f_2 are two symmetric functions satisfying (1.4) and (1.5), then the function f_1^α with $\alpha > 0$ and the product $f_1 f_2$ also satisfy (1.4) and (1.5). Note that the Cauchy–Schwarz inequality and (1.5) imply that any symmetric function f satisfying (1.5) must be inverse concave, i.e., its dual function

$$f_*(z_1, \ldots, z_n) = f(z_1^{-1}, \ldots, z_n^{-1})^{-1}$$

is concave with respect to its argument.

The first result of this paper is the following convergence result for the flow (1.2):

Theorem 1.2. *Let $X_0 : M^n \to \mathbb{H}^{n+1}$ be a smooth embedding such that $M_0 = X_0(M)$ is a closed hypersurface in $\mathbb{H}^{n+1}$ $(n \geq 2)$ with positive sectional curvature. Assume that f satisfies Assumption 1.1, and either*

(i) *f_* vanishes on the boundary of Γ_+, and*

$$\lim_{x \to 0+} f\left(x, \frac{1}{x}, \ldots, \frac{1}{x}\right) = +\infty, \tag{1.6}$$

and $\alpha > 0$, or
(ii) *$n = 2$, $f = (\kappa_1 \kappa_2)^{1/2}$ and $\alpha \in [1/2, 2]$.*

Then the flow (1.2) with global term $\phi(t)$ given by (1.3) has a smooth solution M_t for all time $t \in [0, \infty)$, and M_t has positive sectional curvature for each $t > 0$ and converges smoothly and exponentially to a geodesic sphere of radius r_∞ determined by $\mathrm{Vol}(B(r_\infty)) = \mathrm{Vol}(\Omega_0)$ as $t \to \infty$.

Remark 1.3. Examples of functions f satisfying Assumption 1.1 and condition (i) of Theorem 1.2 include:

(a) $n \geq 2$, $f = n^{-1/k} S_k^{1/k}$ with $k > 0$;
(b) $n \geq 3$, $f = E_k^{1/k}$ with $k = 1, \ldots, n$;
(c) $n = 2$, $f = (\kappa_1 + \kappa_2)/2$.

Remark 1.4. We remark that the contracting curvature flows for surfaces with positive scalar curvature in hyperbolic 3-space $\mathbb{H}^3$ have been studied by the first two authors in a recent work [7].

As a key step in the proof of Theorem 1.2, we prove in §3 that the positivity of sectional curvatures of the evolving hypersurface M_t is preserved along the flow (1.2) with any f satisfying Assumption 1.1 and any $\alpha > 0$. In order to show that the positivity of sectional curvatures are preserved, we consider the sectional curvature as a function on the frame bundle $O(M)$ over M, and apply a maximum principle. This requires a rather delicate computation, using inequalities for the Hessian on the total space of $O(M)$ to show the required inequality for the time derivative at a minimum point. The argument is related to that used by the first author to prove a generalised tensor maximum principle in [4, Theorem 3.2], but cannot be deduced directly from that result. The argument combines the ideas of the generalised tensor maximum principle with those of vector bundle maximum principles for reaction-diffusion equations [8, 20].

We remark that the flow (1.2) with

$$f = \left(\frac{E_k}{E_\ell}\right)^{\frac{1}{k-\ell}}, \qquad 1 \leq \ell < k \leq n, \tag{1.7}$$

and any power $\alpha > 0$ does not preserve positive sectional curvatures: Counterexamples can be produced in the spirit of the constructions in [9, Sections 4–5].

The remaining parts of the proof of Theorem 1.2 will be given in §4. In §4.1, we will derive a uniform estimate on the inner radius and outer radius of the evolving domains Ω_t along the flow (1.2). Recall that the *inner radius* ρ_- and *outer radius* ρ_+ of a bounded domain Ω are defined as

$$\rho_- = \sup \bigcup_{p \in \Omega} \{\rho > 0 : B_\rho(p) \subset \Omega\}, \qquad \rho_+ = \inf \bigcup_{p \in \Omega} \{\rho > 0 : \Omega \subset B_\rho(p)\},$$

where $B_\rho(p)$ denotes the geodesic ball of radius ρ centred at some point p in hyperbolic space. All the previous papers [10–12, 23, 32] on constrained curvature flows in hyperbolic space focus on horospherically convex domains, which have the property that $\rho_+ \leq c(\rho_- + \rho_-^{1/2})$ (see e.g. [12, 23]). However, no such property is known for hypersurfaces with positive sectional curvature. Our idea to overcome this obstacle is to use an Alexandrov reflection argument to bound the diameter of the domain Ω_t enclosed by the flow hypersurface M_t. Then we project the domain Ω_t to the unit ball in the Euclidean space $\mathbb{R}^{n+1}$ via the Klein model of hyperbolic space. The upper bound on the diameter of Ω_t implies that this map has bounded distortion. This together with the preservation of the volume of Ω_t gives a uniform lower bound on the inner radius of Ω_t.

Then in §4.2 we adapt Tso's technique [31] to derive an upper bound on the speed if f satisfies Assumption 1.1, where the positivity of sectional curvatures of M_t will be used to estimate the zero order terms of the evolution equation of the auxiliary function. In §4.3, we will complete the proof of Theorem 1.2 by obtaining uniform bounds on the principal curvatures. In case (i) of Theorem 1.2, the upper bound of f together with the positivity of sectional curvatures imply the uniform two-side positive bound of the principal curvatures of M_t. In case (ii) of Theorem 1.2, the estimate $1 \le \kappa_1\kappa_1 = f(\kappa)^2 \le C$ does not prevent κ_2 from going to infinity. Instead, we will obtain the estimate on the pinching ratio κ_2/κ_1 by applying the maximum principle to the evolution equation of $G(\kappa_1, \kappa_2) = (\kappa_1\kappa_2)^{\alpha-2}(\kappa_2 - \kappa_1)^2$ with $\alpha \in [1/2, 2]$. This idea has been applied by the first two authors in [2,6] to prove the pinching estimate for the contracting flow by powers of Gauss curvature in $\mathbb{R}^3$. Once we have the uniform estimate on the principal curvatures of the evolving hypersurfaces, higher regularity estimates can be derived by a standard argument. A continuation argument then yields the long time existence of the flow, and the Alexandrov reflection argument as in [10, §6] implies the smooth convergence of the flow to a geodesic sphere.

1.2. Alexandrov–Fenchel inequalities

The volume preserving curvature flow is a useful tool in the study of hypersurface geometry. We will illustrate an application of Theorem 1.2 in the proof of Alexandrov–Fenchel type inequalities (involving the quermassintegrals) for hypersurfaces in hyperbolic space. Recall that for a convex domain Ω in hyperbolic space, the quermassintegral $W_k(\Omega)$ is defined as follows (see [26,28]):[1]

$$W_k(\Omega) = \frac{\omega_{k-1}\cdots\omega_0}{\omega_{n-1}\cdots\omega_{n-k}} \int_{\mathcal{L}_k} \chi(L_k \cap \Omega)\, dL_k, \qquad k = 1, \ldots, n, \tag{1.8}$$

where $\mathcal{L}_k$ is the space of k-dimensional totally geodesic subspaces L_k in $\mathbb{H}^{n+1}$ and ω_n denotes the area of the n-dimensional unit sphere in Euclidean space. The function χ is defined to be 1 if $L_k \cap \Omega \neq \emptyset$ and to be 0 otherwise. Furthermore, we set

$$W_0(\Omega) = |\Omega|, \qquad W_{n+1}(\Omega) = |\mathbb{B}^{n+1}| = \frac{\omega_n}{n+1}.$$

If the boundary of Ω is smooth, we can define the principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ and the curvature integrals

$$V_{n-k}(\Omega) = \int_{\partial\Omega} E_k(\kappa)\, d\mu, \qquad k = 0, 1, \ldots, n, \tag{1.9}$$

[1] Note that the definition (1.8) is differs from the definition in [28] by a constant multiple $\frac{n+1-k}{n+1}$.

of the boundary $M = \partial\Omega$. The quermassintegrals and the curvature integrals of a smooth convex domain Ω in $\mathbb{H}^{n+1}$ are related by the following equations (see [28]):

$$V_{n-k}(\Omega) = (n-k)W_{k+1}(\Omega) + kW_{k-1}(\Omega), \qquad k = 1, \ldots, n-1, \tag{1.10}$$

$$V_n(\Omega) = nW_1(\Omega) = |\partial\Omega|, \tag{1.11}$$

$$V_0(\Omega) = \omega_n + nW_{n-1}(\Omega). \tag{1.12}$$

In [32], Wang and Xia proved the Alexandrov–Fenchel inequalities for a smooth h-convex domain Ω in $\mathbb{H}^{n+1}$, which state that

$$W_k(\Omega) \ge f_k \circ f_\ell^{-1}(W_\ell(\Omega)) \tag{1.13}$$

for any $0 \le \ell < k \le n$, with equality if and only if Ω is a geodesic ball, where $f_k : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function defined by $f_k(r) = W_k(B(r))$, the k-th quermassintegral of the geodesic ball of radius r. The proof in [32] is by applying the quermassintegral preserving flow for smooth h-convex hypersurfaces with speed given by the quotient (1.7) and $\alpha = 1$, and is similar to the Euclidean analogue considered by McCoy [24]. The inequality (1.13) implies the inequality

$$\int_{\partial\Omega} E_k \, d\mu \ge |\partial\Omega|(1 + (|\partial\Omega|/\omega_n)^{-2/n})^{k/2} \tag{1.14}$$

for smooth h-convex domains, which compares the curvature integral (1.9) and the boundary area. Note that the inequality (1.14) with $k = 2$ was proved earlier by the third author with Li and Xiong [22] for star-shaped and 2-convex domains using the inverse curvature flow in hyperbolic space. For the other even k, the inequality (1.14) was also proved for smooth h-convex domains using the inverse curvature flow by Ge, Wang and Wu [17]. It is an interesting problem to prove the inequalities (1.13) and (1.14) under an assumption weaker than h-convexity.

Applying the result in Theorem 1.2, we show that the *h-convexity* assumption for the inequality (1.13) can be replaced by the weaker assumption of *positive sectional curvature* in the case $\ell = 0$ and $1 \le k \le n$.

Corollary 1.5. *Let $M = \partial\Omega$ be a smooth closed hypersurface in $\mathbb{H}^{n+1}$ which has positive sectional curvature and encloses a smooth bounded domain Ω. Then for any $n \ge 2$ and $k = 1, \ldots, n$, we have*

$$W_k(\Omega) \ge f_k \circ f_0^{-1}(W_0(\Omega)), \tag{1.15}$$

where $f_k : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function defined by $f_k(r) = W_k(B(r))$, the k-th quermassintegral of the geodesic ball of radius r. Moreover, equality holds in (1.15) if and only if Ω is a geodesic ball.

The quermassintegral $W_k(\Omega_t)$ of the evolving domain Ω_t along the flow (1.2) with $F = E_k^{1/k}$ satisfies (see Lemma 2.3)

$$\frac{d}{dt} W_k(\Omega_t) = \int_{M_t} E_k(\phi(t) - E_k^{\alpha/k}) \, d\mu_t,$$

which is non-positive for each $\alpha > 0$ by the choice (1.3) of $\phi(t)$ and the Hölder inequality. This means that $W_k(\Omega_t)$ is decreasing along the flow (1.2) with $F = E_k^{1/k}$ unless E_k is constant on M_t (which is equivalent to M_t being a geodesic sphere). Then Corollary 1.5 follows from the monotonicity of W_k and the convergence result in Theorem 1.2.

1.3. Volume preserving flow for horospherically convex hypersurfaces

In the second part of this paper, we will consider the flow of h-convex hypersurfaces in hyperbolic space with speed given by functions of the shifted Weingarten matrix $\mathcal{W} - \mathrm{I}$ plus a global term chosen to preserve modified quermassintegrals of the evolving domains. Let us first define the following modified quermassintegrals:

$$\widetilde{W}_k(\Omega) := \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} W_i(\Omega), \qquad k = 0, \ldots, n, \tag{1.16}$$

for an h-convex domain Ω in hyperbolic space. Thus $\widetilde{W}_k$ is a linear combination of the quermassintegrals of Ω. In particular, $\widetilde{W}_0(\Omega) = |\Omega|$ is the volume of Ω. The modified quermassintegrals defined in (1.16) satisfy the following property:

Proposition 1.6. *The modified quermassintegral $\widetilde{W}_k$ is monotone with respect to inclusion for h-convex domains: if Ω_0 and Ω_1 are h-convex domains with $\Omega_0 \subset \Omega_1$, then $\widetilde{W}_k(\Omega_0) \leq \widetilde{W}_k(\Omega_1)$.*

This property is not obvious from the definition (1.16) and its proof will be given in §5. We will first investigate some of the properties of horospherically convex regions in hyperbolic space $\mathbb{H}^{n+1}$. In particular, for such regions we define a *horospherical Gauss map*, which is a map to the unit sphere, and we show that each horospherically convex region is completely described in terms of a scalar function u on the sphere $\mathbb{S}^n$ which we call the *horospherical support function*. There are interesting formal similarities between this situation and that of convex Euclidean bodies. We show that the h-convexity of a region Ω is equivalent to the matrix

$$A_{ij} = \bar{\nabla}_j \bar{\nabla}_k \varphi - \frac{|\bar{\nabla}\varphi|^2}{2\varphi} \bar{g}_{ij} + \frac{\varphi - \varphi^{-1}}{2} \bar{g}_{ij}$$

on the sphere $\mathbb{S}^n$ being positive definite, where $\bar{g}_{ij}$ is the standard round metric on $\mathbb{S}^n$, $\varphi = \mathrm{e}^u$ and u is the horospherical support function of Ω. The shifted Weingarten matrix $\mathcal{W} - \mathrm{I}$ is related to the matrix A_{ij} by

$$A_{ij} = \varphi^{-1} [(\mathcal{W} - \mathrm{I})^{-1}]_i^k \bar{g}_{kj}. \tag{1.17}$$

Using this characterization of h-convex domains, for any two h-convex domains Ω_0 and Ω_1 with $\Omega_0 \subset \Omega_1$ we can find a foliation of h-convex domains Ω_t which is expanding from Ω_0 to Ω_1. This can be used to prove Proposition 1.6 by computing the variation of $\widetilde{W}_k$. We expect that the description of horospherically convex regions which we develop here will be useful in further investigations beyond the scope of this paper.

The flow we will consider is the following:

$$\begin{cases} \frac{\partial}{\partial t}X(x,t) = (\phi(t) - F(\mathcal{W} - \mathrm{I}))\nu(x,t), \\ X(\cdot,0) = X_0(\cdot), \end{cases} \tag{1.18}$$

for a smooth and strictly h-convex hypersurface in hyperbolic space, where F is a smooth, symmetric, degree one homegeneous function of the shifted Weingarten matrix $\mathcal{W} - \mathrm{I} = (h_i^j - \delta_i^j)$. For simplicity, we denote $S_{ij} = h_i^j - \delta_i^j$. Note that the eigenvalues of (S_{ij}) are the shifted principal curvatures $\lambda = (\lambda_1, \ldots, \lambda_n) = (\kappa_1 - 1, \ldots, \kappa_n - 1)$. Again by a theorem of Schwarz [27], $F(\mathcal{W} - \mathrm{I}) = f(\lambda)$, where f is a smooth symmetric function of n variables $\lambda = (\lambda_1, \ldots, \lambda_n)$. We choose the global term $\phi(t)$ in (1.18) as

$$\phi(t) = \left(\int_{M_t} E_l(\lambda)\,d\mu_t\right)^{-1} \int_{M_t} E_l(\lambda) F\,d\mu_t, \qquad l = 0, \ldots, n, \tag{1.19}$$

such that $\widetilde{W}_l(\Omega_t)$ remains constant, where Ω_t is the domain enclosed by the evolving hypersurface M_t.

We will prove the following result for the flow (1.18) with $\phi(t)$ given in (1.19).

Theorem 1.7. *Let $n \geq 2$ and $X_0 : M^n \to \mathbb{H}^{n+1}$ be a smooth embedding such that $M_0 = X_0(M)$ is a smooth closed and strictly h-convex hypersurface in $\mathbb{H}^{n+1}$. If f is a smooth, symmetric, increasing and degree one homogeneous function, and either*

- (i) *f is concave and f approaches zero on the boundary of the positive cone Γ_+, or*
- (ii) *f is concave and inverse concave, or*
- (iii) *f is inverse concave and its dual function f_* approaches zero on the boundary of Γ_+, or*
- (iv) *$n = 2$,*

then the flow (1.18) with the global term $\phi(t)$ given by (1.19) has a smooth solution M_t for all time $t \in [0, \infty)$, and M_t is strictly h-convex for any $t > 0$ and converges smoothly and exponentially to a geodesic sphere of radius r_∞ determined by $\widetilde{W}_l(B(r_\infty)) = \widetilde{W}_l(\Omega_0)$ as $t \to \infty$.

Constrained curvature flows in hyperbolic space by degree one homogeneous, concave and inverse concave function of the principal curvatures were studied by Makowski [23] and Wang and Xia [32]. The quermassintegral preserving flow by any positive power of a degree one homogeneous function f of the principal curvatures, which is inverse concave and its dual function f_* approaches zero on the boundary of the positive cone Γ_+, was studied recently by the first and third authors [10]. Note that the speed function f of the flow (1.18) in Theorem 1.7 is not a homogeneous function of the principal curvatures κ_i and there are essential differences in the analysis compared with the previously mentioned work [10, 23, 32].

The key step in the proof of Theorem 1.7 is a pinching estimate for the shifted principal curvatures λ_i. That is, we will show that the ratio of the largest shifted principal curvature λ_n to the smallest shifted principal curvature λ_1 is controlled by its initial value along the flow (1.18). For the proof, we adapt methods from the proof of pinching estimates of the principal curvatures for contracting curvature flows [1,4,5,9] and the constrained curvature flows in Euclidean space [24,25]. In particular, in case (iii) we define the tensor $T_{ij} = S_{ij} - \varepsilon F \delta_i^j$ and show that the positivity of T_{ij} is preserved by applying the tensor maximum principle (proved by the first author [4]). The inverse concavity is used to estimate the sign of the gradient terms. This case is similar to the pinching estimate for the contracting curvature flow in Euclidean case [9, Lemma 11]. Although the proof there is given in terms of the Gauss map parametrisation of the convex solutions of the flow in Euclidean space, which is not available in hyperbolic space, we can deal with the gradient terms directly using the inverse concavity of f.

To prove Theorem 1.7, we next show that the inner radius and outer radius of the enclosed domain Ω_t of the evolving hypersurface M_t satisfy a uniform estimate $0 < C^{-1} < \rho_-(t) \leq \rho_+(t) \leq C$ for some positive constant C. This relies on the preservation of $\widetilde{W}_l(\Omega_t)$ and the monotonicity of $\widetilde{W}_l$ under inclusion of h-convex domains stated in Proposition 1.6. With the estimate on the inner radius and outer radius, the technique of Tso [31] yields the upper bound on F and the Harnack inequality of Krylov and Safonov [21] yields the lower bound on F. The pinching estimate then gives the estimate on the shifted principal curvatures λ_i. The long time existence and the convergence of the flow follow by a standard argument.

The result in Theorem 1.7 is useful in the study of the geometry of hypersurfaces. The first application of Theorem 1.7 is the following rigidity result.

Corollary 1.8. *Let M be a smooth, closed and strictly h-convex hypersurface in $\mathbb{H}^{n+1}$ with principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ satisfying $f(\lambda) = C$ for some constant $C > 0$, where $\lambda = (\lambda_1, \ldots, \lambda_i)$ with $\lambda_i = \kappa_i - 1$ and f is a symmetric function satisfying the condition of Theorem 1.7. Then M is a geodesic sphere.*

The second application of Theorem 1.7 is a new class of Alexandrov–Fenchel type inequalities between quermassintegrals of h-convex hypersurface in hyperbolic space.

Corollary 1.9. *Let $M = \partial\Omega$ be a smooth, closed and strictly h-convex hypersurface in $\mathbb{H}^{n+1}$. Then for any $0 \leq l < k \leq n$,*

$$\widetilde{W}_k(\Omega) \geq \tilde{f}_k \circ \tilde{f}_l^{-1}(\widetilde{W}_l(\Omega)) \tag{1.20}$$

with equality holding if and only if Ω is a geodesic ball. Here the function $\tilde{f}_k : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\tilde{f}_k(r) = \widetilde{W}_k(B(r))$, which is an increasing function by Proposition 1.6; and $\tilde{f}_l^{-1}$ is the inverse function of $\tilde{f}_l$.

The inequality (1.20) can be obtained by applying Theorem 1.7 with

$$f = \left(\frac{E_k(\lambda)}{E_l(\lambda)}\right)^{\frac{1}{k-l}}, \quad 0 \leq l < k \leq n, \tag{1.21}$$

in the flow (1.18). We see that along the flow (1.18) with this f, the modified quermassintegral $\widetilde{W}_l(\Omega_t)$ remains a constant and $\widetilde{W}_k(\Omega_t)$ is decreasing in time by the Hölder inequality. In fact, by Lemma 2.4 the modified quermassintegral evolves by

$$\frac{d}{dt}\widetilde{W}_k(\Omega_t) = \int_{M_t} E_k(\lambda)\left(\phi(t) - \left(\frac{E_k(\lambda)}{E_l(\lambda)}\right)^{\frac{1}{k-l}}\right) d\mu_t. \tag{1.22}$$

Applying the Hölder inequality to the equation (1.22) shows that $\widetilde{W}_k(\Omega_t)$ is decreasing in time unless $E_k(\lambda) = CE_l(\lambda)$ on M_t (which is equivalent to M_t being a geodesic sphere by Corollary 1.8). Since the flow exists for all time and converges to a geodesic sphere B_r, the inequality (1.20) follows from the monotonicity of $\widetilde{W}_k(\Omega_t)$ and the preservation of $\widetilde{W}_l(\Omega_t)$.

Remark 1.10. We remark that the inequalities (1.20) are new and can be viewed as an improvement of the inequalities (1.13). For example, the inequality (1.20) with $l = 0$ implies that

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\left(W_i(\Omega) - f_i \circ f_0^{-1}(W_0(\Omega))\right) \geq 0. \tag{1.23}$$

By induction on k, (1.23) implies that each $W_i(\Omega) - f_i \circ f_0^{-1}(W_0(\Omega))$ is non-negative for h-convex domains. Thus our inequalities (1.20) imply that the linear combinations of $W_i(\Omega) - f_i \circ f_0^{-1}(W_0(\Omega))$ as in (1.23) are also non-negative for h-convex domains.

2. Preliminaries

In this section we collect some properties of smooth symmetric functions of n variables, and recall the evolution equations of geometric quantities along the flows (1.2) and (1.18).

2.1. *Properties of symmetric functions*

For a smooth symmetric function $F(A) = f(\kappa(A))$, where $A = (A_{ij}) \in \mathrm{Sym}(n)$ is a symmetric matrix and $\kappa(A) = (\kappa_1, \ldots, \kappa_n)$ gives the eigenvalues of A, we denote by $\dot{F}^{ij}$ and $\ddot{F}^{ij,kl}$ the first and second derivatives of F with respect to the components of its argument, so that

$$\frac{\partial}{\partial s}F(A+sB)\Big|_{s=0} = \dot{F}^{ij}(A)B_{ij}, \qquad \frac{\partial^2}{\partial s^2}F(A+sB)\Big|_{s=0} = \ddot{F}^{ij,kl}(A)B_{ij}B_{kl}$$

for any two symmetric matrices A, B. We also use the notation

$$\dot{f}^i(\kappa) = \frac{\partial f}{\partial \kappa_i}(\kappa), \qquad \ddot{f}^{ij}(\kappa) = \frac{\partial^2 f}{\partial \kappa_i \partial \kappa_j}(\kappa)$$

for the first and second derivatives of f with respect to κ. At any diagonal A with distinct eigenvalues $\kappa = \kappa(A)$, the first derivative of F satisfies

$$\dot{F}^{ij}(A) = \dot{f}^i(\kappa)\delta_{ij},$$

and the second derivative of F in direction $B \in \mathrm{Sym}(n)$ is given in terms of $\dot{f}$ and $\ddot{f}$ by (see [4])

$$\ddot{F}^{ij,kl}(A)B_{ij}B_{kl} = \sum_{i,j} \ddot{f}^{ij}(\kappa)B_{ii}B_{jj} + 2\sum_{i>j} \frac{\dot{f}^i(\kappa) - \dot{f}^j(\kappa)}{\kappa_i - \kappa_j} B_{ij}^2. \tag{2.1}$$

This formula makes sense as a limit in the case of any repeated values of κ_i.

From the equation (2.1), we have

Lemma 2.1. *Suppose A has distinct eigenvalues $\kappa = \kappa(A)$. Then F is concave at A if and only if f is concave at κ and*

$$(\dot{f}^k - \dot{f}^l)(\kappa_k - \kappa_l) \le 0, \quad \forall k \ne l. \tag{2.2}$$

In this paper, we also need the inverse concavity of f in many cases. We include the properties of inverse concave functions in the following lemma.

Lemma 2.2 ([4, 10]). (i) *If f is inverse concave, then*

$$\sum_{k,l=1}^n \ddot{f}^{kl} y_k y_l + 2\sum_{k=1}^n \frac{\dot{f}^k}{\kappa_k} y_k^2 \ge 2f^{-1}\Big(\sum_{k=1}^n \dot{f}^k y_k\Big)^2 \tag{2.3}$$

for any $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, and

$$\frac{\dot{f}^k - \dot{f}^l}{\kappa_k - \kappa_l} + \frac{\dot{f}^k}{\kappa_l} + \frac{\dot{f}^l}{\kappa_k} \ge 0, \quad \forall k \ne l. \tag{2.4}$$

(ii) *If $f = f(\kappa_1, \ldots, \kappa_n)$ is inverse concave, homogeneous of degree one and normalized by $f(1, \ldots, 1) = 1$, then*

$$\sum_{i=1}^n \dot{f}^i \kappa_i^2 \ge f^2. \tag{2.5}$$

2.2. *Evolution equations*

Along any smooth flow

$$\frac{\partial}{\partial t} X(x,t) = \varphi(x,t)\nu(x,t) \tag{2.6}$$

of hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$, where φ is a smooth function on the evolving hypersurfaces $M_t = X(M^n, t)$, we have the following evolution equations for the induced metric g_{ij}, the induced area element $d\mu_t$ and the Weingarten matrix $\mathcal{W} = (h_i^j)$ of M_t:

$$\frac{\partial}{\partial t} g_{ij} = 2\varphi h_{ij}, \tag{2.7}$$

$$\frac{\partial}{\partial t} d\mu_t = nE_1\varphi\, d\mu_t, \tag{2.8}$$

$$\frac{\partial}{\partial t} h_i^j = -\nabla^j\nabla_i\varphi - \varphi(h_i^k h_k^j - \delta_i^j). \tag{2.9}$$

From the evolution equations (2.8) and (2.9), we can derive the evolution equation of the curvature integral V_{n-k}:

$$\begin{aligned}\frac{d}{dt}V_{n-k}(\Omega_t) &= \frac{d}{dt}\int_{M_t} E_k\, d\mu_t = \int_{M_t}\Big(\frac{\partial}{\partial t}E_k + nE_1E_k\varphi\Big)\, d\mu_t \\ &= \int_{M_t}\Big(-\frac{\partial E_k}{\partial h_i^j}\nabla^j\nabla_i\varphi - \varphi\frac{\partial E_k}{\partial h_i^j}(h_i^k h_k^j - \delta_i^j) + nE_1E_k\varphi\Big)\, d\mu_t \\ &= \int_{M_t}\varphi\big((n-k)E_{k+1} + kE_{k-1}\big)\, d\mu_t, \end{aligned} \tag{2.10}$$

where we used integration by parts and the fact that $\frac{\partial E_k}{\partial h_i^j}$ is divergence free. Since the quermassintegrals are related to the curvature integrals by (1.10)–(1.12), applying an induction argument to (2.10) yields

Lemma 2.3 (cf. [10,32]). *Along the flow (2.6), the quermassintegral W_k of the evolving domain Ω_t satisfies*

$$\frac{d}{dt}W_k(\Omega_t) = \int_{M_t} E_k(\kappa)\varphi\, d\mu_t, \qquad k = 0, \dots, n.$$

We can also derive the following evolution equation for the modified quermassintegrals.

Lemma 2.4. *Along the flow (2.6), the modified quermassintegral $\widetilde{W}_k$ of the evolving domain Ω_t satisfies*

$$\frac{d}{dt}\widetilde{W}_k(\Omega_t) = \int_{M_t} E_k(\lambda)\varphi\, d\mu_t, \qquad k = 0, \dots, n,$$

where $\lambda = (\lambda_1, \dots, \lambda_n) = (\kappa_1 - 1, \dots, \kappa_n - 1)$ are the shifted principal curvatures of M_t.

Proof. Firstly, we derive the formula for $\sigma_k(\lambda)$ in terms of $\sigma_i(\kappa)$, $i = 0, \dots, k$. By the definition of the elementary symmetric polynomials, we have

$$\prod_{i=1}^n (t + \lambda_i) = \sum_{k=0}^n \sigma_k(\lambda) t^{n-k}.$$

On the other hand,

$$\begin{aligned}\prod_{i=1}^n (t + \lambda_i) &= \prod_{i=1}^n (t - 1 + \kappa_i) = \sum_{l=0}^n \sigma_l(\kappa)(t-1)^{n-l} \\ &= \sum_{l=0}^n \sigma_l(\kappa)\sum_{i=0}^{n-l}\binom{n-l}{i} t^i(-1)^{n-l-i} \\ &= \sum_{k=0}^n\Big(\sum_{i=0}^k \binom{n-i}{k-i}(-1)^{k-i}\sigma_i(\kappa)\Big)t^{n-k}.\end{aligned}$$

Comparing the coefficients of t^{n-k}, we have

$$\begin{aligned}\sigma_k(\lambda) &= \sum_{i=0}^{k}\binom{n-i}{k-i}(-1)^{k-i}\sigma_i(\kappa) = \sum_{i=0}^{k}\binom{n-i}{k-i}(-1)^{k-i}\binom{n}{i}E_i(\kappa)\\ &= \binom{n}{k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}E_i(\kappa).\end{aligned}$$

Equivalently,

$$E_k(\lambda) = \binom{n}{k}^{-1}\sigma_k(\lambda) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}E_i(\kappa). \tag{2.11}$$

Then by the definition (1.16) of $\widetilde{W}_k$ and Lemma 2.3,

$$\begin{aligned}\frac{d}{dt}\widetilde{W}_k(\Omega_t) &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{d}{dt}W_i(\Omega_t)\\ &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\int_{M_t}E_i(\kappa)\varphi\, d\mu_t = \int_{M_t}E_k(\lambda)\varphi\, d\mu_t.\end{aligned}$$ □

If we consider the flow (1.2), i.e., $\varphi = \phi(t) - \Psi(\mathcal{W})$, using (2.9) and Simons' identity we have the evolution equations for the curvature function $\Psi = \Psi(\mathcal{W})$ and the Weingarten matrix $\mathcal{W} = (h_i^j)$ of M_t (see [10]):

$$\frac{\partial}{\partial t}\Psi = \dot{\Psi}^{kl}\nabla_k\nabla_l\Psi + (\Psi - \phi(t))(\dot{\Psi}^{ij}h_i^k h_k^j - \dot{\Psi}^{ij}\delta_i^j) \tag{2.12}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}h_i^j = {}& \dot{\Psi}^{kl}\nabla_k\nabla_l h_i^j + \ddot{\Psi}^{kl,pq}\nabla_i h_{kl}\nabla^j h_{pq} + (\dot{\Psi}^{kl}h_k^r h_{rl} + \dot{\Psi}^{kl}g_{kl})h_i^j\\ & - \dot{\Psi}^{kl}h_{kl}(h_i^p h_p^j + \delta_i^j) + (\Psi - \phi(t))(h_i^k h_k^j - \delta_i^j),\end{aligned} \tag{2.13}$$

where ∇ denotes the Levi-Civita connection with respect to the induced metric g_{ij} on M_t, and $\dot{\Psi}^{kl}$, $\ddot{\Psi}^{kl,pq}$ denote the derivatives of Ψ with respect to the components of the Weingarten matrix $\mathcal{W} = (h_i^j)$.

If we consider the flow (1.18) of h-convex hypersurfaces, i.e., $\varphi = \phi(t) - F(\mathcal{W} - \mathrm{I})$, we have the similar evolution equation for the curvature function F:

$$\frac{\partial}{\partial t}F = \dot{F}^{kl}\nabla_k\nabla_l F + (F - \phi(t))(\dot{F}^{ij}h_i^k h_k^j - \dot{F}^{ij}\delta_i^j), \tag{2.14}$$

and a parabolic-type equation for the Weingarten matrix $\mathcal{W} = (h_i^j)$ of M_t:

$$\begin{aligned}\frac{\partial}{\partial t}h_i^j = {}& \dot{F}^{kl}\nabla_k\nabla_l h_i^j + \ddot{F}^{kl,pq}\nabla_i h_{kl}\nabla^j h_{pq} + (\dot{F}^{kl}h_k^r h_{rl} + \dot{F}^{kl}g_{kl})h_i^j\\ & - \dot{F}^{kl}h_{kl}(h_i^p h_p^j + \delta_i^j) + (F - \phi(t))(h_i^k h_k^j - \delta_i^j).\end{aligned} \tag{2.15}$$

However, we observe that $\dot{F}^{kl}$, $\ddot{F}^{kl,pq}$ in (2.14) and (2.15) denote the derivatives of F with respect to the components of the shifted Weingarten matrix $\mathcal{W} - \mathrm{I}$, so the homogeneity of F implies that $\dot{F}^{kl}(h_k^l - \delta_k^l) = F$. Denote the components of the shifted Weingarten matrix by $S_{ij} = h_i^j - \delta_i^j$. Then the equation (2.15) implies that

$$\frac{\partial}{\partial t} S_{ij} = \dot{F}^{kl}\nabla_k\nabla_l S_{ij} + \ddot{F}^{kl,pq}\nabla_i h_{kl}\nabla^j h_{pq} + (\dot{F}^{kl}S_{kr}S_{rl} + 2F - 2\phi(t))S_{ij} - (\phi(t) + \dot{F}^{kl}\delta_k^l)S_{ik}S_{kj} + \dot{F}^{kl}S_{kr}S_{rl}\delta_i^j. \tag{2.16}$$

2.3. A generalised maximum principle

In §6.1, we will use the tensor maximum principle to prove the pinching estimate along the flow (1.18). For the convenience of the readers, we include here the statement of the tensor maximum principle, which was first proved by Hamilton [19] and was generalised by the first author [4].

Theorem 2.5 ([4]). *Let S_{ij} be a smooth time-varying symmetric tensor field on a compact manifold M, satisfying*

$$\frac{\partial}{\partial t} S_{ij} = a^{kl}\nabla_k\nabla_l S_{ij} + u^k\nabla_k S_{ij} + N_{ij},$$

where a^{kl} and u are smooth, ∇ is a (possibly time-dependent) smooth symmetric connection, and a^{kl} is positive definite everywhere. Suppose that

$$N_{ij}v^i v^j + \sup_{\Lambda} 2a^{kl}(2\Lambda_k^p\nabla_l S_{ip}v^i - \Lambda_k^p\Lambda_l^q S_{pq}) \geq 0 \tag{2.17}$$

whenever $S_{ij} \geq 0$ and $S_{ij}v^j = 0$. If S_{ij} is positive definite everywhere on M at $t = 0$ and on ∂M for $0 \leq t \leq T$, then it is positive on $M \times [0, T]$.

3. Preserving positive sectional curvature

In this section, we will prove that the flow (1.2) preserves the positivity of sectional curvature if $\alpha > 0$ and f satisfies Assumption 1.1.

Theorem 3.1. *If the initial hypersurface M_0 has positive sectional curvature, then along the flow (1.2) in $\mathbb{H}^{n+1}$ with f satisfying Assumption 1.1 and any power $\alpha > 0$ the evolving hypersurface M_t has positive sectional curvature for $t > 0$.*

Proof. The sectional curvature defines a smooth function on the Grassmannian bundle of two-dimensional subspaces of TM. For convenience we lift it to a function on the orthonormal frame bundle $O(M)$ over M: Given a point $x \in M$ and $t \geq 0$, and a frame

$\mathbb{O} = \{e_1, \ldots, e_n\}$ for T_xM which is orthonormal with respect to $g(x,t)$, we define

$$G(x,t,\mathbb{O}) = h_{(x,t)}(e_1,e_1)h_{(x,t)}(e_2,e_2) - h_{(x,t)}(e_1,e_2)^2 - 1.$$

We consider a point (x_0,t_0) and a frame $\mathbb{O}_0 = \{\bar{e}_1,\ldots,\bar{e}_n\}$ at which a new minimum of the function G is attained, so that $G(x,t,\mathbb{O}) \geq G(x_0,t_0,\mathbb{O}_0)$ for all $x \in M$, all $t \in [0,t_0]$, and all $\mathbb{O} \in F(M)_{(x,t)}$. The fact that $\mathbb{O}_0$ achieves the minimum of G over the fibre $F(M)_{(x_0,t_0)}$ implies that $\bar{e}_1$ and $\bar{e}_2$ can be rotated so as to be the eigenvectors of $h_{(x_0.t_0)}$ corresponding to κ_1 and κ_2, where $\kappa_1 \leq \cdots \leq \kappa_n$ are the principal curvatures at (x_0,t_0). Since G is invariant under rotation in the subspace orthogonal to $\bar{e}_1$ and $\bar{e}_2$, we can assume that $h(\bar{e}_i,\bar{e}_i) = \kappa_i$ and $h(\bar{e}_i,\bar{e}_j) = 0$ for $i \neq j$.

The time derivative of G at $(x_0,t_0,\mathbb{O}_0)$ is given by (2.13), noting that the frame $\mathbb{O}(t)$ for T_xM defined by $\frac{d}{dt}e_i(t) = (F^\alpha - \phi)\mathcal{W}(e_i)$ remains orthonormal with respect to $g(x,t)$ if $e_i(t_0) = \bar{e}_i$ for each i. This yields

$$\begin{aligned}
\frac{\partial}{\partial t}G\bigg|_{(x_0,t_0,\mathbb{O}_0)} &= \kappa_1\frac{\partial}{\partial t}h_2^2 + \kappa_2\frac{\partial}{\partial t}h_1^1 \\
&= \kappa_1\dot{\Psi}^{kl}\nabla_k\nabla_l h_{22} + \kappa_2\dot{\Psi}^{kl}\nabla_k\nabla_l h_{11} + \kappa_1\ddot{\Psi}(\nabla_2 h,\nabla_2 h) + \kappa_2\ddot{\Psi}(\nabla_1 h,\nabla_1 h) \\
&\quad + 2(\dot{\Psi}^{kl}h_k^r h_{rl} + \dot{\Psi}^{kl}g_{kl})\kappa_1\kappa_2 - (\alpha-1)\Psi\kappa_1\kappa_2(\kappa_1+\kappa_2) \\
&\quad - (\alpha+1)\Psi(\kappa_1+\kappa_2) - \phi(t)(\kappa_1\kappa_2 - 1)(\kappa_1+\kappa_2). \qquad (3.1)
\end{aligned}$$

Since $\Psi = f^\alpha$, the zero order terms in (3.1) satisfy

$$\begin{aligned}
&2(\dot{\Psi}^{kl}h_k^r h_{rl} + \dot{\Psi}^{kl}g_{kl})\kappa_1\kappa_2 - (\alpha-1)\Psi\kappa_1\kappa_2(\kappa_1+\kappa_2) \\
&\qquad - (\alpha+1)\Psi(\kappa_1+\kappa_2) - \phi(t)(\kappa_1\kappa_2-1)(\kappa_1+\kappa_2) \\
&\quad = 2\alpha f^{\alpha-1}\sum_k \dot{f}^k(\kappa_k-\kappa_2)(\kappa_k-\kappa_1) \\
&\qquad + G\Big(f^{\alpha-1}\sum_k \dot{f}^k\kappa_k(2\alpha\kappa_k - (\alpha-1)(\kappa_1+\kappa_2)) - \phi(t)(\kappa_1+\kappa_2)\Big) \\
&\quad \geq -CG,
\end{aligned}$$

where C is a bound for the smooth function in the last bracket. To estimate the remaining terms in (3.1), we consider the second derivatives of G along a curve on $O(M)$ defined as follows: We let γ be any geodesic of $g(t_0)$ in M with $\gamma(0) = x_0$, and define a frame $\mathbb{O}(s) = (e_1(s),\ldots,e_n(s))$ at $\gamma(s)$ by taking $e_i(0) = \bar{e}_i$ for each i, and $\nabla_s e_i(s) = \Gamma_{ij}e_j(s)$ for some constant antisymmetric matrix Γ. Then we compute

$$\begin{aligned}
\frac{d^2}{ds^2}G(x(s),t_0,\mathbb{O}(s))\bigg|_{s=0} &= \kappa_2\nabla_s^2 h_{11} + \kappa_1\nabla_s^2 h_{22} + 2(\nabla_s h_{22}\nabla_s h_{11} - (\nabla_s h_{12})^2) \\
&\quad + 4\sum_{p>2}\Gamma_{1p}\kappa_2\nabla_s h_{1p} + 4\sum_{p>2}\Gamma_{2p}\kappa_1\nabla_s h_{2p} \\
&\quad + 2\sum_{p>2}\Gamma_{1p}^2\kappa_2(\kappa_p-\kappa_1) + 2\sum_{p>2}\Gamma_{2p}^2\kappa_1(\kappa_p-\kappa_2). \qquad (3.2)
\end{aligned}$$

Since G has a minimum at $(x_0, t_0, \mathbb{O}_0)$, the right-hand side of (3.2) is non-negative for any choice of Γ. Minimizing over Γ gives

$$\begin{aligned} 0 \le \kappa_2 \nabla_s^2 h_{11} + \kappa_1 \nabla_s^2 h_{22} + 2(\nabla_s h_{22} \nabla_s h_{11} - (\nabla_s h_{12})^2) \\ - 2\sum_{p>2} \frac{\kappa_2}{\kappa_p - \kappa_1} (\nabla_s h_{1p})^2 - 2\sum_{p>2} \frac{\kappa_1}{\kappa_p - \kappa_2} (\nabla_s h_{2p})^2, \end{aligned} \tag{3.3}$$

with the terms on the last line regarded as vanishing if the denominators vanish (since the corresponding component of ∇h vanishes in that case). This gives

$$\begin{aligned} \frac{\partial}{\partial t} G\Big|_{(x_0,t_0,\mathbb{O}_0)} &\ge \kappa_1 \ddot{\Psi}(\nabla_2 h, \nabla_2 h) + \kappa_2 \ddot{\Psi}(\nabla_1 h, \nabla_1 h) - 2\sum_k \dot{\Psi}^k (\nabla_k h_{22} \nabla_k h_{11} - (\nabla_k h_{12})^2) \\ &+ 2\sum_k \dot{\Psi}^k \Big(\sum_{p>2} \frac{\kappa_2}{\kappa_p - \kappa_1} (\nabla_k h_{1p})^2 + \sum_{p>2} \frac{\kappa_1}{\kappa_p - \kappa_2} (\nabla_k h_{2p})^2 \Big) - CG. \end{aligned} \tag{3.4}$$

The right-hand side can be expanded using $\Psi = f^\alpha$ and the identity (2.1):

$$\begin{aligned} &\frac{f^{1-\alpha}}{\alpha} \Big(\frac{d}{dt} G + CG \Big) \\ &\quad \ge \kappa_2 \Big(\sum_{k,l} \ddot{f}^{kl} \nabla_1 h_{kk} \nabla_1 h_{ll} + (\alpha - 1) \frac{(\nabla_1 f)^2}{f} + \sum_{k \ne l} \frac{\dot{f}^k - \dot{f}^l}{\kappa_k - \kappa_l} (\nabla_1 h_{kl})^2 \Big) \\ &\qquad + \kappa_1 \Big(\sum_{k,l} \ddot{f}^{kl} \nabla_2 h_{kk} \nabla_2 h_{ll} + (\alpha - 1) \frac{(\nabla_2 f)^2}{f} + \sum_{k \ne l} \frac{\dot{f}^k - \dot{f}^l}{\kappa_k - \kappa_l} (\nabla_2 h_{kl})^2 \Big) \\ &\qquad - 2\sum_k \dot{f}^k (\nabla_k h_{22} \nabla_k h_{11} - (\nabla_k h_{12})^2) \\ &\qquad + 2\sum_k \dot{f}^k \Big(\sum_{p>2} \frac{\kappa_2}{\kappa_p - \kappa_1} (\nabla_k h_{1p})^2 + \sum_{p>2} \frac{\kappa_1}{\kappa_p - \kappa_2} (\nabla_k h_{2p})^2 \Big). \end{aligned}$$

Note that by assumption the function f satisfies the inequalities (1.4) and (1.5). By the inequality (1.4), for any $k \ne l$ we have

$$\frac{\dot{f}^k - \dot{f}^l}{\kappa_k - \kappa_l} + \frac{\dot{f}^k}{\kappa_l} = \frac{\dot{f}^k \kappa_k - \dot{f}^l \kappa_l}{(\kappa_k - \kappa_l)\kappa_l} \ge 0.$$

The inequality (1.5) is equivalent to

$$\sum_{k,l} \ddot{f}^{kl} y_k y_l \ge f^{-1} \Big(\sum_{k=1}^n \dot{f}^k y_k \Big)^2 - \sum_{k=1}^n \frac{\dot{f}^k}{\kappa_k} y_k^2$$

for all $(y_1, \dots, y_n) \in \mathbb{R}^n$. These imply that

$$
\begin{aligned}
&\frac{f^{1-\alpha}}{\alpha}\left(\frac{dG}{dt}+CG\right) \geq \kappa_2\Big(\alpha\frac{(\nabla_1 f)^2}{f} - \sum_{k=1}^n \frac{\dot f^k}{\kappa_k}(\nabla_1 h_{kk})^2 - \sum_{k\neq l}\frac{\dot f^k}{\kappa_l}(\nabla_1 h_{kl})^2\Big)\\
&\quad + \kappa_1\left(\alpha\frac{(\nabla_2 f)^2}{f} - \sum_{k=1}^n \frac{\dot f^k}{\kappa_k}(\nabla_2 h_{kk})^2 - \sum_{k\neq l}\frac{\dot f^k}{\kappa_l}(\nabla_2 h_{kl})^2\right)\\
&\quad + 2\sum_k \dot f^k\left(-\nabla_k h_{11}\nabla_k h_{22} + (\nabla_k h_{12})^2\right) + 2\sum_{k=1}^n\sum_{p>2}\frac{\dot f^k}{\kappa_p}(\kappa_2(\nabla_1 h_{kp})^2 + \kappa_1(\nabla_2 h_{kp})^2)\\
&= \kappa_2\alpha\frac{(\nabla_1 f)^2}{f} + \kappa_1\alpha\frac{(\nabla_2 f)^2}{f} + \sum_{k,p=3}^n \frac{\dot f^k}{\kappa_p}(\kappa_2(\nabla_1 h_{kp})^2 + \kappa_1(\nabla_2 h_{kp})^2)\\
&\quad - \kappa_2\left(\frac{\dot f^1}{\kappa_1}(\nabla_1 h_{11})^2 + \frac{\dot f^2}{\kappa_2}(\nabla_1 h_{22})^2 + \frac{\dot f^1}{\kappa_2}(\nabla_1 h_{12})^2 + \frac{\dot f^2}{\kappa_1}(\nabla_1 h_{21})^2\right)\\
&\quad - \kappa_1\left(\frac{\dot f^1}{\kappa_1}(\nabla_2 h_{11})^2 + \frac{\dot f^2}{\kappa_2}(\nabla_2 h_{22})^2 + \frac{\dot f^1}{\kappa_2}(\nabla_2 h_{12})^2 + \frac{\dot f^2}{\kappa_1}(\nabla_2 h_{21})^2\right)\\
&\quad + 2\dot f^1(-\nabla_1 h_{11}\nabla_1 h_{22} + (\nabla_1 h_{12})^2) + 2\dot f^2(-\nabla_2 h_{11}\nabla_2 h_{22} + (\nabla_2 h_{12})^2)\\
&\quad + 2\dot f^1\sum_{p>2}\left(\frac{\kappa_2}{\kappa_p}(\nabla_1 h_{1p})^2 + \frac{\kappa_1}{\kappa_p}(\nabla_2 h_{1p})^2\right) + 2\dot f^2\sum_{p>2}\left(\frac{\kappa_2}{\kappa_p}(\nabla_1 h_{2p})^2 + \frac{\kappa_1}{\kappa_p}(\nabla_2 h_{2p})^2\right)\\
&\quad + 2\sum_{k>2}\dot f^k(-\nabla_k h_{11}\nabla_k h_{22} + (\nabla_k h_{12})^2).
\end{aligned}
\tag{3.5}
$$

Since $(x_0, \mathbb{O}_0)$ is a minimum point of G at time t_0, we have $\nabla_i G = 0$ for $i = 1, \dots, n$, so

$$\kappa_2\nabla_i h_{11} + \kappa_1\nabla_i h_{22} = 0, \qquad i = 1, \dots, n. \tag{3.6}$$

After substituting (3.6) into (3.5), the second to the fourth lines on the right of (3.5) vanish, the last line becomes $2\sum_{k>2}\dot f^k\big(\frac{\kappa_1}{\kappa_2}(\nabla_k h_{22})^2 + (\nabla_k h_{12})^2\big) \geq 0$, and the remaining terms are non-negative.

We conclude that $\frac{\partial}{\partial t}G \geq -CG$ at a spatial minimum point, and so by the maximum principle [20, Lemma 3.5] we have $G \geq \mathrm{e}^{-Ct}\inf_{t=0} G > 0$ under the flow (1.2). □

4. Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2.

4.1. Shape estimate

First, we show that the preservation of the volume of Ω_t, together with a reflection argument, implies that the inner radius and outer radius of Ω_t are uniformly bounded from above and below by positive constants.

Lemma 4.1. *Denote by $\rho_-(t)$, $\rho_+(t)$ the inner radius and outer radius of Ω_t, the domain enclosed by M_t. Then there exist positive constants c_1, c_2 depending only on n, M_0 such that*

$$0 < c_1 \leq \rho_-(t) \leq \rho_+(t) \leq c_2 \tag{4.1}$$

for all time $t \in [0, T)$.

Proof. We first use the Alexandrov reflection method to estimate the diameter of Ω_t. In [10], the first and third authors have already used the Alexandrov reflection method in the proof of convergence of the flow. Let γ be a geodesic line in $\mathbb{H}^{n+1}$, and let $H_{\gamma(s)}$ be the totally geodesic hyperbolic n-plane in $\mathbb{H}^{n+1}$ which is perpendicular to γ at $\gamma(s)$, $s \in \mathbb{R}$. We use the notation H_s^+ and H_s^- for the half-spaces in $\mathbb{H}^{n+1}$ determined by $H_{\gamma(s)}$:

$$H_s^+ := \bigcup_{s' \geq s} H_{\gamma(s')}, \qquad H_s^- := \bigcup_{s' \leq s} H_{\gamma(s')}.$$

For a bounded domain Ω in $\mathbb{H}^{n+1}$, denote $\Omega^+(s) = \Omega \cap H_s^+$ and $\Omega^-(s) = \Omega \cap H_s^-$. The reflection map across $H_{\gamma(s)}$ is denoted by $R_{\gamma,s}$. We define

$$S_\gamma^+(\Omega) := \inf\{s \in \mathbb{R} \mid R_{\gamma,s}(\Omega^+(s)) \subset \Omega^-(s)\}.$$

It has been proved in [10] that for any geodesic line γ in $\mathbb{H}^{n+1}$, $S_\gamma^+(\Omega_t)$ is strictly decreasing along the flow (1.2) unless $R_{\gamma,\bar{s}}(\Omega_t) = \Omega_t$ for some $\bar{s} \in \mathbb{R}$. Note that to prove this property, we only need the convexity of the evolving hypersurface $M_t = \partial\Omega_t$, which is guaranteed by the positivity of the sectional curvature. The readers may refer to [13, 14] for more details on the Alexandrov reflection method.

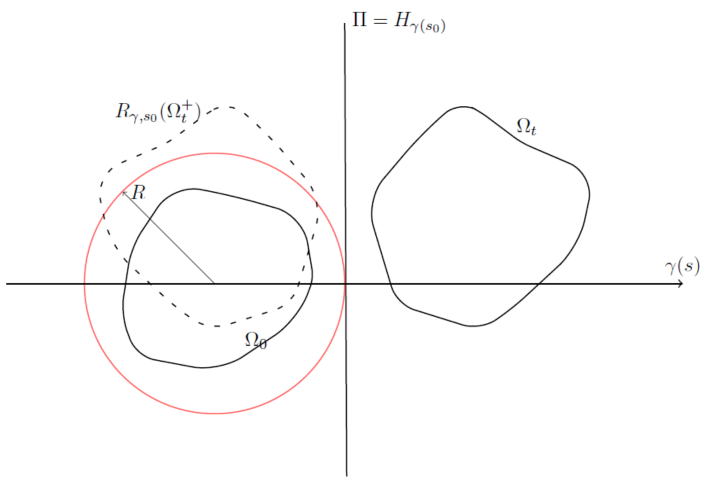

Fig. 1. Ω_t cannot leave B_R.

Choose $R > 0$ such that the initial domain Ω_0 is contained in some geodesic ball $B_R(p)$ of radius R and centred at some point p in hyperbolic space. The above reflection property implies that $\Omega_t \cap B_R(p) \neq \emptyset$ for any $t \in [0, T)$. If not, there exists some time t such that Ω_t does not intersect the geodesic ball B_R. Choose a geodesic line $\gamma(s)$ with the property that there exists a geodesic hyperplane $\Pi = H_{\gamma(s_0)}$ which is perpendicular to $\gamma(s)$ and is tangent to the geodesic sphere ∂B_R, and the domain Ω_t lies in the half-space $H^+_{s_0}$. Then $R_{\gamma,s_0}(\Omega_0^+) = \emptyset \subset \Omega_0^-$. Since $S^+_\gamma(\Omega_t)$ is decreasing, we have $R_{\gamma,s_0}(\Omega_t^+) \subset \Omega_t^-$. However, this is not possible because $\Omega_t^- = \Omega_t \cap H^-_{s_0} = \emptyset$ and $R_{\gamma,s_0}(\Omega_t^+)$ is obviously not empty.

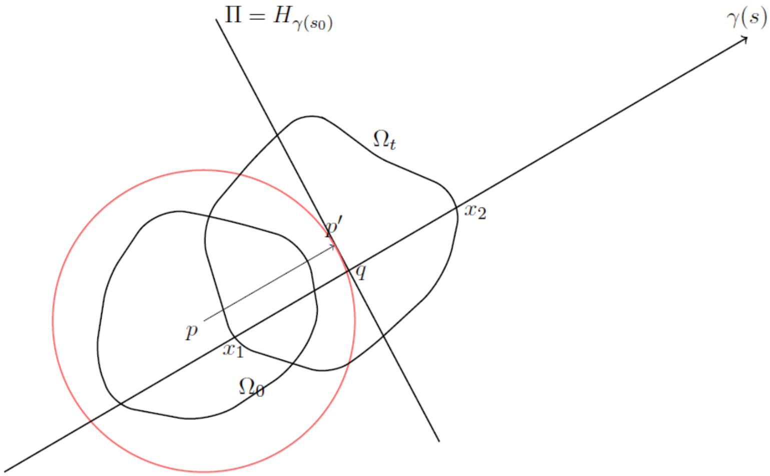

Fig. 2. Diameter of Ω_t is bounded.

For any $t \in [0, T)$, let x_1, x_2 be points on $M_t = \partial\Omega_t$ such that $d(p, x_1) = \min\{d(p, x) : x \in M_t\}$ and $d(p, x_2) = \max\{d(p, x) : x \in M_t\}$, where $d(\cdot, \cdot)$ is the distance in hyperbolic space. Since Ω_0 is contained in the geodesic ball $B_R(p)$ and $\Omega_t \cap B_R(p) \neq \emptyset$, we deduce from $|\Omega_t| = |\Omega_0|$ that $x_1 \in B_R(p)$. If $x_2 \in B_R(p)$, then the diameter of Ω_t is bounded from above by R. Therefore it suffices to study the case $x_2 \notin B_R(p)$. Let $\gamma(s)$ be the geodesic line passing through x_1 and x_2, i.e., there are $s_1 < s_2 \in \mathbb{R}$ such that $\gamma(s_1) = x_1$ and $\gamma(s_2) = x_2$. We choose the geodesic plane $\Pi = H_{\gamma(s_0)}$ for some $s_0 \in (s_1, s_2)$ such that Π is perpendicular to γ and is tangent to the boundary of $B_R(p)$ at $p' \in \partial B_R(p)$. Let $q = \gamma(s_0)$ be the intersection point $\gamma \cap \Pi$. By the Alexandrov reflection property, $d(x_2, q) \leq d(q, x_1)$. Then the triangle inequality implies

$$\begin{aligned} d(p, x_2) &\leq d(p, x_1) + d(x_1, x_2) \leq d(p, x_1) + 2d(q, x_1) \\ &\leq d(p, x_1) + 2(d(q, p') + d(p', p) + d(p, x_1)) \leq 7R, \end{aligned}$$

where we used the fact $x_1 \in B_R(p)$. This shows that the diameter of Ω_t is uniformly bounded along the flow (1.2).

To estimate the lower bound of the inner radius of Ω_t, we project the domain Ω_t in the hyperbolic space $\mathbb{H}^{n+1}$ to the unit ball in the Euclidean space $\mathbb{R}^{n+1}$ as in [10, §5]. Denote by $\mathbb{R}^{1,n+1}$ the Minkowski spacetime, that is, the vector space $\mathbb{R}^{n+2}$ endowed with the Minkowski spacetime metric $\langle \cdot, \cdot \rangle$ given by $\langle X, X\rangle = -X_0^2 + \sum_{i=1}^n X_i^2$ for any vector $X = (X_0, X_1, \ldots, X_n) \in \mathbb{R}^{n+2}$. Then the hyperbolic space is characterized as

$$\mathbb{H}^{n+1} = \{X \in \mathbb{R}^{1,n+1} \mid \langle X, X\rangle = -1,\ X_0 > 0\}.$$

An embedding $X : M^n \to \mathbb{H}^{n+1}$ induces an embedding $Y : M^n \to B_1(0) \subset \mathbb{R}^{n+1}$ by

$$X = \frac{(1, Y)}{\sqrt{1 - |Y|^2}}.$$

The induced metrics g_{ij}^X and g_{ij}^Y of $X(M^n) \subset \mathbb{H}^{n+1}$ and $Y(M^n) \subset \mathbb{R}^{n+1}$ are related by

$$g_{ij}^X = \frac{1}{1 - |Y|^2}\left(g_{ij}^Y + \frac{\langle Y, \partial_i Y\rangle\langle Y, \partial_j Y\rangle}{(1 - |Y|^2)}\right).$$

Let $\tilde{\Omega}_t \subset B_1(0)$ be the corresponding image of Ω_t in $B_1(0) \subset \mathbb{R}^{n+1}$, and observe that $\tilde{\Omega}_t$ is a convex Euclidean domain. Then the diameter bound of Ω_t implies the diameter bound on $\tilde{\Omega}_t$. In particular, $|Y| \le C < 1$ for some constant C. This implies that the induced metrics g_{ij}^X and g_{ij}^Y are comparable. So the volume of $\tilde{\Omega}_t$ is also bounded below by a constant depending on the volume of Ω_0 and the diameter of Ω_t. Let $\omega_{\min}(\tilde{\Omega}_t)$ be the minimal width of $\tilde{\Omega}_t$. Then the volume of $\tilde{\Omega}_t$ is bounded by a constant times $\omega_{\min}(\tilde{\Omega}_t)(\mathrm{diam}(\tilde{\Omega}_t))^n$, since $\tilde{\Omega}_t$ is contained in a spherical prism of height $\omega_{\min}(\tilde{\Omega}_t)$ and radius $\mathrm{diam}(\tilde{\Omega}_t)$. It follows that $\omega_{\min}(\tilde{\Omega}_t)$ is bounded from below by a positive constant C. Since $\tilde{\Omega}_t$ is strictly convex, an estimate of Steinhagen [29] implies that the inner radius $\tilde{\rho}_-(t)$ of $\tilde{\Omega}_t$ is bounded below by $\tilde{\rho}_-(t) \ge c(n)\omega_{\min} \ge C > 0$, from which we obtain the uniform positive lower bound on the inner radius $\rho_-(t)$ of Ω_t. This finishes the proof. □

By (4.1), the inner radius of Ω_t is bounded below by a positive constant c_1. This implies that there exists a geodesic ball of radius c_1 contained in Ω_t for each $t \in [0, T)$. The same argument as in [10, Lemma 4.2] yields the existence of a geodesic ball with fixed centre enclosed by the flow hypersurface on a suitable fixed time interval.

Lemma 4.2. *Let M_t be a smooth solution of the flow (1.2) on $[0, T)$ with the global term $\phi(t)$ given by (1.3). For any $t_0 \in [0, T)$, let $B_{\rho_0}(p_0)$ be an inball of Ω_{t_0}, where $\rho_0 = \rho_-(t_0)$. Then*

$$B_{\rho_0/2}(p_0) \subset \Omega_t, \qquad t \in [t_0, \min\{T, t_0 + \tau\}), \tag{4.2}$$

for some τ depending only on n, α, Ω_0.

4.2. Upper bound of F

Now we can use the technique of Tso [31] as in [10] to prove the upper bound of F along the flow (1.2) provided that F satisfies Assumption 1.1. The inequality (1.4) and the fact that each M_t has positive sectional curvature are crucial in the proof.

Theorem 4.3. *Assume that F satisfies Assumption 1.1. Then along the flow (1.2) with any $\alpha > 0$, we have $F \leq C$ for any $t \in [0, T)$, where C depends on n, α, M_0 but not on T.*

Proof. For any given $t_0 \in [0, T)$, let $B_{\rho_0}(p_0)$ be the inball of Ω_{t_0}, where $\rho_0 = \rho_-(t_0)$. Consider the support function $u(x,t) = \sinh r_{p_0}(x)\langle \partial r_{p_0}, \nu\rangle$ of M_t with respect to the point p_0, where $r_{p_0}(x)$ is the distance function in $\mathbb{H}^{n+1}$ from the point p_0. Since M_t is strictly convex, by (4.2),

$$u(x,t) \geq \sinh(\rho_0/2) =: 2c \tag{4.3}$$

on M_t for any $t \in [t_0, \min\{T, t_0+\tau\})$. On the other hand, the estimate (4.1) implies that $u(x,t) \leq \sinh(2c_2)$ on M_t for all $t \in [t_0, \min\{T, t_0+\tau\})$. Recall that the support function $u(x,t)$ evolves by

$$\frac{\partial}{\partial t}u = \dot{\Psi}^{kl}\nabla_k\nabla_l u + \cosh r_{p_0}(x)(\phi(t) - \Psi - \dot{\Psi}^{kl}h_{kl}) + \dot{\Psi}^{ij}h_i^k h_{kj}u, \tag{4.4}$$

as we computed in [10], where $\Psi = F^\alpha(\mathcal{W})$. Define the auxiliary function

$$W(x,t) = \frac{\Psi(x,t)}{u(x,t)-c},$$

which is well-defined on M_t for all $t \in [t_0, \min\{T, t_0+\tau\})$. Combining (2.12) and (4.4), we have

$$\begin{aligned}
\frac{\partial}{\partial t}W = {} & \dot{\Psi}^{ij}\left(\nabla_j\nabla_i W + \frac{2}{u-c}\nabla_i u\nabla_j W\right)\\
& - \frac{\phi(t)}{u-c}\big(\dot{\Psi}^{ij}(h_i^k h_k^j - \delta_i^j) + W\cosh r_{p_0}(x)\big)\\
& + \frac{\Psi}{(u-c)^2}(\Psi + \dot{\Psi}^{kl}h_{kl})\cosh r_{p_0}(x) - \frac{c\Psi}{(u-c)^2}\dot{\Psi}^{ij}h_i^k h_k^j - W\dot{\Psi}^{ij}\delta_i^j.
\end{aligned}$$

By homogeneity of Ψ and the inverse concavity of F, we have $\Psi + \dot{\Psi}^{kl}h_{kl} = (1+\alpha)\Psi$ and $\dot{\Psi}^{ij}h_i^k h_k^j \geq \alpha F^{\alpha+1}$. Moreover, by (1.4) and the fact that $\kappa_1\kappa_2 > 1$,

$$\begin{aligned}
\dot{\Psi}^{ij}(h_i^k h_k^j - \delta_i^j) &= \alpha f^{\alpha-1}\sum_{i=1}^n \dot{f}^i(\kappa_i^2 - 1) \geq \alpha f^{\alpha-1}\big(\dot{f}^2(\kappa_2^2-1) + \dot{f}^1(\kappa_1^2-1)\big)\\
&\geq \alpha f^{\alpha-1}\dot{f}^1\left(\frac{\kappa_1}{\kappa_2}(\kappa_2^2-1) + (\kappa_1^2-1)\right)\\
&= \alpha f^{\alpha-1}\dot{f}^1\kappa_2^{-1}(\kappa_1\kappa_2 - 1)(\kappa_1+\kappa_2) \geq 0,
\end{aligned}$$

where we used $\kappa_i \geq 1$ for $i = 2, \dots, n$ in the first inequality. Then we arrive at

$$\frac{\partial}{\partial t} W \leq \dot{\Psi}^{ij} \left(\nabla_j \nabla_i W + \frac{2}{u - c} \nabla_i u \nabla_j W \right) + (\alpha + 1) W^2 \cosh r_{p_0}(x) - \alpha c W^2 F. \tag{4.5}$$

The remaining proof of Theorem 4.3 is the same as in [10, §4]. We include it here for convenience of the readers. Using (4.3) and the upper bound $r_{p_0}(x) \leq 2c_2$, we find from (4.5) that

$$\begin{aligned} \frac{\partial}{\partial t} W \leq \dot{\Psi}^{ij} & \left(\nabla_j \nabla_i W + \frac{2}{u - c} \nabla_i u \nabla_j W \right) \\ & + W^2 \big((\alpha + 1) \cosh(2c_2) - \alpha c^{1+1/\alpha} W^{1/\alpha} \big) \end{aligned} \tag{4.6}$$

on $[t_0, \min\{T, t_0 + \tau\})$. Let $\tilde{W}(t) = \sup_{M_t} W(\cdot, t)$. Then (4.6) implies that

$$\frac{d}{dt} \tilde{W}(t) \leq \tilde{W}^2 \big((\alpha + 1) \cosh(2c_2) - \alpha c^{1+1/\alpha} \tilde{W}^{1/\alpha} \big),$$

from which it follows by the maximum principle that

$$\tilde{W}(t) \leq \max \left\{ \left(\frac{2(1 + \alpha) \cosh(2c_2)}{\alpha} \right)^{\alpha} c^{-(\alpha+1)}, \left(\frac{2}{1 + \alpha} \right)^{\frac{\alpha}{1+\alpha}} c^{-1} (t - t_0)^{-\frac{\alpha}{1+\alpha}} \right\}. \tag{4.7}$$

Then the upper bound on F follows by (4.7) and the facts that

$$c = \tfrac{1}{2} \sinh(\rho_0/2) \geq \tfrac{1}{2} \sinh(c_1/2)$$

and $u - c \leq 2c_2$, where c_1, c_2 are the constants in (4.1) depending only on n, M_0. □

4.3. Long time existence and convergence

In this subsection, we complete the proof of Theorem 1.2. Firstly, in the case (i) of Theorem 1.2, we can directly deduce a uniform estimate on the principal curvatures of M_t. In fact, since $f(\kappa)$ is bounded from above by Theorem 4.3,

$$C \geq f(\kappa_1, \kappa_2, \dots, \kappa_n) \geq f(\kappa_1, 1/\kappa_1, \dots, 1/\kappa_1), \tag{4.8}$$

where in the second inequality we used the facts that f is increasing in each argument and $\kappa_i \kappa_1 > 1$ for $i = 2, \dots, n$. Combining (4.8) and (1.6) gives $\kappa_1 \geq C > 0$ for some uniform constant C. Since the dual function f_* of f vanishes on the boundary of the positive cone Γ_+ and $f_*(\tau_i) = 1/f(\kappa_i) \geq C > 0$ by Theorem 4.3, the upper bound on $\tau_i = 1/\kappa_i \leq C$ gives a lower bound on τ_i, which is equivalent to an upper bound on κ_i. In summary, we obtain a uniform two-sided positive bound on the principal curvatures of M_t along the flow (1.2) in case (i) of Theorem 1.2.

The examples of f satisfying Assumption 1.1 and condition (i) of Theorem 1.2 include

(a) $n \geq 2$, $f = n^{-1/k} S_k^{1/k}$ with $k > 0$;
(b) $n \geq 3$, $f = E_k^{1/k}$ with $k = 1, \dots, n$;
(c) $n = 2$, $f = (\kappa_1 + \kappa_2)/2$.

We next consider case (ii) of Theorem 1.2, i.e., $n = 2$, $f = (\kappa_1\kappa_2)^{1/2}$. In general, the estimate $1 \le \kappa_1\kappa_2 = f(\kappa) \le C$ cannot prevent κ_2 from going to infinity. Instead, we will prove that the pinching ratio κ_2/κ_1 is bounded from above along the flow (1.2) with $f = (\kappa_1\kappa_2)^{1/2}$ and $\alpha \in [1/2, 2]$. This together with the estimate $1 \le \kappa_1\kappa_2 \le C$ yields a uniform estimate on κ_1 and κ_2.

Lemma 4.4. *In the case $n = 2$, $f = (\kappa_1\kappa_2)^{1/2}$ and $\alpha \in [1/2, 2]$, the principal curvatures κ_1, κ_2 of M_t satisfy*

$$0 < 1/C \le \kappa_1 \le \kappa_2 \le C, \qquad \forall t \in [0, T), \tag{4.9}$$

for some positive constant C along the flow (1.2).

Proof. In this case, $\Psi(\mathcal{W}) = \psi(\kappa) = K^{\alpha/2}$, where $K = \kappa_1\kappa_2$ is the Gauss curvature. The derivatives of ψ with respect to κ_i are listed in the following:

$$\dot\psi^1 = \frac{\alpha}{2}K^{\alpha/2-1}\kappa_2, \qquad \dot\psi^2 = \frac{\alpha}{2}K^{\alpha/2-1}\kappa_1, \tag{4.10}$$

$$\ddot\psi^{11} = \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)K^{\alpha/2-2}\kappa_2^2, \qquad \ddot\psi^{22} = \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)K^{\alpha/2-2}\kappa_1^2, \tag{4.11}$$

$$\ddot\psi^{12} = \ddot\psi^{21} = \frac{\alpha^2}{4}K^{\alpha/2-1}. \tag{4.12}$$

Then

$$\dot\Psi^{ij}h_i^k h_k^j = \sum_{i=1}^n \dot\psi^i\kappa_i^2 = \frac{\alpha}{2}K^{\alpha/2}H, \tag{4.13}$$

$$\dot\Psi^{ij}\delta_i^j = \sum_{i=1}^n \dot\psi^i = \frac{\alpha}{2}K^{\alpha/2-1}H, \qquad \dot\Psi^{ij}h_i^j = \alpha K^{\alpha/2}, \tag{4.14}$$

where $H = \kappa_1 + \kappa_2$ is the mean curvature.

To prove (4.9), we define

$$G = K^{\alpha-2}(H^2 - 4K)$$

and aim to prove that $G(x,t) \le \max_{M_0} G(x,0)$ by the maximum principle. Using (4.13) and (4.14), the evolution equations (2.12) and (2.13) imply that

$$\frac{\partial}{\partial t}K = \dot\Psi^{kl}\nabla_k\nabla_l K + \left(\frac{\alpha}{2}-1\right)K^{-1}\dot\Psi^{kl}\nabla_k K\nabla_l K + (K^{\alpha/2} - \phi(t))(K-1)H \tag{4.15}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}H = {}& \dot\Psi^{kl}\nabla_k\nabla_l H + \ddot\Psi^{kl,pq}\nabla_i h_{kl}\nabla^i h_{pq} \\ & + K^{\alpha/2-1}\left(K + \frac{\alpha}{2}(1-K)\right)(H^2-4K) + 2K^{\alpha/2}(K-1) \\ & - \phi(t)(H^2 - 2K - 2).\end{aligned} \tag{4.16}$$

By a direct computation using (4.15) and (4.16), we obtain the evolution equation of G:

$$\begin{aligned}\frac{\partial}{\partial t}G = {} & \dot{\Psi}^{kl}\nabla_k\nabla_l G - 2(\alpha-2)K^{-1}\dot{\Psi}^{kl}\nabla_k K\nabla_l G\\ & + (\alpha-2)(3\alpha/2-2)K^{\alpha-4}\dot{\Psi}^{kl}\nabla_k K\nabla_l K(H^2-4K)\\ & - 2K^{\alpha-2}\dot{\Psi}^{kl}\nabla_k H\nabla_l H - 2(\alpha-2)K^{\alpha-3}\dot{\Psi}^{kl}\nabla_k K\nabla_l K\\ & + 2HK^{\alpha-2}\ddot{\Psi}^{kl,pq}\nabla_i h_{kl}\nabla^i h_{pq}\\ & + 2HK^{3\alpha/2-3}(H^2-4K) - HK^{\alpha-3}(H^2-4K)(\alpha K+2-\alpha)\phi(t). \qquad (4.17)\end{aligned}$$

We will apply the maximum principle to prove that $\max_{M_t} G$ is non-increasing in time along the flow (1.2). We first look at the zero order terms of (4.17), i.e., the terms in the last line of (4.17), which we denote by Q_0. Since $K = \kappa_1\kappa_2 > 1$ by Theorem 3.1, we have

$$\phi(t) = \frac{1}{|M_t|}\int_{M_t} K^{\alpha/2} \geq 1 \quad \text{and} \quad \alpha K + 2 - \alpha > 2.$$

We also have $H^2 - 4K = (\kappa_1-\kappa_2)^2 \geq 0$. Then

$$\begin{aligned}Q_0 &\leq 2HK^{3\alpha/2-3}(H^2-4K) - HK^{\alpha-3}(H^2-4K)(\alpha K+2-\alpha)\\ &= HK^{\alpha-3}(H^2-4K)(2K^{\alpha/2}-\alpha K+\alpha-2).\end{aligned}$$

For any $K > 1$, denote $f(\alpha) = 2K^{\alpha/2} - \alpha K + \alpha - 2$. Then $f(2) = f(0) = 0$ and $f(\alpha)$ is a convex function of α. Therefore $f(\alpha) \leq 0$ and $Q_0 \leq 0$ provided that $\alpha \in [0, 2]$.

At the critical point of G, we have $\nabla_i G = 0$ for all $i = 1, 2$, which is equivalent to

$$2H\nabla_i H = \big(4(\alpha-1) - (\alpha-2)K^{-1}H^2\big)\nabla_i K. \qquad (4.18)$$

Then the gradient terms (denoted by Q_1) of (4.17) at the critical point of G satisfy

$$\begin{aligned}Q_1 K^{3-\alpha} = {} & \left(-8(\alpha-1)^2\frac{K}{H^2} - 2(\alpha-1)(\alpha-2) + (\alpha-1)(\alpha-2)\frac{H^2}{K}\right)\\ & \times \dot{\Psi}^{kl}\nabla_k K\nabla_l K + 2HK\ddot{\Psi}^{kl,pq}\nabla_i h_{kl}\nabla^i h_{pq}. \qquad (4.19)\end{aligned}$$

Using (4.10)–(4.12), we have

$$\dot{\Psi}^{kl}\nabla_k K\nabla_l K = \frac{\alpha}{2}K^{\alpha/2-1}(\kappa_2(\nabla_1 K)^2 + \kappa_1(\nabla_2 K)^2),$$

$$\begin{aligned}\ddot{\Psi}^{kl,pq}\nabla_i h_{kl}\nabla^i h_{pq} = {} & \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)K^{\alpha/2-2}\sum_{i=1}^{2}(\kappa_2^2(\nabla_i h_{11})^2 + \kappa_1^2(\nabla_i h_{22})^2)\\ & + \frac{\alpha^2}{2}K^{\alpha/2-1}\sum_{i=1}^{2}\nabla_i h_{11}\nabla_i h_{22} - \alpha K^{\alpha/2-1}\sum_{i=1}^{2}(\nabla_i h_{12})^2\\ = {} & \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)K^{\alpha/2-2}((\nabla_1 K)^2 + (\nabla_2 K)^2)\\ & + \alpha K^{\alpha/2-1}\big(\nabla_1 h_{11}\nabla_1 h_{22} - (\nabla_1 h_{12})^2 + \nabla_2 h_{11}\nabla_2 h_{22} - (\nabla_2 h_{12})^2\big).\end{aligned}$$

The equation (4.18) implies that $\nabla_i h_{11}$ and $\nabla_i h_{22}$ are linearly dependent, i.e., there exist functions g_1, g_2 such that

$$g_1 \nabla_i h_{11} = g_2 \nabla_i h_{22}. \tag{4.20}$$

The functions g_1, g_2 can be expressed explicitly as follows:

$$\begin{aligned} g_1 &= 2HK - (4(\alpha-1)K + (2-\alpha)H^2)\kappa_2, \\ g_2 &= -2HK + (4(\alpha-1)K + (2-\alpha)H^2)\kappa_1. \end{aligned}$$

Without loss of generality, we can assume that both g_1 and g_2 are non-zero at the critical point of G. In fact, if $g_1 = 0$, then

$$\begin{aligned} 0 = g_1 &= \big((\alpha-2)(H^2-4K) + 2H\kappa_1 - 4K\big)\kappa_2 \\ &= \big((\alpha-2)(\kappa_1-\kappa_2) + 2\kappa_1\big)(\kappa_1-\kappa_2)\kappa_2. \end{aligned} \tag{4.21}$$

Since $\alpha \le 2$, we have $(\alpha-2)(\kappa_1-\kappa_2) + 2\kappa_1 \ge 2\kappa_1 > 0$. Thus (4.21) is equivalent to $\kappa_2 = \kappa_1$ and we have nothing to prove.

By (4.20), we have

$$\begin{aligned} (\nabla_1 K)^2 &= (\kappa_2 + g_2^{-1} g_1 \kappa_1)^2 (\nabla_1 h_{11})^2 = g_2^{-2} 4H^2K^2(H^2-4K)(\nabla_1 h_{11})^2, \\ (\nabla_2 K)^2 &= (\kappa_2 g_1^{-1} g_2 + \kappa_1)^2 (\nabla_2 h_{22})^2 = g_1^{-2} 4H^2K^2(H^2-4K)(\nabla_2 h_{22})^2. \end{aligned}$$

In view of the Codazzi identity, the equation (4.20) also implies that

$$\begin{aligned} &\nabla_1 h_{11}\nabla_1 h_{22} - (\nabla_1 h_{12})^2 + \nabla_2 h_{11}\nabla_2 h_{22} - (\nabla_2 h_{12})^2 \\ &\quad = \nabla_1 h_{11}\nabla_1 h_{22} - (\nabla_1 h_{22})^2 + \nabla_2 h_{11}\nabla_2 h_{22} - (\nabla_2 h_{11})^2 \\ &\quad = g_2^{-2} g_1 (g_2 - g_1)(\nabla_1 h_{11})^2 + g_1^{-2} g_2 (g_1 - g_2)(\nabla_2 h_{22})^2 \\ &\quad = (2-\alpha)(H^2-4K)\big(g_2^{-2} g_1 (\nabla_1 h_{11})^2 + g_1^{-2} g_2 (\nabla_2 h_{22})^2\big). \end{aligned}$$

Therefore we can write the right-hand side of (4.19) as a linear combination of $(\nabla_1 h_{11})^2$ and $(\nabla_2 h_{22})^2$:

$$\frac{2}{\alpha} Q_1 K^{4-3\alpha/2} = q_1 g_2^{-2} (\nabla_1 h_{11})^2 + q_2 g_1^{-2} (\nabla_2 h_{22})^2, \tag{4.22}$$

where the coefficients q_1, q_2 satisfy

$$\begin{aligned} q_1 &= \left(-8(\alpha-1)^2 \frac{K^2}{H^2} + 2(\alpha-1)(\alpha-2)K + (\alpha-2)H^2\right) 4H^2K(H^2-4K)\kappa_2 \\ &\quad + 4(2-\alpha)H^3K^2(H^2-4K) \\ &= -32(\alpha-1)^2K^3(H^2-4K)\kappa_2 - 4(2-\alpha)(2\alpha-1)H^2K^2(H^2-4K)\kappa_2 \\ &\quad - 4(2-\alpha)H^2K(H^2-4K)\kappa_2^3 \end{aligned}$$

and

$$q_2 = -32(\alpha-1)^2K^3(H^2-4K)\kappa_1 - 4(2-\alpha)(2\alpha-1)H^2K^2(H^2-4K)\kappa_1 - 4(2-\alpha)H^2K(H^2-4K)\kappa_1^3.$$

It can be checked directly that q_1 and q_2 are both non-positive if $\alpha \in [1/2, 2]$. Thus the gradient terms Q_1 of (4.17) are non-positive at a critical point of G if $\alpha \in [1/2, 2]$. The maximum principle implies that $\max_{M_t} G$ is non-increasing in time. It follows that $G(x,t) \le \max_{M_0} G(x,0)$. Since $1 < K \le C$ for some constant $C > 0$ by Theorems 3.1 and 4.3, we have

$$H^2 = 4K + K^{\alpha-2}G \le C. \tag{4.23}$$

Finally, the estimate (4.9) follows from (4.23) and $K > 1$ immediately. □

Now we have proved that the principal curvatures κ_i of M_t satisfy the uniform estimate $0 < 1/C \le \kappa_i \le C$ for some constant $C > 0$, which is equivalent to the C^2 estimate for M_t. Since the functions f we considered in Theorem 1.2 are inverse concave, we can apply an argument similar to that in [10, §5] to derive higher regularity estimates. The standard continuation argument then implies the long time existence of the flow, and the argument in [10, §6] implies the smooth convergence to a geodesic sphere as time goes to infinity.

5. Horospherically convex regions

In this section we will investigate some of the properties of horospherically convex regions in hyperbolic space (that is, regions which are given by the intersection of a collection of horoballs). In particular, for such regions we define a horospherical Gauss map, which is a map to the unit sphere, and we show that each horospherically convex region is completely described in terms of a scalar function on the sphere which we call the *horospherical support function*. There are interesting formal similarities between this situation and that of convex Euclidean bodies. For the purposes of this paper the main result we need is that the modified quermassintegrals are monotone with respect to inclusion for horospherically convex domains. However we expect that the description of horospherically convex regions which we develop here will be useful in further investigations beyond the scope of this paper.

We remark that a similar development is presented in [15], but in a slightly different context: In that paper the 'horospherically convex' regions are those which are intersections of complements of horoballs (corresponding to principal curvatures greater than -1 everywhere, while we deal with regions which are intersections of horoballs, corresponding to principal curvatures greater than 1. Our condition is more stringent but is more useful for the evolution equations we consider here.

5.1. *The horospherical Gauss map*

The horospheres in hyperbolic space are the submanifolds with constant principal curvatures equal to 1 everywhere. If we identify $\mathbb{H}^{n+1}$ with the future time-like hyperboloid in the Minkowski space $\mathbb{R}^{n+1,1}$, then the condition of constant principal curvatures equal to 1 implies that the null vector $\bar{\mathbf{e}} := X - \nu$ is constant on the hypersurface, since we have $\mathcal{W} = \mathrm{I}$, and hence

$$D_v \bar{\mathbf{e}} = D_v(X - \nu) = DX((\mathrm{I} - \mathcal{W})(v)) = 0$$

for all tangent vectors v. Then we observe that

$$X \cdot \bar{\mathbf{e}} = X \cdot (X - \nu) = -1,$$

from which it follows that the horosphere is the intersection of the null hyperplane $\{X \mid X \cdot \bar{\mathbf{e}} = -1\}$ with the hyperboloid $\mathbb{H}^{n+1}$. The horospheres are therefore in one-to-one correspondence with points $\bar{\mathbf{e}}$ in the future null cone which are given by $\{\bar{\mathbf{e}} = \lambda(\mathbf{e}, 1) \mid \mathbf{e} \in S^n,\ \lambda > 0\}$, and there is a one-parameter family of these for each $\mathbf{e} \in S^n$. For convenience we parametrise these by their signed geodesic distance s from the 'north pole' $N = (0, 1) \in \mathbb{H}^{n+1}$, satisfying $-1 = \lambda(\cosh(s)N + \sinh(s)(\mathbf{e}, 0)) \cdot (\mathbf{e}, 1) = -\lambda \mathrm{e}^s$. It follows that $\lambda = \mathrm{e}^{-s}$. Thus we denote by $H_{\mathbf{e}}(s)$ the horosphere $\{X \in \mathbb{H}^{n+1} \mid X \cdot (\mathbf{e}, 1) = -\mathrm{e}^s\}$. The interior region (called a *horoball*) is denoted by

$$B_{\mathbf{e}}(s) = \{X \in \mathbb{H}^{n+1} \mid 0 > X \cdot (\mathbf{e}, 1) > -\mathrm{e}^s\}.$$

A region Ω in $\mathbb{H}^{n+1}$ is *horospherically convex* (or *h-convex* for convenience) if every boundary point p of $\partial\Omega$ has a supporting horoball, i.e. a horoball B such that $\Omega \subset B$ and $p \in \partial B$. If the boundary of Ω is a smooth hypersurface, then this implies that every principal curvature of $\partial\Omega$ is greater than or equal to 1 at p. We say that Ω is *uniformly h-convex* if there is $\delta > 0$ such that all principal curvatures exceed $1 + \delta$.

Let $M^n = \partial\Omega$ be a hypersurface which is the boundary of a horospherically convex region Ω. Then the *horospherical Gauss map* $\mathbf{e} : M \to S^n$ assigns to each $p \in M$ the point $\mathbf{e}(p) = \pi(X(p) - \nu(p)) \in S^n$, where $\pi(x, y) = x/y$ is the radial projection from the future null cone onto the sphere $S^n \times \{1\}$. We observe that the derivative of $\mathbf{e}$ is non-singular if M is uniformly h-convex: If v is a tangent vector to M, then

$$D\mathbf{e}(v) = D\pi|_{X-\nu}((\mathcal{W} - \mathrm{I})(v)).$$

Here $\tilde{v} = (\mathcal{W} - \mathrm{I})(v)$ is a non-zero tangent vector to M since the eigenvalues κ_i of $\mathcal{W}$ are greater than 1. In particular $\tilde{v}$ is spacelike. On the other hand the kernel of $D\pi|_{X-\nu}$ is the line $\mathbb{R}(X - \nu)$ consisting of null vectors. Therefore $D\pi(\tilde{v}) \neq 0$. Thus $D\mathbf{e}$ is an injective linear map, hence an isomorphism. It follows that $\mathbf{e}$ is a diffeomorphism from M to S^n.

5.2. *The horospherical support function*

Let $M^n = \partial\Omega$ be the boundary of a compact h-convex region. Then for each $\mathbf{e} \in S^n$ we define the *horospherical support function* of Ω (or M) *in direction* $\mathbf{e}$ by

$$u(\mathbf{e}) := \inf\{s \in \mathbb{R} \mid \Omega \subset B_{\mathbf{e}}(s)\}.$$

Alternatively, define $f_{\mathbf{e}}:\mathbb{H}^{n+1}\to\mathbb{R}$ by $f_{\mathbf{e}}(\xi)=\log(-\xi\cdot(\mathbf{e},1))$. This is a smooth function on $\mathbb{H}^{n+1}$, and we have the alternative characterisation

$$u(\mathbf{e})=\sup\{f_{\mathbf{e}}(\xi):\xi\in\Omega\}. \tag{5.1}$$

The function u is called the *horospherical support function* of the region Ω, and $B_{\mathbf{e}}(u(\mathbf{e}))$ is the *supporting horoball in direction* $\mathbf{e}$. The support function completely determines a horospherically convex region Ω, as an intersection of horoballs:

$$\Omega=\bigcap_{\mathbf{e}\in S^n}B_{\mathbf{e}}(u(\mathbf{e})). \tag{5.2}$$

5.3. Recovering the region from the support function

If the region is uniformly h-convex, in the sense that all principal curvatures are greater than 1, then there is a unique point of M in the boundary of the supporting horoball $B_{\mathbf{e}}(u(\mathbf{e}))$. We denote this point by $\bar{X}(\mathbf{e})$. We observe that $\bar{X}=X\circ\mathbf{e}^{-1}$, so if M is smooth and uniformly h-convex (so that $\mathbf{e}$ is a diffeomorphism) then $\bar{X}$ is a smooth embedding.

We will show that $\bar{X}$ can be written in terms of the support function u, as follows: Choose local coordinates $\{x^i\}$ for S^n near $\mathbf{e}$. We write $\bar{X}(\mathbf{e})$ as a linear combination of the basis consisting of the two null elements $(\mathbf{e},1)$ and $(-\mathbf{e},1)$, together with $(\mathbf{e}_j,0)$, where $\mathbf{e}_j=\frac{\partial\mathbf{e}}{\partial x^j}$ for $j=1,\ldots,n$:

$$\bar{X}(\mathbf{e})=\alpha(-\mathbf{e},1)+\beta(\mathbf{e},1)+\gamma^j(\mathbf{e}_j,0)$$

for some coefficients α, β, γ^j. Since $\bar{X}(\mathbf{e})\in\mathbb{H}^{n+1}$ we have $|\gamma|^2-4\alpha\beta=-1$, so that $\beta=\frac{1+|\gamma|^2}{4\alpha}$. We also know that $\bar{X}(\mathbf{e})\cdot(\mathbf{e},1)=-\mathrm{e}^{u(\mathbf{e})}$ since $\bar{X}(\mathbf{e})\in H_{\mathbf{e}}(u(\mathbf{e}))$, implying that $\alpha=\frac{1}{2}\mathrm{e}^u$. This gives

$$\bar{X}(\mathbf{e})=\tfrac{1}{2}\mathrm{e}^{u(\mathbf{e})}(-\mathbf{e},1)+\tfrac{1}{2}\mathrm{e}^{-u(\mathbf{e})}(1+|\gamma|^2)(\mathbf{e},1)+\gamma^j(\mathbf{e}_j,0).$$

Furthermore, the normal to M at the point $\bar{X}(\mathbf{e})$ must coincide with the normal to the horosphere $H_{\mathbf{e}}(u(\mathbf{e}))$, which is given by

$$\nu=\bar{X}-\bar{e}=\bar{X}-\mathrm{e}^{-u(\mathbf{e})}(\mathbf{e},1). \tag{5.3}$$

Since $|\bar{X}|^2=-1$ we have $\partial_j\bar{X}\cdot\bar{X}=0$, and hence

$$0=\partial_j\bar{X}\cdot\nu=\partial_j\bar{X}\cdot(\bar{X}-\mathrm{e}^{-u}(\mathbf{e},1))=-\mathrm{e}^{-u}\partial_jX\cdot(\mathbf{e},1).$$

Observing that $(\mathbf{e},1)\cdot(\mathbf{e},1)=0$ and $(\mathbf{e}_i,0)\cdot(\mathbf{e},1)=0$, and that $\partial_j\mathbf{e}_i=-\bar{g}_{ij}\mathbf{e}$ and $\partial_j\mathbf{e}=\mathbf{e}_j$, the condition becomes

$$0=\partial_jX\cdot(\mathbf{e},1)=\left(\tfrac{1}{2}\mathrm{e}^u u_j(-\mathbf{e},1)-\gamma_j(\mathbf{e},0)\right)\cdot(\mathbf{e},1)=-\mathrm{e}^u u_j-\gamma_j,$$

where $\gamma_j=\gamma^i\bar{g}_{ij}$ and $\bar{g}$ is the standard metric on S^n. It follows that we must have $\gamma_j=-\mathrm{e}^u u_j$. This gives the following expressions for $\bar{X}$:

$$\bar{X}(\mathbf{e})=\left(-\mathrm{e}^u\bar{\nabla}u+\left(\tfrac{1}{2}\mathrm{e}^u|\bar{\nabla}u|^2-\sinh u\right)\mathbf{e},\ \tfrac{1}{2}\mathrm{e}^u|\bar{\nabla}u|^2+\cosh u\right) \tag{5.4}$$

$$=-\mathrm{e}^u u_p\bar{g}^{pg}(\mathbf{e}_q,0)+\tfrac{1}{2}(\mathrm{e}^u|\bar{\nabla}u|^2+\mathrm{e}^{-u})(\mathbf{e},1)+\tfrac{1}{2}\mathrm{e}^u(-\mathbf{e},1). \tag{5.5}$$

5.4. A condition for horospherical convexity

Given a smooth function u, we can use the expression (5.4) to define a map to hyperbolic space. In this section we determine when the resulting map is an embedding defining a horospherically convex hypersurface.

In order for $\bar{X}$ to be an immersion, we require the derivatives $\partial_j \bar{X}$ to be linearly independent. Since we have constructed $\bar{X}$ in such a way that $\partial_j X$ is orthogonal to the normal vector ν to the horosphere $B_{\mathbf{e}}(u(\mathbf{e}))$, $\partial_j \bar{X}$ is a linear combination of the basis for the space orthogonal to ν and $\bar{X}$ given by the projections E_k of $(e_k, 0)$, $k = 1, \ldots, n$. Computing explicitly, we find

$$E_k = (\mathbf{e}_k, 0) - u_k(\mathbf{e}, 1). \tag{5.6}$$

The immersion condition is therefore equivalent to invertibility of the matrix A defined by

$$A_{jk} = -\partial_j \bar{X} \cdot E_k.$$

Given that A is non-singular, we infer that $\bar{X}$ is an immersion with unit normal vector $\nu(\mathbf{e})$, and we can differentiate the equation $X - \nu = \mathrm{e}^{-u}(\mathbf{e}, 1)$ to obtain

$$-(h_j^p - \delta_j^p)\partial_p X = -u_j \mathrm{e}^{-u}(\mathbf{e}, 1) + \mathrm{e}^{-u}(\mathbf{e}_j, 0).$$

Taking the inner product with E_k using (5.6), we obtain

$$(h_j^p - \delta_j^p)A_{pk} = \mathrm{e}^{-u}\bar{g}_{jk}. \tag{5.7}$$

It follows that A is non-singular precisely when $\mathcal{W} - \mathrm{I}$ is non-singular, and is given by

$$A_{jk} = \mathrm{e}^{-u}[(\mathcal{W} - \mathrm{I})^{-1}]_j^p \bar{g}_{pk}. \tag{5.8}$$

In particular, A is symmetric, and $\mathcal{W} - \mathrm{I}$ is positive definite (corresponding to uniform h-convexity) if and only if the matrix A is positive definite. We conclude that if u is a smooth function on S^n, then the map X defines an embedding to the boundary of a uniformly h-convex region if and only if the tensor A computed from u is positive definite.

Computing A explicitly using (5.5), we obtain

$$\begin{aligned} A_{jk} &= \big((\bar{\nabla}_j(\mathrm{e}^u \bar{\nabla} u), 0) - \mathrm{e}^u u_j(\mathbf{e}, 0) - \tfrac{1}{2}\partial_j(\mathrm{e}^u|\bar{\nabla} u|^2 + \mathrm{e}^{-u})(\mathbf{e}, 1) \\ &\quad - \tfrac{1}{2}\mathrm{e}^u u_j(-\mathbf{e}, 1) - \big(\tfrac{1}{2}\mathrm{e}^u|\nabla u|^2 - \sinh u\big)(\mathbf{e}_j, 0)\big) \cdot ((\mathbf{e}_k, 0) - u_k(\mathbf{e}, 1)) \\ &= \bar{\nabla}_j(\mathrm{e}^u \bar{\nabla}_k u) - \tfrac{1}{2}\mathrm{e}^u|\bar{\nabla} u|^2 \bar{g}_{jk} + \sinh u\, \bar{g}_{jk}. \end{aligned}$$

It is convenient to write this in terms of the function $\varphi = \mathrm{e}^u$:

$$A_{jk} = \bar{\nabla}_j \bar{\nabla}_k \varphi - \frac{|\bar{\nabla}\varphi|^2}{2\varphi}\bar{g}_{jk} + \frac{\varphi - \varphi^{-1}}{2}\bar{g}_{jk}. \tag{5.9}$$

5.5. Monotonicity of the modified quermassintegrals

We will prove that the modified quermassintegrals $\tilde{W}_k$ are monotone with respect to inclusion by making use of the following result:

Proposition 5.1. *Suppose that $\Omega_1 \subset \Omega_2$ are smooth, strictly h-convex domains in $\mathbb{H}^{n+1}$. Then there exists a smooth map $X : S^n \times [0, 1] \to \mathbb{H}^{n+1}$ such that*

(1) *$X(\cdot, t)$ is a uniformly h-convex embedding of S^n for each t;*
(2) *$X(S^n, 0) = \partial\Omega_0$ and $X(S^n, 1) = \partial\Omega_1$;*
(3) *the hypersurfaces $M_t = X(S^n, t)$ are expanding, in the sense that $\frac{\partial X}{\partial t} \cdot \nu \geq 0$, equivalently, the enclosed regions Ω_t are nested: $\Omega_s \subset \Omega_t$ for each $s \leq t$ in $[0, 1]$.*

Proof. Let u_0 and u_1 be the horospherical support functions of Ω_0 and Ω_1 respectively, The inclusion $\Omega_0 \subset \Omega_1$ implies that $u_0(\mathbf{e}) \leq u_1(\mathbf{e})$ for all $\mathbf{e} \in S^n$, by (5.1).

We define $X(\mathbf{e}, t) = \bar{X}[u(\mathbf{e}, t)]$ according to (5.4), where

$$e^{u(\mathbf{e},t)} = \varphi(\mathbf{e}, t) := (1 - t)\varphi_0(\mathbf{e}) + t\varphi_1(\mathbf{e}),$$

where $\varphi_i = e^{u_i}$ for $i = 0, 1$. Then $u(\mathbf{e}, t)$ is increasing in t, and it follows that the regions Ω_t are nested, by (5.2).

We check that each Ω_t is a strictly h-convex region, by showing that the matrix A constructed from $u(\cdot, t)$ is positive definite for each t: We have

$$\begin{aligned} A_{jk}[u(\cdot, t)] &= \bar{\nabla}_j \bar{\nabla}_k \varphi_t - \frac{|\bar{\nabla}\varphi_t|^2}{2\varphi_t}\bar{g}_{jk} + \frac{\varphi_t - \varphi_t^{-1}}{2}\bar{g}_{jk} \\ &= (1-t)A_{jk}[u_0] + tA_{jk}[u_1] \\ &\quad + \frac{1}{2}\left(-\frac{|(1-t)\bar{\nabla}\varphi_0 + t\bar{\nabla}\varphi_1|^2}{(1-t)\varphi_0 + t\varphi_1} + (1-t)\frac{|\bar{\nabla}\varphi_0|^2}{\varphi_0} + t\frac{|\bar{\nabla}\varphi_1|^2}{\varphi_1}\right)\bar{g}_{jk} \\ &\quad + \frac{1}{2}\left(-\frac{1}{(1-t)\varphi_0 + t\varphi_1} + \frac{1-t}{\varphi_0} + \frac{t}{\varphi_1}\right)\bar{g}_{jk} \\ &= (1-t)A_{jk}[u_0] + tA_{jk}[u_1] + t(1-t)\frac{|\varphi_0\bar{\nabla}\varphi_1 - \varphi_1\bar{\nabla}\varphi_0|^2 + |\varphi_1 - \varphi_0|^2}{2\varphi_0\varphi_1((1-t)\varphi_0 + t\varphi_1)}\bar{g}_{jk} \\ &\geq (1-t)A_{jk}[u_0] + tA_{jk}[u_1]. \end{aligned}$$

Since $A_{jk}[u_0]$ and $A_{jk}[u_1]$ are positive definite, so is $A_{jk}[u_t]$ for each $t \in [0, 1]$, and we conclude that the region Ω_t is uniformly h-convex. □

Corollary 5.2. *The modified quermassintegral $\widetilde{W}_k$ is monotone with respect to inclusion for h-convex domains: if Ω_0 and Ω_1 are h-convex domains with $\Omega_0 \subset \Omega_1$, then $\widetilde{W}_k(\Omega_0) \leq \widetilde{W}_k(\Omega_1)$.*

Proof. We use the map X constructed in Proposition 5.1. By Lemma 2.4 we have

$$\frac{d}{dt}\widetilde{W}_k(\Omega_t) = \int_{M_t} E_k(\lambda)\frac{\partial X}{\partial t} \cdot \nu \, d\mu_t.$$

Since each M_t is h-convex, we have $\lambda_i > 0$ and hence $E_k(\lambda) > 0$, and from Proposition 5.1 we have $\frac{\partial X}{\partial t} \cdot \nu \geq 0$. It follows that $\frac{d}{dt}\widetilde{W}_k(\Omega_t) \geq 0$ for each t, and hence $\widetilde{W}_k(\Omega_0) \leq \widetilde{W}_k(\Omega_1)$ as claimed. □

5.6. Evolution of the horospherical support function

We end this section with the following observation that the flow (1.18) of h-convex hypersurfaces is equivalent to an initial value problem for the horospherical support function.

Proposition 5.3. *The flow* (1.18) *of h-convex hypersurfaces in* $\mathbb{H}^{n+1}$ *is equivalent to the following initial value problem:*

$$\begin{cases} \frac{\partial}{\partial t}\varphi = -F((A_{ij})^{-1}) + \varphi\phi(t), \\ \varphi(\cdot, 0) = \varphi_0(\cdot) \end{cases} \tag{5.10}$$

on $S^n \times [0, T)$*, where* $\varphi = \mathrm{e}^u$ *and* A_{ij} *is the matrix defined in* (5.9).

Proof. Suppose that $X(\cdot, t) : M \to \mathbb{H}^{n+1}, t \in [0, T)$, is a family of smooth, closed and strictly h-convex hypersurfaces satisfying the flow (1.18). Then as explained in §5.1, the horospherical Gauss map $\mathbf{e}$ is a diffeomorphism from $M_t = X(M, t)$ to S^n. We can reparametrise M_t so that $\bar{X} = X \circ \mathbf{e}^{-1}$ is a family of smooth embeddings from S^n to $\mathbb{H}^{n+1}$. Then

$$\frac{\partial}{\partial t}\bar{X}(z, t) = \frac{\partial}{\partial t}X(p, t) + \frac{\partial X}{\partial p_i}\frac{\partial p^i}{\partial t},$$

where $z \in S^n$ and $p = \mathbf{e}^{-1}(z) \in M_t$. Since $\frac{\partial X}{\partial p_i}$ is tangent to M_t, we have

$$\frac{\partial}{\partial t}\bar{X}(z, t) \cdot \nu(z, t) = \frac{\partial}{\partial t}X(p, t) \cdot \nu(z, t) = \phi(t) - F(\mathcal{W} - \mathrm{I}). \tag{5.11}$$

On the other hand, by (5.3) we have

$$\bar{X}(z, t) - \nu(z, t) = \mathrm{e}^{-u(z,t)}(z, 1), \tag{5.12}$$

where $u(\cdot, t)$ is the horospherical support function of M_t and $(z, 1) \in \mathbb{R}^{n+1,1}$ is a null vector. Differentiating (5.12) in time gives

$$\frac{\partial}{\partial t}\bar{X}(z, t) - \frac{\partial}{\partial t}\nu(z, t) = -\mathrm{e}^{-u(z,t)}\frac{\partial u}{\partial t}(z, 1).$$

Then

$$\begin{aligned} \frac{\partial}{\partial t}\bar{X}(z, t) \cdot \nu(z, t) &= -\mathrm{e}^{-u(z,t)}\frac{\partial u}{\partial t}(z, 1) \cdot \nu(z, t) \\ &= -\mathrm{e}^{-u(z,t)}\frac{\partial u}{\partial t}(z, 1) \cdot (\bar{X}(z, t) - \mathrm{e}^{-u(z,t)}(z, 1)) \\ &= -\mathrm{e}^{-u(z,t)}\frac{\partial u}{\partial t}(z, 1) \cdot \frac{1}{2}\mathrm{e}^{u(z,t)}(-z, 1) \\ &= \frac{\partial u}{\partial t}, \end{aligned} \tag{5.13}$$

where we used (5.3) and (5.5). Combining (5.11) and (5.13) implies that

$$\frac{\partial u}{\partial t} = \phi(t) - F(\mathcal{W} - \mathrm{I}). \tag{5.14}$$

Therefore $\varphi = \mathrm{e}^u$ satisfies

$$\frac{\partial \varphi}{\partial t} = \mathrm{e}^u \phi(t) - F(\mathrm{e}^u(\mathcal{W} - \mathrm{I})) = \varphi\phi(t) - F((A_{ij})^{-1}) \tag{5.15}$$

with A_{ij} defined in (5.9).

Conversely, suppose that we have a smooth solution $\varphi(\cdot, t)$ of the initial value problem (5.10) with A_{ij} positive definite. Then by the discussion in §5.4, the map $\bar{X}$ given in (5.5) using the function $u = \log\varphi$ defines a family of smooth h-convex hypersurfaces in $\mathbb{H}^{n+1}$. We claim that we can find a family of diffeomorphisms $\xi(\cdot, t) : S^n \to S^n$ such that $X(z, t) = \bar{X}(\xi(z, t), t)$ solves the flow equation (1.18). Since

$$\begin{aligned}
\frac{\partial}{\partial t} X(z, t) &= \frac{\partial}{\partial t}\bar{X}(\xi, t) + \partial_i \bar{X}\frac{\partial \xi^i}{\partial t}\\
&= \left(\frac{\partial}{\partial t}\bar{X}(\xi, t) \cdot \nu(\xi, t)\right)\nu(\xi, t) + \left(\frac{\partial}{\partial t}\bar{X}(\xi, t)\right)^{\top} + \partial_i \bar{X}\frac{\partial \xi^i}{\partial t}\\
&= (\phi(t) - F(\mathcal{W} - I))\nu(\xi, t) + \left(\frac{\partial}{\partial t}\bar{X}(\xi, t)\right)^{\top} + \partial_i \bar{X}\frac{\partial \xi^i}{\partial t},
\end{aligned}$$

where $(\cdot)^{\top}$ denotes the tangential part, it suffices to find a family of diffeomorphisms $\xi(\cdot, t) : S^n \to S^n$ such that

$$\left(\frac{\partial}{\partial t}\bar{X}(\xi, t)\right)^{\top} + \partial_i \bar{X}\frac{\partial \xi^i}{\partial t} = 0,$$

which is equivalent to

$$\left(\frac{\partial}{\partial t}\bar{X}(\xi, t)\right)^{\top} \cdot E_j - A_{ij}\frac{\partial \xi^i}{\partial t} = 0. \tag{5.16}$$

By the assumption that A_{ij} is positive definite on $S^n \times [0, T)$, the standard theory of ordinary differential equations implies that the system (5.16) has a unique smooth solution for the initial condition $\xi(z, 0) = z$. This completes the proof. □

6. Proof of Theorem 1.7

In this section, we will give the proof of Theorem 1.7.

6.1. Pinching estimate

Firstly, we prove the following pinching estimate for the shifted principal curvatures of the evolving hypersurfaces along the flow (1.18).

Proposition 6.1. *Let M_t be a smooth solution to the flow (1.18) on $[0, T)$ and assume that F satisfies the assumption in Theorem 1.7. Then there exists a constant $C > 0$ depending only on M_0 such that*

$$\lambda_n \leq C\lambda_1 \tag{6.1}$$

for all $t \in [0, T)$, where $\lambda_n = \kappa_n - 1$ is the largest shifted principal curvature and $\lambda_1 = \kappa_1 - 1$ is the smallest shifted principal curvature.

Proof. We consider the four cases of F separately.

(i) F is concave and F vanishes on the boundary of the positive cone Γ_+. Define $G = F^{-1}\operatorname{tr}(S)$ on $M \times [0, T)$. Then (2.14) and (2.16) imply that

$$\begin{aligned}\frac{\partial}{\partial t}G &= F^{-1}\frac{\partial}{\partial t}\operatorname{tr}(S) - F^{-2}\operatorname{tr}(S)\frac{\partial}{\partial t}F \\ &= \dot{F}^{kl}\nabla_k\nabla_l G + 2F^{-1}\dot{F}^{kl}\nabla_k F\nabla_l G + F^{-1}\sum_{i=1}^{n}\ddot{F}^{kl,pq}\nabla_i h_{kl}\nabla^i h_{pq} \\ &\quad + \phi(t)f^{-2}\Big(\operatorname{tr}(S)\sum_k \dot{f}^k\lambda_k^2 - f|S|^2\Big) + f^{-1}\Big(n\sum_k \dot{f}^k\lambda_k^2 - |S|^2\sum_k \dot{f}^k\Big).\end{aligned} \tag{6.2}$$

Since F is concave, by the inequality (2.2) we have

$$\begin{aligned}\operatorname{tr}(S)\sum_k \dot{f}^k\lambda_k^2 - f|S|^2 &= \sum_{k,l}(\dot{f}^k\lambda_k^2\lambda_l - \dot{f}^k\lambda_k\lambda_l^2) \\ &= \frac{1}{2}\sum_{k,l}(\dot{f}^k - \dot{f}^l)(\lambda_k - \lambda_l)\lambda_k\lambda_l \leq 0,\end{aligned}$$

and

$$\begin{aligned}n\sum_k \dot{f}^k\lambda_k^2 - |S|^2\sum_k \dot{f}^k &= \sum_{k,l}(\dot{f}^k\lambda_k^2 - \dot{f}^k\lambda_l^2) \\ &= \frac{1}{2}\sum_{k,l}(\dot{f}^k - \dot{f}^l)(\lambda_k^2 - \lambda_l^2) \leq 0.\end{aligned}$$

Thus the zero order terms on the right of (6.2) are always non-positive. The concavity of F also implies that the third term there is non-positive. Then we have

$$\frac{\partial}{\partial t}G \leq \dot{F}^{kl}\nabla_k\nabla_l G + 2F^{-1}\dot{F}^{kl}\nabla_k F\nabla_l G. \tag{6.3}$$

The maximum principle implies that the supremum of G over M_t is decreasing in time along the flow (1.18). The assumption that f approaches zero on the boundary of the positive cone Γ_+ then guarantees that the region $\{G(t) \leq \sup_{t=0} G\} \subset \Gamma_+$ does not touch the boundary of Γ_+. Since G is homogeneous of degree zero with respect to λ_i, this implies that $\lambda_n \leq C\lambda_1$ for some constant $C > 0$ depending only on M_0 for all $t \in [0, T)$.

(ii) F is concave and inverse concave. Define a tensor $T_{ij} = S_{ij} - \varepsilon\operatorname{tr}(S)\delta_i^j$, where ε is chosen such that T_{ij} is positive definite initially. Clearly, $0 < \varepsilon \leq 1/n$. The evolution equation (2.16) implies that

$$\frac{\partial}{\partial t}T_{ij} = \dot F^{kl}\nabla_k\nabla_l T_{ij} + \ddot F^{kl,pq}\nabla_i h_{kl}\nabla_j h_{pq} - \varepsilon\Big(\sum_{i=1}^n \ddot F^{kl,pq}\nabla_i h_{kl}\nabla^i h_{pq}\Big)\delta_i^j$$
$$+\Big(\sum_{k=1}^n \dot f^k\lambda_k^2 + 2f - 2\phi(t)\Big)T_{ij} - \Big(\phi(t)+\sum_{k=1}^n \dot f^k\Big)(T_i^kT_{kj} + 2\varepsilon\,\mathrm{tr}(S)T_{ij})$$
$$+\varepsilon\Big(\phi(t)+\sum_{k=1}^n \dot f^k\Big)(|S|^2 - \varepsilon(\mathrm{tr}(S))^2)\delta_i^j + \sum_{k=1}^n \dot f^k\lambda_k^2(1-\varepsilon n)\delta_i^j. \tag{6.4}$$

We will apply the tensor maximum principle in Theorem 2.5 to show that T_{ij} is positive definite for $t>0$. If not, there exists a first time $t_0>0$ and some point $x_0\in M_{t_0}$ such that T_{ij} has a null vector $v\in T_{x_0}M_{t_0}$, i.e., $T_{ij}v^j=0$ at (x_0,t_0). The second line of (6.4) satisfies the null vector condition and can be ignored. The last line of (6.4) is also non-negative, since $0<\varepsilon<1/n$ and $|S|^2\ge(\mathrm{tr}(S))^2/n$. For the gradient terms in (6.4), Theorem 4.1 of [4] implies that

$$\ddot F^{kl,pq}\nabla_i h_{kl}\nabla_j h_{pq}v^iv^j - \varepsilon\Big(\sum_{i=1}^n \ddot F^{kl,pq}\nabla_i h_{kl}\nabla^i h_{pq}\Big)|v|^2$$
$$+\sup_\Lambda 2a^{kl}(2\Lambda_k^p\nabla_l T_{ip}v^i - \Lambda_k^p\Lambda_l^q T_{pq})\ge 0$$

for the null vector v provided that F is concave and inverse concave. Thus by Theorem 2.5, the tensor T_{ij} is positive definite for $t\in[0,T)$. Equivalently,

$$\lambda_1\ge\varepsilon(\lambda_1+\cdots+\lambda_n)$$

for any $t\in[0,T)$, which implies the pinching estimate (6.1).

(iii) F is inverse concave and F_* approaches zero on the boundary of Γ_+. In this case, we define $T_{ij}=S_{ij}-\varepsilon F\delta_i^j$, where ε is chosen such that T_{ij} is positive definite initially. By (2.14) and (2.16),

$$\frac{\partial}{\partial t}T_{ij} = \dot F^{kl}\nabla_k\nabla_l T_{ij} + \ddot F^{kl,pq}\nabla_i h_{kl}\nabla^j h_{pq} + (\dot f^k\lambda_k^2 + 2f - 2\phi(t))S_{ij}$$
$$-(\phi(t)+\sum_{k=1}^n \dot f^k)S_{ik}S_{kj} + \dot f^k\lambda_k^2\delta_i^j - \varepsilon(F-\phi(t))\dot f^k\lambda_k(\lambda_k+2)\delta_i^j. \tag{6.5}$$

Suppose $v=e_1$ is the null eigenvector of T_{ij} at (x_0,t_0) for some first time $t_0>0$. Denote the zero order terms of (6.5) by Q_0. At the point (x_0,t_0), εF is the smallest eigenvalue of S_{ij} with corresponding eigenvector v. Then

$$\begin{aligned}Q_0v^iv^j &= (\dot f^k\lambda_k^2+2f-2\phi(t))\varepsilon f|v|^2 + (f-\phi(t)-\dot f^k\kappa_k)\varepsilon^2f^2|v|^2\\ &\quad+\dot f^k\lambda_k^2|v|^2 - \varepsilon(f-\phi(t))\dot f^k\lambda_k(\lambda_k+2)|v|^2\\ &= \dot f^k\lambda_k^2(1+\varepsilon\phi(t))|v|^2 - \varepsilon^2f^2\Big(\sum_k \dot f^k+\phi(t)\Big)|v|^2\\ &= |v|^2\Big(\dot f^k\lambda_k\varepsilon(\lambda_k-\varepsilon f)\phi(t) + \sum_k \dot f^k(\lambda_k^2-\varepsilon^2f^2)\Big)\ge 0.\end{aligned}$$

By Theorem 2.5, to show that T_{ij} remains positive definite for $t > 0$, it suffices to show that

$$Q_1 := \ddot{F}^{kl,pq}\nabla_1 h_{kl}\nabla_1 h_{pq} + 2\sup_{\Lambda} \dot{F}^{kl}(2\Lambda_k^p \nabla_l T_{1p} - \Lambda_k^p \Lambda_l^q T_{pq}) \geq 0.$$

Noting that $T_{11} = 0$ and $\nabla_k T_{11} = 0$ at (x_0, t_0), the supremum over Λ can be computed exactly as follows:

$$\begin{aligned} 2\dot{F}^{kl}(2\Lambda_k^p \nabla_l T_{1p} - \Lambda_k^p \Lambda_l^q T_{pq}) &= 2\sum_{k=1}^n \sum_{p=2}^n \dot{f}^k (2\Lambda_k^p \nabla_k T_{1p} - (\Lambda_k^p)^2 T_{pp}) \\ &= 2\sum_{k=1}^n \sum_{p=2}^n \dot{f}^k \left(\frac{(\nabla_k T_{1p})^2}{T_{pp}} - \left(\Lambda_k^p - \frac{\nabla_k T_{1p}}{T_{pp}}\right)^2 T_{pp}\right). \end{aligned}$$

It follows that the supremum is obtained by choosing $\Lambda_k^p = \frac{\nabla_k T_{1p}}{T_{pp}}$. The required inequality for Q_1 becomes

$$Q_1 = \ddot{F}^{kl,pq}\nabla_1 h_{kl}\nabla_1 h_{pq} + 2\sum_{k=1}^n \sum_{p=2}^n \dot{f}^k \frac{(\nabla_k T_{1p})^2}{T_{pp}} \geq 0.$$

Using (2.1) to express the second derivatives of F and noting that $\nabla_k T_{1p} = \nabla_k h_{1p} - \varepsilon \nabla_k F \delta_1^p = \nabla_k h_{1p}$ at (x_0, t_0) for $p \neq 1$, we have

$$Q_1 = \ddot{f}^{kl}\nabla_1 h_{kk}\nabla_1 h_{ll} + 2\sum_{k>l} \frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l}(\nabla_1 h_{kl})^2 + 2\sum_{k=1}^n \sum_{l=2}^n \frac{\dot{f}^k}{\lambda_l - \varepsilon F}(\nabla_1 h_{kl})^2. \tag{6.6}$$

Since f is inverse concave, the inequality (2.3) implies that the first term of the right-hand side of (6.6) satisfies

$$\begin{aligned} \ddot{f}^{kl}\nabla_1 h_{kk}\nabla_1 h_{ll} &\geq 2f^{-1}\Big(\sum_{k=1}^n \dot{f}^k \nabla_1 h_{kk}\Big)^2 - 2\sum_k \frac{\dot{f}^k}{\lambda_k}(\nabla_1 h_{kk})^2 \\ &= 2f^{-1}(\nabla_1 F)^2 - 2\sum_k \frac{\dot{f}^k}{\lambda_k}(\nabla_1 h_{kk})^2. \end{aligned}$$

Then

$$\begin{aligned} Q_1 \geq\ & 2f^{-1}(\nabla_1 F)^2 - 2\sum_k \frac{\dot{f}^k}{\lambda_k}(\nabla_1 h_{kk})^2 \\ &+ 2\sum_{k>l} \frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l}(\nabla_1 h_{kl})^2 + 2\sum_{k=1}^n \sum_{l=2}^n \frac{\dot{f}^k}{\lambda_l - \varepsilon F}(\nabla_1 h_{kl})^2 \\ \geq\ & 2f^{-1}(\nabla_1 F)^2 - 2\frac{\dot{f}^1}{\lambda_1}(\nabla_1 h_{11})^2 - 2\sum_{k>1} \frac{\dot{f}^k}{\lambda_k}(\nabla_1 h_{kk})^2 \\ &+ 2\sum_{k>1} \frac{\dot{f}^k - \dot{f}^1}{\lambda_k - \lambda_1}(\nabla_k h_{11})^2 - 2\sum_{k\neq l>1} \frac{\dot{f}^k}{\lambda_l}(\nabla_1 h_{kl})^2 \\ &+ 2\sum_{k>1} \frac{\dot{f}^1}{\lambda_k - \varepsilon F}(\nabla_k h_{11})^2 + 2\sum_{k>1,l>1} \frac{\dot{f}^k}{\lambda_l - \varepsilon F}(\nabla_1 h_{kl})^2 \end{aligned}$$

$$= 2f^{-1}(\nabla_1 F)^2 - 2\frac{\dot f^1}{\lambda_1}(\nabla_1 h_{11})^2 + 2\sum_{k>1}\frac{\dot f^k}{\lambda_k - \lambda_1}(\nabla_k h_{11})^2$$
$$+ 2\sum_{k>1,l>1}\dot f^k\left(\frac{1}{\lambda_l - \varepsilon F} - \frac{1}{\lambda_l}\right)(\nabla_1 h_{kl})^2$$
$$\geq 2\left(\frac{1}{\varepsilon^2 F} - \frac{\dot f^1}{\lambda_1}\right)(\nabla_1 h_{11})^2 = 2\left(\frac{\sum_{k=1}^n \dot f^k\lambda_k}{\varepsilon^2 F^2} - \frac{\dot f^1}{\lambda_1}\right)(\nabla_1 h_{11})^2 \geq 0,$$

where we used $\lambda_1 = \varepsilon F$ and $\nabla_k h_{11} = \varepsilon\nabla_k F$ at (x_0, t_0), and the inequality in (2.4) due to the inverse concavity of f. Theorem 2.5 implies that T_{ij} remains positive definite for $t \in [0, T)$. Equivalently,

$$\frac{1}{\lambda_1} \leq \frac{1}{\varepsilon} f(\lambda)^{-1} = \frac{1}{\varepsilon} f_*\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right) \tag{6.7}$$

for all $t \in [0, T)$. Since f_* approaches zero on the boundary of the positive cone Γ_+, the estimate (6.7) and Lemma 12 of [9] give the pinching estimate (6.1).

(iv) $n = 2$. In this case, we do not need any second derivative condition on F. Define

$$G = \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}\right)^2.$$

Then G is homogeneous of degree zero in the shifted principal curvatures λ_1, λ_2. The evolution equation (2.16) implies that

$$\frac{\partial}{\partial t}G = \dot F^{kl}\nabla_k\nabla_l G + (\dot G^{ij}\ddot F^{kl,pq} - \dot F^{ij}\ddot G^{kl,pq})\nabla_i S_{kl}\nabla^j S_{pq}$$
$$-\Big(\phi(t) + \sum_k \dot f^k\Big)\dot G^{ij} S_{ik}S_{kj} + \Big(\sum_k \dot f^k\lambda_k^2\Big)\dot G^{ij}\delta_i^j. \tag{6.8}$$

The zero order terms of (6.8) are equal to

$$Q_0 = -4G\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\Big(\phi(t) + \sum_k \dot f^k\Big) - \frac{4G}{\lambda_1 + \lambda_2}\sum_k \dot f^k\lambda_k^2 \leq 0.$$

The same argument as in [5] shows that the gradient terms of (6.8) are non-positive at the critical point of G. Then the maximum principle implies that the supremum of G over M_t is non-increasing in time along the flow (1.18). This gives the pinching estimate (6.1) and the strict h-convexity of M_t for all $t \in [0, T)$. □

6.2. *Shape estimate*

Denote by $\rho_-(t)$, $\rho_+(t)$ the inner radius and outer radius of Ω_t. Then there exist two points $p_1, p_2 \in \mathbb{H}^{n+1}$ such that $B_{\rho_-(t)}(p_1) \subset \Omega_t \subset B_{\rho_+(t)}(p_2)$. By Corollary 5.2, the modified quermassintegral $\widetilde W_l$ is monotone under inclusion of *h-convex* domains in $\mathbb{H}^{n+1}$. This implies that

$$\tilde f_l(\rho_-(t)) = \widetilde W_l(B_{\rho_-(t)}(p_1)) \leq \widetilde W_l(\Omega_t) \leq \widetilde W_l(B_{\rho_+(t)}(p_2)) = \tilde f_l(\rho_+(t)).$$

Along the flow (1.18), $\widetilde{W}_l(\Omega_t) = \widetilde{W}_l(\Omega_0)$ is a fixed constant. Therefore,

$$\rho_-(t) \le C \le \rho_+(t),$$

where $C = \tilde{f}_l^{-1}(\widetilde{W}_l(\Omega_0)) > 0$ depends only on l, n and Ω_0.

On the other hand, since each Ω_t is *h-convex*, the inner radius and outer radius of Ω_t satisfy $\rho_+(t) \le c(\rho_-(t) + \rho_-(t)^{1/2})$ for some uniform positive constant c (see [12, 23]). Thus there exist positive constants c_1, c_2 depending only on n, l, M_0 such that

$$0 < c_1 \le \rho_-(t) \le \rho_+(t) \le c_2 \tag{6.9}$$

for all $t \in [0, T)$.

6.3. C^2 estimate

Proposition 6.2. *Under the assumptions of Theorem 1.7 with $\phi(t)$ given in (1.19), we have $F \le C$ for any $t \in [0, T)$, where C depends on n, l, M_0 but not on T.*

Proof. For any given $t_0 \in [0, T)$, let $B_{\rho_0}(p_0)$ be the inball of Ω_{t_0}, where $\rho_0 = \rho_-(t_0)$. Then a similar argument to that in [10, Lemma 4.2] yields

$$B_{\rho_0/2}(p_0) \subset \Omega_t, \qquad t \in [t_0, \min\{T, t_0 + \tau\}), \tag{6.10}$$

for some positive τ depending only on n, l, Ω_0. Consider the support function $u(x,t) = \sinh r_{p_0}(x) \langle \partial r_{p_0}, \nu \rangle$ of M_t with respect to the point p_0. Then (6.10) implies that

$$u(x,t) \ge \sinh(\rho_0/2) =: 2c \tag{6.11}$$

on M_t for any $t \in [t_0, \min\{T, t_0 + \tau\})$. On the other hand, the estimate (6.9) implies that $u(x,t) \le \sinh(2c_2)$ on M_t for all $t \in [t_0, \min\{T, t_0 + \tau\})$. Define the auxiliary function

$$W(x,t) = \frac{F(\mathcal{W} - \mathrm{I})}{u(x,t) - c}$$

on M_t for $t \in [t_0, \min\{T, t_0 + \tau\})$. Combining (2.14) and the evolution equation (4.4) for the support function, we see that the function W evolves by

$$\begin{aligned}\frac{\partial}{\partial t} W &= \dot{F}^{ij}\Big(\nabla_j \nabla_i W + \frac{2}{u-c}\nabla_i u \nabla_j W\Big) \\ &\quad - \frac{\phi(t)}{u-c}\big(\dot{F}^{ij}(h_i^k h_k^j - \delta_i^j) + W \cosh r_{p_0}(x)\big) \\ &\quad + \frac{F}{(u-c)^2}(F + \dot{F}^{kl}h_{kl})\cosh r_{p_0}(x) - \frac{cF}{(u-c)^2}\dot{F}^{ij}h_i^k h_k^j - W\dot{F}^{ij}\delta_i^j. \end{aligned} \tag{6.12}$$

The second line of (6.12) involves the global term $\phi(t)$ and is clearly non-positive by the h-convexity of the evolving hypersurface. By the homogeneity of f with respect to $\lambda_i = \kappa_i - 1$, we have $F + \dot{F}^{kl}h_{kl} = 2F + \sum_{k=1}^n \dot{f}^k$ and

$$\dot{F}^{ij}h_i^k h_k^j = \dot{f}^k(\lambda_k + 1)^2 = \dot{f}^k\lambda_k^2 + 2f + \sum_k \dot{f}^k \ge Cf^2,$$

where the last inequality is due to the pinching estimate (6.1). The last term of (6.12) is non-positive and can be thrown away. In summary, we arrive at

$$\frac{\partial}{\partial t}W \leq \dot{\Psi}^{ij}\left(\nabla_j\nabla_i W + \frac{2}{u-c}\nabla_i u\nabla_j W\right) + W^2\left(2+F^{-1}\sum_{k=1}^n \dot{f}^k\right)\cosh r_{p_0}(x) - c^2 C W^3.$$

Noting that $\dot{f}^k$ is homogeneous of degree zero, the pinching estimate (6.1) implies that each $\dot{f}^k$ is bounded from above and below by positive constants. Then without loss of generality we can assume that $F^{-1}\sum_{k=1}^n \dot{f}^k \leq 1$ since otherwise $F \leq \sum_{k=1}^n \dot{f}^k \leq C$ for some constant $C > 0$. By the upper bound $r_{p_0}(x) \leq 2c_2$, we obtain the estimate

$$\frac{\partial}{\partial t}W \leq \dot{\Psi}^{ij}\left(\nabla_j\nabla_i W + \frac{2}{u-c}\nabla_i u\nabla_j W\right) + W^2(3\cosh(2c_2) - c^2 C W)$$

on $[t_0, \min\{T, t_0+\tau\})$. Then the maximum principle implies that W is uniformly bounded from above and the upper bound on F follows by the upper bound on the outer radius in (6.9). ◻

Proposition 6.3. *There exists a positive constant C, independent of time T, such that $F \geq C > 0$.*

Proof. Since the evolving hypersurface M_t is strictly h-convex, for each time $t_0 \in [0, T)$ there exists a point $p \in \mathbb{H}^{n+1}$ and $x_0 \in M_{t_0}$ such that $\Omega_{t_0} \subset B_{\rho_+(t_0)}(p)$ and $\Omega_{t_0} \cap B_{\rho_+(t_0)}(p) = x_0$. By the estimate (6.9) on the outer radius, the value of F at the point (x_0, t_0) satisfies

$$F(x_0, t_0) \geq \coth \rho_+(t_0) \geq \coth c_2.$$

Recall that the function F satisfies the evolution equation (2.14):

$$\frac{\partial}{\partial t}F = g^{ik}\dot{F}^{ij}\nabla_k\nabla_j F + (F - \phi(t))(\dot{F}^{ij}h_i^k h_k^j - \dot{F}^{ij}\delta_i^j). \tag{6.13}$$

By the pinching estimate (6.1) and the upper bound on the curvature proved in Proposition 6.2, the equation (6.13) is uniformly parabolic and the coefficient of the gradient terms and the lower order terms in (6.13) have bounded C^0 norm. Then there exists $r > 0$ depending only on the bounds on the coefficients of (6.13) such that we can apply the Harnack inequality of Krylov and Safonov [21] to (6.13) in a space-time neighbourhood $B_r(x_0) \times (t_0 - r^2, t_0]$ of x_0 and deduce the lower bound $F \geq CF(x_0, t_0) \geq C > 0$ in a smaller neighbourhood $B_{r/2}(x_0) \times (t_0 - r^2/4, t_0]$. Note that the diameter r of the space-time neighbourhood is independent of the point (x_0, t_0). Consider the boundary point $x_1 \in \partial B_{r/2}(x_0)$. We can look at the equation (6.13) in a neighborhood $B_r(x_1) \times (t_0 - r^2, t_0]$ of the point (x_1, t_0). The Harnack inequality implies that $F \geq CF(x_1, t_0) \geq C > 0$ in $B_{r/2}(x_1) \times (t_0 - r^2/4, t_0]$. Since the diameter of each M_{t_0} is uniformly bounded from above, after a finite number of iterations we conclude that $F \geq C > 0$ on M_{t_0} for a uniform constant C independent of t_0. ◻

The pinching estimate (6.1) together with the bounds on F proven in Propositions 6.2 and 6.3 implies that the shifted principal curvatures $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfy

$$0 < C^{-1} \le \lambda_i \le C$$

for some constant $C > 0$ and $t \in [0, T)$. This gives the uniform C^2 estimate of the evolving hypersurfaces M_t. Moreover, the global term $\phi(t)$ given in (1.19) satisfies $0 < C^{-1} \le \phi(t) \le C$ for some constant $C > 0$.

6.4. Long time existence and convergence

If F is inverse concave, by applying a similar argument to [10, 25] (see also [30]) to the equation (5.10), we can first derive a $C^{2,\alpha}$ estimate and then a $C^{k,\alpha}$ estimate for all $k \ge 2$. If F is concave or $n = 2$, we write the flow (1.18) as a scalar parabolic PDE for the radial function as follows: Since each M_t is strictly h-convex, we write M_t as a radial graph over a geodesic sphere for a smooth function ρ on S^n. Let $\{\theta^i\}$, $i = 1, \ldots, n$, be a local coordinate system on S^n. The induced metric on M_{t_0} from $\mathbb{H}^{n+1}$ takes the form

$$g_{ij} = \bar{\nabla}_i \rho \bar{\nabla}_j \rho + \sinh^2 \rho \bar{g}_{ij},$$

where $\bar{g}_{ij}$ denotes the round metric on S^n. Up to a tangential diffeomorphism, the flow equation (1.18) is equivalent to the scalar parabolic equation

$$\frac{\partial}{\partial t}\rho = (\phi(t) - F(\mathcal{W} - \mathrm{I}))\sqrt{1 + |\bar{\nabla}\rho|^2/\sinh^2 \rho} \tag{6.14}$$

for the smooth function $\rho(\cdot, t)$ on S^n. The Weingarten matrix $\mathcal{W} = (h_i^j)$ can be expressed as

$$h_i^j = \frac{\coth \rho}{v}\delta_i^j + \frac{\coth \rho}{v^3 \sinh^2 \rho}\bar{\nabla}^j \rho \bar{\nabla}_i \rho - \frac{\tilde{\sigma}^{jk}}{v \sinh^2 \rho}\bar{\nabla}_k \bar{\nabla}_i \rho,$$

where

$$v = \sqrt{1 + |\bar{\nabla}\rho|^2/\sinh^2 \rho} \quad \text{and} \quad \tilde{\sigma}^{jk} = \sigma^{jk} - \frac{\bar{\nabla}^j \rho \bar{\nabla}^k \rho}{v^2 \sinh^2 \rho}.$$

Thus we can apply an argument as in [3, 24] to derive a higher regularity estimate. Therefore, for any F satisfying the assumption of Theorem 1.7, the solution of the flow (1.18) exists for all time $t \in [0, \infty)$ and remains smooth and strictly h-convex. Moreover, the Alexandrov reflection argument as in [10, §6] implies that the flow converges smoothly as $t \to \infty$ to a geodesic sphere ∂B_{r_∞} which satisfies $\widetilde{W}_l(B_{r_\infty}) = \widetilde{W}_l(\Omega_0)$. This finishes the proof of Theorem 1.7.

7. Conformal deformation in the conformal class of $\bar{g}$

In this section we mention an interesting connection (closely related to the results of [15]) between flows of h-convex hypersurfaces in hyperbolic space by functions of principal curvatures, and conformal flows of conformally flat metrics on S^n. This allows us to translate some of our results to convergence theorems for metric flows, and our isoperimetric inequalities to corresponding results for conformally flat metrics.

The crucial observation is that there is a correspondence between conformally flat metrics on S^n satisfying a certain curvature inequality, and horospherically convex hypersurfaces. To describe this, we recall that the isometry group of $\mathbb{H}^{n+1}$ coincides with $O_+(n+1,1)$, the group of future preserving linear isometries of Minkowski space. This also coincides with the Möbius group of conformal diffeomorphisms of S^n, by the following correspodence: If $L \in O_+(n+1,1)$, we define a map ρ_L from S^n to S^n by

$$\rho_L(\mathbf{e}) = \pi(L(\mathbf{e},1)),$$

where $\pi(x,y) = x/y$ is the radial projection from the future null cone to the sphere at height 1. This defines a group homomorphism from $O_+(n+1,1)$ to the group of Möbius transformations. We have the following result:

Proposition 7.1. *If $L \in O_+(n+1,1)$ and $M \subset \mathbb{H}^{n+1}$ is a horospherically convex hypersurface with horospherical support function $u : S^n \to \mathbb{R}$, denote by u_L the horospherical support function of $L(M)$. Then ρ_L is an isometry from $\mathrm{e}^{-2u}\bar{g}$ to $\mathrm{e}^{-2u_L}\bar{g}$. That is,*

$$\mathrm{e}^{-2u(\mathbf{e})}\bar{g}_{\mathbf{e}}(v_1,v_2) = \mathrm{e}^{-2u_L(\rho_L(\mathbf{e}))}\bar{g}_{\rho_L(\mathbf{e})}(D\rho_L(v_1), D\rho_L(v_2))$$

for all $\mathbf{e} \in S^n$ and $v_1, v_2 \in T_{\mathbf{e}}S^n$.

Proof. We compute

$$\begin{aligned}
\mathrm{e}^{-u(\mathbf{e})} &= -X\cdot(\mathbf{e},1) = -L(X)\cdot L(\mathbf{e},1)\\
&= -L(X)\cdot\mu(\mathbf{e}_L,1) \quad \text{where } \mu = |L(\mathbf{e},1)\cdot(0,1)|\\
&= \mu\mathrm{e}^{-u_L(\mathbf{e}_L)}.
\end{aligned}$$

On the other hand, the Möbius transformation ρ_L is a conformal transformation with conformal factor $\mu = |L(\mathbf{e},1)\cdot(0,1)|$. The result follows directly. □

Corollary 7.2. *Isometry invariants of a horospherically convex hypersurface M are Möbius invariants of the conformally flat metric $\tilde{g} = \mathrm{e}^{-2u}\bar{g}$, and vice versa. In particular, Riemannian invariants of g are isometry invariants of M.*

Computing explicitly, we find that for $n > 2$ the Schouten tensor

$$\tilde{S}_{ij} = \frac{1}{n-2}\left(\tilde{R}_{ij} - \frac{\tilde{R}}{2(n-1)}\tilde{g}_{ij}\right)$$

of $\tilde{g}$ (which completely determines the curvature tensor for a conformally flat metric) is given by

$$\begin{aligned}
\tilde{S}_{ij} &= \tfrac{1}{2}\bar{g}_{ij} + \bar{\nabla}_i\bar{\nabla}_j u + u_i u_j - \tfrac{1}{2}|\bar{\nabla}u|^2\bar{g}_{ij} = \mathrm{e}^{-u}A_{ij} + \tfrac{1}{2}\tilde{g}_{ij}\\
&= [(\mathcal{W}-\mathrm{I})^{-1}]_i^p\,\tilde{g}_{pk} + \tfrac{1}{2}\tilde{g}_{ij}.
\end{aligned}$$

It follows that the eigenvalues of $\tilde{S}_{ij}$ (with respect to $\tilde{g}_{ij}$) are $\frac{1}{2} + \frac{1}{\lambda_i}$, where $\lambda_i = \kappa_i - 1$. When $n = 2$ the tensor $\tilde{S}_{ij}$ defined by the right-hand side of the above equation is by construction Möbius-invariant, and so gives a Möbius invariant of $\tilde{g}$ which is not a Riemannian invariant. This tensor still has the same relation to the principal curvatures of the corresponding h-convex hypersurface.

We observe that this connection between the Schouten tensor of $\tilde{g}$ and the Weingarten map of the hypersurface leads to a coincidence between the corresponding evolution equations: If a family of h-convex hypersurfaces $M_t = X(M,t)$ evolves according to a curvature-driven evolution equation of the form

$$\frac{\partial X}{\partial t} = -F(\mathcal{W} - \mathrm{I}, t)\nu$$

then the metric $\tilde{g}$ satisfies $\tilde{S} > \frac{1}{2}\tilde{g}$, and evolves according to the parabolic conformal flow

$$\frac{\partial \tilde{g}}{\partial t} = 2F\left(\left(\tilde{S} - \frac{1}{2}\tilde{g}\right)^{-1}, t\right)\tilde{g}.$$

In particular the convergence theorems for hypersurface flows correspond to convergence theorems for the corresponding conformal flows, and the resulting geometric inequalities for hypersurfaces imply corresponding geometric inequalities for the metric $\tilde{g}$.

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References

[1] Andrews, B.: Contraction of convex hypersurfaces in Euclidean space. Calc. Var. Partial Differential Equations **2**, 151–171 (1994) Zbl 0805.35048 MR 1385524

[2] Andrews, B.: Gauss curvature flow: the fate of the rolling stones. Invent. Math. **138**, 151–161 (1999) Zbl 0936.35080 MR 1714339

[3] Andrews, B.: Fully nonlinear parabolic equations in two space variables. arXiv:0402235 (2004)

[4] Andrews, B.: Pinching estimates and motion of hypersurfaces by curvature functions. J. Reine Angew. Math. **608**, 17–33 (2007) Zbl 1129.53044 MR 2339467

[5] Andrews, B.: Moving surfaces by non-concave curvature functions. Calc. Var. Partial Differential Equations **39**, 649–657 (2010) Zbl 1203.53062 MR 2729317

[6] Andrews, B., Chen, X.: Surfaces moving by powers of Gauss curvature. Pure Appl. Math. Quart. **8**, 825–834 (2012) Zbl 1263.53058 MR 2959911

[7] Andrews, B., Chen, X.: Curvature flow in hyperbolic spaces. J. Reine Angew. Math. **729**, 29–49 (2017) Zbl 1371.53060 MR 3680369

[8] Andrews, B., Hopper, C.: The Ricci Flow in Riemannian Geometry. Lecture Notes in Math. 2011, Springer, Heidelberg (2011) Zbl 1214.53002 MR 2760593

[9] Andrews, B., McCoy, J., Zheng, Y.: Contracting convex hypersurfaces by curvature. Calc. Var. Partial Differential Equations **47**, 611–665 (2013) Zbl 1288.35292 MR 3070558

[10] Andrews, B., Wei, Y.: Quermassintegral preserving curvature flow in hyperbolic space. Geom. Funct. Anal. **28**, 1183–1208 (2018) Zbl 1401.53047 MR 3856791

[11] Bertini, M. C., Pipoli, G.: Volume preserving non-homogeneous mean curvature flow in hyperbolic space. Differential Geom. Appl. **54**, 448–463 (2017) Zbl 1372.53064 MR 3693942

[12] Cabezas-Rivas, E., Miquel, V.: Volume preserving mean curvature flow in the hyperbolic space. Indiana Univ. Math. J. **56**, 2061–2086 (2007) Zbl 1130.53045 MR 2359723

[13] Chow, B.: Geometric aspects of Aleksandrov reflection and gradient estimates for parabolic equations. Comm. Anal. Geom. **5**, 389–409 (1997) Zbl 0899.53044 MR 1483984

[14] Chow, B., Gulliver, R.: Aleksandrov reflection and nonlinear evolution equations. I. The n-sphere and n-ball. Calc. Var. Partial Differential Equations **4**, 249–264 (1996) Zbl 0851.58041 MR 1386736

[15] Espinar, J. M., Gálvez, J. A., Mira, P.: Hypersurfaces in $\mathbb{H}^{n+1}$ and conformally invariant equations: the generalized Christoffel and Nirenberg problems. J. Eur. Math. Soc. **11**, 903–939 (2009) Zbl 1203.53057 MR 2538508

[16] Gao, S., Li, H., Ma, H.: Uniqueness of closed self-similar solutions to σ_k^α-curvature flow. Nonlinear Differential Equations Appl. **25**, art. 45, 26 pp. (2018) Zbl 1409.53057 MR 3845754

[17] Ge, Y., Wang, G., Wu, J.: Hyperbolic Alexandrov–Fenchel quermassintegral inequalities II. J. Differential Geom. **98**, 237–260 (2014) Zbl 1301.53077 MR 3263518

[18] Guan, P., Ma, X.-N.: The Christoffel–Minkowski problem. I. Convexity of solutions of a Hessian equation. Invent. Math. **151**, 553–577 (2003) Zbl 1213.35213 MR 1961338

[19] Hamilton, R. S.: Three-manifolds with positive Ricci curvature. J. Differential Geom. **17**, 255–306 (1982) Zbl 0504.53034 MR 664497

[20] Hamilton, R. S.: Four-manifolds with positive curvature operator. J. Differential Geom. **24**, 153–179 (1986) Zbl 0628.53042 MR 862046

[21] Krylov, N. V., Safonov, M. V.: A property of the solutions of parabolic equations with measurable coefficients. Izv. Akad. Nauk SSSR Ser. Mat. **44**, 161–175, 239 (1980) (in Russian) Zbl 0464.35035 MR 563790

[22] Li, H., Wei, Y., Xiong, C.: A geometric inequality on hypersurface in hyperbolic space. Adv. Math. **253**, 152–162 (2014) Zbl 1316.53077 MR 3148549

[23] Makowski, M.: Mixed volume preserving curvature flows in hyperbolic space. arXiv:1208.1898 (2012)

[24] McCoy, J. A.: Mixed volume preserving curvature flows. Calc. Var. Partial Differential Equations **24**, 131–154 (2005) Zbl 1079.53099 MR 2164924

[25] McCoy, J. A.: More mixed volume preserving curvature flows. J. Geom. Anal. **27**, 3140–3165 (2017) Zbl 1079.53099 MR 3708009

[26] Santaló, L. A.: Integral Geometry and Geometric Probability. 2nd ed., Cambridge Math. Library, Cambridge Univ. Press, Cambridge (2004) Zbl 1116.53050 MR 2162874

[27] Schwarz, G. W.: Smooth functions invariant under the action of a compact Lie group. Topology **14**, 63–68 (1975) Zbl 0297.57015 MR 0370643

[28] Solanes, G.: Integral geometry and the Gauss–Bonnet theorem in constant curvature spaces. Trans. Amer. Math. Soc. **358**, 1105–1115 (2006) Zbl 1082.53075 MR 2187647

[29] Steinhagen, P.: Über die größte Kugel in einer konvexen Punktmenge. Abh. Math. Sem. Univ. Hamburg **1**, 15–26 (1922) JFM 48.0837.03 MR 3069385

[30] Tian, G., Wang, X.-J.: A priori estimates for fully nonlinear parabolic equations. Int. Math. Res. Notices **2013**, 3857–3877 Zbl 1334.35109 MR 3096911

[31] Tso, K.: Deforming a hypersurface by its Gauss–Kronecker curvature. Comm. Pure Appl. Math. **38**, 867–882 (1985) Zbl 0612.53005 MR 812353

[32] Wang, G., Xia, C.: Isoperimetric type problems and Alexandrov–Fenchel type inequalities in the hyperbolic space. Adv. Math. **259**, 532–556 (2014) Zbl 1292.52008 MR 3197666