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Harmonic quasi-isometric maps II: negatively curved manifolds

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Abstract. We prove that a quasi-isometric map, and more generally a coarse embedding, between pinched Hadamard manifolds is within bounded distance of a unique harmonic map.

Keywords. Harmonic map, harmonic measure, quasi-isometric map, coarse embedding, boundary map, Hadamard manifold, negative curvature

1. Introduction

The aim of this article, which is a sequel to [3], is to prove the following theorem.

Theorem 1.1. *Let $f : X \to Y$ be a quasi-isometric map between two pinched Hadamard manifolds. Then there exists a unique harmonic map $h : X \to Y$ which stays within bounded distance of f, i.e.*

$$\sup_{x \in X} d(h(x), f(x)) < \infty.$$

We first recall a few definitions. A *pinched Hadamard manifold* X is a complete simply connected Riemannian manifold of dimension at least 2 whose sectional curvature is pinched between two negative constants: $-b^2 \le K_X \le -a^2 < 0$. A map $f : X \to Y$ between two metric spaces X and Y is said to be *quasi-isometric* if there exist constants $c \ge 1$ and $C \ge 0$ such that f is (c, C)-*quasi-isometric*, which means that

$$c^{-1} d(x, x') - C \le d(f(x), f(x')) \le c\, d(x, x') + C \qquad (1.1)$$

for all x, x' in X. A $\mathcal{C}^2$ map $h : X \to Y$ between Riemannian manifolds X and Y is said to be *harmonic* if it satisfies the elliptic nonlinear partial differential equation $\operatorname{tr}(D^2 h) = 0$ where $D^2 h$ is the second covariant derivative of h.

Partial results towards the existence statement were obtained in [31], [41], [17], [27], [5]. A major breakthrough was achieved by Markovic who solved the Schoen conjecture, i.e. the case where $X = Y$ is the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$, and by Lemm–Markovic who proved the existence for $X = Y = \mathbb{H}^k_{\mathbb{R}}$ in [28], [29] and [23]. The existence when

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both X and Y are rank one symmetric spaces, which was conjectured by Li and Wang [25, Introduction], was proved in our paper [3]. We refer to [3, Section 1.2] for more motivations and a precise historical perspective on this result.

As explained in [11], the harmonic map h is not always a diffeomorphism even when f is a diffeomorphism.

Partial results towards the uniqueness statement were obtained by Li and Tam [24], and by Li and Wang [25]. All these papers were dealing with rank one symmetric spaces.

Note that Theorem 1.1 was conjectured by Markovic during a 2016 Summer School in Grenoble. According to our knowledge, Theorem 1.1 is new even in the case where both X and Y are assumed to be surfaces.

The strategy of the proof of the existence follows the lines of the proof in [3]. As in [3], we replace the quasi-isometric map f by a $\mathcal{C}^\infty$ map whose first two covariant derivatives are bounded. But we need to modify the barycenter argument we used in [3] for this smoothing step. See Subsection 2.2.1 for more details on this step. As in [3], we then introduce the harmonic maps h_R that coincide with f on a sphere of X with large radius R and we need a uniform bound for the distances between the maps h_R and f. The heart of our argument is in Section 3 which contains the boundary estimates, and in Section 4 which contains the interior estimates, for $d(h_R, f)$. The proof of the interior estimates is based on a new simplification of an idea by Markovic [29]. Indeed we will introduce a point x where $d(h_R(x), f(x))$ is maximal and focus on a subset U_{ℓ_0} of a sphere $S(x, \ell_0)$ whose definition (4.10) is much simpler than in [29] or [3]. This simplification is the key point which allows us to extend the arguments of [3] to pinched Hadamard manifolds. In this proof we use uniform control on the harmonic measures on all the spheres of X, which is given in Proposition 4.9. We refer to Section 4.1 for more details on our strategy of the proof of existence.

In order to prove uniqueness, we need to introduce Gromov–Hausdorff limits of the pointed metric spaces X and Y with respect to base points going to infinity and therefore to deal with $\mathcal{C}^2$ Riemannian manifolds with $\mathcal{C}^1$ metrics. This will be done in Section 5. We refer to Subsection 5.1 for more details on our strategy of the proof of uniqueness.

In Section 7, we extend Theorem 1.1 to coarse embeddings (see Definition 6.2 and Theorem 7.1). The proof is similar but relies on the existence of a boundary map for coarse embeddings. We also show that Theorem 1.1 cannot be extended to Lipschitz maps (Example 7.3).

Section 6 is dedicated to the existence of this boundary map which, for a coarse embedding, is well-defined outside a set of zero Hausdorff dimension (Theorem 6.5). The existence of such a boundary map seems to be new.

2. Smoothing

In this section, we recall a few basic facts on Hadamard manifolds, and we explain how to replace our quasi-isometric map f by a $\mathcal{C}^\infty$ map whose first two covariant derivatives are bounded.

2.1. The geometry of Hadamard manifolds

We first recall basic estimates on Hadamard manifolds for triangles, for images of triangles under quasi-isometric maps, and for the Hessian of the distance function.

All the Riemannian manifolds will be assumed to be connected. We will denote by d their distance function.

A *Hadamard manifold* is a complete simply connected Riemannian manifold X of dimension $k \geq 2$ whose curvature is non-positive, $K_X \leq 0$. For instance, the Euclidean space $\mathbb{R}^k$ is a Hadamard manifold with zero curvature 0, and the real hyperbolic space $\mathbb{H}^k_{\mathbb{R}}$ is a Hadamard manifold with constant curvature -1. We will say that X is *pinched* if there exist constants $a, b > 0$ such that

$$-b^2 \leq K_X \leq -a^2 < 0.$$

For instance, non-compact rank one symmetric spaces are pinched Hadamard manifolds.

Let x_0, x_1, x_2 be three points on a Hadamard manifold X. The *Gromov product* of the points x_1 and x_2 seen from x_0 is defined as

$$(x_1|x_2)_{x_0} := (d(x_0, x_1) + d(x_0, x_2) - d(x_1, x_2))/2. \tag{2.1}$$

We recall the basic comparison lemma which is one of the motivations for introducing the Gromov product.

Lemma 2.1. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq -a^2 < 0$. Let T be a geodesic triangle in X with vertices x_0, x_1, x_2, and let θ_0 be the angle of T at x_0. Then:*

(a) $(x_0|x_2)_{x_1} \geq d(x_0, x_1)\ \sin^2(\theta_0/2)$.
(b) $\theta_0 \leq 4e^{-a(x_1|x_2)_{x_0}}$.
(c) *If* $\min((x_0|x_1)_{x_2}, (x_0|x_2)_{x_1}) \geq b^{-1}$, *one has* $\theta_0 \geq e^{-b(x_1|x_2)_{x_0}}$.

Proof. This is classical. See for instance [3, Lemma 2.1]. □

We now recall the effect of a quasi-isometric map on the Gromov product.

Lemma 2.2. *Let X, Y be Hadamard manifolds with $-b^2 \leq K_X, K_Y \leq -a^2 < 0$, and let $f : X \to Y$ be a (c, C)-quasi-isometric map. There exists $A = A(a, b, c, C) > 0$ such that, for all x_0, x_1, x_2 in X,*

$$c^{-1}(x_1|x_2)_{x_0} - A \ \leq\ (f(x_1)|f(x_2))_{f(x_0)} \ \leq\ c(x_1|x_2)_{x_0} + A. \tag{2.2}$$

Proof. This is a general property of quasi-isometric maps between Gromov δ-hyperbolic spaces which is due to M. Burger. See [13, Prop. 5.15]. □

When x_0 is a point in a Riemannian manifold X, we denote by d_{x_0} the distance function defined by $d_{x_0}(x) = d(x_0, x)$ for x in X. We denote by $d^2_{x_0}$ the square of this function. When $F : X \to \mathbb{R}$ is a $\mathcal{C}^2$ function, we denote by DF its differential and by D^2F its second covariant derivative.

Lemma 2.3. *Let X be a Hadamard manifold and $x_0 \in X$. Assume that $-b^2 \leq K_X \leq -a^2 \leq 0$. The Hessian of the distance function d_{x_0} satisfies*

$$a\coth(ad_{x_0})g_0 \leq D^2 d_{x_0} \leq b\coth(bd_{x_0})g_0 \tag{2.3}$$

on $X \setminus \{x_0\}$, where $g_0 := g_X - Dd_{x_0} \otimes Dd_{x_0}$ and g_X is the Riemannian metric on X.

When $a = 0$ the left-hand side of (2.3) must be interpreted as $d_{x_0}^{-1} g_0$.

Proof. This is classical. See for instance [3, Lemma 2.3]. □

2.2. *Smoothing rough Lipschitz maps*

The following proposition will allow us to assume in Theorem 1.1 that the quasi-isometric map f we start with is $\mathcal{C}^\infty$ with bounded derivative and bounded second covariant derivative.

2.2.1. Rough Lipschitz maps. A map $f : X \to Y$ between metric spaces X and Y is said to be *rough Lipschitz* if there exist constants $c \geq 1$ and $C \geq 0$ such that, for all x, x' in X,

$$d(f(x), f(x')) \leq cd(x, x') + C. \tag{2.4}$$

Proposition 2.4. *Let X, Y be Hadamard manifolds with bounded curvatures, $-b^2 \leq K_X, K_Y \leq 0$. Let $f : X \to Y$ be a rough Lipschitz map. Then there exists a $\mathcal{C}^\infty$ map $\widetilde{f} : X \to Y$ within bounded distance of f and whose first two covariant derivatives $D\widetilde{f}$ and $D^2\widetilde{f}$ are bounded on X.*

We denote $k = \dim X$ and $k' = \dim Y$. We will first construct in 2.2.2 a regularized map $\widetilde{f} : X \to Y$ which is Lipschitz continuous. This construction is the same as for rank one symmetric spaces in [3, Proposition 2.4]. The construction will not allow us to control the second covariant derivative, hence we will have to combine this first construction with an iterative smoothing process in local charts that we will explain in 2.2.3.

2.2.2. Lipschitz continuity. The first part of the proof of Proposition 2.4 relies on the following classical lemma (see [20, Section 2]).

Lemma 2.5. *Let Y be a Hadamard manifold.*

(a) *Let μ be a positive finite Borel measure on Y supported by a closed ball $B(y_0, R)$. The function Q_μ on Y defined by*

$$Q_\mu(y) = \int_Y d(y, w)^2 \, \mathrm{d}\mu(w)$$

has a unique minimum point y_μ in Y, called the center of mass of μ; it belongs to $B(y_0, R)$.

(b) *Let μ_1, μ_2 be positive finite Borel measures on Y. Assume that*

(i) $\mu_1(Y) \geq m$ *and* $\mu_2(Y) \geq m$ *for some* $m > 0$,

(ii) *both* μ_1 *and* μ_2 *are supported on* $B(y_0, R)$,

(iii) $\|\mu_1 - \mu_2\| \leq \varepsilon$.

Then

$$d(y_{\mu_1}, y_{\mu_2}) \leq 4\varepsilon R/m. \tag{2.5}$$

Proof. (a) Since Y is a proper space, i.e. its balls are compact, the function Q_μ is proper and admits a minimum, say at y_μ. Since Y has non-positive curvature, the median inequality holds: for all y, y_1, y_2, y_3 in Y where y_3 is the midpoint of y_1 and y_2,

$$\tfrac{1}{2} d(y_1, y_2)^2 \leq d(y, y_1)^2 + d(y, y_2)^2 - 2d(y, y_3)^2. \tag{2.6}$$

Integrating (2.6) with respect to μ, one checks that Q_μ has the following uniform convexity property: if y_3 is the midpoint of y_1 and y_2 then

$$\tfrac{m}{2} d(y_1, y_2)^2 \leq Q_\mu(y_1) + Q_\mu(y_2) - 2Q_\mu(y_3).$$

Applying this inequality with $y_1 = y_\mu$ and $y_2 = y$ one gets, for each y in Y,

$$\tfrac{m}{2} d(y_\mu, y)^2 \leq Q_\mu(y) - Q_\mu(y_\mu), \tag{2.7}$$

so that y_μ is the unique minimum point of Q_μ.

We now check that $y_\mu \in B(y_0, R)$. By the median inequality (2.6), the ball $B(y_0, R)$ is convex, every point y in Y admits a unique nearest point y' in $B(y_0, R)$, and this point y' also satisfies the inequality

$$d(y', w) \leq d(y, w) \qquad \text{for all } w \text{ in } B(y_0, R).$$

Therefore, $Q_\mu(y') \leq Q_\mu(y)$. This proves that the center of mass y_μ belongs to $B(y_0, R)$.

(b) Applying (2.7) twice, one gets

$$\tfrac{m}{2} d(y_{\mu_1}, y_{\mu_2})^2 \leq Q_{\mu_1}(y_{\mu_2}) - Q_{\mu_1}(y_{\mu_1}),$$
$$\tfrac{m}{2} d(y_{\mu_1}, y_{\mu_2})^2 \leq Q_{\mu_2}(y_{\mu_1}) - Q_{\mu_2}(y_{\mu_2}).$$

Summing these inequalities yields

$$\begin{aligned} m\, d(y_{\mu_1}, y_{\mu_2})^2 &\leq (Q_{\mu_1} - Q_{\mu_2})(y_{\mu_2}) - (Q_{\mu_1} - Q_{\mu_2})(y_{\mu_1}) \\ &\leq \varepsilon \sup_{w \in B(y_0, R)} |d(y_{\mu_1}, w)^2 - d(y_{\mu_2}, w)^2| \leq 4\varepsilon R d(y_{\mu_1}, y_{\mu_2}), \end{aligned}$$

which proves (2.5). □

We now choose a non-negative C^∞ function $\chi : \mathbb{R} \to \mathbb{R}$ with support in $]-1, 1[$, which is equal to 1 on a neighborhood of $[-1/2, 1/2]$ and satisfies $|\chi'| \leq 4$.

Proof of Proposition 2.4. First step: Lipschitz continuity. We now explain this first construction. We can assume $b = 1$. Since a rough Lipschitz map $f : X \to Y$ is always within bounded distance of a Borel measurable map, we can assume that f itself is Borel measurable. For x in X, we introduce the positive finite measure μ_x on Y such that

$$\mu_x(\varphi) = \int_X \varphi(f(z))\, \chi(d(x,z))\, \mathrm{dvol}_X(z)$$

for any positive function φ on Y. The measure μ_x is the image by f of a measure supported in $B(x,1)$. We define $\widetilde{f}(x) \in Y$ to be the center of mass of μ_x. Lemma 2.5(a) tells us that the map $x \mapsto \widetilde{f}(x)$ is well-defined. The Lipschitz continuity of $\widetilde{f}$ will follow from Lemma 2.5(b) applied to $\mu_1 := \mu_{x_1}$ and $\mu_2 := \mu_{x_2}$ with x_1, x_2 in X. Let us check that the three assumptions in Lemma 2.5(b) are satisfied.

(i) Because of the pinching of the curvature of X, the Bishop volume estimates tell us that there exist positive constants $0 < m_0 < M_0$ such that, for all x,

$$m_0 \le \mathrm{vol}(B(x,1/2)) \le \mu_x(Y) \le \mathrm{vol}(B(x,1)) \le M_0.$$

(ii) When $x_1, x_2 \in X$ with $d(x_1,x_2) \le 1$, the bound (2.4) ensures that both μ_{x_1} and μ_{x_2} are supported in $B(f(x_1), 2c + C)$.

(iii) We have

$$\|\mu_{x_1} - \mu_{x_2}\| \le M_0 \sup_{z\in X} |\chi(d(x_1,z)) - \chi(d(x_2,z))| \le 4M_0\, d(x_1,x_2).$$

Thus Lemma 2.5 applies and yields a bound on the Lipschitz constant of $\widetilde{f}$, namely

$$\mathrm{Lip}(\widetilde{f}) := \sup_{x_1\neq x_2} d(\widetilde{f}(x_1), \widetilde{f}(x_2))/d(x_1,x_2) \le \frac{16(2c+C)M_0}{m_0}. \qquad \square$$

2.2.3. Bound on the second derivative. The second step of the proof of Proposition 2.4 relies on three lemmas. The first lemma provides a nice system of charts on Y.

Lemma 2.6. *Let Y be a Hadamard manifold with $-b^2 \le K_Y \le 0$ and $k' = \dim Y$. There exist constants $r_0 = r_0(k',b) > 0$ and $c_0 = c_0(k',b) > 1$ such that, for each y in Y, there exists a $\mathcal{C}^\infty$ chart Φ_y for the open ball,*

$$\Phi_y : \mathring{B}(y,r_0) \xrightarrow{\sim} U_y \subset \mathbb{R}^{k'} \quad \textit{with} \quad \Phi_y(y) = 0, \tag{2.8}$$

such that

$$\|D\Phi_y\| \le c_0, \quad \|D\Phi_y^{-1}\| \le c_0, \quad \|D^2\Phi_y\| \le c_0, \quad \|D^2\Phi_y^{-1}\| \le c_0. \tag{2.9}$$

In particular, for all $r < r_0$,

$$\Phi_y(B(y, c_0^{-1}r)) \subset B(0,r) \quad \textit{and} \quad B(0, c_0^{-1}r) \subset \Phi_y(B(y,r)). \tag{2.10}$$

We have endowed $\mathbb{R}^{k'}$ with the standard Euclidean structure.

Proof of Lemma 2.6. This is classical. One can for instance choose the so-called almost linear coordinates, as in [19, Section 2] or [32, Section 3]. They are defined in the following way. We fix an orthonormal basis $(e_i)_{1\leq i\leq k'}$ for the tangent space T_yY and set $y_i := \exp_y(-e_i) \in Y$. The map Φ_y is defined by the formula

$$\Phi_y(z) = (d(z, y_1)-1, \dots, d(z, y_{k'})-1),$$

where z belongs to a sufficiently small ball $\mathring{B}(y, r_0)$. See [19, pp. 43 and 58] for a detailed proof. ◻

There exist better systems of coordinates, the so-called harmonic coordinates. We will not need them in this section, but we will need them in Section 5 to prove uniqueness (see Lemma 5.2).

The second lemma explains how to modify a Lipschitz map g inside a tiny ball $B(x, r)$ of X so that the new map $g_{x,r}$ is constant on $B(x, r/2)$ and the first two derivatives of $g_{x,r}$ are controlled by those of g. We recall that $\chi : \mathbb{R} \to \mathbb{R}$ is a non-negative $\mathcal{C}^\infty$ function with support included in $]-1, 1[$, which is equal to 1 on a neighborhood of $[-1/2, 1/2]$ and 4-Lipschitz, i.e. $|\chi'| \leq 4$.

Lemma 2.7. *Let X and Y be Hadamard manifolds with bounded curvatures $-b^2 \leq K_X, K_Y \leq 0$. Let $r_0 > 0$ and $c_0 \geq 1$ be as in Lemma 2.6. Let $g : X \to Y$ be a Lipschitz map, x a point in X, $y = g(x)$ and $0 < r < r_0$. Assume that*

$$\operatorname{Lip}(g) < \frac{r_0}{c_0^2 r}. \tag{2.11}$$

Then the following formula defines a Lipschitz map $g_{r,x} : X \to Y$:

$$g_{r,x}(z) = \begin{cases} g(x) & \text{if } d(z,x) \leq r/2,\\ \Phi_y^{-1}\Big(\big(1-\chi\big(\tfrac{d(z,x)}{r}\big)\big)\Phi_y(g(z))\Big) & \text{if } r/2 \leq d(z,x) \leq r,\\ g(z) & \text{if } d(z,x) \geq r. \end{cases}$$

We have

$$\operatorname{Lip}_{B(x,r)}(g_{r,x}) \leq 5c_0^2 \operatorname{Lip}_{B(x,r)}(g). \tag{2.12}$$

In particular,

$$\operatorname{Lip}(g_{r,x}) \leq 5c_0^2 \operatorname{Lip}(g). \tag{2.13}$$

Moreover, if g is $\mathcal{C}^2$ in a neighborhood of a point z in X, then $g_{r,x}$ is also $\mathcal{C}^2$ in this neighborhood and

$$\|D^2 g_{r,x}(z)\| \leq \big(\|D^2 g(z)\| + \operatorname{Lip}_{B(x,r)}(g)^2 + 1\big) M_r, \tag{2.14}$$

where the constant $M_r \geq 1$ depends only on r, b, k, k′ and χ.

Proof. Condition (2.11) ensures that, for any z in $B(x, r)$, the image $g(z)$ belongs to $\mathring{B}(y, c_0^{-2} r_0)$. Therefore, by (2.10), the vector $\Phi_y(g(z))$ belongs to $\mathring{B}(0, c_0^{-1} r_0) \subset \mathbb{R}^{k'}$. When we multiply this vector by the scalar $1 - \chi(\cdot)$, the new vector is still in the same ball. That is why, using again (2.10), the element $g_{r,x}(z)$ is well-defined and belongs to $B(y, r_0)$.

The upper bound (2.12) follows from the chain rule. Indeed, when z is a point in $B(x,r)$ where g is differentiable, the bound (2.9) yields

$$\begin{aligned}\|Dg_{r,x}(z)\| &\le c_0\left(\frac{4}{r}\|\Phi_y(g(z))\| + \|D(\Phi_y\circ g)(z)\|\right)\\ &\le 5c_0\,\mathrm{Lip}_{B(x,r)}(\Phi_y\circ g) \le 5c_0^2\,\mathrm{Lip}_{B(x,r)}(g).\end{aligned}$$

The upper bound (2.14) follows from similar and longer computations left to the reader, which also use the bounds (2.3) for $D^2 d_x$. □

We will also need a third lemma. We recall that a subset X_0 of a metric space X is said to be r-*separated* if the distance between two distinct points of X_0 is at least r.

Lemma 2.8. *Let X be a Hadamard manifold with $-b^2 \le K_X \le 0$. Let $k = \dim X$ and $N_0 := 100^k$. There exists a radius $r_0 = r_0(k,b) > 0$ such that, for any $r < r_0$, every $r/2$-separated subset X_0 of X can be decomposed as a union of at most N_0 subsets which are $2r$-separated.*

Proof. The bound on the curvature of X and the Bishop volume estimates ensure that we can choose $r_0 > 0$ such that

$$\mathrm{vol}(B(x,4r)) \le N_0\,\mathrm{vol}(B(x,r/4)) \quad \text{for all } r < r_0 \text{ and } x \text{ in } X. \tag{2.15}$$

This r_0 works. Indeed, let $X_1,\dots,X_{N_0}$ be a sequence of disjoint $2r$-separated subsets of X_0 with X_1 maximal in X_0, X_2 maximal in $X_0 \smallsetminus X_1$, and so on. Every point x of X_0 must be in one of the X_i's with $i \le N_0$ because if not, each X_i contains a point in $B(x,2r)$, contradicting (2.15). □

Proof of Proposition 2.4. Second step: bound on $D^2\widetilde{f}$. According to the first step of this proof, we can now assume that the map $f : X \to Y$ is c-Lipschitz with $c \ge 1$.

We can choose a new radius $r_0 = r_0(k,k',b)$ that satisfies both conclusions of Lemma 2.8 for X and of Lemma 2.6 for Y. We will freely use the notations of these two lemmas. Now let

$$r_1 = \frac{r_0}{5^{N_0}c_0^{2N_0+2}c}$$

and pick a maximal $r_1/4$-separated subset X_0 of X. Thanks to Lemma 2.8, we write this set X_0 as a union

$$X_0 = X_1 \cup \dots \cup X_{N_0}$$

of N_0 subsets X_i which are $2r_1$-separated.

In order to construct $\widetilde{f}$ from f, we will use a finite iterative process based on Lemma 2.7. Starting with $f_0 = f$, we construct by induction a finite sequence of maps f_i for $i \le N_0$ and we set $\widetilde{f} := f_{N_0}$. In the notations of Lemma 2.7, the map f_i is defined from f_{i-1} by letting

$$f_i(z) = \begin{cases}(f_{i-1})_{r_1,x}(z) & \text{if } d(z,x) \le r_1 \text{ for some } x \text{ in } X_{i+1},\\ f_{i-1}(z) & \text{otherwise,}\end{cases}$$

so that the Lipschitz constants of these maps satisfy

$$\mathrm{Lip}(f_i) \le 5c_0^2\,\mathrm{Lip}(f_{i-1}) \le 5^i c_0^{2i} c. \tag{2.16}$$

Indeed, once f_i is known to be well-defined and to satisfy (2.16), it also satisfies the bound (2.11): $\mathrm{Lip}(f_i) < \frac{r_0}{c_0^2 r_1}$. Therefore Lemma 2.7 ensures that f_{i+1} is well-defined and, using (2.12), that f_{i+1} also satisfies (2.16):

$$\mathrm{Lip}(f_{i+1}) \leq 5c_0^2 \,\mathrm{Lip}(f_i) \leq 5^{i+1} c_0^{2(i+1)} c.$$

Let $\Lambda := M_{r_1} + 25c_0^4 + 1$. By (2.14) and (2.16), for any $i \leq N_0$ and z in X,

$$\|D^2 f_i(z)\| + \mathrm{Lip}(f_i)^2 + 1 \leq \Lambda\big(\|D^2 f_{i-1}(z)\| + \mathrm{Lip}(f_{i-1})^2 + 1\big). \tag{2.17}$$

Since X_0 is a maximal $r_1/4$-separated subset of X, every z in X belongs to at least one ball $\mathring{B}(x, r/2)$ where x is in one of the sets X_{i_0}. But then the function f_{i_0} is constant in a neighborhood of z. Therefore, using (2.16) and applying the bound (2.17) $N_0 - i_0$ times one deduces that $\widetilde{f}$ is a $\mathcal{C}^2$ map that satisfies the uniform upper bound

$$\|D^2 \widetilde{f}(z)\| \leq ((5^{i_0} c_0^{2i_0} c)^2 + 1)\Lambda^{N_0 - i_0} \leq \Lambda^{N_0} c^2. \qquad \square$$

3. Harmonic maps

In this section we begin the proof of the existence part in Theorem 1.1. We first recall basic facts concerning harmonic maps. We explain why a standard compactness argument reduces this existence part to proving a uniform upper bound on the distance between f and the harmonic map h_R which is equal to f on the sphere $S(O, R)$. Then we provide this upper bound near $S(O, R)$.

3.1. Harmonic functions and the distance function

We recall basic facts on the Laplace operator on Hadamard manifolds.

The Laplace–Beltrami operator Δ on a Riemannian manifold X is defined as the trace of the Hessian. In local coordinates, the Laplacian of a function φ is

$$\Delta\varphi = \mathrm{tr}(D^2\varphi) = \frac{1}{v} \sum_{i,j} \frac{\partial}{\partial x_i}\left(v g_X^{ij} \frac{\partial}{\partial x_j} \varphi \right) \tag{3.1}$$

where $v = \sqrt{\det(g_{X\,ij})}$ is the volume density. The function φ is said to be *harmonic* if $\Delta\varphi = 0$ and *subharmonic* if $\Delta\varphi \geq 0$.

We will need the following basic lemma.

Lemma 3.1. *Let X be a Hadamard manifold with $K_X \leq -a^2 \leq 0$ and let $x_0 \in X$. Then the function d_{x_0} is subharmonic. More precisely, the distribution $\Delta d_{x_0} - a$ is non-negative.*

Proof. This is [3, Lemma 2.5]. $\square$

3.2. Harmonic maps and the distance function

In this subsection, we recall two useful facts satisfied by a harmonic map h: the subharmonicity of the functions $d_{y_0} \circ h$, and Cheng's estimate for the differential Dh.

Definition 3.2. Let $h : X \to Y$ be a $\mathcal{C}^2$ map between Riemannian manifolds. The *tension field* of h is the trace of the second covariant derivative, $\tau(h) := \mathrm{tr}(D^2 h)$. The map h is said to be *harmonic* if $\tau(h) = 0$.

Note that the tension field $\tau(h)$ is a *Y-valued vector field on X*, i.e. it is a section of the pull-back of the tangent bundle $TY \to Y$ under the map $h : X \to Y$.

For instance, an isometric immersion with minimal image is always harmonic. The problem of the existence, regularity and uniqueness of harmonic maps under various boundary conditions is a very classical topic (see [10], [38], [19], [9], [40], [37] or [26]). In particular, when Y is simply connected and has non-positive curvature, a harmonic map is always $\mathcal{C}^\infty$, and is a minimum of the energy functional among maps that agree with h outside a compact subset of X.

Lemma 3.3. *Let $h : X \to Y$ be a harmonic $\mathcal{C}^\infty$ map between Riemannian manifolds. Let $y_0 \in Y$ and let $\rho_h := d_{y_0} \circ h : X \to \mathbb{R}$. If Y is Hadamard, then the continuous function ρ_h is subharmonic on X.*

Proof. See [3, Lemma 3.2]. □

Another crucial property of harmonic maps is the following bound for their differential due to Cheng.

Lemma 3.4. *Let X, Y be Hadamard manifolds with $-b^2 \le K_X \le 0$. Let $k = \dim X$, $z \in X$, $r > 0$ and let $h : B(z,r) \to Y$ be a harmonic $\mathcal{C}^\infty$ map such that $h(B(z,r))$ lies in a ball of radius R_0. Then*

$$\|Dh(z)\| \le 2^5 k \frac{1+br}{r} R_0.$$

In applications, we will use this inequality with $r = b^{-1}$.

Proof. This is a simplified version of [8, Formula 2.9]. □

3.3. Existence of harmonic maps

In this subsection we prove Theorem 1.1, taking for granted Proposition 3.5 below.

Let X and Y be Hadamard manifolds with $-b^2 \le K_X, K_Y \le -a^2 < 0$. Let $k = \dim X$ and $k' = \dim Y$. Let $f : X \to Y$ be a (c, C)-quasi-isometric $\mathcal{C}^\infty$ map whose first two covariant derivatives are bounded.

We fix a point O in X. For $R > 0$, we write $B_R := B(O, R)$. Since Y is a Hadamard manifold, according to Hamilton [16] (see also Schoen and Uhlenbeck [35], [36]) there exists a unique harmonic map $h_R : B_R \to Y$ which is $\mathcal{C}^\infty$ on B_R and satisfies the Dirichlet condition $h_R = f$ on ∂B_R. We denote

$$d(h_R, f) = \sup_{x \in B(O,R)} d(h_R(x), f(x)).$$

The main step for proving existence in Theorem 1.1 is the following uniform estimate.

Proposition 3.5. *There exists a constant $\rho \ge 1$ such that $d(h_R, f) \le \rho$ for any $R \ge 1$.*

The constant ρ is a function of a, b, c, C, k and k'. More precisely, when f satisfies (4.1), ρ only needs to satisfy (4.6)–(4.8).

We briefly recall the classical argument used to deduce Theorem 1.1 from this proposition.

Proof of Theorem 1.1. As explained in Proposition 2.4, we may assume that the (c, C)-quasi-isometric map f is $\mathcal{C}^\infty$ with bounded first two covariant derivatives. Pick an unbounded increasing sequence of radii R_n and let $h_{R_n} : B_{R_n} \to Y$ be the harmonic $\mathcal{C}^\infty$ map that agrees with f on the sphere ∂B_{R_n}. Proposition 3.5 ensures that the sequence of maps (h_{R_n}) is locally uniformly bounded. Using the Cheng Lemma 3.4 it follows that their first derivatives are also locally uniformly bounded. The Ascoli–Arzelà theorem implies that, after extracting a subsequence, the sequence (h_{R_n}) converges uniformly on every ball B_S towards a continuous map $h : X \to Y$. Using the Schauder estimates, one also gets a uniform bound for the $\mathcal{C}^{2,\alpha}$-norms of h_{R_n} on B_S. These classical estimates will be recalled in formulas (5.32) and (5.33) in Section 5.6. Therefore, by the Ascoli–Arzelà theorem again, the sequence (h_{R_n}) converges in the $\mathcal{C}^2$-norm and the limit map h is $\mathcal{C}^2$ and harmonic. By construction, this limit harmonic map h stays within bounded distance of the quasi-isometric map f. □

Remark 3.6. By the uniqueness part of our Theorem 1.1 that we will prove in Section 5, the harmonic map h which stays within bounded distance of f is unique. Hence the above argument also proves that the whole family of harmonic maps h_R converges to h uniformly on compact subsets of X when R goes to infinity.

3.4. Boundary estimate

In this subsection, we begin the proof of Proposition 3.5: we bound the distance between h_R and f near the sphere ∂B_R.

Proposition 3.7. *Let X, Y be Hadamard manifolds and $k = \dim X$. Assume moreover that $K_X \le -a^2 < 0$ and $-b^2 \le K_Y \le 0$. Let $c \ge 1$ and $f : X \to Y$ be a $\mathcal{C}^\infty$ map with $\|Df(x)\| \le c$ and $\|D^2 f(x)\| \le bc^2$. Let $O \in X$, $R > 0$ and set $B_R := B(O, R)$. Let $h_R : B_R \to Y$ be the harmonic $\mathcal{C}^\infty$ map whose restriction to the sphere ∂B_R is equal to f. Then, for every x in B_R,*

$$d(h_R(x), f(x)) \le \frac{3kbc^2}{a} d(x, \partial B_R). \tag{3.2}$$

An important feature of this upper bound is that it does not depend on the radius R, provided the distance $d(x, \partial B_R)$ remains bounded. This is why we call (3.2) the *boundary estimate*. The proof relies on an idea of Jost [19, Section 4].

Proof of Proposition 3.7. This proposition is already in [3, Proposition 3.8]. We give a slightly shorter proof. Let $x \in B_R$ and let $y \in Y$ be chosen so that $d(y, f(B_R)) \ge b^{-1}$ and

$$d_y(h_R(x)) - d_y(f(x)) = d(f(x), h_R(x)). \tag{3.3}$$

This point y is far away on the geodesic ray starting at $h_R(x)$ and containing $f(x)$. Let φ be the $\mathcal{C}^\infty$ function on the ball B_R defined by

$$\varphi(z) := d_y(h_R(z)) - d_y(f(z)) - \frac{3kbc^2}{a}(R - d_O(z)) \quad \text{for all } z \text{ in } B_R. \tag{3.4}$$

This is the sum of three functions, $\varphi = \varphi_1 + \varphi_2 + \varphi_3$.

The first function $\varphi_1 : z \mapsto d_y(h_R(z))$ is subharmonic on B_R, i.e. $\Delta\varphi_1 \geq 0$. This follows from Lemma 3.3 and the harmonicity of the map h_R.

The second function $\varphi_2 : z \mapsto -d_y(f(z))$ has a bounded Laplacian, $|\Delta\varphi_2| \leq 3kbc^2$. Indeed, since y is far away, formula (2.3) yields the bound $\|D^2 d_y\| \leq 2b$ on $f(B_R)$ so that

$$|\Delta\varphi_2| = |\Delta(d_y \circ f)| \leq k\|D^2 d_y\|\,\|Df\|^2 + k\|Dd_y\|\,\|D^2 f\| \leq 3kbc^2.$$

The third function $\varphi_3 : z \mapsto -\frac{3kbc^2}{a}(R - d_O(z))$ has a Laplacian bounded below, $\Delta\varphi_3 \geq 3kbc^2$. This follows from Lemma 3.1 which says that $\Delta d_O \geq a$.

Hence the function φ is subharmonic: $\Delta\varphi \geq 0$. Since φ is zero on ∂B_R, one gets $\varphi(x) \leq 0$ as required. □

4. Interior estimate

In this section we complete the proof of Proposition 3.5.

4.1. *Strategy*

We first explain more precisely the notations and the assumptions that we will use in the whole section.

Let X and Y be Hadamard manifolds whose curvatures are pinched, $-b^2 \leq K_X, K_Y \leq -a^2 < 0$. Let $k = \dim X$ and $k' = \dim Y$. We start with a $\mathcal{C}^\infty$ quasi-isometric map $f : X \to Y$ whose first and second covariant derivatives are bounded. We fix constants $c \geq 1$ and $C > 0$ such that, for all x, x' in X:

$$\|Df(x)\| \leq c, \qquad \|D^2 f(x)\| \leq bc^2, \tag{4.1}$$

$$c^{-1}d(x,x') - C \leq d(f(x), f(x')) \leq c\,d(x,x'). \tag{4.2}$$

Note that the additive constant C in the right-hand side term of (1.1) has been removed since the derivative of f is now bounded by c.

4.1.1. Choosing the radius ℓ_0. We fix a point O in X. We introduce a fixed radius ℓ_0 depending only on a, b, k, k', c and C. This radius ℓ_0 is only required to satisfy the three inequalities (4.3)–(4.5) that will be needed later on.

The first condition we impose on the radius ℓ_0 is

$$b\ell_0 > 1. \tag{4.3}$$

The second condition is

$$\ell_0 > \frac{(A+b^{-1})c}{\sin^2(\varepsilon_0/2)} \quad \text{where} \quad \varepsilon_0 := (3c^2M)^{-N}, \tag{4.4}$$

where A is the constant given by Lemma 2.2, and M, N are the constants given by Proposition 4.9. The third condition we impose on ℓ_0 is

$$16e^{aC/2}e^{-a\ell_0/(4c)} < \theta_0 \quad \text{where} \quad \theta_0 := e^{-bA}(\varepsilon_0/4)^{bc/a}. \tag{4.5}$$

4.1.2. Assuming ρ to be large. We want to prove Proposition 3.5. For $R > 0$, recall that $h_R : B(O, R) \to Y$ is the harmonic $\mathcal{C}^\infty$ map whose restriction to the sphere $\partial B(O, R)$ is equal to f. We let

$$\rho := \sup_{x \in B(O,R)} d(h_R(x), f(x)).$$

We argue by way of contradiction. If this supremum ρ is not uniformly bounded with respect to R, we can fix a radius R such that ρ satisfies the three inequalities (4.6)–(4.8) below that we will use later on.

The first condition we impose on the radius ρ is

$$a\rho > 8kbc^2\ell_0. \tag{4.6}$$

The second condition is

$$\frac{2^7(a\rho)^2}{\sinh(a\rho/2)} < \theta_0. \tag{4.7}$$

The third condition is

$$\rho > 4c\ell_0 M(2^{10}e^{b\ell_0}k)^N \tag{4.8}$$

where M, N are the constants given by Proposition 4.9.

We denote by x a point of $B(O, R)$ where the supremum (4.1.2) is achieved:

$$d(h_R(x), f(x)) = \rho.$$

According to the boundary estimate in Proposition 3.7, condition (4.6) yields

$$d(x, \partial B(O, R)) \geq \frac{a\rho}{3kbc^2} \geq 2\ell_0.$$

Combined with (4.3), this ensures that $B(x, \ell_0) \subset B(O, R - b^{-1})$. This inclusion will allow us to apply Cheng's Lemma 3.4 at each point z of $B(x, \ell_0)$.

4.1.3. Getting a contradiction. We will focus on the restrictions of f and h_R to $B(x, \ell_0)$. We set $y := f(x)$. For y_1, y_2 in $Y \setminus \{y\}$, we denote by $\theta_y(y_1, y_2)$ the angle at y of the geodesic triangle with vertices y, y_1, y_2. For $z \in S(x, \ell_0)$, we will analyze the triangle inequality

$$\theta_y(f(z), h_R(x)) \leq \theta_y(f(z), h_R(z)) + \theta_y(h_R(z), h_R(x)) \tag{4.9}$$

and prove that on a subset U_{ℓ_0} of the sphere, each term on the right-hand side is small (Lemmas 4.5 and 4.6) while the measure of U_{ℓ_0} is large enough (Lemma 4.4) to ensure that the left-hand side is not that small (Lemma 4.8), giving rise to a contradiction. These arguments rely on uniform lower and upper bounds for the harmonic measures on spheres of X that will be given in Proposition 4.9.

We denote by ρ_h the function on $B(x,\ell_0)$ given by $\rho_h(z) = d(y, h_R(z))$ where again $y = f(x)$. By Lemma 3.3, this function is subharmonic.

Definition 4.1. We set

$$U_{\ell_0} = \{z \in S(x,\ell_0) \mid \rho_h(z) \geq \rho - \ell_0/(2c)\}. \tag{4.10}$$

4.2. *Measure estimate*

We first observe that one can control the size of $\rho_h(z)$ and of $Dh_R(z)$ on $B(x,\ell_0)$. We then derive a lower bound for the measure of U_{ℓ_0}.

Lemma 4.2. *For z in $B(x,\ell_0)$, one has*

$$\rho_h(z) \leq \rho + c\ell_0.$$

Proof. The triangle inequality and (4.2) give, for any z in $B(x,\ell_0)$,

$$\rho_h(z) \leq d(h_R(z), f(z)) + d(f(z), y) \leq \rho + c\ell_0. \qquad \square$$

Lemma 4.3. *For z in $B(x,\ell_0)$, one has*

$$\|Dh_R(z)\| \leq 2^8 kb\rho.$$

Proof. For all z, z' in $B(O,R)$ with $d(z,z') \leq b^{-1}$, the triangle inequality and (4.2) yield

$$\begin{aligned} d(h_R(z), h_R(z')) &\leq d(h_R(z), f(z)) + d(f(z), f(z')) + d(f(z'), h_R(z')) \\ &\leq \rho + b^{-1}c + \rho \leq 2\rho + c\ell_0 \leq 3\rho. \end{aligned}$$

For these last two inequalities, we have used (4.3) and (4.6). Applying Cheng's Lemma 3.4 with $R_0 = 3\rho$ and $r = b^{-1}$, one then gets for all z in $B(O, R - b^{-1})$ the bound $\|Dh_R(z)\| \leq 2^8 kb\rho$. $\square$

We now give a lower bound for the measure of U_{ℓ_0}.

Lemma 4.4. *Let $\sigma = \sigma_{x,\ell_0}$ be the harmonic measure on the sphere $S(x,\ell_0)$ at the center point x. Then*

$$\sigma(U_{\ell_0}) \geq \frac{1}{3c^2}. \tag{4.11}$$

Proof. By Lemma 3.3, the function ρ_h is subharmonic on $B(x,\ell_0)$. Hence ρ_h is not larger than the harmonic function on the ball with the same boundary values on $S(x,\ell_0)$. Comparing these functions at the center x, one gets

$$\int_{S(x,\ell_0)} (\rho_h(z) - \rho)\, \mathrm{d}\sigma(z) \geq 0. \tag{4.12}$$

By Lemma 4.2, the function ρ_h is bounded by $\rho + c\ell_0$. Hence (4.12) and the definition of U_{ℓ_0} imply

$$c\ell_0\sigma(U_{\ell_0}) - \frac{\ell_0}{2c}(1 - \sigma(U_{\ell_0})) \geq 0$$

so that $\sigma(U_{\ell_0}) \geq \frac{1}{3c^2}$. $\square$

4.3. Upper bound for $\theta_y(f(z), h_R(z))$

For all z in U_{ℓ_0}, we give an upper bound for the angle between $f(z)$ and $h_R(z)$ seen from the point $y = f(x)$.

Lemma 4.5. *For z in U_{ℓ_0}, one has*

$$\theta_y(f(z), h_R(z)) \leq 4e^{aC/2}e^{-a\ell_0/(4c)}. \tag{4.13}$$

Proof. For z in U_{ℓ_0}, we consider the triangle with vertices y, $f(z)$ and $h_R(z)$. Its side lengths satisfy

$$d(h_R(z), f(z)) \leq \rho, \quad d(y, f(z)) \geq \frac{\ell_0}{c} - C, \quad d(y, h_R(z)) \geq \rho - \frac{\ell_0}{2c},$$

where we use successively the definition of ρ, the quasi-isometry lower bound (4.2) and the definition of U_{ℓ_0}. Hence, one gets the following lower bound for the Gromov product:

$$(f(z)|h_R(z))_y \geq \frac{\ell_0}{4c} - \frac{C}{2}.$$

Since $K_Y \leq -a^2$, Lemma 2.1 now yields (4.13). □

4.4. Upper bound for $\theta_y(h_R(z), h_R(x))$

For all z in $S(x, \ell_0)$, we give an upper bound for the angle between $h_R(z)$ and $h_R(x)$ seen from $y = f(x)$.

Lemma 4.6. *For all z in $S(x, \ell_0)$, one has*

$$\theta_y(h_R(z), h_R(x)) \leq \frac{2^5(a\rho)^2}{\sinh(a\rho/2)}. \tag{4.14}$$

The proof will rely on the following lemma which also ensures that $\theta_y(h_R(z), h_R(x))$ is well-defined.

Lemma 4.7. *For all z in $B(x, \ell_0)$, one has $\rho_h(z) \geq \rho/2$.*

Proof. Assume by way of contradiction that there exists a point z_1 in $B(x, \ell_0)$ such that $\rho_h(z_1) = \rho/2$. Set $r_1 := d(x, z_1)$. One has $0 < r_1 \leq \ell_0$. According to Lemma 4.3, one can bound the differential of h_R on $B(x, \ell_0)$ as

$$\sup_{B(x,\ell_0)} \|Dh_R\| \leq 2^8 kb\rho.$$

Hence

$$\rho_h(z) \leq 3\rho/4 \quad \text{for all } z \text{ in } S(x, r_1) \cap B\big(z_1, \tfrac{1}{2^{10}kb}\big).$$

By comparison with the hyperbolic plane with curvature $-b^2$, this intersection contains the trace on the sphere $S(x, r_1)$ of a cone C_α with vertex x and angle α as long as $\sin(\alpha/2) \leq \frac{\sinh(2^{-11}/k)}{\sinh(br_1)}$. For instance we will choose $\alpha := e^{-b\ell_0}2^{-10}/k$.

Let $\sigma' = \sigma_{x,r_1}$ be the harmonic measure on $S(x,r_1)$ at the center point x. Using the subharmonicity of ρ_h as in the proof of Lemma 3.3, one gets

$$\int_{S(x,r_1)} (\rho_h(z) - \rho)\, \mathrm{d}\sigma'(z) \geq 0. \tag{4.15}$$

By Lemma 4.2, the function ρ_h is bounded by $\rho + c\ell_0$. Using the bound $\rho_h(z) \leq \frac{3}{4}\rho$ when z is in the cone C_α, (4.15) now implies that

$$c\ell_0 - \frac{\rho}{4}\sigma'(C_\alpha) \geq 0.$$

Using the uniform lower bounds for the harmonic measures on the spheres of X in Proposition 4.9, one gets

$$\rho \leq 4c\ell_0 M\alpha^{-N} = 4c\ell_0 M(2^{10}e^{b\ell_0}k)^N,$$

which contradicts (4.8). □

Proof of Lemma 4.6. Let us first sketch the proof. Let $z \in S(x,\ell_0)$. We denote by $t \mapsto z_t$, for $0 \leq t \leq \ell_0$, the geodesic segment between x and z. By Lemma 4.7, the curve $t \mapsto h_R(z_t)$ lies outside $B(y,\rho/2)$ and by Cheng's bound on $\|Dh_R(z_t)\|$ one controls the length of this curve.

We now detail the argument. We denote by $(\rho(y'), v(y')) \in\,]0,\infty[\times T^1_y Y$ the polar exponential coordinates centered at y. For a point y' in $Y \smallsetminus \{y\}$, they are defined by the equality $y' = \exp_y(\rho(y')v_\rho(y'))$. Since $K_Y \leq -a^2$, the Alexandrov comparison theorem for infinitesimal triangles and the Gauss lemma [12, 2.93] yield

$$\sinh(a\rho(y'))\|Dv(y')\| \leq a.$$

Writing $v_h := v \circ h_R$, we thus have, for any z' in $B(x,\ell_0)$,

$$\sinh(a\rho_h(z'))\, \|Dv_h(z')\| \leq a\|Dh_R(z')\|.$$

Hence, Lemma 4.7 yields

$$\theta_y(h_R(z), h_R(x)) \leq \ell_0 \sup_{0\leq t\leq \ell_0} \|Dv_h(z_t)\| \leq \frac{a\ell_0}{\sinh(a\rho/2)} \sup_{0\leq t\leq \ell_0} \|Dh_R(z_t)\|.$$

Therefore, using Lemma 4.3 and (4.6), one gets

$$\theta_y(h_R(z), h_R(x)) \leq \frac{2^8 kb\rho\, a\ell_0}{\sinh(a\rho/2)} \leq \frac{2^5(a\rho)^2}{\sinh(a\rho/2)}.$$ □

4.5. Lower bound for $\theta_y(f(z), h_R(x))$

We find a point z in U_{ℓ_0} for which the angle between $f(z)$ and $h(x)$ seen from $y = f(x)$ has an explicit lower bound.

Lemma 4.8. *There exist points z_1, z_2 in U_{ℓ_0} such that*

$$\theta_y(f(z_1), f(z_2)) \geq \theta_0,$$

where θ_0 is the angle given by (4.5).

Proof. Let $\sigma_0 := \frac{1}{3c^2}$. According to Lemma 4.4, one has $\sigma(U_{\ell_0}) \geq \sigma_0 > 0$ Thus, using the uniform upper bounds for harmonic measures on spheres of X in Proposition 4.9, one can find z_1, z_2 in U_{ℓ_0} such that

$$\sigma_0 \leq M\theta_x(z_1, z_2)^{1/N}.$$

This can be rewritten as

$$\theta_x(z_1, z_2) \geq \varepsilon_0, \tag{4.16}$$

where ε_0 is the angle introduced in (4.4) by the equality $\sigma_0 = M\varepsilon_0^{1/N}$. Therefore, using Lemma 2.1(a) and (4.4), we get the following lower bound on the Gromov products:

$$\min((x|z_1)_{z_2}, (x|z_2)_{z_1})) \geq \ell_0 \sin^2(\varepsilon_0/2) \geq (A + b^{-1})c.$$

Using then Lemma 2.2, one gets

$$\min((y|f(z_1))_{f(z_2)}, (y|f(z_2))_{f(z_1)}) \geq b^{-1}. \tag{4.17}$$

This inequality allows us to apply Lemma 2.1(c), which gives

$$\theta_y(f(z_1), f(z_2)) \geq e^{-b(f(z_1)|f(z_2))_y}.$$

Therefore, by Lemma 2.2,

$$\theta_y(f(z_1), f(z_2)) \geq e^{-bA}e^{-bc\,(z_1|z_2)_x}.$$

Using Lemma 2.1(b) and (4.16), one gets

$$\theta_y(f(z_1), f(z_2)) \geq e^{-bA}(\theta_x(z_1, z_2)/4)^{bc/a} \geq e^{-bA}(\varepsilon_0/4)^{bc/a} = \theta_0,$$

according to the definition (4.5) of θ_0. □

End of proof of Proposition 3.5. Using Lemmas 4.5 and 4.6 and the triangle inequality (4.9) one gets, for any two points $z_i = z_1$ or z_2 in U_{ℓ_0},

$$\begin{aligned}\theta_y(f(z_i), h_R(x)) &\leq 4e^{aC/2}e^{-a\ell_0/(4c)} + \frac{2^5(a\rho)^2}{\sinh(a\rho/2)} \\ &< \tfrac{1}{2}\theta_0 \quad \text{by (4.5) and (4.7).}\end{aligned}$$

Therefore, using again a triangle inequality, one has $\theta_y(f(z_1), f(z_2)) < \theta_0$, which contradicts Lemma 4.8. □

4.6. Harmonic measures

The following proposition gives the uniform lower and upper bounds for the harmonic measure on a sphere at the center which were used in the proof of Lemmas 4.7 and 4.8.

Proposition 4.9. *Let $0 < a < b$ and $k \geq 2$ be an integer. There exist positive constants M, N depending only on a, b, k such that for every k-dimensional Hadamard manifold X with pinched curvature $-b^2 \leq K_X \leq -a^2$, for every point x in X, every radius $r > 0$ and every angle $\theta \in [0, \pi]$ one has*

$$\tfrac{1}{M}\theta^N \leq \sigma_{x,r}(C_{x,\theta}) \leq M\theta^{1/N} \tag{4.18}$$

where $\sigma_{x,r}$ denotes the harmonic measure on $S(x, r)$ at the point x and where $C_{x,\theta}$ stands for any cone with vertex x and angle θ.

We recall that, by definition, $\sigma_{x,r}$ is the unique probability measure on $S(x, r)$ such that, for every continuous function h on $B(x, r)$ which is harmonic in the interior $\mathring{B}(x, r)$, one has

$$h(x) = \int_{S(x,r)} h(z)\, \mathrm{d}\sigma_{x,r}(z).$$

A proof of Proposition 4.9 is given in [4]. It relies on various technical tools of potential theory on pinched Hadamard manifolds: the Harnack inequality, the barrier functions constructed by Anderson and Schoen [2] and upper and lower bounds for the Green functions due to Ancona [1]. Related estimates are the one by Kifer–Ledrappier [21, Theorem 3.1 and 4.1] where (4.18) is proven for the sphere at infinity or by Ledrappier–Lim [22, Proposition 3.9] where the Hölder regularity of the Martin kernel is proven.

5. Uniqueness of harmonic maps

In this section we prove the uniqueness part in Theorem 1.1.

5.1. Strategy

In other words we will prove the following proposition.

Proposition 5.1. *Let X, Y be pinched Hadamard manifolds and let $h_0, h_1 : X \to Y$ be quasi-isometric harmonic maps that stay within bounded distance of one another:*

$$\sup_{x \in X} d(h_0(x), h_1(x)) < \infty.$$

Then $h_0 = h_1$.

When $X = Y = \mathbb{H}^2$, this proposition was proven by Li and Tam [24]. When both X and Y admit a cocompact group of isometries, it was proven by Li and Wang [25, Theorem 2.3]. The aim of this subsection is to explain how to get rid of these extra assumptions.

Note that the assumption that the h_i are quasi-isometric is useful. Indeed, there do exist non-constant bounded harmonic functions on X. Note that there also exist bounded harmonic maps with open images. Here is a very simple example. Let $0 < \lambda < 1$. The map h_λ from the Poincaré unit disk $\mathbb{D}$ of $\mathbb{C}$ into itself given by $z \mapsto \lambda z$ is harmonic. More generally, for any harmonic map $h : \mathbb{D} \to \mathbb{D}$, the map $h_\lambda : \mathbb{D} \to \mathbb{D} : z \mapsto h(\lambda z)$ is harmonic with bounded image.

Before going into technical details, we first explain the strategy of the proof of uniqueness.

Strategy of proof of Proposition 5.1. We recall that $x \mapsto d(h_0(x), h_1(x))$ is a subharmonic function on X and that, by the maximum principle, a subharmonic function that achieves its maximum value is constant. Unfortunately since X is non-compact we cannot a priori ensure that this bounded function achieves its maximum. That is why we will use a recentering argument. This will force us to deal with Riemannian manifolds which are not $\mathcal{C}^\infty$ (see Section 5.4).

We assume, towards a contradiction, that $h_0 \neq h_1$, we choose a sequence of points p_n in X for which

$$d(h_0(p_n), h_1(p_n)) \to \delta := \sup_{x \in X} d(h_0(x), h_1(x)) > 0 \tag{5.1}$$

and we set $q_n := h_0(p_n)$.

The pinching conditions on X and Y ensure that, after extracting a subsequence, the pointed metric spaces (X, p_n) and (Y, q_n) converge in the Gromov–Hausdorff topology to pointed metric spaces (X_∞, p_∞) and (Y_∞, q_∞) which are $\mathcal{C}^2$ Hadamard manifolds with $\mathcal{C}^1$ Riemannian metrics satisfying the same pinching conditions (Proposition 5.14). Moreover, extracting again a subsequence, the harmonic map h_0 (resp. h_1) seen as a sequence of maps between the pointed Hadamard manifolds (X, p_n) and (Y, q_n) converges locally uniformly to a map $h_{0,\infty}$ (resp $h_{1,\infty}$) between the pointed C^2 Hadamard manifolds (X_∞, p_∞) and (Y_∞, q_∞). These harmonic maps $h_{0,\infty}$ and $h_{1,\infty}$ are still harmonic quasi-isometric maps (Lemma 5.15).

The limit distance function $x \mapsto d(h_{0,\infty}(x), h_{1,\infty}(x))$ is a subharmonic function on X_∞ that now achieves its maximum $\delta > 0$ at the point p_∞. Hence, by the maximum principle, this distance function is constant and equal to δ (Lemma 5.16). Generalizing [25, Lemma 2.2], we will see in Corollary 5.19 that this equidistance property implies that both $h_{0,\infty}$ and $h_{1,\infty}$ take their values in a geodesic of Y_∞. This contradicts the fact that $h_{0,\infty}$ and $h_{1,\infty}$ are quasi-isometric maps, and concludes this description of the strategy of proof. □

In the following subsections of Section 5, we fill in the details of the proof.

5.2. *Harmonic coordinates*

We first introduce the so-called harmonic coordinates, which improve the quasilinear coordinates introduced in Lemma 2.6. We refer to [15] or [19] for more details.

The harmonic coordinates have been introduced by DeTurk and Kazdan and extensively used by Cheeger, Jost, Karcher, Petersen and others to prove various compactness results for compact Riemannian manifolds. Besides being harmonic, the main advantage of these coordinates is that, for every $\alpha \in]0,1[$, they are uniformly bounded in the $\mathcal{C}^{2,\alpha}$-norm, i.e. they are uniformly bounded in the $\mathcal{C}^2$-norm and one also has uniform control of the α-Hölder norm of their second covariant derivatives. Moreover, one has uniform control on the size of balls on which these harmonic charts are defined. This is what the following lemma tells us.

We endow $\mathbb{R}^k$ with the standard Euclidean structure.

Lemma 5.2. *Let X be a k-dimensional Hadamard manifold with bounded curvature, $-1 \le K_X \le 0$. Let $0 < \alpha < 1$. There exist constants $r_0 = r_0(k) > 0$ and $c_0 = c_0(k,\alpha) > 0$ such that, for every x in X, there exists a $\mathcal{C}^\infty$ diffeomorphism*

$$\Psi_x : \mathring{B}(x,r_0) \xrightarrow{\sim} U_x \subset \mathbb{R}^k \quad \text{with} \quad \Psi_x(x) = 0, \tag{5.2}$$

$$\|D\Psi_x\| \le c_0, \qquad \|D\Psi_x^{-1}\| \le c_0, \qquad \|D^2\Psi_x\| \le c_0, \qquad \|D^2\Psi_x^{-1}\| \le c_0 \tag{5.3}$$

and such that each component $z_1, \dots, z_k$ of Ψ_x is a harmonic function.

In particular, for all $r < r_0$ one has

$$\Psi_x(B(x, c_0^{-1} r)) \subset B(0,r) \quad \text{and} \quad B(0, c_0^{-1} r) \subset \Psi_x(B(x,r)). \tag{5.4}$$

The second covariant derivatives of all Ψ_x are also uniformly α-Hölder:

$$\|D^2\Psi_x\|_{\mathcal{C}^\alpha} \le c_0. \tag{5.5}$$

This α-Hölder seminorm $\|D^2\Psi_x\|_{\mathcal{C}^\alpha}$ is defined as follows. Using the vector fields $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_k}$ on $\mathring{B}(x,r_0)$ associated to our coordinate system $\Psi_x = (z_1,\dots,z_k)$, we reinterpret the tensor $D^2\Psi_x$ as a family of vector valued functions on $\mathring{B}(x,r_0)$. Indeed, we set

$$T_x^{ij}(z) = D^2\Psi_x(z)\big(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j}\big) \in \mathbb{R}^k \quad \text{for } i, j \text{ in } \{1,\dots,k\},$$

and the bound (5.5) means that

$$\|D^2\Psi_x\|_{\mathcal{C}^\alpha} := \max_{i,j} \sup_{z,z'} \frac{\|T_x^{ij}(z) - T_x^{ij}(z')\|}{d(z,z')^\alpha} \le c_0. \tag{5.6}$$

The uniform bounds (5.3) and (5.5) have three consequences.

First, in the harmonic coordinate systems $\Psi_x = (z_1,\dots,z_k)$, the Christoffel coefficients Γ_{ij}^ℓ are uniformly bounded in the $\mathcal{C}^\alpha$-norm. Indeed, these coefficients $(\Gamma_{ij}^\ell)_{1\le\ell\le k}$ are the components of the vector $-D^2\Psi_x\big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\big) \in \mathbb{R}^k$.

Second, on their domain of definition, the transition functions

$$\Psi_{x'} \circ \Psi_x^{-1} \text{ are uniformly bounded in the } \mathcal{C}^{2,\alpha}\text{-norm.} \tag{5.7}$$

Third, in the coordinate systems $\Psi_x = (z_1,\dots,z_k)$, the coefficients of the metric tensor

$$g_{ij} := g\big(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j}\big) \text{ are uniformly bounded in the } \mathcal{C}^{1,\alpha}\text{-norm.} \tag{5.8}$$

Proof of Lemma 5.2. See [19, pp. 62 and 65] or [32, Section 4]. □

5.3. *Gromov–Hausdorff convergence*

In this subsection, we recall the definition of Gromov–Hausdorff convergence for pointed metric spaces and some of its key properties. We refer to [7] for more details.

5.3.1. Definition. When X is a metric space, we will denote by d or d_X the distance on X. Recall that $B(x, R)$ denotes the closed ball with center x and radius R, and $\mathring{B}(x, R)$ the open ball. Also recall that a metric space X is *proper* if all its balls are compact or, equivalently, if X is complete and for all $R > 0$ and $\varepsilon > 0$ every ball of radius R can be covered by finitely many balls of radius ε.

We also recall the notion of Gromov–Hausdorff distance between two (isometry classes of proper) pointed metric spaces.

Definition 5.3. The *Gromov–Hausdorff distance* between pointed metric spaces (X, p) and (Y, q) is the infimum of the $\varepsilon > 0$ for which there exists a subset $\mathcal{R}$ of $X \times Y$, called a *correspondence*, such that

(i) the correspondence $\mathcal{R}$ contains the pair (p, q),
(ii) for all x in $B(p, \varepsilon^{-1})$, there exists y in Y with (x, y) in $\mathcal{R}$,
(iii) for all y in $B(q, \varepsilon^{-1})$, there exists x in X with (x, y) in $\mathcal{R}$,
(iv) for all (x, y) and (x', y') in $\mathcal{R}$, one has $|d(x, x') - d(y, y')| \le \varepsilon$.

Heuristically, this correspondence $\mathcal{R}$ can be thought of as an ε-rough isometry between these two balls with radius ε^{-1}.

Based on this definition, a sequence (X_n, p_n) of pointed metric spaces converges to a pointed metric space (X_∞, p_∞) if, for all $\varepsilon > 0$, there exists n_0 such that for $n \ge n_0$, there exists a map $f_n : B(p_n, \varepsilon^{-1}) \to X_\infty$ such that

(α) $d(f_n(p_n), p_\infty) \le \varepsilon$,
(β) $|d(f_n(x), f_n(x')) - d(x, x')| \le \varepsilon$ for all x, x' in $B(p_n, \varepsilon^{-1})$,
(γ) the ε-neighborhood of $f_n(B(p_n, \varepsilon^{-1}))$ contains $B(p_\infty, \varepsilon^{-1} - \varepsilon)$.

Definition 5.3 is only useful for complete metric spaces. Indeed, the Gromov–Hausdorff topology does not distinguish between a metric space and its completion. It does not distinguish either between two pointed metric spaces that are isometric: it is a distance on the set of isometry classes of proper pointed metric spaces. See [7, Theorem 8.1.7].

The following equivalent definition of Gromov–Hausdorff convergence is useful when one wants to get rid of the ambiguity coming from the group of isometries of (X_∞, p_∞).

Fact 5.4. *Let (X_n, p_n), for $n \ge 1$, and (X_∞, p_∞) be pointed proper metric spaces. The sequence (X_n, p_n) converges to (X_∞, p_∞) if and only if there exists a complete metric space Z containing isometrically all the metric spaces X_n and X_∞ as disjoint closed subsets, and such that*

(a) *p_n converges to p_∞ in Z,*
(b) *X_n converges to X_∞ in the Hausdorff topology.*

Statement (b) means that

- every point z of X_∞ is the limit of a sequence $(x_n)_{n\geq 1}$ with $x_n \in X_n$,
- every cluster point $z \in Z$ of a sequence $(x_n)_{n\geq 1}$ with $x_n \in X_n$ belongs to X_∞.

Sketch of proof of Fact 5.4. Assume that (X_n, p_n) converges to (X_∞, p_∞). We want to construct the metric space Z. We choose a sequence $\varepsilon_n \searrow 0$ and correspondences $\mathcal{R}_n$ on $X_n \times X_\infty$ as in Definition 5.3 with $p = p_n$, $q = p_\infty$ and $\varepsilon = \varepsilon_n$. This allows us to construct, for every $n \geq 1$, a metric space Y_n which is the disjoint union of X_n and X_∞, which contains isometrically both X_n and X_∞ and such that the distance between x in X_n and y in X_∞ is given by

$$d_{Y_n}(x, y) = \inf\{d_{X_n}(x, x') + \varepsilon + d_{X_\infty}(y', y)\}, \tag{5.9}$$

where the infimum is over all pairs (x', y') which belong to $\mathcal{R}_n$.

The space Z is defined as the disjoint union of all the X_n and of X_∞. The distance on Z is given on each union $Y_n := X_n \cup X_\infty$ by (5.9) and the distance between x in X_m and z in X_n with $m \neq n$ is

$$d_Z(x, z) = \inf\{d_{Y_m}(x, y) + d_{Y_n}(y, z)\}, \tag{5.10}$$

where the infimum is over all y in X_∞.

Then (a) follows from (i), and (b) follows from (ii)–(iv). □

The choice of such isometric embeddings of all X_n and X_∞ in a fixed metric space Z will be called a *realization* of Gromov–Hausdorff convergence. Such a realization is not unique. It is useful since it allows us to define the notion of a converging sequence of points x_n in X_n to a limit x_∞ in X_∞ by the condition $d_Z(x_n, x_\infty) \xrightarrow[n\to\infty]{} 0$.

5.3.2. Compactness criterion. A fundamental tool in this topic is the following compactness result for *uniformly proper* pointed metric spaces due to Cheeger–Gromov:

Fact 5.5. *Let $(X_n, p_n)_{n\geq 1}$ be a sequence of pointed proper metric spaces. Suppose that, for all $R > 0$ and $\varepsilon > 0$, there exists an integer $N = N(R, \varepsilon)$ such that, for all $n \geq 1$, the ball $B(p_n, R)$ of X_n can be covered by N balls of radius ε. Then there exists a subsequence of (X_n, p_n) which converges to a proper pointed metric space (X_∞, p_∞).*

For the proof see [7, Theorem 8.1.10].

The following lemma gives a compactness property for sequences of Lipschitz functions between pointed metric spaces.

Lemma 5.6. *Let $(X_n, p_n)_{n\geq 1}$ and $(Y_n, q_n)_{n\geq 1}$ be sequences of pointed proper metric spaces which converge respectively to proper pointed metric spaces (X_∞, p_∞) and (Y_∞, q_∞). As in Fact 5.4, we choose metric spaces Z_X and Z_Y which realize these Gromov–Hausdorff convergences as Hausdorff convergences.*

Let $c > 1$ and let $(f_n : X_n \to Y_n)_{n\geq 1}$ be a sequence of c-Lipschitz maps such that $f_n(p_n) = q_n$. Then there exists a c-Lipschitz map $f_\infty : X_\infty \to Y_\infty$ such that, after extracting a subsequence, the sequence of maps f_n converges to f_∞. This means that for each sequence $x_n \in X_n$ which converges to $x_\infty \in X_\infty$, the sequence $f_n(x_n) \in Y_n$ converges to $f_\infty(x_\infty) \in Y_\infty$.

Proof. This follows from basic topological arguments.

First step. We first choose a point x_∞ in X_∞ and a sequence x_n in X_n converging to x_∞. Since the metric space Z_Y is proper and the sequence $f_n(x_n)$ is bounded in Z_Y, we can assume after extracting a subsequence that $f_n(x_n)$ converges to a point $y_\infty \in Y_\infty$. Since the f_n are c-Lipschitz, this limit y_∞ does not depend on the choice of the sequence x_n converging to x_∞. We define $f_\infty(x_\infty) := y_\infty$.

Second step. We choose a countable dense subset $S_\infty \subset X_\infty$ and use Cantor's diagonal argument to ensure that the first step is valid simultaneously for all x_∞ in S_∞.

Last step. One checks that the limit map $f_\infty : S_\infty \to Y_\infty$ is c-Lipschitz. Hence it extends uniquely as a c-Lipschitz map $f_\infty : X_\infty \to Y_\infty$ and the sequence f_n converges locally uniformly to f_∞. □

5.3.3. Length spaces and Alexandrov spaces. We recall a few well-known definitions (see [7]).

A *length space* is a complete metric space for which the distance δ between two points is the infimum of the lengths of curves joining them. When X is proper, any two points at distance δ can be joined by a curve of length δ. Such a curve is called a *geodesic segment*.

Let $K \leq 0$. A CAT(K)*-space* or CAT*-space with curvature at most* K is a length space in which any geodesic triangle (P, Q, R) is thinner than a comparison triangle $(\overline{P}, \overline{Q}, \overline{R})$ in the plane $\overline{X}$ of constant curvature K. Let us explain what this means. A *comparison triangle* is a triangle in $\overline{X}$ with the same side lengths. For every point P' on the geodesic segment $[P, Q]$ we denote by $\overline{P}'$ the corresponding point on the geodesic segment $[\overline{P}, \overline{Q}]$, i.e. the point such that $d(P, P') = d(\overline{P}, \overline{P}')$. *Thinner* means that always $d(P', R) \leq d(\overline{P}', \overline{R})$. Note that a CAT(0)-space is always simply connected (see [6, Corollary II.1.5]). We also recall that in a proper CAT(0)-space, any two points can be joined by a unique geodesic.

Similarly, a *metric space with curvature at least* K is a length space in which any geodesic triangle (P, Q, R) is thicker than a comparison triangle $(\overline{P}, \overline{Q}, \overline{R})$ in the plane $\overline{X}$ of constant curvature K. *Thicker* means that always $d(P', R) \geq d(\overline{P}', \overline{R})$.

The following proposition tells us that these properties are closed for the Gromov–Hausdorff topology.

Fact 5.7. *Let* $(X_n, p_n)_{n\geq 1}$ *and* (X_∞, p_∞) *be pointed proper metric spaces. Let* $K \leq 0$. *Assume that the sequence* (X_n, p_n) *converges to* (X_∞, p_∞).

(i) *If the* X_n*'s are length spaces, then* X_∞ *is also a length space.*
(ii) *If the* X_n*'s are* CAT(K)*-spaces, then* X_∞ *is also a* CAT(K)*-space.*
(iii) *If the* X_n*'s have curvature at least* K, *then* X_∞ *too.*

Proof. For (i), see [7, Theorem 8.1.9]; for (ii), see [6, Corollary II.3.10]; and for (iii), see [7, Theorem 10.7.1]. □

5.4. Hadamard manifolds with $\mathcal{C}^1$ metrics

In this subsection we focus on $\mathcal{C}^2$ Hadamard manifolds when the Riemannian metric is only assumed to be $\mathcal{C}^1$. These Hadamard manifolds will occur in Subsection 5.5 as Gromov–Hausdorff limits of pinched $\mathcal{C}^\infty$ Hadamard manifolds.

5.4.1. Definition. We need first to clarify the definitions. We will deal with $\mathcal{C}^2$ manifolds X. This means that X has a system of charts $x \mapsto (x_1, \dots, x_k)$ into $\mathbb{R}^k$ for which the transition functions are of class $\mathcal{C}^2$. These manifolds will be endowed with a $\mathcal{C}^1$ Riemannian metric g. This means that in any $\mathcal{C}^2$ chart, the functions $g\big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big)$ are continuously differentiable.

In general, on such a Riemannian manifold, there might exist two different geodesics which are tangent at some point (see [18] for an example with a $\mathcal{C}^{1,\alpha}$ Riemannian metric). The following lemma tells us that this kind of example will not occur here since we are dealing only with CAT(0)-spaces whose curvature is bounded below. Note that since the metric tensor is not assumed to be twice differentiable, the expression "curvature bounded below" refers to the definitions in Section 5.3.

Definition 5.8. By a $\mathcal{C}^2$ *Hadamard manifold with a* $\mathcal{C}^1$ *metric*, we mean a $\mathcal{C}^2$ manifold endowed with a $\mathcal{C}^1$ Riemannian metric which is CAT(0) and complete.

5.4.2. Exponential map

Lemma 5.9. *Let X be a $\mathcal{C}^2$ Hadamard manifold with a $\mathcal{C}^1$ metric of bounded curvature.*

(a) *For all x in X and v in T_xX there is a unique geodesic $t \mapsto \exp_x(tv)$ starting from x at speed v. This geodesic is of class $\mathcal{C}^2$.*

(b) *This exponential map induces a homeomorphism $\Psi : TX \xrightarrow{\sim} X \times X$ given by $\Psi(x, v) = (x, \exp_x(v))$ for x in X and v in T_xX.*

Proof. This lemma looks very familiar. But, since the Christoffel coefficients might not be Lipschitz continuous, we cannot apply the Cauchy–Lipschitz theorem on existence and uniqueness of solutions of differential equations.

(a) Since the Christoffel coefficients are continuous, we can apply the Peano–Arzelà theorem. It tells us that there exists at least one geodesic of class $\mathcal{C}^2$ starting from x at speed v. Uniqueness follows from the lower bound on the curvature.

(b) Since X is CAT(0), the map Ψ is a bijection. Since a uniform limit of geodesics on X is also a geodesic, the map Ψ is continuous. This map Ψ is also proper, so it is a homeomorphism. □

5.4.3. Geodesic interpolation of h_0 and h_1. In the rest of this section we prove a few technical properties of the interpolation h_t of two equidistant Lipschitz maps h_0 and h_1 with values in a Hadamard manifold (Lemma 5.10). In Section 5.8, we will apply this lemma to two equidistant harmonic maps h_0 and h_1 obtained by a limit process. Lemma 5.10 will be used to compare the energy of h_0 and h_1 with the energy of some small perturbations of h_0 and h_1. However, in Section 5.4, we do not need to assume that h_0 and h_1 are harmonic. Here are the precise assumptions and notations for Lemma 5.10.

Let X be a $\mathcal{C}^2$ Riemannian manifold with $\mathcal{C}^1$ metric and Y be a $\mathcal{C}^2$ Hadamard manifold with $\mathcal{C}^1$ metric. Let $h_0, h_1 : X \to Y$ be $\mathcal{C}^1$ maps such that

$$d(h_0(x), h_1(x)) = 1 \quad \text{for all } x \text{ in } X. \tag{5.11}$$

Since Y is a Hadamard manifold, there exists a unique map

$$h : [0, 1] \times X \to Y, \quad (t, x) \mapsto h(t, x) = h_t(x), \tag{5.12}$$

such that, for all x in X, the path $t \mapsto h_t(x)$ is the unit-speed geodesic joining $h_0(x)$ and $h_1(x)$. This map h is called the *geodesic interpolation* of h_0 and h_1. By convexity of the distance function, h is Lipschitz continuous. Therefore, by Rademacher's theorem, the map h is differentiable on a subset of full measure (with respect to the Riemannian measure on X). In particular, there exists a subset $X' \subset X$ of full measure such that, for all x in X', the map h is differentiable at (x, t) for almost all t in $[0, 1]$. In particular, for all tangent vectors $V \in T_x X$ at a point $x \in X'$, the derivative

$$t \mapsto J_V(t) := D_x h_t(V) \in T_{h_t(x)} Y \tag{5.13}$$

is well-defined for almost all t in $[0, 1]$. Such a measurable vector field J_V on the geodesic $t \mapsto h_t(x)$ will be called a *Jacobi field*. We denote by

$$t \mapsto \tau_x(t) := \partial_t h_t(x) \in T_{h_t(x)} Y \tag{5.14}$$

the unit tangent vector to the geodesic $t \mapsto h_t(x)$.

Lemma 5.10. *We keep the above assumptions and notations. Let $x \in X'$ and $V \in T_x X$.*

(a) *There exists a constant $\alpha_V \in \mathbb{R}$ such that*

$$\langle J_V(t), \tau_x(t) \rangle = \alpha_V \quad \textit{for all } t \textit{ in } [0, 1] \textit{ where } J_V(t) \textit{ is defined.} \tag{5.15}$$

(b) *There exists a convex function $t \mapsto \varphi_V(t)$ on $[0, 1]$ such that*

$$\varphi_V(t) = \| J_V(t) \| \quad \textit{for all } t \textit{ in } [0, 1] \textit{ where } J_V(t) \textit{ is defined.} \tag{5.16}$$

(c) *The function $\psi_V := (\varphi_V^2 - \alpha_V^2)^{1/2}$ is also convex on $[0, 1]$.*

Proof. When Y is a $\mathcal{C}^\infty$ Hadamard manifold, the vector field J_V is a classical Jacobi field and this lemma is well-known. Indeed, ψ_V is the norm of the orthogonal component K_V of the Jacobi field J_V, and (5.12) follows from the Jacobi equation satisfied by K_V. We now explain how to adapt the classical proof when Y is only assumed to be a $\mathcal{C}^2$ Hadamard manifold with a $\mathcal{C}^1$ metric.

(a) Since $t \mapsto h_t(x)$ is a unit-speed geodesic, one has $d(h_s(x), h_t(x)) = |t - s|$ for all s, t in $[0, 1]$. Differentiating this equality gives, when $J_V(s)$ and $J_V(t)$ are defined,

$$\langle J_V(s), \tau_x(s) \rangle = \langle J_V(t), \tau_x(t) \rangle.$$

Hence this scalar product is almost surely constant.

(b) Let $c : [-\varepsilon_0, \varepsilon_0] \to X$ be a $\mathcal{C}^1$ curve with $c(0) = x$ and $\partial_s c(0) = V$. Since the space Y is CAT(0), when $s > 0$ the functions

$$t \mapsto \varphi_s(t) := \frac{1}{s} d(h_t(c(0)), h_t(c(s)))$$

are convex on $[0, 1]$. The set $S_V := \{t \in [0, 1] \mid J_V(t) \text{ is defined}\}$ has full measure and contains the endpoints 0 and 1. For all t in S_V, one can compute the limit $\lim_{s\to 0} \varphi_s(t) = \|J_V(t)\|$. Since the functions φ_s are convex, the limit $\varphi_V(t) := \lim_{s\to 0} \varphi_s(t)$ exists for all t in $[0, 1]$ and is a convex function.

(c) We slightly change the parametrization of the geodesic interpolation: the function $k : (t, s) \mapsto k_t(s) := h_{t-s\alpha_V}(c(s))$ is well-defined when $t - s\alpha_V$ is in $[0, 1]$, and the paths $t \mapsto k_t(s)$ are also unit-speed geodesics. Hence, for almost all t in $[0, 1]$, the vector field

$$t \mapsto K_V(t) := \partial_s k_t(0) \in T_{k_t(0)} Y \tag{5.17}$$

is well-defined and one has the orthogonal decomposition

$$J_V(t) = K_V(t) + \alpha_V \tau_x(t).$$

In particular,

$$\psi_V(t) = \|K_V(t)\|. \tag{5.18}$$

The same argument as in (b) with the Jacobi field K_V proves that ψ_V is also convex. □

5.4.4. Geodesic interpolation in negative curvature. Lemma 5.11 below improves Lemma 5.10 when the curvature of Y is uniformly negative. Indeed, it tells us that the norm $t \mapsto \psi_V(t)$ of the Jacobi field K_V is uniformly convex.

Lemma 5.11. *We keep the assumptions and notations of Lemma 5.10. Moreover, assume that Y is a* CAT$(-a^2)$*-space with $a > 0$. Then the function ψ_V has the following uniform convexity property:*

$$\psi_V(t) \le \frac{\sinh(a(1-t))}{\sinh(a)} \psi_V(0) + \frac{\sinh(at)}{\sinh(a)} \psi_V(1) \quad \textit{for all } t \textit{ in } [0, 1]. \tag{5.19}$$

Remark 5.12. One can reformulate (5.19) as the following inequality between positive measures:

$$\tfrac{d^2}{dt^2} \psi_V \ge a^2 \psi_V.$$

Proof of Lemma 5.11. The inequality (5.19) will follow from an upper bound for the norm of the Jacobi field $t \mapsto K_V(t)$ by the norm of a well-chosen Jacobi field $t \mapsto \overline{K}(t)$ along a geodesic segment in the hyperbolic plane of curvature $-a^2$. Here are the details of the construction of $\overline{K}$.

Using a slight rescaling, we can assume without loss of generality that the geodesics $t \mapsto k_t(s)$ are defined for t in $[0, 1]$ and that the Jacobi field $K_V(t)$ is well-defined for $t = 0$ and for $t = 1$. We choose $s > 0$. Later on we will let s go to 0. We set $P_t := k_t(0)$ and $Q_{s,t} := k_t(s)$, and we apply Reshetnyak's Lemma 5.13 below to the four points P_0, P_1, $Q_{s,1}$, $Q_{s,0}$. According to that lemma, there exists a convex quadrilateral $\overline{C}_s$ in the hyperbolic plane $\overline{Y}$ of curvature $-a^2$ with vertices $\overline{P}_0$, $\overline{P}_1$, $\overline{Q}_{s,1}$, $\overline{Q}_{s,0}$, and a 1-Lipschitz map $j : \overline{C}_s \to Y$ whose restriction to each of the four geodesic sides $\overline{P}_0\overline{P}_1$, $\overline{P}_1\overline{Q}_{s,1}$,

$\overline{Q}_{s,1}\overline{Q}_{s,0}$, $\overline{Q}_{s,0}\overline{P}_0$ is an isometry onto each of the four geodesic segments P_0P_1, $P_1Q_{s,1}$, $Q_{s,1}Q_{s,0}$, $Q_{s,0}P_0$. Indeed, since $d(\overline{P}_0, \overline{P}_1) = 1$, we can assume that the vertices $\overline{P}_0$ and $\overline{P}_1$ do not depend on s and that the quadrilateral $\overline{C}_s$ is positively oriented.

Since the vectors $K_V(0)$ and $K_V(1)$ are orthogonal to the geodesic segment $t \mapsto k_t(0)$, by Lemma 5.9 each of the four successive angles θ_i (for $i = 1, \dots, 4$) between the four successive geodesic segments P_0P_1, $P_1Q_{s,1}$, $Q_{s,1}Q_{s,0}$, $Q_{s,0}P_0$ in Y is equal to $\pi/2 + o(1)$, where $o(1)$ goes to 0 when s goes to 0. Since j is 1-Lipschitz, each of the corresponding angles $\overline{\theta}_i$ between the geodesic sides $\overline{P}_0\overline{P}_1$, $\overline{P}_1\overline{Q}_{s,1}$, $\overline{Q}_{s,1}\overline{Q}_{s,0}$, $\overline{Q}_{s,0}\overline{P}_0$ in the hyperbolic plane $\overline{Y}$ is no smaller than θ_i. Since the sum of the angles $\overline{\theta}_i$ is bounded above by 2π, each of them also satisfies, when s goes to 0,

$$\overline{\theta}_i = \pi/2 + o(1). \tag{5.20}$$

Denote by $t \mapsto \overline{P}_t$ and $t \mapsto \overline{Q}_{s,t}$ the unit-speed parametrizations of the sides $\overline{P}_0\overline{P}_1$ and $\overline{Q}_0\overline{Q}_1$. For t in [0, 1], one has $j(\overline{P}_t) = P_t$ and $j(\overline{Q}_{s,t}) = Q_{s,t}$, and also

$$d(P_t, Q_{s,t}) \le d(\overline{P}_t, \overline{Q}_{s,t}) \tag{5.21}$$

with equality when $t = 0$ or 1:

$$d(P_0, Q_{s,0}) = d(\overline{P}_0, \overline{Q}_{s,0}) \quad \text{and} \quad d(P_1, Q_{s,1}) = d(\overline{P}_1, \overline{Q}_{s,1}). \tag{5.22}$$

We now focus on these convex quadrilaterals $\overline{C}_s$ in the hyperbolic plane $\overline{Y}$ of curvature $-a^2$. We write $\overline{Q}_{s,t} = \exp_{\overline{P}_t}(s\overline{K}_{s,t})$ where $\overline{K}_{s,t}$ belongs to $T_{\overline{P}_t}\overline{Y}$. Since $K_V(0)$ and $K_V(1)$ are well-defined, by (5.17), (5.18), (5.20) and (5.22) the limits

$$\overline{K}(0) = \lim_{s\to 0} \overline{K}_{s,0} \quad \text{and} \quad \overline{K}(0) = \lim_{s\to 0} \overline{K}_{s,1}$$

exist and satisfy

$$\|\overline{K}(0)\| = \psi_V(0) \quad \text{and} \quad \|\overline{K}(1)\| = \psi_V(1). \tag{5.23}$$

Therefore, the limit

$$\overline{K}(t) = \lim_{s\to 0} \overline{K}_{s,t}$$

exists for all t in [0, 1]. Moreover, by (5.17), (5.18) and (5.21),

$$\psi_V(t) \le \|\overline{K}(t)\|. \tag{5.24}$$

Since $t \mapsto \overline{K}(t)$ is a Jacobi field along the geodesic segment $t \mapsto \overline{P}_t$, which is orthogonal to the tangent vector, its norm

$$\overline{\psi}(t) := \|\overline{K}(t)\|$$

satisfies the Jacobi differential equation

$$\frac{d^2}{dt^2}\overline{\psi} = a^2\overline{\psi}.$$

Hence,

$$\overline{\psi}(t) = \frac{\sinh(a(1-t))}{\sinh(a)}\overline{\psi}(0) + \frac{\sinh(at)}{\sinh(a)}\overline{\psi}(1) \quad \text{for all } t \text{ in } [0,1]. \tag{5.25}$$

We now deduce (5.19) directly from (5.23)–(5.25). □

We have used the following existence result for a majorizing quadrilateral, due to Reshetnyak [34]. More precisely we have used the boundary of the majorizing quadrilateral $\overline{C}$.

Lemma 5.13. *Let Y be a* $\mathrm{CAT}(-a^2)$*-space and $\overline{Y}$ be the hyperbolic plane of curvature $-a^2$. Then, for any four points P_0, P_1, Q_1, Q_0 in Y there exists a convex quadrilateral $\overline{C}$ in $\overline{Y}$ with vertices $\overline{P}_0, \overline{P}_1$, $\overline{Q}_1, \overline{Q}_0$ and a 1-Lipschitz map $j : \overline{C} \to Y$ which is an isometry on each of the four geodesic sides of $\overline{C}$, and which sends each of these four vertices $\overline{R}_i$ to the corresponding given point R_i in Y.*

5.5. Limits of Hadamard manifolds

In this subsection we describe the Gromov–Hausdorff limits of pinched Hadamard manifolds.

The following proposition is a variation on the Cheeger compactness theorem.

Proposition 5.14. *Let $(X_n, p_n)_{n\geq 1}$ be a sequence of k-dimensional pointed Hadamard manifolds with pinched curvature $-1 \leq K_{X_n} \leq -a^2 \leq 0$.*

(a) *There exists a subsequence of (X_n, p_n) which converges to a pointed proper* CAT*-space (X_∞, p_∞) with curvature between -1 and $-a^2$.*
(b) *The space X_∞ has the structure of a $\mathcal{C}^2$ Hadamard manifold such that the distance on X_∞ comes from a $\mathcal{C}^1$ Riemannian metric.*

The same proof shows that X_∞ has the structure of a $\mathcal{C}^{2,\alpha}$ Hadamard manifold with a $\mathcal{C}^{1,\alpha}$ Riemannian metric, for every $0 < \alpha < 1$. We will not use this improvement.

Even though this proposition follows from [33, Theorem 72, p. 311], we give a sketch of proof below.

Proof of Proposition 5.14. (a) The assumption on the curvature of X_n ensures that for each $R > 0$, one has uniform estimates for the volumes of balls with radius R in X_n: for all $n \geq 1$ and x in X_n, one has

$$\mathrm{vol}(B_{\mathbb{R}^k}(O, R)) \leq \mathrm{vol}(B_{X_n}(x, R)) \leq \mathrm{vol}(B_{\mathbb{H}^k}(O, R)).$$

Therefore, for each $0 < \varepsilon < R$, there exists an integer $N = N(R, \varepsilon)$ such that every ball $B_{X_n}(p_n, R)$ can be covered by N balls of radius ε. Hence, according to Fact 5.5, there exists a subsequence of (X_n, p_n) which converges to a proper pointed metric space (X_∞, p_∞). According to Fact 5.7, X_∞ is a CAT-space with curvature between -1 and $-a^2$.

(b) It remains to check that X_∞ is a $\mathcal{C}^2$ manifold with a $\mathcal{C}^1$ Riemannian metric. We isometrically imbed the converging sequence (X_n, p_n) in a proper metric space Z as in Fact 5.4. We fix $r_0, c_0 > 0$ as in Lemma 5.2 where we introduced the harmonic coordinates, and we choose a maximal $\frac{r_0}{2c_0}$-separated subset S_∞ of X_∞. For each x_∞ in S_∞, we choose a sequence x_n of points in X_n that converges to x_∞. By (5.3), the harmonic charts

$$\Psi_{x_n} : \mathring{B}(x_n, r_0/c_0) \to \mathbb{R}^k \tag{5.26}$$

are uniformly bi-Lipschitz. More precisely, for all z, z' in $\mathring{B}(x_n, r_0/c_0)$,

$$c_0^{-1} d(z, z') \leq \|\Psi_{x_n}(z) - \Psi_{x_n}(z')\| \leq c_0\, d(z, z').$$

Hence after extracting a subsequence, Ψ_{x_n} converges to a bi-Lipschitz map

$$\Psi_{x_\infty} : \mathring{B}(x_\infty, r_0/c_0) \to \mathbb{R}^k. \tag{5.27}$$

The extraction can be done simultaneously for all the points x_∞ in the countable set S_∞. The collection of maps Ψ_{x_∞} endows X_∞ with the structure of a Lipschitz manifold.

We now prove that X_∞ is a $\mathcal{C}^2$ manifold. Indeed, we will check that, for any x_∞ and x'_∞ in S_∞, the transition functions $\Phi_{x'_\infty} \circ \Phi_{x_\infty}^{-1}$ are of class $\mathcal{C}^2$. This just follows from the fact that these functions are uniform limits on compact sets of the transition functions $\Phi_{x'_n} \circ \Phi_{x_n}^{-1}$ which are, by (5.7), uniformly bounded in the $\mathcal{C}^{2,\alpha}$-norm.

Finally, we check that the distance d on X_∞ comes from a $\mathcal{C}^1$ Riemannian metric on X_∞. By (5.8), the Riemannian metrics $(g_n)_{ij}$ on X_n, seen as functions in the charts Ψ_{x_n} of X_n, are uniformly bounded in the $\mathcal{C}^{1,\alpha}$-norm. Extracting again a subsequence, there exists a $\mathcal{C}^1$ Riemannian metric $(g_\infty)_{ij}$ in the charts Ψ_{x_∞} of X_∞ such that

$$(g_n)_{ij} \text{ converges to } (g_\infty)_{ij} \text{ in the } \mathcal{C}^1 \text{ topology.} \tag{5.28}$$

Let d_∞ be the distance on X_∞ associated with g_∞. We check that $d_\infty = d$ on X_∞. Let x'_∞ and x''_∞ be points in X_∞. They are limits of points x'_n and x''_n in X_n. Let c_n be the geodesic segment joining x'_n to x''_n. Extracting once more a subsequence, we find that the curves c_n converge uniformly to a curve joining x'_∞ and x''_∞. This curve must be a geodesic for g_∞. This proves that $d_\infty(x'_\infty, x''_\infty) = d(x'_\infty, x''_\infty)$. □

5.6. *Convergence of harmonic maps*

We now explain how to obtain the limit harmonic maps.

We first notice that we can extend Definition 3.2: A $\mathcal{C}^2$ map $h : X \to Y$ between $\mathcal{C}^2$ Riemannian manifolds with $\mathcal{C}^1$ metrics X and Y is said to be *harmonic* if its tension field is zero, $\tau(h) := \mathrm{tr}(D^2 h) = 0$. Indeed, the tension field of a $\mathcal{C}^2$ map h at a point x depends only on the 2-jet of h and on the 1-jet of the metrics of X and Y at x and $h(x)$. More precisely, if we write h in a coordinate system, $h : (x_1, \ldots, x_k) \mapsto (h_1, \ldots, h_{k'})$, the equation $\mathrm{tr}\, D^2 h = 0$ reads

$$\Delta h_\lambda = -\sum_{ij\mu\nu} g^{ij} \Gamma^\lambda_{\mu\nu} \frac{\partial h_\mu}{\partial x_i} \frac{\partial h_\nu}{\partial x_j} \quad (\lambda \leq k') \tag{5.29}$$

where $\Gamma^\lambda_{\mu\nu}$ are the Christoffel coefficients on Y and where Δ is the Laplace operator on X defined as in (3.1):

$$\Delta : \varphi \mapsto \frac{1}{v} \frac{\partial}{\partial x_i} \left(v g^{ij} \frac{\partial \varphi}{\partial x_j} \right) \tag{5.30}$$

where $v = \sqrt{\det(g_{ij})}$ denotes the volume density on X. See [19, Section 1.3] for more details.

Lemma 5.15. *Let $(X_n, p_n)_{n\geq 1}$ and $(Y_n, q_n)_{n\geq 1}$ be sequences of equidimensional pointed Hadamard manifolds with curvature between -1 and 0. Let $c, C > 0$ and let $h_n : X_n \to Y_n$ be a sequence of (c, C)-quasi-isometric harmonic maps such that $\sup_n d(h_n(p_n), q_n) < \infty$. After extracting a subsequence, the sequences of pointed metric spaces (X_n, p_n) and (Y_n, q_n) converge respectively to pointed $\mathcal{C}^2$ manifolds with $\mathcal{C}^1$ Riemannian metrics (X_∞, p_∞) and (Y_∞, q_∞), and h_n converges to a c-quasi-isometric map $h_\infty : X_\infty \to Y_\infty$. The map h_∞ is of class $\mathcal{C}^2$ and is harmonic.*

Proof. Being harmonic, the maps h_n are $\mathcal{C}^\infty$. Since they are also (c, C)-quasi-isometric, according to Cheng's Lemma 3.4 there exists some constant $C' > 0$ such that the maps h_n are C'-Lipschitz. The first two statements then follow from Proposition 5.14 and Lemma 5.6.

It remains to show that the limit map h_∞ is of class $\mathcal{C}^2$ and harmonic. The key point will be a uniform bound for the $\mathcal{C}^{2,\alpha}$-norm of h_n in suitable harmonic coordinates. Let $k := \dim X_n$ and $k' := \dim Y_n$. Let x_∞ be a point in X_∞ and $y_\infty := h_\infty(x_\infty)$. Let x_n be a sequence in X_n converging to x_∞ and let $y_n := h_n(x_n)$.

We look at the maps h_n through the harmonic charts Ψ_{x_n} of X_n and Ψ_{y_n} of Y_n as in (5.26). By (5.27), these charts converge respectively to charts Ψ_{x_∞} of X_∞ and Ψ_{y_∞} of Y_∞. By (5.28), in these charts, the Riemannian metrics of X_n and Y_n converge to the Riemannian metrics of X_∞ and Y_∞ in the $\mathcal{C}^{1,\alpha}$-norm.

Let $0 < \alpha < 1$. Writing (5.29) for $h = h_n$ in these harmonic coordinates on a small open ball $\Omega := \mathring{B}\big(0, \frac{r_0}{c_0 C'}\big)$ of $\mathbb{R}^k$ that does not depend on n, one gets

$$\sum_{ij} g^{ij} \frac{\partial^2 h_\lambda}{\partial z_i \partial z_j} = - \sum_{ij\mu\nu} g^{ij} \Gamma^\lambda_{\mu\nu} \frac{\partial h_\mu}{\partial z_i} \frac{\partial h_\nu}{\partial z_j}. \tag{5.31}$$

The coefficients of this equation depend on n, but Lemma 5.2 ensures that they are uniformly bounded in the $\mathcal{C}^\alpha$-norm. The Schauder estimates for functions u on Ω and compact subsets K of Ω as in [33, Theorem 70, p. 303] thus tell us that

$$\|u\|_{\mathcal{C}^{1,\alpha},K} \leq M(\|\Delta u\|_{\mathcal{C}^0,\Omega} + \|u\|_{\mathcal{C}^\alpha,\Omega}), \tag{5.32}$$

$$\|u\|_{\mathcal{C}^{2,\alpha},K} \leq M(\|\Delta u\|_{\mathcal{C}^\alpha,\Omega} + \|u\|_{\mathcal{C}^\alpha,\Omega}), \tag{5.33}$$

for some constant $M = M(k, \Omega, K)$. Therefore, since the maps h_n are C'-Lipschitz, combining (5.29), (5.32) and (5.33) yields a uniform bound for the $\mathcal{C}^{2,\alpha}$-norm of the maps h_n. Hence the Ascoli lemma ensures that, after extracting a subsequence, h_n converges to a $\mathcal{C}^2$ map in the $\mathcal{C}^2$ topology. This proves that the limit map h_∞ is $\mathcal{C}^2$ and is harmonic. □

5.7. Construction of the limit equidistant harmonic maps

We now explain why the limit harmonic maps $h_{0,\infty}$ and $h_{1,\infty}$ constructed in the strategy of Proposition 5.1 are equidistant.

We first sum up the construction of these limit maps.

We start with two Hadamard manifolds X, Y of bounded curvature, and with two distinct quasi-isometric harmonic maps $h_0, h_1 : X \to Y$ such that $\delta := d(h_0, h_1)$ is finite and non-zero. We choose a sequence of points p_n in X such that $d(h_0(p_n), h_1(p_n))$ converges to δ and we set $q_{0,n} := h_0(p_n)$ and $q_{1,n} := h_1(p_n)$. We will frequently replace this sequence by subsequences without mentioning it. By Proposition 5.14, there exist $\mathcal{C}^2$ Hadamard manifolds with $\mathcal{C}^1$ metrics (X_∞, p_∞) and $(Y_\infty, q_{0,\infty})$ which are the Gromov–Hausdorff limits of the pointed metric spaces (X, p_n) and $(Y, q_{0,n})$. These limit Hadamard manifolds also have bounded curvature. We denote by $q_{1,\infty}$ the limit in Y_∞ of the sequence $q_{1,n}$. By the Cheng Lemma 3.4, the harmonic quasi-isometric maps h_0 and h_1 are Lipschitz continuous. By Lemma 5.6, there exists a limit map $h_{0,\infty} : (X_\infty, p_\infty) \to (Y_\infty, q_{0,\infty})$ of the sequence of Lipschitz continuous maps $h_0 : (X, p_n) \to (Y, q_{0,n})$. There also exists a limit map $h_{1,\infty} : (X_\infty, p_\infty) \to (Y_\infty, q_{1,\infty})$ of the sequence of Lipschitz continuous maps $h_1 : (X, p_n) \to (Y, q_{1,n})$. By Lemma 5.15, these limit maps $h_{0,\infty}$ and $h_{1,\infty}$ are still harmonic quasi-isometric maps.

Lemma 5.16. *With the above notation, the two limit harmonic quasi-isometric maps $h_{0,\infty}$, $h_{1,\infty}$ are equidistant. More precisely, for all x in X_∞, one has $d(h_{0,\infty}(x), h_{1,\infty}(x)) = \delta > 0$ where $\delta := d(h_0, h_1)$.*

We will apply this lemma to two pinched Hadamard manifolds X, Y. In this case, the limit $\mathcal{C}^2$ Hadamard manifolds X_∞, Y_∞ will also be pinched.

Proof of Lemma 5.16. Let Δ_∞ be the Laplace operator on X_∞ defined as in (5.30). We first check that the function $\varphi_\infty : x \mapsto d(h_{0,\infty}(x), h_{1,\infty}(x))$ is subharmonic on X_∞. This means that $\Delta_\infty \varphi_\infty$ is a positive measure on X_∞. Assume first that the Riemannian metric on Y_∞ is $\mathcal{C}^\infty$. In this case, φ_∞ is the composition of a harmonic map $h = (h_0, h_1) : X_\infty \to Y_\infty \times Y_\infty$ and of a convex $\mathcal{C}^\infty$ function $F = d : Y_\infty \times Y_\infty \to \mathbb{R}$, so that the function φ_∞ is subharmonic on X_∞ because of the formula

$$\Delta_\infty(F \circ h) = \sum_{i=1}^{k} D^2 F(D_{e_i} h, D_{e_i} h) + \langle DF, \tau(h)\rangle,$$

where $(e_i)_{i=1}^k$ is an orthonormal basis of the tangent space to X.

Since the Riemannian metric on Y might not be of class $\mathcal{C}^\infty$, we will use instead a limit argument. We fix a point x_∞ in X_∞. In a chart $(x_1, \ldots, x_k)$, the Laplace operator Δ_∞ of the Riemannian metric $(g_\infty)_{ij}$ of X_∞ reads

$$\psi \mapsto \Delta_\infty \psi = \frac{1}{v_\infty} \frac{\partial}{\partial x_i} \left(v_\infty g_\infty^{ij} \frac{\partial \psi}{\partial x_j} \right), \tag{5.34}$$

where v_∞ is the volume density. We want to prove that for every $\mathcal{C}^2$ function $\psi \geq 0$ with compact support in a small neighborhood of x_∞, one has

$$\int_{\mathbb{R}^k} \varphi_\infty \, \Delta_\infty \psi \, v_\infty \, \mathrm{d}x \geq 0. \tag{5.35}$$

The function φ_∞ on the pointed metric space (X_∞, p_∞) is equal to the limit of the sequence of functions $\varphi_n : x \mapsto d(h_0(x), h_1(x))$ on the pointed metric spaces (X, p_n), as defined in Lemma 5.6. Note that the dependence on n comes from the base point p_n

which varies with n. We choose a sequence x_n in X_n converging to x_∞. As in the proof of Lemma 5.15, we look at the functions φ_n through the harmonic charts Ψ_{x_n} of X_n. By (5.27), these charts converge to a chart Ψ_{x_∞} of X_∞. By (5.28), in these charts $(x_1, \dots, x_k)$ the Riemannian metrics $(g_n)_{ij}$ of X_n converge to the Riemannian metric $(g_\infty)_{ij}$ of X_∞ in the $\mathcal{C}^1$ topology.

Since, by the above argument, the functions φ_n are subharmonic for the metric $(g_n)_{ij}$, for every $\mathcal{C}^2$ function $\psi \geq 0$ with compact support in these charts one has, at each step n,

$$\int \varphi_n \Delta_n \psi \, v_n \, \mathrm{d}x \geq 0 \tag{5.36}$$

where Δ_n and v_n are the Laplace operator and the volume density of the metric $(g_n)_{ij}$. Letting n go to ∞ in (5.36) gives (5.35). This proves that the function φ_∞ is subharmonic.

By construction, this subharmonic function φ_∞ on X_∞ achieves its maximum $\delta > 0$ at the point p_∞. By (5.34), the Laplace operator is an elliptic linear differential operator with continuous coefficients. Hence, by the strong maximum principle in [14, Theorem 8.19, p. 198], this function φ_∞ is constant and equal to δ. □

The aim of Subsections 5.8 and 5.9 is to prove that such equidistant harmonic quasi-isometric maps $h_{0,\infty}$ and $h_{1,\infty}$ cannot exist (Corollary 5.19) when Y_∞ is pinched. This will conclude the proof of Proposition 5.1.

5.8. Equidistant harmonic maps

We first study equidistant harmonic maps without any pinching assumption.

The following lemma extends [25, Lemma 2.2] to the case where the source space X is only assumed to be a $\mathcal{C}^2$ Hadamard manifold. We include a complete proof to deal with this weaker regularity assumption.

Lemma 5.17. *Let X, Y be $\mathcal{C}^2$ Hadamard manifolds with $\mathcal{C}^1$ Riemannian metrics of bounded curvature. Let $h_0, h_1 : X \to Y$ be harmonic maps such that the distance function $x \mapsto d(h_0(x), h_1(x))$ is constant. For t in $[0, 1]$, let h_t be the geodesic interpolation of h_0 and h_1 as in (5.12). Then for almost all x in X, t in $[0, 1]$ and V in $T_x X$, one has*

$$\|Dh_0(V)\| = \|Dh_t(V)\| = \|Dh_1(V)\|. \tag{5.37}$$

Note that we cannot conclude that (5.37) is valid for all x and t since the interpolation h_t might not be of class $\mathcal{C}^1$.

We will use the following straightforward inequality for convex functions.

Lemma 5.18. *Let $t \mapsto \Phi_t$ be a non-negative convex function on $[0, 1]$. Then, for all t in $[0, 1/2]$, one has*

$$\Phi_t + \Phi_{1-t} \leq \Phi_0 + \Phi_1 - 2t(\Phi_0 + \Phi_1 - 2\Phi_{1/2}). \tag{5.38}$$

Proof. We just add the following two convexity inequalities: $\Phi_t \leq (1-2t)\Phi_0 + 2t\Phi_{1/2}$ and $\Phi_{1-t} \leq (1-2t)\Phi_1 + 2t\Phi_{1/2}$. □

Proof of Lemma 5.17. The idea is to construct two small perturbations f_ε and g_ε of the harmonic maps h_0 and h_1 with support in a compact set K of X, and to compare the sum of the energies of f_ε and g_ε with the sum of the energies of h_0 and h_1.

Let $0 \leq \varepsilon \leq 1$. Here is the definition of $f_\varepsilon, g_\varepsilon : X \to Y$. We fix a $\mathcal{C}^1$ cut-off function $\eta : X \to [0, 1]$, $x \mapsto \eta_x$, with compact support K, and we let, for all x in X,

$$f_\varepsilon(x) := h_{\varepsilon\eta_x}(x) \quad \text{and} \quad g_\varepsilon(x) := h_{1-\varepsilon\eta_x}(x). \tag{5.39}$$

These functions are Lipschitz continuous, hence almost everywhere differentiable. In order to compute their differentials, we use the notations (5.13) and (5.14): for all x in a subset $X' \subset X$ of full measure, all V in T_xX, and almost all t in $[0, 1]$, we let

$$J_V(t) := D_xh_t(V) \quad \text{and} \quad \tau_x(t) := \partial_t h_t(x).$$

For such a tangent vector V, it follows from Lemma 5.10(b) that there exists a convex function $t \mapsto \varphi_V(t)$ such that $\varphi_V(t) = \|J_V(t)\|$ for all t where the derivative $J_V(t)$ exists. By the chain rule, for almost all ε in $[0, 1]$, the differentials of f_ε and g_ε are given, for almost all x in X and all V in T_xX, by

$$Df_\varepsilon(V) = J_V(\varepsilon\eta_x) + \varepsilon\, V.\eta\, \tau_x(\varepsilon\eta_x), \tag{5.40}$$

$$Dg_\varepsilon(V) = J_V(1-\varepsilon\eta_x) - \varepsilon\, V.\eta\, \tau_x(1-\varepsilon\eta_x) \tag{5.41}$$

where $V.\eta = d\eta(V)$ is the derivative of the function η in the direction V.

According to Lemma 5.10(a), for almost all x in X and all V in T_xX, the scalar product $\langle J_V(t), \tau_x(t)\rangle$ is almost surely constant. Therefore, for almost all ε in $[0, 1]$, x in X and V in the unit tangent bundle T^1_xX, one has the equality

$$\|Df_\varepsilon(V)\|^2 + \|Dg_\varepsilon(V)\|^2 = \varphi_V(\varepsilon\eta_x)^2 + \varphi_V(1-\varepsilon\eta_x)^2 + 2\varepsilon^2(V.\eta)^2. \tag{5.42}$$

We introduce the convex function $t \mapsto \Phi^V_t := \varphi_V(t)^2$. Using (5.38), one gets for almost all ε in $[0, 1]$, x in X and V in T^1_xX the bound

$$\|Df_\varepsilon(V)\|^2 + \|Dg_\varepsilon(V)\|^2 \leq \Phi^V_0 + \Phi^V_1 - 2\varepsilon\eta_x(\Phi^V_0 + \Phi^V_1 - 2\Phi^V_{1/2}) + 2\varepsilon^2(V.\eta)^2.$$

We recall that the energy over K of a Lipschitz map $h : X \to Y$ is

$$E_K(h) := \int_K \|D_xh\|^2 \,\mathrm{d}x = \int_{T^1K} \|Dh(V)\|^2 \,\mathrm{d}V,$$

where $\mathrm{d}x$ is the Riemannian measure on X and $\mathrm{d}V$ the Riemannian measure on T^1X. Integrating the previous inequality on the unit tangent bundle of K, one gets the following inequality relating the energy over K of f_ε, g_ε, h_0 and h_1:

$$E_K(f_\varepsilon) + E_K(g_\varepsilon) - E_K(h_0) - E_K(h_1) \leq -\varepsilon \int_{T^1K} A(V)\,\mathrm{d}V + O(\varepsilon^2) \tag{5.43}$$

where A is the function on T^1X defined, for almost all x in X and V in T^1_xX, by

$$A(V) := 2\eta_x(\Phi^V_0 + \Phi^V_1 - 2\Phi^V_{1/2}).$$

Since the harmonic maps h_0 and h_1 are critical points for the energy functional, (5.43) implies that

$$\int_{T^1K} A(V)\,\mathrm{d}V \le 0. \tag{5.44}$$

Since Φ^V is convex, the function A is non-negative. Therefore (5.44) implies that A is almost surely zero. Since the function η was arbitrary, this tells us that, for almost all V in T^1X, one has

$$2\Phi^V_{1/2} = \Phi^V_0 + \Phi^V_1.$$

Since Φ^V is the square of the convex function φ_V, it follows that for almost all V in TX, the function φ_V is constant. This proves (5.37). □

5.9. Equidistant harmonic maps in negative curvature

The following corollary improves the conclusion of Lemma 5.17 when the curvature of Y is uniformly negative.

Corollary 5.19. *Let $a > 0$. Let X, Y be $\mathcal{C}^2$ Hadamard manifolds with $\mathcal{C}^1$ Riemannian metrics. Assume moreover that Y is* CAT$(-a^2)$. *Let $h_0, h_1 : X \to Y$ be harmonic maps such that $x \mapsto d(h_0(x), h_1(x))$ is constant. Then either $h_0 = h_1$, or*

$$h_0 \textit{ and } h_1 \textit{ take their values in the same geodesic } \Gamma \textit{ of } Y. \tag{5.45}$$

This means that, when $h_0 \ne h_1$, there exists a geodesic $t \mapsto \gamma(t)$ in Y and harmonic functions u_0, u_1 on X such that $h_0 = \gamma \circ u_0$, $h_1 = \gamma \circ u_1$ and $u_1 - u_0$ is a bounded harmonic function on X.

Note that this case is ruled out when h_0 and h_1 are within bounded distance of a quasi-isometric map $f : X \to Y$ since X has dimension $k \ge 2$.

Proof of Corollary 5.19. We can assume that the distance between h_0 and h_1 is equal to 1. We recall a few notations that we have already used. For t in $[0, 1]$, let h_t be the geodesic interpolation of h_0 and h_1. For x in X, let $\tau_x(t) := \partial_t h_t(x)$. Since the map $(t, x) \mapsto h_t(x)$ is Lipschitz continuous, the vector $J_V(t) := Dh_t(V)$ is well-defined for almost all t in $[0, 1]$, x in X and V in T_xX. For all such t, x, V, we set

$$\alpha_V(t) := \langle J_V(t), \tau_x(t)\rangle, \quad \varphi_V(t) := \|J_V(t)\|, \quad \psi_V(t) := (\varphi_V(t)^2 - \alpha_V(t)^2)^{1/2}.$$

By Lemmas 5.10(a) and 5.17, one has

$$\alpha_V(0) = \alpha_V(t) = \alpha_V(1) \quad \text{and} \quad \varphi_V(0) = \varphi_V(t) = \varphi_V(1) \tag{5.46}$$

for almost all t in $[0, 1]$ and almost all V in TX, so that

$$\psi_V(0) = \psi_V(t) = \psi_V(1).$$

Comparing these equalities with the uniform convexity of the function ψ_V in (5.19), one infers that $\psi_V(t) = 0$. Hence, when $J_V(t)$ is defined, one has

$$J_V(t) = \alpha_V(0)\tau_x(t). \tag{5.47}$$

We now explain why (5.47) implies (5.45). It is enough to check that, for every $\mathcal{C}^1$ curve

$$c : [0, 1] \to X, \qquad s \mapsto c_s,$$

with speed at most $1/3$, the images

$$h_0(c_{[0,1]}) \text{ and } h_1(c_{[0,1]}) \text{ are both included in the geodesic } \Gamma \tag{5.48}$$

of Y containing both $h_0(c_0)$ and $h_1(c_0)$.

The idea is to construct an auxiliary curve C with zero derivative. Let $\beta : [0, 1] \to [-1/3, 1/3]$ be given by $s \mapsto \beta_s := \int_0^s \alpha_{c'_r}(0)\,\mathrm{d}r$. For t_0 in $[1/3, 2/3]$, consider the curve

$$C : [0, 1] \to Y, \qquad s \mapsto C(s) := h_{t_0-\beta_s}(c_s).$$

Since the speed of c is bounded by $1/3$, the curve C is well-defined. By construction, C is a Lipschitz continuous path, and by (5.46) and (5.47), for almost all s, its derivative is

$$C'(s) = \big(\alpha_{c'_s}(t_0 - \beta_s) - \alpha_{c'_s}(0)\big)\tau_{c_s}(t_0 - \beta_s) = 0.$$

Therefore, $C(s) = C(0)$ for all s in $[0, 1]$, that is,

$$h_{t_0-\beta_s}(c_s) = h_{t_0}(c_0).$$

Using this equality for two distinct values of t_0, we deduce that the geodesic segments $h_{[0,1]}(c_0)$ and $h_{[0,1]}(c_s)$ meet in at least two points. This proves (5.48) and ends the proof of Corollary 5.19. □

This also ends the proof of Proposition 5.1.

6. Boundary maps for weakly coarse embeddings

This section is independent of the previous ones. We prove that a weakly coarse embedding between pinched Hadamard manifolds admits a boundary map which is well-defined outside a set of zero Hausdorff dimension. We prove that the fibers of this boundary map also have zero Hausdorff dimension (Theorem 6.5). More precisely, we will prove quantitative versions of these facts (Propositions 6.13 and 6.15) that we will use in Section 7.

6.1. *Weakly coarse embeddings*

In this subsection, we introduce various classes of rough Lipschitz maps $f : X \to Y$ between pinched Hadamard manifolds generalizing quasi-isometric maps.

Let X and Y be Hadamard manifolds with pinched sectional curvatures, $-b^2 \leq K_X, K_Y \leq -a^2 < 0$. Let $k = \dim X$ and $k' = \dim Y$.

Definition 6.1. Let $c > 0$. A map $f : X \to Y$ is *rough c-Lipschitz* if for all $x, x' \in X$ with $d(x, x') \leq 1$ one has $d(f(x), f(x')) \leq c$.

When $f : X \to Y$ is a rough c-Lipschitz map, one has, for all x, x' in X,

$$d(f(x), f(x')) \leq c d(x, x') + c.$$

Definition 6.2. A map $f : X \to Y$ is a *coarse embedding* if there exist non-decreasing unbounded functions φ_1, φ_2 such that, for all $x, x' \in X$,

$$\varphi_1(d(x, x')) \leq d(f(x), f(x')) \leq \varphi_2(d(x, x')). \tag{6.1}$$

Note that a map which is within bounded distance of a coarse embedding is also a coarse embedding. In Definition 6.2 one may always assume that φ_2 is affine, that is, f is rough Lipschitz. A quasi-isometric map is a special case of a coarse embedding, where φ_1 is also an affine function.

Definition 6.3. A *weakly coarse embedding* is a rough Lipschitz map $f : X \to Y$ for which there exist $c_0, C_0 > 0$ such that, for all x, x' in X,

$$d(f(x), f(x')) \leq c_0 \implies d(x, x') \leq C_0. \tag{6.2}$$

Equivalently, this means that there exist non-decreasing non-negative and non-zero functions φ_1, φ_2 such that (6.1) holds. Of course, any coarse embedding $f : X \to Y$ is a weakly coarse embedding.

Example 6.4. There exist many coarse and weakly coarse embeddings f from $\mathbb{H}^2$ to $\mathbb{H}^3$. More precisely, for any non-decreasing 1-Lipschitz function $\varphi_1 : [0, \infty[\to [0, \infty[$ with $\varphi_1(0) = 0$ one can choose a 1-Lipschitz map f for which φ_1 is the best lower bound in (6.1).

Proof. Indeed, one first constructs a unit-speed $\mathcal{C}^1$ curve $f_0 : \mathbb{R} \to \mathbb{H}^2$ such that $\varphi_1(t) = \min_{s\in\mathbb{R}} d(f_0(s+t), f_0(s))$ for every $t \geq 0$. We set $\mathbb{H}^1 := \mathbb{R}$ and, for $k \geq 1$, we embed each space $\mathbb{H}^k$ as a totally geodesic hyperplane in $\mathbb{H}^{k+1}$ and denote by $x \mapsto n_x$ a unit normal vector field to $\mathbb{H}^k$ in $\mathbb{H}^{k+1}$. We now define the Lipschitz map $f : \mathbb{H}^2 \to \mathbb{H}^3$ as $f(\exp(t n_s)) := \exp(t n_{f_0(s)})$ for all s in $\mathbb{H}^1$ and $t \in \mathbb{R}$. □

For any point $x_0 \in X$ and $r > 0$, we identify through the exponential map each sphere $S(x_0, r)$ with the unit tangent sphere

$$S_{x_0} := \{\xi \in T_{x_0}X \mid \|\xi\| = 1\}.$$

More precisely, when $\xi \in S_{x_0}$, we denote by $r \mapsto \xi_r := \exp_{x_0}(r\xi)$ the corresponding unit-speed geodesic ray (so that $\xi_0 = x_0$).

We denote by $\overline{X} = X \cup \partial X$ the visual compactification of X. The boundary ∂X is the set of (equivalence classes of) rays in X. The map $\psi_{x_0} : \xi \mapsto \lim_{r\to\infty} \xi_r$ gives a homeomorphism from the unit tangent sphere S_{x_0} onto the sphere at infinity ∂X. We say that a subset A of ∂X has *zero Hausdorff dimension* if, seen in S_{x_0}, it has zero Hausdorff dimension. One can check that this property does not depend on the choice of x_0, because for any other point $x_1 \in X$, the homeomorphism $\psi_{x_1}^{-1} \circ \psi_{x_0}$ is bi-Hölder.

In this subection we will prove the following theorem.

Theorem 6.5. *Let $f : X \to Y$ be a weakly coarse embedding between pinched Hadamard manifolds.*

(a) *There exists a subset $A \subset \partial X$ of zero Hausdorff dimension such that, for all $\xi \in \partial X \setminus A$, the limit $\partial f(\xi) := \lim_{r\to\infty} f(\xi_r)$ exists in ∂Y.*
(b) *For every $\xi \in \partial X \setminus A$, the fiber $\{\eta \in \partial X \setminus A \mid \partial f(\eta) = \partial f(\xi)\}$ has zero Hausdorff dimension.*

The map $\partial f : \partial X \setminus A \to \partial Y$ is called the *boundary map* of f.

The proof of Theorem 6.5 will last up to the end of this section. The quantitative estimates (6.8) and (6.10) that we will obtain during this proof will be used again in Section 7.

6.2. *Hausdorff dimension and Frostman measures*

In this subsection we introduce classical notations and definitions from geometric measure theory.

Definition 6.6. Let $M, \nu > 0$. A Borel probability measure σ on a compact metric space S is said to be (M, ν)-*Frostman* if, for all $\xi \in S$ and all $r > 0$,

$$\sigma(B(\xi, r)) \le M r^{\nu}. \tag{6.3}$$

Proposition 4.9 tells us that all the harmonic measures $\sigma_{x,r}$ of a pinched Hadamard manifold are $(M, 1/N)$-Frostman, where the constants (M, N) do not depend on the center x or the radius $r > 0$.

Let $\nu, \delta > 0$. For a subset $A \subset S$, we denote

$$H^{\nu}_{\delta}(A) = \inf\Big\{\sum_{i\ge1} \operatorname{diam}(U_i)^{\nu} \;\Big|\; A \subset \bigcup_i U_i,\ \operatorname{diam}(U_i) \le \delta\Big\}.$$

When $\delta = \infty$, we denote similarly

$$H^{\nu}_{\infty}(A) = \inf\Big\{\sum_{i\ge1} \operatorname{diam}(U_i)^{\nu} \;\Big|\; A \subset \bigcup_i U_i\Big\}. \tag{6.4}$$

We recall that the ν-*dimensional Hausdorff measure* of A is defined as

$$H^{\nu}(A) = \sup_{\delta>0} H^{\nu}_{\delta}(A)$$

and the *Hausdorff dimension* of A is

$$\dim_H(A) = \inf\{\nu > 0 \mid H^{\nu}(A) = 0\}.$$

Observe that also

$$\dim_H(A) = \inf\{\nu > 0 \mid H^{\nu}_{\infty}(A) = 0\}. \tag{6.5}$$

The following easy lemma relates $H^{\nu}_{\infty}(A)$ to Frostman measures.

Lemma 6.7. *Let σ be a (M, ν)-Frostman measure on a compact metric space S and $A \subset S$. Then $\sigma(A) \le M H^{\nu}_{\infty}(A)$.*

Proof. Observe that $\sigma(A) \le \sum_{i\ge1} \sigma(U_i) \le M \sum_{i\ge1} \operatorname{diam}(U_i)^{\nu}$ for any covering (U_i) of A. □

6.3. *Image of a large sphere*

In this subsection we focus on those points of a sphere $S(x_0, r)$ whose images under a weakly coarse embedding are too close to a given point.

The following definition will play a key role in the proof of Theorem 6.5.

Definition 6.8. Let c, C_1, $C_2 > 0$. A rough c-Lipschitz map $f : X \to Y$ has *property* $\mathcal{C}_{C_1,C_2}$ if, for all $x_0 \in X$, $y_0 \in Y$ and $r, s > 0$, the set

$$A_{x_0,y_0,r,s} := \{\xi \in S_{x_0} \mid d(y_0, f(\xi_r)) \le s\} \tag{6.6}$$

can be covered by at most $C_1 e^{bk's}$ balls of radius $C_2 e^{-ar}$, where $k' = \dim Y$.

If such constants C_1, C_2 exist, we say that f has *property* $\mathcal{C}$.

In this definition the unit-sphere S_{x_0} is endowed with the distance induced by the Riemannian norm on $T_{x_0}X$.

The bound on the size of a covering of the set (6.6) will be very useful for Hausdorff dimension estimations. The precise value bk' for the exponential growth in Definition 6.8 is not particularly important. It is obtained in the next proposition and it merely avoids the introduction of another parameter.

Proposition 6.9. *Every weakly coarse embedding $f : X \to Y$ has property $\mathcal{C}$.*

In particular, Propositions 6.13 and 6.15 below apply to all weakly coarse embeddings f.

We will use the Bishop volume estimates (see for example [12]) which compare the volume of balls in X and in the hyperbolic space $\mathbb{H}^k$.

Lemma 6.10. *Let X be a pinched Hadamard manifold with dimension k and sectional curvature $-b^2 \le K_X \le -a^2 < 0$. Then, for $R > 0$,*

$$a^{-k}\,\mathrm{vol}(B_{\mathbb{H}^k}(O, aR)) \le \mathrm{vol}(B_X(x, R)) \le b^{-k}\,\mathrm{vol}(B_{\mathbb{H}^k}(O, bR)).$$

We will also need to bound angles by Gromov products as in Lemma 2.1.

Lemma 6.11. *Let Y be a Hadamard manifold with $K_Y \le -a^2 < 0$. Then, for all $y_0 \in Y$ and $y_1, y_2 \in Y \smallsetminus \{y_0\}$,*

$$\theta_{y_0}(y_1, y_2) \le 4e^{-a(y_1,y_2)_{y_0}},$$

where $\theta_{y_0}(y_1, y_2)$ is the angle at y_0 of the geodesic triangle (y_0, y_1, y_2) and $(y_1, y_2)_{y_0} := \frac{1}{2}(d(y_0, y_1) + d(y_0, y_2) - d(y_1, y_2))$ is the Gromov product.

Proof of Proposition 6.9. We will see that f has property $\mathcal{C}_{C_1,C_2}$ where the constants C_1, C_2 depend only on a, b, k', and on c_0, C_0 from (6.2).

It follows from the volume estimates of Lemma 6.10 that there exists a constant $C_1 > 0$ such that for each ball $B(y_0, s) \subset Y$ ($s > 0$) and each covering of minimal cardinality of this ball by balls with radii $c_0/2$,

$$B(y_0, s) \subset \bigcup_{i\in I} B(y_i, c_0/2),$$

this cardinality is at most $C_1 e^{bk's}$.

Since f is a (c_0, C_0)-weakly coarse embedding, for each $i \in I$ the inverse image $f^{-1}(B(y_i, c_0/2))$ is either empty or lies in $B(x_i, C_0) \subset X$. By Lemma 6.11, the set $B(x, C_0) \cap S(x_0, r)$ lies in a cone with vertex x_0 and angle $\theta_r = C_2 e^{-ar}$. □

Remark 6.12. Any map $\widetilde{f} : X \to Y$ within bounded distance of a map $f : X \to Y$ with property $\mathcal{C}$ also has property $\mathcal{C}$.

6.4. Construction of the boundary map

We now investigate the long-term behavior of the images of geodesic rays under a rough Lipschitz map satisfying property $\mathcal{C}$.

Let X, Y be pinched Hadamard manifolds and $f : X \to Y$ be a rough Lipschitz map with property $\mathcal{C}$. Proposition 6.13 below tells us that, except for a set of rays of zero Hausdorff dimension, the image under f of a ray goes to infinity in Y at positive speed and this image converges to a point in ∂Y.

We need some notations. For $x_0 \in X$, let A_{x_0} be the set of rays whose image does not go to infinity at positive speed:

$$A_{x_0} := \left\{ \xi \in S_{x_0} \;\middle|\; \liminf_{n\to\infty} \tfrac{1}{n} d(f(x_0), f(\xi_n)) = 0 \right\}.$$

Then $A_{x_0} = \bigcap_{\alpha>0} A_{x_0,\alpha}$, where, for $\alpha > 0$,

$$A_{x_0,\alpha} := \left\{ \xi \in S_{x_0} \;\middle|\; \liminf_{n\to\infty} \tfrac{1}{n} d(f(x_0), f(\xi_n)) < \alpha \right\}.$$

One has $A_{x_0,\alpha} \subset \bigcap_{n_0\geq 1} A_{x_0,\alpha}(n_0)$, where, for $n_0 \geq 1$,

$$A_{x_0,\alpha}(n_0) := \{\xi \in S_{x_0} \mid d(f(x_0), f(\xi_n)) \leq n\alpha \text{ for some } n \geq n_0\}.$$

With the definition (6.6), one has $A_{x_0,\alpha}(n_0) = \bigcup_{n\geq n_0} A_{x_0,f(x_0),n,n\alpha}$.

Proposition 6.13. *Let X, Y be pinched Hadamard manifolds with sectional curvatures $-b^2 \leq K \leq -a^2 < 0$. Let $c, C_1, C_2 > 0$ and $f : X \to Y$ be a rough c-Lipschitz map with property $\mathcal{C}_{C_1,C_2}$. Let $\alpha > 0$, $k' = \dim Y$ and $\nu_\alpha := bk'\alpha/a$. For $\nu > \nu_\alpha$, set $C_{3,\alpha,\nu} := C_1 C_2^{\nu}/(1 - e^{-a(\nu-\nu_\alpha)})$. Then for any $x_0 \in X$ and $n_0 \geq 1$:*

(a) *One has*

$$H^{\nu}_{\infty}(A_{x_0,\alpha}(n_0)) \leq C_{3,\alpha,\nu} e^{-a(\nu-\nu_\alpha)n_0}. \tag{6.7}$$

(b) *For every (M, ν)-Frostman measure σ on S_{x_0},*

$$\sigma(A_{x_0,\alpha}(n_0)) \leq M C_{3,\alpha,\nu} e^{-a(\nu-\nu_\alpha)n_0}. \tag{6.8}$$

(c) $\dim_H(A_{x_0,\alpha}) \leq \nu_\alpha$.
(d) $\dim_H(A_{x_0}) = 0$.
(e) *For every $\xi \in S_{x_0} \smallsetminus A_{x_0}$, the limit $\partial f(\xi) := \lim_{r\to\infty} f(\xi_r)$ exists in ∂Y.*

The bound (6.8) can be interpreted as a large deviation inequality for the random path $f(\xi_t)$ when the ray ξ is chosen randomly with law σ. A key point is that the constants involved in (6.8) do not depend on the (M, ν)-Frostman measure σ. We will apply it later to various harmonic measures $\sigma = \sigma_{x_0,r}$ on X.

Proof of Proposition 6.13. (a) Since f has property $\mathcal{C}_{C_1,C_2}$,

$$H^\nu_\infty(A_{x_0,\alpha}(n_0)) \leq \sum_{n\geq n_0} H^\nu_\infty(A_{x_0,f(x_0),n,n\alpha}) \leq \sum_{n\geq n_0} C_1 e^{a\nu_\alpha n}\, C_2^\nu e^{-a\nu n} = C_{3,\alpha,\nu} e^{-a(\nu-\nu_\alpha)n_0}.$$

(b) follows from (a) and Lemma 6.7.

(c) Letting n_0 go to infinity in (6.7), one gets $H^\nu_\infty(A_{x_0,\alpha}) = 0$ for all $\nu > \nu_\alpha$. Therefore, (6.5) yields $\dim_H(A_{x_0,\alpha}) \leq \nu_\alpha$.

(d) One has $\dim_H(A_{x_0}) \leq \inf_{\alpha>0} \dim_H(A_{x_0,\alpha}) = 0$.

(e) Since f is rough Lipschitz, one may assume that the parameters r are integers and apply Lemma 6.14 below to the sequence $y_n = f(\xi_n)$. □

Lemma 6.14. *Let Y be a Hadamard manifold with $K_Y \leq -a^2 < 0$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y such that*

$$\sup_{n\geq 0} d(y_n, y_{n+1}) < \infty \quad \textit{and} \quad \liminf_{n\to\infty} \frac{1}{n} d(y_0, y_n) > 0.$$

Then the limit $y_\infty := \lim_{n\to\infty} y_n$ exists in the visual boundary ∂Y.

Proof. Choose $c, \alpha > 0$ and $n_0 \geq 1$ such that

$$d(y_n, y_{n+1}) \leq c \quad \text{and} \quad d(y_0, y_n) \geq n\alpha \quad \text{for all } n \geq n_0.$$

By Lemma 6.11, $\theta_{y_0}(y_n, y_{n+1}) \leq 4e^{ac/2}e^{-a\alpha n}$ for any $n \geq n_0$. Since this series converges, there exists a geodesic ray $\gamma_+ \subset Y$ with origin y_0 such that $\lim_{n\to\infty} \theta_{y_0}(y_n, \gamma_+) = 0$. □

Unlike quasi-isometric maps, a coarse embedding may not have boundary values in every direction. See Example 6.4 where we could begin with a curve f_0 that spirals away in $\mathbb{H}^2$.

6.5. *The fibers of the boundary map*

We now investigate the fibers of the boundary map ∂f of a rough Lipschitz map with property $\mathcal{C}$.

Proposition 6.15 below tells us that the fibers of the boundary map have zero Hausdorff dimension.

We keep the notations of Subsection 6.4 and introduce more notations. As before, X, Y are pinched Hadamard manifolds and $f : X \to Y$ is a rough c-Lipschitz map with property $\mathcal{C}$. For $x_0 \in X$ and $\xi \in S_{x_0}$, let $B^{\xi}_{x_0}$ be the set of rays η that "do not go away from ξ at positive speed":

$$B^{\xi}_{x_0} := \Big\{\eta \in S_{x_0} \;\Big|\; \lim_{n_0\to\infty} \inf_{n,p\geq n_0} \tfrac{1}{n+p} d(f(\xi_n), f(\eta_p)) = 0\Big\}.$$

Then $B^{\xi}_{x_0} = \bigcap_{\alpha>0} B^{\xi}_{x_0,\alpha}$, where, for $\alpha > 0$, we set $\beta_\alpha := \frac{\alpha^2}{2\alpha+c}$ and let

$$B^{\xi}_{x_0,\alpha} := \Big\{\eta \in S_{x_0} \;\Big|\; \lim_{n_0\to\infty} \inf_{n,p\geq n_0} \tfrac{1}{n+p}\, d(f(\xi_n), f(\eta_p)) < \beta_\alpha\Big\}.$$

Then $B^{\xi}_{x_0,\alpha} \subset \bigcap_{n_0\geq 1} B^{\xi}_{x_0,\alpha}(n_0)$, where for any $n_0 \geq 1$ we let

$$B^{\xi}_{x_0,\alpha}(n_0) := \{\eta \in S_{x_0} \mid d(f(\xi_n), f(\eta_p)) \leq (n+p)\beta_\alpha \text{ for some } n, p \geq n_0\}.$$

This specific value for β_α has been chosen in order to obtain the same exponent in (6.7) and in (6.9) below.

Proposition 6.15. *Let X, Y be pinched Hadamard manifolds with sectional curvatures $-b^2 \leq K \leq -a^2 < 0$. Let $c, C_1, C_2 > 0$ and $f : X \to Y$ be a rough c-Lipschitz map with property $\mathcal{C}_{C_1,C_2}$. Let $\alpha > 0$, $k' = \dim Y$, $\nu_\alpha := bk'\alpha/a$ and $\beta_\alpha := \alpha^2/(2\alpha + c)$. For $\nu > \nu_\alpha$, set $C_{4,\alpha,\nu} := \frac{C_1 C_2^{\nu}}{(1-e^{-bk'\beta_\alpha})(1-e^{-a(\nu-\nu_\alpha)})}$. Then for any $x_0 \in X$ and $n_0 \geq 1$ one has:*

(a) *For $\xi \in S_{x_0} \smallsetminus A_{x_0,\alpha}(n_0)$,*

$$H^{\nu}_{\infty}(B^{\xi}_{x_0,\alpha}(n_0)) \leq C_{4,\alpha,\nu} e^{-a(\nu-\nu_\alpha)n_0}. \tag{6.9}$$

(b) *For $\xi \in S_{x_0} \smallsetminus A_{x_0,\alpha}(n_0)$ and any (M, ν)-Frostman measure σ on S_{x_0},*

$$\sigma(B^{\xi}_{x_0,\alpha}(n_0)) \leq M C_{4,\alpha,\nu} e^{-a(\nu-\nu_\alpha)n_0}. \tag{6.10}$$

(c) *For $\xi \in S_{x_0} \smallsetminus A_{x_0,\alpha}$, one has $\dim_H(B^{\xi}_{x_0,\alpha}) \leq \nu_\alpha$.*

(d) *For $\xi \in S_{x_0} \smallsetminus A_{x_0}$, one has $\dim_H(B^{\xi}_{x_0}) = 0$.*

(e) *Assume $n_0 \geq \frac{4e^{2ac}}{1-e^{-a\beta_\alpha}}$. For $\xi, \eta \in S_{x_0} \smallsetminus A_{x_0,\alpha}(n_0)$ with $\eta \notin B^{\xi}_{x_0,\alpha}(n_0)$ and for all $n, p \geq \ell_0 := 4n_0c/\alpha$,*

$$\theta_{f(x_0)}(f(\xi_n), f(\eta_p)) \geq \tfrac{1}{2} e^{-2n_0bc}. \tag{6.11}$$

(f) *For $\xi, \eta \in S_{x_0} \smallsetminus A_{x_0}$ with $\eta \notin B^{\xi}_{x_0}$, one has $\partial f(\eta) \neq \partial f(\xi)$.*

We begin with a technical covering lemma.

Lemma 6.16. *We keep the notations of Proposition 6.15. Fix $n_0 \geq 1$. For $\xi \in S_{x_0}$ and $p \geq n_0$, let*

$$B^{\xi}_{x_0,\alpha,p}(n_0) := \{\eta \in S_{x_0} \mid (f(\xi_n), f(\eta_p)) \leq (n+p)\beta_\alpha \text{ for some } n \geq n_0\}.$$

If $\xi \notin A_{x_0,\alpha}(n_0)$, then $B^{\xi}_{x_0,\alpha,p}(n_0)$ can be covered by at most $\frac{C_1 e^{bk'\alpha p}}{1-e^{-bk'\beta_\alpha}}$ balls of radius $C_2 e^{-ap}$.

Proof. Using the notation (6.6), we have

$$B^{\xi}_{x_0,\alpha,p}(n_0) = \bigcup_{n\ge n_0} A_{x_0,f(\xi_n),p,(n+p)\beta_\alpha}.$$

The key point is that, since f is rough c-Lipschitz and $\xi \notin A_{x_0,\alpha}(n_0)$, this union is finite. Indeed, assume that an integer $n \ge n_0$ satisfies

$$d(f(\xi_n), f(\eta_p)) \le (n+p)\beta_\alpha$$

for some $\eta \in S_{x_0}$. Since $d(f(x_0), f(\xi_n)) \ge n\alpha$ and $d(f(x_0), f(\eta_p)) \le p\,c$, one must have

$$n\alpha - pc \le (n+p)\beta_\alpha.$$

By our choice of β_α, this is equivalent to

$$(n+p)\beta_\alpha \le p\alpha.$$

Therefore, using Definition 6.8, one can cover $B^{\xi}_{x_0,\alpha,p}(n_0)$ by at most $C_1 \sum_n e^{bk'(n+p)\beta_\alpha}$ balls of radius $C_2 e^{-ap}$, where the sum is over $n \ge n_0$ such that $(n+p)\beta_\alpha \le p\alpha$. Computing this sum, one deduces that $B^{\xi}_{x_0,\alpha,p}$ can be covered by at most $\frac{C_1 e^{bk'\alpha p}}{1-e^{-bk'\beta_\alpha}}$ balls of radius $C_2 e^{-ap}$. □

Proof of Proposition 6.15. (a) Since $B^{\xi}_{x_0,\alpha}(n_0) = \bigcup_{p\ge n_0} B^{\xi}_{x_0,\alpha,p}(n_0)$, Lemma 6.16 yields

$$\begin{aligned} H^{\nu}_{\infty}(B^{\xi}_{x_0,\alpha}(n_0)) &\le \sum_{p\ge n_0} H^{\nu}_{\infty}(B^{\xi}_{x_0,\alpha,p}(n_0)) \\ &\le \sum_{p\ge n_0} \frac{C_1 e^{a\nu_\alpha p}}{1-e^{-\beta_\alpha bk'}}\, C_2^{\nu} e^{-a\nu p} = C_{4,\alpha,\nu} e^{-a(\nu-\nu_\alpha)n_0}. \end{aligned}$$

(b) follows from (a) and Lemma 6.7.

(c) Letting n_0 go to infinity in (6.9) one gets $H^{\nu}_{\infty}(B^{\xi}_{x_0,\alpha}) = 0$ for all $\nu > \nu_\alpha$. Therefore, using (6.5), it follows that $\dim_H(B^{\xi}_{x_0,\alpha}) \le \nu_\alpha$.

(d) One has $\dim_H(B^{\xi}_{x_0}) \le \inf_{\alpha>0} \dim_H(B^{\xi}_{x_0,\alpha}) = 0$.

(e) This is a consequence of Lemma 6.17 below applied to the sequences $y_n = f(\xi_n)$ and $z_p = f(\eta_p)$.

(f) This follows from (e). □

6.6. *Two sequences going away from one another*

The aim of this subsection is to prove the following lemma which provides, in a pinched Hadamard manifold, a lower bound for the angle between points in two sequences with bounded speed that "go away from one another at positive speed".

Lemma 6.17. *Let Y be a Hadamard manifold with $-b^2 \leq K_Y \leq -a^2 < 0$. Let $c \geq \alpha \geq \beta > 0$ and $n_0 \geq \frac{4e^{2ac}}{1-e^{-a\beta}}$. Let $(y_n)_{n\in\mathbb{N}}$ and $(z_p)_{p\in\mathbb{N}}$ be two sequences of points in Y with $y_0 = z_0$ such that*

$$d(y_n, y_{n+1}) \leq c \quad \text{and} \quad d(z_p, z_{p+1}) \leq c \quad \text{for } n, p \geq 0, \tag{6.12}$$

$$d(y_0, y_n) \geq n\alpha, \quad d(y_0, z_p) \geq p\alpha \quad \text{and} \quad d(y_n, z_p) \geq (n+p)\beta \quad \text{for } n, p \geq n_0. \tag{6.13}$$

Then, for any integers $n, p \geq \ell_0 := 4n_0c/\alpha$,

$$\theta_{y_0}(y_n, z_p) \geq \tfrac{1}{2}e^{-2n_0bc}. \tag{6.14}$$

We will need two geometric lemmas.

We know that the orthogonal projection from a Hadamard manifold onto a geodesic is a 1-Lipschitz map. The following lemma gives more precise information when the curvature is bounded from above.

Lemma 6.18. *Let Y be a Hadamard manifold with $K_Y \leq -a^2 < 0$. Let $\gamma \subset Y$ be a geodesic. Then the orthogonal projection $\pi : Y \to \gamma$ is smooth and, for $y \in Y$,*

$$\|D_y\pi\| \leq \frac{1}{\cosh(ad(y,\gamma))} \leq 2e^{-ad(y,\gamma)}.$$

Proof. The proof relies on a Jacobi field estimate (see [12]).

Let $y \in Y \setminus \gamma$, let $\bar{y} = \pi(y) \in \gamma$ and $\ell = d(y,\gamma) = d(y,\bar{y})$. Denote by $c : s \in [0,\ell] \to c(s) \in Y$ the unit-speed parametrization of the geodesic segment $[\bar{y}, y]$ with $c(0) = \bar{y}$ and $c(\ell) = y$.

Let $v \in T_yY$. We want to bound $\|D_y\pi(v)\|/\|v\|$. We may assume that v is orthogonal to $\operatorname{Ker} D_y\pi$, i.e. to the geodesic c at y.

Choose a smooth curve $t \mapsto y(t) \in Y$ with $y(0) = y$ and $y'(0) = v$, and let $\bar{y}(t) = \pi(y(t)) \in \gamma$. We can assume that $d(y(t), \bar{y}(t)) = \ell$ for all t. For each parameter t, introduce the constant-speed geodesic $c_t : [0,\ell] \to Y$ such that $c_t(0) = \bar{y}(t)$ and $c_t(\ell) = y(t)$. By construction, each vector $u(t) := \frac{d}{ds}c_t(s)\big|_{s=0} \in T_{\bar{y}(t)}Y$ is normal to γ at the point $\bar{y}(t)$.

The map $(s,t) \mapsto c_t(s)$ is a variation of geodesics, so that $J : [0,\ell] \to \frac{d}{dt}c_t(s)\big|_{t=0} \in [0,\ell] \in T_{c(s)}Y$ is a Jacobi field along the geodesic c. We have $J(0) = D_y\pi(v)$ and $J(\ell) = v$. Since both $J(0)$ and $J(\ell)$ are normal to c, it follows that J is a normal Jacobi field. Since γ is a geodesic and each $u(t)$ is normal to γ, we infer from the equality $J'(0) = u'(0)$ that $J'(0)$ is normal to γ, i.e. orthogonal to $J(0)$. The Jacobi field equation $J'' + R(c', J)c' = 0$ and the hypothesis on the curvature now yield

$$(\|J\|^2)'' = 2\|J'\|^2 - 2R(c', J, c', J) \geq 2(\|J\|')^2 + 2a^2\|J\|^2$$

and therefore

$$\|J\|'' \geq a^2\|J\|.$$

Since $\|J\|'(0) = \langle J(0), J'(0)\rangle/\|J(0)\| = 0$, one deduces that $\|J(t)\| \geq \|J(0)\|\cosh(at)$ for all $t \geq 0$. In particular, $\|D_y\pi(v)\| \leq \|v\|/\cosh(a\ell)$. □

The second lemma is an easy angle comparison lemma.

Lemma 6.19. *Let Y be a Hadamard manifold with $-b^2 \leq K_Y \leq 0$. Let $\gamma \subset Y$ be a geodesic, $y_0 \in \gamma$, $y \in Y$ and $\bar{y} = \pi(y)$ be the projection of y on γ. Assume that $d(y_0, \bar{y}) \leq R$ and $d(\bar{y}, y) \geq R$. Then $\theta_{y_0}(y, \bar{y}) \geq \frac{1}{2}e^{-bR}$.*

Proof. The angles of a triangle in $\mathbb{H}^2(-b^2)$ with the same side lengths are smaller than the angles of the triangle $(y_0 y \bar{y})$. It follows that $\theta_{y_0}(y, \bar{y}) \geq \varphi$, where φ is the angle of an isosceles right triangle in $\mathbb{H}^2(-b^2)$ with adjacent sides of length R, which is $\varphi = \arctan\left(\frac{1}{\cosh(bR)}\right) \geq \frac{1}{2}e^{-bR}$. □

Proof of Lemma 6.17. Let γ_+ be a geodesic ray starting from $y_0 = z_0$. Denote by $\pi : Y \to \gamma$ the orthogonal projection onto the geodesic γ that contains γ_+. Identify $\gamma \sim \mathbb{R}$ so that $\gamma_+ \sim [0, \infty[$. Introduce, for $n, p \in \mathbb{N}$, the points $\bar{y}_n = \pi(y_n)$ and $\bar{z}_p = \pi(z_p)$, and the subintervals $I_n = [\bar{y}_n, \bar{y}_{n+1}]$ and $J_p = [\bar{z}_p, \bar{z}_{p+1}]$ of γ.

Let $R := 2n_0 c$. We claim that

$$\min(\bar{y}_N, \bar{z}_P) \leq R \quad \text{for all } N, P \geq 0. \tag{6.15}$$

According to (6.12), $\max(\bar{y}_{n_0}, \bar{z}_{n_0}) \leq n_0 c$. Hence it is enough to check that the interval $\mathcal{I} := [\bar{y}_{n_0}, \bar{y}_N] \cap [\bar{z}_{n_0}, \bar{z}_P]$ has length $|\mathcal{I}| \leq n_0 c$.

Let $q \in \mathcal{I}$. This point lies in some non-empty interval $I_n \cap J_p$ with $n, p \geq n_0$. Since the projection π is 1-Lipschitz, using (6.12) again yields $d(\bar{y}_n, \bar{z}_p) \leq 2c$. According to (6.13) one has $d(y_n, z_p) \geq \beta(n + p)$ so that

$$\text{either} \quad d(y_n, \bar{y}_n) \geq n\beta - c \quad \text{or} \quad d(z_p, \bar{z}_p) \geq p\beta - c,$$

and Lemma 6.18 now provides a bound for the length of one of the intervals I_n or J_p:

$$\text{either} \quad |I_n| \leq 2ce^{2ac - na\beta} \quad \text{or} \quad |J_p| \leq 2ce^{2ac - pa\beta}.$$

It follows that

$$|\mathcal{I}| \leq \sum_{n \geq n_0} 2ce^{2ac - na\beta} + \sum_{p \geq n_0} 2ce^{2ac - pa\beta}$$
$$\leq \frac{4ce^{2ac}}{1 - e^{-a\beta}} e^{-n_0 a\beta} \leq n_0 c.$$

This proves (6.15).

Now, let $n, p \geq \ell_0 := 4n_0 c/\alpha$ so that, by (6.13), one has $d(y_0, y_n) \geq 2R$ and $d(y_0, z_p) \geq 2R$. The claim (6.15) tells us that

$$\text{either} \quad d(y_0, \bar{y}_n) \leq R \quad \text{or} \quad d(y_0, \bar{z}_p) \leq R.$$

Hence by Lemma 6.19,

$$\text{either} \quad \theta_{y_0}(y_n, \gamma_+) \geq \tfrac{1}{2}e^{-bR} \quad \text{or} \quad \theta_{y_0}(z_p, \gamma_+) \geq \tfrac{1}{2}e^{-bR}.$$

Since this is true for any ray γ_+ based at y_0, one gets $\theta_{y_0}(y_n, z_p) \geq \frac{1}{2}e^{-bR}$. □

Proof of Theorem 6.5. Point (a) follows from Propositions 6.13(d, e); and (b) follows from Propositions 6.15(d, f). □

Remark 6.20. It follows from the proof that Theorem 6.5 also holds for any rough Lipschitz map $f : X \to Y$ between pinched Hadamard manifolds with property $\mathcal{C}$.

7. Beyond quasi-isometric maps

The aim of this subsection is the following extension of Theorem 1.1 to all weakly coarse embeddings f, and in particular to all coarse embeddings f (see Definitions 6.2 and 6.3).

7.1. *Weakly coarse embeddings and harmonic maps*

Theorem 7.1. *Every weakly coarse embedding $f : X \to Y$ between pinched Hadamard manifolds is within bounded distance of a unique harmonic map $h : X \to Y$.*

Indeed, we will prove a more general proposition using Definition 6.8.

Proposition 7.2. *Every rough Lipschitz map $f : X \to Y$ with property $\mathcal{C}$ between pinched Hadamard manifolds is within bounded distance of a unique harmonic map $h : X \to Y$.*

The main new ingredients in the proof are the construction and properties of a boundary map of f. Those new ingredients which do not involve harmonic maps were explained in Section 6. We now explain how to adapt the proof of Theorem 1.1 using these new ingredients.

7.2. *Rough Lipschitz harmonic maps*

We first want to point out that Theorem 7.1 cannot be extended to all rough Lipschitz maps.

Example 7.3. There exists an injective Lipschitz map $f : \mathbb{H}^2 \to \mathbb{H}^2$ from the hyperbolic plane to itself that extends continuously to the visual boundary as the identity map, and which is not within bounded distance of any harmonic map.

Proof. We will consider a map $f : \mathbb{H}^2 \to \mathbb{H}^2$ that commutes with a parabolic subgroup of $\mathrm{Isom}(\mathbb{H}^2)$. Let us work in the upper half-plane model. The map f is defined by

$$f(u, v) = (u, v + v^2), \qquad u \in \mathbb{R},\ v > 0,$$

so that $f \circ s_t = s_t \circ f$ where $s_t(u, v) = (t - u, v)$ for any $t \in \mathbb{R}$. Observe that f extends continuously to the visual compactification of $\mathbb{H}^2$ by the identity, and that f is 2-Lipschitz.

Assume by way of contradiction that there exists a harmonic map $h : \mathbb{H}^2 \to \mathbb{H}^2$ within bounded distance of f.

First case: the map h is unique. In this case h also commutes with the isometries s_t, so that there exists a continuous function $g : [0, \infty] \to [0, \infty]$ such that

$$h(u, v) = (u, g(v)), \qquad u \in \mathbb{R}, v > 0,$$

and $g(0) = 0$, $g(\infty) = \infty$. Saying that h is harmonic is equivalent to requiring that g satisfies the differential equation

$$gg'' = (g')^2 - 1.$$

It follows that the harmonic map h coincides with one of the maps $h_a : \mathbb{H}^2 \to \mathbb{H}^2$ defined by

$$h_a(u, v) = \big(u, \tfrac{1}{a}\sinh(av)\big)$$

for some constant $a \geq 0$. Observe that none of the maps h_a is within bounded distance of f, hence the contradiction.

Second case: the map h is not unique. Let h_0, h_1 be two distinct harmonic maps within bounded distance of f. We want again to find a contradiction. We will use arguments similar to those in Section 5. Let $x_0 := (0, 1) \in \mathbb{H}^2$. We choose a sequence of points x_n in $\mathbb{H}^2$ for which

$$d(h_0(x_n), h_1(x_n)) \to \delta := \sup_{x\in\mathbb{H}^2} d(h_0(x), h_1(x)) > 0$$

and we set $y_n := f(x_n)$. Let φ_n and ψ_n be the isometries of $\mathbb{H}^2$ fixing $\infty \in \partial\mathbb{H}^2$ and such that $\varphi_n(x_0) = x_n$ and $\psi_n(x_0) = y_n$. After passing to a subsequence, $\psi_n^{-1} \circ f \circ \varphi_n$ converges to one of the maps $f_\beta : \mathbb{H}^2 \to \mathbb{H}^2$ with $\beta \in [0, \infty]$ where

$$\begin{aligned} f_\beta &: (u, v) \mapsto \left(\frac{u}{1+\beta}, \frac{v+\beta v^2}{1+\beta}\right) && \text{when } 0 \leq \beta < \infty, \\ f_\infty &: (u, v) \mapsto (0, v^2) && \text{when } \beta = \infty. \end{aligned}$$

For $i = 0$ and 1, the sequence of harmonic maps $h_{i,n} := \psi_n^{-1} \circ h_i \circ \varphi_n$ converges, after extraction, to a harmonic map $h_{i,\infty} : \mathbb{H}^2 \to \mathbb{H}^2$ within bounded distance of f_β. The subharmonic function $x \mapsto d(h_{0,\infty}(x), h_{1,\infty}(x))$ achieves its maximum value at $x = x_0$, hence is a constant function equal to δ. Therefore, by Corollary 5.19, the harmonic maps $h_{0,\infty}$ and $h_{1,\infty}$ take their values in the same geodesic Γ. This forces $\beta = \infty$ and the geodesic Γ is the image of f_∞. Now we write

$$f_\infty(u, v) = (0, e^{2F_\infty(u,v)}) \quad \text{and} \quad h_{0,\infty}(u, v) = (0, e^{2H_{0,\infty}(u,v)}),$$

where $F_\infty(u, v) = \log v$ and where $H_{0,\infty}$ is a harmonic function.

The function $G_\infty := F_\infty - H_{0,\infty}$ is then a bounded function on $\mathbb{H}^2$ such that $\Delta G_\infty = 1$. Such a function G_∞ does not exist. Indeed, $G : x \mapsto 2\log(\cosh(d(x_0, x)/2))$ also satisfies $\Delta G = 1$ and the function $G - G_\infty$ would be proper and harmonic, contradicting the maximum principle. □

7.3. An overview of the proof of Proposition 7.2

Proof of Proposition 7.2. The strategy is the same as for Theorem 1.1:

Step 1: smoothing f out. By Proposition 2.4 there exists a smooth map $\widetilde{f} : X \to Y$ within bounded distance of f and whose first and second covariant derivatives are bounded on X. This function $\widetilde{f}$ is Lipschitz and still has property $\mathcal{C}$. Hence we can assume that $f = \widetilde{f}$.

Step 2: solving a bounded Dirichlet problem. We fix $O \in X$. For any radius R we consider the unique harmonic map $h_R : B(O, R) \to Y$ satisfying the Dirichlet condition $h_R = f$ on $S(O, R)$.

Step 3: estimating $d(h_R, f)$. In Subsection 7.4 we will check

Proposition 7.4. *There exists a constant $\rho \geq 1$ such that $d(h_R, f) \leq \rho$ for any $R \geq 1$.*

Step 4: letting $h_R \to h$. We prove this convergence as in Section 3.3. □

The proofs of Steps 1, 2 and 4, as well as the proof of uniqueness, require only minor modifications of the ones for quasi-isometric maps. Thus, the remainder of this paper will be devoted to the proof of Step 3.

7.4. Interior estimate for rough Lipschitz

In this subsection we complete the proof of Proposition 7.4 whose structure is exactly the same as the proof of Proposition 3.5. We will just quickly repeat the arguments of Section 4 pointing out the changes in the choice of the numerous constants involved in the proof.

7.4.1. Strategy. Let X and Y be Hadamard manifolds whose curvatures are pinched, $-b^2 \leq K \leq -a^2 < 0$. Let $k = \dim X$ and $k' = \dim Y$. We fix constants $M, N > 0$ as in Proposition 4.9. We set $\alpha = a/(2bk'N)$ so that, with the notation of Propositions 6.13 and 6.15, one has $\nu_\alpha = 1/(2N)$. We set $\nu = 2\nu_\alpha = 1/N$.

We start with a $\mathcal{C}^\infty$ Lipschitz map $f : X \to Y$ whose first and second covariant derivatives are bounded. We fix constants $c, C_1, C_2 \geq 1$ such that f has property $\mathcal{C}_{C_1,C_2}$ as in Definition 6.8 and for all x in X,

$$\|Df(x)\| \leq c, \qquad \|D^2 f(x)\| \leq bc^2. \tag{7.1}$$

We let $C_3 = C_{3,\alpha,\nu} \leq C_4 = C_{4,\alpha,\nu}$ be as in Proposition 6.13 and 6.15:

$$C_3 = \frac{C_1 C_2^{1/N}}{1 - e^{-a/(2N)}}, \qquad C_4 = \frac{C_1 C_2^{1/N}}{(1 - e^{-bk'\beta})(1 - e^{-a/(2N)})} \quad \text{where} \quad \beta = \frac{\alpha^2}{2\alpha + c}.$$

Choosing ℓ_0 very large. We fix O in X. We introduce a fixed integer radius ℓ_0 depending only on a, b, k, k', c, C_1 and C_2. The integer $\ell_0 \geq 1$ is only required to satisfy (7.2)–(7.4):

$$b\ell_0 > 1, \tag{7.2}$$

$$\ell_0 > 4n_0 c/\alpha, \qquad \text{where } n_0 \geq \frac{4e^{2ac}}{1 - e^{-a\beta}} \text{ is chosen with } MC_4 e^{-an_0\alpha} \leq \frac{\alpha}{8c}, \tag{7.3}$$

$$16e^{-a\alpha\ell_0/4} < \theta_0 \qquad \text{where } \theta_0 := e^{-2n_0 bc}/2. \tag{7.4}$$

Choosing ρ very large. For $R > 0$, let $h_R : B(O, R) \to Y$ be the harmonic $\mathcal{C}^\infty$ map whose restriction to $\partial B(O, R)$ is f. We let $\rho := \sup_{x \in B(O,R)} d(h_R(x), f(x))$. If this supremum ρ is not uniformly bounded, we can fix a radius R such that ρ satisfies the inequalities (4.6)–(4.8), which we rewrite below:

$$a\rho > 8kbc^2\ell_0, \tag{7.5}$$

$$\frac{2^7(a\rho)^2}{\sinh(a\rho/2)} < \theta_0. \tag{7.6}$$

$$\rho > 4c\ell_0 M(2^{10}e^{b\ell_0}k)^N. \tag{7.7}$$

We denote by x a point of $B(O, R)$ where the supremum is achieved: $d(h_R(x), f(x)) = \rho$. According to the boundary estimate (3.2) one has, using (7.5),

$$d(x, \partial B(O, R)) \geq \frac{a\rho}{3kbc^2} \geq 2\ell_0.$$

Getting a contradiction. We focus on the restrictions of f and h_R to $B(x, \ell_0)$. Set $y := f(x)$. For ξ on the unit tangent sphere S_x, we analyze the triangle inequality

$$\theta_y(f(\xi_{\ell_0}), h_R(x)) \leq \theta_y(f(\xi_{\ell_0}), h_R(\xi_{\ell_0})) + \theta_y(h_R(\xi_{\ell_0}), h_R(x)), \tag{7.8}$$

and prove that on a subset $U_{\ell_0} \smallsetminus A_{x,\alpha}(n_0)$ of the sphere, each term on the right-hand side is small (Lemmas 7.9 and 7.10) while the left-hand side is not always that small (Lemma 7.12), giving rise to a contradiction.

Definition 7.5. Let $U_{\ell_0} = \{\xi \in S_x \mid d(y, h_R(\xi_{\ell_0})) \geq \rho - \ell_0\alpha/2\}$.

7.4.2. Measure estimate

Lemma 7.6. *For ξ in S_x, one has $d(y, h_R(\xi_{\ell_0})) \leq \rho + c\ell_0$.*

Proof. This is Lemma 4.2. □

Lemma 7.7. *For ξ in S_x, and $r \leq \ell_0$, one has $\|Dh_R(\xi_r)\| \leq 2^8 kb\rho$.*

Proof. This is Lemma 4.3. It uses (7.2) and (7.5). □

Lemma 7.8. *Let $\sigma = \sigma_{x,\ell_0}$ be the harmonic measure on the sphere $S_x \simeq S(x, \ell_0)$ at the center point x. Then $\sigma(U_{\ell_0}) \geq \alpha/(3c)$.*

Proof. Same as that of Lemma 4.4, using Lemma 7.6. □

7.4.3. Estimating the angles

Lemma 7.9. *For ξ in $U_{\ell_0} \smallsetminus A_{x,\alpha}(n_0)$, one has $\theta_y(f(\xi_{\ell_0}), h_R(\xi_{\ell_0})) \leq 4e^{-a\alpha\ell_0/4} < \theta_0/4$.*

Proof. Same as that of Lemma 4.5, using (7.4). □

Lemma 7.10. *For ξ in S_x, one has*

$$\theta_y(h_R(\xi_{\ell_0}), h_R(x)) \leq \frac{2^5(a\rho)^2}{\sinh(a\rho/2)} < \frac{\theta_0}{4}.$$

Proof. Same as that of Lemma 4.6, relying on Lemma 7.11 and using (7.5) and (7.6). □

Lemma 7.11. *For all ξ in S_x and $r \le \ell_0$, one has $d(y, h_R(\xi_r)) \ge \rho/2$.*

Proof. Same as that of Lemma 4.7, using Lemma 7.7 and (7.7). □

Lemma 7.12. *There exist ξ, η in $U_{\ell_0} \smallsetminus A_{x,\alpha}(n_0)$ with $\theta_y(f(\xi_{\ell_0}), f(\eta_{\ell_0})) \ge \theta_0$.*

Proof. Recall that $\sigma := \sigma_{x,\ell_0}$ denotes the harmonic measure at x for $S(x, \ell_0)$. Let $\sigma_0 := \alpha/(4c)$. According to Lemma 7.8, one has

$$\sigma(U_{\ell_0}) > \sigma_0 > 0.$$

Since the harmonic measure σ is $(M, 1/N)$-Frostman (Proposition 4.9), one may apply (6.8) of Proposition 6.13 to σ and get, using (7.3),

$$\sigma(A_{x,\alpha}(n_0)) \le MC_3 e^{-\frac{\alpha n_0}{2N}} \le \frac{\alpha}{8c} = \sigma_0/2.$$

Therefore, there exists an element $\xi \in U_{\ell_0} \smallsetminus A_{x,\alpha}(\ell_0)$. On may now apply (6.10) to the harmonic measure $\sigma = \sigma_{x,\ell_0}$ to get, using (7.3) again,

$$\sigma(B^{\xi}_{x,\alpha}(n_0)) \le MC_4 e^{-\frac{\alpha n_0}{2N}} \le \frac{\alpha}{8c} = \sigma_0/2.$$

Therefore, there exists an element $\eta \in U_{\ell_0} \smallsetminus (A_{x,\alpha}(n_0) \cup B^{\xi}_{x,\alpha}(n_0))$. It satisfies

$$\theta_y(f(\xi_{\ell_0}), f(\eta_{\ell_0}) \ge e^{-2n_0 bc}/2 = \theta_0$$

because of (7.3), (7.4) and Proposition 6.15(e). □

End of proof of Proposition 7.4. Let $\xi, \eta \in U_{\ell_0} \smallsetminus A_{x,\alpha}(n_0)$ be given by Lemma 7.12. Applying Lemmas 7.9 and 7.10 to ξ and η, one gets

$$\theta_y(f(\xi_{\ell_0}), f(\eta_{\ell_0})) \le \theta_y(f(\xi_{\ell_0}), h_R(x)) + \theta_y(h_R(x), f(\eta_{\ell_0})) < \theta_0,$$

which contradicts Lemma 7.12. □

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