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Star flows and multisingular hyperbolicity

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Abstract. A vector field X is called a star flow if every periodic orbit of any vector field C^1-close to X is hyperbolic. It is known that the chain recurrence classes of a generic star flow X on a 3- or 4-manifold are either hyperbolic, or *singular hyperbolic* (see [MPP] for 3-manifolds and [LGW] for 4-manifolds).

As it is defined, the notion of singular hyperbolicity forces the singularities in the same class to have the same index. However in higher dimensions (i.e. ≥ 5), [dL1] shows that singularities of different indices may be robustly in the same chain recurrence class of a star flow. Therefore the usual notion of singular hyperbolicity is not enough for characterizing the star flows.

We present a form of hyperbolicity (called *multisingular hyperbolicity*) which makes the hyperbolic structure of regular orbits compatible with the one of singularities even if they have different indices. We show that multisingular hyperbolicity implies that the flow is star, and conversely we prove that there is a C^1-open and dense subset of the open set of star flows which are multisingular hyperbolic.

More generally, for most of the hyperbolic structures (dominated splitting, partial hyperbolicity etc.) well defined on regular orbits, we propose a way of generalizing it to a compact set containing singular points.

Keywords. Singular hyperbolicity, dominated splitting, linear Poincaré flow, star flows

1. Introduction

1.1. General setting and historical presentation

Considering the infinite diversity of dynamical behaviors, it is natural to have a special interest in *robust properties*, that is, properties that cannot be broken by small perturbations of the system; in other words, a dynamical property is robust if it holds on a (non-empty) open set of diffeomorphisms or flows.

One important starting point in dynamical systems has been the characterization of structural stability (i.e. systems whose topological dynamics is unchanged under small perturbations) by hyperbolicity (i.e. a global structure expressed in terms of transversality

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and of uniform expansion and contraction). This characterization, first stated in the stability conjecture [PaSm], was proven for diffeomorphisms in the C^1 topology by Robin and Robinson [R1], [R2] (hyperbolic systems are structurally stable) and Mañé [Ma2] (structurally stable systems are hyperbolic). The equivalent result for flows (also for the C^1-topology) leads to extra difficulties and was proven in [H2].

We can see in this case how the robustness of the properties is related to the structure in the tangent space: in this case, a very strong robust property is related to a very strong uniform structure. However, hyperbolic systems are not dense in the set of diffeomorphisms or flows; instability and non-hyperbolicity may be robust. In order to describe a larger set of systems, one can consider less rigid robust properties, and try to characterize them by (weaker) structures that limit the effect of small perturbations.

In this spirit there are several results for diffeomorphisms in the C^1-topology:

1. A system is *robustly transitive* if every C^1-close system is transitive. [Ma] proves that robustly transitive surface diffeomorphisms are globally hyperbolic (i.e. are Anosov diffeomorphisms). This is no longer true in higher dimensions (see examples in [Sh, Ma1]). [DPU, BDP] show that robustly transitive diffeomorphisms admit a structure called *dominated splitting*, and their *finest dominated splitting* is *volume partially hyperbolic*. This result extends to *robustly transitive sets*, and to *robustly chain recurrent sets*.
2. One says that a system is *star* if all periodic orbits are hyperbolic in a robust fashion: every periodic orbit of every C^1-close system is hyperbolic. For a diffeomorphism, to be star is equivalent to be hyperbolic (an important step is done in [Ma] and has been completed in [H1]).

Now, what is the situation for flows? The dynamics of flows seems to be closely related to the dynamics of diffeomorphisms. Even more, the dynamics of vector fields in dimension n looks like that of diffeomorphisms in dimension $n-1$. Several results can be translated from one setting to the other, for instance by considering suspension. For example, [D] proved that robustly transitive flows on 3-manifolds are Anosov flows, generalizing Mañé's result for surface diffeomorphisms. More generally, in any dimension, if a vector field is robustly transitive (or chain recurrent) then [Vi] shows that it is non-singular, and its *linear Poincaré flow* (that is, the natural action of the differential on the normal bundle) admits a dominated splitting which is volume partially hyperbolic. On the other hand, if one considers the suspension of a robustly transitive diffeomorphism without a partially hyperbolic splitting (as built in [BV]) one gets a robustly transitive vector field X whose flow $\{\phi^t\}$ does not admit any dominated splitting. This leads to the fundamental observation that

> *for flows, hyperbolic structures live on the normal bundle for the linear Poincaré flow, and not on the tangent bundle.*

However, there is a phenomenon which is really specific to vector fields: the existence of singularities (zeros of the vector field) accumulated, in a robust way, by regular recurrent orbits. Then, some of the previously mentioned results may fail to translate to the flow setting.

The first example with this behavior has been indicated by Lorenz [Lo]. Then [GW] constructs a C^1-open set of vector fields in a 3-manifold, having a topologically transitive attractor containing periodic orbits (which are all hyperbolic) and one singularity. The examples in [GW] are known as the geometric Lorenz attractors.

The Lorenz attractor is also an example of a robustly non-hyperbolic star flow, showing that the result in [H1] is no longer true for flows. In dimension 3 the difficulties introduced by the robust coexistence of singularities and periodic orbits are now almost fully understood. In particular, Morales, Pacifico and Pujals [MPP] defined the notion of *singular hyperbolicity*, which requires some compatibility between the hyperbolicity of the singularity and the hyperbolicity of the regular orbits. They prove the following

- For C^1-generic star flows on 3-manifolds, every chain recurrence class is singular hyperbolic. It was conjectured in [GWZ] that the same result could hold without the generic assumption. However, [BaMo] built a star flow on a 3-manifold having a chain recurrence class which is not singular hyperbolic, contradicting the conjecture. We exhibit a very simple such example in Section 10.
- Any robustly transitive set containing a singular point of a flow on a 3-manifold is either a singular hyperbolic attractor or a singular hyperbolic repeller.

The singular hyperbolic structure for a compact invariant set K of a vector field X on a 3-manifold is equivalent to the existence of a volume partially hyperbolic splitting of the tangent bundle for the flow ϕ^t, for $t \neq 0$.

Let us make two observations:

- The singular hyperbolic structure lives on the tangent bundle (and not on the normal bundle), contradicting our fundamental observation above.
- When the compact set is singular, the splitting has only two bundles, one of dimension 1 and the other of dimension 2; this asymmetry forces all singularities contained in K to have the same index. In other words, all singularities contained in a singular hyperbolic compact set have the same index, by definition of singular hyperbolicity. The examples of star flows in [BaMo], as well as the examples presented in Section 10, contain singular points with distinct indices, and therefore are not singular hyperbolic.

1.2. Star flows and hyperbolic structures

The aim of this paper is to propose a new way to define hyperbolic structures that overcomes the difficulties introduced by the existence of singularities in chain recurrence sets. In order to illustrate what we are aiming at, we will present informally one of the main corollaries of this paper, which is a necessary and sufficient condition for a generic flow to be a star flow.

As mentioned earlier, we want to look at hyperbolic structures in the normal space and for the linear Poincaré flow. But our chain recurrence classes might have singularities. As in [LGW], we define a way to extend the linear Poincaré flow to the singularities.

For a maximal invariant set Λ of a vector field X we denote by $\Lambda_{U,\mathbb{P}}(X)$ the closure in $\mathbb{P}M$ of $\{\langle X(x)\rangle : x \in \Lambda(X,U) \setminus \mathrm{Sing}(X)\}$; it is a $\phi^t_{\mathbb{P}}$-invariant compact set, but in general it fails to vary upper semicontinuously with X.

The smallest compact set satisfying all the required properties is

$$\widetilde{\Lambda(X,U)} = \limsup_{Y \xrightarrow{C^1} X} \Lambda_{U,\mathbb{P}}(Y).$$

Now Conley theory asserts that any chain recurrence class C admits a basis of neighborhoods which are nested filtrating neighborhoods $U_{n+1} \subset U_n$ with $C = \bigcap U_n$ (see Section 3.1 for the definitions). We define

$$\widetilde{\Lambda(C)} = \bigcap_n \widetilde{\Lambda(X,U_n)}.$$

Over this set of directions we can define a *normal bundle* $\mathcal{N}$: the fiber over $L \in \mathbb{P}M$ (corresponding to a line $L \subset T_xM$) is the quotient $N_L = T_xM/L$. The derivative of the flow $D\phi^t$ of X passes to the quotient on the normal bundle $\mathcal{N}$ in a linear cocycle over $\phi^t_{\mathbb{P}}$, called the *extended linear Poincaré flow* and denoted by $\psi^t_{\mathcal{N}}$.

But by looking at the linear Poincaré flow, we lose some of the information about expansion rates along the flow direction, which might be unimportant away from singularities but, in our case, play a crucial role. To recover this information we need a corollary of one of our main results.

Let X be a vector field with a singular chain recurrence class C with a set of singularities S.

For every singularity $\sigma \in S$ we consider a neighborhood U_σ of σ such that $\{\sigma\}$ is the maximal invariant set in it. We also consider a Riemannian metric $\|\cdot\|$ such that $\|(D_x\phi^t)|_L\| = 1$ in the complement of $\bigcup_{\sigma\in S} U_\sigma$.

Corollary 1. *There exists a multiplicative cocycle*

$$h_\sigma = \{h^t_\sigma\} : \widetilde{\Lambda(X,U)} \times \mathbb{R} \to \mathbb{R}$$

such that if x and $\phi^t(x)$ are in U_σ then $h^t_\sigma(L) = \|(D_x\phi^t)|_L\|$, and otherwise $h^t_\sigma(L) = 1$.

Definition 2. We say that a flow is *multisingular hyperbolic* in C if there is an invariant continuous splitting $\mathcal{N} = E \oplus_{\prec} F$ for $\psi^t_{\mathcal{N}}$ over $\widetilde{\Lambda(C)}$ and there are sets of singularities $S_1 \subset C \cap \mathrm{Sing}(X)$ and $S_2 \subset C \cap \mathrm{Sing}(X)$ such that the vectors in E are uniformly contracted by the flow

$$\Big(\prod_{\sigma_i\subset S_1} h^t_{\sigma_i}\Big)\cdot \psi^t_{\mathcal{N}}$$

and the ones in F are uniformly expanded by the flow

$$\Big(\prod_{\sigma_i\subset S_2} h^t_{\sigma_i}\Big)\cdot \psi^t_{\mathcal{N}}.$$

A *multisingular hyperbolic flow* is a flow that is multisingular hyperbolic in all its chain recurrence classes.

Note that with this definition all multisingular hyperbolic flows are star flows.

With all of these definitions we are now able to state the corollary regarding star flows:

Corollary 3. *There is an open and dense set $\mathcal{R}$ of star flows that are multisingular hyperbolic.*

However, this corollary is not as satisfying as one would hope, in the following sense: *One cannot tell whether a given vector field is or not a multisingular hyperbolic vector field without additional information on perturbations of the vector field.*

This is not a problem of the definition of multisingular hyperbolicity itself but rather a problem of the set over which we define the multisingular hyperbolicity.

One of the difficulties this paper deals with is defining a bigger set of directions over the singularities, which varies upper semicontinuously with the flow, but such that the corollary stated above still holds. Also we discuss the following question: To what extent are these two ways of extending the linear Poincaré flow to the singularities different?

Note that the problem of extending the linear Poincaré flow to the singularities is not a problem that is only useful for star flows but rather a way to deal with any hyperbolic structure in a chain recurrence class with singularities. In fact, the idea of recovering the information on the expansion along the orbit by multiplying the flow with a cocycle like the one illustrated above is also applicable to many other settings. We will later give a definition of hyperbolic structure for singular chain classes involving these concepts.

1.3. Discussion of the notion of singular hyperbolicity in dimensions > 3

1.3.1. The natural generalization of singular hyperbolicity. The notion of singular hyperbolicity defined by [MPP] admits a straightforward generalization in higher dimensions: following [LGW, GWZ, SGW], a chain recurrence class is called *singular hyperbolic* if the tangent bundle over this class admits a dominated, partially hyperbolic splitting into two bundles, one uniformly contracting (resp. expanding) and the other expanding (resp. contracting) area on any two-dimensional subspace. If instead of area expansion we ask for volume expansion of the non-uniform bundle, then as in [MM] this is called *sectional hyperbolicity*.

These notions have been very helpful for the study of singular star flows. If the chain recurrence set of a vector field X can be covered by filtrating sets U_i in which the maximal invariant set Λ_i is singular hyperbolic, then X is a star flow. Conversely, [LGW] and [GWZ] prove that this property characterizes the generic star flows on 4-manifolds. In [SGW] the authors prove the singular hyperbolicity of generic star flows in any dimension assuming an extra property: if two singularities are in the same chain recurrence class then they must have the same *s-index* (dimension of the stable manifold).

However, in dimension ≥ 4, singularities of different indices may coexist C^1-robustly in the same class, and these classes may have a robust property which requires a notion of hyperbolicity. For instance:

- In dimension 4, [BLY] built a flow having a robustly chain recurrent attractor containing saddles of different indices. In particular, this attractor is not singular hyperbolic in the sense of [SGW].

- In [dL1], an example is announced of a star flow in dimension 5 admitting singularities of different indices which belong robustly to the same chain recurrence class. This example cannot satisfy the singular hyperbolicity used in [SGW].

1.3.2. A local solution for a local problem. If one wants to explain robust properties of chain recurrence classes containing a singularity, one needs to understand the interaction between the hyperbolic structure on the regular orbits and the local dynamics in the neighborhood of the singular point:

> *Why do the regular orbits not lose their hyperbolic structure when crossing a small neighborhood of the singularity?*

That is a local problem.

Singular hyperbolicity, as defined in [GWZ, SGW], and sectional hyperbolicity as in [MM] are global ways for fixing this local problem. As a consequence, if several singularities coexist in the same class, the global solution needs to solve the local problem corresponding to each singularity; as a consequence, singular hyperbolicity implies that the singularities have the same local behavior. This explains why singular hyperbolicity could not characterize all the star flows but only those for which singular points of distinct indices are assumed to belong in distinct chain recurrence classes.

This paper provides a local answer to this local problem: the way for fixing the hyperbolic structure of the regular orbits with the one of a given singular point needs to be independent of what we do in the neighborhood of the other singular points. For that:

- The main new tool will be Theorem 1 which associates a cocycle to any singularity of a vector field.
- Another important tool built in [LGW] is the generalized linear Poincaré flow, and we need to recall its construction to present our results.
- The last tool will be the notion of extended maximal invariant set. Such a notion has already been defined and used in [LGW, SGW]; we propose here a slightly different notion and we compare it (see Theorem 2) with the one in [LGW, SGW].

Given any usual notion of hyperbolic structure (hyperbolicity, partial hyperbolicity, volume hyperbolicity, etc.), well defined on compact invariant sets far from the singularities, we propose a notion of *(multi)singular hyperbolic structure* generalizing it to compact invariant sets containing singular points.

Then we will illustrate the power of this notion by paying special attention to star flows. In this particular setting, the usual structure (for regular orbits) one wants to generalize to the singular setting is uniform hyperbolicity. In order to avoid confusion with the singular structure defined in [LGW], we will call our way of generalizing uniform hyperbolicity to singular sets *multisingular hyperbolicity*. Then Theorem 3 proves that multisingular hyperbolicity characterizes the star flows in any dimension:

> *Multisingular hyperbolic flows are star flows, and conversely, a C^1-open and dense subset of the star flows consists of multisingular hyperbolic flows.*

In the same spirit, generalizing the results in [BDP], the second author announced in [dL2] that every C^1-robustly chain recurrence class of a singular flow is *singular volume partially hyperbolic*; in particular, the example of robustly chain recurrent attractors in [BLY] is singular volume partially hyperbolic.

2. Presentation of our results

2.1. The extended linear Poincaré flow

The hyperbolic structure we will define does not lie on the tangent bundle, but on the normal bundle. However, the flows we consider are singular and so the normal bundle (and therefore the linear Poincaré flow) is not defined at the singularities. In [LGW], the authors define the notion of *extended linear Poincaré flow* defined on some sort of blow-up of the singularities. Our notion of *multisingular hyperbolicity* will be expressed in terms of this extended linear Poincaré flow (see the precise definition in Section 4); we present it roughly below.

- We denote by $\mathbb{P}M$ the projective tangent bundle of M, that is, a point L of $\mathbb{P}M$ corresponds to a line of the tangent space at a point of M.
- We denote by $\Lambda_X \subset \mathbb{P}M$ the union

$$\Lambda_X = \{\langle X(x)\rangle \in \mathbb{P}T_xM : x \in M \setminus \mathrm{Sing}(X)\} \cup \bigcup_{y\in\mathrm{Sing}(X)} \mathbb{P}T_yM.$$

  It is a compact set, invariant under the topological flow $\phi^t_{\mathbb{P}}$.
- The restriction of $\psi^t_{\mathcal{N}}$ to the fibers of $\mathbb{P}M$ over $\{\langle X(x)\rangle \in \mathbb{P}T_xM : x \in M\setminus\mathrm{Sing}(X)\}$ is naturally conjugate to the linear Poincaré flow over $M\setminus\mathrm{Sing}(X)$; thus the restriction of $\psi^t_{\mathcal{N}}$ to Λ_X is a natural extension, over the singular points, of the linear Poincaré flow.

2.2. A local cocycle associated to a singular point

Let X be a vector field and ϕ^t its flow. A (multiplicative) *cocycle* over X is a continuous map $h\colon \Lambda_X \times \mathbb{R} \to \mathbb{R}$, $h(L,t) = h^t(L)$, satisfying the cocycle relation

$$h^{r+s}(L) = h^r(\phi^s_{\mathbb{P}}(L)) \cdot h^s(L).$$

Remark 4. For instance, fix a Riemannian metric $\|\cdot\|$ on M. For $L \in \mathbb{P}T_xM$ and $t \in \mathbb{R}$ set

$$h^t_X(L) = \|(D_x\phi^t)|_L\|,$$

where $(D_x\phi^t)|_L$ is the restriction to L of the derivative at x of the flow ϕ^t. The map $h_X\colon \Lambda_X \times \mathbb{R} \to \mathbb{R}$, $(L,t) \mapsto h^t_X(L)$, is a cocycle that we will call the *expansion in the direction of the flow*.

Let $\sigma \in \mathrm{Sing}(X)$ be an isolated singular point. We will say that $h = \{h^t\}$ is a *local cocycle* at σ if for any neighborhood U of $\mathbb{P}T_\sigma M$ in Λ_X there is a constant $C > 1$ such that

$$1/C < h^t(L) < C$$

for every $(L,t) \in \Lambda_X \times \mathbb{R}$ with $L \notin U$ and $\phi^t_{\mathbb{P}}(L) \notin U$.

Definition 5. Let $\sigma \in \mathrm{Sing}(X)$ be an isolated singular point. A *local reparametrization cocycle associated to* σ is a cocycle $h = \{h^t\}\colon \Lambda_X \times \mathbb{R} \to \mathbb{R}$ such that:

- $\{h^t\}$ is a local cocycle at σ;
- there is a neighborhood U of σ and $C > 1$ such that
$$\frac{1}{C} < \frac{h^t(L)}{h^t_X(L)} < C$$
for any $t \in \mathbb{R}$ and $L \in \Lambda_X$ such that $L \in \mathbb{P}T_xM$ with $x \in U$ and $\phi^t(x) \in U$, where $\{h^t_X\}$ is the cocycle of expansion in the flow direction.

The following result is central to this paper:

Theorem 1. *Let X be a vector field on a closed manifold and let σ be a simple singularity of X (that is, the derivative of X at σ is invertible). Then:*

- *There is a local reparametrization cocycle $h_\sigma\colon \Lambda_X \times \mathbb{R} \to \mathbb{R}$ associated to σ.*
- *If h'_σ is another reparametrization cocycle then $h^t_\sigma/(h'_\sigma)^t$ is uniformly bounded (thus h_σ is unique up to a bounded cocycle).*
- *There is a C^1-neighborhood $\mathcal{U}$ of X and a continuous map $\mathcal{U} \ni Y \mapsto h_{Y,\sigma_Y}$ where σ_Y is a continuation of σ on Y and h_{Y,σ_Y} is a reparametrization cocycle for Y and σ_Y.*

The proof of Theorem 1 is the aim of Section 6.

Notice that the product of two cocycles is a cocycle, and the power of a cocycle is a cocycle.

Definition 6. Let X be a vector field on a compact manifold such that the zeros of X are all simple. We say that a cocycle h is a *reparametrization cocycle* if, for every σ in $\mathrm{Sing}(X)$, there exists a choice of a local reparametrization cocycle h_σ associated to σ and a positive number $\alpha(\sigma)$ such that
$$h^t = \prod_{\sigma \in \mathrm{Sing}(X)} (h^t_\sigma)^{\alpha(\sigma)}.$$

2.3. Hyperbolic structures

2.3.1. Hyperbolic structures over compact subsets of $\mathbb{P}M$. Consider now a vector field X on a compact manifold M and $\Lambda_X \subset \mathbb{P}M$. We assume that every singularity of X is simple. Let $K \subset \Lambda_X$ be a $\phi^t_{\mathbb{P}}$-invariant compact set. A *singular hyperbolic structure on K* is a dominated splitting
$$\mathcal{N} = E_1 \oplus_{\prec} \cdots \oplus_{\prec} E_k$$
of the normal bundle over K for the extended linear Poincaré flow, with the following additional property:

For some of the bundles E_i there exists a number $1 \le d_i \le \dim E_i$ and a reparametrization cocycle $h^t_i = \prod_{\sigma \in \mathrm{Sing}(X)} (h^t_\sigma)^{\alpha_i(\sigma)}$ such that
$$J(h^t_i \cdot (\psi^t_{\mathcal{N}}|_{D_i}))$$
is a uniform contraction or expansion for any subspace $D_i \subset E_i$ of dimension d_i.

We show that singular hyperbolic structures are robust in the following sense:

Lemma 7. *Let X be a vector field on a compact manifold. If $K \subset \Lambda_X \subset \mathbb{P}M$ is a $\phi^t_{\mathbb{P}}$-invariant compact set admitting a singular hyperbolic structure, then there is a C^1-neighborhood of X and a neighborhood U of K in $\mathbb{P}M$ such that for any Y in $\mathcal{U}$ the maximal invariant set of $\phi_{Y,\mathbb{P}}$ in $\Lambda_Y \cap U$ admits the same singular hyperbolic structure.*

This lemma is a straightforward consequence of the fact that the reparametrization cocycles used to define the singular hyperbolic structures admit a continuous choice with respect to the vector field (last item of Theorem 1).

2.3.2. Multisingular hyperbolicity. One of the many possible motivations for looking for new definitions of hyperbolic structures in the case of singular flows is understanding what is the type of hyperbolicity that a typical star flow carries (and that allows for singularities of different indices in the same invariant compact connected set). With the above way of defining a singular hyperbolic structure we next define our candidate for hyperbolicity of a typical star flow in dimensions more than three.

Definition 8. Let X be a vector field on a compact manifold. If $K \subset \Lambda_X \subset \mathbb{P}M$ is a $\phi_{\mathbb{P}}$-invariant compact set we say that K is *multisingular hyperbolic* if there is a dominated splitting $\mathcal{N} = E \oplus_{\prec} F$ for $\psi^t_{\mathcal{N}}$ and there are two reparametrization cocycles h^t_s and h^t_u such that the vectors in E are uniformly contracted by the flow $h^t_s \cdot \psi^t_{\mathcal{N}}$ and the ones in F are uniformly expanded by the flow $h^t_u \cdot \psi^t_{\mathcal{N}}$.

Note that this definition is equivalent to hyperbolicity away from singularities and choosing h^t_u as in Remark 4 and h^t_s as the identity we get a hyperbolic structure that is equivalent to (positive) singular hyperbolicity.

In a very similar way it is possible to generalize partial hyperbolicity or volume partial hyperbolicity.

2.4. The extended maximal invariant set

The next difficulty is to define the set on which we would like to define a hyperbolic structure.

We are interested in the dynamics of X in a compact region U on M, that is, to describe the maximal invariant set $\Lambda(X, U)$ in U. An important property is that the maximal invariant set depend upper semicontinuously on the vector field X. This property is fundamental for a hyperbolic structure to be a robust property.

Therefore we need to consider a compact part of $\mathbb{P}M$, as small as possible, such that:

- It is invariant under the flow $\phi^t_{\mathbb{P}}$.
- It contains all the directions spanned by $X(x)$ for $x \in \Lambda(X, U) \setminus \mathrm{Sing}(X)$.
- It varies upper semicontinuously with X.

We denote by $\Lambda_{U,\mathbb{P}}(X)$ the closure in $\mathbb{P}M$ of $\{\langle X(x)\rangle : x \in \Lambda(X, U) \setminus \mathrm{Sing}(X)\}$; it is a $\phi^t_{\mathbb{P}}$-invariant compact set, but in general it fails to vary upper semicontinuously with X.

The smallest compact set satisfying all the required properties is

$$\widetilde{\Lambda(X,U)} = \limsup_{Y\to X} \Lambda_{U,\mathbb{P}}(Y).$$

Definition 9. We will say that X *has a singular hyperbolic structure* in a compact region U if the compact set $\widetilde{\Lambda(X,U)} \subset \Lambda_X \subset \mathbb{P}M$ has a singular hyperbolic structure, as defined in Section 2.3.1.

As a straightforward consequence of the upper semicontinuous dependence of $\widetilde{\Lambda(X,U)}$ on the vector field X one gets the robustness of the singular hyperbolic structure of X in U.

Lemma 10. *If X has a singular hyperbolic structure in a compact region U then the same singular hyperbolic structure holds for every vector field C^1-close to X.*

Remark 11. If X is non-singular on U then a singular hyperbolic structure of X is equivalent to the corresponding (non-singular) hyperbolic structure.

More generally, if X has a singular hyperbolic structure on U then every ϕ_t-invariant compact set $K \subset U \setminus \mathrm{Sing}(X)$ has a corresponding (non-singular) hyperbolic structure.

The set $\widetilde{\Lambda(X,U)}$ is a fundamental tool for defining singular hyperbolic structures. However, it may be hard to calculate because it depends not only on X but also on all C^1-small perturbations of X. That is a little unsatisfactory: hyperbolic structures have been invented to control the effect of small perturbations. However, in order to know whether $\widetilde{\Lambda(X,U)}$ admits a hyperbolic structure, we need to understand the effect of perturbations of X.

In what follows, we propose another set, much simpler to compute, since it does not depend on perturbations of X.

In Section 5.2 we define the notion of *central space $E^c_{\sigma,U}$ of a singular point $\sigma \in U$*. Then the *extended maximal invariant set* is the set $B(X,U) \subset \mathbb{P}M$ of all lines L such that either

- L is contained in the central space of a singular point in $\bar{U}$, or
- L is directed by the vector $X(x)$ at the regular point $x \in \Lambda(X,U) \setminus \mathrm{Sing}(X)$.

Proposition 41 proves that $B(X,U)$ varies upper semicontinuously with the vector field X. In particular, once again, the existence of a dominated splitting $\mathcal{N}_L = E_L \oplus F_L$ of the normal bundle $\mathcal{N}$ over $B(X,U)$ is a robust property, as also is the existence of a singular hyperbolic structure. Furthermore, Remark 42 shows it is larger than $\widetilde{\Lambda(X,U)}$:

$$\widetilde{\Lambda(X,U)} \subset B(X,U).$$

2.5. *Hyperbolic structures over a chain recurrence class*

Other sets one is interested in when defining hyperbolic structures are the chain recurrence classes $C(\sigma)$ of singular points σ. Conley theory asserts that any chain recurrence class C admits a basis of nested filtrating neighborhoods $U_{n+1} \subset U_n$, $C = \bigcap U_n$ (see Section 3.1

for the definitions). We define

$$\widetilde{\Lambda(C)} = \bigcap_n \widetilde{\Lambda(X, U_n)} \quad \text{and} \quad B(C) = \bigcap_n B(X, U_n).$$

These two sets are independent of the choice of the sequence U_n. Clearly $\widetilde{\Lambda(C)} \subset B(C)$.

Definition 12. We say that a chain recurrence class C has a given singular hyperbolic structure if $\widetilde{\Lambda(C)}$ carries that structure.

Remark 13. If C is a chain recurrence class which has a singular hyperbolic structure then X has this structure on a small filtrating neighborhood of C.

If $\sigma \in \mathrm{Sing}(X)$ is a hyperbolic singular point, we define $E^c_\sigma = \bigcap_n E^c_{\sigma,U_n}$ and we call it the *center space* of σ. We denote by $\mathbb{P}^c_\sigma = \mathbb{P}E^c_\sigma$ its projective space.

Remark 14. Consider the open and dense set of vector fields whose singular points are all hyperbolic. In this open set the singularities depend continuously on the field. Then for every singular point σ, the projective center space $\mathbb{P}^c_\sigma$ varies upper semicontinuously, and in particular the dimension $\dim E^c_\sigma$ varies upper semicontinuously. As it is a non-negative integer, it is locally minimal and locally constant on an open and dense subset.

We will say that such a singular point has *locally minimal center space*.

We prove

Theorem 2. *Let X be a vector field on a closed manifold, whose singular points are hyperbolic, with locally minimal center spaces, and such that the finest dominated splitting of the center spaces is into one- or two-dimensional subspaces. Then for every $\sigma \in \mathrm{Sing}(X)$, every hyperbolic structure on $\widetilde{\Lambda(C(\sigma))}$ extends to $B(C(\sigma))$.*

2.6. Multisingular hyperbolicity and star flows

We say that a vector field X whose singularities are all hyperbolic is *multisingular hyperbolic* if every chain recurrence class is multisingular hyperbolic. Recall that a vector field is a *star flow* if it belongs to the C^1-interior of the set of vector fields whose periodic orbits are all hyperbolic.

Remark 15.
- If C is a non-singular chain recurrence class which is multisingular hyperbolic then it is uniformly hyperbolic and therefore is a hyperbolic basic set and a homoclinic class.
- If C is a chain recurrence class which is multisingular hyperbolic then X is multisingular hyperbolic on a small filtrating neighborhood of C.

One may check easily

Lemma 16. *If X is multisingular hyperbolic, then X is a star flow.*

Conversely, we will show

Theorem 3. *There is a C^1-open and dense subset $\mathcal{U}$ of $\mathcal{X}^1(M)$ such that if $X \in \mathcal{U}$ is a star flow then the chain recurrent set $\mathcal{R}(X)$ is contained in the union of finitely many pairwise disjoint filtrating regions in each of which X is multisingular hyperbolic.*

Indeed, we will get a more precise result: our notion of singular hyperbolic structure allows many possible choices of reparametrization cocycles. However, in the setting of star flows, some of the results in [SGW] allow us to fix *a priori* the reparametrization cocycle. More precisely, according to [SGW] for an open and dense subset of the set of star flows, one has the following properties:

1. Any chain recurrence class C admits a (unique) dominated splitting $\mathcal{N} = E \oplus F$ for the extended linear Poincaré flow on $\widetilde{\Lambda(C)}$ which is the limit of the hyperbolic splittings of the periodic orbits for C^1-nearby flows.
2. The set $\mathrm{Sing}(X) \cap C$ is the union of two sets S_E and S_F, where:
   - $\sigma \in S_E$ if the stable space E^s_σ has the same dimension as the bundle E of the dominated splitting of the extended linear Poincaré flow over $\Lambda\widetilde{(C(\sigma))}$ (and thus $\dim E^u_\sigma = \dim F + 1$).
   - $\sigma \in S_F$ if $\dim E^u_\sigma = \dim F$ and $\dim E^s_\sigma = \dim E + 1$.

In particular, the indices of the singularities in a given chain recurrence class may differ by at most 1 from each other.

Then one considers the reparametrization cocycles h^t_E and h^t_F defined as

$$h^t_E = \prod_{\sigma \in S_E} h^t_\sigma \quad \text{and} \quad h^t_F = \prod_{\sigma \in S_F} h^t_\sigma.$$

Now, Theorem 3 is a straightforward corollary of

Theorem 4. *There is a C^1-open and dense subset $\mathcal{U}$ of the open set of star flows such that for any X in $\mathcal{U}$ every chain recurrence class admits a dominated splitting $\mathcal{N} = E \oplus_{\prec} F$ for the extended linear Poincaré flow $\psi^t_{\mathcal{N}}$ over $B(C)$ and such that the reparametrized flow*

$$(h^t_E \psi^t_{\mathcal{N}}|_E, h^t_F \psi^t_{\mathcal{N}}|_F)$$

is uniformly hyperbolic.

In other words, X is multisingular hyperbolic and its reparametrization cocycles are (h^t_E, h^t_F).

Remark 17. If all the singular points in a chain recurrence class C have the same index, that is, if S_E or S_F is empty, then multisingular hyperbolicity is the same as singular hyperbolicity as in [SGW].

The proof of Theorem 4 follows closely the proof in [SGW] that star flows with only singular points of the same index are singular hyperbolic.

Question 1. *Can we remove the generic assumption, at least in dimension 3, in Theorem 3? In other words, is it true that, given any star flow X (for instance on a 3-manifold) every chain recurrence class of X is multisingular hyperbolic?*

3. Basic definitions and preliminaries

3.1. Chain recurrent set

The following notions and theorems are due to Conley [Co] and they can be found in several other references (for example [AN]).

- We say that a pair of sequences $\{x_i\}_{0\le i\le k}$ and $\{t_i\}_{0\le i\le k-1}$, $k \ge 1$, are an *ε-pseudo orbit from x_0 to x_k* for a flow ϕ if for every $0 \le i \le k-1$ one has

$$t_i - t_{i-1} \ge 1 \quad \text{and} \quad d(x_{i+1}, \phi^{t_i}(x_i)) < \varepsilon.$$

- A compact invariant set Λ is called *chain transitive* if for any $\varepsilon > 0$ and any $x, y \in \Lambda$ there is an ε-pseudo orbit from x to y.
- We say that $x \in M$ is *chain recurrent* if for every $\varepsilon > 0$ there is an ε-pseudo orbit from x to x. We call the set of chain recurrent points the *chain recurrent set* and denote it by $\mathfrak{R}(M)$.
- We say that $x, y \in \mathfrak{R}(M)$ are *chain related* if, for every $\varepsilon > 0$, there are ε-pseudo orbits from x to y and from y to x. This is an equivalence relation. The equivalence classes of this relation are called *chain recurrence classes*.

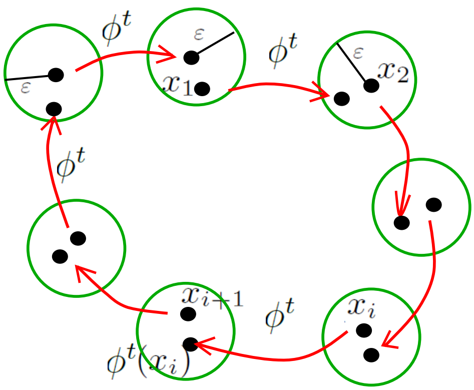


Fig. 1. An ε-pseudo orbit.

Definition 18. • An *attracting region* (also called a *trapping region*) is a compact set U such that $\phi^t(U)$ is contained in the interior of U for every $t > 0$. The maximal invariant set in an attracting region is called an *attracting set*. A *repelling region* is an attracting region for $-X$, and the maximal invariant set is called a *repeller*.

- A *filtrating region* is the intersection of an attracting region with a repelling region.
- Let C be a chain recurrence class of M for the flow ϕ. A *filtrating neighborhood* of C is a (compact) neighborhood which is a filtrating region.

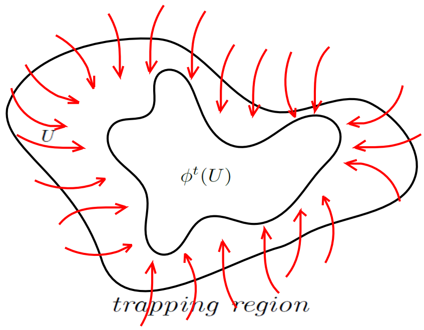


Fig. 2. A trapping region or attracting region.

Definition 19. Let $\{\phi^t\}$ be a flow on a Riemannian manifold M. A *complete Lyapunov function* is a continuous function $\mathcal{L} : M \to \mathbb{R}$ such that:

- $\mathcal{L}(\phi^t(x))$ is decreasing for t if $x \in M \setminus \mathfrak{R}(M)$.
- Two points $x, y \in \mathfrak{R}(M)$ are chain related if and only if $\mathcal{L}(x) = \mathcal{L}(y)$.
- $\mathcal{L}(\mathfrak{R}(M))$ is nowhere dense.

The next result is called the fundamental theorem of dynamical systems by some authors:

Theorem 5 (Conley [Co]). *Let X be a C^1 vector field on a compact manifold M. Then its flow $\{\phi^t\}$ admits a complete Lyapunov function.*

The next corollary will be used often in this paper:

Corollary 20. *Let ϕ be a C^1 vector field on a compact manifold M. Every chain recurrence class C of X admits a basis of filtrating neighborhoods, that is, every neighborhood of C contains a filtrating neighborhood of C.*

Lemma 21 (Connecting lemma [BC]). *Let ϕ_t be a flow induced by a vector field $X \in \mathcal{X}^1(M)$ such that all periodic orbits of X are hyperbolic. For any C^1-neighborhood $\mathcal{U}$ of X and $x, y \in M$, if y is in the same chain recurrence class as x, then there exist $Y \in \mathcal{U}$ and $t > 0$ such that $\phi_t^Y(x) = y$. Moreover, for any $k \geq 1$, let $\{x_{i,k}, t_{i,k}\}_{i=0}^{n_k}$ be a $(1/k)$-pseudo orbit from x to y and define*

$$\Delta_k = \bigcup_{i=0}^{n_k-1} \phi_{[0,t_{i,k}]}(x_{i,k}).$$

Let Δ be the upper Hausdorff limit of Δ_k. Then for any neighborhood U of Δ, there exists $Y \in \mathcal{U}$ with $Y = X$ on $M \setminus U$ and $t > 0$ such that $\phi_t^Y(x) = y$.

For a generic vector field $X \in \mathcal{X}^1(M)$ we also have:

Theorem 6 ([C]). *There exists a generic set $G_{\text{approx}} \subset \mathcal{X}^1(M)$ such that for every $X \in G_{\text{approx}}$ and for every chain recurrence class C there exists a sequence of periodic orbits γ_n which converges to C in the Hausdorff topology.*

3.2. *Linear cocycles*

Let $\phi = \{\phi^t\}_{t\in\mathbb{R}}$ be a topological flow on a compact metric space K. A *linear cocycle over* (K, ϕ) is a continuous map $\{A^t\}\colon E \times \mathbb{R} \to E$ defined by

$$A^t(x, v) = (\phi^t(x), A_t(x)v),$$

where:

- $\pi : E \to K$ is a d-dimensional vector bundle over K.
- $A_t : K \times \mathbb{R} \ni (x, t) \mapsto \mathrm{GL}(E_x, E_{\phi^t(x)})$ is a continuous map that satisfies the *cocycle relation*

$$A_{t+s}(x) = A_t(\phi^s(x))A_s(x) \qquad \text{for any } x \in K \text{ and } t, s \in \mathbb{R}.$$

Note that $\mathcal{A} = \{A^t\}_{t\in\mathbb{R}}$ is a flow on the space E which projects on ϕ^t:

$$\begin{array}{ccc} E & \xrightarrow{A^t} & E \\ \downarrow & & \downarrow \\ K & \xrightarrow{\phi^t} & K \end{array}$$

If $\Lambda \subset K$ is a ϕ-invariant subset, then $\pi^{-1}(\Lambda) \subset E$ is $\mathcal{A}$-invariant, and we call the restriction of $\{A^t\}$ to $\pi^{-1}(\Lambda)$ *the restriction of* $\mathcal{A}$ *to* Λ.

3.3. *Hyperbolicity, dominated splitting on linear cocycles*

Definition 22. Let ϕ be a topological flow on a compact metric space Λ. We consider a vector bundle $\pi : E \to \Lambda$ and a linear cocycle $\mathcal{A} = \{A^t\}$ over (Λ, X).

We say that $\mathcal{A}$ admits a *dominated splitting over* Λ if:

- There exists a splitting $E = E^1 \oplus \cdots \oplus E^k$ over λ into k subbundles.
- The dimension of the subbundles is constant, i.e. $\dim E^i_x = \dim E^i_y$ for all $x, y \in \Lambda$ and $i \in \{1, \ldots, k\}$.
- The splitting is invariant, i.e. $A^t(x)(E^i_x) = E^i_{\phi^t(x)}$ for all $i \in \{1, \ldots, k\}$.
- There exists a $t > 0$ such that for every $x \in \Lambda$ and any pair of non-zero vectors $v \in E^i_x$ and $u \in E^j_x$, $i < j$, one has

$$\frac{\|A^t(u)\|}{\|u\|} \le \frac{1}{2}\,\frac{\|A^t(v)\|}{\|v\|}. \tag{1}$$

We denote $E = E^1 \oplus_{\prec} \cdots \oplus_{\prec} E^k$. The notation $\oplus_{\prec}$ is used to highlight the fact that, in addition to the fact that E can be expressed as a direct sum of the spaces E^i, these spaces are ordered so that each is dominated by the next one.

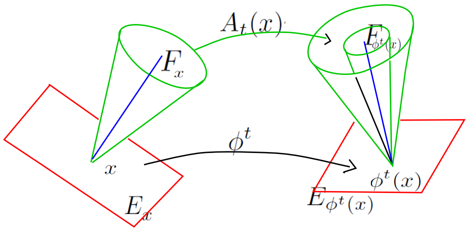
A classical result (see for instance [BDV, Appendix B]) asserts that the bundles of a dominated splitting are always continuous. A given cocycle may admit several dominated splittings. However, the dominated splitting is unique if one prescribes the dimensions $\dim E^i$.

Associated to the dominated splitting we define a family of cone fields C_a^{iu} around each space $E^i \oplus \cdots \oplus E^k$ as follows. Let us write the vectors $v \in E$ as $v = (v_1, v_2)$ with $v_1 \in E^1 \oplus \cdots \oplus E^{i-1}$ and $v_2 \in E^i \oplus \cdots \oplus E^k$. Then the cone field C_a^{iu} is the set

$$C_a^{iu} = \{v = (v_1, v_2) : \|v_1\| < a\|v_2\|\}.$$

These are called the *family of unstable cone fields* and the domination implies that they are strictly invariant for times larger than t: the cone C_a^{iu} at $T_x M$ is taken by A^t to the interior of the cone C_a^{iu} at $T_{\phi^t x} M$.

Analogously we define the *stable family of cone fields* C_a^{is} around $E^1 \oplus \cdots \oplus E^i$ and the domination implies that they are strictly invariant for negative times smaller than $-t$.



One says that one of the bundles E^i is (*uniformly*) *contracting* (resp. *expanding*) if there is $t > 0$ such that for every $x \in \Lambda$ and every non-zero vector $u \in E_x^i$ one has $\|A^t(u)\|/\|u\| < 1/2$ (resp. $\|A^{-t}(u)\|/\|u\| < 1/2$). In both cases one says that E^i is *hyperbolic*.

Notice that if E^j is contracting (resp. expanding) then the same holds for any E^i with $i < j$ (resp. $j < i$) as a consequence of the domination.

Definition 23. We say that the linear cocycle $\mathcal{A}$ is *hyperbolic over* Λ if there is a dominated splitting $E = E^s \oplus_{\prec} E^u$ over Λ into two hyperbolic subbundles such that E^s is uniformly contracting and E^u is uniformly expanding.

One says that E^s is the *stable bundle*, and E^u is the *unstable bundle*.

The existence of a dominated splitting or of a hyperbolic structure is an open property in the following sense:

Proposition 24. *Let K be a compact metric space, $\pi : E \to K$ a d-dimensional vector bundle, and $\mathcal{A}$ a linear cocycle over K. Let Λ_0 be a ϕ-invariant compact set. Assume that the restriction of $\mathcal{A}$ to Λ_0 admits a dominated splitting $E^1 \oplus_{\prec} \cdots \oplus_{\prec} E^k$, for some $t > 0$.*

Then there is a compact neighborhood U of Λ_0 with the following property. Let $\Lambda = \bigcap_{t\in\mathbb{R}} \phi^t(U)$ be the maximal invariant set of ϕ in U. Then the dominated splitting admits a unique extension as a dominated splitting over Λ for $2t > 0$. Furthermore if one of the subbundles E^i is hyperbolic over Λ_0, it is still hyperbolic over Λ.

As a consequence, if $\mathcal{A}$ has a hyperbolic structure over Λ_0 then (up to shrinking U if necessary) it also has a hyperbolic structure over Λ.

3.4. Robustness of hyperbolic structures

The aim of this section is to explain that Proposition 24 can be seen as a robustness property.

Let M be a manifold and ϕ_n a sequence of flows in M tending to ϕ_0 as $n \to \infty$, in the C^0-topology on compact subsets: for any compact set $K \subset M$ and any $T > 0$, the restriction of ϕ_n^t to K, $t \in [-T, T]$, tends uniformly (in $x \in K$ and $t \in [-T, T]$) to ϕ_0^t.

Let Λ_n be compact ϕ_n-invariant subsets of M, and assume that the upper limit of Λ_n for the Hausdorff topology is contained in Λ_0: more precisely, any neighborhood of Λ_0 contains all but finitely many of the Λ_n's. One can also see this property in another way: Consider the subset $\mathcal{I} = \{0\} \cup \{1/n : n \in \mathbb{N} \setminus \{0\}\} \subset \mathbb{R}$ endowed with the induced topology. Set $M_\infty = M \times \mathcal{I}$ and

$$\Lambda_\infty = \Lambda_0 \times \{0\} \cup \bigcup_{n>0} \Lambda_n \times \{1/n\} \subset M_\infty.$$

With this notation, the upper limit of the Λ_n is contained in Λ_0 if and only if Λ_∞ is a compact subset.

Let $\pi : E \to M$ be a vector bundle. We denote by $E_\infty = E \times \mathcal{I}$ the vector bundle $\pi_\infty : E \times \mathcal{I} \to M \times \mathcal{I}$. We denote by $E_\infty|_{\Lambda_\infty}$ the restriction of E_∞ to the compact subset Λ_∞.

Assume now that $\mathcal{A}_n$ are linear cocycles over the restriction of E to Λ_n. We denote by $\mathcal{A}_\infty$ the map defined on the restriction $E_\infty|_{\Lambda_\infty}$ by

$$\begin{aligned} A^t_\infty(x, 0) &= (A^t_0(x), 0) && \text{for } (x, 0) \in \Lambda_0 \times \{0\}, \\ A^t_\infty(x, 1/n) &= (A^t_n(x), 1/n) && \text{for } (x, 1/n) \in \Lambda_n \times \{1/n\}. \end{aligned}$$

Note that $\mathcal{A}_\infty$ is a cocycle over Λ_∞ and hence a map on $E_\infty|_{\Lambda_\infty} \times \mathbb{R}$.

Definition 25. With the notation above, we say that the family of cocycles $\mathcal{A}_n$ tends to $\mathcal{A}_0$ as $n \to \infty$ if the map $\mathcal{A}_\infty$ is continuous, and therefore is a linear cocycle.

As a consequence of Proposition 24 we get

Corollary 26. *Let $\pi : E \to M$ be a linear cocycle over a manifold M and let ϕ_n be a sequence of flows on M converging to ϕ_0 as $n \to \infty$. Let Λ_n be a sequence of ϕ_n-invariant compact subsets so that the upper limit of the Λ_n, as $n \to \infty$, is contained in Λ_0.*

Let $\mathcal{A}_n$ be a sequence of linear cocycles over ϕ_n defined on the restriction of E to Λ_n. Assume that $\mathcal{A}_n$ tends to $\mathcal{A}_0$ as $n \to \infty$.

Suppose that $\mathcal{A}_0$ admits a dominated splitting $E = E^1 \oplus_\prec \cdots \oplus_\prec E^k$ over Λ_0. Then, for any n large enough, $\mathcal{A}_n$ admits a dominated splitting with the same number of subbundles and the same dimensions of the subbundles. Furthermore, if E^i was hyperbolic (contracting or expanding) over Λ_0, it is still hyperbolic (contracting or expanding, respectively) for $\mathcal{A}_n$ over Λ_n.

The proof just consists in applying Proposition 24 to a neighborhood of $\Lambda_0 \times \{0\}$ in Λ_∞.

3.5. Reparametrized cocycles and hyperbolic structures

Let $\mathcal{A} = \{A^t(x)\}$ and $\mathcal{B} = \{B^t(x)\}$ be two linear cocycles on the same vector bundle $\pi : \mathcal{E} \to \Lambda$ and over the same flow ϕ^t on a compact invariant set Λ of a manifold M. We say that $\mathcal{B}$ is a *reparametrization* of $\mathcal{A}$ if there is a continuous map $h = \{h^t\} \colon \Lambda \times \mathbb{R} \to (0, +\infty)$ such that for every $x \in \Lambda$ and $t \in \mathbb{R}$ one has

$$B^t(x) = h^t(x)A^t(x).$$

The reparametrizing map h^t satisfies the cocycle relation

$$h^{r+s}(x) = h^r(x)h^s(\phi^r(x)),$$

and is called a *cocycle*.

One can easily check the following lemma:

Lemma 27. *Let $\mathcal{A}$ be a linear cocycle and $\mathcal{B}$ be a reparametrization of $\mathcal{A}$. Then any dominated splitting for $\mathcal{A}$ is a dominated splitting for $\mathcal{B}$.*

Remark 28.
- If h^t is a cocycle, then for any $\alpha \in \mathbb{R}$ the power $(h^t)^\alpha : x \mapsto (h^t(x))^\alpha$ is a cocycle.
- If f^t and g^t are cocycles then $h^t = f^t \cdot g^t$ is a cocycle.

A cocycle h^t is called a *coboundary* if there is a continuous function $h \colon \Lambda \to (0, +\infty)$ such that

$$h^t(x) = \frac{h(\phi^t(x))}{h(x)}.$$

A coboundary cocycle is uniformly bounded. Two cocycles g^t, h^t are called *cohomologous* if g^t/h^t is a coboundary.

Remark 29. The cohomology relation (where two cocycles are related if they are cohomologous) is an equivalence relation among the cocycles and is compatible with the product: if g_1^t and g_2^t are cohomologous and h_1^t and h_2^t are cohomologous, then $g_1^t h_1^t$ and $g_2^t h_2^t$ are cohomologous.

Lemma 30. *Let $\mathcal{A} = \mathcal{A}^t$ be a linear cocycle, and $h = h^t$ be a cocycle which is bounded. Then $\mathcal{A}$ is uniformly contracted (resp. expanded) if and only if the cocycle $\mathcal{B} = h \cdot \mathcal{A}$ is uniformly contracted (resp. expanded).*

As a consequence one gets

Corollary 31. *If g and h are cohomologous, then $g \cdot \mathcal{A}$ is hyperbolic if and only if $h \cdot \mathcal{A}$ is hyperbolic.*

4. The extended linear Poincaré flow

4.1. The linear Poincaré flow

Let X be a C^1 vector field on a compact manifold M. We denote by ϕ^t the flow of X.

Definition 32. The *normal bundle* of X is the vector subbundle N_X over $M \setminus \mathrm{Sing}(X)$ defined as follows: the fiber $N_X(x)$ of $x \in M \setminus \mathrm{Sing}(X)$ is the quotient space of T_xM by the vector line $\mathbb{R}.X(x)$.

Note that if M is endowed with a Riemannian metric, then $N_X(x)$ is canonically identified with the space orthogonal to $X(x)$:

$$N_X = \{(x, v) \in TM : v \perp X(x)\}$$

Consider $x \in M \setminus \mathrm{Sing}(X)$ and $t \in \mathbb{R}$. Thus $D\phi^t(x) : T_xM \to T_{\phi^t(x)}M$ is a linear isomorphism mapping $X(x)$ onto $X(\phi^t(x))$. Therefore $D\phi^t(x)$ passes to the quotient as a linear isomorphism $\psi^t(x)\colon N_X(x) \to N_X(\phi^t(x))$:

$$\begin{array}{ccc} T_xM & \xrightarrow{D\phi^t} & T_{\phi^t(x)}M \\ \downarrow & & \downarrow \\ N_X(x) & \xrightarrow{\psi^t} & N_X(\phi^t(x)) \end{array}$$

where the vertical arrows are the canonical projections.

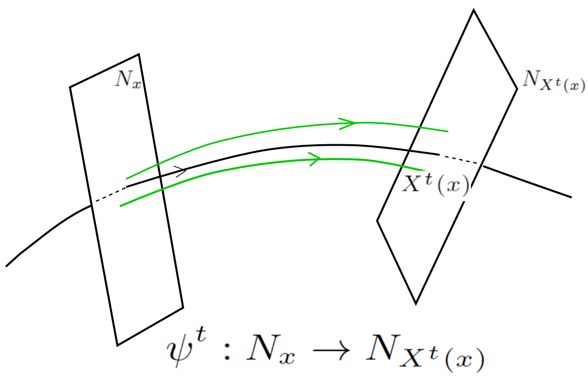


Fig. 3. ψ^t is the differential of the holonomy or Poincaré map.

Proposition 33. *Let X be a C^1 vector field on a manifold M, and Λ be a compact invariant set of X. Assume that Λ does not contain any singularity of X. Then Λ is hyperbolic if and only if the linear Poincaré flow over Λ is hyperbolic.*

Notice that the notion of dominated splitting for non-singular flows is sometimes better expressed in terms of the linear Poincaré flow: for instance, the linear Poincaré flow of a robustly transitive vector field always admits a dominated splitting, while the flow by itself may not admit any dominated splitting. See for instance the suspension of the example in [BV].

4.2. The extended linear Poincaré flow

We are dealing with singular flows, and the linear Poincaré flow is not defined on the singularities of the vector field X. However, we can extend it to a flow on a larger set for which the singularities of X do not play a specific role, as in [LGW]. We call this the *extended linear Poincaré flow*.

This flow will be a linear cocycle defined on certain vector bundles over a manifold, which we define now.

Definition 34. Let M be a d-dimensional manifold.

- We define *the projective tangent bundle of* M to be the fiber bundle $\Pi_{\mathbb{P}}\colon \mathbb{P}M \to M$ whose fiber $\mathbb{P}_x$ is the projective space of the tangent space T_xM; in other words, a point $L_x \in \mathbb{P}_x$ is a one-dimensional vector subspace of T_xM.
- *The tautological bundle of* $\mathbb{P}M$ is the one-dimensional vector bundle over $\mathbb{P}M$, $\Pi_{\mathcal{T}}\colon \mathcal{T}M \to \mathbb{P}M$, whose fiber $\mathcal{T}_L$ over $L \in \mathbb{P}M$ is the line L itself.
- The *normal bundle of* $\mathbb{P}M$ is the $(d-1)$-dimensional vector bundle over $\mathbb{P}M$, $\Pi_{\mathcal{N}}\colon \mathcal{N} \to \mathbb{P}M$, whose fiber $\mathcal{N}_L$ over $L \in \mathbb{P}_x$ is the quotient space T_xM/L.
  If we endow M with a Riemannian metric, then $\mathcal{N}_L$ is identified with the hyperplane orthogonal to L in T_xM.

Let X be a C^r vector field on a compact manifold M, and ϕ^t its flow. The natural actions of the derivative of ϕ^t on $\mathbb{P}M$, $\mathcal{T}M$ and $\mathcal{N}$ define flows on these manifolds. More precisely, for any $t \in \mathbb{R}$:

- We denote by $\phi^t_{\mathbb{P}}\colon \mathbb{P}M \to \mathbb{P}M$ the flow defined by

$$\phi^t_{\mathbb{P}}(L_x) = D\phi^t(L_x) \in \mathbb{P}_{\phi^t(x)}.$$

- We denote by $\phi^t_{\mathcal{T}}\colon \mathcal{T}M \to \mathcal{T}M$ the topological flow whose restriction to a fiber $\mathcal{T}_L$ is the linear isomorphism onto $\mathcal{T}_{\phi^t_{\mathbb{P}}(L)}$ which is the restriction of $D\phi^t$ to the line $\mathcal{T}_L$.
- We denote by $\psi^t_{\mathcal{N}}\colon \mathcal{N} \to \mathcal{N}$ the flow whose restriction to a fiber $\mathcal{N}_L$, $L \in \mathbb{P}_x$, is the linear isomorphism onto $\mathcal{N}_{\phi^t_{\mathbb{P}}(L)}$ defined as follows: $D\phi^t(x)$ is a linear isomorphism from T_xM to $T_{\phi^t(x)}M$ which maps the line $\mathcal{T}_L \subset T_xM$ onto the line $\mathcal{T}_{\phi^t_{\mathbb{P}}(L)}$. Therefore it passes to the quotient as the announced linear isomorphism.

$$\begin{array}{ccc} T_xM & \xrightarrow{D\phi^t} & T_{\phi^t(x)}M \\ \downarrow & & \downarrow \\ \mathcal{N}_L & \xrightarrow{\psi^t_{\mathcal{N}}} & \mathcal{N}_{\phi^t_{\mathbb{P}}(L)} \end{array}$$

Note that $\phi^t_{\mathbb{P}}$, $t \in \mathbb{R}$, defines a flow on $\mathbb{P}_M$ which is a cocycle over ϕ^t whose action on the fibers is by projective maps.

The one-parameter families $\phi^t_{\mathcal{T}}$ and $\psi^t_{\mathcal{N}}$ define flows on $\mathcal{T}M$ and $\mathcal{N}$, respectively, which are linear cocycles over $\phi^t_{\mathbb{P}}$. We call $\phi^t_{\mathcal{T}}$ the *tautological flow*, and $\psi^t_{\mathcal{N}}$ the *extended* *linear Poncaré flow*. We can summarize this by the following diagrams:

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\psi^t_{\mathcal{N}}} & \mathcal{N} \\ \downarrow & & \downarrow \\ \mathbb{P}M & \xrightarrow{\phi^t_{\mathbb{P}}} & \mathbb{P}M \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi^t} & M \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{T}M & \xrightarrow{\phi^t_{\mathcal{T}}} & \mathcal{T}M \\ \downarrow & & \downarrow \\ \mathbb{P}M & \xrightarrow{\phi^t_{\mathbb{P}}} & \mathbb{P}M \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi^t} & M \end{array}$$

Remark 35. The extended linear Poincaré flow is really an extension of the linear Poincaré flow defined in the previous section; more precisely:

Let $S_X: M \setminus \mathrm{Sing}(X) \to \mathbb{P}M$ be the section of the projective bundle such that $S_X(x)$ is the line $\langle X(x)\rangle \in \mathbb{P}_x$ generated by $X(x)$. Then:

- The fibers $N_X(x)$ and $\mathcal{N}_{S_X(x)}$ are canonically identified.
- The linear isomorphisms $\psi^t: N_X(x) \to N_X(\phi^t(x))$ and $\psi^t_{\mathcal{N}}: \mathcal{N}_{S_X(x)} \to \mathcal{N}_{S_X(\phi^t(x))}$ are equal (under the identification of the fibers).

4.3. Maximal invariant set and lifted maximal invariant set

Let X be a vector field on a manifold M and $U \subset M$ be a compact region. The *maximal invariant set* $\Lambda = \Lambda_U$ of X in U is the intersection

$$\Lambda(X, U) = \bigcap_{t\in\mathbb{R}} \phi^t(U).$$

We say that a compact X-invariant set K is *locally maximal* if there exists an open neighborhood U of K such that $K = \Lambda(X, U)$.

Definition 36. The *lifted maximal invariant set in* U, denoted by $\Lambda_{\mathbb{P},U} \subset \mathbb{P}M$ (or simply $\Lambda_{\mathbb{P}}$ if one may omit the dependence on U), is the closure of the set of lines $\langle X(x)\rangle$ for regular $x \in \Lambda_U$:

$$\Lambda_{\mathbb{P},U} = \overline{S_X(\Lambda_U \setminus \mathrm{Sing}(X))} \subset \mathbb{P}M,$$

where $S_X: M \setminus \mathrm{Sing}(X) \to \mathbb{P}M$ is the section defined by X.

5. The extended maximal invariant set

5.1. Strong stable, strong unstable and center spaces associated to a hyperbolic singularity

Let X be a vector field and $\sigma \in \mathrm{Sing}(X)$ be a hyperbolic singular point of X. Let $\lambda^s_k < \cdots < \lambda^s_2 < \lambda^s_1 < 0 < \lambda^u_1 < \lambda^u_2 < \cdots < \lambda^u_l$ be the Lyapunov exponents of ϕ_t at σ and let

$$E^s_k \oplus_{\prec} \cdots \oplus_{\prec} E^s_2 \oplus_{\prec} E^s_1 \oplus_{\prec} E^u_1 \oplus_{\prec} E^u_2 \oplus_{\prec} \cdots \oplus_{\prec} E^s_l$$

be the corresponding (finest) dominated splitting over σ.

A subspace F of $T_\sigma M$ is called a *center subspace* if it is of one of the forms below:

- $F = E^s_i \oplus_\prec \cdots \oplus_\prec E^s_2 \oplus_\prec E^s_1$;
- $F = E^u_1 \oplus_\prec E^u_2 \oplus_\prec \cdots \oplus_\prec E^s_j$;
- $F = E^s_i \oplus_\prec \cdots \oplus_\prec E^s_1 \oplus_\prec E^u_1 \oplus_\prec \cdots \oplus_\prec E^s_j$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$.

A subspace

$$E^{ss}_i(\sigma) = E^s_k \oplus_\prec \cdots \oplus_\prec E^s_{i+1} \oplus_\prec E^s_i, \quad 1 \leq i \leq k,$$

of $T_\sigma M$ is called a *strong stable space*.

A classical result from hyperbolic dynamics asserts that for any i there is a unique injectively immersed manifold $W^{ss}_i(\sigma)$ in M, called a *strong stable manifold*, with tangent space $E^{ss}_i(\sigma)$ and invariant by the flow of X.

We define analogously the *strong unstable spaces* $E^{uu}_j(\sigma)$ and the *strong unstable manifolds* $W^{uu}_j(\sigma)$ for $j = 1, \dots, l$.

5.2. The lifted maximal invariant set and the singular points

The aim of this section is to add to the lifted maximal invariant set $\Lambda_{\mathbb{P},U}$ some set over the singular points in order to recover some upper semicontinuity properties. As mentioned in Section 2.4, we want to define a set that is as small as possible, but which can be defined without any information on the perturbations of our vector field.

We define the *escaping stable space* $E^{ss}_{\sigma,U}$ as the biggest strong stable space $E^{ss}_j(\sigma)$ such that the corresponding strong stable manifold $W^{ss}_j(\sigma)$ is *escaping*, that is,

$$\Lambda_{X,U} \cap W^{ss}_j(\sigma) = \{\sigma\}.$$

We define the *escaping unstable space* analogously.

We define the *central space* $E^c_{\sigma,U}$ *of* σ *in* U as the center space such that

$$T_\sigma M = E^{ss}_{\sigma,U} \oplus E^c_{\sigma,U} \oplus E^{uu}_{\sigma,U}.$$

We denote by $\mathbb{P}^i_{\sigma,U}$ the projective space of $E^i(\sigma, U)$ where $i \in \{ss, uu, c\}$.

Lemma 37. *Let U be a compact region and X a vector field whose singular points are hyperbolic. Then, for any $\sigma \in \mathrm{Sing}(X) \cap U$,*

$$\Lambda_{\mathbb{P},U} \cap \mathbb{P}^{ss}_{\sigma,U} = \Lambda_{\mathbb{P},U} \cap \mathbb{P}^{uu}_{\sigma,U} = \emptyset.$$

Proof. Suppose (towards a contradiction) that $L \in \Lambda_{\mathbb{P},U} \cap \mathbb{P}^{ss}_{\sigma,U}$. This means that there exists a sequence $x_n \in \Lambda_{X,U} \setminus \mathrm{Sing}(X)$ converging to σ such that L_{x_n} converges to L, where L_{x_n} is the line $\mathbb{R}X(x_n) \in \mathbb{P}_{x_n}$.

We fix a small neighborhood V of σ endowed with local coordinates such that the vector field is very close to its linear part in these coordinates: in particular, there is a small cone $C^{ss} \subset V$ around $W^{ss}_{\sigma,U}$ whose complement is strictly invariant in the following sense: the positive orbit of a point outside C^{ss} remains outside C^{ss} until it leaves V. For n large enough the points x_n belong to V.

As $\mathbb{R}X(x_n)$ tends to L, this implies that the points x_n, for n large, are contained in the cone C^{ss}. In particular, they cannot belong to $W^u(\sigma)$. Therefore they admit negative iterates $y_n = \phi^{-t_n}(x_n)$ with the following properties:

- $\phi^{-t}(x_n) \in V$ for all $t \in [0, t_n]$.
- $\phi^{-t_n-1}(x_n) \notin V$.
- $t_n \to \infty$.

Up to considering a subsequence, one may assume that the points y_n converge to a point y, and one may easily check that $y \in W^s(\sigma)\setminus\{\sigma\}$. Furthermore all the points y_n are in $\Lambda_{X,U}$, so that $y \in \Lambda_{X,U}$.

We conclude the proof by showing that $y \in W^{ss}_{\sigma,U}$, which contradicts the definition of $W^{ss}_{\sigma,U}$. If $y \notin W^{ss}_{\sigma,U}$ then its positive orbit arrives at σ tangentially to weaker stable spaces: in particular, there is $t > 0$ such that $\phi^t(y)$ does not belong to the cone C^{ss}.

Consider n large, in particular t_n is larger than t and $\phi^t(y_n)$ is so close to y that $\phi^t(y_n) \notin C^{ss}$: this contradicts the fact that $x_n = \phi^{t_n}(y_n) \in C^{ss}$.

We have proved $\Lambda_{\mathbb{P},U} \cap \mathbb{P}^{ss}_{\sigma,U} = \emptyset$; the proof that $\Lambda_{\mathbb{P},U} \cap \mathbb{P}^{uu}_{\sigma,U} = \emptyset$ is analogous. □

As a consequence we get the following characterization of the central space of σ in U:

Lemma 38. *Let U be a compact region and X a vector field whose singular points are hyperbolic. Then for any $\sigma \in \mathrm{Sing}(X) \cap U$ the central space $E^c_{\sigma,U}$ is the smallest center space containing $\Lambda_{\mathbb{P},U} \cup \mathbb{P}_\sigma M$.*

Proof. The proof that $E^c_{\sigma,U}$ contains $\Lambda_{\mathbb{P},U} \cap \mathbb{P}_\sigma$ is very similar to the end of the proof of Lemma 37 and we just sketch it: by definition of the strong escaping manifolds, the points x_n admit a neighborhood of a fundamental domain which is disjoint from the maximal invariant set. This implies that any point in $\Lambda_{X,U}$ close to σ lies outside of arbitrarily large cones around the escaping strong direction. Therefore the vector X at these points is almost tangent to $E^c_{\sigma,U}$.

Assume now for instance that:

- $E^c_{\sigma,U} = E^s_i \oplus_\prec \cdots \oplus_\prec E^s_1 \oplus_\prec E^u_1 \oplus_\prec \cdots \oplus_\prec E^u_j$: in particular $W^{ss}_{i+1}(\sigma)$ is the escaping strong stable manifold.
- $\Lambda_{\mathbb{P},U} \cap \mathbb{P}_\sigma$ is contained in the smaller center space

$$E^s_{i-1} \oplus_\prec \cdots \oplus_\prec E^s_1 \oplus_\prec E^u_1 \oplus_\prec \cdots \oplus_\prec E^u_j.$$

We will show that the strong stable manifold $W^{ss}_i(\sigma)$ is escaping, contradicting the maximality of $W^{ss}_{i+1}(\sigma)$. Suppose there is $x \in (W^{ss}_i(\sigma) \setminus \{\sigma\}) \cap \Lambda_{X,U}$. The positive orbit of x tends to σ tangentially to $E^s_k \oplus_\prec \cdots \oplus_\prec E^s_i$ and thus $X(\phi^t(x))$ for t large is almost tangent to $E^s_k \oplus_\prec \cdots \oplus_\prec E^s_i$; this implies that $\Lambda_{\mathbb{P},U} \cap \mathbb{P}_\sigma$ contains at least a direction in $E^s_k \oplus_\prec \cdots \oplus_\prec E^s_i$, contradicting the hypothesis. □

Lemma 39. *Let U be a compact region. Given a hyperbolic singular point σ in U, and its continuation σ_Y for vector fields Y in a C^1-neighborhood of X, both escaping strong stable and unstable spaces $E^{ss}_{\sigma_Y,U}$ and $E^{uu}_{\sigma_Y,U}$ depend lower semicontinuously on Y.*

As a consequence, the central space $E^c_{\sigma_Y,U}$ depends upper semicontinuously on Y, and so does its projective space $\mathbb{P}^{ss}_{\sigma_Y,U}$.

Proof. We will only give the proof for the escaping strong stable space, as the proof for the escaping strong unstable space is identical.

As σ is in the interior of U, there is $\delta > 0$ and a C^1-neighborhood $\mathcal{U}$ of X such that, for any $Y \in \mathcal{U}$:

- σ has a hyperbolic continuation σ_Y for Y.
- The finest dominated splitting of σ_X for X has a continuation for σ_Y which is a dominated splitting (but maybe not the finest).
- The local stable manifold of size δ of σ_Y is contained in U and depends continuously on Y.
- For any strong stable space $E^{ss}(\sigma)$, the corresponding local strong stable manifold $W^{ss}(\sigma_Y)$ varies continuously with $Y \in \mathcal{U}$.

Let E^{ss} denote the escaping strong stable space of σ and let $W^{ss}_\delta(\sigma)$ be the corresponding local strong stable manifold. We fix a sphere S_X embedded in the interior of $W^{ss}_\delta(\sigma)$, transverse to X and intersecting every orbit in $W^{ss}_\delta(\sigma) \setminus \{\sigma\}$. By definition of escaping strong stable manifold, for every $x \in S_X$ there is $t(x) > 0$ such that $\phi^{t(x)}(x) \notin U$.

As S_X is compact and the complement of U is open, there is a finite family t_i, $i = 0, \dots, k$, an open covering $V_0, \dots, V_k$ and a C^1-neighborhood $\mathcal{U}_1$ of X such that, for every $x \in U_i$ and every $Y \in \mathcal{U}_1$ the point $\phi_Y^{t_i}(x)$ does not belong to U.

For Y in a smaller neighborhood $\mathcal{U}_2$ of X, the union of the V_i's covers a sphere $S_Y \subset W^{ss}_\delta(\sigma_Y, Y)$ intersecting every orbit in $W^{ss}_\delta(\sigma_Y, Y) \setminus \{\sigma_Y\}$.

This shows that $W^{ss}_\delta(\sigma_Y, Y)$ is contained in the escaping strong stable manifold of σ_Y, proving the lower semicontinuity. □

5.3. *The extended maximal invariant set*

We are now able to define the subset of $\mathbb{P}M$ which extends the lifted maximal invariant set and depends upper semicontinuously on X.

Definition 40. Let U be a compact region and X a vector field whose singular points are hyperbolic. Then the set

$$B(X, U) = \Lambda_{\mathbb{P},U} \cup \bigcup_{\sigma \in \mathrm{Sing}(X)\cap U} \mathbb{P}^c_{\sigma,U} \subset \mathbb{P}M$$

is called the *extended maximal invariant set of X in U*.

Proposition 41. *Let U be a compact region and X a vector field whose singular points are hyperbolic. Then the extended maximal invariant set $B(X, U)$ of X in U is a compact subset of $\mathbb{P}M$, invariant under the flow $\phi^t_{\mathbb{P}}$. Furthermore, the map $X \mapsto B(X, U)$ depends upper semicontinuously on X.*

Proof. First notice that the set of singular points of Y in U consists of finitely many hyperbolic singularities varying continuously with Y in a neighborhood of X. The extended maximal invariant set is compact, being the union of finitely many compact sets.

Let Y_n be a sequence of vector fields tending to X in the C^1-topology, and let $(x_n, L_n) \in B(Y_n, U)$. Up to considering a subsequence we may assume that (x_n, L_n) tends to a point $(x, L) \in \mathbb{P}M$, and we need to prove that (x, L) belongs to $B(X, U)$.

First assume that $x \notin \mathrm{Sing}(X)$. Then, for n large, x_n is not a singular point for Y_n so that $L_n = \langle Y_n(x_n)\rangle$ and therefore $L = \langle X(x)\rangle$ belongs to $B(X, U)$ as desired.

Thus we may assume $x = \sigma \in \mathrm{Sing}(X)$. First notice that if, for infinitely many n, x_n is a singularity of Y_n then L_n belongs to $\mathbb{P}^c_{\sigma_{Y_n},U}$. As $\mathbb{P}^c_{\sigma_Y,U}$ varies upper semicontinuously with Y, we deduce that L belongs to $\mathbb{P}^c_{\sigma_X,U}$, as desired. So we may assume that $x_n \notin \mathrm{Sing}(Y_n)$.

We fix a neighborhood V of σ endowed with coordinates, so that X (and therefore Y_n for large n) is very close to its linear part in V. Let $S_X \subset W^s_{\mathrm{loc}}(\sigma)$ be a sphere transversal to X and intersecting every orbit in $W^s_{\mathrm{loc}}(\sigma) \setminus \{\sigma\}$, and let W be a small neighborhood of S_X.

Assume that $x_n \notin W^u(\sigma_{Y_n})$ for infinitely many n. There is a sequence $t_n > 0$ with the following properties:

- $\phi^{-t}_{Y_n}(x_n) \in V$ for all $t \in [0, t_n]$.
- $\phi^{-t_n}_{Y_n}(x_n) \in W$.
- t_n tends to $+\infty$ as $n \to \infty$.

Up to considering a subsequence, one may assume that the points $y_n = \phi^{-t_n}_{Y_n}(x_n)$ tend to a point $y \in W^s(\sigma)$.

Claim. *The point y does not belong to $W^{ss}_{\sigma,U}$.*

Proof. By definition of the escaping strong stable manifold, for every $y \in W^{ss}_{\sigma,U}$ there is t such that $\phi^t(y) \notin U$; thus $\phi^t_{Y_n}(y_n) \notin U$ for y_n close enough to y; in particular $y_n \notin \Lambda_{Y_n,U}$. □

Since $y \notin W^{ss}_{\sigma,U}$ for $T > 0$ large enough the line $\langle X(z)\rangle$, $z = \phi^T(y)$, is almost tangent to $E^{cu} = E^c_{\sigma,U} \oplus E^{uu}_{\sigma,U}$. As a consequence, for n large, $\langle Y_n(z_n)\rangle$, where $z_n = \phi^T_{Y_n}(y_n)$, is almost tangent to the continuation E^{cu}_n of E^c for σ_n, Y_n. As $x_n = \phi^{t_n-T}_{Y_n}(y_n)$, and as $t_n - T \to \infty$, the dominated splitting implies that $L_n = \langle Y_n(x_n)\rangle$ is almost tangent to E^{cu}_n.

This shows that $L \subset E^{cu}$. Notice that this also holds if x_n belongs to the unstable manifold of σ_{Y_n}. Arguing analogously we find that $L \subset E^{cs} = E^c_{\sigma,U} \oplus E^{ss}_{\sigma,U}$. Thus $L \subset E^c_{\sigma,U}$, concluding the proof. □

Remark 42. The lower semicontinuity of the strong escaping stable and unstable manifolds of a vector field X, and the upper semicontinuity of E^c_σ, imply that there is a C^1-neighborhood $\mathcal{U}$ of X such that for any Y in $\mathcal{U}$ there are no regular orbits approaching the singularity σ tangent to the escaping spaces. In fact, domination implies that any regular orbit approaching σ becomes tangent to E^c_σ. This implies that

$$\widetilde{\Lambda(X, U)} \subset B(X, U).$$

6. Reparametrizing cocycle associated to a singular point

Let X be a C^1 vector field, ϕ^t its flow, and σ a hyperbolic singularity of X. We denote by $\Lambda_X \subset \mathbb{P}M$ the union

$$\Lambda_X = \overline{\{\mathbb{R}X(x) : x \notin \mathrm{Sing}(X)\}} \cup \bigcup_{x \in \mathrm{Sing}(X)} \mathbb{P}_x M.$$

It can be shown easily that this set is upper semicontinuous in X, as in the case of $B(X, U)$ (see Proposition 41).

Lemma 43. *Λ_X is a compact subset of $\mathbb{P}M$ invariant under the flow $\phi^t_{\mathbb{P}}$, and the map $X \mapsto \Lambda_X$ is upper semicontinuous. Finally, if the singularities of X are hyperbolic then $B(X, U) \subset \Lambda_X$ for any compact region U.*

Let U_σ be a compact neighborhood of σ on which $\{\sigma\}$ is the maximal invariant set. Let V_σ be a compact neighborhood of $\mathrm{Sing}(X) \setminus \{\sigma\}$ such that $V_\sigma \cap U_\sigma = \emptyset$. We fix a (C^1) Riemannian metric $\|\cdot\|$ on M such that

$$\|X(x)\| = 1 \quad \text{for all } x \in M \setminus (U_\sigma \cup V_\sigma).$$

Consider the map $h_\sigma : \Lambda_X \times \mathbb{R} \to (0, +\infty)$, $h_\sigma(L, t) = h^t_\sigma(L)$, defined as follows:

- If $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma$ and $\phi^t(x) \notin U_\sigma$, then $h^t_\sigma(L) = 1$.
- If $L \in \mathbb{P}T_x M$ with $x \in U_\sigma$ and $\phi^t(x) \notin U_\sigma$ then $L = \mathbb{R}X(x)$ and $h^t_\sigma(L) = 1/\|X(x)\|$.
- If $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma$ and $\phi^t(x) \in U_\sigma$ then $L = \mathbb{R}X(x)$ and $h^t_\sigma(L) = \|X(\phi^t(x))\|$.
- If $L \in \mathbb{P}T_x M$ with $x \in U_\sigma$ and $\phi^t(x) \in U_\sigma$ but $x \neq \sigma$ then $L = \mathbb{R}X(x)$ and $h^t_\sigma(L) = \|X(\phi^t(x)\|/\|X(x)\|$.
- If $L \in \mathbb{P}T_\sigma M$ then $h^t_\sigma(L) = \|\phi^t_{\mathbb{P}}(u)\|/\|u\|$ where u is a vector in L.

Note that the case in which x is not the singularity and $x \in U_\sigma$ can be written as in the last item by taking $u = X(x)$.

Lemma 44. *Let X be a C^1 vector field, ϕ^t its flow, and σ a hyperbolic singularity of X. Define the sets Λ_X, U_σ, V_σ and the map h_σ as above. Then h_σ is a (continuous) cocycle on Λ_X.*

Proof. The continuity of h_σ comes from the continuity of the norm and the fact that the compact neighborhood U_σ contains only one singularity. Now for $L \in \Lambda_X$ we aim to show that h_σ satisfies the cocycle relation

$$h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L) = h^{t+s}_\sigma(L).$$

- If $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma$, $\phi^s(x) \notin U_\sigma$ and $\phi^{s+t}(x) \notin U_\sigma$, then $h^{t+s}_\sigma(L) = h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L) = 1$.
- Let $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma$, $\phi^s(x) \notin U_\sigma$ and $\phi^{s+t}(x) \in U_\sigma$. Then x is not singular and $L = \mathbb{R}X(x)$. Since $h^s_\sigma(L) = 1$, we have

$$h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L) = \|X(\phi^t(\phi^s(x)))\| = \|X(\phi^{t+s}(x))\| = h^{t+s}_\sigma(L).$$

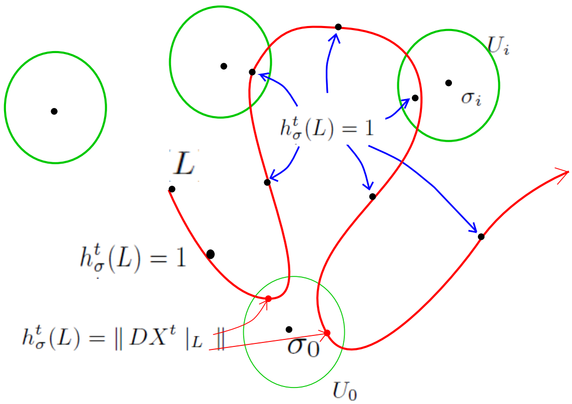


Fig. 4. The local cocycle h^t_σ associated to the singularity $\sigma = \sigma_0$.

- If $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma$, $\phi^s(x) \in U_\sigma$, and $\phi^{t+s}(x) \notin U_\sigma$, then $L = \mathbb{R}X(x)$, $h^s_\sigma(L) = \|X(\phi^s(x))\|$, and

$$h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L) = \frac{1}{\|X(\phi^s(x))\|}\|X(\phi^s(x))\| = 1 = h^{t+s}_\sigma(L).$$

- If $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma$, $\phi^s(x) \in U_\sigma$, and $\phi^{t+s}(x) \in U_\sigma$, then $L = \mathbb{R}X(x)$, $h^s_\sigma(L) = \|X(\phi^s(x))\|$, and

$$\begin{aligned} h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L) &= \frac{\|X(\phi^t(\phi^s(x)))\|}{\|X(\phi^s(x)\|}\|X(\phi^s(x))\| = \|X(\phi^t(\phi^s(x)))\| \\ &= \|X(\phi^{t+s}(x))\| = h^{t+s}_\sigma(L). \end{aligned}$$

- If $L \in \mathbb{P}T_x M$ with $x \in U_\sigma$, $\phi^s(x) \notin U_\sigma$, and $\phi^{s+t}(x) \notin U_\sigma$, then $h^t_\sigma(\phi^s_{\mathbb{P}}(L)) = 1$ and

$$h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L) = \frac{1}{\|X(x)\|} = h^{t+s}_\sigma(L).$$

- Let $L \in \mathbb{P}T_x M$ with $x \in U_\sigma$, $\phi^s(x) \notin U_\sigma$ and $\phi^{s+t}(x) \in U_\sigma$. Since $h^s_\sigma(L) = 1/\|X(x)\|$, we have

$$h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L) = \|X(\phi^t(\phi^s(x)))\|\frac{1}{\|X(x)\|} = \frac{\|X(\phi^{t+s}(x))\|}{\|X(x)\|} = h^{t+s}_\sigma(L).$$

- If $L \in \mathbb{P}T_x M$ with $x \in U_\sigma$, $\phi^s(x) \in U_\sigma$ and $\phi^{t+s}(x) \notin U_\sigma$, then $L = \mathbb{R}X(x)$, $h^t_\sigma(\phi^s_{\mathbb{P}}(L)) = 1/\|X(\phi^s(x))\|$ and

$$h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L) = \frac{\|X(\phi^s(x))\|}{\|X(x)\|}\frac{1}{\|X(\phi^s(x))\|} = \frac{1}{\|X(x)\|} = h^{t+s}_\sigma(L).$$

- If $L \in \mathbb{P}T_x M$ with $x \in U_\sigma \setminus \{\sigma\}$, $\phi^s(x) \in U_\sigma$ and $\phi^{t+s}(x) \in U_\sigma$ then $L = \mathbb{R}X(x)$ and

$$\begin{aligned} h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L) &= \frac{\|X(\phi^t(\phi^s(x)))\|}{\|X(\phi^s(x))\|}\,\frac{\|X(\phi^s(x))\|}{\|X(x)\|} = \frac{\|X(\phi^t(\phi^s(x)))\|}{\|X(x)\|} \\ &= \frac{\|X(\phi^{t+s}(x))\|}{\|X(x)\|} = h^{t+s}_\sigma(L). \end{aligned}$$

- If $L \in \mathbb{P}T_\sigma M$, let u be a vector in L; then

$$\begin{aligned} h^{t+s}_\sigma(L) &= \frac{\|D\phi^{t+s}_{\mathbb{P}}(u)\|}{\|u\|} = \frac{\|D\phi^{t+s}_{\mathbb{P}}(u)\|}{\|D\phi^s_{\mathbb{P}}(u)\|}\,\frac{\|D\phi^s_{\mathbb{P}}(u)\|}{\|u\|} \\ &= \frac{\|D\phi^t_{\mathbb{P}}(D\phi^s_{\mathbb{P}}(u))\|}{\|D\phi^s_{\mathbb{P}}(u)\|}\,\frac{\|D\phi^s_{\mathbb{P}}(u)\|}{\|u\|} = h^t_\sigma(\phi^s_{\mathbb{P}}(L))h^s_\sigma(L). \end{aligned}$$ □

Lemma 45. *The cohomology class of a cocycle h defined as above is independent of the choice of the metric* $\|\cdot\|$ *and of the neighborhoods* U_σ *and* V_σ.

Proof. Let $\|\cdot\|$ and $\|\cdot\|'$ be two different metrics. Let U_σ, V_σ and U'_σ, V'_σ be two different sets of neighborhoods of σ and $\mathrm{Sing}(X) \setminus \{\sigma\}$ such that:

- $V_\sigma \cap U_\sigma = \emptyset$.
- $V'_\sigma \cap U'_\sigma = \emptyset$.
- $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$.
- $\|X(x)\| = 1$ for all $x \in M \setminus (U_\sigma \cup V_\sigma)$.
- $\|X(x)\|' = 1$ for all $x \in M \setminus (U'_\sigma \cup V'_\sigma)$.

We define h_σ as before for the metric $\|\cdot\|$ and h'_σ as before for the metric $\|\cdot\|'$. We define a function $g\colon B(X, U) \to (0, +\infty)$ such that:

- If $L \in \mathbb{P}T_x M$ with $x \notin V'_\sigma \cup V_\sigma$, then $g(L) = \|u\|'/\|u\|$ for a non-zero vector u in L.
- If $L \in \mathbb{P}T_x M$ with $x \in V'_\sigma \cup V_\sigma$, then $g(L) = 1$.

Claim. *The function* $g\colon B(X, U) \to (0, +\infty)$ *defined above is continuous.*

Proof. The continuity of the norms $\|\cdot\|$ and $\|\cdot\|'$, and the fact that they are 1 outside $V_\sigma \cup V'_\sigma$ and $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$, gives us the continuity on the boundary of $U_\sigma \cup U'_\sigma$. □

The following claim will show us that the functions h_σ and h'_σ differ by a coboundary defined as $g^t(L) = \frac{g(D\phi^t_{\mathbb{P}}(u))}{g(u)}$.

Claim. *The functions* h_σ *and* h'_σ *are such that*

$$h'^t_\sigma(u) = h^t_\sigma(u)\frac{g(D\phi^t_{\mathbb{P}}(u))}{g(u)}.$$

Proof. • For the metric $\|\cdot\|'$ and $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma \cup U'_\sigma$ and $\phi^t(x) \notin U_\sigma \cup U'_\sigma$, one has $g^t(L) = 1$. On the other hand $h'^t(L) = 1$ as desired.

- If $L \in \mathbb{P}T_xM$ with $x \in U_\sigma \cup U'_\sigma$ and $\phi^t(x) \notin U_\sigma \cup U'_\sigma$ then $g^t(L) = \|u\|/\|u\|'$. Take $u = X(x)$. Then $h^t(L) = 1/\|X(x)\|$ and

$$h'^t(L) = h^t(L)\frac{\|X(x)\|}{\|X(x)\|'}.$$

- If $L \in \mathbb{P}T_xM$ with $x \notin U_\sigma \cup U'_\sigma$ and $\phi^t(x) \in U_\sigma \cup U'_\sigma$ then $L = \mathbb{R}X(x)$. Take $u = X(x)$. Then $g^t(L) = \frac{\|D\phi^t_{\mathbb{P}}(u)\|'}{\|D\phi^t_{\mathbb{P}}(u)\|}$ and since $h^t(L) = \|D\phi^t_{\mathbb{P}}(u)\|$,

$$h'^t(L) = h^t(L)\frac{\|D\phi^t_{\mathbb{P}}(u)\|'}{\|D\phi^t_{\mathbb{P}}(u)\|}.$$

.

- If $L \in \mathbb{P}T_xM \cap B(X, U)$ with $x \in U_\sigma$ and $\phi^t(x) \in U_\sigma$, then taking $u = X(x)$, one has

$$g^t(L)\frac{\|D\phi^t_{\mathbb{P}}(u)\|'\|u\|}{\|D\phi^t_{\mathbb{P}}(u)\|\|u\|'},$$

  and $h^t(L) = D\phi^t_{\mathbb{P}}(u)/\|u\|$. So $h'^t(L) = h^t(L)g^t(L)$. □

Now in order to finish the proof we need to show that the condition that $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$ does not restrict generality. For this, suppose we started with any other norm $\|\cdot\|''$ and that there exist neighborhoods V''_σ, U''_σ such that:

- $V''_\sigma \cap U''_\sigma = \emptyset$.
- $\|X(x)\|'' = 1$ for all $x \in M \setminus (U''_\sigma \cup V''_\sigma)$.

Choose a smaller neighborhood $V'_\sigma \subset V''_\sigma$. Then $V'_\sigma \cap U''_\sigma = \emptyset$. Analogously $U'_\sigma \subset U''_\sigma$ will satisfy $V''_\sigma \cap U'_\sigma = \emptyset$. Now if we choose the neighborhoods V'_σ and U'_σ as small as we want, and a norm $\|\cdot\|'$ such that $\|X(x)\|' = 1$ for all $x \in M \setminus (U''_\sigma \cup V''_\sigma)$, the claims above imply that the corresponding h'' and h' differ by a coboundary. Therefore h' can be chosen so that h'' and h differ by a coboundary. □

We denote by $[h(X, \sigma)]$ the cohomology class of any cocycle defined as h above.

Lemma 46. *Consider a vector field X and a hyperbolic singularity σ of X. Then there is a C^1-neighborhood $\mathcal{U}$ of X such that σ has a well defined hyperbolic continuation σ_Y for Y in $\mathcal{U}$, and for any $Y \in \mathcal{U}$ there is a map $h_Y : \Lambda_Y \times \mathbb{R} \to (0, +\infty)$ such that:*

- *h_Y is a cocycle belonging to the cohomology class $[h(Y, \sigma_Y)]$.*
- *h_Y depends continuously on Y: if $Y_n \in \mathcal{U}$ converges to $Z \in \mathcal{U}$ for the C^1-topology and if $L_n \in \Lambda_{Y_n}$ converges to $L \in \Lambda_Z$, then $h^t_{Y_n}(L_n)$ tends to $h^t_Z(L)$ for every $t \in \mathbb{R}$; furthermore, this convergence is uniform in $t \in [-1, 1]$.*

Proof. The manifold M is endowed with a Riemannian metric $\|\cdot\|$. We fix the neighborhoods U_σ and V_σ for X and $\mathcal{U}$ is a C^1-neighborhood of X such that σ_Y is the maximal invariant set for Y in U_σ and $\mathrm{Sing}(Y) \setminus \{\sigma_Y\}$ is contained in the interior of V_σ. Up to shrinking $\mathcal{U}$ if necessary, we also assume that there are compact neighborhoods $\tilde{U}_\sigma$ of σ_Y contained in the interior of U_σ and $\tilde{V}_\sigma$ of $\mathrm{Sing}(Y) \setminus \{\sigma_Y\}$ contained in the interior of V_σ.

We fix a continuous function $\xi: M \to [0,1]$ so that $\xi(x) = 1$ for $x \in M \setminus (U_\sigma \cup V_\sigma)$ and $\xi(x) = 0$ for $x \in \tilde{U}_\sigma \cup \tilde{V}_\sigma$.

For any $Y \in \mathcal{U}$ we consider the map $\eta_Y: M \to (0, +\infty)$ defined by

$$\eta_Y(x) = \frac{\xi(x)}{\|X(x)\|} + 1 - \xi(x).$$

This map is a priori not defined on $\mathrm{Sing}(Y)$ but extends by continuity to $y \in \mathrm{Sing}(Y)$ by $\eta_Y(y) = 1$, and is continuous.

This map depends continuously on Y. Now we consider the metric $\|\cdot\|_Y = \eta_Y \|\cdot\|$. Note that $\|Y(x)\|_Y = 1$ for $x \in M \setminus (U_\sigma \cup V_\sigma)$. Now h_Y is the cocycle built in Lemma 45 for U_σ, V_σ and $\|\cdot\|_Y$. □

Notice that, according to Remark 29, if $\sigma_1, \dots, \sigma_k$ are hyperbolic singularities of X, the homology class of the product cocycle $h^t_{\sigma_1} \cdots h^t_{\sigma_k}$ is well defined, and admits representatives varying continuously with the flow.

7. Extension of the dominated splitting

7.1. The dominated splittings over the singularities

The aim of this section is to prove Theorem 2.

Remark 47. Suppose that the finest Lyapunov decomposition of the singularity is

$$T_\sigma M = E_1 \oplus \cdots \oplus E_i \oplus \cdots \oplus E_j \oplus \cdots \oplus E_l.$$

If we pick a direction $L \in \mathbb{P}_\sigma M$ such that the closure of its orbit under $\phi^t_{\mathbb{P}}$, denoted $\overline{O(L)}$, is contained in $E_i \oplus \cdots \oplus E_j$, then the angle between $\phi^t_{\mathbb{P}}(L)$ and any space

$$\sum_{h<i} E_h \quad \text{or} \quad \sum_{h>j} E_h$$

is uniformly away from zero.

Recall that $\pi_L : T_x M \to \mathcal{N}_L$ is the projection associated to the normal bundle. We can identify $\pi_L(E_h)$ with E_h for all $h < i$ and $h > j$, and also we can identify

$$\phi^t_{\mathbb{P}}\Big(\sum_{h<k} E_h\Big) \text{ with } \sum_{h<k} E_h \quad \text{ for } k < i$$

and

$$\phi^t_{\mathbb{P}}\Big(\sum_{k<h} E_h\Big) \text{ with } \sum_{k<h} E_h \quad \text{ for } k > j.$$

Lemma 48. *Let X be a vector field on a d-dimensional manifold M with a hyperbolic singularity σ where the finest splitting of the tangent space is*

$$T_\sigma M = E_1 \oplus \cdots \oplus E_l.$$

Consider a Riemannian metric such that $E_i \perp E_j$ for all $i \neq j$. Let L be such that $L = \langle u \rangle$ where u belongs to some E_i. Then

$$\mathcal{N}_L = E_1 \oplus \cdots \oplus \pi_L(E_i) \oplus \cdots \oplus E_l$$

is the finest dominated splitting over the closure of the orbit of L.

Proof. Suppose that $\lambda < 0$; the other case is analogous. Let $w \in \pi_L(E_i)$. Since $D\phi^t(u)$ and $\psi^t_{\mathcal{N}}(w)$ are perpendicular, we have

$$J(D\phi^t)|_{E_i} = J(\psi^t_{\mathcal{N}})|_{\pi_L(E_i)} \|D\phi^t(u)\|.$$

If $\dim E_i = d$ we have $J(D\phi^t)|_{E_i} = \mathrm{e}^{t\lambda d}$ and the largest Lyapunov exponent of $\psi^t_{\mathcal{N}}$ cannot be greater than λ. Therefore there are two possibilities: either

1. there exists a constant C such that $J(\psi^t_{\mathcal{N}}) = C\mathrm{e}^{t\lambda(d-1)}$ for t large enough, or
2. the ratio of contraction of u is greater than λ.

Since $D\phi^t|_{E_i}$ has all the Lyapunov exponents equal to λ, for t large enough we have

$$\|D\phi^t(u)\| \leq C_1 \|u\| e^{\lambda t},$$

reaching a contradiction. □

The following lemma together with Lemma 53 are very similar to Lemma 4.3 in [LGW]. Since the context is slightly different and the statement is split into two parts, we add the proof anyway.

Lemma 49. *Let X be a vector field with a hyperbolic singularity σ where the finest hyperbolic decomposition of the tangent space is*

$$T_\sigma M = E_1 \oplus \cdots \oplus E_i \oplus \cdots \oplus E_j \oplus \cdots \oplus E_l.$$

Suppose E_i is a stable space, and E_j an unstable space. Define $k_i = \sum_{k=1}^{i-1} \dim E_k$ and $h_j = \sum_{k=j+1}^{l} \dim E_k$.

Consider a direction $L = \langle u \rangle$, where u is a vector in $(E_i \oplus E_j) \setminus (E_i \cup E_j)$. Assume $E \oplus_{\prec} F$ is a dominated splitting for $\psi_{\mathcal{N}}$ over the closure $\overline{O(L)}$ of the orbit of L for $\phi_{\mathbb{P}}$. Then either

- $\dim E \leq k_i$*, in which case there is $1 \leq i' < i$ such that, for any $L' \in \overline{O(L)}$,*

$$E = \pi_{L'}\Big(\sum_{k=1}^{i'} E_k\Big) \simeq \sum_{k=1}^{i'} E_k \quad \text{and} \quad F = \pi_{L'}\Big(\sum_{k=i'+1}^{l} E_k\Big),$$

  in particular F contains the projection of the sum of the E_k for $k \geq i$; or

- $\dim F \le h_j$, *in which case there is* $j < j' \le l$ *such that, for any* $L' \in \overline{O(L)}$,

$$F = \pi_{L'}\Big(\sum_{k=j'}^{l} E_k\Big) \simeq \sum_{k=j'}^{l} E_k \quad \textit{and} \quad E = \pi_{L'}\Big(\sum_{k=1}^{j'-1} E_k\Big),$$

in particular E *contains the projection of the sum of the* E_k *for* $k \le j$.

Proof. First note that, as L is contained in $E_i \oplus E_j$ but not in E_i or E_j, the ω-limit of L for $\phi_{\mathbb{P}}$ is contained in $\mathbb{P}E_j$ and its α-limit is contained in $\mathbb{P}E_i$.

Towards a contradiction, assume that there is a dominated splitting $E \oplus F$ over $\overline{O(L)}$ such that $\dim E > k_i$ and $\dim F > h_j$.

According to Lemma 48, for any $L_\omega \in \overline{O(L)} \cap \mathbb{P}E_j$, the finest dominated splitting of $\overline{O(L_\omega)}$ is obtained from $E_1 \oplus \cdots \oplus E_i \oplus \cdots \oplus E_j \oplus \cdots \oplus E_l$ by replacing the space E_j by its projection and keeping all the others unchanged (modulo their identification with their projections). Thus the splitting $E \oplus F$ is just given by the dimension. So there is $i \le r < j$ such that for any such L_ω one has

$$E(L_\omega) = E_1 \oplus \cdots \oplus E_r \quad \text{and} \quad F(L_\omega) = E_{r+1} \oplus \cdots \oplus E_{j-1} \oplus \pi_{L_\omega}(E_j) \oplus \cdots \oplus E_l.$$

The same argument shows that there is $i < s \le j$ such that for any L_α in $\mathbb{P}E_i \cap \overline{O(L)}$ (α-limit of L) one has

$$E(L_\alpha) = E_1 \oplus \cdots \oplus E_{i-1} \oplus \pi_{L_\alpha}(E_i) \oplus \cdots \oplus E_s \quad \text{and} \quad F(L_\alpha) = E_{s+1} \oplus \cdots \oplus E_l.$$

This allows us to find the spaces $E(L)$ and $F(L)$. For that, consider an unstable cone around the space $F(L_\omega)$ and extend it by continuity to a small neighborhood of the ω-limit of L. Then $E(L)$ is exactly the set of vectors which do not enter the unstable cone for large positive iterates of the extended linear Poincaré flow. One deduces that

$$E(L) = E_1 \oplus \cdots \oplus E_{i-1} \oplus \pi_L(E_i \oplus L) \oplus E_{i+1} \oplus \cdots \oplus E_r.$$

In the same way, $F(L)$ consists of the vectors which do not enter the stable cone defined on the α-limit set of L under large negative iterates of the extended linear Poincaré flow. One deduces that

$$F(L) = E_s \oplus \cdots \oplus E_{j-1} \oplus \pi_L(E_j \oplus L) \oplus E_{j+1} \oplus \cdots \oplus E_l.$$

Consider the positive iterates $\psi^t_{\mathcal{N}}(F(L)) = F(\phi^t_{\mathbb{P}}(L))$ of $F(L)$. Denote $L_t = \phi^t_{\mathbb{P}}(L)$. Then $F(L_t)$ contains $\pi_{L_t}(E_j \oplus L_t)$, which has the same dimension as E_j. Recall that, by hypothesis, L_t is contained in $E_i \oplus E_j$. Thus $\pi_{L_t}(E_j \oplus L_t)$ converges in $\mathcal{N}(L_\omega)$ to some subspace of $\pi_{L_\omega}(E_i \oplus E_j) \simeq E_i \oplus \pi_{L_\omega}(E_j)$ containing $\pi_{L_\omega}(E_j)$ and having the same dimension as E_j. This implies that the limit of $\pi_{L_t}(E_j \oplus L_t)$ as $t \to +\infty$ contains vectors in E_i but is contained in $F(L_\omega)$. This contradicts the fact that $E_i \subset E(L_\omega)$.

This contradiction implies that $\dim E \le k_i$ or $\dim F \le h_j$. We now conclude the proof in the first case, the other case being similar.

Assume $\dim E \le k_i$. Then looking at the finest dominated splitting at L_ω one deduces that there is $1 \le i' < i$ such that $\dim E = \sum_{k=1}^{i'} \dim E_k$. Then the splitting $\mathcal{N}(L') = \tilde{E}(L') \oplus \tilde{F}(L')$ defined as

$$\tilde{E}(L') = \pi_{L'}\Big(\sum_{k=1}^{i'} E_k\Big) \simeq \sum_{k=1}^{i'} E_k \quad \text{and} \quad \tilde{F}(L') = \pi_{L'}\Big(\sum_{k=i'+1}^{l} E_k\Big)$$

is invariant, has constant dimension for $L' \in \overline{O(L)}$ and coincides with a dominated splitting over the ω-limit set and over the α-limit set. Therefore it is a dominated splitting so that $\dim \tilde{E} = \dim E$ and so $\tilde{E} = E$ and $\tilde{F} = F$, concluding the proof. □

7.2. *Relating the central space of the singularities with the dominated splitting on* $\widetilde{\Lambda}$

Now let us go back to our dynamical context. Let X be a vector field with a chain recurrence class C and a singularity $\sigma \in C$. We consider the following splitting of the tangent space M:

$$E^{ss} \oplus E^c \oplus E^{uu},$$

into the stable escaping space, the central space and the unstable escaping space. We suppose that the singularities are hyperbolic and that the dimension of the central space is locally constant. These are open and dense conditions. Let us consider the hyperbolic eigenvalues of the hyperbolic splitting restricted to the central space:

$$\lambda_1 < \cdots < \lambda_l$$

and the associated spaces:

$$E^c = E_1 \oplus \cdots \oplus E_l.$$

Note that from Remark 42 we know that $\widetilde{\Lambda} \subset B(X, U)$, and from Theorem 6 we have $\Lambda_{\mathbb{P}}(X, U) \subset \widetilde{\Lambda}$.

Remark 50. By Lemma 39 there is a C^1-open and dense set such that the dimension of the central space is locally constant. By definition of central space there is always a direction L_1 in $\widetilde{\Lambda} \cap \mathbb{P}_\sigma M$ such that $L_1 = \langle u \rangle$ where u belongs to E_1 and L_l in $\widetilde{\Lambda} \cap \mathbb{P}_\sigma M$, such that $L_l = \langle v \rangle$ where v belongs to E_l.

Lemma 51. *Consider a vector field X such that:*

- *There is a hyperbolic singularity σ and the splitting of the tangent space of the singularity into escaping spaces and central space is*

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu}.$$

- *The central space splits as $E^c = E_1 \oplus \cdots \oplus E_l$.*
- *The chain recurrence class of σ, $C(\sigma)$, is not trivial.*
- *The dimension of E^c is locally constant (i.e. the dimension of $E^c(\sigma, Y)$ is constant for Y in a C^1-open neighborhood of X).*

Then for any C^1-neighborhood $\mathcal{U}$ of X, there is Y in $\mathcal{U}$ such that there is a homoclinic orbit $\gamma \subset C(\sigma)$ that approaches the singularity tangent to the E_1 direction for the future and tangent to the E_l direction for the past.

Proof. Let us consider the finest hyperbolic decomposition of the central space of σ for this vector field:

$$E^c = E_1 \oplus \cdots \oplus E_l.$$

By definition, there is an orbit in the stable manifold tangent to $E^{ss} \oplus E_1$ that is contained in $C(\sigma)$ and there is an orbit in the unstable manifold tangent to $E_l \oplus E^{uu}$ that is contained in $C(\sigma)$. In the open set around X such that the dimension of the central space is constant, we choose Y such that all periodic orbits of Y are hyperbolic, and the orbit in the stable manifold tangent to $E^{ss} \oplus E_1$ approaches the singularity in the direction of E_1, while the orbit in the unstable manifold tangent to $E_l \oplus E^{uu}$ approaches the singularity in the direction of E_l. By Theorem 21 we can get another vector field Y_1 arbitrarily close to Y that has a homoclinic orbit Γ that approaches the singularity in the direction of E_1 for the future and in the direction of E_l for the past (observe that for Y_1 the dimension of the central space is the same as for X). □

Corollary 52. *Consider a vector field X such that:*

- *There is a singularity σ and the tangent space of the singularity splits into escaping spaces and central space as follows:*

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu}.$$

- *The central space splits as $E^c = E_1 \oplus \cdots \oplus E_l$.*
- *The chain recurrence class $C(\sigma)$ is not trivial.*
- *The dimension of E^c is locally constant (i.e. the dimension of $E^c(\sigma, Y)$ is constant for Y in a C^1-open neighborhood of X).*

Then:

- *There is $L_1 \in \widetilde{\Lambda}(C(\sigma)) \cap \pi_{\mathbb{P}}(E_1)$ and $L_l \in \widetilde{\Lambda}(C(\sigma)) \cap \pi_{\mathbb{P}}(E_l)$.*
- *There is $L \in \widetilde{\Lambda}(C(\sigma))$ such that $L = \langle u \rangle$ where u is a vector in $(E_1 \oplus E_l) \setminus (E_1 \cup E_l)$.*

Proof. The first item is a direct consequence of Lemma 51.

For the second item, from Lemma 51, we can find a vector field Y having a homoclinic orbit γ that approaches the singularity σ tangent to L_1 and it approaches the singularity for the past, tangent to a direction L_l in E_l. We may assume that Y is linearizable in a neighborhood of σ. We now consider a linearized neighborhood of the singularity that we call U_σ, and choose two regular points x, y such that $x \in W^s_{\text{loc}}(\sigma) \cap \gamma$ and $y \in W^u_{\text{loc}}(\sigma) \cap \gamma$. Then we can choose $x_n \to x$ and $y_n \to y$ such that $\phi_{t_n}(x_n) = y_n$ and $\{\phi_t(x_n) : 0 \le t \le t_n\}$ is tangent to $E_1 \oplus E_l$; note that for n large enough we can suppose that the segment of orbit from x_n to y_n is in U_σ and in the linearized neighborhood, so actually if $L_n = \langle Y(x_n) \rangle$ then

$$\{\phi^t_{\mathbb{P}}(L_n) : 0 \le t \le t_n\} \subset \pi_{\mathbb{P}}(E_1 \oplus E_l \setminus (E_1 \cup E_l)).$$

We now perturb our vector field Y to a new vector field $X_n \to Y$ so that there is a closed orbit γ_n formed by the segment of orbit between x_n and y_n in U_σ, and the segment of orbit of γ outside U_σ (see Figure 5).

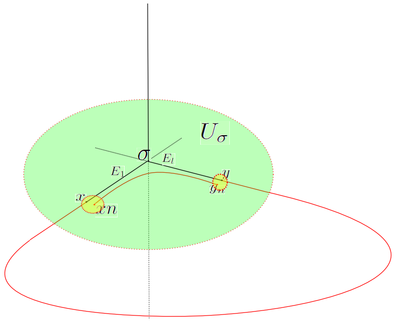


Fig. 5. Perturbation to get γ_n.

We can now find a p_n in γ_n satisfying $p_n \to \sigma$ and if $L_{p_n} = \langle X_n(p_n) \rangle$, then the upper limit of L_{p_n} is a subset of $\widetilde{\Lambda}(C(\sigma))$ (i.e. all limit points of L_{p_n} are in $\widetilde{\Lambda}(C(\sigma))$). Taking a subsequence if necessary, we may assume that $L_{p_n} \to L$ where $L = \langle u \rangle$ and $u \in (E_1 \oplus E_l) \setminus (E_1 \cup E_l)$. □

In this section we suppose that the extended linear Poincaré flow over $\widetilde{\Lambda}(C(\sigma))$ has a dominated splitting,

$$\mathcal{N}_L = \mathcal{N}^E \oplus \mathcal{N}^F,$$

where L is a direction in $\widetilde{\Lambda}(C(\sigma))$.

We denote by $\pi_L : T_x M \to \mathcal{N}_L$ where $L \in \mathbb{P}_x M$ the projection onto the normal space at a given direction L.

Lemma 53. *Let X be a vector field having a singular chain recurrence class $C(\sigma)$. Denote $S = \mathrm{Sing}(X) \cap C(\sigma)$ and suppose that:*

- *Every $\sigma \in S$ is hyperbolic.*
- *The dimension of the central space of $\sigma \in S$ is locally constant.*
- *The extended linear Poincaré flow over $\widetilde{\Lambda}(C(\sigma))$ has a dominated splitting,*

$$\mathcal{N}_L = \mathcal{N}^E \oplus \mathcal{N}^F,$$

  where L is any direction in $\widetilde{\Lambda}(C(\sigma))$.

Let L be a direction in $\widetilde{\Lambda} \cap \mathbb{P}_\sigma M$. Then either

$$\pi_L(E^c_\sigma) \subset \mathcal{N}^E_L \quad \text{or} \quad \pi_L(E^c_\sigma) \subset \mathcal{N}^F_L.$$

Proof. Since σ is hyperbolic we can suppose that the tangent space at σ splits as

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu},$$

into the escaping spaces and the central space. We also consider the finest hyperbolic splitting $E^c = E_1 \oplus \cdots \oplus E_l$.

Suppose that $\dim \mathcal{N}_L^E = n$. If $\dim E^{ss} \geq n$, then since $\dim E^{ss} = \dim \pi_L(E^{ss})$ we have

$$\mathcal{N}_L^E \subset \pi_L(E^{ss}).$$

This implies that $\pi_L(E_\sigma^c) \subset \mathcal{N}_L^F$.

Suppose that $\dim \mathcal{N}_L^F = m$. If $\dim E^{uu} \geq m$, then since $\dim E^{uu} = \dim \pi_L(E^{uu})$ we have

$$\mathcal{N}_L^F \subset \pi_L(E^{uu}).$$

This implies that $\pi_L(E_\sigma^c) \subset \mathcal{N}_L^E$.

Suppose now that

$$\dim E^{ss} < \dim \mathcal{N}_L^E. \tag{2}$$

From Corollary 52, for every L we have $\pi_L(E_1) \subset \mathcal{N}_L^E$. Since the singularity is not isolated, E_1 is a contracting space and E_l is expanding.

Suppose for contradiction that there exists a direction $L_u = \langle u \rangle$ such that $\mathcal{N}_{L_u}$ contains some v such that $\langle v \rangle = L_v \in \pi_{L_u}(E_1 \oplus \cdots \oplus E_l)$ with $v \notin \mathcal{N}_{L_u}^E$.

We can assume without loss of generality that $v \in E_l$ and $u \in E_1$. Then Lemma 48 gives $\mathcal{N}_{L_u}^E \cap E_l = \emptyset$. This implies that $\dim \mathcal{N}_L^E < \dim \pi_L(E^{ss} \oplus E^c)$ for any L in $\widetilde{\Lambda}(C(\sigma))$, and therefore

$$\dim \mathcal{N}_L^F > \dim \pi_L(E^{uu}) \tag{3}$$

for any L in $\widetilde{\Lambda}(C(\sigma))$.

On the other hand, we are under the hypotheses of Corollary 52. Thus there is $L \in \widetilde{\Lambda}(C(\sigma))$ such that $L = \langle w \rangle$ where w $(E_1 \oplus E_l) \setminus (E_1 \cup E_l)$. Taking E_1 for E_i and E_l for E_j we see from Lemma 49 that either

$$\dim E^{ss} \geq \dim \mathcal{N}_L^E \quad \text{or} \quad \dim \mathcal{N}_L^F \leq \dim \pi_L(E^{uu}),$$

contradicting (2) and (3). This allows us to conclude the proof. We can do this for all singularities in S. ◻

Corollary 54. *Let X be a vector field having a singular chain recurrence class $C(\sigma)$. Let $S = \operatorname{Sing}(X) \cap C(\sigma)$, and suppose that:*

- *All $\sigma \in S$ are hyperbolic.*
- *The dimension of the central space of any singularity σ is locally constant.*

Then the extended linear Poincaré flow has a dominated splitting over $\widetilde{\Lambda}(C(\sigma))$ if and only if it has a dominated splitting over $B(C(\sigma))$ of the same dimension.

Proof. Suppose that $B(C(\sigma))$ has a dominated splitting. Then $\widetilde{\Lambda}(C(\sigma))$ has a dominated splitting of the same dimension, since it is a compact invariant subset.

Conversely, suppose that there is a dominated splitting of the normal bundle in $\widetilde{\Lambda}(C(\sigma))$,

$$\mathcal{N}_L = \mathcal{N}^E \oplus \mathcal{N}^F.$$

Then, according to the previous lemma, we have two possibilities:

$$\pi_L(E_\sigma^c) \subset \mathcal{N}_L^E \quad \text{or} \quad \pi_L(E_\sigma^c) \subset \mathcal{N}_L^F.$$

The tangent space at σ splits as

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu},$$

into the escaping spaces and the central space. We also consider the finest hyperbolic splittings

$$E^c = E_1 \oplus \cdots \oplus E_l, \quad E^{ss} = E_{s1} \oplus \cdots \oplus E_{sk}, \quad E^{uu} = E_{u1} \oplus \cdots \oplus E_{ur}.$$

So if $\pi_L(E^c_\sigma) \subset \mathcal{N}^E_L$, Lemma 48 implies that there exists i such that

$$\mathcal{N}^F_L = E_{ui} \oplus \cdots \oplus E_{ur}, \quad \mathcal{N}^E_L = E_{ss} \oplus \pi_L(E^c) \oplus E_{u1} \oplus \cdots \oplus E_{ui-1}.$$

The same dominated splitting can be defined for any $L \in B(C(\sigma))$. The other case is analogous.

We can do the same for every singularity in the class. □

Lemma 55. *Consider a chain recurrence class $C(\sigma)$ of a singularity σ where*

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu}.$$

Consider the finest Lyapunov splitting $E^c = E_1 \oplus \cdots \oplus E_l$. If the dimension of the central space is locally constant then

$$\pi_{\mathbb{P}}(E_i) \cap \widetilde{\Lambda}(C(\sigma)) \neq \emptyset$$

for all the Lyapunov spaces E_i of the hyperbolic splitting. Moreover if all the spaces E_i are one- or two-dimensional, then

$$\pi_{\mathbb{P}}(E_i) \subset \widetilde{\Lambda}(C(\sigma))$$

for all $1 \le i \le l$.

Proof. Let us consider $\pi_{\mathbb{P}}(E_1)$ in $\mathbb{P}^c_\sigma$. By definition of central space, there is an orbit γ_1 tangent to $E^{ss} \oplus E_1$ that is not tangent to E^{ss}. This implies that $\pi_{\mathbb{P}}(E_1) \cap \widetilde{\Lambda}(C(\sigma)) \neq \emptyset$.

Consider a small filtrating neighborhood U of $C(\sigma)$. First we perturb X to a vector field Y' that is Kupka–Smale. We can make the perturbation small enough so that the vector field Y' satisfies the hypotheses of the lemma as well, since our assumptions are robust.

By Lemma 21 we perturb Y' to Y so that γ_1 is a homoclinic connection of the singularity and without changing the property that γ_1 becomes tangent to $E^{ss} \oplus E_1$ as it approaches the singularity. Now we perturb Y to Y_1 breaking the homoclinic connection in the direction of E_2 so that it is no longer tangent to $E^{ss} \oplus E_1$ but it is tangent to $E^{ss} \oplus E_1 \oplus E_2$. The domination implies that the orbit will become tangent to E_2 as it approaches σ. We can do this perturbation so that γ_1 remains the same outside the linear neighborhood of the singularity and such that the α-limit also remains the same (σ). Therefore, γ_1 still belongs to $C(\sigma)_{Y_1}$. Thus there is a direction $L_2 \in E_2$ such that $L_2 \in \Lambda_{\mathbb{P},U}(Y_1)$ for any U. We can continue this process for all $1 \le i \le l$.

We conclude that in any small enough C^1-neighborhood of X there are vector fields Y_{i-1} such that

$$\Lambda_{\mathbb{P},U}(Y_{i-1}) \cap \pi_{\mathbb{P}}(E_i) \neq \emptyset.$$

Since the C^1-neighborhood of X can be taken arbitrarily small, we have

$$\widetilde{\Lambda}(X, U) \cap E_i \neq \emptyset.$$

Since this is true for any small enough filtrating neighborhood, it follows that

$$\widetilde{\Lambda}(C(\sigma)) \cap E_i \neq \emptyset.$$

If the central space splits into only one- or two-dimensional spaces, let us take $L \in \pi_{\mathbb{P}}(E_i) \cap \widetilde{\Lambda}$ where E_i is two-dimensional with complex Lyapunov exponents. Since $\widetilde{\Lambda}(C(\sigma))$ is invariant, the orbit of L under $\phi^t_{\mathbb{P}}$, denoted $O(L)$, is such that $O(L) \subset \widetilde{\Lambda}(C(\sigma))$. Since E_i has complex Lyapunov exponents, the direction L is not invariant and $O(L)$ covers all directions of E_i and therefore $\pi_{\mathbb{P}}(E_i) \subset \Lambda_{\mathbb{P}}(X, U)$. □

The next corollary implies Theorem 2:

Corollary 56. *Let X be a vector field having a singular chain recurrence class $C(\sigma)$. Let $S = \mathrm{Sing}(X) \cap C(\sigma)$ and suppose that:*

- *Every $\sigma \in S$ is hyperbolic.*
- *The dimension of the central space of $\sigma \in S$ is locally constant, and the finest Lyapunov splitting is into one- or two-dimensional spaces.*

If $\widetilde{\Lambda}(C(\sigma))$ has a hyperbolic structure on the normal bundle $\mathcal{N}_L = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_i \oplus \cdots \oplus \mathcal{N}_r$ for the extended linear Poincaré flow, then $B(C(\sigma))$ has the same hyperbolic structure.

Proof. From Corollary 54, the dominated splitting in $\widetilde{\Lambda}$ extends to $B(X, U)$. So let us consider the space $\mathcal{N}_i$ and the number d_i such that

$$J(h^t_i \cdot (\psi^t_{\mathcal{N}}|_{D_i}))$$

is a uniform contraction or expansion for any subspace $D_i \subset \mathcal{N}_i$ of dimension d_i over the orbits in $\widetilde{\Lambda}$. We suppose, without loss of generality, that it is a contraction.

Since $B(C(\sigma))$ and $\widetilde{\Lambda}$ coincide on the directions that are not over the singularities, we only need to check that for every σ and every orbit of an $L \in B(C(\sigma)) \cap \mathbb{P}_\sigma M$ the Jacobian $J(h^t_i \cdot (\psi^t_{\mathcal{N}}|_{D_i}))$ contracts uniformly for any subspace $D_i \subset E_i$ of dimension d_i.

The tangent space of σ splits as

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu},$$

into the escaping spaces and the central space. We also consider the finest Lyapunov splitting $E^c = E_1 \oplus \cdots \oplus E_l$. From Lemma 55 we know that $\pi_{\mathbb{P}}(E_i) \subset \widetilde{\Lambda}$ for every $i \in \{1, \ldots, l\}$.

Then we consider $L \in B(X, U)$ and a vector u in the direction of L. In coordinates of the central space, $u = (u_1, \ldots, u_i, \ldots, u_j, \ldots, u_l)$. We suppose that u_h is the first non-zero coordinate of u and u_j is the last. Domination implies that for t sufficiently negatively large, $\phi^t_{\mathbb{P}}(L)$ is in a small cone around $\pi_{\mathbb{P}}(E_h)$ and remains there thereafter. For the future, $\phi^t_{\mathbb{P}}(L)$ is in a small cone around $\pi_{\mathbb{P}}(E_j)$ and remains there thereafter. Since the contraction and expansion rates extend to the cones around $\pi_{\mathbb{P}}(E_j)$ and $\pi_{\mathbb{P}}(E_h)$, and the orbit is outside these cones only for a finite time, we get our conclusion.

We are now ready to define our notion of multisingular hyperbolicity.

8. Multisingular hyperbolicity

8.1. Definition of multisingular hyperbolicity

Definition 57. Let X be a C^1 vector field on a compact manifold and let U be a compact region. One says that X is *multisingular hyperbolic* on U if:

(1) Every singularity of X in U is hyperbolic. We denote $S = \mathrm{Sing}(X) \cap U$.
(2) The restriction of the extended linear Poincaré flow $\{\psi^t_{\mathcal{N}}\}$ to the extended maximal invariant set $B(X, U)$ admits a dominated splitting $\mathcal{N}_L = E_L \oplus F_L$.
(3) There is a subset $S_E \subset S$ such that the reparametrized cocycle $h^t_E \psi^t_{\mathcal{N}}$ is uniformly contracted when restricted to the bundles E over $B(X, U)$, where $h_E = \prod_{\sigma \in S_E} h_\sigma$. (If S_E is empty, one may assume that $h_E = 1$.)
(4) There is a subset $S_F \subset S$ such that the reparametrized cocycle $h^t_F \psi^t_{\mathcal{N}}$ is uniformly expanded in restriction to the bundles F over $B(X, U)$, where $h_F = \prod_{\sigma \in S_F} h_\sigma$. (If S_F is empty, one may assume that $h_F = 1$.)

Remark 58. The subsets S_E and S_F are not necessarily uniquely defined, leading to several notions of multisingular hyperbolicity. We can also slightly modify this definition allowing the product of powers of the h_σ. In that case $\tilde{h}_E$ would be of the form

$$h^t_E = \prod_{\sigma \in S_E} (h^t_\sigma)^{\alpha_E(\sigma)}$$

for some $\alpha_E(\sigma) \in \mathbb{R}$.

Theorem 7. *Let X be a C^1 vector field on a compact manifold M and let $U \subset M$ be a compact region. Assume that X is multisingular hyperbolic on U. Then X is a star flow on U, that is, there is a C^1-neighborhood $\mathcal{U}$ of X such that every periodic orbit contained in U of a vector field $Y \in \mathcal{U}$ is hyperbolic. Furthermore every $Y \in \mathcal{U}$ is multisingular hyperbolic in U.*

Proof. Recall that the extended maximal invariant set $B(Y, U)$ varies upper semicontinuously with Y for the C^1-topology. Therefore, according to Proposition 24 there is a C^1-neighborhood $\mathcal{U}_0$ of X such that, for every $Y \in \mathcal{U}_0$, the extended linear Poincaré flow $\psi^t_{\mathcal{N},Y}$ admits a dominated splitting $E \oplus_{\prec} F$ over $B(Y, U)$, whose dimensions are independent of Y and whose bundles vary continuously with Y.

Let S_E and S_F be the sets of singular points of X in the definition of singular hyperbolicity. Lemma 46 allows us to choose two families of cocycles $h^t_{E,Y}$ and $h^t_{F,Y}$ depending continuously on Y in a small neighborhood $\mathcal{U}_1$ of X and which belong to the product of the cohomology classes of cocycles associated to the singularities in S_E and S_F, respectively. Thus the linear cocycle

$$h^t_{E,Y} \cdot \psi^t_{\mathcal{N},Y}|_{E,Y} \quad \text{over } B(Y, U)$$

varies continuously with Y in $\mathcal{U}_1$, and is uniformly contracted for X. Thus, it is uniformly contracted for Y in a C^1-neighborhood of X.

One shows in the same way that

$$h^t_{F,Y} \cdot \psi^t_{\mathcal{N},Y}|_{F,Y} \quad \text{over } B(Y, U)$$

is uniformly expanded for Y in a small neighborhood of X.

We have just proved that there is a neighborhood $\mathcal{U}$ of X such that $Y \in \mathcal{U}$ is multisingular hyperbolic in U.

Consider a (regular) periodic orbit γ of Y and let π be its period. Just by construction of the cocycles h_E and h_F, one can check that

$$h^\pi_E(\gamma(0)) = h^\pi_F(\gamma(0)) = 1.$$

One deduces that the linear Poincaré flow is uniformly hyperbolic along γ so that γ is hyperbolic, ending the proof. □

8.2. Multisingular hyperbolic structures over a singular point

The aim of this subsection is to prove

Proposition 59. *Let X be a C^1 vector field on a compact manifold and $U \subset M$ a compact region. Assume that X is multisingular hyperbolic in U and let i denote the dimension of the stable bundle of the reparametrized extended linear Poincaré flow. Let σ be a singularity of X. Then either*

- *at least one entire invariant (stable or unstable) manifold of σ is escaping from U, or*
- *σ is Lorenz-like. More precisely, either*
  - *the stable index is $i+1$, the central space $E^c_{\sigma,U}$ contains exactly one stable direction E^s_1 ($\dim E^s_1 = 1$), and $E^s_1 \oplus E^u(\sigma)$ is sectionally dissipative, in which case $\sigma \in S_F$; or*
  - *the stable index is i, the central space $E^c_{\sigma,U}$ contains exactly one unstable direction E^u_1 ($\dim E^u_1 = 1$), and $E^s(\sigma) \oplus E^u_1$ is sectionally contracting, in which case $\sigma \in S_E$.*

Note that in the first case of this proposition the class of the singularity in U must be trivial. If it were not, the regular orbits of the class that accumulate on σ, entering U, would accumulate on an orbit of the stable manifold. Therefore the stable manifold could not be completely escaping. The same reasoning holds for the unstable manifold.

Let $E^s_k \oplus_\prec \cdots \oplus_\prec E^s_1 \oplus_\prec E^u_1 \oplus_\prec \cdots \oplus_\prec E^u_\ell$ be the finest dominated splitting of the flow over σ. For the proof, we will assume in the rest of the section that the class of σ is not trivial, and therefore we are not in the first case of the proposition. In other words, we assume that there are $a, b > 0$ such that

$$E^c_{\sigma,U} = E^s_a \oplus_\prec \cdots \oplus_\prec E^s_1 \oplus_\prec E^u_1 \oplus_\prec \cdots \oplus_\prec E^u_b.$$

We assume that X is multisingular hyperbolic of s-index i and we denote by $E \oplus_\prec F$ the corresponding dominated splitting of the extended linear Poincaré flow over $B(X, U)$.

The following lemma is a direct consequence of Lemma 53:

Lemma 60. *Let X be a C^1 vector field on a compact manifold and $U \subset M$ a compact region. Assume that X is multisingular hyperbolic in U and let i denote the dimension of the stable bundle of the reparametrized extended linear Poincaré flow. Let σ be a singularity of X. Then with the notation above, either*

- $i = \dim E \leq \dim E^s_k \oplus \cdots \oplus E^s_{a+1}$ (*i.e. the dimension of E is smaller than or equal to the dimension of the biggest stable escaping space*); *or*
- $\dim M - i - 1 = \dim F \leq \dim E^u_\ell \oplus \cdots \oplus E^u_{b+1}$ (*i.e. the dimension of F is smaller than or equal to the dimension of the biggest unstable escaping space*).

According to Lemma 60 we now assume that $i \leq \dim E^s_k \oplus \cdots \oplus E^s_{a+1}$ (the other case is analogous, with X replaced by $-X$).

Lemma 61. *With the hypotheses above, for every $L \in \mathbb{P}^c_{\sigma,U}$ the projection of $E^c_{\sigma,U}$ on the normal space $\mathcal{N}_L$ is contained in $F(L)$.*

Proof. This is because the projection of $E^s_k \oplus \cdots \oplus E^s_{a+1}$ has dimension at least the dimension i of E and hence contains $E(L)$. Thus the projection of $E^c_{\sigma,U}$ is transverse to E. As the projection of $E^s_{\sigma,U}$ on $\mathcal{N}_L$ defines a $\psi^t_\mathcal{T}$-invariant bundle over the $\phi^t_\mathcal{T}$-invariant compact set $\mathbb{P}^c_{\sigma,U}$, one concludes that the projection is contained in F. □

As a consequence, the bundle F is not uniformly expanded on $\mathbb{P}^c_{\sigma,U}$ for the extended linear Poincaré flow. As it is expanded by the reparametrized flow, this implies $\sigma \in S_F$.

Consider now $L \in E^s_a$. Then $\psi^t_\mathcal{N}$ restricted to the projection of $E^c_{\sigma,U}$ on $\mathcal{N}_L$ consists in multiplying the natural action of the derivative by the exponential contraction along L. As it is included in F, multisingular hyperbolicity implies that it is a uniform expansion; this means that:

- L is the unique contracting direction in $E^s_{\sigma,U}$; in other words, $a = 1$ and $\dim E^s - \dim E^s_a = 1$.
- Contraction along E^s_a is less than expansion in E^u_j, $j > 1$; in other words $E^c_{\sigma,U}$ is sectionally expanding.

To finish the proof of Proposition 59, it remains to check the s-index of σ: for $L \in E^s_a$ one finds that $F(L)$ is isomorphic to $E^u_1 \oplus \cdots \oplus E^u_\ell$ so that the s-index of σ is $i + 1$, ending the proof.

9. Multisingular hyperbolicity is a necessary condition for star flows: Proof of Theorem 4

The aim of this section is to prove

Lemma 62. *Let X be a generic star vector field on M. Consider a chain recurrence class C of X. Then there is a filtrating neighborhood U of C such that the extended maximal invariant set $B(X, U)$ is multisingular hyperbolic.*

Notice that, as the multisingluar hyperbolicity of $B(X, U)$ is a robust property, Lemma 62 implies Theorem 3.

As already mentioned, the proof of Lemma 62 consists essentially in recovering the results in [SGW] and adjusting a few of them to the new setting. So we start by recalling several of the results from or used in [SGW].

We begin by stating the following properties of star flows:

Lemma 63 ([L], [Ma2]). *For any star vector field X on a closed manifold M, there is a C^1-neighborhood $\mathcal{U}$ of X and numbers $\eta > 0$ and $T > 0$ such that, for any periodic orbit γ of a vector field $Y \in \mathcal{U}$ and any integer $m > 0$ the following holds: Let $N = N_s \oplus N_u$ be the stable-unstable splitting of the normal bundle N for the linear Poincaré flow ψ_t^Y. Then:*

- Domination: *For every $x \in \gamma$ and $t \geq T$, one has*

$$\frac{\|\psi_t^Y|_{N_s}\|}{\min(\psi_t^Y|_{N_u})} \leq e^{-2\eta t}.$$

- Uniform hyperbolicity at the period: *If the period $\pi(\gamma)$ is larger than T then, for every $x \in \gamma$, one has:*

$$\prod_{i=0}^{[m\pi(\gamma)/T]-1} \|\psi_t^Y|_{N_s}(\phi_{iT}^Y(x))\| \leq e^{-m\eta\pi(\gamma)},$$

$$\prod_{i=0}^{[m\pi(\gamma)/T]-1} \min(\psi_t^Y|_{N_u}(\phi_{iT}^Y(x))) \geq e^{m\eta\pi(\gamma)}.$$

Here $\min(A)$ *is the mini-norm of A, i.e.,* $\min(A) = \|A^{-1}\|^{-1}$.

Now we need some generic properties for flows:

Lemma 64 ([C], [BGY]). *There is a C^1-dense G_δ subset $\mathcal{G}$ of the C^1-open set of star flows of M such that, for every $X \in \mathcal{G}$, one has:*

- *Every critical element (zero or periodic orbit) of X is hyperbolic and therefore admits a well defined continuation in a C^1-neighborhood of X.*
- *For every critical element p of X, the chain recurrence class $C(p)$ is continuous in X in the Hausdorff topology.*

- *If p and q are two critical elements of X such that $C(p) = C(q)$ then there is a C^1-neighborhood $\mathcal{U}$ of X such that the chain recurrence classes of p and q still coincide for every $Y \in \mathcal{U}$.*
- *For any non-trivial chain recurrence class C of X, there exists a sequence of periodic orbits Q_n such that Q_n tends to C in the Hausdorff topology.*

Lemma 65 ([SGW, Lemma 4.2]). *Let X be a star flow in M and $\sigma \in \mathrm{Sing}(X)$. Assume that the Lyapunov exponents of $\phi_t(\sigma)$ are*

$$\lambda_1 \leq \cdots \leq \lambda_{s-1} \leq \lambda_s < 0 < \lambda_{s+1} \leq \lambda_{s+1} \leq \cdots \leq \lambda_d.$$

If the chain recurrence class $C(\sigma)$ is non-trivial, then:

- *Either $\lambda_{s-1} \neq \lambda_s$ or $\lambda_{s+1} \neq \lambda_{s+2}$.*
- *If $\lambda_{s-1} = \lambda_s$, then $\lambda_s + \lambda_{s+1} < 0$.*
- *If $\lambda_{s+1} = \lambda_{s+2}$, $\lambda_s + \lambda_{s+1} > 0$.*
- *If $\lambda_{s-1} \neq \lambda_s$ and $\lambda_{s+1} \neq \lambda_{s+2}$, then $\lambda_s + \lambda_{s+1} \neq 0$.*

We say that a singularity σ satisfying the conditions of the previous lemma is *Lorenz-like* of index s, and we define the *saddle value* of σ as

$$\mathrm{sv}(\sigma) = \lambda_s + \lambda_{s+1}.$$

Consider a Lorenz-like singularity σ.

- If $\mathrm{sv}(\sigma) > 0$, we consider the splitting

  $$T_\sigma M = G^{ss}_\sigma \oplus G^{cu}_\sigma$$

  where (using the notations of Lemma 65) the space G^{ss}_σ corresponds to the Lyapunov exponents λ_1 to λ_{s-1}, and G^{cu}_σ corresponds to $\lambda_s, \ldots, \lambda_d$.
- If $\mathrm{sv}(\sigma) < 0$, we consider the splitting

  $$T_\sigma M = G^{cs}_\sigma \oplus G^{uu}_\sigma$$

  where the space G^{cs}_σ corresponds to λ_1 to λ_{s+1}, and G^{uu}_σ corresponds to the Lyapunov exponents $\lambda_{s+2}, \ldots, \lambda_d$.

Corollary 66. *Let X be a vector field, σ a Lorenz-like singularity of X, and $h_\sigma : \Lambda_X \times \mathbb{R} \to (0, +\infty)$ a cocycle in the cohomology class $[h_\sigma]$ defined in Section 6.*

(1) *If $\mathrm{Ind}(\sigma) = s + 1$ and $\mathrm{sv}(\sigma) > 0$, then the restriction of $\psi_{\mathcal{N}}$ to $\mathbb{P}G^{cu}_\sigma$ admits a dominated splitting $N_L = E_L \oplus F_L$ with $\dim E_L = s$ for $L \in \mathbb{P}G^{cu}_\sigma$. Furthermore:*
   - *E is uniformly contracting for $\psi_{\mathcal{N}}$.*
   - *F is uniformly expanding for the reparametrized extended linear Poincaré flow $h_\sigma \cdot \psi_{\mathcal{N}}$.*

(2) *If $\mathrm{Ind}(\sigma) = s$ and $\mathrm{sv}(\sigma) < 0$, then the restriction of $\psi_{\mathcal{N}}$ to $\mathbb{P}G^{su}_\sigma$ admits a dominated splitting $N_L = E_L \oplus F_L$ with $\dim E_L = s$ for $L \in \mathbb{P}G^{cu}_\sigma$. Furthermore:*
   - *F is uniformly expanding for $\psi_{\mathcal{N}}$.*
   - *E is uniformly contracting for the reparametrized extended linear Poincaré flow $h_\sigma \cdot \psi_{\mathcal{N}}$.*

Proof. We only consider the first case $\mathrm{Ind}(\sigma) = s + 1$ and $\mathrm{sv}(\sigma) > 0$; the other is analogous and can be deduced by reversing the time.

We consider the restriction of $\psi_{\mathcal{N}}$ to $\mathbb{P}G^{cu}_{\sigma}$, that is, for points $L \in \tilde{\Lambda}_X$ corresponding to lines contained in G^{cu}_{σ}. Therefore the normal space N_L can be identified, up to a projection which is uniformly bounded, to the direct sum of G^{ss}_{σ} with the normal space of L in G^{cu}_{σ}.

Now we fix $E_L = G^{ss}_{\sigma}$ and F_L is the normal space of L in G^{cu}_{σ}. As G^{ss}_{σ} and G^{cu}_{σ} are invariant under the derivative of the flow ϕ_t, one sees that the splitting $N_L = E_L \oplus F_L$ is invariant under the extended linear Poincaré flow over $\mathbb{P}G^{cu}_{\sigma}$. Let us first prove that this splitting is dominated:

By Lemma 65 if we choose a unit vector v in E_L we know that for any $t > 0$ one has

$$\|\psi^t_{\mathcal{N}}(v)\| \leq K e^{t\lambda_{s-1}}.$$

Now let us choose a unit vector u in F_L, and consider $w_t = \psi^t_{\mathcal{N}}(u) \in F_{\phi^t_{\mathbb{P}}(L)}$. Then for any $t > 0$, one has

$$\|D\phi^{-t}(w_t)\| \leq K' e^{t(-\lambda_s)} \|w_t\|.$$

The extended linear Poincaré flow is obtained by projecting the image by the derivative of the flow on the normal bundle. Since the projection on the normal space does not increase the norm of the vectors, one gets

$$\|\psi^{-t}_{\mathcal{N}}(w_t)\| \leq K' e^{t(-\lambda_s)} \|w_t\|,$$

in other words

$$\frac{1}{\|\psi^t_{\mathcal{N}}(u)\|} \leq K' e^{t(-\lambda_s)}.$$

Putting together these inequalities one gets

$$\frac{\|\psi^t_{\mathcal{N}}(v)\|}{\|\psi^t_{\mathcal{N}}(u)\|} \leq K K' e^{t(\lambda_{s-1}-\lambda_s)}.$$

This provides the domination as $\lambda_{s-1} - \lambda_s < 0$.

Notice that $E_L = G^{ss}_{\sigma}$ is uniformly contracted by the extended linear Poincaré flow $\psi_{\mathcal{N}}$, because it coincides, on G^{ss}_{σ} and for $L \in \mathbb{P}G^{cu}_{\sigma}$, with the differential of the flow ϕ^t. To finish the proof, it remains to show that the reparametrized extended linear Poincaré flow $h_{\sigma} \cdot \psi_{\mathcal{N}}$ expands uniformly the vectors in F_L for $L \in \mathbb{P}G^{cu}_{\sigma}$.

Notice that, over the whole projective space $\mathbb{P}_{\sigma}$, the cocycle $h_{\sigma,t}(L)$ is the rate of expansion of the derivative of ϕ_t in the direction of L. Therefore $h_{\sigma} \cdot \psi_{\mathcal{N}}$ is defined as follows: Consider a line $D \subset N_L$. Then the expansion rate of the restriction of $h_{\sigma} \cdot \psi_{\mathcal{N}}$ to D is the expansion rate of the area on the plane spanned by L and D by the derivative of ϕ_t.

The hypothesis $\lambda^s + \lambda^{s+1} > 0$ implies that the derivative of ϕ_t expands uniformly the area on the planes contained in G^{cu}_{σ}, concluding the proof. □

Lemma 67 ([SGW, Lemma 4.5 and Theorem 5.7]). *Let X be a C^1 generic star vector field and let $\sigma \in \mathrm{Sing}(X)$. Then there is a filtrating neighborhood U of $C(\sigma)$ such that, for any two periodic points $p, q \in U$,*

$$\mathrm{Ind}(p) = \mathrm{Ind}(q).$$

Furthermore, for any singularity σ' in U,

$$\mathrm{Ind}(\sigma') = \begin{cases} \mathrm{Ind}(q) & \text{if } \mathrm{sv}(\sigma') < 0, \\ \mathrm{Ind}(q) + 1 & \text{if } \mathrm{sv}(\sigma') > 0. \end{cases}$$

Lemma 68. *There is a dense G_δ set $\mathcal{G}$ in the set of star flows on M with the following properties: Let X be in $\mathcal{G}$, and let C be a chain recurrence class of X. Then there is a (small) filtrating neighborhood U of C such that the lifted maximal invariant set $\widetilde{\Lambda}(X, U)$ of X in U has a dominated splitting $\mathcal{N} = E \oplus_{\prec} F$ for the extended linear Poincaré flow such that E extends the stable bundle for every periodic orbit γ contained in U.*

Proof. According to Lemma 67, the class C admits a filtrating neighborhood U in which the periodic orbits are hyperbolic and with the same index. On the other hand, according to Lemma 64, every chain recurrence class in U is accumulated by periodic orbits. Since X is a star flow, Lemma 63 asserts that the normal bundle over the union of these periodic orbits admits a dominated splitting for the linear Poincaré flow, corresponding to their stable/unstable splitting. It follows that the union of the corresponding orbits in the lifted maximal invariant set has a dominated splitting for $\mathcal{N}$. Since any dominated splitting defined on an invariant set extends to the closure of this set, we have a dominated splitting on the closure of the lifted periodic orbits, and hence on the whole $\widetilde{\Lambda}(X, U)$. □

Lemma 68 asserts that the lifted maximal invariant set $\tilde{\Lambda}(X, U)$ admits a dominated splitting. What we need now is to extend this dominated splitting to the extended maximal invariant set

$$B(X, U) = \tilde{\Lambda}(X, U) \cup \bigcup_{\sigma_i \in \mathrm{Sing}(X) \cap U} \mathbb{P}^c_{\sigma_i, U}.$$

We need the following theorem to have more information on the projective center spaces $\mathbb{P}^c_{\sigma_i, U}$.

Lemma 69 ([SGW, Lemma 4.7]). *Let X be a star flow in M and σ be a singularity of X such that $C(\sigma)$ is non-trivial.*

- *If $\mathrm{sv}(\sigma) > 0$, then*
$$W^{ss}(\sigma) \cap C(\sigma) = \{\sigma\},$$
where $W^{ss}(\sigma)$ is the strong stable manifold associated to the space G^{ss}_σ.
- *If $\mathrm{sv}(\sigma) < 0$, then*
$$W^{uu}(\sigma) \cap C(\sigma) = \{\sigma\},$$
where $W^{uu}(\sigma)$ is the strong unstable manifold associated to the space G^{uu}_σ.

Remark 70. Consider a vector field X and a hyperbolic singularity σ of X. Assume that $W^{ss}(\sigma) \cap C(\sigma) = \{\sigma\}$, where $C(\sigma)$ is the chain recurrence class of σ. Then there is a filtrating neighborhood U of $C(\sigma)$ on which the strong stable manifold $W^{ss}(\sigma)$ is escaping from U (see the definition in Section 5.2).

Proof. Each orbit in $W^{ss}(\sigma) \setminus \{\sigma\}$ goes out of some filtrating neighborhood of $C(\sigma)$ and the nearby orbits exit the same filtrating neighborhood. Notice that the space of orbits in $W^{ss}(\sigma) \setminus \{\sigma\}$ is compact, so that we can consider a finite cover of it by open sets for which the corresponding orbits exit the same filtrating neighborhood of $C(\sigma)$. The announced filtrating neighborhood is the intersection of these finitely many filtrating neighborhoods. □

Remark 70 allows us to consider the *escaping strong stable manifold* and the *strong unstable manifold* of a singularity σ without referring to a specific filtrating neighborhood U of the class $C(\sigma)$: these notions do not depend on U small enough. Hence the notion of the *center space* $E^c_\sigma = E^c(\sigma, U)$ is also independent of U for U small enough. Thus we will denote

$$\mathbb{P}^c_\sigma = \mathbb{P}^c_{\sigma,U}$$

for a sufficiently small neighborhood U of the chain recurrence class $C(\sigma)$.

Remark 71. Lemma 69 together with Remark 70 implies that:

- If $\mathrm{sv}(\sigma) > 0$, then $E^c_\sigma \subset G^{cu}$.
- If $\mathrm{sv}(\sigma) < 0$, then $E^c_\sigma \subset G^{cs}$.

Lemma 72. *Let X be a generic star vector field on M. Consider a chain recurrence class C of X. Then there is a neighborhood U of C such that the extended maximal invariant set $B(X, U)$ has a dominated splitting for the extended linear Poincaré flow,*

$$\mathcal{N}_{B(X,U)} = E \oplus_{\prec} F,$$

which extends the stable-unstable bundle defined on the lifted maximal invariant set $\widetilde{\Lambda}(X, U)$.

Proof. The case where C is not singular is already done. According to Lemma 67 there exist an integer s and a neighborhood U of C such that every periodic orbit in U has index s and every singular point σ in U is Lorenz-like, and either its index is s and $\mathrm{sv}(\sigma) < 0$, or σ has index $s + 1$ and $\mathrm{sv}(\sigma) > 0$.

According to Remark 71,

$$B(X, U) \subset \tilde{\Lambda}(X, U) \cup \bigcup_{\mathrm{sv}(\sigma_i)<0} \mathbb{P}G^{cs}_{\sigma_i} \cup \bigcup_{\mathrm{sv}(\sigma_i)>0} \mathbb{P}G^{cu}_{\sigma_i}$$

By Corollary 66 and Lemma 68 each of these sets admits a dominated splitting $E \oplus F$ for the extended linear Poincaré flow $\psi_{\mathcal{N}}$ with $\dim E = s$.

The uniqueness of dominated splittings for prescribed dimensions implies that these dominated splittings coincide on the intersections, concluding the proof. □

We already proved the existence of a dominated splitting $E \oplus F$, with $\dim E = s$, for the extended linear Poincaré flow over $B(X, U)$ for a small filtrating neighborhood of C, where s is the index of any periodic orbit in U. It remains to show that the extended linear Poincaré flow admits a reparametrization which contracts uniformly the bundle E and a reparametrization which expands the bundle F.

Lemma 65 divides the set of singularities into two kinds: those with positive saddle value and those with negative saddle value. We denote

$$S_E := \{x \in \mathrm{Sing}(X) \cap U : \mathrm{sv}(x) < 0\},$$
$$S_F := \{x \in \mathrm{Sing}(X) \cap U : \mathrm{sv}(x) > 0\}.$$

Recall that Section 6 associated to every singular point σ a cocycle $h_\sigma : \Lambda_X \times \mathbb{R} \to \mathbb{R}$, whose cohomology class is well defined. Define

$$h_E = \prod_{\sigma \in S_E} h_\sigma \quad \text{and} \quad h_F = \prod_{\sigma \in S_F} h_\sigma .$$

Now Lemma 62, and therefore Theorem 3, is a direct consequence of the next lemma:

Lemma 73. *Let X be a generic star vector field on M. Consider a chain recurrence class C of X. Then there is a neighborhood U of C such that the extended maximal invariant set $B(X, U)$ is such that the normal space has a dominated splitting $\mathcal{N}_{B(X,U)} = E \oplus_\prec F$ such that the space E (resp. F) is uniformly contracting (resp. expanding) for the reparametrized extended linear Poincaré flow $h_E^t \cdot \psi_\mathcal{N}^t$ (resp. $h_F^t \cdot \psi_\mathcal{N}^t$).*

The proof uses the following theorem by Gan, Shi and Wen, which describes the ergodic measures for a star flow. Given a C^1 vector field X, an ergodic measure μ for the flow ϕ^t is said to be *hyperbolic* if either μ is supported on a hyperbolic singularity, or μ has exactly one zero Lyapunov exponent, whose invariant subspace is spanned by X.

Theorem 8 ([SGW, Lemma 5.6]). *Let X be a star flow. Any invariant ergodic measure μ of the flow ϕ^t is a hyperbolic measure. Moreover, if μ is not the atomic measure on any singularity, then* $\mathrm{supp}(\mu) \cap H(P) \neq \emptyset$, *where P is a periodic orbit with the index of μ, i.e., the number of negative Lyapunov exponents of μ (with multiplicity), and $H(P)$ is the homoclinic class of P.*

Proof of Lemma 73. Towards a contradiction, assume that the bundle E is uniformly contracting for $h_E \cdot \psi_\mathcal{N}^t$ over $B(X, U)$ for no filtrating neighborhood U of the class C.

One deduces the following claim:

Claim. *Let $\tilde{C} \subset \tilde{\Lambda}(X)$ be the closure in $\mathbb{P}M$ of the lift of $C \setminus \mathrm{Sing}(X)$ and let $S = \mathrm{Sing}(X) \cap C$. Then, for every $T > 0$, there exists an ergodic invariant measure μ_T whose support is contained in*

$$\bigcup_{s \in S} \mathbb{P}_s^c \cup \tilde{C}$$

such that

$$\int \log \|h_E^T \cdot \psi_\mathcal{N}^T|_E\| \, d\mu(x) \geq 0.$$

Proof. For each U, there exists an ergodic measure μ_T whose support is contained in $B(X, U)$ such that

$$\int \log \|h_E^T \cdot \psi_{\mathcal{N}}^T|_E\| \, d\mu_T(x) \geq 0.$$

But a priori the class C need not be a maximal invariant set in U. We fix this by observing that

$$\bigcup_{s \in S} \mathbb{P}_s^c \cup \tilde{C} \subset B(X, U)$$

for any U as small as we want and actually we can choose a sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods such that $U_n \to C$ and therefore

$$\bigcup_{s \in S} \mathbb{P}_s^c \cup \tilde{C} = \bigcap_{n \in \mathbb{N}} B(X, U_n).$$

This defines a sequence $\mu_T^n \to \mu_T^0$ of measures such that

$$\int \log \|h_E^T \cdot \psi_{\mathcal{N}}^T|_E\| \, d\mu_T^n(x) \geq 0,$$

and with supports converging to $\bigcup_{s\in S} \mathbb{P}_s^c \cup \tilde{C}$. The resulting limit measure μ_T^0, whose support is contained in $\bigcup_{s\in S} \mathbb{P}_s^c \cup \tilde{C}$, might not be ergodic but it is invariant. We can decompose it as a sum of ergodic measures, and so if

$$\int \log \|h_E^T \cdot \psi_{\mathcal{N}}^T|_E\| \, d\mu_T^0(x) \geq 0,$$

there must exist an ergodic measure μ_T in the ergodic decomposition of μ_T^0 such that

$$\int \log \|h_E^T \cdot \psi_{\mathcal{N}}^T|_E\| \, d\mu_T(x) \geq 0,$$

and the support of μ_T is contained in $\bigcup_{s\in S} \mathbb{P}_s^c \cup \tilde{C}$. □

Recall that for generic star flows, every chain recurrence class in $B(X, U)$ is the Hausdorff limit of periodic orbits of the same index and which satisfy the conclusion of Lemma 63. Let $\eta > 0$ and $T_0 > 0$ be given by Lemma 63. We consider an ergodic measure $\mu = \mu_T$ for some $T > T_0$.

Claim. *Let ν_n be a measure supported on a periodic orbit γ_n with period $\pi(\gamma_n) > T$. Then $\int \log h_E^T \, d\nu_n(x) = 0$.*

Proof. By definition of h_E^T,

$$\log h_E^T = \log \prod_{\sigma_i \in S_E} \|h_{\sigma_i}^T\|,$$

so it suffices to prove the claim for a given $h^T_{\sigma_i}$. For every x in γ, by the cocycle condition in Lemma 44 we have

$$\prod_{i=0}^{\lfloor m\pi(\gamma)/T\rfloor-1} h^T_{\sigma_i}(\phi^Y_{iT}(x)) = h^{\lfloor m\pi(\gamma)/T\rfloor-1}_{\sigma_i}(x).$$

The norm of the vector field restricted to γ is bounded, and therefore $h^{\lfloor m\pi(\gamma)/T\rfloor-1}_{\sigma_i}(x)$ is bounded for $m \in \mathbb{N}$ going to infinity. Then this is also true for h^T_E. Since ν_n is an ergodic measure, we have

$$\begin{aligned}\int \log h^T_E\, d\nu_n(x) &= \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{\lfloor m\pi(\gamma)/T\rfloor-1} \log(h^T_E(\phi^Y_{iT}(x)))\\ &= \lim_{m\to\infty} \frac{1}{m} \log\Big(\prod_{i=0}^{\lfloor m\pi(\gamma)/T\rfloor-1} h^T_E(\phi^Y_{iT}(x))\Big)\\ &= \lim_{m\to\infty} \frac{1}{m} \log(h_E^{\lfloor m\pi(\gamma)/T\rfloor-1}(x)) = 0.\end{aligned}$$

□

Claim. *There is a singular point σ_i such that μ is supported on $\mathbb{P}^c_{\sigma_i}$.*

Proof. Suppose that $\mu(\bigcup_{\sigma_i\in \mathrm{Sing}(X)} \mathbb{P}^c_{\sigma_i}) = 0$. Then μ projects to an ergodic measure ν on M supported on the class C and such that it is the singularities, for which

$$\int \log \|h_E \cdot \psi^T_{\mathcal{N}}|_E\|\, d\nu(x) \geq 0.$$

(Recall that ψ^T is the linear Poincaré flow, and h^T_E can be defined as a function of $x \in M$ instead of a function of $L \in \mathbb{P}M$ outside an arbitrarily small neighborhood of the singularities.)

However, as X is generic, the ergodic closing lemma implies that ν is the weak-* limit of measures ν_n supported on periodic orbits γ_n which converge for the Hausdorff distance to the support of ν. Therefore, for n large enough, the γ_n are contained in a small filtrating neighborhood of C, hence

$$\int \log \|h^T_E \cdot \psi^T|_E\|\, d\nu_n(x) \leq -\eta.$$

The map $\log \|h^T_E \cdot \psi^T|_E\|$ is not continuous. Nevertheless, it is uniformly bounded and the unique discontinuity points are the singularities of X. These singularities have (by assumption) weight 0 for ν and thus admit neighborhoods with arbitrarily small weight. Outside such a neighborhood the map is continuous. One deduces that

$$\int \log \|h^T_E \cdot \psi^T|_E\|\, d\nu(x) = \lim \int \log \|h_E \cdot \psi^T|_E\|\, d\nu_n(x)$$

and therefore it is strictly negative, contradicting the assumption. This contradiction proves the claim. □

On the other hand, Corollary 66 asserts that:

- If σ_i is such that $\mathrm{sv}(\sigma_i) < 0$, then $\sigma_i \in S_E$ and the restriction of $\psi_{\mathcal{N}}$ to $\mathbb{P}G^{sc}_{\sigma_i}$ is such that h_E coincides with h_{σ_i} and $h_E \cdot \psi_{\mathcal{N}}$ uniformly contracts the bundle E.
- If $\mathrm{sv}(\sigma_i) > 0$, then $\sigma_i \notin S_E$ and the restriction of $\psi_{\mathcal{N}}$ to $\mathbb{P}G^{cu}_{\sigma_i}$ uniformly contracts E. Note that in this case h^t_E is constant equal to 1.

Recall that $\mathbb{P}^c_{\sigma_i}$ is contained in $\mathbb{P}G^{cs}_{\sigma_i}$ (resp. $\mathbb{P}G^{cu}_{\sigma_i}$) if $\mathrm{sv}(\sigma_i) < 0$ (resp. $\mathrm{sv}(\sigma_i) > 0$). One deduces that there are $T_1 > 0$ and $\varepsilon > 0$ such that

$$\log \|h_E \cdot \psi^T_{\mathcal{N}}|_{E_L}\| \leq -\varepsilon, \qquad \forall L \in \mathbb{P}^c_{\sigma_i} \text{ and } T > T_1.$$

Therefore the measures μ_T, for $T > \max\{T_0, T_1\}$ cannot be supported on $\mathbb{P}^c_{\sigma_i}$, leading to a contradiction. The expansion for F is proved analogously.

This finishes the proof of Lemma 73 and therefore the proof of Lemma 62 and Theorem 3. □

10. A multisingular hyperbolic set in $\mathbb{R}^3$

In this section we will build a chain recurrence class in M^3 containing two singularities of different indices that will be multisingular hyperbolic. However, this will not be a robust class, and the singularities will not be robustly related. Other examples of this kind are in [BaMo]. A robust example on a 5-dimensional manifold is given in [dL1]

Theorem 9. *There exists a vector field X on $S^2 \times S^1$ with an isolated chain recurrence class Λ such that:*

- *There are two singularities in Λ. They are Lorenz-like and of different indices.*
- *There is a cycle between the singularities. The cycle and the singularities are the only orbits in Λ.*
- *The set Λ is multisingular hyperbolic.*

To begin the proof, let us construct a vector field X that we will later show to have the properties of Theorem 9.

Consider a vector field in S^2 having:

- A source f_0 such that the basin of repulsion of f_0 is a disc bounded by a cycle Γ formed by the unstable manifold of a saddle s_0 and a sink σ_0.
- A source α_0 in the other component limited by Γ.
- We require that the tangent at σ_0 splits into two spaces, one having a stronger contraction than the other. The strongest direction is tangent to Γ at σ_0.

Note that the unstable manifold of s_0 is formed by two orbits. These orbits have their ω-limit in σ_0, and as they approach σ_0, they become tangent to the weak stable direction (see Figure 6).

Now we consider S^2 embedded in S^3, and we define a vector field X_0 on S^3 that is normally hyperbolic on S^2, in fact we have S^2 times a strong expansion, and two extra sinks that we call ω_0 and P_0 completing the dynamics (see Figure 7).

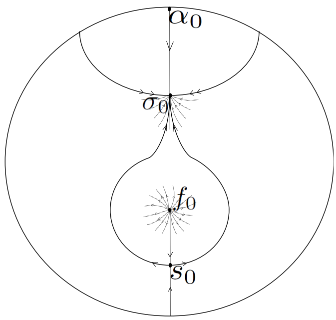


Fig. 6. The vector field in S^2.

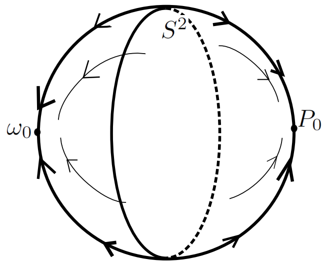


Fig. 7. S^2 normally repelling in S^3.

Note that σ_0 is now a saddle. We require that the weaker contraction at σ_0 is weaker than the expansion. So σ_0 is Lorenz-like.

Now we remove a neighborhood of f_0 and P_0. The resulting manifold is diffeomorphic to $S^2 \times [-1, 1]$ and the vector field X_0 will be pointing outwards on $A_0 = S^2 \times \{1\}$ (corresponding to removing a neighborhood of P_0) and entering on $B_0 = S^2 \times \{-1\}$ (corresponding to removing a neighborhood of f_0) (see Figure 8).

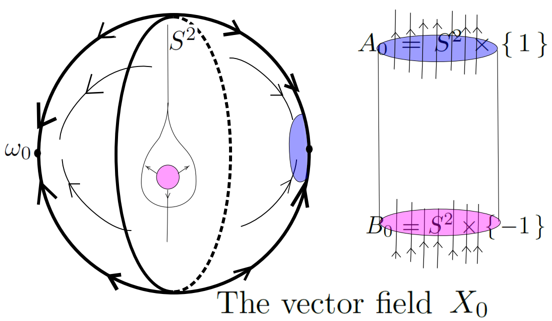


Fig. 8. Removing a neighborhood of f_0 and P_0.

Now we consider another copy of $S^2 \times [-1, 1]$ with a vector field X_1 that is the time reversal of X_0. Therefore X_1 has a saddle called σ_1 that has a strong expansion, a weaker expansion and a contraction, and is Lorenz-like. It also has a sink called α_1, a source called ω_1, and a saddle called s_1.

The vector field X_1 is entering on $A_1 = S^2 \times \{1\}$ and pointing outwards on $B_1 = S^2 \times \{-1\}$.

We can now paste X_1 and X_0 together along their boundaries (A_0 with A_1 and the other two). Since both vector fields are transversal to the boundaries we can obtain a C^1 vector field X in the resulting manifold that is diffeomorphic to $S^2 \times S^1$.

We do not paste any of the boundaries using the identity. We first describe the map gluing A_0 with A_1. We paste them by a rotation so that

$$(\overline{W^u(\alpha_0)} \cap A_0)^c \quad \text{and} \quad W^u(s_0) \cap A_0$$

are mapped to

$$W^s(\alpha_1) \cap (A_1).$$

We also require that $W^u(\sigma_0) \cap A_0$ is mapped to $W^s(\sigma_1) \cap A_1$, We will later require an extra condition on this gluing map, which is a generic condition, and which will guarantee multisingular hyperbolicity.

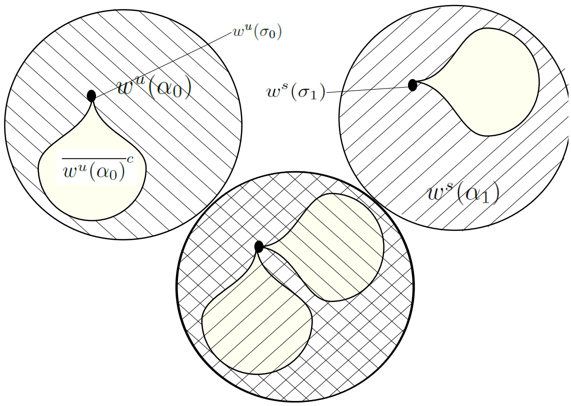


Fig. 9. Pasting $S^2 \times \{1\}$ and $S^2 \times \{1\}$.

To glue B_0 to B_1, let us first observe that these boundaries were formed by removing a neighborhood of f_0 and f_1. Then by construction $\overline{W^s(\sigma_0)} \cap B_0$ is a circle that we will call $\mathcal{C}_0$. We can also define the corresponding $\mathcal{C}_1$. Note that all points in $\mathcal{C}_0$, except one, are in $W^s(\sigma_0)$, while there is one point l in $\mathcal{C}_0$ that is in $W^s(s_0)$. We paste B_0 and B_1, mapping $\mathcal{C}_0$ so as to intersect $\mathcal{C}_1$ transversally at points of $W^s(\sigma_0)$ and $W^u(\sigma_1)$.

Note that the resulting vector field X has a cycle between two Lorenz-like singularities σ_0 and σ_1.

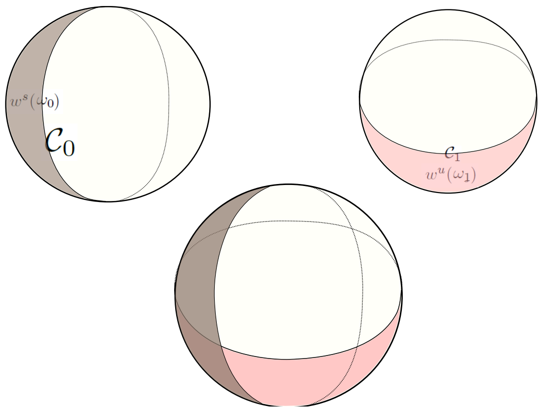


Fig. 10. Pasting $S^2 \times \{-1\}$ and $S^2 \times \{-1\}$.

Lemma 74. *The vector field X defined above is such that the cycle and the singularities are the only chain recurrent points.*

Proof.
- All the recurrent orbits of X_0 in S^3 are the singularities. Once we remove the neighborhoods of the two singularities we obtain the manifold with boundary $S^2 \times [-1, 1]$. The the only other orbits of the vector field X (that results from pasting X_0 and X_1) that may be recurrent have to intersect the boundaries.
- The points in A_0 that are in $\overline{W^u(\alpha_0)}^c \cup W^u(\sigma_0)$ are wandering since they are mapped to the stable manifold of the sink α_1.
- The points in $B_0 \cap W^u(\alpha_0)$ are wandering.

As a conclusion, the only point in A_0 whose orbit could be recurrent is the one in

$$B_0 \cap W^u(\sigma_0).$$

Let us now look at the points in B_0. There is a circle $\mathcal{C}_0$, corresponding to $\overline{W^s(\sigma_0)} \cap B_0$ that divides B_0 into two components. One of these components is the basin of the sink ω_0 and the other is what used to be the basin of P_0. So we have the following possibilities:

- The points that are in the basin of the sink ω_0 are not chain recurrent.
- The points that are in what used to be the basin of P_0 are either mapped into the basin of ω_1 or sent to what used to be the basin of P_1. Note that these points cross A_0 for the past, and since they are not in the stable manifold of σ_1, they are wandering.
- Some points in $\mathcal{C}_0$ will be mapped to the basin of ω_1, others to what used to be the basin of P_1, and others to $\mathcal{C}_1$. In the first two cases those points are wandering.

To sum up:

- The only recurrent orbits that cross B_0 are in the intersection of $\mathcal{C}_0$ with $\mathcal{C}_1$.
- The only recurrent orbits that cross A_0 are in the intersection of $W^u(\sigma_0)$ with $W^s(\sigma_1)$.

- The only recurrent orbits that do not cross the boundaries of $S^2 \times [-1, 1]$ are singularities.

This proves the lemma. □

For the Lorenz singularity σ_0 of X which is a positive saddle and such that $T_{\sigma_0}M = E^{ss} \oplus E^s \oplus E^{uu}$, we define $B_{\sigma_0} \subset \mathbb{P}M$ as

$$B_{\sigma_0} = \pi_{\mathbb{P}}(E^s \oplus E^{uu}).$$

For the Lorenz singularity σ_1 of X which is a negative saddle and such that $T_{\sigma_1}M = E^{ss} \oplus E^u \oplus E^{uu}$, we define $B_{\sigma_1} \subset \mathbb{P}M$ as

$$B_{\sigma_1} = \pi_{\mathbb{P}}(E^{ss} \oplus E^u).$$

Let a, b and c be points that are one in each of the three regular orbits forming the cycle between the two singularities of X; assume a is the one such that the α-limit is σ_0. We define $L_a = S_X(a)$, $L_b = S_X(b)$ and $L_c = S_X(c)$. We also denote by $O(L_a)$, $O(L_b)$ and $O(L_c)$ the orbits of L_a, L_b and L_c under $\phi^t_{\mathbb{P}}$.

Proposition 75. *Suppose that X is a vector field defined as above. Then there exist an open set U containing the orbits of a, b and c and the saddles σ_0 and σ_1 such that*

$$B(U, X) = B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a) \cup O(L_b) \cup O(L_c).$$

Proof. The two orbits of the strong stable manifold of σ_0 go by construction to α_0 for the past. This implies that the strong stable manifold is escaping. The fact that there is a cycle tells us that there are no other escaping directions, therefore the center space is formed by the weak stable and the unstable spaces. By definition $B_{\sigma_0} = \mathbb{P}^c_{\sigma_0}$. Analogously we see that $B_{\sigma_1} = \mathbb{P}^c_{\sigma_1}$. Since the cycle formed by the orbits of a, b and c and the saddles σ_0 and σ_1 is an isolated chain recurrence class, we can choose U small enough so that this class is the maximal invariant set in U. This proves our proposition. □

Lemma 76. *We can choose a vector field X defined as above in such a way that it is multisingular hyperbolic in U.*

Proof. The reparametrized linear Poincaré flow is hyperbolic when restricted to the bundle over $B_{\sigma_0} \cup B_{\sigma_1}$ and of index 1. We consider the set $B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a)$.

The strong stable space at σ_0 is the stable space for the reparametrized linear Poincaré flow. There is a well defined stable space in the linearized neighborhood of σ_0, and since the stable space is invariant for the future, there is a one-dimensional stable flag that extends along the orbit of a. We can reason analogously with the strong unstable manifold of σ_1 and conclude that there is an unstable flag extending through the orbit of a. We can choose the gluing maps of $S^2 \times \{-1\}$ to $S^2 \times \{-1\}$ so that the stable and unstable flags in the orbit of a intersect transversally. This is because this condition is open and dense in the set of possible gluing maps with the properties mentioned above. Therefore the set $B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a)$ is hyperbolic for the reparametrized linear Poincaré flow.

Analogously we prove that $B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a) \cup O(L_b) \cup O(L_c)$ is hyperbolic for the reparametrized linear Poincaré flow, and since from Proposition 75 there exists U such that

$$B_{\sigma_1} \cup B_{\sigma_2} \cup O(L_a) \cup O(L_b) \cup O(L_c) = B(U, X),$$

it follows that X is multisingular hyperbolic in U. □

The example in [BaMo] consists of two singular (negatively and positively) hyperbolic sets H_- and H_+ of different indices, and wandering orbits going from one to the other. Since they are singular hyperbolic, H_- and H_+ are multisingular hyperbolic sets of the same index. Moreover, the stable and unstable flags (for the reparametrized linear Poincaré flow) along the orbits joining H_- and H_+ intersect transversally. This is also true for H_-.

With all these ingredients we can prove (in a similar way to what we just did with the simpler example above) that the chain recurrence class containing H_- and H_+ in [BaMo] is multisingular hyperbolic, while it was shown by the authors that it is not singular hyperbolic.

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