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A compactness theorem for the fractional Yamabe problem, Part I: The nonumbilic conformal infinity

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Abstract. Assume that (X, g^+) is an asymptotically hyperbolic manifold, $(M, [\bar{h}])$ is its conformal infinity, ρ is the geodesic boundary defining function associated to $\bar{h}$ and $\bar{g} = \rho^2 g^+$. For any γ in $(0, 1)$, we prove that the solution set of the γ-Yamabe problem on M is compact in $C^2(M)$ provided that convergence of the scalar curvature $R[g^+]$ of (X, g^+) to $-n(n+1)$ is sufficiently fast as ρ tends to 0 and the second fundamental form on M never vanishes. Since most of the arguments in the blow-up analysis performed here are insensitive to the geometric assumption imposed on X, our proof also provides a general scheme toward other possible compactness theorems for the fractional Yamabe problem.

Keywords. Fractional Yamabe problem, nonumbilic conformal infinity, compactness, blow-up analysis

1. Introduction

Given any $n \in \mathbb{N}$, let (X^{n+1}, g^+) be an asymptotically hyperbolic manifold with conformal infinity $(M^n, [\bar{h}])$. According to [33, 61, 42, 12, 30], there exists a family of self-adjoint conformally covariant pseudo-differential operators $P^\gamma[g^+, \bar{h}]$ on M defined for generic γ and whose principal symbols are the same as those of $(-\Delta_{\bar{h}})^\gamma$. If (X, g^+) is Poincaré–Einstein and $\gamma \in \mathbb{N}$, the operator $P^\gamma[g^+, \bar{h}]$ coincides with the GJMS operator constructed by Graham et al. [32] via the ambient metric; refer to Graham and Zworski [33]. In particular, $P^\gamma[g^+, \bar{h}]$ is equal to the conformal Laplacian or the Paneitz operator for $\gamma = 1$ or 2, respectively.

Let us call $Q^\gamma[g^+, \bar{h}] = P^\gamma[g^+, \bar{h}](1)$ the *γ-scalar curvature*. One natural question is if there exists a metric $\bar{h}'$ conformal to $\bar{h}$ on M whose γ-scalar curvature $Q^\gamma[g^+, \bar{h}']$ is constant. By virtue of the conformal covariance property of P^γ, this is reduced to

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searching a smooth solution of the equation

$$P^\gamma[g^+, \bar{h}]u = cu^p \quad \text{and} \quad u > 0 \quad \text{on } (M^n, \bar{h}) \tag{1.1}$$

for some constant $c \in \mathbb{R}$ provided $n > 2\gamma$ and $p = 2^*_{n,\gamma} - 1 = (n + 2\gamma)/(n - 2\gamma)$.

For $\gamma = 1$, the study of the existence of a solution to (1.1) was initiated by Yamabe [75] and completely solved through the successive works of Trudinger [73], Aubin [5] and Schoen [67]. See also Lee and Parker [48] and Bahri [6] where a unified proof based on the use of conformal normal coordinates and a proof not depending on the positive mass theorem are devised, respectively. If $\gamma = 2$ (and $n \geq 5$), existence theory of (1.1) becomes considerably harder because of the lack of a maximum principle. Up to now, only partial results are available such as those by Qing and Raske [65], Gursky and Malchiodi [36] and Hang and Yang [38]. In [38], the authors could treat a general class of manifolds having the property that the Yamabe constant (1.4) is positive and there exists a representative of the conformal class $[\bar{h}]$ with semipositive Q-curvature Q^2. It is worth mentioning a related work of Gursky et al. [35] which examined a conformally invariant condition for the existence of a conformal metric having positive scalar and Q-curvatures. Meanwhile, equation (1.1) with $\gamma = 1/2$ has a deep relation to the boundary Yamabe problem (or the higher-dimensional Riemann mapping theorem) formulated by Escobar [23]; refer also to Remark 1.2(1). The existence theory for the latter problem has been completed due to the effort of Cherrier [14], Escobar [23, 24], Marques [57, 58], Almaraz [1], Chen [13] and Mayer and Ndiaye [59].

If $\gamma \notin \mathbb{N}$, it is not a simple task to solve (1.1) directly, since the operator $P^\gamma[g^+, \bar{h}]$ is nonlocal and defined in a rather abstract way. However, Chang and González [12] discovered that (1.1) can be interpreted as a Caffarelli–Silvestre-type degenerate elliptic equation of [9], which is indeed local, for which a variety of well-known techniques like constraint minimization and the Moser iteration technique can be applied; see Proposition 2.1 for a more precise description. From this observation, González and Qing [30] succeeded in finding solutions to (1.1) for $\gamma \in (0, 1)$ under the hypothesis that $M = \partial X$ is nonumbilic and of sufficiently large dimension. Their approach was further developed in the works of González and Wang [31] and the present authors [46], which cover most cases when the local geometry dominates. In [46], the authors also established the existence result for 2-dimensional or locally conformally flat manifolds provided that a certain technical assumption on the Green's function of P^γ holds. Recently, Mayer and Ndiaye [60] and Daskalopoulos et al. [16] pursued the critical point at infinity approach and the flow approach, respectively, removing the technical condition imposed on the Green's function.

Furthermore, Case and Chang [11] obtained an extension result for $\gamma \in (1, n/2)$ which generalizes [12]. By utilizing it and adapting the argument of Gursky and Malchiodi [36], they also deduced that $P^\gamma[g^+, \bar{h}]$ satisfies a strong maximum principle for $\gamma \in (1, \min\{2, n/2\})$ when $[\bar{h}]$ has a metric whose scalar curvature is nonnegative and the γ-curvature is semipositive. It is plausible that this with some further ideas in [36] may allow one to get certain existence results for (1.1) under the prescribed conditions.

In many cases, (1.1) may have higher Morse index (or energy) solutions as shown in [68, 64] for $\gamma = 1$. If (X, g^+) is the Poincaré ball whose conformal infinity is the standard sphere, it follows immediately from the noncompactness of the conformal group that the solution set to (1.1) is unbounded in $L^\infty(M)$ for any $\gamma > 0$. Indeed, classification results by Obata [63] and Jin et al. [40, Theorem 1.8] show that there is a correspondence between an element of the conformal group and a solution to (1.1) for $\gamma = 1$ and $\gamma \in (0, 1)$, respectively.

In this connection, for manifolds M that are not conformally diffeomorphic to the standard sphere, Schoen [70] raised the question of $C^2(M)$-compactness of the solution set to (1.1) with $\gamma = 1$ and suggested a general strategy towards its proof. The first affirmative answer was given by Schoen himself [69] in the locally conformally flat case. Li and Zhu [55] obtained it for $n = 3$ and Druet [22] did it for $n \leq 5$. If $n \geq 6$, the analysis is more delicate because one needs to prove that the Weyl tensor vanishes to the order greater than $\lfloor (n-6)/2 \rfloor$ at a blow-up point. By solving this technical difficulty, Marques [56] and Li and Zhang [52] could deal with the situation that either $n \leq 7$ or the Weyl tensor never vanishes on M. Assuming the validity of the positive mass theorem and performing a refined blow-up analysis on the basis of the linearized problem (as in our Section 4), Li and Zhang [53] extended the result up to dimension 11, and Khuri et al. [43] finally verified it for $n \leq 24$. Surprisingly, according to Brendle [7] and Brendle and Marques [8], there are C^∞-metrics on the sphere S^n with $n \geq 25$ such that even though they are not conformally equivalent to the canonical metric, a blowing-up family of solutions to (1.1) does exist.

For $\gamma = 2$, Y. Li and Xiong [51] obtained the $C^4(M)$-compactness result assuming that the kernel of the Paneitz operator P^2 is trivial, its Green function is positive, the positive mass type theorem holds for P^2 and one of the following assumptions holds: $5 \leq n \leq 9$, or M is locally conformally flat, or $n \geq 8$ and the Weyl tensor never vanishes. See also the previous works [39, 65, 49]. On the other hand, Wei and Zhao [74] established a noncompactness result for $n \geq 25$. While it is expected that $C^4(M)$-compactness holds in general up to dimension 24, a rigorous proof is not known yet.

For the boundary Yamabe problem, corresponding to the case $\gamma = 1/2$, compactness results were deduced when X^{n+1} is locally conformally flat [28], $n = 3$ [29, 4] and $n = 4, 5$ [47]. Compactness of the solution set also follows under the assumption that $n \geq 6$ and the second fundamental form on M vanishes nowhere [1, 47]; refer to [37, 21] for more results. Almaraz [3] showed that a blow-up phenomenon still happens if $n+1 \geq 25$.

If 2γ is a noninteger value, only a little has been revealed up to now. As far as we know, the only article that investigates compactness of the solution set to (1.1) for γ in $[1, n/2)$ is [66] due to Qing and Raske, which concerns locally conformally flat manifolds M with positive Yamabe constant and with Poincaré exponent less than $(n-2\gamma)/2$. As for noncompactness, the present authors [45] constructed asymptotic hyperbolic manifolds which are small perturbations of the Poincaré ball and exhibit a blow-up phenomenon for $n \geq 24$ if $\gamma \in (0, \gamma^*)$ and $n \geq 25$ if $\gamma \in [\gamma^*, 1)$ where γ^* is a number close to 0.940197. However, the compactness issue for (1.1) with $\gamma \in (0, 1)$ has not been discussed in the literature so far, unless the underlying manifold is the Poincaré ball; see [40].

In this paper, we are concerned with the compactness of the solution set to the γ-*Yamabe problem* (1.1) provided $\gamma \in (0, 1)$ and $c > 0$. As can be predicted from the representation theorem of Palais–Smale sequences associated with fractional Yamabe-type equations in [27], the conformal covariance property of P^γ enables us to perform a local analysis even though it is a nonlocal operator.

We will state the main theorem in a slightly more general setting; more precisely, we will allow p to be subcritical. Since we always assume that the metric g^+ in X is fixed, we write $P^\gamma_{\bar h} = P^\gamma[g^+, \bar h]$ and $Q^\gamma_{\bar h} = Q^\gamma[g^+, \bar h]$.

Theorem 1.1. *Let* $\gamma \in (0, 1)$,

$$n \geq \begin{cases} 7 & \text{for } 0 < \gamma \leq \sqrt{1/19}, \\ 6 & \text{for } \sqrt{1/19} < \gamma \leq 1/2, \\ 5 & \text{for } 1/2 < \gamma \leq \sqrt{5/11}, \\ 4 & \text{for } \sqrt{5/11} < \gamma < 1, \end{cases} \tag{1.2}$$

and (X^{n+1}, g^+) *be an asymptotically hyperbolic manifold with conformal infinity* $(M^n, [\bar h])$. *Denote by* ρ *a geodesic defining function associated to* M, *i.e., a unique defining function splitting the metric* $\bar g = \rho^2 g^+$ *on the closure* $\overline{X}$ *of* X *as* $d\rho^2 + h_\rho$ *near* M *where* $\{h_\rho\}_\rho$ *is a family of metrics on* M *such that* $h_0 = \bar h$.

Assume that $\bar g \in C^4(\overline{X}, \mathbb{R}^{(n+1)\times(n+1)})$, *the first* $L^2(X)$*-eigenvalue* $\lambda_1(-\Delta_{g^+})$ *of the Laplace–Beltrami operator* $-\Delta_{g^+}$ *satisfies*

$$\lambda_1(-\Delta_{g^+}) > n^2/4 - \gamma^2, \tag{1.3}$$

the γ*-Yamabe constant defined as*

$$\Lambda^\gamma(M, [\bar h]) = \inf_{u \in H^\gamma(M)\setminus\{0\}} \frac{\int_M u P^\gamma_{\bar h} u \, dv_{\bar h}}{\left(\int_M |u|^{\frac{2n}{n-2\gamma}} dv_{\bar h}\right)^{\frac{n-2\gamma}{n}}} \tag{1.4}$$

is positive, and

$$R[g^+] + n(n+1) = o(\rho^2) \quad \text{as } \rho \to 0 \text{ uniformly on } M \tag{1.5}$$

where $R[g^+]$ *is the scalar curvature of* (X, g^+).

If the second fundamental form π *of* $(M, \bar h)$ *as a submanifold of* $(\overline{X}, \bar g)$ *never vanishes, then for any* $\varepsilon_0 > 0$ *small, there exists a constant* $C > 1$ *depending only on* X^{n+1}, g^+, $\bar h$, γ *and* ε_0 *such that*

$$C^{-1} \leq u \leq C \quad \text{on } M \quad \text{and} \quad \|u\|_{C^2(M)} \leq C \tag{1.6}$$

for any solution in $H^\gamma(M)$ *to* (1.1) *with* $1 + \varepsilon_0 \leq p \leq 2^*_{n,\gamma} - 1$.

Here $H^\gamma(M)$ is the fractional Sobolev space defined via a partition of unity on M. Also, the values of $\sqrt{1/19}$ and $\sqrt{5/11}$ are approximately 0.229 and 0.674, respectively.

A couple of remarks regarding Theorem 1.1 are in order.

Remark 1.2. (1) It is remarkable that the dimension restriction (1.2) is exactly the same as the one appearing in the existence result [46, Corollary 2.7] for equation (1.1) on nonumbilic conformal infinities.

(2) Condition (1.5) makes the boundary Yamabe problem and the 1/2-Yamabe problem identical (modulo the remainder term) in view of the energy expansion. This fact is essentially contained in [34, Section 4]. Also, (1.5) and the property that $\pi \neq 0$ at each point of M are intrinsic in the sense that they do not depend on the choice of a representative of the class $[\bar{h}]$. Refer to [46, Lemma 2.1 and Subsections 2.3, 2.4].

(3) Hypothesis (1.5) implies that the mean curvature H is identically 0 on M; see [46, Lemma 2.3]. As a particular consequence, $\pi = \pi - H\bar{g}$ on M, the latter tensor being the trace-free second fundamental form. Thus our theorem generalizes the result of Almaraz [2, Theorem 1.2] on the boundary Yamabe problem (corresponding to the case $\gamma = 1/2$) under the further assumption that $H = 0$ on M.

(4) If $R[g^+] = -n(n+1)$ in X, which is a stronger condition than (1.5), one can check that $\Lambda^\gamma(M, [\bar{h}]) > 0$ implies (1.3); see [10, Proposition 5.1]. It is an interesting open question whether (1.3) follows from $\Lambda^\gamma(M, [\bar{h}]) > 0$ and (1.5).

(5) The standard transversality argument implies that the set of Riemannian metrics on $\overline{X}$ whose trace-free second fundamental form is nonzero everywhere is open and dense in the space of all Riemannian metrics on $\overline{X}$. On the other hand, there exists an asymptotically hyperbolic manifold X^{n+1} that can be realized as a small perturbation of the Poincaré half-space, for which the solution set of the γ-Yamabe problem is noncompact provided that $n \geq 24$ or $n \geq 25$ according to the magnitude of $\gamma \in (0, 1)$; refer to [45]. In this example, the conformal infinity M is the totally geodesic (in particular, umbilic) boundary of $\overline{X}$.

(6) While the condition that M is nowhere umbilic is generic and conformally covariant, it does not hold for typical asymptotically hyperbolic metrics such as Poincaré–Einstein metrics. We expect that for such metrics, a compactness result analogous to Theorem 1.1 continues to hold under suitable conditions. For example, in light of the existence result [46, Corollary 3.4], imposing the conditions that M is umbilic, the Weyl tensor on M never vanishes, and $n \geq 7$ for $\gamma \in [1/2, 1)$ and $n \geq 8$ for $\gamma \in (0, 1)$ seems enough. Also, if a suitable condition on the Green's function on $P^\gamma_{\bar{h}}$ is given, a compactness result may be obtained provided that M is either locally conformally flat or 2-dimensional, as in [28, 29]. An interesting question is whether the dimension assumption in [45] is optimal for a blow-up phenomenon of the solution set to (1.1) to occur. To answer it, one needs a positive mass type theorem for $P^\gamma_{\bar{h}}$, which is out of reach at the moment. On the other hand, by applying the extension theorems and the energy identities established in [11, 10] and the methods of this paper, it should be possible to obtain a compactness result for $\gamma > 1$.

One can deduce Theorem 1.1 from the next theorem and elliptic regularity theory; see Subsection 6.2 and Appendix A.

Theorem 1.3 (Vanishing theorem for the second fundamental form). *For $\gamma \in (0, 1)$ and $n \in \mathbb{N}$ satisfying (1.2), let (X^{n+1}, g^+) be an asymptotically hyperbolic manifold with conformal infinity $(M^n, [\bar{h}])$ such that (1.3) is valid. Moreover assume that ρ is a* *geodesic defining function associated to* M, $\bar{g} = \rho^2 g^+$, $\Lambda^\gamma(M, [\bar{h}]) > 0$, *and* (1.5) *holds. If* $\{(u_m, y_m)\}_{m \in \mathbb{N}}$ *is a sequence of pairs in* $C^\infty(M) \times M$ *such that each* u_m *is a solution of* (1.1), y_m *is a local maximum point of* u_m *satisfying* $u_m(y_m) \to \infty$ *and* $y_m \to y_0 \in M$ *as* $m \to \infty$, *then the second fundamental form* π *at* y_0 *vanishes.*

As a corollary of Theorem 1.1, we can compute the Leray–Schauder degree of all solutions to equation (1.1) if every hypothesis imposed in the theorem holds. Since $P^\gamma_{\bar{h}}$ is a self-adjoint operator as shown in [33], any $L^{p+1}(M)$-normalized minimizer of

$$\inf_{u \in H^\gamma(M) \setminus \{0\}} \frac{\int_M u P^\gamma_{\bar{h}} u \, dv_{\bar{h}}}{(\int_M |u|^{p+1} \, dv_{\bar{h}})^{2/(p+1)}} \quad \text{for } 1 \le p \le 2^*_{n,\gamma} - 1$$

(which is the same as (1.4) if $p = 2^*_{n,\gamma} - 1$) solves

$$P^\gamma_{\bar{h}} u = \mathcal{E}(u)|u|^{p-1}u \quad \text{on } (M, \bar{h}) \quad \text{where} \quad \mathcal{E}(u) = \int_M u P^\gamma_{\bar{h}} u \, dv_{\bar{h}} \quad (\text{the energy of } u). \tag{1.7}$$

Furthermore, if the γ-Yamabe constant $\Lambda^\gamma(M, [\bar{h}])$ is positive, then it is easy to check that the operator $T : L^{2n/(n+2\gamma)}(M) \to H^\gamma(M)$ is well-defined by the relation

$$T(v) = u \quad \text{if and only if} \quad P^\gamma_{\bar{h}} u = v \text{ on } M.$$

Hence, it is natural to define a map $\mathcal{F}_p : D_\Lambda \to L^\infty(M)$ by $\mathcal{F}_p(u) = u - T(\mathcal{E}(u)u^p)$ where

$$D_\Lambda = \{u \in L^\infty(M) : u > \Lambda^{-1} \text{ and } \|u\|_{L^\infty(M)} < \Lambda\} \quad \text{for each } \Lambda > 1;$$

this map has the property that $\mathcal{F}_p(u) = 0$ if and only if u is a solution of (1.1). The elliptic estimate in Lemma A.2 below implies that $\mathcal{F}_p$ is the sum of the identity and a compact map. Moreover we infer from Lemma 6.6, a consequence of Theorem 1.1, that $0 \notin \mathcal{F}_p(\partial D_\Lambda)$ for all $1 \le p \le 2^*_{n,\gamma} - 1$ if Λ is sufficiently large. Therefore the Leray–Schauder degree $\deg(\mathcal{F}_p, D_\Lambda, 0)$ of the map $\mathcal{F}_p$ in the domain D_Λ with respect to the point $0 \in L^\infty(M)$ is well-defined.

Theorem 1.4. *Under the assumptions of Theorem* 1.1,

$$\deg(\mathcal{F}_p, D_\Lambda, 0) = -1.$$

In particular, the fractional Yamabe equation (1.1) *has a solution.*

Theorem 1.4 gives a new proof of the existence of a solution to (1.1) under rather restrictive assumptions. Compare it with [30, 46]. We also expect that a strong Morse inequality holds in our framework; refer to [43, Theorem 1.4].

Our proof of the main theorems relies on Schoen's argument [70] yielding the compactness theorem for the classical Yamabe problem. It has been further developed through the works of Li and Zhu [55], Druet [22], Marques [56], Khuri et al. [43] (for the Yamabe problem), Han and Li [37], Felli and Ould Ahmedou [28, 29], Almaraz [2], Almaraz et al. [4], Kim et al. [47] (for the boundary Yamabe problem), G. Li [49], Y. Y. Li and Xiong [51] (for the Q-curvature problem), Schoen and Zhang [71], Li [50], Jin et al. [41] (for the classical and fractional Nirenberg problem) and Niu et al. [62] (for the critical Lane–Emden equation involving the regional fractional Laplacian) among others.

Although certain parts of the proof can be obtained by minor modifications of the classical arguments, there are still plenty of technical difficulties which demand new ideas. We will pay attention to, for instance, the following features.

- For the fractional Yamabe problem, it is not the best idea to apply conformal changes on the whole manifold $\overline{X}$, because one needs to properly control the interior geometry, which is a nontrivial issue. Therefore we will control only the boundary metric through conformal changes. This restriction does not allow us to use the standard conformal Fermi coordinates on $\overline{X}$ (given in [57, Proposition 3.1]), and forces us to work with more geometric quantities, especially those appearing in the low-order terms of some asymptotic expansion of a Pohozaev-type identity; refer to Section 5. To handle the interior of the manifold, we will employ the geometric assumption (1.5) and examine the first-order partial differential equation satisfied by the geodesic defining function.
- We largely depend on the extension result of Chang and González [12] to analyze solutions. Because of the degeneracy of the extended problem (2.4), it is not easy to study the asymptotic behavior of the Green's function near its singularity; see Appendix B.1 where some of its qualitative properties are obtained. Hence, in showing the decay property of rescaled solutions, we do not use potential analysis, but iteratively apply the rescaling argument based on the maximum principle.
- Regularity theory that we require is technically more difficult to deduce than ones for the classical local problems, or even nonlocal problems on the Euclidean space.
- Suppose that $\gamma \in (0, 1)\setminus\{1/2\}$. In this case, it is not easy to compute integrals involving the bubbles by using their integral representations (related to the Poisson or Green kernels). We will solve this technical issue by further developing the Fourier transform technique due to González and Qing [30].

To reduce overlaps, we will omit the proofs of several intermediate results which closely follow standard arguments, giving appropriate references. Our main concern is to clarify the novelty of the nonlocal problems defined on general conformal infinities.

The paper is organized as follows: In Section 2, we recall some analytic and geometric tools necessary to investigate the fractional Yamabe problem (1.1). In Section 3, we introduce some concepts regarding a blowing-up sequence $\{u_m\}_{m\in\mathbb{N}}$ of solutions to (1.1) and perform an asymptotic analysis near each blow-up point of $\{u_m\}_{m\in\mathbb{N}}$. Section 4 is devoted to deducing a sharp pointwise estimate of u_m near each isolated simple blow-up point. This allows one to establish the vanishing theorem for the second fundamental form at any isolated simple blow-up point, which is discussed in Section 5. Finally, the main theorems are proved in Section 6 with the aid of a local Pohozaev sign condition which guarantees that every blow-up point is isolated simple. In the appendices, we provide technical results needed in the main body of the proof as well as their proofs. Firstly, in Appendix A, we present several elliptic regularity results. Then we study the asymptotic behavior of the Green's function near its singularity in Appendix B.1. We also derive a fractional Bôcher's theorem in Appendix B.2. Finally, a number of integrals involving the *standard bubble* $W_{1,0}$, whose precise definition is given in Subsection 2.2, will be computed in Appendix C.

Notations

• The Einstein convention is adopted throughout the paper. We shall use the indices $1 \le i, j, k, l \le n$.

• For any $t \in \mathbb{R}$, set $t_+ = \max\{t, 0\}$ and $t_- = \max\{-t, 0\}$. Clearly, $t = t_+ - t_-$.

• Let $N = n + 1$. Also, for any $x \in \mathbb{R}^N_+ = \{(x_1, \dots, x_n, x_N) \in \mathbb{R}^N : x_N > 0\}$, we denote $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n \simeq \partial\mathbb{R}^N_+$.

• For any $\bar{x} \in \mathbb{R}^n$, $x = (\bar{x}, 0) \in \partial\mathbb{R}^N_+$ and $r > 0$, $B^n(\bar{x}, r)$ signifies the n-dimensional ball with center $\bar{x}$ and radius r. Similarly, $B^N_+(x, r)$ is the N-dimensional upper half-ball centered at x having radius r. We often identify $B^n(\bar{x}, r) = \partial B^N_+(x, r) \cap \partial\mathbb{R}^N_+$. Set $\partial_I B^N_+((\bar{x}, 0), r) = \partial B^N_+(x, r) \cap \mathbb{R}^N_+$.

• For a function f on $\mathbb{R}^N_+$, we often write $\partial_i f = \frac{\partial f}{\partial x_i}$ and $\partial_N f = \frac{\partial f}{\partial x_N}$.

• $|\mathbb{S}^{n-1}|$ is the surface area of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$.

• The spaces $W^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})$ and $D^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})$ are the completions of $C^\infty_c(\overline{\mathbb{R}^N_+})$ with respect to the norms

$$\|U\|_{W^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})} = \left(\int_{\mathbb{R}^N_+} x_N^{1-2\gamma} (|\nabla U|^2 + U^2)\, dx\right)^{1/2},$$

$$\|U\|_{D^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})} = \left(\int_{\mathbb{R}^N_+} x_N^{1-2\gamma} |\nabla U|^2\, dx\right)^{1/2},$$

respectively. The natural function space $W^{1,2}(X; \rho^{1-2\gamma})$ for the fractional Yamabe problem (2.4) is analogously defined.

• For any $\beta \in (0, \infty) \setminus \mathbb{N}$ and domain Ω, we write $C^\beta(\Omega)$ for the Hölder space $C^{\lfloor\beta\rfloor, \beta - \lfloor\beta\rfloor}(\Omega)$ where $\lfloor\beta\rfloor$ is the greatest integer that does not exceed β.

• Assume that $(M, \bar{h})$ and $(\overline{X}, \bar{g})$ are compact Riemannian manifolds. Then $B_{\bar{h}}(y, r) \subset (M, \bar{h})$ stands for the geodesic ball centered at $y \in M$ of radius $r > 0$. Moreover, $dv_{\bar{g}}$ is the volume form of $(\overline{X}, \bar{g})$ and $d\sigma$ represents a surface measure.

• $C > 0$ denotes a generic constant possibly depending on the dimension n of an underlying manifold M, the order γ of the conformal fractional Laplacian P^γ and so on. It may vary from line to line. Moreover, a notation $C(\alpha, \beta, \dots)$ means that the constant C depends on $\alpha, \beta, \dots$.

Remark 1.5. By [26, Lemma 3.1] (or we may follow [20, proof of Proposition 2.1.1] by replacing [25, Theorem 1.2] with [72, Lemma 2.2]) and [9, Section 3.2], we have

$$D^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma}) \hookrightarrow L^{\frac{2(n-2\gamma+2)}{n-2\gamma}}(\mathbb{R}^N_+; x_N^{1-2\gamma}) \quad \text{and} \quad D^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma}) \hookrightarrow H^\gamma(\mathbb{R}^n). \tag{1.8}$$

Hence we infer from [19, Corollary 7.2] that the trace embedding $D^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma}) \hookrightarrow L^q(\Omega)$ is compact for any $q \in [1, 2^*_{n,\gamma})$ and a smooth bounded domain $\Omega \subset \mathbb{R}^n$.

2. Preliminaries

2.1. Geometric background

We recall the extension result involving the conformal fractional Laplacian P^γ obtained by Chang and González [12]; see also [9, 30].

Proposition 2.1. *Suppose that $\gamma \in (0,1)$, $n > 2\gamma$ and (X, g^+) is an asymptotically hyperbolic manifold with conformal infinity $(M, [\bar{h}])$. Also, assume that ρ is a geodesic defining function associated to M, $\bar{g} = \rho^2 g^+$ and the mean curvature H is 0 on M. Set $s = n/2 + \gamma$ and*

$$E_{\bar{g}}(\rho) = \rho^{-1-s}(-\Delta_{g^+} - s(n-s))\rho^{n-s} \quad \text{in } X.$$

Then

$$\begin{aligned} E_{\bar{g}}(\rho) &= \frac{n-2\gamma}{4n}\left[R[\bar{g}] - (n(n+1) + R[g^+])\rho^{-2}\right]\rho^{1-2\gamma} \\ &= -\frac{n-2\gamma}{2} \cdot \frac{\partial_\rho \sqrt{|\bar{g}|}}{\sqrt{|\bar{g}|}}\rho^{-2\gamma} \qquad (\text{by [46, (2.5)]}) \end{aligned} \tag{2.1}$$

in $M \times (0, r_0)$ for some small $r_0 > 0$, where $R[\bar{g}]$ and $R[g^+]$ are the scalar curvatures of $(\overline{X}, \bar{g})$ and (X, g^+), respectively, and $|\bar{g}|$ is the determinant of $\bar{g}$.

(1) *Define*

$$\kappa_\gamma = \frac{2^{-(1-2\gamma)}\Gamma(\gamma)}{\Gamma(1-\gamma)} > 0 \quad \text{and} \quad \partial^\gamma_\nu U = -\kappa_\gamma \lim_{\rho \to 0+} \rho^{1-2\gamma}\frac{\partial U}{\partial \rho} \quad \text{on } M \tag{2.2}$$

where ν denotes the outward unit normal vector with respect to X and Γ is the Gamma function. If a positive function $U \in W^{1,2}(X; \rho^{1-2\gamma})$ satisfies

$$\begin{cases} -\mathrm{div}_{\bar{g}}(\rho^{1-2\gamma}\nabla U) + E_{\bar{g}}(\rho)U = 0 & \text{in } (X, \bar{g}), \\ U = u & \text{on } M, \end{cases} \tag{2.3}$$

then

$$\partial^\gamma_\nu U = P^\gamma_{\bar{h}} u \quad \text{on } M.$$

(2) *If* (1.3) *is also true, then there is a special defining function* ρ^* *such that* $E_{\bar{g}^*}(\rho^*) = 0$ *in* X *and* $\rho^*(\rho) = \rho(1 + O(\rho^{2\gamma}))$ *near* M. *Moreover the function* $U^* = (\rho/\rho^*)^{(n-2\gamma)/2} U$ *solves*

$$\begin{cases} -\mathrm{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma}\nabla U^*) = 0 & \text{in } (X, \bar{g}^*), \\ \partial_\nu^\gamma U^* = P_{\bar{h}}^\gamma u - Q_{\bar{h}}^\gamma u & \text{on } M. \end{cases}$$

Here $\bar{g}^* = (\rho^*)^2 g^+$ *and* $Q_{\bar{h}}^\gamma = P_{\bar{h}}^\gamma(1)$ *are called the* adapted metric *on* $\overline{X}$ *and the* fractional scalar curvature *on* $(M, \bar{h})$, *respectively.*

The spectral requirement (1.3) in the second assertion was pointed out in [11, Section 6], from which the term "adapted metric" comes.

Note that the condition $\Lambda^\gamma(M, [\bar{h}]) > 0$ (see (1.4)) implies that the functional

$$J^\gamma(U) = \int_X (\rho^{1-2\gamma}|\nabla U|_{\bar{g}}^2 + E_{\bar{g}}(\rho)U^2)\, dv_{\bar{g}} \quad \text{for } U \in W^{1,2}(X; \rho^{1-2\gamma})$$

is coercive, that is, there exists $C > 0$ independent of U such that $J^\gamma(U) \geq C\|U\|^2_{W^{1,2}(X;\rho^{1-2\gamma})}$. See [18, Lemma 2.5] for the proof. Therefore, given any $u \in H^\gamma(M)$, the standard minimization argument guarantees the existence and uniqueness of the extension $U \in W^{1,2}(X; \rho^{1-2\gamma})$ of u which satisfies (2.3). Furthermore, testing (2.3) with u_-, we easily observe that if $u \geq 0$ on M, then $U \geq 0$ in X. If $u > 0$ on M, then the strong maximum principle for elliptic operators gives $U > 0$ on $\overline{X}$.

On the other hand, without loss of generality, we can always assume that the constant $c > 0$ in equation (1.1) is exactly 1. As a result, (1.1) is equivalent to the degenerate elliptic problem

$$\begin{cases} -\mathrm{div}_{\bar{g}}(\rho^{1-2\gamma}\nabla U) + E_{\bar{g}}(\rho)U = 0 & \text{in } (X, \bar{g}), \\ U > 0 & \text{on } \overline{X}, \\ U = u & \text{on } M, \\ \partial_\nu^\gamma U = u^p & \text{on } M. \end{cases} \tag{2.4}$$

Next, choose any $y \in M$ and let $x = (\bar{x}, x_N) \in \mathbb{R}^N_+$ be *Fermi coordinates* on $\overline{X}$ around y, i.e., $\bar{x} = (x_1, \ldots, x_n)$ are normal coordinates on M at y and $x_N = \rho$. In [23, Lemma 3.1], the following expansion of the metric $\bar{g}$ near y is given.

Lemma 2.2. *In terms of Fermi coordinates* x *on* $\overline{X}$ *around* $y \in M$,

$$\sqrt{|\bar{g}|}(x) = 1 - nHx_N + \tfrac{1}{2}(n^2H^2 - \|\pi\|^2 - R_{NN}[\bar{g}])x_N^2 - nH_{,i}x_ix_N - \tfrac{1}{6}R_{ij}[\bar{h}]x_ix_j + O(|x|^3)$$

and

$$\bar{g}^{ij}(x) = \delta_{ij} + 2\pi_{ij}x_N + \tfrac{1}{3}R_{ikjl}[\bar{h}]x_kx_l + \bar{g}^{ij}{}_{,Nk}x_Nx_k + (3\pi_{ik}\pi_{kj} + R_{iNjN}[\bar{g}])x_N^2 + O(|x|^3).$$

Here

- δ_{ij} *is the Kronecker delta;*
- $\|\pi\|^2 = \bar{h}^{ik}\bar{h}^{jl}\pi_{ij}\pi_{kl}$ *is the square of the norm of the second fundamental form* π;

- *$R_{ikjl}[\bar{h}]$ is a component of the Riemannian curvature tensor on M and $R_{iNjN}[\bar{g}]$ is that of the Riemannian curvature tensor on $\overline{X}$;*
- *$R_{ij}[\bar{h}] = R_{ikjk}[\bar{h}]$ and $R_{NN}[\bar{g}] = R_{NiNi}[\bar{g}]$.*

Every tensor in the expansion is evaluated at $y = 0$ and commas denote partial differentiation.

Suppose that a term consists of exactly one copy of x_N but any number of x_i's, as in x_N, $x_i x_N$ and $x_i x_j x_N$. If $H = 0$ on M, the coefficient of this term in the expansion of $\sqrt{|\bar{g}|}$ is 0. In particular, condition (A.5) holds.

As explained in the introduction, in dealing with the fractional Yamabe problem, it is better to control only the boundary metric through conformal changes and then to work with special metrics on $\overline{X}$ described in Proposition 2.1. This is a distinguishing property compared to the boundary Yamabe problem. The following lemma is a reformulation of [46, Lemmas 2.4 and 3.2].

Lemma 2.3. *Let (X, g^+) be an asymptotically hyperbolic manifold such that (1.5) holds. Then the conformal infinity $(M, [\bar{h}])$ admits a representative $\tilde{h} \in [\bar{h}]$, a corresponding geodesic boundary defining function $\tilde{\rho}$ and the metric $\tilde{g} = \tilde{\rho}^2 g^+$ such that*

(1) $R_{ij}[\tilde{h}](y) = R_{ij;k}[\tilde{h}](y) + R_{jk;i}[\tilde{h}](y) + R_{ki;j}[\tilde{h}](y) = 0$;

(2) $H = 0$ *on M and* $R_{\tilde{\rho}\tilde{\rho}}[\tilde{g}](y) = \dfrac{1-2n}{2(n-1)}\|\pi(y)\|^2$,

for a fixed point $y \in M$. Here the semicolon designates covariant differentiation.

2.2. *Definition and properties of bubbles*

Suppose that $\gamma \in (0, 1)$ and $n > 2\gamma$. For arbitrary $\lambda > 0$ and $\sigma \in \mathbb{R}^n$, let $w_{\lambda,\sigma}$ be the *bubble* defined as

$$w_{\lambda,\sigma}(\bar{x}) = \alpha_{n,\gamma}\left(\frac{\lambda}{\lambda^2 + |\bar{x} - \sigma|^2}\right)^{\frac{n-2\gamma}{2}} \quad \text{for } \bar{x} \in \mathbb{R}^n, \quad \alpha_{n,\gamma} = 2^{\frac{n-2\gamma}{2}}\left(\frac{\Gamma\left(\frac{n+2\gamma}{2}\right)}{\Gamma\left(\frac{n-2\gamma}{2}\right)}\right)^{\frac{n-2\gamma}{4\gamma}}. \tag{2.5}$$

We also introduce the γ*-harmonic extension* $W_{\lambda,\sigma}$ of $w_{\lambda,\sigma}$, the unique solution of

$$\begin{cases} -\operatorname{div}(x_N^{1-2\gamma}\nabla W_{\lambda,\sigma}) = 0 & \text{in } \mathbb{R}^N_+, \\ W_{\lambda,\sigma} = w_{\lambda,\sigma} & \text{on } \mathbb{R}^n. \end{cases} \tag{2.6}$$

Then it is well-known that

$$\partial^\gamma_\nu W_{\lambda,\sigma} = -\kappa_\gamma \lim_{x_N\to 0+} x_N^{1-2\gamma}\partial_N W_{\lambda,\sigma} = (-\Delta)^\gamma w_{\lambda,\sigma} = w_{\lambda,\sigma}^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{on } \mathbb{R}^n \tag{2.7}$$

where κ_γ is the positive number appearing in (2.2) and ν is the outward unit normal vector to $\mathbb{R}^N_+$.

Lemma 2.4. (1) [Symmetry] *The value of* $W_{1,0}(\bar{x}, x_N)$ *for* $(\bar{x}, x_N) \in \mathbb{R}^N_+$ *is determined by* $|\bar{x}|$ *and* x_N. *In particular,* $\partial_i W_{1,0}(\bar{x}, x_N) = -\partial_i W_{1,0}(-\bar{x}, x_N)$ *for each* $1 \le i \le n$.

(2) [Decay] *There exists a constant* $C > 0$ *depending only on n,* γ *and* ℓ *such that*

$$|\nabla^\ell_{\bar{x}} W_{1,0}(x)| \le \frac{C}{1+|x|^{n-2\gamma+\ell}} \quad \text{and} \quad |x_N^{1-2\gamma}\partial_N W_{1,0}(x)| \le \frac{C}{1+|x|^n} \tag{2.8}$$

for all $x \in \mathbb{R}^N_+$ *and* $\ell \in \mathbb{N} \cup \{0\}$.

(3) [Classification] *Suppose that* $\Phi \in W^{1,2}_{\text{loc}}(\mathbb{R}^N_+; x_N^{1-2\gamma})$ *is a nontrivial solution of*

$$\begin{cases} -\operatorname{div}(x_N^{1-2\gamma}\nabla\Phi) = 0 & \text{in } \mathbb{R}^N_+, \\ \Phi \ge 0 & \text{in } \mathbb{R}^N_+, \\ \partial^\gamma_\nu \Phi = \Phi^p & \text{on } \mathbb{R}^n. \end{cases}$$

Then $p \ge 2^*_{n,\gamma} - 1$. *Moreover, if* $p = 2^*_{n,\gamma} - 1$, *then* $\Phi(x) = W_{\lambda,\sigma}(x)$ *for all* $x \in \mathbb{R}^N_+$ *and some* $(\lambda, \sigma) \in (0, \infty) \times \mathbb{R}^n$.

(4) [Nondegeneracy] *The solution space of the linear problem*

$$\begin{cases} -\operatorname{div}(x_N^{1-2\gamma}\nabla\Phi) = 0 & \text{in } \mathbb{R}^N_+, \\ \partial^\gamma_\nu \Phi = \frac{n+2\gamma}{n-2\gamma} w_{\lambda,\sigma}^{\frac{4\gamma}{n-2\gamma}} \Phi & \text{on } \mathbb{R}^n, \\ \|\Phi(\cdot, 0)\|_{L^\infty(\mathbb{R}^n)} < \infty \end{cases}$$

is spanned by

$$Z^1_{\lambda,\sigma} = \frac{\partial W_{\lambda,\sigma}}{\partial \sigma_1}, \ \ldots, \ Z^n_{\lambda,\sigma} = \frac{\partial W_{\lambda,\sigma}}{\partial \sigma_n} \quad \text{and} \quad Z^0_{\lambda,\sigma} = -\frac{\partial W_{\lambda,\sigma}}{\partial \lambda}. \tag{2.9}$$

Proof. Since $w_{1,0}(\bar{x})$ depends only on $|\bar{x}|$, claim (1) follows from the uniqueness of γ-extension. Moreover the sharp decay estimate [46, Section A] for $W_{1,0}$ gives (2). Assertions (3) and (4) are implied by the results of Jin et al. [40, Theorem 1.8, Remark 1.9] and Dávila et al. [17], respectively. □

2.3. *Modification of* (2.4)

Suppose that (X, g^+) is an asymptotically hyperbolic manifold with conformal infinity $(M, [\bar{h}])$. Consider sequences of parameters $\{p_m\}_{m\in\mathbb{N}} \subset [1+\varepsilon_0, 2^*_{n,\gamma} - 1]$ for any fixed $\varepsilon_0 > 0$, metrics $\{\bar{h}_m\}_{m\in\mathbb{N}} \subset [\bar{h}]$ on M, corresponding geodesic boundary defining functions $\{\rho_m\}_{m\in\mathbb{N}}$ and positive functions $\{f_m\}_{m\in\mathbb{N}}$ on M. Set $\bar{g}_m = \rho_m^2 g^+$ and $\delta_m = (2^*_{n,\gamma} - 1) - p_m \ge 0$. It is convenient to deal with the following form of the equation:

$$\begin{cases} -\operatorname{div}_{\bar{g}_m}(\rho_m^{1-2\gamma}\nabla U_m) + E_{\bar{g}_m}(\rho_m) U_m = 0 & \text{in } (X, \bar{g}_m), \\ U_m > 0 & \text{on } \overline{X}, \\ U_m = u_m & \text{on } M, \\ \partial^\gamma_\nu U_m = f_m^{-\delta_m} u_m^{p_m} & \text{on } M, \end{cases} \tag{2.10}$$

rather than (2.4).

We further assume that $p_m \to p_0$, $\bar{g}_m = \rho_m^2 g^+ \to \bar{g}_0$ in $C^4(\overline{X}, \mathbb{R}^{N\times N})$ for a metric $\bar{g}_0$ on $\overline{X}$ and $f_m \to f_0 > 0$ in $C^2(M)$ as $m \to \infty$. Then, in particular, the sequence $\{E_{\bar{g}_m}(\rho_m)\}_{m\in\mathbb{N}}$ is bounded in $C^2(\overline{X})$. This property will be needed when we apply Lemmas A.3 and A.4.

Suppose that $\tilde{h}_m = w_m^{4/(n-2\gamma)}\bar{h}_m$ on M for a positive function w_m on M such that

$$w_m(y_m) = 1, \qquad \frac{\partial w_m}{\partial x_i}(y_m) = 0 \quad \text{for each } i = 1, \dots, n. \tag{2.11}$$

If $\tilde{\rho}_m$ is the geodesic boundary defining function associated to $\tilde{h}_m$ and $\tilde{g}_m = \tilde{\rho}_m^2 g^+$, then $\tilde{g}_m = (\tilde{\rho}_m/\rho_m)^2 \bar{g}_m$ on $\overline{X}$. Furthermore, a direct computation using [12, Lemma 4.1] shows that $\widetilde{U}_m = (\rho_m/\tilde{\rho}_m)^{(n-2\gamma)/2} U_m$ solves

$$\begin{cases} -\mathrm{div}_{\tilde{g}_m}(\tilde{\rho}_m^{1-2\gamma}\nabla\widetilde{U}_m) + E_{\tilde{g}_m}(\tilde{\rho}_m)\widetilde{U}_m = 0 & \text{in } (X, \tilde{g}_m), \\ \widetilde{U}_m > 0 & \text{on } \overline{X}, \\ \widetilde{U}_m = \tilde{u}_m & \text{on } M, \\ \partial_\nu^\gamma \widetilde{U}_m = w_m^{-\frac{n+2\gamma}{n-2\gamma}} \partial_\nu^\gamma U_m = \tilde{f}_m^{-\delta_m} \tilde{u}_m^{p_m} & \text{on } M, \end{cases} \tag{2.12}$$

where $\tilde{f}_m = w_m f_m$, which is the same form as that of (2.10).

As a matter of fact, the resemblance of (2.10) and (2.12) is no accident. The differential operator in each equation can be realized as a weighted conformal Laplacian on a smooth metric measure space and its associated conformally covariant boundary operator. Their forms do not change under conformal changes. For more detailed accounts, refer to [11, 10].

2.4. Pohozaev's identity

Pick a small $r_1 \in (0, r_0)$ (see (2.1)) such that the $\bar{g}_m$-Fermi coordinates centered at $y \in M$ are well-defined in the closed geodesic half-ball $\overline{B_+^N(y, r_1)} \subset \overline{X}$ for every $m \in \mathbb{N}$ and $y \in M$.

In this subsection, we provide a local version of Pohozaev's identity for

$$\begin{cases} -\mathrm{div}(x_N^{1-2\gamma}\nabla U) = x_N^{1-2\gamma} Q & \text{in } B_+^N(0, r_1) \subset \mathbb{R}_+^N, \\ U = u > 0 & \text{on } B^n(0, r_1) \subset \mathbb{R}^n, \\ \partial_\nu^\gamma U = f^{-\delta} u^p & \text{on } B^n(0, r_1) \end{cases} \tag{2.13}$$

where $p \in [1, 2_{n,\gamma}^* - 1]$, $Q \in L^\infty(B_+^N(0, r_1))$ and $f \in C^1(B^n(0, r_1))$.

Lemma 2.5. *Let $U \in W^{1,2}(B_+^N(0, r_1); x_N^{1-2\gamma})$ be a solution to (2.13) such that U, $\partial_i U$ and $x_N^{1-2\gamma}\partial_N U$ are Hölder continuous on $\overline{B_+^N(0, r_1)}$. Given any $r \in (0, r_1)$, define*

$$\begin{aligned} \mathcal{P}(U, r) = \kappa_\gamma \int_{\partial_I B_+^N(0,r)} x_N^{1-2\gamma} \left[\frac{n-2\gamma}{2} u \frac{\partial u}{\partial \nu} - \frac{r}{2}|\nabla u|^2 + r\left|\frac{\partial u}{\partial \nu}\right|^2\right] d\sigma_x \\ + \frac{r}{p+1} \int_{\partial B^n(0,r)} f^{-\delta} u^{p+1}\, d\sigma_{\bar{x}} \end{aligned} \tag{2.14}$$

where κ_γ is the positive number in (2.2) and ν is the outward unit normal vector to $\partial_I B^N_+(0,r)$. Then

$$\mathcal{P}(U,r) = -\kappa_\gamma \int_{B^N_+(0,r)} x_N^{1-2\gamma} Q \cdot \left[x_i \partial_i U + x_N \partial_N U + \frac{n-2\gamma}{2} U \right] dx$$
$$- \frac{\delta}{p+1} \int_{B^n(0,r)} x_i \partial_i f \, f^{-(\delta+1)} u^{p+1} \, d\bar{x} + \left(\frac{n}{p+1} - \frac{n-2\gamma}{2} \right) \int_{B^n(0,r)} f^{-\delta} u^{p+1} \, d\bar{x}$$

for all $r \in (0, r_1)$.

Proof. The proof is similar to that of [40, Proposition 4.7]. □

3. Basic properties of blow-up points

3.1. Various types of blow-up points

We start by recalling the notion of blow-up, isolated blow-up and isolated simple blow-up. Our definition is a slight modification of the one introduced in [2, Section 4] (cf. [40, 43, 51]).

Definition 3.1. As before, let (X, g^+) be an asymptotically hyperbolic manifold with conformal infinity $(M, [\bar{h}])$. Here we use the notations of Subsection 2.3 and the small number $r_1 > 0$ picked in Subsection 2.4.

(1) $y_0 \in M$ is called a *blow-up point* of $\{U_m\}_{m\in\mathbb{N}} \subset W^{1,2}(X; \rho^{1-2\gamma})$ if there exists a sequence $\{y_m\}_{m\in\mathbb{N}} \subset M$ such that y_m is a local maximum point of $u_m = U_m|_M$ satisfying $u_m(y_m) \to \infty$ and $y_m \to y_0$ as $m \to \infty$. For simplicity, we will often say that $y_m \to y_0 \in M$ is a blow-up point of $\{U_m\}_{m\in\mathbb{N}}$.

(2) $y_0 \in M$ is an *isolated blow-up point* of $\{U_m\}_{m\in\mathbb{N}}$ if y_0 is a blow-up point such that

$$u_m(y) \le C d_{\bar{h}_m}(y, y_m)^{-\frac{2\gamma}{p_m - 1}} \quad \text{for any } y \in M \setminus \{y_m\} \text{ with } d_{\bar{h}_m}(y, y_m) < r_2 \tag{3.1}$$

for some $C > 0$, $r_2 \in (0, r_1]$ where $\bar{h}_m = \bar{g}_m|_{TM}$ and $d_{\bar{h}_m}$ is the distance function in the metric $\bar{h}_m$.

(3) Define a weighted spherical average of u_m by

$$\bar{u}_m(r) = r^{\frac{2\gamma}{p_m-1}} \frac{\int_{\partial B^n(y_m,r)} u_m \, d\sigma_{\bar{h}_m}}{\int_{\partial B^n(y_m,r)} d\sigma_{\bar{h}_m}}, \quad r \in (0, r_1). \tag{3.2}$$

We say that an isolated blow-up point y_0 of $\{U_m\}_{m\in\mathbb{N}}$ is *simple* if there exists r_3 in $(0, r_2]$ such that $\bar{u}_m$ has exactly one critical point in $(0, r_3)$ for large $m \in \mathbb{N}$.

Roughly speaking, item (2) (or (3), respectively) in the above definition describes the situation when *clustering of bubbles* (or *bubble towers*, respectively) is excluded among various blow-up scenarios.

Hereafter, we always assume that $\{(u_m, y_m)\}_{m\in\mathbb{N}}$ is a sequence of pairs in $C^\infty(M) \times M$ such that u_m is a solution to (1.1) with $c = 1$, and y_m is a local maximum point of u_m satisfying $u_m(y_m) \to \infty$ and $y_m \to y_0 \in M$ as $m \to \infty$. Then $y_m \to y_0 \in M$ becomes a blow-up point of a sequence $\{U_m\}_{m\in\mathbb{N}} \subset W^{1,2}(X; \rho^{1-2\gamma})$ where each U_m is a solution to (2.10) with $\bar{g}_m = \bar{g}$, $\bar{h}_m = \bar{h}$, $\rho_m = \rho$ and $f_m = 1$. Set $M_m = u_m(y_m)$ and $\epsilon_m = M_m^{-(p_m-1)/(2\gamma)}$ for each $m \in \mathbb{N}$. Obviously, $M_m \to \infty$ and $\epsilon_m \to 0$ as $m \to \infty$.

Also, we denote by $\tilde{h}_m$ a representative of the class $[\bar{h}_m]$ satisfying properties (1) and (2) in Lemma 2.3 with $y = y_m$, and by $\widetilde{U}_m$ a solution to (2.12). Inspection of the proof of Lemma 2.3 (found in [46]) shows that one can choose representatives $\{\tilde{h}_m\}_{m\in\mathbb{N}}$ in such a way that $\tilde{h}_m \to \tilde{h}_0$ in $C^4(M)$ as $m \to \infty$ for some metric $\tilde{h}_0$ on M. Then the C^4-smoothness of the metric $\bar{g}$ on $\overline{X}$ and regularity of noncharacteristic first-order partial differential equations imply that $\tilde{g}_m \to \tilde{g}_0$ in $C^4(M \times [0, r])$ for some metric $\tilde{g}_0$ and a small $r > 0$. Refer to Step 5 of the proof of Proposition 3.7 where the proof is essentially given. By extending $\tilde{g}_m$ in a suitable manner, we may assume that $\tilde{g}_m \to \tilde{g}_0$ in $C^4(\overline{X})$.

We shall often use $x \in \mathbb{R}^N_+$ to denote $\tilde{g}_m$-Fermi coordinates on $\overline{X}$ around y_m so that $\widetilde{U}_m$ can be regarded as a function on $\mathbb{R}^N_+$ near the origin.

3.2. Blow-up analysis

We study the asymptotic behavior of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.10) near blow-up points.

Proposition 3.2. *Assume that $p \in [1 + \varepsilon_0, 2^*_{n,\gamma} - 1]$. For any small $\varepsilon_1 > 0$ and large $R > 0$, there are constants C_0, $C_1 > 0$ depending only on (X, g^+), $\bar{h}$, n, γ, ε_0, ε_1 and R such that if $U \in W^{1,2}(X; \rho^{1-2\gamma})$ is a solution to (2.4) with $\max_M U \geq C_0$, then $(2^*_{n,\gamma} - 1) - p < \varepsilon_1$ and $U|_M$ has local maximum points $y_1, \ldots, y_{\mathcal{N}} \in M$ for some $1 \leq \mathcal{N} = \mathcal{N}(U) \in \mathbb{N}$, for which the following statements hold:*

(1) *Let $\hat{r}_m = R\,\alpha_{n,\gamma}^{(p-1)/(2\gamma)} u(y_m)^{-(p-1)/(2\gamma)}$ where $\alpha_{n,\gamma}$ is the positive number defined in (2.5). Then*

$$\overline{B_{\bar{h}}(y_{m_1}, \hat{r}_{m_1})} \cap \overline{B_{\bar{h}}(y_{m_2}, \hat{r}_{m_2})} = \emptyset \quad \text{for } 1 \leq m_1 \neq m_2 \leq \mathcal{N}.$$

(2) *For each $m = 1, \ldots, \mathcal{N}$ and some $\beta = \beta(N, \gamma) \in (0, 1)$,*

$$\left\| \alpha_{n,\gamma} U(y_m)^{-1} U\left(\alpha_{n,\gamma}^{\frac{p-1}{2\gamma}} U(y_m)^{-\frac{p-1}{2\gamma}} \cdot\right) - W_{1,0} \right\|_{C^\beta(\overline{B^N_+(0,2R)})} + \left\| \alpha_{n,\gamma} u(y_m)^{-1} u\left(\alpha_{n,\gamma}^{\frac{p-1}{2\gamma}} u(y_m)^{-\frac{p-1}{2\gamma}} \cdot\right) - w_{1,0} \right\|_{C^{2+\beta}(\overline{B^n(0,2R)})} \leq \varepsilon_1 \tag{3.3}$$

in $\bar{g}$-Fermi coordinates centered at y_m.

(3) *We have*

$$U(y) d_{\bar{h}}(y, \{y_1, \ldots, y_{\mathcal{N}}\})^{\frac{2\gamma}{p-1}} \leq C_1 \quad \text{for } y \in M.$$

Proof. The validity of this proposition comes from a Liouville-type theorem [40, Theorem 1.8] (see our Lemma 2.4(3)) and an induction argument. See [55, proof of Proposition 5.1] for a detailed account for the Yamabe problem. □

We have a remark on (3.3): According to Proposition A.8, only C^β-convergence is guaranteed on the closed half-ball $\overline{B_+^N(0,2R)}$. However, we have $C^{2+\beta}$-convergence on its bottom $\overline{B^n(0,2R)}$.

Lemma A.2 and the standard rescaling argument readily give the annular Harnack inequality around an isolated blow-up point.

Lemma 3.3. *Suppose that $y_m \to y_0 \in M$ is an isolated blow-up point of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.10). In the $\bar{g}_m$-Fermi coordinate system centered at y_m, there exists $C > 0$ independent of $m \in \mathbb{N}$ and $r > 0$ such that*

$$\max_{B_+^N(0,2r)\setminus B_+^N(0,r/2)} U_m \le C \min_{B_+^N(0,2r)\setminus B_+^N(0,r/2)} U_m$$

for any $r \in (0, r_2/3)$ where $r_2 > 0$ is defined in Definition 3.1(2).

Proof. The proof is similar to that of [40, Lemma 4.3]. □

If $y_m \to y_0 \in M$ is an isolated blow-up point of solutions $\{U_m\}_{m\in\mathbb{N}}$ to (2.10), Proposition 3.2 can be extended in the following manner.

Lemma 3.4. *Let $y_m \to y_0 \in M$ be an isolated blow-up point of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.10) with $f_m > 0$ in $B^n(0,r_2)$. In addition, suppose that $\{R_m\}_{m\in\mathbb{N}}$ and $\{\tau_m\}_{m\in\mathbb{N}}$ are arbitrary sequences of positive numbers such that $R_m \to \infty$ and $\tau_m \to 0$ as $m \to \infty$. Then $p_m \to 2_{n,\gamma}^* - 1$, and $\{U_\ell\}_{\ell\in\mathbb{N}}$ and $\{p_\ell\}_{\ell\in\mathbb{N}}$ have subsequences $\{U_{\ell_m}\}_{m\in\mathbb{N}}$ and $\{p_{\ell_m}\}_{m\in\mathbb{N}}$ such that for some $\beta \in (0,1)$,*

$$\left\| \hat{\epsilon}_{\ell_m}^{\frac{2\gamma}{p_{\ell_m}-1}} U_{\ell_m}(\hat{\epsilon}_{\ell_m}\cdot) - W_{1,0} \right\|_{C^\beta(\overline{B_+^N(0,R_m)})} + \left\| \hat{\epsilon}_{\ell_m}^{\frac{2\gamma}{p_{\ell_m}-1}} u_{\ell_m}(\hat{\epsilon}_{\ell_m}\cdot) - w_{1,0} \right\|_{C^{2+\beta}(\overline{B^n(0,R_m)})} \le \tau_m \tag{3.4}$$

in $\bar{g}_m$-Fermi coordinates centered at y_m and $R_m\hat{\epsilon}_{\ell_m} \to 0$ as $m \to \infty$. Here

$$\hat{\epsilon}_{\ell_m} = \alpha_{n,\gamma}^{\frac{p_{\ell_m}-1}{2\gamma}} M_{\ell_m}^{-\frac{p_{\ell_m}-1}{2\gamma}} = \alpha_{n,\gamma}^{\frac{p_{\ell_m}-1}{2\gamma}} u_{\ell_m}(y_{\ell_m})^{-\frac{p_{\ell_m}-1}{2\gamma}}$$

for all $m \in \mathbb{N}$.

In order to prove this, we first need the following analogue of the Hopf lemma.

Lemma 3.5. *Suppose that $\bar{g}$ is a smooth metric on $\overline{B_+^N(0,1)}$, $A \in C^0(\overline{B_+^N(0,1)})$ and $U \in W^{1,2}(B_+^N(0,1); x_N^{1-2\gamma})$ is a solution to*

$$\begin{cases} -\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla U) + x_N^{1-2\gamma} AU = 0 & \text{in } B_+^N(0,1), \\ U \ge c_0 > 0 & \text{on } \overline{B_+^N(0,1)}, \end{cases}$$

such that $x_N^{1-2\gamma}\partial_N U \in C^0(\overline{B_+^N(0,1)})$*. Assume also that there exist a small* $r>0$ *and* $\bar{x}_0 \in B^n(0,r)\setminus\overline{B^n(0,r/2)}$ *such that* $U(\bar{x}_0,0)=c_0$ *and* $U(\bar{x},0)>c_0$ *on* $\{\bar{x}\in\mathbb{R}^n : |\bar{x}|=r/2\}$*. Then*

$$\lim_{x_N\to 0} x_N^{1-2\gamma}\partial_N U(\bar{x}_0,x_N)>0.$$

Proof. Our proof is in the spirit of those in [30, Theorem 3.5] and [11, Proposition 7.1]. Let

$$W(x)=x_N^{-(1-2\gamma)}(x_N+\widetilde{C}_1x_N^2)(e^{-\widetilde{C}_2|\bar{x}|}-e^{-\widetilde{C}_2 r}) \quad \text{in } B_+^N(0,1)$$

with $\widetilde{C}_1,\widetilde{C}_2>0$ sufficiently large. Then there exists a small $\delta>0$ such that

$$\begin{cases} -\operatorname{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla W)+x_N^{1-2\gamma}AW\le 0 & \text{in } B_+^N(0,1),\\ U-\delta W\ge c_0/2>0 & \text{on } \overline{B_+^N(0,1)}.\end{cases}$$

Therefore

$$\begin{cases} -\operatorname{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla(U-\delta W))+x_N^{1-2\gamma}A_+(U-\delta W)\ge 0 & \text{in } \Gamma_1,\\ U-\delta W\ge c_0 & \text{on } \partial\Gamma_1,\end{cases}$$

where $\Gamma_1=(B^n(0,1)\setminus\overline{B^n(0,1/2)})\times(0,1/2)$. By the maximum principle, $U-\delta W>c_0$ in Γ_1. Since $(U-\delta W)(\bar{x}_0,0)=c_0$, the assertion follows. □

Proof of Lemma 3.4. We set

$$V_m(x)=\hat{\epsilon}_m^{\frac{2\gamma}{p_m-1}}U_m(\hat{\epsilon}_m x) \quad \text{for all } x\in B_+^N(0,r_2\hat{\epsilon}_m^{-1}).$$

Then we infer from (2.10) and (3.1) that

$$\begin{cases} -\operatorname{div}_{\bar{g}_m(\hat{\epsilon}_m\cdot)}(x_N^{1-2\gamma}\nabla V_m)+\hat{\epsilon}_m^2x_N^{1-2\gamma}A_m(\hat{\epsilon}_m\cdot)V_m=0 & \text{in } B_+^N(0,r_2\hat{\epsilon}_m^{-1}),\\ \partial_\nu^\gamma V_m=f_m^{-\delta_m}(\hat{\epsilon}_m\cdot)V_m^{p_m} & \text{on } B^n(0,r_2\hat{\epsilon}_m^{-1}),\end{cases} \tag{3.5}$$

where

$$A_m=x_N^{-(1-2\gamma)}E_{\bar{g}_m}(x_N^{1-2\gamma})$$

in $\bar{g}_m(\hat{\epsilon}_m\cdot)$-coordinates centered at y_m, and

$$V_m(\bar{x},0)\le C|\bar{x}|^{-\frac{2\gamma}{p_m-1}} \quad \text{for all } \bar{x}\in B^n(0,r_2\hat{\epsilon}_m^{-1}).$$

Thus Lemma 3.3 yields

$$V_m(x)\le C|x|^{-\frac{2\gamma}{p_m-1}} \quad \text{for all } x\in B_+^N(0,r_2\hat{\epsilon}_m^{-1}). \tag{3.6}$$

Also, by Definition 3.1(1),

$$V_m(0)=\alpha_{n,\gamma}, \quad \nabla_{\bar{x}}V_m(0)=0 \quad \text{for all } m\in\mathbb{N}. \tag{3.7}$$

On the other hand, Lemma 3.5 implies

$$\inf_{x\in\partial_I B^N_+(0,r)} V_m = \inf_{x\in B^N_+(0,r)} V_m \quad \text{for each } r\in(0,1]. \tag{3.8}$$

Indeed, since V_m is positive on its domain, we have

$$-\mathrm{div}_{\bar{g}_m(\hat{\epsilon}_m\cdot)}(x_N^{1-2\gamma}\nabla V_m)+\hat{\epsilon}_m^2 x_N^{1-2\gamma}(A_m)_+(\hat{\epsilon}_m\cdot)V_m\ge 0 \quad \text{in } B^N_+(0,1).$$

Because of the classical maximum principle, V_m does not attain its infimum in the interior of $B^N_+(0,r)$ for any $r\in(0,1]$, unless it is a constant function. However, it cannot be constant, because otherwise we get the absurd relation

$$0=\partial^\gamma_\nu V_m=f_m^{-\delta_m}(\hat{\epsilon}_m\cdot)V_m^{p_m}>0 \quad \text{on } B^n(0,r).$$

Moreover, the infimum of V_m is not achieved on the bottom $B^n(0,r)$, because the existence of a minimum point $\bar{x}_m\in B^n(0,r)$ of V_m and the Hopf lemma produce the contradictory relation

$$0>\partial^\gamma_\nu V_m(\bar{x}_m,0)=f_m^{-\delta_m}(\hat{\epsilon}_m\bar{x}_m)V_m^{p_m}(\bar{x}_m,0)>0.$$

Therefore (3.8) must be true.

Now one observes from (3.7), (3.8) and Lemma 3.3 that $V_m(x)\le C$ for $|x|\le 1$. In light of (3.6), this reads

$$V_m(x)\le C \quad \text{for } x\in B^N_+(0,r_2\hat{\epsilon}_m^{-1}) \tag{3.9}$$

where $C>0$ is a constant independent of $m\in\mathbb{N}$.

Accordingly, by making use of (3.5), (3.7), (3.9), (A.3), (A.4), (A.6), (A.17) and Lemma 2.4(3), we deduce the existence of $\beta\in(0,1)$ such that

$$p_m\to 2^*_{n,\gamma}-1,\qquad V_m\to W_{1,0}\ \text{ in } C^\beta_{\mathrm{loc}}(\mathbb{R}^N_+)\quad\text{and}\quad V_m(\cdot,0)\to w_{1,0}\text{ in } C^{2+\beta}_{\mathrm{loc}}(\mathbb{R}^n)$$

after passing to a subsequence. The assertion of the lemma is true. □

Keeping in mind that our proof is not affected by picking a subsequence of $\{U_\ell\}_{\ell\in\mathbb{N}}$, we always select $\{R_m\}_{m\in\mathbb{N}}$ first and then $\{U_{\ell_m}\}_{m\in\mathbb{N}}$ satisfying (3.4) and $R_m\hat{\epsilon}_{\ell_m}\to 0$. From now on, we write $\{U_m\}_{m\in\mathbb{N}}$ to denote $\{U_{\ell_m}\}_{m\in\mathbb{N}}$ to simplify notation.

The next result is a simple consequence of the previous lemma for $\tau_m=w_{1,0}(R_m)/2$.

Corollary 3.6. (1) *Suppose that $y_m\to y_0\in M$ is an isolated blow-up point of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.10). If $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$ is a sequence of solutions to (2.12) constructed as in Subsection 3.1, then $y_m\to y_0\in M$ is an isolated blow-up point of $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$.*

(2) *Assume that $y_0\in M$ is an isolated blow-up point of $\{U_m\}_{m\in\mathbb{N}}$. Then the function $\bar{u}_m$ defined in (3.2) has exactly one critical point in $(0,R_m\hat{\epsilon}_m)$ for large $m\in\mathbb{N}$. In particular, if the isolated blow-up point $y_0\in M$ of $\{U_m\}_{m\in\mathbb{N}}$ is also simple, then $\bar{u}_m'(r)<0$ for all $r\in[R_m\hat{\epsilon}_m,r_3)$; see Definition 3.1(3).*

Proof. Choose $l, L > 0$ such that $l \le w_m^{-1} = \tilde{u}_m/u_m \le L$ on M for all $m \in \mathbb{N}$. If we use the normal coordinates on M at y_m, it follows from (3.4) that

$$lM_m(w_{1,0}(\hat{\epsilon}_m^{-1}\cdot)-\tau_m) \le \alpha_{n,\gamma}\, \tilde{u}_m \le LM_m(w_{1,0}(\hat{\epsilon}_m^{-1}\cdot)+\tau_m) \quad \text{in } \overline{B^n(0, R_m\hat{\epsilon}_m)}. \tag{3.10}$$

Hence there exists a sufficiently large $R > 0$ independent of $m \in \mathbb{N}$ such that $R < R_m$ and $\tilde{u}_m(\cdot) \le \tilde{u}_m(0)/2$ on $\partial B^n(0, R\hat{\epsilon}_m)$ for each large $m \in \mathbb{N}$, from which we infer that $\tilde{u}_m$ has a local maximum point $\tilde{y}_m$ on M satisfying $d_{\tilde{h}_m}(y_m, \tilde{y}_m) = |\tilde{y}_m| \le R\hat{\epsilon}_m$. Now it is easy to check that $\tilde{y}_m \to y_0$ is a blow-up point of $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$. Furthermore, since (3.10) implies the existence of a constant $C > 0$ depending on $R > 0$ such that

$$\tilde{u}_m \le CM_m \le C(|y| + |\tilde{y}_m|)^{-\frac{2\gamma}{p_m-1}} \le C|y - \tilde{y}_m|^{-\frac{2\gamma}{p_m-1}} \quad \text{if } |y| \le R\hat{\epsilon}_m,$$

one finds

$$\tilde{u}_m(y) \le Cd_{\tilde{h}_m}(y, \tilde{y}_m)^{-\frac{2\gamma}{p_m-1}} \quad \text{for any } y \in M \setminus \{\tilde{y}_m\} \text{ with } d_{\tilde{h}_m}(y, \tilde{y}_m) < r_2$$

where the magnitude of $r_2 > 0$ may be reduced if necessary. As a result, an application of the proof of Lemma 3.4 to $\tilde{u}_m$ shows that for $R' \gg R$ large enough, $\tilde{u}_m$ is $C^2(\overline{B_{\tilde{h}_m}(\tilde{y}_m, R'\hat{\epsilon}_m)})$-close to a suitable rescaling of the standard bubble $w_{1,0}$ so that it has a unique critical point on $B_{\tilde{h}_m}(\tilde{y}_m, R'\hat{\epsilon}_m)$, the local maximum point $\tilde{y}_m$. However, by (2.11), $y_m \in B_{\tilde{h}_m}(\tilde{y}_m, R'\hat{\epsilon}_m)$ is already a critical point of $\tilde{u}_m$, and so it is equal to $\tilde{y}_m$. This completes the proof of (1). The verification of (2) is plain. □

3.3. *Isolated simple blow-up points*

Let $y_m \to y_0 \in M$ be an isolated simple blow-up point of $\{U_m\}_{m\in\mathbb{N}}$. By Corollary 3.6(1), $y_m \to y_0$ is an isolated blow-up point of $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$. The objective of this subsection is to show that the behavior of each $\widetilde{U}_m$ in the geodesic half-ball $B_{\bar{g}_m}(y_m, r) \cap \overline{X}$ can be controlled whenever $r > 0$ is chosen to be sufficiently small. We will use $\tilde{g}_m$-Fermi coordinates centered at y_m, so $B_{\tilde{g}_m}(y_m, r) \cap X$ is identified with $B_+^N(0, r) \subset \mathbb{R}_+^N$.

Proposition 3.7. *Assume $n > 2 + 2\gamma$ and $y_m \to y_0 \in M$ is an isolated simple blow-up point of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.10). Then one can choose $C > 0$ large and $r_4 \in (0, \min\{r_3, R_0\}]$ small (refer to Definition 3.1(3) and Proposition B.2), independent of $m \in \mathbb{N}$, such that*

$$\begin{cases} M_m|\nabla_{\bar{x}}^{\ell}\widetilde{U}_m(x)| \le C|x|^{-(n-2\gamma+\ell)} \ (\ell = 0, 1, 2), \\ M_m|x_N^{1-2\gamma}\partial_N\widetilde{U}_m(x)| \le C|x|^{-n} \end{cases} \quad \text{for } 0 < |x| \le r_4 \tag{3.11}$$

where $M_m = u_m(y_m)$ and $\widetilde{U}_m$ is the function constructed in Subsection 3.1.

Proof. The proof consists of six steps. Let $o(1)$ denote any sequence tending to 0 as $m \to \infty$.

Step 1 (Rough upper decay estimate of $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$). For any fixed sufficiently small $\eta > 0$, set $\lambda_m = (n-2\gamma-\eta)(p_m-1)/(2\gamma) - 1$. We shall show that there exist $r_4' = r_4'(\eta) \in (0, r_3]$ and a large constant $C > 0$ independent of $m \in \mathbb{N}$ such that

$$M_m^{\lambda_m} \widetilde{U}_m(x) \le C|x|^{-(n-2\gamma)+\eta} \quad \text{in } \Gamma_2 = B_+^N(0, r_4') \setminus B_+^N(0, R_m\hat{\epsilon}_m). \tag{3.12}$$

This can be proved as in [28, Lemma 2.7] and [40, Lemma 4.6]. However, since this is one of the places where hypothesis (1.5) is used, we sketch the proof.

Because $\widetilde{U}_m \le CU_m$ in $B_+^N(0, r_3)$ for some $C > 0$, it suffices to verify that

$$M_m^{\lambda_m} U_m(x) \le C|x|^{-(n-2\gamma)+\eta} \quad \text{in } \Gamma_2. \tag{3.13}$$

Thanks to Lemma 3.3 and Corollary 3.6(2), it turns out that

$$U_m^{p_m-1}(x) \le CR_m^{-2\gamma+o(1)}|x|^{-2\gamma} \quad \text{in } B_+^N(0, r_3) \setminus B_+^N(0, R_m\hat{\epsilon}_m). \tag{3.14}$$

Let

$$\begin{cases} \mathfrak{L}_m(U) = -\mathrm{div}_{\bar{g}_m}(x_N^{1-2\gamma}\nabla U) + E_{\bar{g}_m}(x_N)U & \text{in } B_+^N(0, r_3), \\ \mathfrak{B}_m(U) = \partial_\nu^\gamma U - f_m^{-\delta_m} u_m^{p_m-1} u & \text{on } B^n(0, r_3), \end{cases}$$

where $u = U$ on $B^n(0, r_3)$. By (2.10), we clearly have $U_m > 0$, $\mathfrak{L}_m(U_m) = 0$ in $B_+^N(0, r_3)$ and $\mathfrak{B}_m(U_m) = 0$ in $B^n(0, r_3)$.

Assume that $0 \le \mu \le n - 2\gamma$. Then one can calculate

$$\mathfrak{L}_m(|x|^{-\mu}) = x_N^{1-2\gamma}(\mu(n-2\gamma-\mu) + O(|x|))|x|^{-(\mu+2)}. \tag{3.15}$$

Moreover, [46, Lemma 2.3] tells us that (1.5) ensures $H = 0$ on M. Hence $\partial_N\sqrt{|\bar{g}_m|} = O(x_N)$ by Lemma 2.2 and

$$\begin{aligned} &\mathrm{div}_{\bar{g}_m}(x_N^{1-2\gamma}\nabla(x_N^{2\gamma}|x|^{-(\mu+2\gamma)})) - \mathrm{div}(x_N^{1-2\gamma}\nabla(x_N^{2\gamma}|x|^{-(\mu+2\gamma)})) \\ &= x_N^{1-2\gamma}[O(|x|)x_N^{2\gamma}\partial_{ij}|x|^{-(\mu+2\gamma)} + O(|x|)x_N^{2\gamma}\partial_i|x|^{-(\mu+2\gamma)} + O(x_N)(\partial_N x_N^{2\gamma})|x|^{-(\mu+2\gamma)}] \\ &= x_N^{1-2\gamma}[O(|x|)x_N^{2\gamma}|x|^{-(\mu+2\gamma+2)}] \end{aligned}$$

(cf. (5.2)). This implies that

$$\mathfrak{L}_m(x_N^{2\gamma}|x|^{-(\mu+2\gamma)}) = x_N^{1-2\gamma}((\mu+2\gamma)(n-\mu) + O(|x|))|x|^{-(\mu+2)}(x_N/|x|)^{2\gamma}. \tag{3.16}$$

From (3.15), (3.16) and the computation

$$\begin{aligned} \mathfrak{B}_m(|x|^{-\mu} - \zeta x_N^{2\gamma}|x|^{-(\mu+2\gamma)}) &= |x|^{-(\mu+2\gamma)}[2\gamma\kappa_\gamma\zeta + f_m^{-\delta_m}u_m^{p_m-1}(\zeta x_N^{2\gamma} - |x|^{2\gamma})] \\ &= |x|^{-(\mu+2\gamma)}[2\gamma\kappa_\gamma\zeta + O(R_m^{-2\gamma+o(1)})] \quad \text{(by (3.14))} \end{aligned}$$

in $B^N_+(0, r_3) \setminus B^N_+(0, R_m\hat{\epsilon}_m)$ for a fixed $\zeta \in \mathbb{R}$, we observe that the function

$$\Phi_{1m}(x) = L_m(|x|^{-\eta} - \zeta x_N^{2\gamma}|x|^{-(\eta+2\gamma)}) + L_0 M_m^{-\lambda_m}(|x|^{-(n-2\gamma)+\eta} - \zeta x_N^{2\gamma}|x|^{-n+\eta}),$$

with a suitable choice of $L_0,\ L_m > 0$ large and $\zeta,\ \eta > 0$ small, satisfies

$$\mathfrak{L}_m(\Phi_{1m}) \geq 0 \quad \text{in } \Gamma_2, \quad \mathfrak{B}_m(\Phi_{1m}) \geq 0 \quad \text{on } \partial\Gamma_2 \cap \mathbb{R}^n \quad \text{and} \quad U_m \leq \Phi_{1m} \quad \text{on } \partial\Gamma_2 \setminus \mathbb{R}^n.$$

Consequently, the generalized maximum principle (Lemma A.5) yields $U_m \leq \Phi_{1m}$ in Γ_2. From this and the assumption that $y_m \to y_0$ is an isolated simple blow-up point of $\{U_m\}_{m\in\mathbb{N}}$, we infer that (3.13) or (3.12) holds.

Step 2 (Lower decay estimate of $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$). We claim that there is a large constant $C > 0$ such that

$$M_m\widetilde{U}_m(x) \geq C^{-1}|x|^{-(n-2\gamma)} \quad \text{in } \Gamma_2 \tag{3.17}$$

where the magnitude of r_4' is reduced if necessary.

Let G_m be the Green's function that solves (B.1) provided $\bar{g} = \bar{g}_m$ and $B_R = B^N_+(0, r_4')$. By (2.10), (3.4) and (B.3), we find that $U = M_m U_m - C^{-1}G_m$ solves

$$\begin{cases} -\text{div}_{\bar{g}_m}(x_N^{1-2\gamma}\nabla U) + x_N^{1-2\gamma}A_m U = 0 & \text{in } \Gamma_2, \\ \partial^\gamma_\nu U = M_m f_m^{-\delta_m} u_m^{p_m} \geq 0 & \text{on } \partial\Gamma_2 \cap \mathbb{R}^n, \\ U \geq 0 & \text{on } \partial\Gamma_2 \setminus \mathbb{R}^n, \end{cases}$$

provided that $C > 0$ is large enough. Hence the weak maximum principle discussed in Remark A.6 shows that $U \geq 0$ in Γ_2. Inequality (3.17) now follows from the inequality $U_m \leq C\widetilde{U}_m$ in $B^N_+(0, r_4')$ and (B.3).

Step 3 (Rough upper decay estimate of derivatives of $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$). We assert

$$\begin{cases} M_m^{\lambda_m}|\nabla^\ell_{\bar{x}}\widetilde{U}_m(x)| \leq C(1+\hat{\epsilon}_m^{o(1)})|x|^{-(n-2\gamma+\ell)+\eta} \ (\ell = 1, 2), \\ M_m^{\lambda_m}|x_N^{1-2\gamma}\partial_N\widetilde{U}_m(x)| \leq C(1+\hat{\epsilon}_m^{o(1)})|x|^{-n+\eta} \end{cases} \quad \text{in } \Gamma_2. \tag{3.18}$$

We apply the standard rescaling argument described, e.g., in [29, proof of Lemma 2.6]. Given any $m \in \mathbb{N}$ and $R \in [2R_m\hat{\epsilon}_m, r_4'/2]$, set

$$\mathcal{U}_R(x) = M_m^{\lambda_m} R^{n-2\gamma-\eta}\widetilde{U}_m(Rx) \quad \text{in } \Gamma_3 = B^N_+(0,2) \setminus B^N_+(0,1/2),$$

which solves

$$\begin{cases} -\text{div}_{\bar{g}_m(R\cdot)}(x_N^{1-2\gamma}\nabla\mathcal{U}_R) + E_{\bar{g}_m(R\cdot)}(x_N)\mathcal{U}_R = 0 & \text{in } \Gamma_3, \\ \partial^\gamma_\nu\mathcal{U}_R = (\hat{\epsilon}_m/R)^{2\gamma\lambda_m}\tilde{f}_m^{-\delta_m}(R\,\cdot)\mathcal{U}_R^{p_m} & \text{on } \partial\Gamma_3' = \partial\Gamma_3 \cap \mathbb{R}^n. \end{cases} \tag{3.19}$$

Inequalities (3.18) will be valid if there exists $C > 0$ independent of m and R such that

$$\sum_{\ell=1}^{2}|\nabla^\ell_{\bar{x}}\mathcal{U}_R(x)| + |x_N^{1-2\gamma}\partial_N\mathcal{U}_R(x)| \leq C(1+\hat{\epsilon}_m^{o(1)}) \quad \text{for } |x| = 1. \tag{3.20}$$

Because of relatively poor regularity property of degenerate elliptic equations, especially when $\gamma \in (0, 1)$ is small, derivation of (3.20) is rather technical. In particular, as we will see shortly, it requires the lower estimate (3.17) of $M_m^{\lambda_m} \widetilde{U}_m$ in contrast to the local case $\gamma = 1$.

In light of (3.12), we have $\mathcal{U}_R(x) \le C$ in Γ_3. Applying the Hölder estimate (A.3), a bootstrap argument with the Schauder estimate (A.17), and the derivative estimate (A.6) for (3.19), we obtain

$$\|\nabla_{\bar{x}} \mathcal{U}_R\|_{C^0(K)} \le C[1 + (\hat{\epsilon}_m/R)^{2\gamma\lambda_m}] \le C \tag{3.21}$$

for any proper compact subset K of $\Gamma_3 \cup \partial\Gamma_3'$. Furthermore, (3.17) yields

$$(\hat{\epsilon}_m/R)^{2\gamma\lambda_m} \mathcal{U}_R^{p_m-2}(x) \le \begin{cases} C & \text{for } n < 6\gamma, \\ (\hat{\epsilon}_m/R)^{2\gamma\lambda_m - \eta(2-p_m)} \hat{\epsilon}_m^{o(1)} \le C\hat{\epsilon}_m^{o(1)} & \text{for } n \ge 6\gamma, \end{cases}$$

on $\partial\Gamma_3'$. This together with (A.17) and (3.21) gives

$$\begin{aligned} \|\mathcal{U}_R\|_{C^{\beta'}(\widetilde{K}')} &\le C[1 + (\hat{\epsilon}_m/R)^{2\gamma\lambda_m}(1 + \|\mathcal{U}_R\|_{C^{\beta}(\widetilde{K})} + \|\mathcal{U}_R^{p_m-2} \nabla_{\bar{x}} \mathcal{U}_R\|_{C^0(\widetilde{K})})] \\ &\le C(1 + \hat{\epsilon}_m^{o(1)}) \end{aligned} \tag{3.22}$$

for all compact subsets $\widetilde{K}' \subsetneq \widetilde{K}$ of Γ_3' and exponents $1 \le \beta < \beta' \le \min\{2, \beta + 2\gamma\}$. The desired inequality (3.20) is now derived from (3.21), (3.22), (A.6) with $\ell_0 = 2$ and (A.11).

Step 4 (Estimate of δ_m). For $\delta_m = (2^*_{n,\gamma} - 1) - p_m \ge 0$,

$$\delta_m = O\left(M_m^{-\frac{2}{n-2\gamma} + o(1)}\right) \quad \text{and} \quad M_m^{\delta_m} \to 1 \quad \text{as } m \to \infty. \tag{3.23}$$

The proof makes use of Pohozaev's identity in Lemma 2.5 and is analogous to that in [40, Lemma 4.8]. Hence we omit it.

Step 5 (Estimate of $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$ on $\{|x| = r_4\}$). We demonstrate

$$\max_{|x|=r_4} M_m \widetilde{U}_m(x) \le C(r_4) \tag{3.24}$$

for any sufficiently small $r_4 \in (0, r_4']$.

Suppose this does not hold. Then there exists a sequence $\{z_m\}_{m\in\mathbb{N}}$ of points on the half-sphere $\{x \in \mathbb{R}^N_+ : |x| = r_4\}$ such that

$$M_m U_m(z_m) \to \infty \quad \text{as } m \to \infty.$$

Let $E_{\bar{g}_m}(x_N) = x_N^{1-2\gamma} A_m$ where $\{A_m\}_{m\in\mathbb{N}}$ is a family of functions whose C^2-norm is uniformly bounded. We divide into cases according to the sign of A_m.

Case 1: $A_m \ge 0$ in $B^N_+(0, r_4')$ for all $m \in \mathbb{N}$. In this case, one can argue as in [56, Proposition 4.5] or [2, Proposition 4.3] to reach a contradiction. The proof is omitted.

Case 2: $A_m \geq 0$ in $B^N_+(0, r'_4)$ is violated for some $m \in \mathbb{N}$. In this situation, we will recover positivity of A_m by employing a conformal change of the metric $\bar{g}_m$ on M. Owing to (1.5) (or the condition $H = 0$ on M), (2.1) and Lemma 2.2, we have

$$x_N^{1-2\gamma} A_m = E_{\bar{g}_m}(x_N) \geq \frac{n-2\gamma}{2} x_N^{1-2\gamma} \left[1 - 2r'_4 \left\| \nabla \sqrt{|\bar{g}_m|} \right\|_{L^\infty(B^N_+(0,r'_4))}\right] \\ \times \left[\|\pi_m(0)\|^2 + R_{NN}[\bar{g}_m](0) - r'_4 \left\| \nabla \left(x_N^{-1} \partial_N \sqrt{|\bar{g}_m|}\right) \right\|_{L^\infty(B^N_+(0,r'_4))}\right]$$

in $B^N_+(0, r'_4)$, where π_m is the second fundamental form of $(M, \bar{h}_m) \subset (\overline{X}, \bar{g}_m)$. Also, inspecting the proof of [46, Lemma 2.4], we see that

$$\|\pi_m(0)\|^2 + R_{NN}[\bar{g}_m](0) = \frac{1}{2(n-1)} \left(R[\bar{h}_m](0) - \|\pi_m(0)\|^2\right).$$

We want to find a representative $\check{h}_m$ of the conformal class $[\bar{h}_m]$ and a small $r''_4 \in (0, r'_4]$ such that

$$E_{\check{g}_m}(x_N) = x_N^{1-2\gamma} \check{A}_m \geq 0 \quad \text{in } B^N_+(0, r''_4)$$

for all $m \in \mathbb{N}$, where $\check{g}_m$ is the metric on $\overline{X}$ defined via the geodesic boundary defining function associated to $\check{h}_m$. To this end, it suffices to confirm that given a fixed small number $\varepsilon > 0$,

$$R[\check{h}_m](0) - \|\check{\pi}_m(0)\|^2 \geq 1/\varepsilon \tag{3.25}$$

and

$$r''_4 \left\| \nabla \sqrt{|\check{g}_m|} \right\|_{L^\infty(B^N_+(0,r''_4))} \leq \varepsilon, \quad r''_4 \left\| \nabla \left(x_N^{-1} \partial_N \sqrt{|\check{g}_m|}\right) \right\|_{L^\infty(B^N_+(0,r''_4))} \leq \frac{1}{2\varepsilon}. \tag{3.26}$$

Here $\check{\pi}_m$ is the second fundamental form of $(M, \check{h}_m) \subset (\overline{X}, \check{g}_m)$.

Set $f_m(\bar{x}) = -K|\bar{x}|^2$ in $B^n(0, r'_4)$ for some large $K > 0$ and then extend it to M suitably so that $f_m \in C^\infty(M)$. If we let $\check{h}_m = e^{2f_m} \bar{h}_m$, then the transformation law of the scalar curvature and the umbilic tensor under a conformal change (see [23, (1.1)] and [46, (2.2)]) gives

$$\begin{aligned} &R[\check{h}_m](0) - \|\check{\pi}_m(0)\|^2 \\ &\quad = e^{-2f_m(0)} \left(R[\bar{h}_m] - \|\pi_m\|^2 - 2(n-1)\Delta_{\bar{x}} f_m - (n-1)(n-2)|\nabla_{\bar{x}} f_m|^2\right)(0) \\ &\quad \geq 4n(n-1)K - \sup_{m \in \mathbb{N}} \left(|R[\bar{h}_m]| + \|\pi_m\|^2\right)(0) \geq 2n(n-1)K > 0, \end{aligned}$$

which establishes (3.25).

Verifying (3.26) requires a little more work. We extend f_m on M to its collar neighborhood $M \times [0, r'_4)$ by solving a first-order partial differential equation

$$\langle df_m, d\rho_m \rangle_{\bar{g}_m} + \frac{\rho_m}{2} |df_m|^2_{\bar{g}_m} = 0 \quad \text{on } M \times [0, r'_4).$$

Since the equation is noncharacteristic, a solution exists and is unique provided r_4' small. Locally, it is written as

$$\frac{\partial f_m}{\partial x_N} + \frac{x_N}{2}\left[\bar{g}_m^{ij}\frac{\partial f_m}{\partial x_i}\frac{\partial f_m}{\partial x_j} + \left(\frac{\partial f_m}{\partial x_N}\right)^2\right] = 0 \quad \text{in } B_+^N(0, r_4'). \tag{3.27}$$

We easily see that $\check{g}_m = e^{2f_m}\bar{g}_m$ on $M \times [0, r_4')$. Hence, by the assumption $H = 0$ on M and (3.27), it is sufficient to find a small $r_4'' = r_4''(K) > 0$ such that

$$\|\nabla f_m\|_{L^\infty(B_+^N(0,r_4''))} \le \varepsilon \quad \text{and} \quad \|\nabla^2 f_m\|_{L^\infty(B_+^N(0,r_4''))} \le C/\varepsilon \quad \text{for all } m \in \mathbb{N}$$

so as to ensure the validity of (3.26). Given any $\bar{x} \in B^n(0, 2r_4'')$, the characteristic equation of (3.27) is the system of $2N + 1$ ordinary differential equations for the functions

$$\mathbf{p} = \mathbf{p}(s; \bar{x}) = (p_1, \dots, p_N)(s; \bar{x}), \qquad z = z(s; \bar{x}), \qquad \mathbf{x} = \mathbf{x}(s; \bar{x}) = (x_1, \dots, x_N)(s; \bar{x})$$

defined as

$$\begin{cases} \dot{\mathbf{p}} = -\left(\dfrac{x_N}{2}\partial_1 \bar{g}_m^{ij}(\mathbf{x})p_i p_j, \dots, \dfrac{x_N}{2}\partial_n \bar{g}_m^{ij}(\mathbf{x})p_i p_j, \frac{1}{2}(\bar{g}_m^{ij}(\mathbf{x})p_i p_j + p_N^2)\right), \\ \dot{z} = x_N \bar{g}_m^{ij}(\mathbf{x})p_i p_j + p_N(1 + x_N p_N), \\ \dot{\mathbf{x}} = (x_N \bar{g}_m^{1i}(\mathbf{x})p_i, \dots, x_N \bar{g}_m^{ni}(\mathbf{x})p_i, 1 + x_N p_N), \\ \mathbf{p}(0; \bar{x}) = (-2K\bar{x}, 0), \quad z(0; \bar{x}) = -K|\bar{x}|^2, \quad \mathbf{x}(0; \bar{x}) = (\bar{x}, 0). \end{cases}$$

Here the dot stands for differentiation with respect to s and the domain of the functions $(\mathbf{p}, z, \mathbf{x})(\cdot; \bar{x})$ is assumed to be $[0, 2r_4'')$. The asymptotic analysis of the system indicates

$$\|\nabla f_m\|_{L^\infty(B_+^N(0,r_4''))} \le \sup_{\bar{x}\in B^n(0,2r_4'')} \|\mathbf{p}(\cdot; \bar{x})\|_{L^\infty([0,2r_4''))} \le 5K^{-1}$$

and

$$\|\nabla^2 f_m\|_{L^\infty(B_+^N(0,r_4''))} \le 2 \sup_{\bar{x}\in B^n(0,2r_4'')} (\|\nabla_{\bar{x}}\mathbf{p}(\cdot; \bar{x})\|_{L^\infty([0,2r_4''))} + \|\dot{\mathbf{p}}(\cdot; \bar{x})\|_{L^\infty([0,2r_4''))}) \le 5K$$

for any fixed $r_4'' \in (0, K^{-2})$, thereby establishing the desired inequalities.

Now, with the fact that $x_N^{1-2\gamma}\check{A}_m \ge 0$ in $B_+^N(0, r_4'')$, we may consider the family $\{\check{U}_m\}_{m\in\mathbb{N}}$ of solutions to (2.10) in which the tildes are replaced with checks. Notice that since $H = 0$ is an intrinsic condition that comes from (1.5), the new metrics $\check{g}_m$ on $\overline{X}$ still satisfy necessary conditions for the regularity results in Appendices A and B.2. Hence our situation is reduced to Case 1 and we get the same contradiction.

Step 6 (Completion of the proof). Finally, reasoning as in [56, Proposition 4.5] or [2, Proposition 4.3] with estimates (3.23) and (3.24), we get the desired inequality $M_m\widetilde{U}_m(x) \le C|x|^{-(n-2\gamma)}$ in $B_+^N(0, r_4) \setminus \{0\}$. The other estimates in (3.11) are established as in Step 3. This finishes the proof. □

4. Linear theory and refined blow-up analysis

4.1. Linear theory

Let $\chi : [0, \infty) \to [0, 1]$ be a smooth function such that $\chi(t) = 1$ on $[0, 1]$ and 0 in $[2, \infty)$. Set also $\chi_\epsilon(t) = \chi(\epsilon t)$ for any $\epsilon > 0$.

Proposition 4.1. *Let $n > 2 + 2\gamma$, $\epsilon > 0$, and π be a symmetric 2-tensor (that is, an $n \times n$-matrix) whose trace $\mathrm{tr}(\pi)$ is 0. Also, suppose that $W_{1,0}$ and $w_{1,0}$ are the standard bubbles appearing in (2.5) and (2.6), respectively. Then there exists a solution $\Psi \in D^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})$ to the linear equation*

$$\begin{cases} -\mathrm{div}(x_N^{1-2\gamma}\nabla\Psi) = x_N^{1-2\gamma} \cdot 2\epsilon x_N \chi_\epsilon(|x|)\pi_{ij}\partial_{ij}W_{1,0} & \text{in } \mathbb{R}^N_+, \\ \partial_\nu^\gamma \Psi = \frac{n+2\gamma}{n-2\gamma} w_{1,0}^{\frac{4\gamma}{n-2\gamma}} \Psi & \text{on } \mathbb{R}^n, \end{cases} \tag{4.1}$$

such that

$$|\nabla_{\bar{x}}^\ell \Psi(x)| \le \frac{C\epsilon|\pi|_\infty}{1 + |x|^{n-2\gamma-1+\ell}}, \qquad |x_N^{1-2\gamma}\partial_N\Psi(x)| \le \frac{C\epsilon|\pi|_\infty}{1+|x|^{n-1}} \tag{4.2}$$

for any $x \in \mathbb{R}^N_+$, $\ell \in \mathbb{N} \cup \{0\}$ and some $C > 0$ independent of $\epsilon > 0$,

$$\Psi(0) = \frac{\partial\Psi}{\partial x_1}(0) = \cdots = \frac{\partial\Psi}{\partial x_n}(0) = 0 \tag{4.3}$$

and

$$\int_{\mathbb{R}^N_+} x_N^{1-2\gamma}\nabla\Psi \cdot \nabla W_{1,0}\, dx = \int_{\mathbb{R}^n} w_{1,0}^{\frac{n+2\gamma}{n-2\gamma}} \Psi \, d\bar{x} = 0. \tag{4.4}$$

Here $|\pi|_\infty = \max_{i,j=1,\dots,n} |\pi_{ij}|$.

Proof. Given a fixed $\epsilon > 0$, let $Q \in L^{\frac{2(n-2\gamma+2)}{n-2\gamma+4}}(\mathbb{R}^N_+; x_N^{1-2\gamma})$ be defined by

$$Q(x) = 2\epsilon x_N \chi_\epsilon(|x|)\pi_{ij}\partial_{ij}W_{1,0}(x) \quad \text{for } x = (\bar{x}, x_N) \in \mathbb{R}^N_+.$$

By the symmetry of the functions $Z^0_{1,0}, \dots, Z^n_{1,0}$ given in (2.9) and the assumption that $\mathrm{tr}(\pi) = 0$, we see that

$$\int_{\mathbb{R}^N_+} x_N^{1-2\gamma} Q Z^0_{1,0}\, dx = \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} Q Z^1_{1,0}\, dx = \cdots = \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} Q Z^n_{1,0}\, dx = 0.$$

Therefore, from the nondegeneracy result in Lemma 2.4(4) and the Fredholm alternative, we get a unique solution $\widetilde{\Psi} \in D^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})$ to (4.1) satisfying

$$\begin{aligned} \int_{\mathbb{R}^N_+} x_N^{1-2\gamma}\nabla\widetilde{\Psi} \cdot \nabla Z^0_{1,0}\, dx &= \int_{\mathbb{R}^N_+} x_N^{1-2\gamma}\nabla\widetilde{\Psi} \cdot \nabla Z^1_{1,0}\, dx = \cdots \\ &= \int_{\mathbb{R}^N_+} x_N^{1-2\gamma}\nabla\widetilde{\Psi} \cdot \nabla Z^n_{1,0}\, dx = 0. \end{aligned}$$

Furthermore, by repetitive applications of the maximum principle and the scaling method with the help of Lemmas A.3 and A.4, we can prove that $\widetilde{\Psi}$ satisfies (4.2). See [18, proof of Lemma 3.3] for the details. Multiplying the first equation and (2.6)–(2.7) by $W_{1,0}$ and $\widetilde{\Psi}$, respectively, also reveals that

$$\frac{n+2\gamma}{n-2\gamma}\int_{\mathbb{R}^n} w_{1,0}^{\frac{n+2\gamma}{n-2\gamma}}\widetilde{\Psi}\,d\bar{x} = \kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{1-2\gamma}\nabla\widetilde{\Psi}\cdot\nabla W_{1,0}\,dx = \int_{\mathbb{R}^n} w_{1,0}^{\frac{n+2\gamma}{n-2\gamma}}\widetilde{\Psi}\,d\bar{x}.$$

Hence (4.4) holds for the function $\widetilde{\Psi}$.

Now, if we set

$$\Psi = \widetilde{\Psi} - \frac{2\widetilde{\Psi}(0)}{\alpha_{n,\gamma}(n-2\gamma)}Z^0_{1,0} - \sum_{i=1}^n \frac{\partial_i\widetilde{\Psi}(0)}{\alpha_{n,\gamma}(n-2\gamma)}Z^i_{1,0},$$

then it can be easily shown from (2.8) that Ψ is the desired function that satisfies (4.1)–(4.4). This concludes the proof. □

4.2. *Refined blow-up analysis*

As before, let $y_m \to y_0 \in M$ be an isolated simple blow-up point of $\{U_m\}_{m\in\mathbb{N}}$. In view of Corollary 3.6(1) and (2.11), $y_m \to y_0$ is an isolated blow-up point of $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$ and $M_m = \widetilde{U}_m(y_m)$ ($= \widetilde{U}_m(0)$ if the $\tilde{g}_m$-Fermi coordinate system around y_m is used). Also, Proposition 3.7 ensures the validity of the pointwise estimate (3.11) for $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$ near y_0. The objective of this subsection is to refine it by analyzing the ϵ_m-order terms. Recall the functions $W_{1,0}$ and Ψ_m defined in (2.6) and constructed in Proposition 4.1 (where the tensor π_m is replaced by the second fundamental form $\tilde{\pi}_m(y_m)$ at y_m of $(M, \tilde{h}_m) \subset (\overline{X}, \tilde{g}_m)$), respectively.

Proposition 4.2. *Suppose that $n > 2+2\gamma$. Let $\epsilon_m = M_m^{-(p_m-1)/(2\gamma)}$, $\hat{\epsilon}_m = \alpha_{n,\gamma}^{(p_m-1)/(2\gamma)}\epsilon_m$,*

$$\widetilde{V}_m(x) = \hat{\epsilon}_m^{\frac{2\gamma}{p_m-1}}\widetilde{U}_m(\hat{\epsilon}_m x) \quad \text{in } B^N_+(0, r_4'\hat{\epsilon}_m^{-1}), \tag{4.5}$$

and $\alpha_{n,\gamma}$ and r_4 be the positive constants introduced in (2.5) and Proposition 3.7, respectively. Then one can find $C>0$ and $r_5 \in (0, r_4]$ independent of $m \in \mathbb{N}$ such that

$$|\nabla^\ell_{\bar{x}}\widetilde{V}_m - \nabla^\ell_{\bar{x}}(W_{1,0}+\Psi_m)|(x) \le \frac{C\epsilon_m^2}{1+|x|^{n-2\gamma-2+\ell}} \tag{4.6}$$

for $\ell = 0, 1, 2$ and

$$|x_N^{1-2\gamma}\partial_N\widetilde{V}_m - x_N^{1-2\gamma}\partial_N(W_{1,0}+\Psi_m)|(x) \le \frac{C\epsilon_m^2}{1+|x|^{n-2}} \tag{4.7}$$

for $|x| \le r_5\hat{\epsilon}_m^{-1}$.

Proof. Our main tool will be the maximum principle; compare the proof of [2, Proposition 6.1] for the boundary Yamabe problem which makes use of Green's representation formula. The proof is split into three steps.

Step 1 (An estimate of $\widetilde{V}_m - (W_{1,0} + \Psi_m)$). We assert that

$$|\widetilde{V}_m - (W_{1,0} + \Psi_m)| \le C \max\{\epsilon_m^2, \delta_m\} \quad \text{in } B_+^N(0, r_4\hat{\epsilon}_m^{-1}) \tag{4.8}$$

where $\delta_m = (2_{n,\gamma}^* - 1) - p_m$. Set

$$\Lambda_m = \max_{|x| \le r_4\hat{\epsilon}_m^{-1}} |\widetilde{V}_m - (W_{1,0} + \Psi_m)|(x) = |\widetilde{V}_m - (W_{1,0} + \Psi_m)|(\hat{x}_m).$$

If $|\hat{x}_m| \ge \eta r_4\hat{\epsilon}_m^{-1}$ for any fixed small $\eta \in (0, 1)$, we obtain an inequality

$$\Lambda_m \le C\hat{\epsilon}_m^{n-2\gamma} = o(\epsilon_m^2)$$

stronger than (4.8). Thus we may assume that $|\hat{x}_m| \le \eta r_4\hat{\epsilon}_m^{-1}$. Let

$$\Theta_m = \Lambda_m^{-1}[\widetilde{V}_m - (W_{1,0} + \Psi_m)]$$

and

$$\hat{\mathfrak{L}}_m(\Theta_m) = -\mathrm{div}_{\hat{g}_m}(x_N^{1-2\gamma}\nabla\Theta_m) + \widehat{E}_m(x_N)\Theta_m$$

in $B_+^N(0, r_4\hat{\epsilon}_m^{-1})$. Then

$$\begin{cases} \hat{\mathfrak{L}}_m(\Theta_m) = x_N^{1-2\gamma}\widehat{Q}_{1m} & \text{in } B_+^N(0, r_4\hat{\epsilon}_m^{-1}), \\ \partial_\nu^\gamma\Theta_m - B_m\Theta_m = \widehat{Q}_{2m} & \text{on } B^n(0, r_4\hat{\epsilon}_m^{-1}), \end{cases} \tag{4.9}$$

where $\hat{g}_m = \tilde{g}_m(\hat{\epsilon}_m\cdot)$,

$$\begin{aligned} \widehat{E}_m(x_N) &= \frac{n-2\gamma}{4n}\left[R[\hat{g}_m] - (n(n+1) + R[g^+](\hat{\epsilon}_m\cdot))x_N^{-2}\right]x_N^{1-2\gamma} \\ &= \frac{n-2\gamma}{4n}\hat{\epsilon}_m^2\left[R[\tilde{g}_m](\hat{\epsilon}_m\cdot) + o(1)\right]x_N^{1-2\gamma} \qquad \text{(by (1.5))}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \widehat{Q}_{1m} = \Lambda_m^{-1}\Bigg[&(\hat{g}_m^{ij} - \delta^{ij})\partial_{ij}(W_{1,0} + \Psi_m) + \frac{\partial_i\sqrt{|\hat{g}_m|}}{\sqrt{|\hat{g}_m|}}\hat{g}_m^{ij}\partial_j(W_{1,0} + \Psi_m) \\ &+ \frac{\partial_N\sqrt{|\hat{g}_m|}}{\sqrt{|\hat{g}_m|}}\partial_N(W_{1,0} + \Psi_m) + \partial_i\hat{g}_m^{ij}\partial_j(W_{1,0} + \Psi_m) \\ &- 2\hat{\epsilon}_m x_N\chi_m(|x|)(\pi_m)_{ij}\partial_{ij}W_{1,0} - \widehat{E}_m(x_N)(W_{1,0} + \Psi_m)\Bigg] \end{aligned}$$

$$\begin{aligned} \widehat{Q}_{2m} = \Lambda_m^{-1}\Bigg[&(\hat{f}_m^{-\delta_m} - 1)(w_{1,0} + \Psi_m)^{p_m} + (w_{1,0} + \Psi_m)^{\frac{n+2\gamma}{n-2\gamma}}\{(w_{1,0} + \Psi_m)^{-\delta_m} - 1\} \\ &+ \left\{(w_{1,0} + \Psi_m)^{\frac{n+2\gamma}{n-2\gamma}} - w_{1,0}^{\frac{n+2\gamma}{n-2\gamma}} - \frac{n+2\gamma}{n-2\gamma}w_{1,0}^{\frac{4\gamma}{n-2\gamma}}\Psi_m\right\}\Bigg], \end{aligned}$$

$$B_m = \hat{f}_m^{-\delta_m}\frac{\widetilde{V}_m^{p_m} - (w_{1,0} + \Psi_m)^{p_m}}{\widetilde{V}_m - (w_{1,0} + \Psi_m)}$$

and $\hat{f}_m = f_m(\hat{\epsilon}_m\cdot)$.

As a preliminary step, we first deduce pointwise estimates of the functions $\widehat{Q}_{1m}$, $\widehat{Q}_{2m}$ and B_m. By (3.9), (A.6), (3.11), (2.8) and (4.2),

$$|\nabla_{\bar{x}}^{\ell}\widetilde{V}_m(x)| + |\nabla_{\bar{x}}^{\ell}W_{1,0}(x)| \le \frac{C}{1+|x|^{n-2\gamma+\ell}} \quad \text{and} \quad |\nabla_{\bar{x}}^{\ell}\Psi_m(x)| \le \frac{C\hat{\epsilon}_m}{1+|x|^{n-2\gamma-1+\ell}} \tag{4.11}$$

in $B_+^N(0, r_4\hat{\epsilon}_m^{-1})$ for $\ell = 0, 1$. Moreover, by (3.4) and (3.11),

$$\widetilde{V}_m(\bar{x}) \ge \frac{C}{1+|\bar{x}|^{n-2\gamma}} \quad \text{in } B^n(0, r_4\hat{\epsilon}_m^{-1}). \tag{4.12}$$

From these inequalities and Lemmas 2.2–2.4 (especially, $H = 0$ on M), we discover

$$|\widehat{Q}_{1m}(x)| \le \frac{C\Lambda_m^{-1}\epsilon_m^2}{1+|x|^{n-2\gamma}} \quad \text{in } B_+^N(0, r_4\hat{\epsilon}_m^{-1}), \tag{4.13}$$

$$|\widehat{Q}_{2m}(\bar{x})| \le C\Lambda_m^{-1}\left[\frac{\delta_m \log(1+|\bar{x}|)}{1+|\bar{x}|^{n+2\gamma+o(1)}} + \frac{\epsilon_m^2}{1+|\bar{x}|^{n-2+2\gamma}}\right] \quad \text{on } B^n(0, r_4\hat{\epsilon}_m^{-1}), \tag{4.14}$$

$$|B_m(\bar{x})| \le C[\widetilde{V}_m^{p_m-1} + (w_{1,0}+\Psi_m)^{p_m-1}] \le \frac{C}{1+|\bar{x}|^{4\gamma+o(1)}} \quad \text{on } B^n(0, r_4\hat{\epsilon}_m^{-1}). \tag{4.15}$$

Reducing r_4 if necessary, we have $w_{1,0}(\bar{x}) \ge 2|\Psi_m(\bar{x})|$ on $B^n(0, r_4\hat{\epsilon}_m^{-1})$. Using this fact, (4.11), (4.12) and the inequality

$$|x^p - px + (p-1)| \le \begin{cases} C(1-x)^2 \min\{1, x^{p-2}\} & \text{for } p \in (1,2), \\ C(1-x)^2(1+x^{p-2}) & \text{for } p \ge 2, \end{cases}$$

in $(0, \infty)$, we also deduce that

$$|\nabla_{\bar{x}} B_m(\bar{x})| \le C\left[\frac{\delta_m}{1+|\bar{x}|^{4\gamma+o(1)}} + \frac{1}{1+|\bar{x}|^{1+4\gamma+o(1)}}\right], \tag{4.16}$$

$$|\nabla_{\bar{x}}\widehat{Q}_{2m}(\bar{x})| \le C\Lambda_m^{-1}\left[\frac{\delta_m\epsilon_m}{1+|\bar{x}|^{n+2\gamma+o(1)}} + \frac{\delta_m \log(1+|\bar{x}|)}{1+|\bar{x}|^{n+1+2\gamma+o(1)}} + \frac{\epsilon_m^2}{1+|\bar{x}|^{n-1+2\gamma}}\right], \tag{4.17}$$

on $B^n(0, r_4\hat{\epsilon}_m^{-1})$.

Next, we claim that there is a number $\eta \in (0,1)$ such that

$$|\Theta_m(x)| \le C\left[\frac{1}{1+|x|^{\gamma}} + \frac{\Lambda_m^{-1}(\epsilon_m^2+\delta_m)}{1+|x|^{n-2-2\gamma}}\right] \quad \text{in } B_+^N(0, \eta r_4\hat{\epsilon}_m^{-1}). \tag{4.18}$$

To verify it, we construct a barrier function

$$\Phi_{2m}(x) = \begin{cases} L(1+\Lambda_m^{-1}(\epsilon_m^2+\delta_m))[(2-|x|^2) - \zeta x_N^{2\gamma}(2-|x|^{2-2\gamma})] & \text{for } |x| \le 1, \\ L[(|x|^{-\gamma} - \zeta x_N^{2\gamma}|x|^{-3\gamma}) + \Lambda_m^{-1}(\epsilon_m^2+\delta_m)(|x|^{-(n-2-2\gamma)} - \zeta x_N^{2\gamma}|x|^{-(n-2)})] & \\ \qquad \text{for } 1 < |x| < \eta r_4\hat{\epsilon}_m^{-1}, \end{cases}$$

with $L > 0$ large and $\zeta > 0$ small. Indeed, we see from $H = 0$ on M, (4.9) and (4.11)–(4.15) that

$$\begin{cases} \hat{\mathfrak{L}}_m(\Phi_{2m}) \ge \dfrac{C x_N^{1-2\gamma} \Lambda_m^{-1} \epsilon_m^2}{1+|x|^{n-2\gamma}} \ge \pm x_N^{1-2\gamma} \widehat{Q}_{1m} = \hat{\mathfrak{L}}_m(\pm\Theta_m) & \text{in } B_+^N(0, \eta r_4 \hat{\epsilon}_m^{-1}), \\ \partial_\nu^\gamma \Phi_{2m} \ge C\left[\dfrac{1}{1+|\bar{x}|^{3\gamma}} + \dfrac{\Lambda_m^{-1}(\epsilon_m^2+\delta_m)}{1+|\bar{x}|^{n-2}}\right] \ge |B_m| + |\widehat{Q}_{2m}| & \text{on } B^n(0, \eta r_4 \hat{\epsilon}_m^{-1}), \\ \qquad \ge \pm \partial_\nu^\gamma \Theta_m & \\ \Phi_{2m} \ge C\Lambda_m^{-1} \epsilon_m^{n-2\gamma} \ge \pm\Theta_m & \text{on } \partial_I B_+^N(0, \eta r_4 \hat{\epsilon}_m^{-1}), \end{cases} \tag{4.19}$$

for sufficiently small $\eta \in (0, 1)$. Thus, rescaling (4.19) and employing the weak maximum principle in Remark A.6, we establish that $|\Theta_m| \le \Phi_{2m}$ in $B_+^N(0, \eta r_4 \hat{\epsilon}_m^{-1})$. This implies (4.18).

Suppose now that $\Lambda_m^{-1}(\epsilon_m^2 + \delta_m) \to 0$ as $m \to \infty$. By Lemmas 3.4, A.2, A.7 and (4.13)–(4.17), there exist a function Θ_0 and a number $\beta \in (0, 1)$ such that

$$\Theta_m \to \Theta_0 \quad \text{in } C^\beta_{\text{loc}}(\overline{\mathbb{R}^N_+}) \cap C^1_{\text{loc}}(\mathbb{R}^n) \text{ and weakly in } W^{1,2}_{\text{loc}}(\mathbb{R}^N_+; x_N^{1-2\gamma}) \tag{4.20}$$

along a subsequence, and so

$$\begin{cases} -\operatorname{div}(x_N^{1-2\gamma} \nabla\Theta_0) = 0 & \text{in } \mathbb{R}^N_+, \\ |\Theta_0| \le \dfrac{C}{1+|x|^\gamma} & \text{in } \mathbb{R}^N_+, \\ \partial_\nu^\gamma \Theta_0 = \frac{n+2\gamma}{n-2\gamma} w_{1,0}^{\frac{4\gamma}{n-2\gamma}} \Theta_0 & \text{on } \mathbb{R}^n. \end{cases}$$

Consequently, from the fact that $\widetilde{V}_m(0) = \nabla_{\bar{x}} \widetilde{V}_m(0) = 0$, Lemma 2.4(1), (4.3) and (4.20), we see that

$$\Theta_0(0) = \frac{\partial\Theta_0}{\partial x_1}(0) = \cdots = \frac{\partial\Theta_0}{\partial x_n}(0) = 0.$$

In view of Lemma 2.4(4), $\Theta_0 = 0$ in $\mathbb{R}^N_+$. It follows from (4.20) that $|\hat{x}_m| \to \infty$ as $m \to \infty$. However, the uniform estimate (4.18) on Θ_m then implies $1 = \Theta_m(\hat{x}_m) \to 0$, so we get a contradiction. Estimate (4.8) must be true.

Step 2 (Estimate of δ_m). We assert

$$\delta_m \le C\epsilon_m^2. \tag{4.21}$$

Its proof can be done as in [2, Lemma 6.2] with minor modifications, so is omitted.

As a particular consequence of (4.8) and (4.21), we get

$$|\widetilde{V}_m - (W_{1,0} + \Psi_m)|(x) \le C\epsilon_m^2 \quad \text{for all } x \in B_+^N(0, r_4 \hat{\epsilon}_m^{-1}). \tag{4.22}$$

Step 3 (Completion of the proof). We can now deduce (4.6) with $\ell = 0$. Redefine

$$\Theta_m = \epsilon_m^{-2}[\widetilde{V}_m - (W_{1,0} + \Psi_m)] \quad \text{for } x \in B_+^N(0, r_4 \hat{\epsilon}_m^{-1})$$

so that it solves equation (4.9) once each quantity Λ_m in the definition of $\widehat{Q}_{1m}$ and $\widehat{Q}_{2m}$ is replaced with ϵ_m^2. As in (4.13) and (4.14),

$$|\widehat{Q}_{1m}(x)| \le \frac{C}{1+|x|^{n-2\gamma}} \quad \text{and} \quad |\widehat{Q}_{2m}(\bar{x})| \le \frac{C}{1+|\bar{x}|^{n+2\gamma-2}} \tag{4.23}$$

for $x \in B_+^N(0, r_4\hat{\epsilon}_m^{-1})$ and $\bar{x} \in B^n(0, r_4\hat{\epsilon}_m^{-1})$. By (4.22) and (4.11),

$$|\Theta_m| \le C \quad \text{in } B_+^N(0, r_4\hat{\epsilon}_m^{-1}) \quad \text{and} \quad |\Theta_m| \le C\epsilon_m^{n-2\gamma-2} \quad \text{on } \partial_I B_+^N(0, r_4\hat{\epsilon}_m^{-1}). \tag{4.24}$$

For any $0 < \mu \le n-2\gamma-2$, we define

$$\Phi_{3m;\mu}(x) = \begin{cases} L_\mu[(2-|x|^2) - \zeta_\mu x_N^{2\gamma}(2-|x|^{2-2\gamma})] & \text{for } |x| \le 1, \\ L_\mu(|x|^{-\mu} - \zeta_\mu x_N^{2\gamma}|x|^{-\mu-2\gamma}) & \text{for } 1 < |x| < r_4\hat{\epsilon}_m^{-1}, \end{cases}$$

where L_μ and ζ_μ are a large and a small positive number respectively depending only on μ, n and γ. If we set $\mu_0 = \min\{\gamma, n-2\gamma-2\}$, a direct computation using $H = 0$ on M, (4.9), (4.15), (4.23) and (4.24) shows

$$\begin{cases} \hat{\mathfrak{L}}_m(\Phi_{3m;\mu_0}) \ge \dfrac{Cx_N^{1-2\gamma}}{1+|x|^{n-2\gamma}} \ge \hat{\mathfrak{L}}_m(\pm\Theta_m) & \text{in } B_+^N(0, r_5'\hat{\epsilon}_m^{-1}), \\ \partial_\nu^\gamma \Phi_{3m;\mu_0} \ge \pm\partial_\nu^\gamma \Theta_m & \text{on } B^n(0, r_5'\hat{\epsilon}_m^{-1}), \\ \Phi_{3m;\mu_0} \ge \pm\Theta_m & \text{on } \partial_I B_+^N(0, r_5'\hat{\epsilon}_m^{-1}), \end{cases} \tag{4.25}$$

for $r_5' \in (0, r_4]$ small enough. Hence we deduce from the weak maximum principle in Remark A.6 that

$$|\Theta_m| \le \Phi_{3m;\mu_0} \le \frac{C}{1+|x|^{\mu_0}} \quad \text{in } B_+^N(0, r_5'\hat{\epsilon}_m^{-1}). \tag{4.26}$$

If $\mu_0 = n-2\gamma-2$, we are done. Otherwise, we put (4.26) into the second inequality of (4.25) in order to improve it so that

$$|\Theta_m| \le \Phi_{3m;\mu_1} \le \frac{C}{1+|x|^{\mu_1}} \quad \text{in } B_+^N(0, r_5''\hat{\epsilon}_m^{-1})$$

for $\mu_1 = \min\{2\gamma, n-2\gamma-2\}$ and $r_5'' \in (0, r_5']$. Iterating this process, we can conclude the proof of (4.6) for $\ell = 0$.

The remaining inequalities, i.e., (4.6) for $\ell = 1, 2$ and (4.7), are derived as in the justification of (3.18). Indeed, a tedious but straightforward calculation shows that the second-order derivatives of the functions $B_m(\bar{x})$ and $\widehat{Q}_{2m}(\bar{x})$ have the required decay rate as $|\bar{x}| \to \infty$. The proof is now complete. □

5. Vanishing theorem for the second fundamental form

Let us denote by $(\tilde{g}_0, \tilde{h}_0)$ the C^4-limit of the sequence $\{(\tilde{g}_m, \tilde{h}_m)\}_{m\in\mathbb{N}}$ given in Subsection 3.1. For each $m \in \mathbb{N}$, let $\tilde{\pi}_m$ and $\tilde{\pi}_0$ be the second fundamental forms of $(M, \tilde{h}_m) \subset (\overline{X}, \tilde{g}_m)$ and $(M, \tilde{h}_0) \subset (\overline{X}, \tilde{g}_0)$, respectively. By employing the sharp pointwise estimate of Proposition 4.2, we now prove that $\tilde{\pi}_0 = 0$ at an isolated simple blow-up point $y_0 \in M$ of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.10).

Proposition 5.1. *Suppose that $\gamma \in (0,1)$, $n \in \mathbb{N}$ satisfies the dimension restriction (1.2) and $y_m \to y_0$ is an isolated simple blow-up point of the sequence $\{U_m\}_{m\in\mathbb{N}}$ so that the description for $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$ in the first paragraph of Subsection 4.2 holds. Then*

$$\|\tilde{\pi}_m(y_m)\| \to 0 \quad \text{as } m \to \infty. \tag{5.1}$$

In particular, $\tilde{\pi}_0(y_0) = 0$.

Proof. We will use Lemma 2.3 with $\tilde{g} = \tilde{g}_m$, $\tilde{h} = \tilde{h}_m$ and $y = y_m$, and think as if $\widetilde{U}_m$ is a function in $\mathbb{R}^N_+$ near the origin by applying $\tilde{g}_m$-Fermi coordinates on $\overline{X}$ around y_m.

Denoting $\hat{g}_m = \tilde{g}_m(\hat{\epsilon}_m\,\cdot)$ and $\hat{f}_m = \tilde{f}_m(\hat{\epsilon}_m\,\cdot)$, we set

$$\begin{aligned} Q_{0m}(U)Q_m(U) &= (\hat{g}_m^{ij} - \delta^{ij})\partial_{ij}U + \left[\frac{\partial_i\sqrt{|\hat{g}_m|}}{\sqrt{|\hat{g}_m|}}\hat{g}_m^{ij}\partial_j U + \frac{\partial_N\sqrt{|\hat{g}_m|}}{\sqrt{|\hat{g}_m|}}\partial_N U\right] + \partial_i\hat{g}_m^{ij}\partial_j U \\ &= Q_{1m}(U) + Q_{2m}(U) + Q_{3m}(U). \end{aligned} \tag{5.2}$$

Also, let $\widehat{E}_m$ be the functions introduced in (4.10) so that $\widetilde{V}_m$ in (4.5) is a solution of

$$\begin{cases} -\text{div}(x_N^{1-2\gamma}\nabla\widetilde{V}_m) + \widehat{E}_m(x_N)\widetilde{V}_m = x_N^{1-2\gamma}Q_m(\widetilde{V}_m) & \text{in } B^N_+(0, r_5\hat{\epsilon}_m^{-1}), \\ \partial^\gamma_\nu\widetilde{V}_m = \hat{f}_m^{-\delta_m}\widetilde{V}_m^{p_m} & \text{on } B^n(0, r_5\hat{\epsilon}_m^{-1}). \end{cases}$$

Thus in view of Pohozaev's identity in Lemma 2.5, one can write

$$\mathcal{P}(\widetilde{V}_m, r\hat{\epsilon}_m^{-1}) = \mathcal{P}_{1m}(\widetilde{V}_m, r\hat{\epsilon}_m^{-1}) + \frac{\delta_m}{p_m+1}\mathcal{P}_{2m}(\widetilde{V}_m, r\hat{\epsilon}_m^{-1}) \quad \text{for any } r \in (0, r_5]$$

where

$$\begin{aligned} \mathcal{P}_{1m}(U,r) &= \kappa_\gamma\int_{B^N_+(0,r)} x_N^{1-2\gamma}[Q_{0m}(U) - Q_m(U)] \\ &\qquad\times\left[x_i\partial_i U + x_N\partial_N U + \frac{n-2\gamma}{2}U\right]dx, \\ \mathcal{P}_{2m}(U,r) &= -\int_{B^n(0,r)} x_i\partial_i\hat{f}_m\hat{f}_m^{-(\delta_m+1)}u^{p_m+1}\,d\bar{x} + \frac{n-2\gamma}{2}\int_{B^n(0,r)}\hat{f}_m^{-\delta_m}u^{p_m+1}\,d\bar{x} \end{aligned} \tag{5.3}$$

for $u = U$ on $B^n(0,r)$.

For a fixed $r \in (0, r_5]$, let

$$\widehat{F}_m(V_1, V_2) = \kappa_\gamma \int_{B^N_+(0, r\hat{\epsilon}_m^{-1})} x_N^{1-2\gamma} [Q_{0m}(V_1) - Q_m(V_1)] \times \left[x_i \partial_i V_2 + x_N \partial_N V_2 + \frac{n-2\gamma}{2} V_2 \right] dx.$$

Owing to (4.6) and (4.7), we are led to

$$\begin{aligned} &\mathcal{P}_{1m}(\widetilde{V}_m, r\hat{\epsilon}_m^{-1}) \\ &= \widehat{F}_m(W_{1,0}, W_{1,0}) + \widehat{F}_m(W_{1,0}, \Psi_m) + \widehat{F}_m(\Psi_m, W_{1,0}) + \widehat{F}_m(\Psi_m, \Psi_m) + o(\epsilon_m^2) \end{aligned} \tag{5.4}$$

provided that $n > 2 + 2\gamma$.

We first estimate $\widehat{F}_m(W_{1,0}, W_{1,0})$. This amounts to calculating the integrals

$$\widetilde{F}_{\ell m} = \kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} Q_{\ell m}(W_{1,0})\, Z^0_{1,0}\, dx \quad \text{for } \ell = 0, 1, 2, 3$$

where $Z^0_{1,0} = x \cdot \nabla W_{1,0} + ((n-2\gamma)/2) W_{1,0}$ is the function described in (2.9). For the value $\widetilde{F}_{1m}$, we discover from Lemmas 2.4(1), 2.2 and 2.3 that

$$\begin{aligned} \widetilde{F}_{1m} &= \kappa_\gamma \Bigg[2\hat{\epsilon}_m (\tilde{\pi}_m)_{ij} \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} \partial_{ij} W_{1,0} Z^0_{1,0}\, dx \\ &\quad + \frac{1}{3}\hat{\epsilon}_m^2 R_{ikjl}[\tilde{h}_m] \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} x_k x_l \partial_{ij} W_{1,0} Z^0_{1,0}\, dx \\ &\quad + \hat{\epsilon}_m^2 (\tilde{g}_m)^{ij}{}_{,Nk} \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} x_k \partial_{ij} W_{1,0} Z^0_{1,0}\, dx \\ &\quad + \hat{\epsilon}_m^2 (3(\tilde{\pi}_m)_{ik}(\tilde{\pi}_m)_{kj} + R_{iNjN}[\tilde{g}_m]) \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} \partial_{ij} W_{1,0} Z^0_{1,0}\, dx + o(\epsilon_m^2) \Bigg] \\ &= \kappa_\gamma [0+0+0+\hat{\epsilon}_m^2 (3\|\tilde{\pi}_m\|^2 + R_{NN}[\tilde{g}_m]) \cdot \frac{1}{n} \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} \Delta_{\bar{x}} W_{1,0} Z_{1,0}\, dx + o(\epsilon_m^2)] \\ &\qquad \text{(since } \widetilde{H}_m = R[\tilde{h}_m] = 0) \\ &= \hat{\epsilon}_m^2 \|\tilde{\pi}_m\|^2 \kappa_\gamma \frac{4n-5}{2n(n-1)} \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} \Delta_{\bar{x}} W_{1,0} Z^0_{1,0}\, dx + o(\epsilon_m^2) \end{aligned} \tag{5.5}$$

where $\widetilde{H}_m = \mathrm{tr}(\tilde{\pi}_m)/n$. Similarly,

$$\widetilde{F}_{2m} = \hat{\epsilon}_m^2 \|\tilde{\pi}_m\|^2 \kappa_\gamma \frac{1}{2(n-1)} \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} \partial_N W_{1,0} Z^0_{1,0}\, dx + o(\epsilon_m^2), \tag{5.6}$$

$$\widetilde{F}_{3m} = o(\epsilon_m^2). \tag{5.7}$$

Moreover, the Gauss–Codazzi equation and Lemma 2.3 yield

$$R[\tilde{g}_m] = 2R_{NN}[\tilde{g}_m] + \|\tilde{\pi}_m\|^2 + R[\tilde{h}_m] - \widetilde{H}_m^2 = -\frac{n}{n-1}\|\tilde{\pi}_m\|^2 \quad \text{at } y_m \in M.$$

Hence we find

$$\widetilde{F}_{0m} = -\hat{\epsilon}_m^2 \|\tilde{\pi}_m\|^2 \kappa_\gamma \frac{n-2\gamma}{4(n-1)} \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} W_{1,0} Z_{1,0}^0 \, dx + o(\epsilon_m^2). \tag{5.8}$$

Consequently, by combining (5.5)–(5.8) and employing (C.1), we obtain

$$\widehat{F}_m(W_{1,0}, W_{1,0}) = \hat{\epsilon}_m^2 \|\tilde{\pi}_m\|^2 \kappa_\gamma \frac{3n^2 + n(16\gamma^2 - 22) + 20(1-\gamma^2)}{8n(n-1)(1-\gamma^2)} \mathcal{C}_0 + o(\epsilon_m^2) \tag{5.9}$$

for some constant $\mathcal{C}_0 > 0$ depending only on n and γ. We note that the coefficient of ϵ_m^2 (or $\hat{\epsilon}_m^2$) is positive if and only if n satisfies (1.2) for each $\gamma \in (0, 1)$.

On the other hand,

$$\widehat{F}_m(W_{1,0}, \Psi_m) + \widehat{F}_m(\Psi_m, W_{1,0}) \geq o(\epsilon_m^2), \tag{5.10}$$

whose verification is deferred to the end of the proof. Also, a direct computation using (4.2) and Lemma 2.2 yields

$$\widehat{F}_m(\Psi_m, \Psi_m) = o(\epsilon_m^2). \tag{5.11}$$

By plugging (5.9)–(5.11) into (5.4), we arrive at

$$\mathcal{P}_{1m}(\widetilde{V}_m, r\hat{\epsilon}_m^{-1}) \geq \hat{\epsilon}_m^2 \|\tilde{\pi}_m\|^2 \kappa_\gamma \frac{3n^2 + n(16\gamma^2 - 22) + 20(1-\gamma^2)}{8n(n-1)(1-\gamma^2)} \mathcal{C}_0 + o(\epsilon_m^2).$$

Using (3.11), (3.23) and (4.5), we deduce

$$\begin{cases} |\nabla_{\bar{x}}^{\ell} \widetilde{V}_m(x)| \leq C|x|^{-(n-2\gamma+\ell)} \ (\ell = 0, 1, 2), \\ |x_N^{1-2\gamma} \partial_N \widetilde{V}_m(x)| \leq C|x|^{-n} \end{cases} \quad \text{in } B_+^N(0, r_4\hat{\epsilon}_m^{-1}).$$

Thus, it follows from (5.3) that

$$|\mathcal{P}(\widetilde{V}_m, r\hat{\epsilon}_m^{-1})| \leq C\epsilon_m^{n-2\gamma} \quad \text{and} \quad \mathcal{P}_{2m}(\widetilde{V}_m, r\hat{\epsilon}_m^{-1}) \geq 0$$

if $r > 0$ is selected to be small enough. As a result, estimate (5.1) follows.

Derivation of (5.10). Since $\tilde{\pi}_m \to \tilde{\pi}_0$ in $C^1(M)$, the norm $|\tilde{\pi}_m|_\infty$ (see the statement of Proposition 4.1) is uniformly bounded in $m \in \mathbb{N}$. Thus, by virtue of (5.2), (2.8), (4.2), Lemma 2.2 and integration by parts, we observe

$$\begin{aligned}
&\widehat{F}_m(W_{1,0}, \Psi_m) + \widehat{F}_m(\Psi_m, W_{1,0}) \\
&= -2\hat{\epsilon}_m (\tilde{\pi}_m)_{ij} \kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} \Big[\partial_{ij} W_{1,0} \Big\{ x_k \partial_k \Psi_m + x_N \partial_N \Psi_m + \frac{n-2\gamma}{2} \Psi_m \Big\} \\
&\qquad + \partial_{ij} \Psi_m \Big\{ x_k \partial_k W_{1,0} + x_N \partial_N W_{1,0} + \frac{n-2\gamma}{2} W_{1,0} \Big\} \Big] dx + o(\epsilon_m^2) \\
&= 2\hat{\epsilon}_m (\tilde{\pi}_m)_{ij} \kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} \big[(n - 2\gamma + 2) \partial_i W_{1,0} \partial_j \Psi_m + \partial_i W_{1,0} (x_k \partial_{jk} \Psi_m + x_N \partial_{jN} \Psi_m) \\
&\qquad + (x_k \partial_{ik} W_{1,0} + x_N \partial_{iN} W_{1,0}) \partial_j \Psi_m \big] dx + o(\epsilon_m^2) \\
&= -2\hat{\epsilon}_m (\tilde{\pi}_m)_{ij} \kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} \partial_i W_{1,0} \partial_j \Psi_m \, dx + o(\epsilon_m^2)
\end{aligned}$$

provided that $n > 2 + 2\gamma$. On the other hand, applying another integration by parts and inserting Ψ_m in (4.1) lead to

$$-2\hat{\epsilon}_m(\tilde{\pi}_m)_{ij}\kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{2-2\gamma}\partial_i W_{1,0}\partial_j \Psi_m\,dx + o(\epsilon_m^2)$$
$$= \underbrace{\kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{1-2\gamma}|\nabla\Psi_m|^2\,dx - \frac{n+2\gamma}{n-2\gamma}\int_{\mathbb{R}^n} w_{1,0}^{\frac{4\gamma}{n-2\gamma}}\Psi_m^2\,d\bar{x}}_{=\mathcal{I}} + o(\epsilon_m^2).$$

It is well-known that the Morse index of $w_{1,0} \in H^\gamma(\mathbb{R}^n)$ is 1 due to the contribution of $w_{1,0}$ itself. Hence we see from (4.4) that $\mathcal{I} \geq 0$; see [18, proof of Lemma 4.5] for more explanation. This completes the proof. □

6. Proof of the main theorems

6.1. Exclusion of bubble accumulation

Set

$$\mathcal{P}'(U,r) = \kappa_\gamma \int_{\partial_I B^N_+(0,r)} x_N^{1-2\gamma}\left[\frac{n-2\gamma}{2}u\frac{\partial u}{\partial r} - \frac{r}{2}|\nabla u|^2 + r\left|\frac{\partial u}{\partial r}\right|^2\right] d\sigma_x, \tag{6.1}$$

which is a part of the function $\mathcal{P}$ defined in (2.14).

Lemma 6.1. *Assume that $\gamma \in (0,1)$ and the dimension condition (1.2) holds. Let $y_m \to y_0$ be an isolated simple blow-up point of $\{U_m\}_{m\in\mathbb{N}}$, and $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$ be the sequence of functions constructed in Subsection 3.1. Suppose further that $\tilde{\pi}_0(y_0) \neq 0$. Then, given $m \in \mathbb{N}$ large and $r > 0$ small, there exist universal constants $\mathcal{C}_1, \ldots, \mathcal{C}_4 > 0$ such that*

$$\hat{\epsilon}_m^{n-2\gamma}\mathcal{P}'(\widetilde{U}_m(0)\widetilde{U}_m, r) \geq \hat{\epsilon}_m^2\mathcal{C}_1 - \hat{\epsilon}_m^{2+\eta}r^{2-\eta}\mathcal{C}_2 - \hat{\epsilon}_m^{n-2\gamma}r^{-n+2\gamma+1}\mathcal{C}_3 - \frac{\hat{\epsilon}_m^n r^n\mathcal{C}_4}{\hat{\epsilon}_m^{2n+o(1)} + r^{2n+o(1)}} \tag{6.2}$$

in $\tilde{g}_m$-Fermi coordinates centered at y_m. Here, $\eta > 0$ is an arbitrarily small number.

Proof. We have

$$\mathcal{P}'(\widetilde{U}_m(0)\widetilde{U}_m, r) = \hat{\epsilon}_m^{-(n-2\gamma)+o(1)}\left[\mathcal{P}(\widetilde{V}_m, r\hat{\epsilon}_m^{-1}) - \frac{r\hat{\epsilon}_m^{-1}}{p+1}\int_{\partial B^n(0,r\hat{\epsilon}_m^{-1})} \hat{f}_m^{-\delta_m}\widetilde{V}_m^{p_m+1}\,d\sigma_{\bar{x}}\right]$$

where $\widetilde{V}_m$ is the function defined in (4.5) and $\hat{\epsilon}_m^{o(1)} \to 1$ as $m \to \infty$. Inspecting the proof of Proposition 5.1 and using the assumption that $\tilde{\pi}_0(y_0) \neq 0$, we obtain

$$\mathcal{P}(\widetilde{V}_m, r\hat{\epsilon}_m^{-1}) \geq \hat{\epsilon}_m^2\mathcal{C}_1 - \hat{\epsilon}_m^{2+\eta}r^{2-\eta}\mathcal{C}_2 - \hat{\epsilon}_m^{n-2\gamma}r^{-n+2\gamma+1}\mathcal{C}_3$$

for some $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 > 0$ and small $\eta > 0$. Also, by (4.11),

$$\left|r\hat{\epsilon}_m^{-1}\int_{\partial B^n(0,r\hat{\epsilon}_m^{-1})} \hat{f}_m^{-\delta_m}\widetilde{V}_m^{p_m+1}\,d\sigma_{\bar{x}}\right| \leq \frac{\mathcal{C}_4\hat{\epsilon}_m^n r^n}{\hat{\epsilon}_m^{2n+o(1)} + r^{2n+o(1)}}$$

for some $\mathcal{C}_4 > 0$. Therefore (6.2) holds. □

We shall use the following Liouville-type lemma to prove Lemma 6.3.

Lemma 6.2. *If $U \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N_+; x_N^{1-2\gamma})$ is a solution to*

$$\begin{cases} -\mathrm{div}(x_N^{1-2\gamma}\nabla U) = 0 & \text{in } \mathbb{R}^N_+, \\ \partial^\gamma_\nu U = 0 & \text{on } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} U(x) \geq 0, \end{cases} \tag{6.3}$$

then it is a nonnegative constant.

Proof. According to the Hölder estimate (A.3) and the asymptotic condition in (6.3), U is in $C^\beta_{\mathrm{loc}}(\overline{\mathbb{R}^N_+})$ and bounded from below. Let $m_0 = \inf_{\mathbb{R}^N_+} U$ and $U_{m_0} = U - m_0 \geq 0$. By the Harnack inequality (A.4) and scaling invariance of (6.3), we have

$$\sup_{B^N_+(0,r)} U_{m_0} \leq C \inf_{B^N_+(0,r)} U_{m_0} \quad \text{for any } r > 0 \tag{6.4}$$

where $C > 0$ is independent of r. Letting $r \to \infty$ in (6.4), we find that $U_{m_0} = 0$ or $U = m_0$ in $\mathbb{R}^N_+$. One more application of the asymptotic condition on U forces $m_0 \geq 0$. □

Lemma 6.3. *Assume that $\gamma \in (0, 1)$ and the dimension condition (1.2) holds. Let $y_m \to y_0 \in M$ be an isolated blow-up point of the sequence $\{U_m\}_{m\in\mathbb{N}}$ and $\pi_0(y_0) \neq 0$. Then y_0 is an isolated simple blow-up point of $\{U_m\}_{m\in\mathbb{N}}$.*

Proof. Thanks to Corollary 3.6(2), the weighted average $\bar{u}_m$ of $u_m = U_m|_M$ (see (3.2) for its precise definition) has exactly one critical point in $(0, R_m\hat{\epsilon}_m)$ for large $m \in \mathbb{N}$. Suppose to the contrary that there exists another critical point ϱ_m of $\bar{u}_m$ such that $R_m\hat{\epsilon}_m \leq \varrho_m \to 0$ as $m \to \infty$. Define

$$\widetilde{T}_m(x) = \varrho_m^{\frac{2\gamma}{p_m-1}} \widetilde{U}_m(\varrho_m x) \quad \text{in } B^N_+(0, \varrho_m^{-1} r_5).$$

Then, using Propositions 3.7 and B.4, Lemma 6.2 and (3.17), one can verify the existence of $c_1 > 0$ and $\beta \in (0, 1)$ such that

$$\widetilde{T}_m(0)\,\widetilde{T}_m \to c_1(|x|^{-(n-2\gamma)} + 1) \quad \text{in } C^\beta_{\mathrm{loc}}(\overline{\mathbb{R}^N_+} \setminus \{0\}) \cap C^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) \tag{6.5}$$

up to a subsequence; see [2, Proposition 8.1]. It follows from (6.2) that

$$\begin{aligned} &\hat{\epsilon}_m^{-(n-2\gamma)}(\hat{\epsilon}_m^2\mathcal{C}_1 - \hat{\epsilon}_m^{2+\eta}\varrho_m^{2-\eta}\mathcal{C}_2 - \hat{\epsilon}_m^{n-2\gamma}\varrho_m^{-n+2\gamma+1}\mathcal{C}_3) - \frac{\hat{\epsilon}_m^{2\gamma}\varrho_m^n\mathcal{C}_4}{\hat{\epsilon}_m^{2n+o(1)} + \varrho_m^{2n+o(1)}} \\ &\qquad \leq \mathcal{P}'(\widetilde{U}_m(0)\widetilde{U}_m, \varrho_m) = \varrho_m^{-(n-2\gamma)+o(1)}\mathcal{P}'(\widetilde{T}_m(0)\widetilde{T}_m, 1). \end{aligned} \tag{6.6}$$

Note that

$$\varrho_m^{n-2\gamma+o(1)} \cdot \hat{\epsilon}_m^{-(n-2\gamma)}(\hat{\epsilon}_m^2\mathcal{C}_1 - \hat{\epsilon}_m^{2+\eta}\varrho_m^{2-\eta}\mathcal{C}_2 - \hat{\epsilon}_m^{n-2\gamma}\varrho_m^{-n+2\gamma+1}\mathcal{C}_3) \geq -2\varrho_m\mathcal{C}_3 \to 0 \tag{6.7}$$

and

$$\varrho_m^{n-2\gamma+o(1)} \cdot \frac{\hat{\epsilon}_m^{2\gamma}\varrho_m^n}{\hat{\epsilon}_m^{2n+o(1)}+\varrho_m^{2n+o(1)}} \le C\left(\frac{\hat{\epsilon}_m}{\varrho_m}\right)^{2\gamma+o(1)} \le \frac{C}{R_m^{2\gamma+o(1)}} \to 0 \tag{6.8}$$

as $m \to \infty$. Hence, taking the limit on (6.6) and employing (6.7), (6.8), (6.5) and (6.1), we obtain

$$\begin{aligned} 0 \le \lim_{m\to\infty} \mathcal{P}'(\widetilde{T}_m(0)\widetilde{T}_m, 1) &= \mathcal{P}'(c_1(|x|^{-(n-2\gamma)}+1), 1) \\ &= -\kappa_\gamma c_1^2 \frac{n-2\gamma}{2} \int_{\partial_I B_+^N(0,1)} x_N^{1-2\gamma}\, d\sigma_x < 0, \end{aligned}$$

which is a contradiction. The assertion in the statement must be true. □

We rule out bubble accumulation by applying Lemma 6.3.

Proposition 6.4. *Assume the hypotheses of Theorem 1.1. Let $\varepsilon_0, \varepsilon_1, R, C_0, C_1$ be positive numbers in the statement of Proposition 3.2. Suppose that $U \in W^{1,2}(X;\rho^{1-2\gamma})$ is a solution to (2.10) and $\{y_1,\dots,y_{\mathcal{N}}\}$ is the set of its local maximum points on M. Then there exists a constant $C_2 > 0$ depending only on (X, g^+), $\bar{h}$, n, γ, ε_0, ε_1 and R such that if $\max_M U \ge C_0$, then $d_{\bar{h}}(y_{m_1}, y_{m_2}) \ge C_2$ for all $1 \le m_1 \ne m_2 \le \mathcal{N}(U)$.*

Proof. By applying Propositions 3.2, 3.7 and B.4, Lemmas 6.1 and 6.3, the maximum principle and the Hopf lemma (Lemma 3.5), one can argue as in [2, Proposition 8.2] or [40, Proposition 5.2]. □

By Proposition 6.4, $\sup_{m\in\mathbb{N}} \mathcal{N}(U_m)$ is bounded. Therefore we have

Corollary 6.5. *Assume the hypotheses of Theorem 1.1. Then the set of blow-up points of $\{U_m\}_{m\in\mathbb{N}}$ is finite and it consists of isolated simple blow-up points.*

6.2. *Proofs of the main theorems*

We are now ready to complete the proofs of the main theorems. All notations used in the proofs are borrowed from Subsection 3.1.

Proof of Theorem 1.3. According to Corollary 6.5, any blow-up point $y_m \to y_0 \in M$ of $\{U_m\}_{m\in\mathbb{N}}$ is isolated simple. Therefore Proposition 5.1 implies the validity of Theorem 1.3. □

Proof of Theorem 1.1. We first claim that $u \le C$ on M. Indeed, if this does not hold, then by Proposition 2.1, there is a sequence $\{U_m\}_{m\in\mathbb{N}} \subset W^{1,2}(X;\rho^{1-2\gamma})$ of solutions to (2.4) which blows up at a point $y_0 \in M$. By applying Theorem 1.3, we conclude that $\pi(y_0) = 0$. However, this contradicts the assumption that π never vanishes on M.

A combination of (1.4), Lemma A.2 and Proposition A.8 now yields the other estimates in (1.6), that is, the lower and $C^{2+\beta}$-estimates of u on M. □

At this stage, only Theorem 1.4 remains to be verified. To define the Leray–Schauder degree $\deg(\mathcal{F}_p, D_\Lambda, 0)$ for all $1 \le p \le 2^*_{n,\gamma} - 1$ and apply its homotopy invariance, we need the following result.

Lemma 6.6. *Assume the hypotheses of Theorem 1.1. Then one can choose a constant* $C = C(X^{n+1}, g^+, \bar{h}, \gamma) > 1$ *such that*

$$C^{-1} \le u \le C \quad \text{on } M$$

for all $1 \le p \le 2^*_{n,\gamma} - 1$ *and* $u > 0$ *satisfying* (1.7).

Proof. We consider the extension problem (2.4) where the fourth line is replaced by $\partial^\gamma_\nu U = \mathcal{E}(u)u^p$ on M. By adapting the proofs of [28, Lemmas 4.1 and 6.5] and applying Theorem 1.1, we get the result. □

Proof of Theorem 1.4. From the previous lemma, we find that $0 \notin \mathcal{F}_p(\partial D_\Lambda)$ for all $1 \le p \le 2^*_{n,\gamma} - 1$ provided $\Lambda > 0$ large enough. Therefore

$$\deg(\mathcal{F}_p, D_\Lambda, 0) = \deg(\mathcal{F}_1, D_\Lambda, 0) \quad \text{for all } 1 \le p \le 2^*_{n,\gamma} - 1.$$

Since $\Lambda^\gamma(M, [\bar{h}]) > 0$, the first $L^2(M)$-eigenvalue of $P^\gamma_{\bar{h}}$ must be positive, for

$$\begin{aligned}\int_M u P^\gamma_{\bar{h}} u \, dv_{\bar{h}} &\ge \Lambda^\gamma(M, [\bar{h}]) \|u\|^2_{L^{\frac{2n}{n-2\gamma}}(M)} \\ &\ge \Lambda^\gamma(M, [\bar{h}]) |M|^{-2\gamma/n} \|u\|^2_{L^2(M)}, \qquad u \in H^\gamma(M).\end{aligned}$$

Also, in [30, Theorem 4.2], it was proved that the first eigenspace of $P^\gamma_{\bar{h}}$ is one-dimensional and spanned by a positive function on M. By $L^2(M)$-orthogonality, the other eigenfunctions must change their signs. Using these characterizations, one can follow the argument in [69], up to minor modifications, to derive $\deg(\mathcal{F}_1, D_\Lambda, 0) = -1$. The proof of Theorem 1.4 is complete. □

Appendix A. Elliptic regularity

For a fixed point $x_0 \in \mathbb{R}^n \simeq \partial\mathbb{R}^N_+$ and $R > 0$, let

$$B_R = B^N_+((x_0, 0), R) \subset \mathbb{R}^N_+, \quad \partial B'_R = B^n(x_0, R) \subset \mathbb{R}^n, \quad \partial B''_R = \partial_I B^N_+((x_0, 0), R), \tag{A.1}$$

so that $\partial B_R = \partial B'_R \cup \partial B''_R$. Suppose also that $\gamma \in (0, 1)$ and

- (g1) $\bar{g}$ is a smooth metric on $\overline{B_R}$ such that $\bar{g}_{iN} = 0$, $\bar{g}_{NN} = 1$ and $\lambda_{\bar{g}}|\xi|^2 \le \bar{g}_{ij}(x)\xi_i\xi_j \le \Lambda_{\bar{g}}|\xi|^2$ on $\overline{B_R}$ for some positive numbers $\lambda_{\bar{g}} \le \Lambda_{\bar{g}}$ and all vectors $\xi \in \mathbb{R}^n$;
- (A1) $A \in L^{2(n-2\gamma+2)/(n-2\gamma+4)}(B_R; x_N^{1-2\gamma})$, $Q \in L^1(B_R; x_N^{1-2\gamma})$, $F = (F_1, \ldots, F_n, F_N) \in L^1(B_R; x_N^{1-2\gamma})$;
- (a1) $a \in L^{2n/(n+2\gamma)}(\partial B'_R)$ and $q \in L^1(\partial B'_R)$.

In this section, we will examine regularity of a weak solution $U \in W^{1,2}(B_R; x_N^{1-2\gamma})$ to a degenerate elliptic equation

$$\begin{cases} -\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma} \nabla U) + x_N^{1-2\gamma} A U = x_N^{1-2\gamma} Q + \mathrm{div}(x_N^{1-2\gamma} F) & \text{in } B_R, \\ U = u & \text{on } \partial B'_R, \\ \partial^\gamma_{\nu,F} U = au + q & \text{on } \partial B'_R, \end{cases} \tag{A.2}$$

where

$$\partial_{\nu,F}^{\gamma} U = \kappa_\gamma \lim_{x_N \to 0+} x_N^{1-2\gamma} \left(F_N - \frac{\partial U}{\partial x_N} \right) = \partial_\nu^\gamma U + \kappa_\gamma \lim_{x_N \to 0+} x_N^{1-2\gamma} F_N.$$

Notice that $\partial_\nu^\gamma = \partial_{\nu,0}^\gamma$. We first recall the precise meaning of a weak solution to (A.2).

Definition A.1. Assume (g1), (A1) and (a1) are valid. A function $U \in W^{1,2}(B_R; x_N^{1-2\gamma})$ is called a *weak solution* to (A.2) if

$$\begin{aligned} &\kappa_\gamma \int_{B_R} x_N^{1-2\gamma} (\langle \nabla U, \nabla \Phi \rangle_{\bar{g}} + AU\Phi) dv_{\bar{g}} \\ &\qquad = \kappa_\gamma \int_{B_R} x_N^{1-2\gamma} (Q\Phi - F_i \partial_i \Phi - F_N \partial_N \Phi) dv_{\bar{g}} + \int_{\partial B_R'} (au + q)\phi \, dv_{\bar{h}} \end{aligned}$$

for every $\Phi \in C^1(\overline{B_R})$ such that $\Phi = \phi$ on $\partial B_R'$ and $\Phi = 0$ on $\partial B_R''$. Here u is the trace of U on $\partial B_R'$ and $\bar{h} = \bar{g}|_{\partial B_R'}$.

A.1. Hölder estimates

By applying the Moser iteration technique, we can deduce Hölder estimates for weak solutions to (A.2).

Lemma A.2. *Assume that the metric $\bar{g}$ satisfies* (g1) *and $U \in W^{1,2}(B_R; x_N^{1-2\gamma})$ is a weak solution to* (A.2). *Suppose also that*

(A2) *$A, Q \in L^{q_1}(B_R; x_N^{1-2\gamma})$ and $F \in L^{q_2}(B_R; x_N^{1-2\gamma})$ for $q_1 > (n - 2\gamma + 2)/2$ and $q_2 > n - 2\gamma + 2$;*
(a2) *$a, q \in L^{q_3}(\partial B_R')$ for $q_3 > n/(2\gamma)$.*

Then $U \in C^\beta(\overline{B_{R/2}})$ and

$$\begin{aligned} \|U\|_{C^\beta(\overline{B_{R/2}})} \le C(&\|U\|_{L^2(B_R; x_N^{1-2\gamma})} + \|Q\|_{L^{q_1}(B_R; x_N^{1-2\gamma})} \\ &+ \|F\|_{L^{q_2}(B_R; x_N^{1-2\gamma})} + \|q\|_{L^{q_3}(\partial B_R')}) \end{aligned} \tag{A.3}$$

where $C > 0$ and $\beta \in (0, 1)$ depend only on n, γ, R, $\lambda_{\bar{g}}$, $\Lambda_{\bar{g}}$, $\|A\|_{L^{q_1}(B_R; x_N^{1-2\gamma})}$ and $\|a\|_{L^{q_3}(\partial B_R')}$. Moreover, if U is nonnegative on $\overline{B_R}$, then also

$$\sup_{B_{R/2}} U \le C\left(\inf_{B_{R/2}} U + \|Q\|_{L^{q_1}(B_R; x_N^{1-2\gamma})} + \|F\|_{L^{q_2}(B_R; x_N^{1-2\gamma})} + \|q\|_{L^{q_3}(\partial B_R')} \right) \tag{A.4}$$

for some $C > 0$ depending only on n, γ, R, $\lambda_{\bar{g}}$, $\Lambda_{\bar{g}}$, $\|A\|_{L^{q_1}(B_R; x_N^{1-2\gamma})}$ and $\|a\|_{L^{q_3}(\partial B_R')}$.

Proof. Derivation of (A.3) and (A.4) can be found in [44, Lemma 5.1 and Remark 5.2]. □

A.2. Derivative estimates

If the functions A, Q, a and q have classical derivatives in the tangential direction, weak solutions to (A.2) have higher differentiability in the same direction. The following result is a huge improvement of [44, Lemma 5.3] in that a much milder condition on $\bar{g}$ is imposed. The reader is advised to see carefully why handling (A.2) becomes more difficult if $\bar{g}$ is non-Euclidean and how it is resolved in the proof.

Lemma A.3. *Assume that the metric $\bar{g}$ satisfies* (g1) *and $U \in W^{1,2}(B_R; x_N^{1-2\gamma})$ is a weak solution to* (A.2)*. Suppose also that ℓ_0 is* 1, 2 *or* 3*, and*

(g2) *we have*

$$|\nabla_{\bar{x}}^{\ell} \partial_N \sqrt{|\bar{g}|}(x)| \le C x_N \quad \text{on } \overline{B_R} \tag{A.5}$$

for any $\ell = 0, \dots, \ell_0$;

(A3a) $A, \dots, \nabla_{\bar{x}}^{\ell_0 - 1} A, Q, \dots, \nabla_{\bar{x}}^{\ell_0 - 1} Q \in L^\infty(B_R)$ *and* $F, \dots, \nabla_{\bar{x}}^{\ell_0 - 1} F \in C^{\beta'}(\overline{B_R})$ *for* $\beta' \in (0, 1)$*;*

(A3b) $\nabla_{\bar{x}}^{\ell_0} A, \nabla_{\bar{x}}^{\ell_0} Q \in L^{q_1}(B_R; x_N^{1-2\gamma})$ *and* $\nabla_{\bar{x}}^{\ell_0} F \in L^{q_2}(B_R; x_N^{1-2\gamma})$ *for* $q_1 > (n - 2\gamma + 2)/2$ *and* $q_2 > n - 2\gamma + 2$*;*

(a3) $a, \dots, \nabla_{\bar{x}}^{\ell_0} a, q, \dots, \nabla_{\bar{x}}^{\ell_0} q \in L^{q_3}(\partial B'_R)$ *for* $q_3 > n/(2\gamma)$*.*

Then $\nabla_{\bar{x}}^{\ell_0} U \in C^\beta(\overline{B_{R/2}})$ and

$$\begin{aligned} \|\nabla_{\bar{x}}^{\ell_0} U\|_{C^\beta(\overline{B_{R/2}})} \le C\Big(& \|U\|_{L^2(B_R; x_N^{1-2\gamma})} + \sum_{\ell=1}^{\ell_0 - 1} \|\nabla_{\bar{x}}^{\ell} A\|_{L^\infty(B_R)} + \|\nabla_{\bar{x}}^{\ell_0} A\|_{L^{q_1}(B_R; x_N^{1-2\gamma})} \\ & + \sum_{\ell=0}^{\ell_0 - 1} \|\nabla_{\bar{x}}^{\ell} Q\|_{L^\infty(B_R)} + \|\nabla_{\bar{x}}^{\ell_0} Q\|_{L^{q_1}(B_R; x_N^{1-2\gamma})} + \sum_{\ell=0}^{\ell_0 - 1} \|\nabla_{\bar{x}}^{\ell} F\|_{C^{\beta'}(\overline{B_R})} \\ & + \|\nabla_{\bar{x}}^{\ell_0} F\|_{L^{q_2}(B_R)} + \sum_{\ell=1}^{\ell_0} \|\nabla_{\bar{x}}^{\ell} a\|_{L^{q_3}(\partial B'_R)} + \sum_{\ell=0}^{\ell_0} \|\nabla_{\bar{x}}^{\ell} q\|_{L^{q_3}(\partial B'_R)}\Big) \end{aligned} \tag{A.6}$$

for $C > 0$ and $\beta \in (0, 1)$ depending only on n, γ, R, $\bar{g}$, A, $\|a\|_{L^{q_3}(\partial B'_R)}$ and $\sum_{\ell=0}^{\ell_0 - 1} \|\nabla_{\bar{x}}^{\ell} U\|_{L^\infty(B_R)}$.

Proof. Assuming that $\ell_0 = 1$, we shall derive (A.6). Given any vector $h \in \mathbb{R}^n$ with $|h|$ small, we define the *difference quotient* $D^h U$ by

$$D^h U(\bar{x}, x_N) = \frac{U(\bar{x} + h, x_N) - U(\bar{x}, x_N)}{|h|} \quad \text{for } (\bar{x}, x_N) \in B_{3R/4}.$$

Then it weakly solves

$$\begin{cases} -\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma} \nabla(D^h U)) + x_N^{1-2\gamma} A(D^h U) = x_N^{1-2\gamma} Q^* + \mathrm{div}(x_N^{1-2\gamma} F^*) & \text{in } B_{3R/4}, \\ \partial^{\gamma}_{\nu, F^*}(D^h U) = a(D^h U) + q^* & \text{on } \partial B'_{3R/4}, \end{cases} \tag{A.7}$$

where $U^h(\bar{x}, x_N) = U(\bar{x} + h, x_N)$,

$$F^* = \big(D^h F_i + D^h(\sqrt{|\bar{g}|}\,\bar{g}^{ij}) \cdot \partial_j U^h,\, D^h F_N + D^h\sqrt{|\bar{g}|} \cdot \partial_N U^h\big),$$
$$Q^* = D^h Q - D^h B \cdot U^h \quad \text{and} \quad q^* = D^h a \cdot U^h + D^h q.$$

The most problematic term in analyzing (A.7) turns out to be $\mathrm{div}(x_N^{1-2\gamma} F^*)$, especially its subterm $\partial_N(x_N^{1-2\gamma} D^h \sqrt{|\bar{g}|} \cdot \partial_N U^h)$. Let us consider it in depth. If we fix a small number $\varepsilon > 0$ and write

$$B_{3R/4,\varepsilon} = B_{3R/4} \cap \{x_N > \varepsilon\} \quad \text{and} \quad \partial B'_{3R/4,\varepsilon} = B_{3R/4} \cap \{x_N = \varepsilon\},$$

then an integration by parts shows

$$\begin{aligned}\int_{B_{3R/4,\varepsilon}} x_N^{1-2\gamma}(D^h\sqrt{|\bar{g}|} \cdot \partial_N U^h)\partial_N \Phi\, dx &= -\int_{B_{3R/4,\varepsilon}} (D^h\sqrt{|\bar{g}|})\partial_N(x_N^{1-2\gamma}\partial_N U^h)\Phi\, dx \\ &- \int_{B_{3R/4,\varepsilon}} x_N^{1-2\gamma}(\partial_N D^h\sqrt{|\bar{g}|})\partial_N U^h \Phi\, dx - \int_{\partial B'_{3R/4,\varepsilon}} x_N^{1-2\gamma}(D^h\sqrt{|\bar{g}|})\partial_N U^h \phi\, d\bar{x}\end{aligned} \tag{A.8}$$

for any $\Phi \in C^1(\overline{B_{3R/4}})$ such that $\Phi = \phi$ on $\partial B'_{3R/4}$ and $\Phi = 0$ on $\partial B''_{3R/4}$. On the other hand, we deduce from (A.2) that

$$\begin{aligned}&\int_{B_{3R/4,\varepsilon}} \sqrt{|\bar{g}|}^h \partial_N(x_N^{1-2\gamma}\partial_N U^h)\Phi\, dx \\ &= -\int_{B_{3R/4,\varepsilon}} x_N^{1-2\gamma}(\sqrt{|\bar{g}|}\,\bar{g}^{ij})^h \partial_i U^h \partial_j \Phi\, dx - \int_{B_{3R/4,\varepsilon}} x_N^{1-2\gamma} A^h U^h \Phi\, dx \\ &\quad + \int_{B_{3R/4,\varepsilon}} x_N^{1-2\gamma} Q^h \Phi\, dx - \int_{B_{3R/4,\varepsilon}} x_N^{1-2\gamma}(F_i)^h \partial_i \Phi\, dx \\ &\quad - \int_{B_{3R/4,\varepsilon}} x_N^{1-2\gamma}(F_N)^h \partial_N \Phi\, dx + \int_{\partial B'_{3R/4,\varepsilon}} x_N^{1-2\gamma}\sqrt{|\bar{g}|}^h \partial_N U^h \phi\, d\bar{x} \\ &\quad + \int_{B_{3R/4,\varepsilon}} x_N^{1-2\gamma}\big(\partial_N\sqrt{|\bar{g}|}^h\big)\partial_N U^h \Phi\, dx + \int_{\partial B'_{3R/4,\varepsilon}} x_N^{1-2\gamma}((F_N)^h - \partial_N U^h)\phi\, d\bar{x}\end{aligned}$$

where $\sqrt{|\bar{g}|}^h(\bar{x}, x_N) = \sqrt{|\bar{g}|}(\bar{x} + h, x_N)$ and so on. Consequently, after substituting

$$\Xi = \big(\sqrt{|\bar{g}|}^h\big)^{-1}(D^h\sqrt{|\bar{g}|})\Phi$$

for Φ in the above identity, combining the result with (A.8) and then taking $\varepsilon \to 0$, we get

$$\begin{aligned}
&\int_{B_{3R/4}} x_N^{1-2\gamma}(D^h\sqrt{|\bar{g}|}\cdot\partial_N U^h)\partial_N\Phi\,dx \\
&\quad= -\int_{B_{3R/4}} x_N^{1-2\gamma}(\sqrt{|\bar{g}|}\bar{g}^{ij})^h\partial_i U^h\partial_j\Xi\,dx - \int_{B_{3R/4}} x_N^{1-2\gamma}A^hU^h\Xi\,dx \\
&\qquad+\int_{B_{3R/4}} x_N^{1-2\gamma}Q^h\Xi\,dx - \int_{B_{3R/4}} x_N^{1-2\gamma}(F_i)^h\partial_i\Xi\,dx \\
&\qquad-\int_{B_{3R/4}} x_N^{1-2\gamma}(F_N)^h\partial_N\Xi\,dx + \int_{B_{3R/4}} x_N^{1-2\gamma}(\partial_N\sqrt{|\bar{g}|})^h\partial_N U^h\Xi\,dx \\
&\quad-\int_{B_{3R/4}} x_N^{1-2\gamma}(\partial_N D^h\sqrt{|\bar{g}|})\partial_N U^h\Phi\,dx + \kappa_\gamma^{-1}\int_{\partial B'_{3R/4}}(au+q)\phi\,d\bar{x}. \qquad \text{(A.9)}
\end{aligned}$$

We have two remarks on (A.9): First, Ξ has the same regularity as Φ and vanishes on $\partial B''_{3R/4}$. Second, by virtue of (g2), there exists a constant $C > 0$ such that

$$\left|\left(\partial_N\sqrt{|\bar{g}|}^h\right)\partial_N U^h\right| + |(\partial_N D^h\sqrt{|\bar{g}|})\partial_N U^h| \le C(x_N\partial_N U)^h \quad \text{in } B_{3R/4}.$$

Furthermore, as pointed out in [44, proof of Lemma 5.3], a rescaling argument gives

$$\|x_N\partial_N U\|_{L^\infty(B_{2R/3})} \le C(\|U\|_{L^\infty(B_{3R/4})} + \|F\|_{C^{\beta'}(\overline{B_R})} + \|Q\|_{L^\infty(B_R)})$$

where $C > 0$ depends only n, R, $\bar{g}$ and $\|A\|_{L^\infty(B_R)}$. Therefore no terms on the right-hand side of (A.9) are harmful.

Now, we introduce a number

$$k = \begin{cases} \|\nabla_{\bar{x}}A\|_{L^{q_1}(B_R;x_N^{1-2\gamma})} + \|\nabla_{\bar{x}}Q\|_{L^{q_1}(B_R;x_N^{1-2\gamma})} + \|\nabla_{\bar{x}}F\|_{L^{q_2}(B_R;x_N^{1-2\gamma})} & \\ \qquad + \|x_N\partial_N U\|_{L^\infty(B_{2R/3})} + \|\nabla_{\bar{x}}a\|_{L^{q_3}(\partial B'_R)} + \|\nabla_{\bar{x}}q\|_{L^{q_3}(\partial B'_R)} & \text{if it is nonzero,} \\ \text{any positive number} & \text{otherwise.} \end{cases}$$

In the latter case, we will let $k \to 0$ at the last stage. For a fixed $K > 0$ and $m \ge 0$, we define

$$V_h = (D^hU)_+ + k, \qquad V_{h,K} = \min\{V_h, K\} \quad \text{and} \quad Z_{h,m} = V_h^{(m+2)/2}.$$

We test (A.7) with $\Phi = \tilde{\chi}(V_{h,K}^m V_h - k^{m+1})$ where $\tilde{\chi} \in C^\infty(\overline{B_R})$ denotes a suitable cut-off function. Employing (A.9), Hölder's inequality, Young's inequality, the weighted Sobolev inequality and the weighted Sobolev trace inequality (see (1.8)) and then letting $K \to \infty$, we derive

$$\begin{aligned}
\|\nabla(\tilde{\chi}Z_{h,m})\|^2_{L^2(B_R;x_N^{1-2\gamma})} &\le Cm^\eta\left[\int_{B_R} x_N^{1-2\gamma}(\tilde{\chi}^2 + |\nabla\tilde{\chi}|^2)Z_{h,m}^2\,dx\right. \\
&\quad\left.+\int_{B_R} x_N^{1-2\gamma}\tilde{\chi}^2V_h^m(|\nabla_{\bar{x}}U_h|^2 + \|x_N\partial_N U\|^2_{L^\infty(B_{2R/3})})\,dx\right] \qquad \text{(A.10)}
\end{aligned}$$

for some $C > 0$ depending only on n, γ, R, $\bar{g}$, A, $\|a\|_{L^{q_3}(\partial B'_R)}$ and $\|U\|_{L^\infty(B_R)}$, and $\eta > 1$ depending only on n and γ; refer to the proofs of [26, Proposition 1] and [44, Lemma 5.3] which provide more detailed descriptions. Combining (A.10) with the corresponding inequality for $(D^h U)_- + k$ and letting $h \to 0$, we see

$$\big\| |\nabla_{\bar{x}} U| + k \big\|^{m+2}_{L^{(m+2)\frac{n-2\gamma+2}{n-2\gamma}}(B_R \cap \{\tilde{\chi}=1\}; x_N^{1-2\gamma})} \le C m^\eta \|\nabla \tilde{\chi}\|^2_{L^\infty(B_R)} \big\| |\nabla_{\bar{x}} U| + k \big\|^{m+2}_{L^{(m+2)}(B_R; x_N^{1-2\gamma})}.$$

Hence the Moser iteration argument implies that

$$\|\nabla_{\bar{x}} U\|_{L^\infty(B_{R/2})} \le (\text{the right-hand side of (A.6) with } \ell_0 = 1).$$

Similarly, one can obtain the weak Harnack inequality as well as the Hölder estimate for $\nabla_{\bar{x}} U$. The cases $\ell_0 = 2$ or 3 can also be treated. We omit the details. □

In the following lemma, we take into account Hölder regularity of the weighted derivative $x_N^{1-2\gamma} \partial_N U$ of a weak solution U to (A.2).

Lemma A.4. *Suppose that the metric $\bar{g}$ satisfies* (g1) *and $U \in W^{1,2}(B_R; x_N^{1-2\gamma})$ is a weak solution to* (A.2) *such that $U, \nabla_{\bar{x}} U, \nabla^2_{\bar{x}} U \in C^\beta(\overline{B_R})$ for some $\beta \in (0,1)$. Furthermore, assume that the following conditions hold:*

(A4) *$A \in C^\beta(\overline{B_R})$, $\sup_{x_N \in (0,R)} (\|Q(\cdot, x_N)\|_{C^\beta(\overline{\partial B'_R})} + \|\partial_i F_i(\cdot, x_N)\|_{C^\beta(\overline{\partial B'_R})}) < \infty$ and $F_N = 0$;*
(a4) *$a, q \in C^\beta(\overline{\partial B'_R})$.*

Then $x_N^{1-2\gamma} \partial_N U \in C^{\min\{\beta, 2-2\gamma\}}(\overline{B_{R/2}})$ and

$$\begin{aligned} \|x_N^{1-2\gamma} \partial_N U\|_{C^{\min\{\beta,2-2\gamma\}}(\overline{B_{R/2}})} &\le C\Big(\sum_{\ell=0}^{2} \|\nabla^\ell_{\bar{x}} U\|_{C^\beta(\overline{B_R})} \\ &\quad + \sup_{x_N \in (0,R)} (\|Q(\cdot, x_N)\|_{C^\beta(\overline{\partial B'_R})} + \|\partial_i F_i(\cdot, x_N)\|_{C^\beta(\overline{\partial B'_R})}) + \|q\|_{C^\beta(\overline{\partial B'_R})}\Big) \end{aligned} \tag{A.11}$$

for $C > 0$ depending only on n, γ, R, $\bar{g}$, $\|A\|_{C^\beta(\overline{B_R})}$ and $\|a\|_{C^\beta(\overline{\partial B'_R})}$.

Proof. Refer to [44, Lemma 5.5]. □

A.3. Two maximum principles

In this part, we list two maximum principles which are used throughout the paper.

The following lemma describes the generalized maximum principle for degenerate elliptic equations.

Lemma A.5. *Suppose that $A \in L^\infty(B_R)$, $a \in L^\infty(\partial B''_R)$ and there exists a function $V \in C^0(\overline{B_R}) \cup C^2(B_R)$ such that $\nabla_{\bar{x}} V$, $x_N^{1-2\gamma} \partial_N V \in C^0(\overline{B_R})$ and*

$$\begin{cases} -\mathrm{div}(x_N^{1-2\gamma} \nabla V) + x_N^{1-2\gamma} A V \geq 0 & \text{in } B_R, \\ V > 0 & \text{on } \overline{B_R}, \\ \partial_\nu^\gamma V + a V \geq 0 & \text{on } \partial B'_R. \end{cases}$$

If $U \in C^0(\overline{B_R}) \cup C^2(B_R)$ satisfies $\nabla_{\bar{x}} U$, $x_N^{1-2\gamma} \partial_N U \in C^0(\overline{B_R})$ and solves

$$\begin{cases} -\mathrm{div}(x_N^{1-2\gamma} \nabla U) + x_N^{1-2\gamma} A U \geq 0 & \text{in } B_R, \\ \partial_\nu^\gamma U + a U \geq 0 & \text{on } \partial B'_R, \\ U \geq 0 & \text{on } \partial B''_R, \end{cases} \tag{A.12}$$

then $U \geq 0$ on $\overline{B_R}$. Furthermore, the same conclusion holds if B_R and $\partial B'_R$ are replaced by $\mathbb{R}^N_+$ and $\mathbb{R}^n$, respectively, and the third inequality in (A.12) is replaced with the condition that $|U(x)|/V(x) \to 0$ uniformly as $|x| \to \infty$.

Proof. Modify suitably the proofs of [40, Lemma A.3] and [45, Lemma 3.7]. □

The next remark concerns the weak maximum principle when the size of the domain is sufficiently small.

Remark A.6. For any fixed $R > 0$, we introduce the space

$$\mathcal{W}_0^{1,2}(B_R; x_N^{1-2\gamma}) = \{U \in W^{1,2}(B_R; x_N^{1-2\gamma}) : U = 0 \text{ on } \partial B''_R\}, \tag{A.13}$$

endowed with the standard $W^{1,2}(B_R; x_N^{1-2\gamma})$-norm. As shown in [20, Lemma 2.1.2], the map $D^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma}) \hookrightarrow L^2(B_R; x_N^{1-2\gamma})$ is compact. Therefore a minimizer of the Rayleigh quotient

$$\lambda_1(R) = \inf_{U \in \mathcal{W}_0^{1,2}(B_R; x_N^{1-2\gamma}) \setminus \{0\}} \frac{\int_{B_R} x_N^{1-2\gamma} |\nabla U|^2 \, dx}{\int_{B_R} x_N^{1-2\gamma} U^2 \, dx}$$

always exists, and so $\lambda_1(R) > 0$. Moreover, we see from dilation symmetry that

$$\lambda_1(R) = (R'/R)^2 \lambda_1(R') \quad \text{for any } 0 < R < R'.$$

Thus, if $|A| \leq \mathcal{M}$ for some constant $\mathcal{M} > 0$, there exists $R'_0 = R'_0(\mathcal{M}, \bar{g}) > 0$ such that

$$\|U\|_* = \left(\int_{B_R} x_N^{1-2\gamma} (|\nabla U|_{\bar{g}}^2 + A U^2) \, dv_{\bar{g}} \right)^{1/2} \quad \text{for } U \in \mathcal{W}_0^{1,2}(B_R; x_N^{1-2\gamma}) \tag{A.14}$$

is a norm equivalent to the $\mathcal{W}_0^{1,2}(B_R; x_N^{1-2\gamma})$-norm for $R \in (0, R'_0)$.

In particular, we have a weak maximum principle: Given any $R \in (0, R_0')$, suppose that $U \in W^{1,2}(B_R; x_N^{1-2\gamma})$ satisfies

$$\begin{cases} -\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla U) + x_N^{1-2\gamma} AU \geq 0 & \text{in } B_R, \\ \partial_\nu^\gamma U \geq 0 & \text{on } \partial B_R', \\ U \geq 0 & \text{on } \partial B_R''. \end{cases} \tag{A.15}$$

Then $U \geq 0$ in B_R. To check it, we just put a test function $U_- \in W^{1,2}(B_R; x_N^{1-2\gamma})$ into (A.15) and use the equivalence between the $*$-norm and the $\mathcal{W}_0^{1,2}(B_R; x_N^{1-2\gamma})$-norm.

A.4. Schauder estimates

In this subsection, we prove the Schauder estimate for solutions to

$$\begin{cases} -\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla U) + x_N^{1-2\gamma} AU = 0 & \text{in } B_R, \\ U = u > 0 & \text{on } \partial B_R', \\ \partial_\nu^\gamma U = q & \text{on } \partial B_R', \end{cases} \tag{A.16}$$

which is a special case of (A.2).

Lemma A.7. *Assume that the metric $\bar{g}$ satisfies* (g1) *and* (g2)*, and $q \in C^\beta(\overline{B_R})$ for some $0 < \beta \notin \mathbb{N}$. Suppose also that*

(g3) *$\bar{g}_{ij}(0) = \delta_{ij}$ where δ_{ij} is the Kronecker delta;*
(A3c) *$A, \ldots, \nabla_{\bar{x}}^{\lceil\beta\rceil} A \in L^\infty(B_R)$ and $\beta' \in (0, \beta + 2\gamma] \cap (0, \lceil\beta\rceil)$.*

If $U \in W^{1,2}(B_R; x_N^{1-2\gamma})$ is a weak solution to (A.16)*, then $u \in C^{\beta'}(\overline{\partial B_{R/2}'})$ and*

$$\|u\|_{C^{\beta'}(\overline{\partial B_{R/2}'})} \leq C\Big(\|U\|_{L^2(B_R; x_N^{1-2\gamma})} + \|q\|_{C^\beta(\overline{\partial B_R'})} + \sum_{\ell=0}^{\lceil\beta\rceil} \|\nabla_{\bar{x}}^\ell A\|_{L^\infty(B_R)}\Big). \tag{A.17}$$

Here $\lceil\beta\rceil$ is the smallest integer exceeding β and $C > 0$ depends only on n, γ, β, R, $\bar{g}$ and A.

Proof. We shall closely follow the argument in [40, proof of Theorem 2.14].

Considering a finite open cover of B_R which consists of balls and half-balls with small diameters, we may assume that $R > 0$ is so small that the $*$-norm in (A.14) is equivalent to the standard $W^{1,2}(B_R; x_N^{1-2\gamma})$-norm. For simplicity, we set

$$M = \|q\|_{C^\beta(\overline{\partial B_R'})} + \sum_{\ell=0}^{\lceil\beta\rceil} \|\nabla_{\bar{x}}^\ell A\|_{L^\infty(B_R)}.$$

Assume that $\beta \in (0, 1)$. For $m \in \mathbb{N}$, let W_m be the unique solution in $W^{1,2}(B_{R/2^m}; x_N^{1-2\gamma})$ to

$$\begin{cases} -\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla W_m) + x_N^{1-2\gamma} AW_m = 0 & \text{in } B_{R/2^m}, \\ \partial_\nu^\gamma W_m = q(0) - q(\bar{x}) & \text{on } \partial B_{R/2^m}', \\ W_m = 0 & \text{on } \partial B_{R/2^m}''. \end{cases} \tag{A.18}$$

Then an application of the weak maximum principle (Remark A.6) to the equation of the function

$$2^{2\gamma m} W_m\left(\frac{x}{2^m}\right) \pm \frac{MR^\beta}{2^{\beta m}}\left[\frac{2R^2-|x|^2}{n+2-2\gamma} + \frac{2R^{2\gamma}-x_N^{2\gamma}}{2\gamma\kappa_\gamma}\right]$$

shows

$$\|W_m\|_{L^\infty(B_{R/2^m})} \le \frac{CM}{2^{(\beta+2\gamma)m}} \tag{A.19}$$

for every $m \in \mathbb{N}$. Define $h_m = W_{m+1} - W_m$. Thanks to Lemmas A.2 and A.3, we have

$$\|\nabla_{\bar{x}}^{\ell}(U+W_0)\|_{L^\infty(B_{R/2})} \le C(\|U\|_{L^2(B_R;x_N^{1-2\gamma})} + M), \tag{A.20}$$

$$\|\nabla_{\bar{x}}^{\ell} h_m\|_{L^\infty(B_{R/2^{m+2}})} \le \frac{CM}{2^{(\beta+2\gamma-\ell)m}}, \tag{A.21}$$

for $\ell = 0, 1$ and all $m \in \mathbb{N}$. By (A.19)–(A.21) and the mean value theorem,

$$\begin{aligned}|u(\bar{x}) - u(0)| &\le |W_m(0,0)| + |W_m(\bar{x},0)| + |(U+W_0)(\bar{x},0) - (U+W_0)(0,0)| \\ &\quad + \sum_{j=0}^{m-1} |h_j(\bar{x},0) - h_j(0,0)| \\ &\le C(\|U\|_{L^2(B_R;x_N^{1-2\gamma})} + M)|\bar{x}|^{\min\{\beta+2\gamma,1\}} \quad \text{for all } \bar{x} \in \partial B'_{R/2},\end{aligned}$$

so that $u \in C^{\beta'}(\overline{\partial B'_{R/2}})$ and (A.17) holds.

Suppose $\beta \in (1, 2)$. In this case, we modify W_m by replacing the second equation of (A.18) with

$$\partial_\nu^\gamma W_m = q(0) + \nabla_{\bar{x}} q(0) \cdot \bar{x} - q(\bar{x}) \quad \text{on } \partial B'_{R/2^m}.$$

Then we use the above argument to conclude that $u \in C^{\beta'}(\overline{\partial B'_{R/2}})$ and (A.17) holds.

The case $\beta > 2$ can be treated similarly. This finishes the proof. □

A.5. Conclusion

From the results obtained in the previous subsections, one gets the following regularity property of solutions to (1.1) and its extension problem (2.4).

Proposition A.8. *Suppose that $U \in W^{1,2}(X; \rho^{1-2\gamma})$ is a weak solution of (2.4) with a fixed $p \in (1, 2^*_{n,\gamma} - 1]$ and condition (1.5) holds. Then the functions U, $\nabla_{\bar{x}} U$, $\nabla_{\bar{x}}^2 U$ and $x_N^{1-2\gamma}\partial_N U$ are Hölder continuous on $\overline{X}$. In particular, the trace $u \in C^2(M)$ of U on M satisfies (1.1) in the classical sense.*

Proof. The above regularity properties are local. Therefore Lemmas A.2, A.3, A.4 and A.7 can be used. □

Appendix B. Green's function and Bôcher's theorem

In this section, we keep the notations of (A.1).

B.1. Green's function

Given a small number $R > 0$, we shall examine the existence and the growth rate of the *Green's function* G in B_R, a solution of

$$\begin{cases} -\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla G) + E_{\bar{g}}(x_N)G = 0 & \text{in } B_R, \\ \partial_\nu^\gamma G = \delta_0 & \text{on } \partial B_R', \\ G = 0 & \text{on } \partial B_R'', \end{cases} \tag{B.1}$$

where $\bar{g}$ is the metric satisfying (g1) and δ_0 is the Dirac measure centered at $0 \in \mathbb{R}^N$. Our argument is based on elliptic regularity theory and does not rely on parametrices.

We start by deriving an auxiliary lemma.

Lemma B.1. *Given a small* $R > 0$*, suppose that* $a = q = 0$ *on* $\partial B_R'$*,* $A \in L^\infty(B_R)$ *and* $Q \in L^{q_1}(B_R; x_N^{1-2\gamma})$ *for*

$$q_1 \in \left[\frac{2(n-2\gamma+2)}{n-2\gamma+4}, \frac{n-2\gamma+2}{2}\right). \tag{B.2}$$

Assume also that $U \in \mathcal{W}_0^{1,2}(B_R; x_N^{1-2\gamma})$ *is a weak solution to* (A.2) *and the* $*$*-norm is equivalent to the* $\mathcal{W}_0^{1,2}(B_R; x_N^{1-2\gamma})$*-norm; see* (A.13) *and* (A.14)*. Then*

$$\|U\|_{L^{q_4}(B_R; x_N^{1-2\gamma})} \le C\|Q\|_{L^{q_1}(B_R; x_N^{1-2\gamma})}$$

for any pair (q_1, q_4) *satisfying* $1/q_1 = 1/q_4 + 2/(n-2\gamma+2)$ *and some constant* $C > 0$ *depending only on* n*,* γ*,* R*,* $\lambda_{\bar{g}}$*,* $\Lambda_{\bar{g}}$*,* q_1 *and* q_4*.*

Proof. One can follow the lines of [15, proof of Lemma 3.3]. The main difference is that we need to apply the Sobolev inequality instead of the Sobolev traced inequality as used in that reference; see (1.8). □

Appealing to the previous lemma, we prove the main result in this subsection.

Proposition B.2. *Assume that* $n \ge 2+2\gamma$*. Then there exists* $0 < R_0 \le \min\{R_0', r_1\}$ *small (refer to Remark* A.6 *and Subsection* 2.4*) such that* (B.1) *possesses a unique solution* $G \in W_{\mathrm{loc}}^{1,2}(B_R \setminus \{0\}; x_N^{1-2\gamma})$ *satisfying*

$$|x|^{n-2\gamma}G(x) \to g_{n,\gamma} \quad \text{uniformly as } |x| \to 0 \quad \text{where } g_{n,\gamma} = \frac{\Gamma\left(\frac{n-2\gamma}{2}\right)}{\pi^{n/2}2^{2\gamma}\Gamma(\gamma)} > 0 \tag{B.3}$$

for any fixed $R \in (0, R_0)$*.*

Proof. The proof is divided into four steps.

Step 1 (Existence). By using (1.5) and (2.1), we rewrite the first equation of (B.1) as

$$-\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla G) + x_N^{1-2\gamma}AG = 0 \quad \text{in } B_R$$

where $A \in C^2(\overline{B_R})$. In view of Remark A.6, there exists $R_0 > 0$ such that $\|\cdot\|_*$ in (A.14) serves as a norm equivalent to the standard $\mathcal{W}^{1,2}(B_R; x_N^{1-2\gamma})$-norm for all $R \in (0, R_0)$. Then a duality argument in [46, proof of Lemma 4.2] shows that the desired function G exists and is contained in $W^{1,2}_{\text{loc}}(B_R \setminus \{0\}; x_N^{1-2\gamma}) \cap W^{1,q}(B_R; x_N^{1-2\gamma})$ for any $1 < q < (n-2\gamma+2)/(n-2\gamma+1)$.

Step 2 (Regularity). Recall that

$$G_{\mathbb{R}^N_+}(x) = \frac{g_{n,\gamma}}{|x|^{n-2\gamma}} \quad \text{in } \mathbb{R}^N_+ \tag{B.4}$$

solves

$$\begin{cases} -\text{div}(x_N^{1-2\gamma} \nabla G_{\mathbb{R}^N_+}) = 0 & \text{in } \mathbb{R}^N_+, \\ \partial^\gamma_\nu G_{\mathbb{R}^N_+} = \delta_0 & \text{on } \mathbb{R}^n. \end{cases}$$

Hence G satisfies (B.1) if and only if $\mathcal{H} = G_{\mathbb{R}^N_+} - G - g_{n,\gamma} R^{-(n-2\gamma)}$ is a solution of

$$\begin{cases} -\text{div}(x_N^{1-2\gamma} \nabla \mathcal{H}) + x_N^{1-2\gamma} A\mathcal{H} = x_N^{1-2\gamma}(\mathcal{Q}(G_{\mathbb{R}^N_+}) - A g_{n,\gamma} R^{-(n-2\gamma)}) & \text{in } B_R, \\ \partial^\gamma_\nu \mathcal{H} = 0 & \text{on } \partial B'_R, \\ \mathcal{H} = 0 & \text{on } \partial B''_R, \end{cases} \tag{B.5}$$

where

$$\mathcal{Q}(U) = -(\bar{g}^{ij} - \delta^{ij})\partial_{ij} U - \frac{\partial_i \sqrt{|\bar{g}|}}{\sqrt{|\bar{g}|}} \bar{g}^{ij} \partial_j U - \frac{\partial_N \sqrt{|\bar{g}|}}{\sqrt{|\bar{g}|}} \partial_N U \\ - \partial_i \bar{g}^{ij} \partial_j U + \frac{n-2\gamma}{4n}(R[\bar{g}] + o(1))U.$$

By a direct calculation, we see that

$$\mathcal{H} \in W^{1,2}_{\text{loc}}(B_R \setminus \{0\}; x_N^{1-2\gamma}) \cap W^{1,q}(B_R; x_N^{1-2\gamma}) \tag{B.6}$$

and

$$|\mathcal{Q}(G_{\mathbb{R}^N_+})| \le \frac{C}{|x|^{n-2\gamma+1}} \in L^q(B_R; x_N^{1-2\gamma}) \tag{B.7}$$

for all $1 < q < (n-2\gamma+2)/(n-2\gamma+1)$. We claim that

$$\|\mathcal{H}\|_{L^{q'}(B_R; x_N^{1-2\gamma})} < \infty \quad \text{whenever } n \ge 2 + 2\gamma \text{ and } 1 < q' < \frac{n-2\gamma+2}{n-2\gamma-1}. \tag{B.8}$$

To justify it, we consider the formal adjoint of (B.5),

$$\begin{cases} -\text{div}(x_N^{1-2\gamma} \nabla U) + x_N^{1-2\gamma} AU = x_N^{1-2\gamma} Q & \text{in } B_R, \\ \partial^\gamma_\nu U = 0 & \text{on } \partial B'_R, \\ U = 0 & \text{on } \partial B''_R, \end{cases} \tag{B.9}$$

where Q is an arbitrary function of class $C^1(\overline{B_R})$. Then

$$(x_N^{1-2\gamma}\partial_N U)(\bar{x}, x_N) - (x_N^{1-2\gamma}\partial_N U)(\bar{x}, \varepsilon) = \int_\varepsilon^{x_N} \partial_N (x_N^{1-2\gamma}\partial_N U)\, dx_N$$
$$= -\int_\varepsilon^{x_N} x_N^{1-2\gamma}(\Delta_{\bar{x}} U - AU + Q)\, dx_N$$

for any small $\varepsilon > 0$. Since $(x_N^{1-2\gamma}\partial_N U)(\bar{x}, \varepsilon) \to 0$ as $\varepsilon \to 0$ thanks to the boundary condition and $\Delta_{\bar{x}} U \in C^0(\overline{B_R})$ in light of Lemma A.3, we observe

$$|\partial_N U(\bar{x}, x_N)| \le C x_N \quad \text{in } B_R$$

and in particular $U \in C^1(\overline{B_R})$. Thus one may use $\mathcal{H}$ and U as test functions for (B.9) and (B.5), respectively. As a consequence,

$$\int_{B_R} x_N^{1-2\gamma} \mathcal{H} Q\, dx = \int_{B_R} x_N^{1-2\gamma} (\nabla\mathcal{H}\cdot\nabla U + A\mathcal{H}U)\, dx$$
$$= \int_{B_R} x_N^{1-2\gamma} \big(\mathcal{Q}(G_{\mathbb{R}^N_+}) - A g_{n,\gamma} r^{-(n-2\gamma)}\big) U\, dx$$
$$\le \|\mathcal{Q}(G_{\mathbb{R}^N_+}) - A g_{n,\gamma} r^{-(n-2\gamma)}\|_{L^{\frac{n-2\gamma+2}{n-2\gamma+1}-\eta}(B_R; x_N^{1-2\gamma})} \|U\|_{L^{n-2\gamma+2+\eta'}(B_R; x_N^{1-2\gamma})}$$
$$\le C\|Q\|_{L^{\frac{n-2\gamma+2}{3}+\eta''}(B_R; x_N^{1-2\gamma})}$$

for any $Q \in C^1(\overline{B_R})$ and small $\eta > 0$. Here η' and η'' are small positive numbers depending only on η. Also, the last inequality is due to (B.7) and Lemma B.1, and the assumption $n \ge 2 + 2\gamma$ is required to ensure that $q_1 = (n - 2\gamma + 2)/3 + \eta''$ satisfies condition (B.2). By duality, the assertion (B.8) follows.

Step 3 (Blow-up rate). We use the rescaling argument of [54, proof of Proposition B.1]. For $0 < R' < R/3$, we define

$$\widetilde{\mathcal{H}}(x) = (R')^{n-2\gamma}\mathcal{H}(R'x) \quad \text{in } B_3 \setminus \overline{B_{1/3}}.$$

It clearly solves

$$\begin{cases} -\mathrm{div}(x_N^{1-2\gamma}\nabla\widetilde{\mathcal{H}}) + x_N^{1-2\gamma}(R')^2 A(R'x)\widetilde{\mathcal{H}} = x_N^{1-2\gamma} O(R') & \text{in } B_3 \setminus \overline{B_{1/3}}, \\ \partial_\nu^\gamma \widetilde{\mathcal{H}} = 0 & \text{on } \partial B_3' \setminus \overline{\partial B_{1/3}'}, \end{cases}$$

where $O(R') \le CR'$. The local integrability condition (B.6), Lemma B.1 and estimate (B.8) show that

$$\|\widetilde{\mathcal{H}}\|_{L^\infty(B_2\setminus B_{1/2})} \le C\Big(\|\widetilde{\mathcal{H}}\|_{L^{\frac{n-2\gamma+2+\eta}{n-2\gamma}}(B_3\setminus\overline{B_{1/3}}; x_N^{1-2\gamma})} + O(R')\Big)$$
$$= C\Big((R')^{\eta'}\|\mathcal{H}\|_{L^{\frac{n-2\gamma+2+\eta}{n-2\gamma}}(B_R; x_N^{1-2\gamma})} + O(R')\Big) \le C(R')^{\eta'}$$

for a small $\eta > 0$ and $\eta' = (n-2\gamma)\eta/(n-2\gamma+2+\eta)$. Therefore

$$|\mathcal{H}(x)| \le C|x|^{-(n-2\gamma)+\eta'} \quad \text{in } B_{2R/3}. \tag{B.10}$$

On the other hand, by virtue of (B.6) and (B.7), we can apply Lemma B.1 to (B.5), getting

$$|\mathcal{H}(x)| \le C \quad \text{in } B_R \setminus \overline{B_{2R/3}}. \tag{B.11}$$

Putting (B.4), (B.10) and (B.11) together gives the blow-up rate (B.3) of G.

Step 4 (Uniqueness). The uniqueness of G follows from Bôcher's theorem stated in Proposition B.4. □

Corollary B.3. *Assume that $n \ge 2+2\gamma$. The regular part $\mathcal{H}$ of the Green's function G defined in the proof of the previous proposition satisfies*

$$|\nabla_{\bar{x}}\mathcal{H}(x)| \le C\frac{|x|^{\eta'}}{|x|^{n-2\gamma+1}} \quad \textit{and} \quad |x_N^{1-2\gamma}\partial_N\mathcal{H}(x)| \le C\frac{|x|^{\eta'}}{|x|^{n}}$$

for any fixed $R \in (0, R_0)$ and small $\eta' > 0$.

Proof. The result follows immediately from (B.10) and the rescaling argument. □

B.2. The proof of Bôcher's theorem

We present the following version of a fractional Bôcher's theorem, which is needed in the proof of Proposition 3.7. The Euclidean case was considered in [40, Lemma 4.10] and [62, Proposition 3.4].

Proposition B.4. *Fix any $R \in (0, R_0)$. Suppose that the metric $\bar{g}$ satisfies* (g1) *and* (g2), $\|A\|_{L^\infty(B_{R_0})} \le \mathcal{M}$, *and $\nabla_{\bar{x}}A \in L^{q_1}(B_{R_0}; x_N^{1-2\gamma})$ for $q_1 > (n-2\gamma+2)/2$. If a function U is nonnegative in $B_R \setminus \{0\}$, belongs to $W^{1,2}(B_R \setminus \overline{B'_\vartheta}; x_N^{1-2\gamma})$ and weakly solves*

$$\begin{cases} -\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla U) + x_N^{1-2\gamma}AU = 0 & \textit{in } B_R \setminus \overline{B'_\vartheta}, \\ \partial_\nu^\gamma U = 0 & \textit{on } \partial B'_R \setminus \overline{\partial B'_\vartheta}, \end{cases}$$

for all $\vartheta \in (0, R)$, then for any $R' \in (0, R)$,

$$U = c_1 G + E \quad \textit{in } B_{R'} \setminus \{0\}.$$

Here c_1 is a nonnegative constant, G is the Green's function that satisfies (B.1) (*where R is replaced with R'*) *and $E \in W^{1,2}(B_R; x_N^{1-2\gamma})$ solves*

$$\begin{cases} -\mathrm{div}_{\bar{g}}(x_N^{1-2\gamma}\nabla E) + x_N^{1-2\gamma}AE = 0 & \textit{in } B_{R'}, \\ \partial_\nu^\gamma E = 0 & \textit{on } \partial B'_{R'}. \end{cases}$$

The numbers R_0 and $\mathcal{M}$ were chosen in Proposition B.2 and Remark A.6. To prove the proposition, we will use the strategy from [55, Section 9] and [62, Section 3]. As a preliminary step, we derive two results.

Lemma B.5. *Assume that U satisfies all the conditions in Proposition B.4. If $U(x) = o(|x|^{-(n-2\gamma)})$ as $|x| \to 0$, then 0 is a removable singularity of U and there exists β in $(0,1)$ such that $U \in W^{1,2}(B_{R'}; x_N^{1-2\gamma}) \cap C^{\beta}(\overline{B_{R'}})$ for any $R' \in (0, R)$.*

Proof. We argue as in [62, Lemma 3.6] with minor modifications. The maximum principle of Remark A.6 combined with the asymptotic behavior (B.3) of G near the origin shows that U is bounded in $B_{R'}$. Owing to the regularity hypotheses on $\bar{g}$ and A, we can apply the scaling method with Lemmas A.3 and A.4, deducing $U \in C^{\beta}(\overline{B_{R'}})$ for some $\beta \in (0,1)$ and

$$|x|\,|\nabla_{\bar{x}} U(x)| + |x|^{2\gamma} |x_N^{1-2\gamma} \partial_N U(x)| \le C \quad \text{in } \overline{B_{R'}}.$$

This in turn implies that $U \in W^{1,2}(B_{R'}; x_N^{1-2\gamma})$. □

Lemma B.6. *Assume that U satisfies all the conditions in Proposition B.4. Then*

$$\limsup_{r \to 0+} \max_{|x|=r} |x|^{n-2\gamma} U(x) < \infty.$$

Proof. This can be checked as in [55, Lemma 9.3] or [62, Lemma 3.7]. □

Proof of Proposition B.4. One can carry out the proof by adapting the ideas of [55, Proposition 9.1] or [62, Proposition 3.4]. Lemmas B.5 and B.6 are required. □

Appendix C. Computation of the integrals involving the standard bubble

We obtain the values of several integrals involving the standard bubble $W_{1,0}$ and its derivatives, which are needed in the proof of the vanishing theorem in Section 5.

Proposition C.1. *Suppose $\gamma \in (0,1)$ and $n > 2 + 2\gamma$. Then there exists a constant $\mathcal{C}_0 > 0$ depending only on n and γ such that*

$$\begin{aligned} \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} \Delta_{\bar{x}} W_{1,0} Z_{1,0}^0 \, dx &= \mathcal{C}_0, \\ \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} \partial_N W_{1,0} Z_{1,0}^0 \, dx &= \frac{3}{2(1+\gamma)} \mathcal{C}_0, \\ \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} W_{1,0} Z_{1,0}^0 \, dx &= -\frac{3}{2(1-\gamma^2)} \mathcal{C}_0. \end{aligned} \tag{C.1}$$

Here $W_{1,0}$ and $Z_{1,0}^0$ are the functions given in (2.6) and (2.9).

Its proof is based on the Fourier transform technique which was introduced by González and Qing [30] and soon improved by González and Wang [31] and Kim et al. [45, 46], where the authors studied the existence and $C^2(M)$-noncompactness of the solution set of (1.1). We first need to recall a lemma obtained in [30, Section 7] and [45, Subsection 4.3].

Lemma C.2. (1) *Assume that* $n \geq 3$ *and* $\gamma \in (0,1)$. *Let* $\widehat{W}_{1,0}(\xi, x_N)$ *be the Fourier transform of* $W_{1,0}(\bar{x}, x_N)$ *in* $\bar{x} \in \mathbb{R}^n$ *for each fixed* $x_N > 0$ *and* K_γ *the modified Bessel function of the second kind of order* γ. *Also choose appropriate numbers* $d_1, d_2 > 0$ *depending only on* n *and* γ *so that the functions* $\varphi(t) = d_1 t^\gamma K_\gamma(t)$ *and* $\hat{w}_{1,0}(t) = d_2 t^{-\gamma} K_\gamma(t)$ *solve*

$$\phi''(t) + \frac{1-2\gamma}{t}\phi'(t) - \phi(t) = 0, \quad \phi(0) = 1 \quad \text{and} \quad \phi(\infty) = 0$$

and

$$\phi''(t) + \frac{1+2\gamma}{t}\phi'(t) - \phi(t) = 0 \quad \text{and} \quad \lim_{t\to 0} t^{2\gamma}\phi(t) + \lim_{t\to\infty} t^{\gamma+1/2} e^t \phi(t) \leq C$$

for some $C > 0$, *respectively. Then*

$$\widehat{W}_{1,0}(\xi, x_N) = \hat{w}_{1,0}(\xi)\varphi(|\xi| x_N) \quad \text{for every } \xi \in \mathbb{R}^n \text{ and } x_N > 0.$$

(2) *Let*

$$\begin{aligned}
\mathcal{A}_\alpha &= \int_0^\infty t^{\alpha-2\gamma}\varphi^2(t)\,dt, \qquad \mathcal{A}'_\alpha = \int_0^\infty t^{\alpha-2\gamma}\varphi(t)\varphi'(t)\,dt, \\
\mathcal{A}''_\alpha &= \int_0^\infty t^{\alpha-2\gamma}(\varphi'(t))^2\,dt, \\
\mathcal{B}_\beta &= \int_0^\infty t^{-\beta+2\gamma}\hat{w}_{1,0}^2(t) t^{n-1}\,dt, \qquad \mathcal{B}'_\beta = \int_0^\infty t^{-\beta+2\gamma}\hat{w}_{1,0}(t)\,\hat{w}'_{1,0}(t) t^{n-1}\,dt \\
\mathcal{B}''_\beta &= \int_0^\infty t^{-\beta+2\gamma}(\hat{w}'_{1,0}(t))^2 t^{n-1}\,dt,
\end{aligned}$$

for $\alpha, \beta \in \mathbb{N} \cup \{0\}$. *Then*

$$\begin{aligned}
\mathcal{A}_\alpha &= \frac{\alpha+2}{\alpha+1}\cdot\left[\left(\frac{\alpha+1}{2}\right)^2 - \gamma^2\right]^{-1}\mathcal{A}_{\alpha+2} = -\left(\frac{\alpha+1}{2} - \gamma\right)^{-1}\mathcal{A}'_{\alpha+1} \\
&= \left(\frac{\alpha+1}{2} - \gamma\right)\left(\frac{\alpha-1}{2} + \gamma\right)^{-1}\mathcal{A}''_\alpha
\end{aligned}$$

for α *odd,* $\alpha \geq 1$ *and*

$$\begin{aligned}
\mathcal{B}_\beta &= \frac{4(n-\beta+1)\mathcal{B}_{\beta-2}}{(n-\beta)(n+2\gamma-\beta)(n-2\gamma-\beta)} = -\frac{2\mathcal{B}'_{\beta-1}}{n+2\gamma-\beta}, \\
\mathcal{B}_{\beta-2} &= \frac{(n-2\gamma-\beta)\mathcal{B}''_{\beta-2}}{n+2\gamma-\beta+2}
\end{aligned}$$

for β *even,* $\beta \geq 2$.

With the help of the previous lemma, we evaluate the following nine integrals.

Lemma C.3. *Assume that $\gamma \in (0,1)$ and $n > 2+2\gamma$. Set $\mathcal{C}_0 = |\mathbb{S}^{n-1}|\mathcal{A}_3\mathcal{B}_2$. Then*

$$\mathcal{I}_1 = \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} W_{1,0}^2 \, dx = \frac{3}{2(1-\gamma^2)}\mathcal{C}_0,$$
$$\mathcal{I}_2 = \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} r W_{1,0}(\partial_r W_{1,0}) \, dx = -\frac{3n}{4(1-\gamma^2)}\mathcal{C}_0,$$
$$\mathcal{I}_3 = \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} W_{1,0}(\partial_N W_{1,0}) \, dx = -\frac{3}{2(1+\gamma)}\mathcal{C}_0,$$
$$\mathcal{I}_4 = \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} r(\partial_r W_{1,0})(\partial_N W_{1,0}) \, dx = \frac{3n-2(1+\gamma)}{4(1+\gamma)}\mathcal{C}_0,$$
$$\mathcal{I}_5 = \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} W_{1,0}(\Delta_{\bar{x}} W_{1,0}) \, dx = -\mathcal{C}_0,$$
$$\mathcal{I}_6 = \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} (\partial_r W_{1,0})^2 \, dx = \mathcal{C}_0,$$
$$\mathcal{I}_7 = \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} (\partial_N W_{1,0})^2 \, dx = \frac{2-\gamma}{1+\gamma}\mathcal{C}_0,$$
$$\mathcal{I}_8 = \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} r(\partial_r W_{1,0})(\partial_{rr} W_{1,0}) \, dx = -\frac{n}{2}\,\mathcal{C}_0,$$
$$\mathcal{I}_9 = \int_{\mathbb{R}^N_+} x_N^{4-2\gamma} (\partial_N W_{1,0})(\Delta_{\bar{x}} W_{1,0}) \, dx = (2-\gamma)\,\mathcal{C}_0,$$

for $r = |\bar{x}|$.

Proof. The quantities $\mathcal{I}_1$, $\mathcal{I}_6$, $\mathcal{I}_7$ were computed in [30, Lemma 7.2]. Moreover, [46, Lemma B.4] provides the value of $\mathcal{I}_8$, and its proof suggests a way to calculate $\mathcal{I}_2$, $\mathcal{I}_3$. Accordingly we only take into account the others. Throughout the proof, we agree that the variable of $\hat{w}_{1,0}$ and $\hat{w}_{1,0}'$ is $|\xi|$ and that of φ and φ' is $|\xi|x_N$. Also, $'$ is used to represent differentiation in the radial variable $|\xi|$.

It follows from Parseval's theorem that

$$\begin{aligned}
\mathcal{I}_4 &= \int_0^\infty x_N^{2-2\gamma}\left(\int_{\mathbb{R}^n} \widehat{x_i \partial_i W_{1,0}} \partial_N \widehat{W}_{1,0} \, d\xi\right) dx_N \\
&= -\int_0^\infty x_N^{2-2\gamma}\left[\int_{\mathbb{R}^n} (n\widehat{W}_{1,0} + \xi_i \partial_i \widehat{W}_{1,0})\partial_N \widehat{W}_{1,0} \, d\xi\right] dx_N \\
&= -\int_0^\infty x_N^{2-2\gamma}\left[\int_{\mathbb{R}^n} (n\hat{w}_{1,0}\varphi + |\xi|\hat{w}_{1,0}'\varphi + x_N|\xi|\hat{w}_{1,0}\varphi')|\xi|\hat{w}_{1,0}\varphi' \, d\xi\right] dx_N \\
&= -|\mathbb{S}^{n-1}|(n\mathcal{A}_2\mathcal{B}_2' + \mathcal{A}_1'\mathcal{B}_2' + \mathcal{A}_3''\mathcal{B}_3).
\end{aligned}$$

Therefore Lemma C.2(2) gives the value in the statement. On the other hand, we observe by applying integration by parts that

$$\mathcal{I}_5 = -\int_{\mathbb{R}^N_+} x_N^{3-2\gamma} |\nabla_{\bar{x}} W_{1,0}|^2 \, dx = -\mathcal{I}_6.$$

Also one can verify

$$\mathcal{I}_9 = -\int_0^\infty x_N^{4-2\gamma}\left[\int_{\mathbb{R}^n} |\xi|^2 \widehat{W}_{1,0}(\partial_N \widehat{W}_{1,0})\, d\xi\right] dx_N = -|\mathbb{S}^{n-1}|\mathcal{A}_4'\mathcal{B}_2,$$

finishing the proof. □

Proof of Proposition C.1. We have

$$Z_{1,0}^0 = r\partial_r W_{1,0} + x_N \partial_N W_{1,0} + \frac{n-2\gamma}{2} W_{1,0},$$
$$\Delta_{\bar{x}} W_{1,0} = \partial_{rr} W_{1,0} + (n-1)r^{-1}\partial_r W_{1,0},$$

for $r = |\bar{x}|$. Hence

$$\int_{\mathbb{R}^N_+} x_N^{3-2\gamma} \Delta_{\bar{x}} W_{1,0} Z_{1,0}^0\, dx = \mathcal{I}_8 + (n-1)\mathcal{I}_6 + \mathcal{I}_9 + \frac{n-2\gamma}{2}\mathcal{I}_5,$$
$$\int_{\mathbb{R}^N_+} x_N^{2-2\gamma} \partial_N W_{1,0} Z_{1,0}^0\, dx = \mathcal{I}_4 + \mathcal{I}_7 + \frac{n-2\gamma}{2}\mathcal{I}_3,$$
$$\int_{\mathbb{R}^N_+} x_N^{1-2\gamma} W_{1,0} Z_{1,0}^0\, dx = \mathcal{I}_2 + \mathcal{I}_3 + \frac{n-2\gamma}{2}\mathcal{I}_1.$$

Now an easy application of Lemma C.3 completes the proof. □

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