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The Lieb–Thirring inequality revisited

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Abstract. We provide new estimates on the best constant of the Lieb–Thirring inequality for the sum of the negative eigenvalues of Schrödinger operators, which significantly improve the so far existing bounds.

Keywords. Lieb–Thirring inequality, Schrödinger operator, Sobolev inequality

1. Introduction

In 1975, Lieb and Thirring [19, 20] proved that the sum of all negative eigenvalues of a Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$, with a real-valued potential $V : \mathbb{R}^d \to \mathbb{R}$, admits the bound

$$\mathrm{Tr}[-\Delta + V]_- \le L_{1,d} \int_{\mathbb{R}^d} V(x)_-^{1+d/2}\, \mathrm{d}x \tag{1}$$

for a finite constant $L_{1,d} > 0$ depending only on the dimension, for all $d \ge 1$. Here we use the convention that $t_\pm = \max\{\pm t, 0\}$.

Inequality (1) should be compared with Weyl's law [18, Theorem 12.12]

$$\mathrm{Tr}[-h^2\Delta + V]_- \approx \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^d\times\mathbb{R}^d} [|hk|^2 + V(x)]_-\, \mathrm{d}k\, \mathrm{d}x = L^{\mathrm{cl}}_{1,d} h^{-d} \int_{\mathbb{R}^d} V(x)_-^{1+d/2}\, \mathrm{d}x \tag{2}$$

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where

$$L_{1,d}^{\mathrm{cl}} = \frac{2}{d+2} \cdot \frac{|B_1|}{(2\pi)^d}$$

with $|B_1|$ the volume of the unit ball in $\mathbb{R}^d$. While (2) is only correct in the semiclassical limit $h \to 0$, the Lieb–Thirring inequality (1) is a universal bound for all finite parameters.

A simpler version of (1) is the following bound for a single eigenvalue:

$$\int_{\mathbb{R}^d} \left(|\nabla u(x)|^2 + V(x)|u(x)|^2\right) \mathrm{d}x \geq -L_{1,d}^{\mathrm{So}} \int_{\mathbb{R}^d} V(x)_-^{1+d/2} \, \mathrm{d}x, \tag{3}$$

which is a consequence of Sobolev's inequality, namely some sort of uncertainty principle. This inequality is essentially due to Keller [14]; see also [4] for a stability analysis. The Lieb–Thirring inequality (1) extends Sobolev's inequality (3) by taking the exclusion principle into account.

The Lieb–Thirring conjecture [20] concerns the best constant in (1) and states that this is given by

$$L_{1,d} = \max\{L_{1,d}^{\mathrm{cl}}, L_{1,d}^{\mathrm{So}}\} = \begin{cases} L_{1,d}^{\mathrm{cl}} & \text{if } d \geq 3, \\ L_{1,2}^{\mathrm{So}} & \text{if } d = 1, 2, \end{cases} \tag{4}$$

with $L_{1,d}^{\mathrm{So}}$ being the best constant in (3). While the lower bound $L_{1,d} \geq \max\{L_{1,d}^{\mathrm{cl}}, L_{1,d}^{\mathrm{So}}\}$ is obvious, proving the matching upper bound is a major challenge in mathematical physics.

The original proof of Lieb and Thirring [19] gave $L_{1,d}/L_{1,d}^{\mathrm{cl}} \leq 4\pi$ in $d = 3$. Since then, there have been many contributions devoted to improving the upper bound on $L_{1,d}$ [17, 8, 3, 13, 6]. The currently best-known result is

$$L_{1,d}/L_{1,d}^{\mathrm{cl}} \leq \pi/\sqrt{3} = 1.814\ldots \tag{5}$$

which was proved for $d = 1$ by Eden–Foias [8] in 1991 and then extended to all $d \geq 1$ by Dolbeault, Laptev and Loss [6] in 2008.

Our new result is

Theorem 1. *For all $d \geq 1$, the best constant in the Lieb–Thirring inequality* (1) *satisfies*

$$L_{1,d}/L_{1,d}^{\mathrm{cl}} \leq 1.456.$$

Our estimate is a significant improvement over (5), but in one dimension is still about 26% bigger than the expected value $L_{1,1}^{\mathrm{So}}/L_{1,1}^{\mathrm{cl}} = 2/\sqrt{3} = 1.155\ldots$ in [20].

Historically, the Lieb–Thirring inequality was invented to prove the stability of matter [19]. In this context, it can be stated as a lower bound on the fermionic kinetic energy,

$$\mathrm{Tr}(-\Delta\gamma) \geq K_d \int_{\mathbb{R}^d} \gamma(x,x)^{1+2/d} \, \mathrm{d}x. \tag{6}$$

Here γ is an arbitrary one-body density matrix on $L^2(\mathbb{R}^d)$, i.e. $0 \leq \gamma \leq 1$ with $\mathrm{Tr}\,\gamma < \infty$, and $\gamma(x,x)$ is the diagonal part of the kernel of γ (which can be defined properly by the spectral decomposition). By a standard duality argument, (1) is equivalent to (6), and the corresponding best constants are related by

$$K_d(1+2/d) = [L_{1,d}(1+d/2)]^{-2/d}. \tag{7}$$

In particular, K_d should be compared with the semiclassical constant

$$K_d^{\mathrm{cl}} = \frac{(2\pi)^2}{|B_1|^{2/d}} \cdot \frac{d}{d+2},$$

which emerges naturally from the lowest kinetic energy of the Fermi gas in a finite volume.

In 2011, Rumin [23] found a direct proof of (6), without using the dual form (1). His method has been used to derive several new estimates, e.g. a positive density analogue of (6) in [10], and it will also be the starting point of our analysis. Note that Rumin's original proof [23] gives $K_d/K_d^{\mathrm{cl}} \geq d/(d+4)$, and hence

$$L_{1,d}/L_{1,d}^{\mathrm{cl}} \leq \left[\frac{d+4}{d}\right]^{d/2}, \tag{8}$$

so $L_{1,1}/L_{1,1}^{\mathrm{cl}} \leq \sqrt{5} = 2.236\ldots$ when $d=1$ and worse estimates in higher dimensions. Therefore, new ideas are needed to push forward the bound.

Our proof of Theorem 1 contains several main ingredients:

- First, we will modify Rumin's proof by introducing an *optimal momentum decomposition*. This gives $L_{1,1}/L_{1,1}^{\mathrm{cl}} \leq 1.618\ldots$ in $d=1$, which is already an improvement over the best-known result (5) in $d=1$.
- Second, we use the Laptev–Weidl *lifting argument* to extend the bound $L_{1,d}/L_{1,d}^{\mathrm{cl}} \leq 1.618\ldots$ to arbitrary dimension d, which is an improvement over the best-known result (5). The idea of lifting with respect to dimension is by now classical [16, 13, 6], but its combination with Rumin's method is not completely obvious and requires an improvement of the bound in [9].
- Third, we take into account a *low momentum averaging*. This improves further the bound to $L_{1,1}/L_{1,1}^{\mathrm{cl}} \leq 1.456$ in $d=1$ (and worse estimates in higher dimensions). This is one of our key ideas and deviates substantially from Rumin's original argument.
- Finally, we transfer the one-dimensional bound in the last step to higher dimensions by the *lifting argument* again.

These steps will be explained in the next four sections. For the proof of Theorem 1 only the last two sections are relevant, but we feel that a slow presentation of the various new ideas might be useful.

As a by-product of our method we obtain Lieb–Thirring inequalities for fractional Schrödinger operators. The inequalities we are interested in have the form

$$\mathrm{Tr}[(-\Delta)^\sigma + V]_- \leq L_{1,d,\sigma} \int_{\mathbb{R}^d} V(x)_-^{1+\frac{d}{2\sigma}}\, \mathrm{d}x \tag{9}$$

and

$$\mathrm{Tr}((-\Delta)^\sigma \gamma) \geq K_{d,\sigma} \int_{\mathbb{R}^d} \gamma(x,x)^{1+2\sigma/d}\, \mathrm{d}x. \tag{10}$$

Again, a duality argument shows that the optimal constants in these two inequalities satisfy the relation

$$K_{d,\sigma}\left(1+\frac{2\sigma}{d}\right) = \left[L_{1,d,\sigma}\left(1+\frac{d}{2\sigma}\right)\right]^{-2\sigma/d}. \tag{11}$$

Finally, the semiclassical constants are given by

$$\begin{aligned} K^{\mathrm{cl}}_{d,\sigma} &= \frac{d}{d+2\sigma}\left(\frac{(2\pi)^d}{|B_1|}\right)^{2\sigma/d}, \\ L^{\mathrm{cl}}_{1,d,\sigma} &= \frac{2\sigma}{d+2\sigma}\,\frac{|B_1|}{(2\pi)^d}. \end{aligned} \tag{12}$$

The main ingredients of the proof of Theorem 1, except the lifting argument, apply equally well to the fractional case. This gives

Theorem 2. *For all $d \geq 1$ and $\sigma > 0$, the best constant in the Lieb–Thirring inequality* (10) *satisfies*

$$K_{d,\sigma}/K^{\mathrm{cl}}_{d,\sigma} \geq \max\left\{\frac{d}{d+4\sigma}\left[\frac{(d+2\sigma)^2\sin\left(\frac{2\pi\sigma}{d+2\sigma}\right)}{2\pi\sigma d}\right]^{1+2\sigma/d}, \frac{d}{d+2\sigma}\left(\frac{2\sigma}{d+2\sigma}\right)^{4\sigma/d}\mathcal{C}_{d,\sigma}^{-2\sigma/d}\right\}$$

where

$$\mathcal{C}_{d,\sigma} := \inf\left\{\left(\int_0^\infty \varphi^2\right)^{\frac{d}{2\sigma}}\frac{d}{2\sigma}\int_0^\infty \frac{(1-\int_0^\infty \varphi(s)f(st)\,\mathrm{d}s)^2}{t^{1+\frac{d}{2\sigma}}}\,\mathrm{d}t\right\} \tag{13}$$

with the infimum taken over all functions $f, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\int_0^\infty f^2 = 1$.

In particular, when $\sigma = 1/2$ and $d = 3$, we have $\mathcal{C}_{3,1/2} \leq 0.046737$ and hence

$$K_{3,1/2}/K^{\mathrm{cl}}_{3,1/2} \geq 0.826.$$

The proof of Theorem 2 is presented in the last section; see also Remark 7 in Section 3.

For $\sigma = 1$ and $d > 1$, the bound from Theorem 2 is not as good as the lower bound in Theorem 1. For all other cases, Theorem 2 yields the best known constants. In particular in the physically relevant case $\sigma = 1/2$ and $d = 3$, i.e., the ultra-relativistic Schrödinger operator in three dimensions, where $K^{\mathrm{cl}}_{3,1/2} = \frac{3}{4}(6\pi^2)^{1/3} = 2.923\ldots$, our result improves significantly the bounds $K_{3,1/2}/K^{\mathrm{cl}}_{3,1/2} \geq 0.6$ in [23, p. 586] and $K_{3,1/2}/K^{\mathrm{cl}}_{3,1/2} \geq 0.558$ in [5, Eq. (3.4)].

An immediate consequence of Theorem 2 is

Corollary 3. *For every fixed $\sigma > 0$, in the limit of large dimensions we have*

$$\limsup_{d\to\infty} L_{1,d,\sigma}/L^{\mathrm{cl}}_{1,d,\sigma} \le e. \tag{14}$$

Indeed, from (11) we have $L_{1,d,\sigma}/L^{\mathrm{cl}}_{1,d,\sigma} = (K^{\mathrm{cl}}_{d,\sigma}/K_{d,\sigma})^{d/(2\sigma)}$. So (14) follows from the first lower bound in Theorem 2 and the fact that $(\sin(t)/t)^{1/t} \to 1$ as $t \to 0$. Note that Rumin's original proof gives a bound similar to (14) but with e replaced by e^2 (see (8)).

As a consequence of (14), we also have

$$\lim_{d\to\infty} K_{d,\sigma}/K^{\mathrm{cl}}_{d,\sigma} = 1. \tag{15}$$

The lower bound $\liminf_{d\to\infty} K_{d,\sigma}/K^{\mathrm{cl}}_{d,\sigma} \ge 1$ follows from (14), and the upper bound $K_{d,\sigma}/K^{\mathrm{cl}}_{d,\sigma} \le 1$ is well-known [9].

Finally, we note that in 2013, Lundholm and Solovej [21] found another direct proof of the kinetic estimate (6). Their approach is based on a local version of the exclusion principle, which is inspired by the first proof of the stability of matter by Dyson and Lenard [7]. Recently, the ideas in [21] have been developed further in [22] to show that

$$\mathrm{Tr}(-\Delta\gamma) \ge (K^{\mathrm{cl}}_d - \varepsilon)\int_{\mathbb{R}^d} \gamma(x,x)^{1+2/d}\,\mathrm{d}x - C_{d,\varepsilon}\int_{\mathbb{R}^d} |\nabla\sqrt{\gamma(x,x)}|^2\,\mathrm{d}x \tag{16}$$

for all $d \ge 1$ and $\varepsilon > 0$ (the gradient error term is always smaller than the kinetic term [11]). Note that from (16), as well as from all existing proofs of the Lieb–Thirring inequality (including the present paper), the real difference between dimensions is not visible. Therefore, new ideas are certainly needed to attack the full conjecture (4).

2. Optimal momentum decomposition

In this section, we use a modified version of Rumin's proof in [23] to prove

Proposition 4. *For $d \ge 1$, the best constant in the Lieb–Thirring inequality (6) satisfies*

$$K_d/K^{\mathrm{cl}}_d \ge \frac{d}{d+4}\left[\frac{(d+2)^2\sin\left(\frac{2\pi}{d+2}\right)}{2\pi d}\right]^{1+2/d}.$$

In particular, when $d = 1$ we get $K_1/K^{\mathrm{cl}}_1 \ge \frac{2187\sqrt{3}}{320\pi^3} \ge 0.381777$ and $L_{1,1}/L^{\mathrm{cl}}_{1,1} \le 1.618435$.

Proof. Let γ be an operator on $L^2(\mathbb{R}^d)$ with $0 \le \gamma \le 1$. By a density argument, it suffices to consider the case when γ is a finite-rank operator with smooth eigenfunctions. For any function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^\infty f^2 = 1$, using the momentum decomposition

$$-\Delta = p^2 = \int_0^\infty f^2(s/p^2)\,\mathrm{d}s, \quad p = -i\nabla,$$

and Fubini's theorem we can write

$$\begin{aligned}\operatorname{Tr}(-\Delta\gamma) &= \int_0^\infty \operatorname{Tr}[f(s/p^2)\gamma f(s/p^2)]\,\mathrm{d}s \\ &= \int_{\mathbb{R}^d}\left[\int_0^\infty (f(s/p^2)\gamma f(s/p^2))(x,x)\,\mathrm{d}s\right]\mathrm{d}x. \end{aligned} \tag{17}$$

Next, we estimate the kernel of $f(s/p^2)\gamma f(s/p^2)$. Using Cauchy–Schwarz and $0 \le \gamma \le 1$, for every $\varepsilon > 0$ we have the operator inequalities

$$\begin{aligned}\gamma &\le (1+\varepsilon) f(s/p^2)\gamma f(s/p^2) + (1+\varepsilon^{-1})(1-f(s/p^2))\gamma(1-f(s/p^2)) \\ &\le (1+\varepsilon) f(s/p^2)\gamma f(s/p^2) + (1+\varepsilon^{-1})(1-f(s/p^2))^2. \end{aligned} \tag{18}$$

This inequality implies for any $x \in \mathbb{R}^d$ the kernel bound

$$\gamma(x,x) \le (1+\varepsilon)(f(s/p^2)\gamma f(s/p^2))(x,x) + (1+\varepsilon^{-1})(1-f(s/p^2))^2(x,x). \tag{19}$$

Optimizing over $\varepsilon > 0$ we obtain

$$\sqrt{\gamma(x,x)} \le \sqrt{(f(s/p^2)\gamma f(s/p^2))(x,x)} + \sqrt{(1-f(s/p^2))^2(x,x)}. \tag{20}$$

Moreover, it is straightforward to see that

$$(1-f(s/p^2))^2(x,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1-f(s/k^2))^2\,\mathrm{d}k = s^{d/2}\frac{|B_1|}{(2\pi)^d}A_f, \tag{21}$$

where

$$A_f := \frac{d}{2}\int_0^\infty \frac{(1-f(t))^2}{t^{1+d/2}}\,\mathrm{d}t. \tag{22}$$

Consequently, we deduce from (20) that

$$(f(s/p^2)\gamma f(s/p^2))(x,x) \ge \left[\sqrt{\gamma(x,x)} - \sqrt{s^{d/2}\frac{|B_1|}{(2\pi)^d}A_f}\right]_+^2. \tag{23}$$

Next, inserting (23) into (17) and integrating over $s > 0$ lead to

$$\operatorname{Tr}(-\Delta\gamma) \ge \left(\int_{\mathbb{R}^d}\gamma(x,x)^{1+2/d}\,\mathrm{d}x\right)\left(\frac{|B_1|}{(2\pi)^d}A_f\right)^{-2/d}\frac{d^2}{(d+2)(d+4)}. \tag{24}$$

Thus,

$$K_d/K_d^{\mathrm{cl}} \ge \frac{d}{d+4}A_f^{-2/d}. \tag{25}$$

Finally, it remains to minimize A_f under the constraint $\int_0^\infty f^2 = 1$. We note that the proof in [23] corresponds to $f(t) = \mathbb{1}(t \leq 1)$ (although the representation there is rather different), which gives $A_f = 1$ but this is not optimal. From Lemma 5 below we have

$$\inf_f A_f = \left[\frac{d}{d+2}\frac{\frac{2\pi}{d+2}}{\sin(\frac{2\pi}{d+2})}\right]^{1+d/2}.$$

Inserting this into (25) we conclude the proof of Proposition 4. □

In the previous proof we needed the following solution of a minimization problem.

Lemma 5. *For any constant* $\beta > 1$*,*

$$\inf\left\{\int_0^\infty (1-f(t))^2 t^{-\beta}\,\mathrm{d}t : f:\mathbb{R}_+\to\mathbb{R}_+, \int_0^\infty f^2\,\mathrm{d}t = 1\right\} = \frac{(\beta-1)^{\beta-1}}{\beta^\beta}\left(\frac{\pi/\beta}{\sin(\pi/\beta)}\right)^\beta$$

and equality is achieved if and only if

$$f(t) = \frac{1}{1+\mu t^\beta} \quad \textit{with} \quad \mu = \left[\frac{\beta-1}{\beta}\cdot\frac{\pi/\beta}{\sin(\pi/\beta)}\right]^\beta.$$

Proof. Heuristically, the optimizer can be found by solving the Euler–Lagrange equation, but to make this rigorous one would have to prove that a minimizer exists. This can be easily done by setting $h(t) = (1-f(t))t^{-\beta/2}$, so the minimization problem is equivalent to

$$\inf\left\{\int_0^\infty h(t)^2\,\mathrm{d}t : h \in \partial C\right\}$$

where $\partial C = \{h:\mathbb{R}_+\to\mathbb{R} : \int_0^\infty (1-t^{\beta/2}h(t))^2\,\mathrm{d}t = 1\}$ is the boundary of the strictly convex set $C = \{h:\mathbb{R}_+\to\mathbb{R} : \int_0^\infty (1-t^{\beta/2}h(t))^2\,\mathrm{d}t \leq 1\}$. Since C is closed, which follows easily from Fatou's lemma, and does not contain the zero function, it contains a function h_* of minimal length. Necessarily $h_* \in \partial C$, otherwise h_* would be in the interior of C and we could shrink it, thus reducing its length a little bit, which is impossible. So $h_*(t) = (1-f_*(t))t^{-\beta/2}$ has minimal L^2 norm under all f with $\int_0^\infty f(t)^2\,\mathrm{d}t = \int_0^\infty (1-t^{\beta/2}h(t))^2\,\mathrm{d}t = 1$. Hence f_* is a minimizer which must obey the Euler–Lagrange equation.

A more direct solution is as follows: Let $f_*(t) = (1+(\mu_* t)^\beta)^{-1}$ with

$$\mu_* = \int_0^\infty \frac{\mathrm{d}t}{(1+t^\beta)^2},$$

so that $t^{-\beta}(1-f_*(t)) = \mu_*^\beta f_*(t)$ and

$$\int_0^\infty f_*(t)^2\,\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}t}{(1+(\mu_* t)^\beta)^2} = \mu_*^{-1}\int_0^\infty \frac{\mathrm{d}t}{(1+t^\beta)^2} = 1.$$

We see that for any $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^\infty f(t)^2\,\mathrm{d}t = 1$,

$$\begin{aligned}\int_0^\infty t^{-\beta}(1-f(t))^2\,\mathrm{d}t &- \int_0^\infty t^{-\beta}(1-f_*(t))^2\,\mathrm{d}t \\ &= 2\int_0^\infty t^{-\beta}(1-f_*(t))(f_*(t)-f(t))\,\mathrm{d}t + \int_0^\infty t^{-\beta}(f(t)-f_*(t))^2\,\mathrm{d}t \\ &= 2\mu_*^\beta \int_0^\infty f_*(t)(f_*(t)-f(t))\,\mathrm{d}t + \int_0^\infty t^{-\beta}(f(t)-f_*(t))^2\,\mathrm{d}t \\ &= \mu_*^\beta \int_0^\infty (f_*(t)-f(t))^2\,\mathrm{d}t + \int_0^\infty t^{-\beta}(f(t)-f_*(t))^2\,\mathrm{d}t \geq 0.\end{aligned}$$

Here we have used $t^{-\beta}(1-f_*(t)) = \mu_*^\beta f_*(t)$ in the second identity and $\int_0^\infty f_*^2 = \int_0^\infty f^2 = \frac{1}{2}\int f_*^2 + \frac{1}{2}\int_0^\infty f^2$ in the last one. This shows that the infimum is attained if and only if $f = f_*$.

It remains to compute the infimum and μ_*. Both follow from the formula [1, 6.2.1 and 6.2.2]

$$\int_0^\infty \frac{u^\zeta}{(1+u)^2}\,\mathrm{d}u = \Gamma(1+\zeta)\Gamma(1-\zeta) \quad \text{if } -1 < \operatorname{Re}\zeta < 1.$$

Alternatively one can use a keyhole type contour encircling the positive real axis and the residue theorem [2, Section 11.1.III] to directly evaluate $\int_0^\infty \frac{u^\zeta}{(1+u)^2}\,\mathrm{d}u$.

Letting $u = t^\beta$, we have

$$\mu_* = \int_0^\infty \frac{\mathrm{d}t}{(1+t^\beta)^2} = \frac{1}{\beta}\int_0^\infty \frac{u^{1/\beta-1}\,\mathrm{d}u}{(1+u)^2} = \frac{\Gamma(1/\beta)\Gamma(2-1/\beta)}{\beta}$$

The functional equations $\Gamma(1+z) = z\Gamma(z)$ and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, the last one again valid for $-1 < \operatorname{Re} z < 1$, yield

$$\mu_* = \frac{1}{\beta}\left(1-\frac{1}{\beta}\right)\Gamma(1/\beta)\Gamma(1-1/\beta) = \left(1-\frac{1}{\beta}\right)\frac{\pi/\beta}{\sin(\pi/\beta)}.$$

Moreover,

$$\int_0^\infty (1-f_*(t))^2 t^{-\beta}\,\mathrm{d}t = \mu_*^\beta \int_0^\infty \frac{(\mu_* t)^\beta\,\mathrm{d}t}{(1+\mu_* t^\beta)^2} = \mu_*^{\beta-1}\int_0^\infty \frac{t^\beta\,\mathrm{d}t}{(1+t^\beta)^2}$$

and

$$\begin{aligned}\int_0^\infty \frac{t^\beta\,\mathrm{d}t}{(1+t^\beta)^2} &= \frac{1}{\beta}\int_0^\infty \frac{u^{1/\beta}\,\mathrm{d}u}{(1+u)^2} = \frac{\Gamma(1+1/\beta)\Gamma(1-1/\beta)}{\beta} = \frac{\Gamma(1/\beta)\Gamma(1-1/\beta)}{\beta^2} \\ &= \frac{1}{\beta}\,\frac{\pi/\beta}{\sin(\pi/\beta)}.\end{aligned}$$

This proves the claimed formula. □

3. Lifting to higher dimensions. I

In dimension $d = 1$ Proposition 4 yields $L_{1,1}/L_{1,1}^{\mathrm{cl}} \leq 1.618435$, which is better than for instance the bound in dimension $d = 3$, namely $L_{1,3}/L_{1,3}^{\mathrm{cl}} \leq 1.994584$. In this section we use a procedure of Laptev and Weidl [15, 16] to show that the higher-dimensional fraction $L_{1,d}/L_{1,d}^{\mathrm{cl}}$ is at least as good as the low-dimensional one.

The idea is to consider potentials V on $\mathbb{R}^d$ that take values in the self-adjoint operators on some separable Hilbert space $\mathcal{H}$. We are looking for an inequality of the form

$$\operatorname{Tr}[-\Delta + V]_- \leq L_{1,d}^{\mathrm{op}} \int_{\mathbb{R}^d} \operatorname{tr}(V(x)_-^{1+d/2})\, \mathrm{d}x, \tag{26}$$

where tr denotes the trace in $\mathcal{H}$, Tr the trace in $L^2(\mathbb{R}^d; \mathcal{H}) = L^2(\mathbb{R}^d) \otimes \mathcal{H}$, the operator $-\Delta$ is interpreted as $-\Delta \otimes \mathbb{1}_{\mathcal{H}}$, and where, by definition, the constant $L_{1,d}^{\mathrm{op}}$ is independent of $\mathcal{H}$. Taking $\mathcal{H}$ one-dimensional we see that (26) coincides with (1) and therefore

$$L_{1,d} \leq L_{1,d}^{\mathrm{op}}. \tag{27}$$

It is not known whether $L_{1,d}$ and $L_{1,d}^{\mathrm{op}}$ coincide, but in this section we will show that the upper bound on $L_{1,d}$ from Proposition 4 is, in fact, also an upper bound on $L_{1,d}^{\mathrm{op}}$.

We show this by using the classical duality argument. This shows the analogue of (7), that is,

$$K_d^{\mathrm{op}}(1 + 2/d) = [L_{1,d}^{\mathrm{op}}(1 + d/2)]^{-2/d}, \tag{28}$$

where K_d^{op} denotes the best constant in the inequality

$$\operatorname{Tr}(-\Delta\gamma) \geq K_d^{\mathrm{op}} \int_{\mathbb{R}^d} \operatorname{tr}(\gamma(x,x)^{1+2/d})\, \mathrm{d}x \tag{29}$$

for all operators γ on $L^2(\mathbb{R}^d; \mathcal{H})$ satisfying $0 \leq \gamma \leq 1$, where $\mathcal{H}$ is an arbitrary (separable) Hilbert space. For such γ, one can consider $\gamma(x,x)$ as a non-negative operator in $\mathcal{H}$.

The following proof improves upon an argument from [9].

Proposition 6. *For $d \geq 1$, the best constant in the Lieb–Thirring inequality (29) satisfies*

$$K_d^{\mathrm{op}}/K_d^{\mathrm{cl}} \geq \frac{d}{d+4}\left[\frac{(d+2)^2 \sin\left(\frac{2\pi}{d+2}\right)}{2\pi d}\right]^{1+2/d}.$$

In particular, when $d = 1$ we get $K_1^{\mathrm{op}}/K_1^{\mathrm{cl}} \geq 0.381777$ and $L_{1,d}^{\mathrm{op}}/L_{1,d}^{\mathrm{cl}} \leq 1.618435$.

Proof. Let γ be an operator on $L^2(\mathbb{R}^d; \mathcal{H})$ with $0 \leq \gamma \leq 1$. By a density argument we may assume that $\mathcal{H}$ is finite-dimensional and that γ is finite rank and with smooth eigenfunctions. The analogue of (17) is

$$\operatorname{Tr}(-\Delta\gamma) = \int_{\mathbb{R}^d} \operatorname{tr}\left[\int_0^\infty (f(s/p^2)\gamma f(s/p^2))(x,x)\, \mathrm{d}s\right] \mathrm{d}x \tag{30}$$

for any $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^\infty f^2 = 1$. The operator inequality (18) implies that for any $x \in \mathbb{R}^d$ one has (19), understood as an operator inequality in $\mathcal{H}$. Denoting by $\lambda_n(T)$ the n-th eigenvalue, in decreasing order and taking multiplicities into account, of a non-negative operator T, we infer from (19), the variational principle and the computation (21) that for any $n \in \mathbb{N}$,

$$\lambda_n(\gamma(x,x)) \leq (1+\varepsilon)\lambda_n\big((f(s/p^2)\gamma f(s/p^2))(x,x)\big) + (1+\varepsilon^{-1})s^{d/2}\frac{|B_1|}{(2\pi)^d}A_f.$$

At this stage we can optimize over $\varepsilon > 0$ and obtain

$$\sqrt{\lambda_n(\gamma(x,x))} \leq \sqrt{\lambda_n\big((f(s/p^2)\gamma f(s/p^2))(x,x)\big)} + \sqrt{(1-f(s/p^2))^2(x,x)}. \tag{31}$$

Thus,

$$\lambda_n\big((f(s/p^2)\gamma f(s/p^2))(x,x)\big) \geq \left[\sqrt{\lambda_n(\gamma(x,x))} - \sqrt{s^{d/2}\frac{|B_1|}{(2\pi)^d}A_f}\right]_+^2. \tag{32}$$

For fixed n (and x) we obtain, after integration over s,

$$\int_0^\infty \lambda_n\big((f(s/p^2)\gamma f(s/p^2))(x,x)\big)\,\mathrm{d}s \geq \lambda_n(\gamma(x,x))^{1+2/d}\left(\frac{|B_1|}{(2\pi)^d}A_f\right)^{-2/d}\frac{d^2}{(d+2)(d+4)}.$$

Summing over n and integrating with respect to x we obtain, by (30),

$$\begin{aligned}\operatorname{Tr}(-\Delta\gamma) &\geq \int_{\mathbb{R}^d}\sum_n\int_0^\infty \lambda_n\big((f(s/p^2)\gamma f(s/p^2))(x,x)\big)\,\mathrm{d}s\,\mathrm{d}x \\ &\geq \int_{\mathbb{R}^d}\operatorname{tr}(\gamma(x,x)^{1+2/d})\,\mathrm{d}x\left(\frac{|B_1|}{(2\pi)^d}A_f\right)^{-2/d}\frac{d^2}{(d+2)(d+4)}.\end{aligned}$$

The proposition now follows in the same way as Proposition 4. □

Remark 7. The same proof yields the operator-valued analogue of Theorem 2. Since there seems to be no analogue of the following proposition for $(-\Delta)^\sigma$ with $\sigma \neq 1$, we do not write this out.

In order to obtain good constants in higher dimensions we recall the following bound which is essentially due to Laptev and Weidl [16]. The extension to $d_1 \geq 2$, which is not needed here, but is interesting in its own right, is due to [12].

Proposition 8. *For any integers* $1 \leq d_1 < d$,

$$L_{1,d}^{\mathrm{op}}/L_{1,d}^{\mathrm{cl}} \leq L_{1,d_1}^{\mathrm{op}}/L_{1,d_1}^{\mathrm{cl}}.$$

In particular, taking $d_1 = 1$ and using the bound from Proposition 6 together with (27) we obtain the following bound.

Corollary 9. *For any* $d \geq 1$, $L_{1,d}/L_{1,d}^{\mathrm{cl}} \leq L_{1,d}^{\mathrm{op}}/L_{1,d}^{\mathrm{cl}} \leq 1.618435$.

The proof of Proposition 8 is by now standard, but we sketch it for the sake of completeness. We need the following more general family of Lieb–Thirring inequalities:

$$\operatorname{Tr}[-\Delta + V]_-^\alpha \leq L_{\alpha,d}^{\mathrm{op}} \int_{\mathbb{R}^d} \operatorname{tr}(V(x)_-^{\alpha+d/2})\, \mathrm{d}x, \tag{33}$$

as well as the semiclassical constant

$$L_{\alpha,d}^{\mathrm{cl}} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^d} (\eta^2 - 1)_-^{\alpha+d/2}\, \mathrm{d}\eta = \frac{\Gamma(\alpha+1)}{(4\pi)^{d/2}\Gamma(\alpha + d/2 + 1)},$$

where again V takes now values in the self-adjoint operators on some auxiliary separable Hilbert space $\mathcal{H}$ and its negative part $V(x)_-$ is in the $\alpha + d/2$ von Neumann–Schatten ideal, tr denotes the trace over $\mathcal{H}$, and Tr the trace over $L^2(\mathbb{R}^d; \mathcal{H}) = L^2(\mathbb{R}^d) \otimes \mathcal{H}$.

The celebrated result by Laptev and Weidl [16] says that $L_{\alpha,d}^{\mathrm{op}} = L_{\alpha,d}^{\mathrm{cl}}$ for any $\alpha \geq 3/2$ and any $d \geq 1$. (For $d = 1$, $\alpha = 3/2$ and in the scalar case, this was shown in the original paper of Lieb and Thirring [20].)

Proof of Proposition 8. We follow the argument in [12] closely: Let $d = d_1 + d_2$ and decompose accordingly $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$ and $-\Delta = -\Delta_1 - \Delta_2$. Let V be a function on $\mathbb{R}^d$ taking values in the self-adjoint operators in some Hilbert space $\mathcal{H}$. For any $x_1 \in \mathbb{R}^{d_1}$ we can consider $W(x_1) = -\Delta_2 + V(x_1, \cdot)$ as a self-adjoint operator in $\tilde{\mathcal{H}} = L^2(\mathbb{R}^d; \mathcal{H})$. Thus, by the operator-valued LT inequality on $\mathbb{R}^{d_1}$,

$$\operatorname{Tr}[-\Delta + V]_- = \operatorname{Tr}_{L^2(\mathbb{R}^{d_1})}[-\Delta_1 + W]_- \leq L_{1,d_1}^{\mathrm{op}} \int_{\mathbb{R}^{d_1}} \operatorname{Tr}_{L^2(\mathbb{R}^{d_2};\mathcal{H})}(W(x_1)_-^{1+d_1/2})\, \mathrm{d}x_1.$$

Since $1 + d_1/2 \geq 3/2$, the bound from [16] implies, for any $x_1 \in \mathbb{R}^{d_1}$,

$$\operatorname{Tr}_{L^2(\mathbb{R}^{d_2};\mathcal{H})}(W(x_1)_-^{1+d_1/2}) \leq L_{1+d_1/2,d_2}^{\mathrm{cl}} \int_{\mathbb{R}^{d_2}} \operatorname{tr}(V(x_1, x_2)_-^{1+d/2})\, \mathrm{d}x_2.$$

Combining the last two inequalities and observing that

$$L_{1,d_1}^{\mathrm{cl}} L_{1+d_1/2,d_2}^{\mathrm{cl}} = L_{1,d}^{\mathrm{cl}}$$

(see [12] for a non-computational proof of this identity), we obtain the claimed inequality. □

4. Low momentum averaging

Our main idea to improve the estimate in Proposition 4 is to average over low momenta $s \leq E$ before using the Cauchy–Schwarz inequality (18). We will actually push forward this idea by adding a weight function. This leads to

Proposition 10. *For $d \geq 1$, the best constant in the Lieb–Thirring inequality (6) satisfies*

$$K_d/K_d^{\mathrm{cl}} \geq \frac{d\, 2^{4/d}}{(d+2)^{1+4/d}\mathcal{C}_d^{2/d}}, \tag{34}$$

where

$$\mathcal{C}_d := \inf\left\{\left(\int_0^\infty \varphi^2\right)^{d/2} \frac{d}{2}\int_0^\infty \frac{(1-\int_0^\infty \varphi(s) f(st)\,\mathrm{d}s)^2}{t^{1+d/2}}\,\mathrm{d}t\right\}, \tag{35}$$

with the infimum taken over all functions $f, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\int_0^\infty f^2 = 1$.

In particular, when $d = 1$ we have $K_1/K_1^{\mathrm{cl}} \geq 0.471851$ and $L_{1,1}/L_{1,1}^{\mathrm{cl}} \leq 1.455786$.

Note that for the infimum in (35) to be finite we need $\int_0^\infty \varphi^2 < \infty$ and, if f is continuous near 0, also $\int_0^\infty \varphi = 1/f(0)$. (The latter implies that $\int_0^\infty \varphi(s) f(st)\,\mathrm{d}s \to 1$ as $t \to 0$.)

Proof of Proposition 10. Let $f, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^\infty f^2 = 1$. Recall the momentum decomposition (17). We have for any $\psi \in L^2(\mathbb{R}^d)$, $s, s' \in (0, \infty)$,

$$\langle \psi, f(s/p^2)\gamma f(s'/p^2)\psi\rangle \leq \sqrt{\langle \psi, f(s/p^2)\gamma f(s/p^2)\psi\rangle}\,\sqrt{\langle \psi, f(s'/p^2)\gamma f(s'/p^2)\psi\rangle},$$

and therefore, for every $E > 0$,

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \varphi(s/E)\langle \psi, f(s/p^2)\gamma f(s'/p^2)\psi\rangle\varphi(s'/E)\,\mathrm{d}s\,\mathrm{d}s' \\
&\qquad\leq \left(\int_0^\infty \varphi(s/E)\sqrt{\langle \psi, f(s/p^2)\gamma f(s/p^2)\psi\rangle}\,\mathrm{d}s\right)^2 \\
&\qquad\leq \left(\int_0^\infty \varphi(s/E)^2\,\mathrm{d}s\right)\left(\int_0^\infty \langle \psi, f(s/p^2)\gamma f(s/p^2)\psi\rangle\,\mathrm{d}s\right).
\end{aligned}$$

This implies that we have the operator inequality

$$\begin{aligned}
&\left(\int_0^\infty \varphi(s)^2\,\mathrm{d}s\right)\left(\int_0^\infty f(s/p^2)\gamma f(s/p^2)\,\mathrm{d}s\right) \\
&\qquad= E^{-1}\left(\int_0^\infty \varphi(s/E)^2\,\mathrm{d}s\right)\left(\int_0^\infty f(s/p^2)\gamma f(s/p^2)\,\mathrm{d}s\right) \\
&\qquad\geq E^{-1}\left(\int_0^\infty \varphi(s/E) f(s/p^2)\,\mathrm{d}s\right)\gamma\left(\int_0^\infty \varphi(s/E) f(s/p^2)\,\mathrm{d}s\right) \\
&\qquad= E g(E/p^2)\gamma g(E/p^2)
\end{aligned} \tag{36}$$

with

$$g(t) := \int_0^\infty \varphi(s) f(st)\,\mathrm{d}s. \tag{37}$$

Next, by the Cauchy–Schwarz estimate similarly to (18) (thanks to $0 \leq \gamma \leq 1$) we have

$$\gamma \leq (1+\varepsilon) g(E/p^2)\gamma g(E/p^2) + (1+\varepsilon^{-1})(1-g(E/p^2))^2 \tag{38}$$

for every $\varepsilon > 0$. Combining (36) and (38) we get

$$E\gamma \leq (1+\varepsilon)\Big(\int_0^\infty \varphi^2\Big)\Big(\int_0^\infty f(s/p^2)\gamma f(s/p^2)\,\mathrm{d}s\Big) + (1+\varepsilon^{-1})E(1-g(E/p^2))^2. \tag{39}$$

Transferring (39) to a kernel bound, using the same computation as in (21)–(22), and then optimizing over $\varepsilon > 0$ we obtain

$$\Big(\int_0^\infty \varphi^2\Big)\int_0^\infty (f(s/p^2)\gamma f(s/p^2))(x,x)\,\mathrm{d}s \geq \Bigg[\sqrt{E\gamma(x,x)} - \sqrt{E^{1+d/2}\frac{|B_1|}{(2\pi)^d}A_g}\Bigg]_+^2. \tag{40}$$

Then optimizing over $E > 0$ leads to

$$\begin{aligned}&\Big(\int_0^\infty \varphi^2\Big)\int_0^\infty (f(s/p^2)\gamma f(s/p^2))(x,x)\,\mathrm{d}s\\ &\geq \sup_{E>0} E\Bigg[\sqrt{\gamma(x,x)} - \sqrt{E^{d/2}\frac{|B_1|}{(2\pi)^d}A_g}\Bigg]_+^2 = \gamma(x,x)^{1+2/d}\frac{(2\pi)^2}{|B_1|^{2/d}}\cdot\frac{2^{4/d}d^2}{(d+2)^{2+4/d}A_g^{2/d}}.\end{aligned} \tag{41}$$

Inserting this into (17) we conclude that

$$\mathrm{Tr}(-\Delta\gamma) \geq \Big(\int_{\mathbb{R}^d}\gamma(x,x)^{1+2/d}\,\mathrm{d}x\Big)\frac{(2\pi)^2}{|B_1|^{2/d}}\cdot\frac{2^{4/d}d^2}{(d+2)^{2+4/d}A_g^{2/d}(\int_0^\infty\varphi^2)}, \tag{42}$$

so the best constant in (6) satisfies

$$K_d/K_d^{\mathrm{cl}} \leq \frac{2^{4/d}d}{(d+2)^{1+4/d}A_g^{2/d}(\int_0^\infty\varphi^2)}.$$

Optimizing over f, φ leads to (34).

When $d=1$, using the upper bound $\mathcal{C}_1 \leq 0.373556$ in Lemma 11 below, we obtain $K_1/K_1^{\mathrm{cl}} \geq 0.471851\ldots$ and $L_{1,1}/L_{1,1}^{\mathrm{cl}} \leq 1.455785\ldots$. □

We end this section with

Lemma 11. *When $d=1$, the constant $\mathcal{C}_d$ in (35) satisfies*

$$1/3 \leq \mathcal{C}_1 \leq 0.373556.$$

Proof. Let $f, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^\infty f^2 = 1$. Take g as in (37) and $a := \int_0^\infty \varphi^2$. By the Cauchy–Schwarz inequality,

$$g(t) = \int_0^\infty \varphi(s)f(st)\,\mathrm{d}s \leq \Big(\int_0^\infty \varphi(s)^2\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^\infty f(ts)^2\,\mathrm{d}s\Big)^{1/2} = \sqrt{a/t}.$$

Therefore, when $d = 1$ we get the desired lower bound

$$a^{1/2}\int_0^\infty \frac{(1-g(t))^2}{2t^{3/2}}\,\mathrm{d}t \geq a^{1/2}\int_0^\infty \frac{[1-\sqrt{a/t}]_+^2}{2t^{3/2}}\,\mathrm{d}t = \frac{1}{3}.$$

The upper bound on $\mathcal{C}_1$ requires an explicit choice of (f, φ). The analysis from Section 2 suggests the choice

$$f(t) = (1+\mu t^{3/2})^{-1}, \quad \mu = \left[\frac{4\pi}{9\sqrt{3}}\right]^{3/2}, \quad \varphi(t) = 5(1-t^{1/4})\mathbb{1}(t \leq 1),$$

which gives $\mathcal{C}_1 \leq 0.381378$. We can do slightly better by taking

$$f(t) = (1+\mu_0 t^{4.5})^{-0.25}, \quad \varphi(t) = c_0\frac{(1-t^{0.36})^{2.1}}{1+t}\mathbb{1}(t \leq 1)$$

with μ_0 and c_0 determined by $\int_0^\infty f^2 = \int_0^\infty \varphi = 1$, leading to $\mathcal{C}_1 \leq 0.373556$. □

5. Lifting to higher dimensions. II

In this section we proceed analogously to Section 3 to extend Proposition 10 to the operator-valued case.

Proposition 12. *For $d \geq 1$, the best constant in the Lieb–Thirring inequality (29) satisfies*

$$K_d^{\mathrm{op}}/K_d^{\mathrm{cl}} \geq \frac{d 2^{4/d}}{(d+2)^{1+4/d}\mathcal{C}_d^{2/d}} \tag{43}$$

with $\mathcal{C}_d$ from (35). In particular, when $d = 1$ we have $K_1^{\mathrm{op}}/K_1^{\mathrm{cl}} \geq 0.471851$ and $L_{1,1}^{\mathrm{op}}/L_{1,1}^{\mathrm{cl}} \leq 1.455786$.

Combining this proposition with Proposition 8 (for $d_1 = 1$) and (27) we obtain Theorem 1. It remains to prove the proposition.

Proof. Let $f, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $\int_0^\infty f^2 = \int_0^\infty \varphi = 1$ and take g as in (37). We follow the proof of Proposition 10 to arrive at the operator inequality (39). As in the proof of Proposition 6 this implies, for any $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$E\lambda_n(\gamma(x,x)) \leq (1+\varepsilon)\left(\int_0^\infty \varphi^2\right)\lambda_n\left(\int_0^\infty (f(s/p^2)\gamma f(s/p^2))(x,x)\,\mathrm{d}s\right) + (1+\varepsilon^{-1})E^{1+d/2}\frac{|B_1|}{(2\pi)^d}A_g.$$

Optimizing over $\varepsilon > 0$ we obtain

$$\left(\int_0^\infty \varphi^2\right)\lambda_n\left(\int_0^\infty (f(s/p^2)\gamma f(s/p^2))(x,x)\,\mathrm{d}s\right) \geq \left[\sqrt{E\lambda_n(\gamma(x,x))} - \sqrt{E^{1+d/2}\frac{|B_1|}{(2\pi)^d}A_g}\right]_+^2.$$

Finally, optimizing over $E > 0$ leads to

$$\left(\int_0^\infty \varphi^2\right)\lambda_n\left(\int_0^\infty (f(s/p^2)\gamma f(s/p^2))(x,x)\,\mathrm{d}s\right)$$
$$\geq \sup_{E>0} E\left[\sqrt{\lambda_n(\gamma(x,x))} - \sqrt{E^{d/2}\frac{|B_1|}{(2\pi)^d}A_g}\right]_+^2$$
$$= \lambda_n(\gamma(x,x))^{1+2/d}\frac{(2\pi)^2}{|B_1|^{2/d}}\cdot\frac{2^{4/d}d^2}{(d+2)^{2+4/d}A_g^{2/d}}.$$

Inserting this into (17) we conclude that

$$\mathrm{Tr}(-\Delta\gamma) \geq \left(\int_{\mathbb{R}^d}\mathrm{tr}(\gamma(x,x)^{1+2/d})\,\mathrm{d}x\right)\frac{(2\pi)^2}{|B_1|^{2/d}}\cdot\frac{2^{4/d}d^2}{(d+2)^{2+4/d}A_g^{2/d}(\int_0^\infty\varphi^2)}.$$

Finally, it remains to optimize over f, φ to obtain (43). The numerical values when $d = 1$ are obtained from the upper bound on $\mathcal{C}_1$ in Lemma 11. □

6. Bounds with fractional operators

The proof of Theorem 2 is essentially the same as that of Theorem 1 (except we do not use the lifting argument) and we only sketch the major steps.

Proof of Theorem 2. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $\int_0^\infty f^2 = 1$. We have the analogue of (17),

$$\mathrm{Tr}((-\Delta)^\sigma\gamma) = \int_{\mathbb{R}^d}\left[\int_0^\infty (f(s/|p|^{2\sigma})\gamma f(s/|p|^{2\sigma}))(x,x)\,\mathrm{d}s\right]\mathrm{d}x. \tag{44}$$

Using the Cauchy–Schwarz inequality as in (18) with a parameter $\varepsilon > 0$ and optimizing over this parameter we obtain a generalization of (20),

$$\sqrt{\gamma(x,x)} \leq \sqrt{(f(s/|p|^{2\sigma})\gamma f(s/|p|^{2\sigma}))(x,x)} + \sqrt{(1-f(s/|p|^{2\sigma}))^2(x,x)} \tag{45}$$

for all $x \in \mathbb{R}^d$. We now compute

$$(1-f(s/|p|^{2\sigma}))^2(x,x) = s^{\frac{d}{2\sigma}}\frac{|B_1|}{(2\pi)^d}A_f^{(\sigma)}, \tag{46}$$

where

$$A_f^{(\sigma)} := \frac{d}{2\sigma}\int_0^\infty\frac{(1-f(t))^2}{t^{1+\frac{d}{2\sigma}}}\,\mathrm{d}t. \tag{47}$$

Consequently, we deduce from (45) that

$$(f(s/|p|^{2\sigma})\gamma f(s/|p|^{2\sigma}))(x,x) \geq \left[\sqrt{\gamma(x,x)} - \sqrt{s^{\frac{d}{2\sigma}}\frac{|B_1|}{(2\pi)^d}A_f^{(\sigma)}}\right]_+^2. \tag{48}$$

Inserting (48) into (44) and integrating over $s > 0$ leads to

$$\operatorname{Tr}((-\Delta)^\sigma \gamma) \geq \left(\int_{\mathbb{R}^d} \gamma(x,x)^{1+2\sigma/d}\, \mathrm{d}x\right)\left(\frac{|B_1|}{(2\pi)^d} A_f^{(\sigma)}\right)^{-2\sigma/d} \frac{d^2}{(d+2\sigma)(d+4\sigma)}. \tag{49}$$

Thus,

$$K_{d,\sigma}/K_{d,\sigma}^{\mathrm{cl}} \geq \frac{d}{d+4\sigma}(A_f^{(\sigma)})^{-2\sigma/d}. \tag{50}$$

Lemma 5 provides the minimum value of $A_f^{(\sigma)}$ optimized over f with $\int_0^\infty f^2 = 1$. This leads to the first desired bound

$$K_{d,\sigma}/K_{d,\sigma}^{\mathrm{cl}} \geq \frac{d}{d+4\sigma}\left[\frac{(d+2\sigma)^2 \sin\left(\frac{2\pi\sigma}{d+2\sigma}\right)}{2\pi\sigma d}\right]^{1+2\sigma/d}. \tag{51}$$

Next, we introduce $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\int_0^\infty \varphi = 1$ and define g as in (37). Then proceeding as in (39) we have the operator inequality

$$\begin{aligned} E\gamma \leq (1+\varepsilon)\left(\int_0^\infty \varphi^2\right)\left(\int_0^\infty f(s/|p|^{2\sigma})\gamma f(s/|p|^{2\sigma})\, \mathrm{d}s\right) \\ + (1+\varepsilon^{-1})E(1-g(E/|p|^{2\sigma}))^2. \end{aligned}$$

Transferring the latter to a kernel bound, using the same computation as in (46)–(47), and optimizing over $\varepsilon > 0$ and then $E > 0$, we obtain the following analogue of (41):

$$\begin{aligned} &\left(\int_0^\infty \varphi^2\right)\int_0^\infty (f(s/|p|^{2\sigma})\gamma f(s/|p|^{2\sigma}))(x,x)\, \mathrm{d}s \\ &\quad \geq \sup_{E>0} E\left[\sqrt{\gamma(x,x)} - \sqrt{E^{\frac{d}{2\sigma}} \frac{|B_1|}{(2\pi)^d} A_g^{(\sigma)}}\right]_+^2 \\ &\quad = \gamma(x,x)^{1+2\sigma/d}\left(\frac{|B_1|}{(2\pi)^d} A_g^{(\sigma)}\right)^{-2\sigma/d}\left(\frac{d}{d+2\sigma}\right)^2\left(\frac{2\sigma}{d+2\sigma}\right)^{4\sigma/d}. \end{aligned} \tag{52}$$

Inserting (52) into (44), and then optimizing over f, φ, we arrive at

$$K_{d,\sigma}/K_{d,\sigma}^{\mathrm{cl}} \geq \frac{d}{d+2\sigma}\left(\frac{2\sigma}{d+2\sigma}\right)^{4\sigma/d}(A_g^{(\sigma)})^{-2\sigma/d}\left(\int_0^\infty \varphi^2\right)^{-1}$$

Optimizing over f, φ gives the second desired estimate

$$K_{d,\sigma}/K_{d,\sigma}^{\mathrm{cl}} \geq \frac{d}{d+2\sigma}\left(\frac{2\sigma}{d+2\sigma}\right)^{4\sigma/d} \mathcal{C}_{d,\sigma}^{-2\sigma/d} \tag{53}$$

with $\mathcal{C}_{d,\sigma}$ given in (13).

Finally, in the physical case $\sigma = 1/2$ and $d = 3$, by taking the trial choice

$$f(t) = (1+\mu_0 t^{10})^{1/4}, \quad \varphi(t) = c_0(1-t^2)^4 \mathbb{1}(t \leq 1)$$

with μ_0 and c_0 determined by $\int_0^\infty f^2 = \int_0^\infty \varphi = 1$, we obtain $\mathcal{C}_{d,\sigma} \leq 0.046736$, which implies $K_{d,\sigma}/K_{d,\sigma}^{\mathrm{cl}} \geq 0.826297$ by (53). □

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