



Mikhail Lyubich · Han Peters

## Structure of partially hyperbolic Hénon maps

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**Abstract.** We describe the structure of “substantially dissipative” complex Hénon maps admitting a dominated splitting on the Julia set. We prove that the Fatou set consists of only finitely many components, each either attracting or parabolic periodic. In particular, there are no wandering components and no rotation domains. Moreover, we show that  $J = J^*$  and the dynamics on  $J$  is hyperbolic away from parabolic cycles.

**Keywords.** Complex dynamics, partial hyperbolicity, dominated splitting, Hénon maps

### 1. Introduction

Dominated splitting is an important classical notion going back to the works of Pliss and Mañé from the ’70s. The dynamics of real surface diffeomorphisms with dominated splitting was described by Pujals and Sambarino [PS09]. Inspired by this result, we considered a similar problem in the complex setting, for dissipative Hénon maps (or more generally, for dissipative polynomial automorphisms of  $\mathbb{C}^2$ ). In this paper we describe the structure of such maps.

Complex Hénon maps are polynomial automorphisms of  $\mathbb{C}^2$  with non-trivial dynamical behavior,

$$f : (x, y) \mapsto (p(x) - by, x), \quad \text{where } \deg p \geq 2, \quad b = \text{Jac } f \neq 0.$$

For a small Jacobian  $b$ , it can be viewed as a perturbation of the one-dimensional polynomial  $p : \mathbb{C} \rightarrow \mathbb{C}$ . Though some initial aspects of the 2D theory resemble the 1D theory, quite quickly it becomes much more difficult, exhibiting various new phenomena.

Dynamics of 1D polynomials on the Fatou set is fully understood, due to the classical work of Fatou, Julia and Siegel, supplemented with Sullivan’s No Wandering Domains Theorem from the early ’80s [Sul85]. This direction of research for Hénon maps was initiated by Bedford and Smillie in the early ’90s. In particular, they gave a description

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M. Lyubich: Institute for Mathematical Sciences, Stony Brook University,  
Stony Brook, NY 11794-3660, United States; e-mail: mlyubich@math.stonybrook.edu

H. Peters (corresponding author): Korteweg de Vries Institute for Mathematics, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands; e-mail: h.peters@uva.nl

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of the dynamics on “recurrent” periodic Fatou components [BS91b]. The “non-recurrent” case was recently treated by the authors [LP14], under an assumption that the Hénon map is “substantially dissipative”, i.e.

$$|\text{Jac } f| < \frac{1}{(\deg p)^2}.$$

This completed the classification of periodic Fatou components in this setting: Any such component is either an attracting or parabolic basin, or a rotation domain, which is analogous to the one-dimensional classification.<sup>1</sup>

The situation with the problem of wandering components is more complicated. In fact, it was recently shown that wandering domains can occur for polynomial maps in two complex dimensions: for polynomial endomorphisms in [ABD<sup>+</sup>16], and for complex Hénon maps in [BB2].

It is quite clear that Sullivan’s proof of the No Wandering Domains Theorem, based upon quasiconformal deformations machinery, is not generalizable to higher dimensions. At the same time, for various special classes of 1D polynomial maps, one can give a direct geometric argument that has a chance to be generalized to the 2D setting. The simplest class of this kind comprises hyperbolic polynomials, for which absence of wandering component was known classically. The Hénon counterpart of this result was established by Bedford and Smillie in the ’90s, resulting in a complete description of the dynamics on the Fatou set for this class [BS91a]: a hyperbolic Hénon map has only finitely many Fatou components, each of which is an attracting basin. Moreover, in this case, the Julia set  $J$  is the closure of saddles:  $J = J^*$ .

Until now, hyperbolic maps have remained the only class of Hénon maps for which these problems were settled down. In this paper, we resolve these problems for substantially dissipative Hénon maps that admit a “dominated splitting” over the Julia set:

**Theorem 1.1.** *Let  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be a substantially dissipative Hénon map that admits a dominated splitting over the Julia set  $J$ . Then:*

- $f$  does not have wandering Fatou components;
- $f$  has only finitely many periodic Fatou components, each being either an attracting or a parabolic basin;
- $J = J^*$ , i.e., the Julia set is the closure of saddles.

Hénon maps with dominated splitting are 2D counterparts of 1D polynomial maps without critical points on the Julia set. Our initial observation was that for such a polynomial, the No Wandering Domains Theorem can be proven by means of Mañé’s techniques (refined in [CJY94] and [ST00]) treating maps with non-recurrent critical points. However, an adaptation of these techniques to the Hénon setting is not straightforward: in particular, it required imposing an assumption of substantial dissipativity, to develop an appropriate version of the  $\lambda$ -lemma, and to bound the iterated degrees of wandering components.

The main work in this paper is to show that, away from the parabolic cycles,  $f$  is expanding in the horizontal direction. (In particular, if there are no parabolic cycles then

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<sup>1</sup> One fine pending issue still unresolved for Hénon maps is whether Herman rings can exist.

$f$  is hyperbolic.) The non-existence of wandering Fatou components and periodic rotation domains follows easily, and it also follows that  $J = J^*$ .

Non-hyperbolic complex Hénon maps admitting a dominated splitting have been constructed by Radu and Tanase [RT19]. These examples are perturbations of 1D parabolic polynomials.

The structure of the paper is as follows. In Section 2 we will review Mañé's Theorem analysing the dynamics of 1D polynomials without recurrent critical points (except possible superattracting cycles). We give a detailed proof that follows ideas from a paper of Shishikura and Tan Lei [ST00]. However, we have chosen to present an argument that will be a closest possible model to the two-dimensional proof we give later. This means that the one-dimensional argument is not the most efficient. For instance, naturally we do not assume the non-existence of wandering components.

In Section 3 we recall Hénon maps and the substantial dissipativity condition, and in Section 4 we define the dominated splitting and make some elementary observations. The dominated splitting on  $J$  induces a lamination on  $J^+ \cap \Delta_R^2$  by vertical disks. In Section 5 this lamination is extended to a neighborhood of  $J^+$ , introducing the artificial vertical lamination. While this lamination is not invariant, it plays an important role in our proofs.

In wandering Fatou components we can consider both the artificial and the dynamical lamination given by strong stable manifolds. These two laminations may not agree on orbits of wandering domains that leave the region of dominated splitting, leading to interpretations of degree, discussed in Sections 6 and 7.

In Section 8 we make the final preparations and in Section 9 we prove the main technical result, Proposition 9.3. In Section 10 we prove the consequences of this proposition, including Theorem 1.1 above.

## 2. The one-dimensional argument

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a polynomial, and assume that there are no critical points on  $J$ , the Julia set of  $f$ . We let  $\Omega$  be a backward invariant open neighborhood of  $J \setminus \{\text{parabolics}\}$ , constructed by removing closed forward invariant sets from a finite number of (pre-) periodic Fatou components. We will assume that  $\Omega$  is arbitrarily thin, i.e. contained in an arbitrarily small neighborhood of the union of  $J \setminus \{\text{parabolics}\}$  with the wandering Fatou components.

The wandering Fatou components can contain only a finite number of critical points  $x_1, \dots, x_\nu$ , having respective local degrees  $d_1, \dots, d_\nu$ . We denote

$$\deg_{\text{crit}} = \prod_{s=1}^{\nu} d_s.$$

The constant  $\deg_{\text{crit}}$  functions as a maximal local degree on the wandering components for all iterates, i.e. if  $V^0, V^1, \dots, V^n$  is an orbit of open connected sets, and each  $V^n$  is contained in a wandering domain and has sufficiently small Euclidean diameter, then  $f^n : V^0 \rightarrow V^n$  has degree at most  $\deg_{\text{crit}}$ .

We will use the following shorthand notation. For a connected set  $V$ , we denote by  $V^{-j}$  a connected component of  $f^{-j}(V)$ . If  $V \subset W$  then we will always assume that  $V^{-j} \subset W^{-j}$ . When working with both  $V^{-i}$  and  $V^{-j}$ , for  $j > i$ , we will assume that  $f^{j-i}(V^{-j}) = V^{-i}$ , i.e.  $V^{-i}$  and  $V^{-j}$  are contained in the same backward orbit.

We will show that there exists  $M \in \mathbb{N}$  such that the following holds whenever  $\Omega$  is sufficiently thin.

**Proposition 2.1.** *Let  $z \in \Omega$  and let  $r > 0$  be such that  $D_{M \cdot r}(z) \subset \Omega$ . Then for every  $j \in \mathbb{N}$  we have*

$$\begin{aligned} \mathbf{deg}(j) : \quad \mathbf{deg}(f^j : D_r^{-j}(z) \rightarrow D_r(z)) &\leq \mathbf{deg}_{\text{crit}}, \\ \mathbf{diam}(j) : \quad \mathbf{diam}_\Omega D_r^{-j}(z) &\leq N_0(2K) \cdot C(1/2, \mathbf{deg}_{\text{crit}}). \end{aligned}$$

The constants  $C(\cdot, \cdot)$  and  $N_0(\cdot)$  will be introduced in Lemmas 2.2 and 2.4 below.

Our proof will closely resemble the proof of a theorem of Mañé, presented in [ST00]. In fact, readers familiar with this reference will likely find our proof needlessly complicated. The reason for these complications is that the proof given here will model the forthcoming 2-dimensional proof. In particular, we will *not* use the fact that there are no wandering domains. In fact, the non-existence of wandering domains follows, in our setting, from the above proposition. The a priori possibility of wandering domains makes the proof significantly more involved.

The fact that Fatou components of one-dimensional polynomials are simply connected is quite useful when dealing with degrees. It follows that if a Fatou component  $U$  does not contain critical points, then  $f : U \rightarrow f(U)$  is univalent. This is another fact that we will not be able to use in higher dimensions, so we will not use it here either. In this respect the setting is more analogous to the iteration of rational functions, where Fatou components may not be simply connected. An elementary proof however shows that for every wandering component  $U$  there exists an  $N \in \mathbb{N}$  such that  $f^n(U)$  is simply connected for  $n \geq N$ , a result known as Baker's Lemma (see for example [Zak]). Thus, if  $f^n(U)$  does not contain critical points, then  $f : f^n(U) \rightarrow f^{n+1}(U)$  is univalent. We can use the fact that  $f : f^n(U) \rightarrow f^{n+1}(U)$  is univalent for  $n$  large enough, because we will prove the corresponding statement for Hénon maps. Note that nowhere else in the one-dimensional argument will we use simple connectedness to conclude univalence.

A third difference from the argument in [ST00] concerns the induction procedure. Instead of applying the induction hypothesis to  $f^n : D_r^{-n}(z) \rightarrow D_r(z)$ , and then mapping backward one more step with  $f : D_r^{-n-1}(z) \rightarrow D_r^{-n}(z)$ , we will first apply one iterate  $f : D_r^{-1}(z) \rightarrow D_r(z)$ , cover  $D_r^{-1}(z)$  with smaller disks  $D_{r_k}(z_k)$ , and applying the induction hypothesis to each  $f^n : D_{r_k}^{-n}(z_k) \rightarrow D_{r_k}(z_k)$ . The reason for this will become apparent when the proof is discussed in the Hénon setting.

### 2.1. Preliminaries

The following basic lemma (see, e.g., [LM97, ST00]) will be used repeatedly:

**Lemma 2.2.** *Let  $d \in \mathbb{N}$  and  $r > 0$ . Then there exists a constant  $C(r, d) > 0$  such that for every proper holomorphic map  $f : \mathbb{D} \rightarrow \mathbb{D}$  of degree at most  $d$ , every connected component of  $f^{-1}D_r(0)$  has hyperbolic diameter at most  $C(r, d)$ . Moreover,  $C(r, d) \rightarrow 0$  as  $r \rightarrow 0$ .*

The first step in the proof is the construction of the backward invariant domain  $\Omega$  where the argument of 2.1 will take place.

**Lemma 2.3.** *Given any  $\epsilon > 0$ , there exists a backward invariant domain  $\Omega$  contained in the  $\epsilon$ -neighborhood of*

$$(J \setminus \{\text{parabolic cycles}\}) \cup \bigcup \{\text{wandering domains}\},$$

*which further has the property that for every  $z \in J$  there exists  $w \notin \Omega$  with  $|z - w| < \epsilon$ .*

*Proof.* We are done if we can remove sufficiently large forward invariant subsets from a finite number of (pre-) periodic Fatou components. By the classification of periodic Fatou components those periodic components are either attracting basins, parabolic basins or Siegel domains. In an immediate basin of an attracting periodic cycle we can construct an arbitrarily large forward invariant compact subset. In a cycle of Siegel domains we can find an arbitrarily large completely invariant compact subset. Finally, in a cycle of parabolic domains we can find an arbitrarily large forward invariant compact subset that intersects  $J$  only in parabolic fixed points. By taking the union of sufficiently large preimages of these subsets we obtain a forward invariant compact subset  $K$  disjoint from  $J \setminus \{\text{parabolic cycles}\}$ , for which  $\Omega = \mathbb{C} \setminus K$  satisfies the required conditions.  $\square$

The domain  $\Omega$  will later be fixed for a constant  $\epsilon > 0$  chosen sufficiently small. In particular, we may assume that the only critical points in  $\Omega$  lie in wandering Fatou components.

**Lemma 2.4.** *For each  $t > 1$  there exists an integer  $N_0 = N_0(t)$  such that for every disk  $D_r(z) \subset D_{t,r}(z) \subset \Omega$  we can cover  $D_r^{-1}(z)$  with at most  $N_0$  disks  $D_{r_k}(z_k)$  satisfying*

$$D_{2t \cdot r_k}(z_k) \subset \Omega.$$

*If  $D_{t,r}(z)$  is contained in a wandering domain  $U^0$ , then the disks  $D_{2t \cdot r_k}(z_k)$  can be chosen so that*

$$D_{2t \cdot r_k}(z_k) \subset U^{-1}.$$

*In all other cases the disks  $D_{2t \cdot r_k}(z_k)$  can be chosen so that*

$$D_{2t \cdot r_k}(z_k) \subset D_{t,r}^{-1}(z).$$

*Proof.* If  $D_r(z)$  is sufficiently small and close to a critical value, it must be contained in a wandering Fatou component  $U$ . The statement follows immediately.

For sufficiently small disks bounded away from critical values the existence of a uniform bound is clear, as the map  $f : D_{t,r}^{-1}(z) \rightarrow D_{t,r}(z)$  is close to linear. Therefore it is sufficient to consider disks  $D_r(z) \subset D_{t,r}(z)$  of radius bounded away from zero. This is a compact family of disks, hence the existence of a uniform  $N_0$  is immediate.  $\square$

We note that while  $N_0(t)$  will play a similar role to the constant  $N_0$  in [ST00], its definition differs as the disks  $D_{r_k}(z_k)$  cover a preimage  $D_r^{-1}(z)$  instead of the original  $D_r(z)$ . In particular, the constant  $N_0$  from [ST00] is universal, while the constants  $N_0(t)$  introduced here depend on  $f$ .

**Remark 2.5.** Since the disks  $D_{2t \cdot r_k}(z)$  are contained in either a wandering domain  $U^{-1}$  or in the preimage  $D_{t \cdot r}^{-1}(z)$ , it follows that the constant  $N_0(t)$  does not depend on  $\Omega$ . To be more precise, when  $\Omega$  is made smaller, the constant  $N_0(t)$  does not need to be changed.

The domain  $\Omega$  is a Riemann surface whose universal cover is the unit disk, hence  $\Omega$  is equipped with a Poincaré metric  $d_\Omega$ . We will prove that all inverse branches  $D_r^{-i}(z)$  of *sufficiently protected* disks, i.e. disks  $D_r(z) \subset D_{2K \cdot r}(z) \subset \Omega$  for a sufficiently large constant  $K$  to be determined later, have Poincaré diameter bounded by

$$\text{diam}_{\max} := N_0(2K) \cdot C(1/2, \text{deg}_{\text{crit}}).$$

By choosing  $\Omega$  sufficiently thin, i.e. the constant  $\epsilon$  in Lemma 2.3 sufficiently small, it follows that the Euclidean diameter of each  $D_r^{-i}(z)$  is arbitrarily small, *unless*  $D_r^{-i}(z)$  is contained in a wandering Fatou component. In that case we cannot control the Euclidean diameter of  $D_r^{-i}(z)$  by making  $\Omega$  thinner.

If  $D_r^{-i}(z)$  does have sufficiently small Euclidean diameter, and is bounded away from the critical values, then it follows that the map  $f : D_r^{-i-1}(z) \rightarrow D_r^{-i}(z)$  is univalent. Notice that simple connectedness is not used here.

By choosing  $\Omega$  sufficiently thin we can guarantee that there are only finitely many wandering domains for which the bound on the Poincaré diameter domain does not imply the necessary bound on the Euclidean diameter.

**Definition 2.6** (Domain with hole). We say that a wandering domain  $U$  is a *domain with hole* if there exists a domain  $V \subset U$  with

$$\text{diam}_\Omega V \leq N_0(2) \cdot C(1/2, \text{deg}_{\text{crit}})$$

for which  $f : V^{-1} \rightarrow V$  is not univalent. Note that in particular any wandering domain that contains a critical value is a wandering domain with hole.

Since the wandering domains are all disjoint and contained in a bounded region, it follows from Lemma 2.3 that if  $\Omega$  is chosen sufficiently thin, then there are only finitely many wandering domains with hole.

**Definition 2.7** (Critical wandering domain). Given a bi-infinite orbit  $(U^j)$  of wandering components, we say that a wandering domain  $U^j$  is *critical* if  $U^{j+1}$  is a wandering component with hole, but  $U^i$  is not for  $i \leq j$ .

Since there are only finitely many domains with hole, there are also only finitely many critical domains. We say that a wandering domain is *post-critical* if it is contained in the forward orbit of a critical domain, and *regular* if it is not. Thus wandering domains in a grand orbit that does not contain critical components are all called regular.

**Definition 2.8** ( $\text{deg}_{\text{max}}$ ). Since there are only finitely many critical domains, and Baker’s Lemma implies that for each orbit  $(U^n)_{n \in \mathbb{Z}}$  of wandering domains the map  $f : U^n \rightarrow U^{n+1}$  is univalent for  $n$  sufficiently large, it follows that there exists an upper bound on the degree of all maps  $f^n : U^0 \rightarrow U^n$  for  $U^0$  critical. We denote this upper bound by  $\text{deg}_{\text{max}}$ .

2.2. *Disks deeply contained in wandering domains*

Let us write  $(U^n)_{n \in \mathbb{Z}}$  for a bi-infinite orbit of wandering Fatou components, i.e.  $f(U^n) = U^{n+1}$ . We will separate several distinct cases. The simplest case occurs when  $D_r(z)$  is contained in what we have called a regular component.

**Lemma 2.9.** *Let  $U$  be a regular wandering component, and consider a protected disk  $D_r(z) \subset D_{2r} \subset U$ . Then*

$$\text{deg}(f^j : D_r^{-j}(z) \rightarrow D_r(z)) = 1$$

and

$$\text{diam}_{\Omega} D_r^{-j}(z) \leq N_0(2) \cdot C(1/2, 1)$$

for all  $j \geq 0$ .

*Proof.* The proof is by induction on  $j$ . Suppose that the statement holds for a certain  $j$ . We can cover  $D_r^{-1}(z)$  with at most  $N_0(2)$  disks  $D_{r_k}(z_k)$  satisfying

$$D_{4r_k}(z_k) \subset D_{2r}^{-1}(z) \subset U^{-1}.$$

Hence we can apply the induction hypothesis for each of the disks  $D_{2r_k}(z_k) \subset D_{4r_k}(z_k)$ , obtaining

$$\text{deg}(f^j : D_{2r_k}^{-j}(z) \rightarrow D_{2r(z)}) = 1,$$

and thus

$$\text{diam}_{\Omega} D_{r_k}^{-i}(z_k) \leq C(1/2, 1)$$

for  $i = 0, \dots, j$ . We claim that for each  $i = 0, \dots, j$  we obtain

$$\text{diam}_{\Omega} D_r^{-i}(z) \leq N_0(2) \cdot C(1/2, 1),$$

and

$$\text{deg}(f^i : D_r^{-i}(z) \rightarrow D_r(z)) = 1.$$

The proof is by induction on  $i$ . Both statements are immediate for  $i = 0$ . Suppose that the statements hold for  $0, \dots, i$ . Since  $U$  is regular, the hyperbolic diameter bound on  $D_r^{-i}(z)$  implies that  $f : D_r^{-i-1}(z) \rightarrow D_r^{-i}(z)$  is univalent. Thus  $D_r^{-i-1}(z)$  is covered by at most  $N_0(2)$  sets  $D_{r_k}^{-i}(z_k)$ . The diameter bound for  $i + 1$  follows, completing the induction step.  $\square$

We now consider the case when  $D_r(z)$  is contained in a wandering domain  $U^n$  that may be post-critical. By renumbering we may assume that  $U^0$  is the critical component. The

definition of  $\text{deg}_{\text{max}}$  immediately gives diameter bounds for preimages of protected disks  $D_r(z) \subset D_{K \cdot r}(z) \subset U^n$ , namely

$$\text{diam}_{\Omega} D_r^{-j}(z) \leq C(1/K, \text{deg}_{\text{max}})$$

for  $j \leq n$ . We obtain the following consequence.

**Lemma 2.10.** *We can choose  $K \in \mathbb{N}$ , independent of the wandering domain  $U^n$ , so that for any  $D_r(z) \subset D_{K \cdot r}(z) \subset U^n$  one has*

$$\text{deg}(f^j : D_r^{-j}(z) \rightarrow D_r(z)) \leq \text{deg}_{\text{crit}}$$

and

$$\text{diam}_{\Omega} D_r^{-j}(z) \leq N_0(2) \cdot C(1/2, \text{deg}_{\text{crit}})$$

for all  $j \in \mathbb{N}$ .

*Proof.* By the previous lemma, we have obtained the required estimates for  $K = 2$  in regular components, i.e. when  $n \leq 0$ .

Let  $n > 0$  and first consider  $j \leq n$ . Since the degree of  $f^j : U^{n-j} \rightarrow U^n$  is bounded by  $\text{deg}_{\text{max}}$  and the disk  $D_{K \cdot r}(z)$  is assumed to lie in  $U^n$  it follows that

$$\text{diam}_{\Omega} D_r^{-j}(z) < C(1/K, \text{deg}_{\text{max}}).$$

The required diameter and degree bounds follow when  $K$  is chosen sufficiently large.

When  $j > n$  we cannot assume a uniform degree bound on the maps  $f^j : U^{n-j} \rightarrow U^{-n}$ . However, since the domain  $U^0$  is simply connected, there is a universal bound from below on the Poincaré distance from the point  $w \in U^0$  to the circle centered at  $w$  of radius  $\frac{1}{2}d(w, \partial U^0)$ . Choosing  $K$  such that  $\text{diam}_{U^0} D_r^{-n}(z)$  is strictly smaller than this universal bound implies that  $D_r^{-n}(z)$  is contained in a disk  $D_{\rho}(w)$  for which  $D_{2\rho}(w) \subset U^0$ . The statement of the previous lemma completes the proof.  $\square$

### 2.3. Disks that are not deeply contained

We start with two definitions.

**The constant  $N_1$ .** Write  $w = K \cdot r$ . Then there exists a constant  $N_1 \in \mathbb{N}$ , which depends only on  $K$ , such that the interval  $[0, w - r/2]$  can be covered by  $N_1$  open disks  $D_{s_v}(w_v)$  satisfying  $\overline{D_{K \cdot s_v}(w_v)} \subset D_{K \cdot r}(0)$ .

**Choice of  $\Omega$ .** We now choose  $\Omega$  sufficiently thin such that the following property is satisfied: Suppose that  $z \in \Omega$  is either not contained in a wandering Fatou component, or is contained in a wandering domain  $U$  but its hyperbolic distance to  $\partial U$  is at most  $\text{diam}_{\Omega} D_{s_v}^{-i}(w_v) \leq N_0(2) \cdot C(1/2, \text{deg}_{\text{crit}})$ . Then the hyperbolic distance between two distinct preimages of  $z$  is at least  $2N_0(2K) \cdot C(1/2, \text{deg}_{\text{crit}})$ .

We restate and prove the main one-dimensional result using the constant  $M = 2K$ .

**Proposition 2.1.** *Let  $z \in \Omega$  and let  $r > 0$  be such that  $D_{2K \cdot r}(z) \subset \Omega$ . Then for every  $j \in \mathbb{N}$  we have*

$$\begin{aligned} \mathbf{deg}(j) : \quad & \text{deg}(f^j : D_r^{-j}(z) \rightarrow D_r(z)) \leq \text{deg}_{\text{crit}}, \\ \mathbf{diam}(j) : \quad & \text{diam}_{\Omega} D_r^{-j}(z) \leq \text{diam}_{\text{max}}. \end{aligned}$$



*Proof.* We assume the statement holds for given  $j$ .

Recall that we have already proved the proposition, with a stronger diameter estimate, in the case where  $D_{K \cdot r}(z)$  is contained in a wandering Fatou component. Thus, we are left with two possibilities: either  $D_r(z)$  is not contained in a wandering Fatou component, or  $D_r(z)$  is contained in a wandering component  $U^n$  but  $D_{K \cdot r}(z)$  is not. We will first prove the induction step for the former case, and then use the conclusion in the proof of the latter case.

Suppose  $z_0 \in D_r(z) \subset D_{2Kr}(z)$  is not contained in a wandering component. Then, by making  $\Omega$  sufficiently thin, it follows that any backward image of  $z_0$  can be assumed to lie arbitrarily close to  $\partial\Omega$ .

Cover  $D_r^{-1}(z)$  by at most  $N_0(2K)$  disks  $D_{r_k}(z_k)$  for which  $D_{4Kr_k}(z_k) \subset D_{2Kr}^{-1}(z)$ . The induction hypothesis gives that

$$\deg(f^j : D_{2r_k}^{-j}(z_k) \rightarrow D_{2r_k}(z_k)) \leq \deg_{\text{crit}},$$

and hence

$$\text{diam}_\Omega D_{r_k}^{-j}(z_k) \leq C(1/2, \deg_{\text{crit}}).$$

We claim that for all  $i = 0, \dots, j$  we have

$$\text{diam}_\Omega D_r^{-i-1}(z) \leq N_0(2K) \cdot C(1/2, \deg_{\text{crit}})$$

and

$$\deg(f^{i+1} : D_r^{-i-1}(z) \rightarrow D_r(z)) = 1,$$

which is proved by induction on  $i$ . The base case  $i = 0$  is immediate, so let us assume these two estimates hold for a certain  $i - 1 < j$ . By the second estimate,  $D_r^{-i}(z)$  is covered by at most  $N_0(2K)$  holomorphic disks  $D_{2r_k}^{-i+1}(z_k)$ , and one of them contains the unique preimage of  $z_0$ , denoted by  $z_0^{-i}$ . Each holomorphic disk  $D_{2r_k}^{-i+1}(z_k)$  has  $d$  preimages, whose hyperbolic diameters in  $\Omega$  are bounded by  $C(1/2, \deg_{\text{crit}})$ . By our choice of  $\Omega$  the  $d$  preimages of  $z_0^{-i}$  have pairwise distances strictly larger than  $2N_0(2K) \cdot C(1/2, \deg_{\text{crit}})$ , hence the  $dN_0(2K)$  preimages of all the disks  $D_{2r_k}^{-i+1}(z_k)$  are grouped into disjoint collections of  $N_0(2K)$  images, which proves the two estimates for  $i$ . The case  $i = j$  completes the proof in the case where  $D_r(z)$  is not contained in a wandering domain.

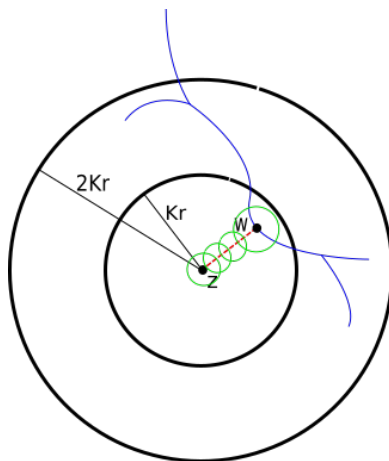
In the remainder of this proof we will therefore assume that  $D_r(z)$  is contained in a Fatou component  $U^n$ , but the larger disk  $D_{Kr}(z)$  is not. Let  $w \in \partial U^n$  be such that  $|z - w|$  is minimal, and write  $[z, w] \subset \mathbb{C}$  for the closed interval (see Figure 1).

Since  $D_{Kr}(w) \subset D_{2Kr}(z)$  it follows that  $D_{Kr}(w) \subset \Omega$ . Hence the disk  $D_{r/2}(w)$  satisfies the conditions of the previously discussed case, and we obtain the estimates

$$\text{diam}_\Omega D_{r/2}^{-i}(w) \leq N_0(2K) \cdot C(1/2, \deg_{\text{crit}})$$

for  $i \leq j + 1$ . By our choice of  $N_1$  the interval  $[z, w]$  can be covered by the disk  $D_{r/2}(w)$  and a bounded number of disks  $D_{s_1}(w_1), \dots, D_{s_{N_1}}(w_{N_1})$  satisfying  $D_{Ks_\nu}(w_\nu) \subset U^n$ . Thus the disks  $D_{s_\nu}(w_\nu)$  satisfy the conditions of Lemma 2.10, and it follows that

$$\text{diam}_\Omega D_{s_\nu}^{-i}(w_\nu) \leq N_0(2) \cdot C(1/2, \deg_{\text{crit}})$$



**Fig. 1.** The interval  $[z, w]$  covered by  $N_1$  smaller disks.

for all  $i \in \mathbb{N}$ . It follows that the hyperbolic distance from each  $f^{-i}(z)$  to  $\partial U^{n-i}$  is bounded by

$$N_1 \cdot N_0(2) \cdot C(1/2, \deg_{\text{crit}}) + N_0(2K) \cdot C(1/2, \deg_{\text{crit}}).$$

By our choice of  $\Omega$  it follows that any two preimages of  $F^{-i}(z)$  have hyperbolic distance at least  $2N_0(2K) \cdot C(1/2, \deg_{\text{crit}})$ . We can conclude the argument by using the same induction on  $i$  as in the previously discussed cases.  $\square$

#### 2.4. Consequences

The degree and diameter estimates obtained imply a number of consequences. The first is the non-existence of wandering domains.

**Lemma 2.11.** *There are no wandering Fatou components.*

*Proof.* Suppose  $U$  is a wandering Fatou component. We can construct the domain  $\Omega$  as above sufficiently thin so that the Poincaré diameter of  $U$  can be made arbitrarily large. In particular, we can find a relatively compact  $L \subset U$  whose Poincaré diameter in  $\Omega$  is strictly larger than  $\text{diam}_{\max}$ . Let  $n_j$  be such that  $f^{n_j}(L)$  converges to a point  $p \in J$ . Without loss of generality we may assume that  $p$  does not lie in a parabolic cycle. Let  $D_r(p)$  be such that  $D_{2K \cdot r}(p) \subset \Omega$ . Then  $f^{n_j}(L) \subset D_r(p)$  for sufficiently large  $j$ , which by Proposition 2.1 implies that  $\text{diam}(L) < \text{diam}_{\max}$ , giving a contradiction.  $\square$

In the absence of wandering domains the proof of Proposition 2.1 becomes considerably simpler. An immediate consequence is the following.

**Corollary 2.12.** *The constant  $\deg_{\text{crit}}$  can be taken equal to 1, and the constant  $M$  equal to 2.*

**Proposition 2.13.** *Suppose  $z \in J$  is not contained in a parabolic cycle, choose  $r > 0$  so that  $D_r(z) \subset D_{2r}(z) \subset \Omega$ , and let  $V^1, V^2, \dots$  be such that  $f(V^1) = D_r$  and  $f(V^{n+1}) = V^n$ . Then*

$$\text{diam}_\Omega V^n \rightarrow 0.$$

*Proof.* By Corollary 2.12 the maps  $f^n : V^n \rightarrow D_r$  are all univalent, hence we can consider the inverse branches  $(f^n)^{-1} : D_r \rightarrow V^n$ . These inverse branches form a bounded, and thus normal, family of holomorphic maps. Let  $h$  be a limit map. Since  $z \in J$  and  $J$  is invariant, the image  $h(D_r)$  must contain a point  $q$  in  $J$ . But since any neighborhood of  $q$  contains points in the basin of infinity,  $h(D_r)$  cannot contain an open neighborhood of  $q$ , and hence  $h$  is constant. □

The non-existence of rotation domains is an immediate consequence.

**Corollary 2.14.** *The polynomial  $f$  does not have any Siegel disks.*

**Corollary 2.15.** *If  $f$  does not have any parabolic cycles then  $f$  is hyperbolic.*

*Proof.* When  $f$  lacks parabolic cycles the set  $J$  is contained in  $\Omega$ . Since  $J$  is compact it follows from Proposition 2.13 that there exist  $N \in \mathbb{N}$  and  $r > 0$  such for any  $z \in D$  and any connected component  $V^N$  of  $(f^N)^{-1}(D_r(z))$  the Euclidean diameter of  $V^N$  is less than  $r/2$ . The Schwarz Lemma therefore implies that  $|(f^N)'| \geq 2$  on  $J$ . □

**Corollary 2.16.** *The polynomial  $f$  lies on the boundary of a hyperbolic component.*

*Proof.* The Fatou set of  $f$  consists of finitely many attracting and/or parabolic basins. By [DH85b], and similarly [Shi87] for rational functions, it follows that a one-parameter family  $f_t$  of perturbations of the polynomial  $f$  can be found with attracting cycles of  $f_t$  close to each of the parabolic cycles of  $f$ . Moreover, for each parabolic cycle  $z_0, \dots, z_p = z_0$  of  $f$ , and each orbit of parabolic petals  $\mathcal{P}_0, \dots, \mathcal{P}_p = \mathcal{P}_0$ , forward invariant in the sense that  $f(\mathcal{P}_j) \subset \mathcal{P}_{j+1}$ , there exist open sets  $V_0, \dots, V_p = V_0$ , with each  $V_j$  arbitrarily close to  $\mathcal{P}_j$ , such that  $f_t$  maps  $V_j$  relatively compactly into  $V_{j+1}$ . By taking sufficiently large inverse images of the collections  $\{V_0, \dots, V_n\}$ , it follows that for any compact subset  $K$  of the Fatou set of  $f$ , there exist for  $f_t$  sufficiently close to  $f$  a forward invariant open set  $V$  in the Fatou set of  $f_t$  which contains  $K$ , and whose image is relatively compact in  $V$ .

Thus the Julia set of  $f_t$  is contained in a small neighborhood of the Julia set of  $f$ . Moreover, there exists an arbitrarily small open neighborhood  $\Omega$  of the Julia set of  $f_t$  which is backward invariant and does not contain any critical points. It follows that  $f_t : f_t^{-1}(\Omega) \rightarrow \Omega$  is expanding in the hyperbolic metric of  $\Omega$ , implying hyperbolicity of  $f_t$ . □

### 3. Hénon maps: Background and preliminaries

Recall from [FM89] that either the dynamical behavior of a polynomial automorphism of  $\mathbb{C}^2$  is dynamically trivial, or the automorphism is conjugate to a finite composition of maps of the form

$$(x, y) \mapsto (p(x) + b \cdot y, x),$$

where  $p$  is a polynomial of degree at least 2 and  $b \neq 0$ . We will refer to such compositions as Hénon maps. Given  $R > 0$  we define the following sets.

$$\Delta_R^2 := \{(x, y) : |x|, |y| < R\},$$

$$V^+ := \{(x, y) : |x| \geq \max(R, |y|)\}, \quad V^- := \{(x, y) : |y| \geq \max(R, |x|)\}.$$

By choosing  $R$  sufficiently large we can ensure that  $f(V^+) \subset V^+$ ,  $f^{-1}(V^-) \subset V^-$  and  $f(\Delta_R^2) \subset \Delta_R^2 \cup V^+$ . One can also guarantee that if  $(x_0, y_0) \in V^+$  and  $(x_1, y_1) = f(x_0, y_0)$  then  $|x_1| > 2|x_0|$ . Similarly one obtains  $|y_{-1}| > 2|y_0|$  for  $(x_0, y_0) \in V^-$ . It follows that every orbit that lands in  $V^+$  must escape to infinity, and every orbit that does not converge to infinity must eventually land in  $\Delta_R^2$  in a finite number of steps.

We write  $K^+$  for the set with bounded forward orbits,  $K^-$  for the set with bounded backward orbits, and  $K = K^+ \cap K^-$ . As usual we define the *forward* and *backward Julia sets* as  $J^\pm = \partial K^\pm$ , and the *Julia set* as  $J = J^+ \cap J^-$ . Let us recall the existence of the Green’s currents  $T^+$  and  $T^-$ , supported on  $J^+$  and  $J^-$ , and the equilibrium measure  $\mu = T^+ \wedge T^-$ , whose support  $J^*$  is contained in  $J$ . Whether  $J^*$  always equals  $J$  is one of the main open questions in the area, and was previously only known for hyperbolic Hénon maps [BS91a].

### 3.1. Wiman Theorem and substantially dissipative Hénon maps

Recall that a subharmonic function  $g : \mathbb{C} \rightarrow \mathbb{R}$  is said to have *order of growth at most  $\rho$*  if

$$g(\zeta) = O(|\zeta|^\rho) \quad \text{as } \zeta \rightarrow \infty.$$

Given  $E \in \mathbb{R}$ , let us call the set  $\{g < E\}$  *subpotential* (of level  $E$ ), and its components *subpotential components*.

**Theorem 3.1** (Wiman [W19]). *Let  $g$  be a non-constant subharmonic function with order of growth strictly less than  $1/2$ . Then subpotential components of any level  $E$  are bounded.*

Let us describe how it was used in the setting of Hénon maps in recent works of the first author and Dujardin [DL15], and in [LP14]. Suppose that  $p \in \mathbb{C}^2$  is a hyperbolic fixed point, and let  $W^s(p)$  be its stable manifold, corresponding to the stable eigenvalue  $\lambda$ . Then there exists a linearization map  $\varphi : \mathbb{C} \rightarrow W^s(p)$  satisfying  $\varphi(\lambda \cdot \zeta) = f(\varphi(\zeta))$ . As usual we let  $G^\pm$  be the plurisubharmonic functions defined by

$$G^\pm(z) = \lim_{n \rightarrow \infty} \frac{1}{(\deg f)^n} \cdot \log^+ \|f^{\pm n}(z)\|.$$

We have the functional equations

$$G^\pm(f(z)) = (\deg f)^{\pm 1} \cdot G^\pm(z).$$

Combining the functional equations for  $G^-$  and  $\phi$  we find that the non-constant subharmonic function  $g = G^- \circ \phi$  satisfies

$$g(\lambda \cdot \zeta) = G^- \circ f \circ \varphi(\zeta) = \frac{1}{\deg f} g(\zeta).$$

Note that  $|\lambda| < |\text{Jac } f|$ . Hence under the assumption that  $|\text{Jac } f| < \frac{1}{\deg f^2}$  it follows that  $g$  is a subharmonic function with order of growth strictly less than  $1/2$ , and therefore according to Wiman’s Theorem all its subpotential components are bounded.

Let us point out that the above discussion also holds when  $p$  is a neutral fixed point, i.e. having one neutral and one attracting multiplier. One considers the strong stable manifold with corresponding eigenvalue  $\lambda$ , satisfying  $|\lambda| = |\text{Jac } f|$ . The subharmonic function  $g$  still has order of growth strictly less than  $1/2$ . The idea can also be applied when  $p$  is not a periodic point but lies in an invariant hyperbolic set, or under the assumption of a dominated splitting, which will be discussed in the next section.

A particular consequence of Wiman’s Theorem is that all connected components of intersections of (strong) stable manifolds with  $\Delta_R^2$  are bounded in the linearization coordinates. By the Maximum Principle they are also simply connected, hence they are disks. The filtration property of Hénon maps tells us that every connected component of the intersection of a stable manifold with  $\Delta_R^2$  is actually a branched cover over the vertical disk  $\Delta_w(R)$ . This will be used heavily in what follows.

### 3.2. Classification of periodic components

3.2.1. *Ordinary components.* We recall the classification of periodic Fatou components  $U$  from [LP14], building upon results of Bedford and Smillie [BS91b]. For a dissipative Hénon map  $f$ , there exist three types of *ordinary* invariant<sup>2</sup> Fatou components  $U$ :

- (i) *Attracting basin:* All orbits in  $U$  converge to an attracting fixed point  $p \in U$ . Moreover,  $U$  is a *Fatou–Bieberbach domain* (i.e., it is biholomorphically equivalent to  $\mathbb{C}^2$ ).
- (ii) *Rotation basin:* All orbits in  $U$  converge to a properly embedded Riemann surface  $\Sigma \subset U$ , which is invariant under  $f$  and biholomorphically equivalent to either an annulus or the unit disk. The biholomorphism can be chosen so that it conjugates the action of  $f|_\Sigma$  to an irrational rotation. The stable manifolds through points in  $\Sigma$  are all embedded complex lines, and the domain  $U$  is biholomorphically equivalent to  $\Sigma \times \mathbb{C}$ .
- (iii) *Parabolic basin:* All orbits in  $U$  converge to a parabolic fixed point  $p \in \partial U$  with the neutral eigenvalue equal to 1. Moreover,  $U$  is a Fatou–Bieberbach domain.

In the literature a periodic point whose multipliers  $\lambda_1$  and  $\lambda_2$  satisfying  $|\lambda_1| < 1$  and  $\lambda_2 = 1$  may be called either semi-parabolic or semi-attracting, depending on context. Since we are working with dissipative Hénon maps, where there is always at least one attracting multiplier, we chose to refer to these points as *parabolic*, and we use analogous terminology for Fatou components. Similarly, we will call a periodic point with one neutral multiplier *neutral*.

In each case, we let  $A = A_U$  be the *attractor* of the corresponding component (i.e., the attracting or parabolic point  $p$ , or the rotational curve  $\Sigma$ ).

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<sup>2</sup> A description of periodic components readily follows. Note also that since  $f$  is invertible, there is no such thing as a “preperiodic” Fatou component.

Along with global Fatou components  $U$ , we will consider *semilocal* ones, which are components  $U^i$  of the intersection  $U \cap \Delta_R^2$ . (Usually there are infinitely many of them.) Each  $U^i$  is mapped under  $f$  into some component  $U^j$ ,  $j = j(i)$ , with “vertical” boundary  $\partial U^i \cap \Delta_R^2$  being mapped into the vertical boundary of  $U^j$ , but the correspondence  $i \mapsto j(i)$  is not in general injective. This dynamical tree of semilocal components resembles closely the one-dimensional picture. In particular, cycles of ordinary semilocal components can be viewed as the *immediate basins* of the corresponding attractors  $A_U$ .

**3.2.2. Absorbing domains.** Given a compact subset  $Q \subset U$ , let us say that an invariant domain  $D \subset U$  is  $Q$ -*absorbing* if there exists  $n \in \mathbb{N}$  such that  $f^n(Q) \subset D$ . If this happens for any  $Q$  (with  $n$  depending on  $Q$ ) then  $D$  is called *absorbing*.

For instance, in the attracting case, any forward invariant neighborhood of the attracting point is absorbing. In the parabolic case, there exists an arbitrary small absorbing “attracting petal”  $P \subset U$  with  $\partial P \cap \partial U = \{p\}$  (see [BSU17]). To construct a  $Q$ -absorbing domain in the rotation case, take a sufficiently large invariant subdomain  $\Sigma' \subset \Sigma$  compactly contained in  $\Sigma$  and let

$$D = \bigcup_{z \in \Sigma'} W_{\text{loc}}^s(z).$$

This implies:

**Lemma 3.2.** *Let  $f$  be a dissipative Hénon map, and let  $U$  be an ordinary invariant Fatou component with an attractor  $A$ . Then any compact set  $Q \subset U$  is contained in a forward invariant domain  $W \subset U$  such that  $\overline{W} \cap \partial U \subset A$ . (In particular,  $W$  is relatively compactly contained in  $U$  in the attracting or rotation cases.)*

Let us say that a subset  $\Omega \subset \Delta_R^2$  is *relatively backward invariant* if

$$f^{-1}(\Omega) \cap \Delta_R^2 \subset \Omega.$$

**Corollary 3.3.** *Given any compact set  $Q$  contained in the union of periodic Fatou components, there exists an open and relatively backward invariant subset  $\Omega$  of  $\Delta_R^2$  containing*

$$\Delta_R^2 \cap (\text{wandering components} \cup J^+ \setminus \overline{\{\text{parabolic cycles}\}})$$

*and avoiding  $Q$ .*

*Proof.* There exist only finitely many periodic Fatou components  $U^i$  intersecting  $Q$ . For each of them, let  $W_i$  be the neighborhood of  $Q \cap U^i$  from Lemma 3.2. Note that  $J^+ \cap \bigcup \overline{W_i}$  is contained in a finite number of parabolic cycles. Take now a small  $\varepsilon > 0$  and let

$$\Omega = \Delta_R^2 \cap \{G^+ < \varepsilon\} \setminus \bigcup \overline{W_i}. \quad \square$$

**3.2.3. Substantially dissipative maps**

**Theorem 3.4** ([LP14]). *For a substantially dissipative Hénon map, any periodic Fatou component is ordinary.*

**Remark 3.5.** In fact, for Hénon maps with dominated splitting, this classification holds without assuming that dissipation is substantial: see Proposition 4.2 below.

### 4. Dominated splitting

4.0.1. *Definition.* We say that a Hénon map  $f$  admits a *dominated splitting* if there is an invariant splitting of the tangent bundle on  $J = J^+ \cap J^-$

$$T_J(\mathbb{C}^2) = E^s \oplus E^c \tag{1}$$

with constants  $0 < \rho < 1$  and  $C > 0$  such that for every  $p \in J$  and any unit vectors  $v \in E_p^s$  and  $w \in E_p^c$  one has

$$\frac{\|df^n v\|}{\|df^n w\|} < C \cdot \rho^n.$$

From now on we will assume that  $f$  is dissipative, from which it immediately follows that  $E^s$  is stable. We cannot conclude that  $E^c$  is unstable, though.

4.0.2. *Cone fields.* Let  $p \in J$  and let  $v \in E_p^s$  have unit length. Given  $0 < \alpha < 1$  we can define the cone

$$C_p^s(\alpha) := \{w \in T_p(\mathbb{C}^2) : |\langle w, v \rangle| \geq \alpha \|w\|\}.$$

It follows from the dominated splitting that we can choose  $\alpha$  continuously, depending on  $p \in J$ , so that

$$df C_p^s(\alpha(p)) \supset C_{f(p)}^s(r \cdot \alpha(f(p)))$$

for some  $r < 1$  which can be chosen independently of  $p \in J$ . We refer to the collection of these cones as the (backward) invariant *vertical cone field* on  $J$ .

Since both  $E_p^s$  and  $\alpha$  vary continuously with  $p$ , and the set  $J$  is closed, we can extend the vertical cone field continuously to  $\Delta_R^2$ . It follows automatically that the extension is backward invariant for points lying in a sufficiently small neighborhood  $\mathcal{N}(J)$ .

Note that all accumulation points of the forward orbit of a point in  $J^+$  must lie in  $K^- = J^-$ , and therefore in  $J = J^+ \cap J^-$ . Writing  $J_R^+ = J^+ \cap \Delta_R^2$  as before, it follows from compactness that there exists an  $N \in \mathbb{N}$  such that  $f^n(J_R^+) \subset \mathcal{N}(J)$  for all  $n \geq N$ . Thus we can pull back the vertical cone field to obtain a backward invariant cone field on a neighborhood of  $J_R^+$ . We will denote this neighborhood by  $\mathcal{N}(J_R^+)$ , and refer to it as the *region of dominated splitting*.

4.0.3. *Strong stable manifolds.* Let us consider the following completely invariant set:

$$\mathcal{V}^+ := \{p : \exists n_0 = n_0(p) \ f^n p \in \mathcal{N}(J) \text{ for } n \geq n_0\}. \tag{2}$$

Let  $U(p)$  be a small ball centered at  $p$ . For  $p \in \mathcal{V}^+$ , consider a straight complex line through  $f^n(p)$  whose tangent space at  $f^n(p)$  is contained in the vertical cone, and pull back this line by  $f^n$ , keeping only the connected component through  $p$  in the neighborhood  $U(p)$ . By the standard graph transform method, this sequence of holomorphic disks converges to a complex submanifold  $W_{loc}^s(p)$ , the so-called *local strong stable manifold* through  $p$ . By pulling back the local stable manifolds through  $f^n(p)$  by  $f^{-n}$  we obtain in the limit the *global strong stable manifold* through  $p$ , denoted by  $W^s(p)$ .

In line with our earlier introduced notation we will write  $W_R^s(p)$  for the connected component of  $W^s(p) \cap \Delta_R^2$  that contains  $p$ , and refer to it as a *semilocal stable manifold*.

**Definition 4.1.** We will refer to the collection of semilocal strong stable manifolds through points whose forward orbit is entirely contained in the domain of dominated splitting as the *(semilocal) dynamical vertical lamination*.

For  $p \in \mathcal{V}^+$ , we let  $E_p^s$  be the tangent line to  $W^s(p)$ , which can also be constructed directly as

$$E_p^s = \bigcap_{n \geq 0} df^{-n} C_{f^n p}^s(\alpha).$$

The lines  $E_p^s$  form the *stable line field* over  $\mathcal{V}^+$ , extending the initial stable line field from (1) over  $J$ .

Similarly to  $\mathcal{V}^+$  we can consider

$$\mathcal{V}^- := \{p : \exists n_0 = n_0(p) \ f^{-n} p \in \mathcal{N}(J) \text{ for } n \geq n_0\}. \quad (3)$$

For  $p \in \mathcal{V}^-$  we cannot guarantee the existence of a horizontal center manifold, but there does exist a unique *central line field*, i.e. a tangent subspace whose pullback under  $f^n$  is contained in the horizontal cone field for all  $n \geq n_0$ . For points  $p \in \mathcal{V}^+ \cap \mathcal{V}^-$  we can consider both the vertical and the central line field. Tangencies between those two line fields play the role of critical orbits. By the dominated splitting these can only occur for orbits that leave and come back to the domain of dominated splitting. A major part of this paper is aimed at obtaining a better understanding of such tangencies.

*4.0.4. Linearization coordinates.* The global strong stable manifolds  $W^s(p)$  of points  $p$  in the dynamical vertical lamination can be uniformized as follows. Denote by  $\pi_p : W^s(p) \rightarrow T_p(W^s(p))$  the projection to the tangent plane. The projection is locally a biholomorphism, as local stable manifolds are graphs over the tangent plane. The size of the local stable manifolds can be taken uniform over all  $p$  in the dynamical vertical lamination. Define  $\varphi_p : W^s(p) \rightarrow T_p(W^s(p))$  by

$$\varphi_p = \lim_{n \rightarrow \infty} [Df^n(p)]^{-1} \circ \pi_{f^n(p)} \circ f^n.$$

Identifying the tangent plane with  $\mathbb{C}$  we can view  $\varphi_p$  as a biholomorphic map from  $W^s(p)$  to  $\mathbb{C}$ . This identification is canonical up to a choice of argument. The identifications can locally be chosen to vary continuously with  $p$ . As the tangent planes to the dynamical vertical lamination vary continuously with  $p$ , and the above convergence to  $\varphi_p$  is uniform over  $p$  in the dynamical vertical lamination, one can locally obtain a continuous family of linearization maps  $\varphi : W^s(p) \rightarrow \mathbb{C}$ .

The composition of the Green's function  $G^-$  with the linearization map gives a subharmonic function in the  $\mathbb{C}$ -coordinates of  $W^s(p)$ , which, provided the neighborhood  $\mathcal{N}_R(J^+)$  is made sufficiently thin, has order of growth strictly less than  $1/2$ . Hence for each  $p \in \mathcal{N}_R(J^+)$  the local stable manifold  $W_R^s(p)$  is a properly embedded disk in  $\Delta_R^2$ , with the projection to the second coordinate giving branched covers of uniformly bounded degrees.



*4.0.5. Fatou components.* While the substantial dissipativity assumption plays an important role in the current paper, the bound on the Jacobian in terms of the degree is not needed for the classification of periodic Fatou components in the dominated splitting setting:

**Proposition 4.2.** *For a dissipative Hénon map with dominated splitting, any periodic Fatou component is an ordinary component.*

*Proof.* In [LP14] the assumption that the Hénon map is substantially dissipative plays a role in only an isolated part of the proof, namely to prove the uniqueness of limit sets on non-recurrent Fatou components. We note that in order to prove this uniqueness, one does not need to assume substantial dissipativity for Hénon maps admitting a dominated splitting. Recall that the only point in the proof where substantial dissipativity is used, is to rule out a one-dimensional limit set  $\Sigma$  contained in the strong stable manifold of a hyperbolic or neutral fixed point. Suppose that there exists a dominated splitting near  $J$ , and that such a  $\Sigma$  does exist. As was pointed out in [LP14], the restriction of  $\{f^n\}$  to  $\Sigma$  is a normal family. Recall also that  $\Sigma$  must lie in  $J$ , hence through each point  $q \in \Sigma$  there exists a strong stable manifold  $W^s(q)$ . If  $\Sigma$  is transverse to the stable field  $\{E^s\}$  at some point  $q \in \Sigma$ , then the union of the stable manifolds contains an open neighborhood of  $q$  on which the family of iterates is necessarily a normal family. This gives a contradiction with  $\Sigma \subset J$ . On the other hand, if  $\Sigma$  is everywhere tangent to the stable field, then for any  $q \in \Sigma$ , it is a domain in the stable manifold  $W^s(q)$ . Being backward invariant,  $\Sigma$  must coincide with  $W^s(q)$ . However,  $W^s(q)$  is conformally equivalent to  $\mathbb{C}$ , while  $\Sigma$  cannot be, giving a contradiction.  $\square$

It is a priori not clear that there are only finitely many periodic components. In the substantially dissipative case the finiteness is a consequence of our main result.

*4.0.6. Rates.* We will show now that the rate of contraction on the central line bundle is subexponential.

**Lemma 4.3.** *Given any  $r_1 < 1$  there exists a  $C > 0$  such that for any  $p \in J$  and any unit vector  $w \in E_p^c$  we have*

$$\|df^n w\| > \frac{1}{C} \cdot r_1^n.$$

*Proof.* For contradiction, suppose that for some  $r_1 < 1$  there exist for arbitrarily large  $n \in \mathbb{N}$  unit vectors  $w_n \in E^c$  with

$$\|df^n w_n\| < r_1^n.$$

Let  $r_1 < r_2 < 1$ . Then there exists an  $\epsilon > 0$  and for every  $n \in \mathbb{N}$  an integer  $k \in \{0, 1, \dots, n-1\}$  such that

$$\|df^j(df^k w_n)\| < r_2^j \cdot \|df^k w_n\|$$

for all  $j \leq \epsilon \cdot n$ . It follows that for every  $m \in \mathbb{N}$  there exists a unit vector  $u_m \in E^c$  for which

$$\|df^j u_m\| < r_2^j$$

for  $j = 0, \dots, m$ . (Here  $u_m$  can be chosen a multiple of a vector  $Df^k w_n$ .)

Since the set of unit vectors in  $E^c$  is compact, there exists an accumulation point  $w \in E^c$  of the sequence  $(u_m)$ . Let  $p \in J$  be such that  $w \in E_p^c$ . By continuity of the differential  $Df$  it follows that

$$\|df^j w\| \leq r_2^j$$

for all  $j \in \mathbb{N}$ . Since  $T_p(\mathbb{C}^2) = E_p^s \oplus E_p^c$ , and by the definition of the dominated splitting, there exists a  $C > 0$  such that

$$\|Df^j(p)\| < C \cdot r_2^j$$

for all  $j \in \mathbb{N}$ . Here we have used that there is a uniform bound from below on the angle between the vertical and horizontal tangent spaces.

Let  $\xi > 1$  be sufficiently small such that  $\xi \cdot r_2 < 1$ . By compactness of  $\overline{\Delta_R^2}$  there exists an  $\epsilon > 0$  such that if  $x, y \in \Delta_R^2$  with  $\|x - y\| < \epsilon$  then

$$\|f(x) - f(y)\| \leq \xi \cdot \|Df(x)\| \cdot \|x - y\|.$$

Let  $z \in \Delta_R^2$  be such that  $\|z - p\| < \epsilon/C$ . Then it follows by induction on  $n$  that

$$\|f^n(z) - f^n(p)\| \leq \xi^n \prod_{j \leq n-1} \|Df(f^j(p))\| \cdot \|z - p\| \leq \epsilon(\xi \cdot r_2)^n$$

for every  $n \in \mathbb{N}$ . Hence there is a neighborhood  $U$  of  $p$  such that

$$\|f^n(z) - f^n(p)\| \rightarrow 0,$$

uniformly over all  $z \in U$ . But then  $\{f^n\}_{n \in \mathbb{N}}$  is a normal family on  $U$ , which contradicts the fact that  $p \in J^+$ . □

It follows that the exponential rate of contraction on the stable subbundle is at least  $|\text{Jac } f|$ .

**Lemma 4.4.** *Given any  $r > |\text{Jac } f|$  we can find  $C > 0$  such that for any unit vector  $v \in E^s$  we have*

$$\|df^n v\| < C \cdot r^n.$$

*Proof.* Write  $v \in E_p^s$ , and let  $w \in E_p^c$  be a unit vector. The inequality follows immediately from Lemma 4.3 and the fact that

$$\|df^n v\| \cdot \|df^n w\| \leq C_1 |\text{Jac } f|^n,$$

where the constant  $C_1$  depends on the minimal angle between the spaces  $E^s$  and  $E^c$ . □

### 5. Dynamical lamination and its extensions

#### 5.1. Dynamical lamination

Now let us assume that the map  $f$  is substantially dissipative. Then the rate  $r$  in Lemma 4.4 can be assumed to be strictly smaller than  $1/d^2$ .

It follows that for any point  $p \in J$ , the composition  $G^- \circ \varphi_p$  is a subharmonic function of order bounded by  $\rho < 1/2$ , so the Wiman Theorem can be applied. It implies that  $W_R^s(p)$  is an embedded holomorphic disk, and that the projection to the second coordinate  $\pi_2 : W_R^s(p) \rightarrow \Delta_R$  gives a branched covering of finite degree.

**Lemma 5.1.** *The degrees of the branched coverings  $\pi_2 : W_R^s(p) \rightarrow \Delta_R$  are uniformly bounded, and*

$$J_R^+ := J^+ \cap \Delta_R^2 = \bigcup_{p \in J} W_R^s(p).$$

*Proof.* Let  $q \in J_R^+$ . Note that the sets

$$V_n(q) = \{(x, y) \in W_R^s(q) : f^{-n}(x, y) \in \overline{\Delta_R^2}\}$$

form a nested sequence of non-empty compact sets, so they have a non-empty intersection. Hence each  $W_R^s(q)$  intersects  $K^- = J^-$ . Therefore we have

$$J_R^+ = \bigcup_{p \in J} W_R^s(p).$$

Note that the degree of  $W_R^s(p)$  depends lower semicontinuously on  $p$ ; the degree may drop at semilocal stable manifolds tangent to the boundary of  $\Delta_R^2$ . However, when we consider the restriction of such a stable manifold to a strictly larger bidisk  $\Delta_{R'}^2$ , its degree, which is still finite, is at least as large as the degree of sufficiently nearby stable manifolds restricted to the smaller bidisk  $\Delta_R^2$ .

To argue that the degrees of the branched coverings are uniformly bounded, suppose for contradiction that there is a sequence  $(W_R^s(p_j))$  for which the degrees converge to infinity. Without loss of generality we may assume that the sequence  $(p_j)$  converges to a point  $p \in J_R^+$ . Let  $R' > R$ . Then for  $j$  sufficiently large the degree of  $W_R^s(p_j)$  is bounded by the degree of  $W_{R'}^s(p)$ , which gives a contradiction. □

#### Construction of artificial vertical lamination: outline

- Step 1. We start with the dynamical vertical lamination from Definition 4.1. This lamination consists of strong stable manifolds through points whose forward orbits are entirely contained in the domain of dominated splitting. The set where the dynamical vertical lamination is defined contains  $J_R^+$  and intersects all parabolic basins and wandering components. However, this set typically does not contain an open neighborhood of  $J_R^+$ .
- Step 2. The dynamical vertical lamination will be extended to the *artificial vertical lamination*, which is defined in an open neighborhood of  $J_R^+$ . However, the artificial vertical lamination is typically not invariant under the dynamics.

Step 3. Finally, the artificial vertical lamination will be adjusted on wandering components, so that it is defined on all semilocal wandering components and on every bi-infinite orbit of semilocal wandering components the artificial vertical lamination is backward invariant except on at most a single component.

In the end we obtain an artificial vertical lamination, defined on a neighborhood of  $J_R^+$  containing all semilocal wandering domains, is invariant on all but finitely many semilocal wandering domains, which agrees with the dynamical vertical lamination wherever the latter is defined, and which has quasiregular transverse regularity.

### 5.2. Local and global extensions of the vertical lamination

We note that the dynamical vertical lamination introduced in Definition 4.1 consists of local leaves  $L_R(a)$  that are connected components of global leaves  $L(a)$  intersected with  $\Delta_R^2$ . The vertical leaves all have natural linearization parametrizations that vary continuously with the base point  $a$ .

Let us recall the  $\lambda$ -lemma, in this version due to Ślodkowski [Sł091].

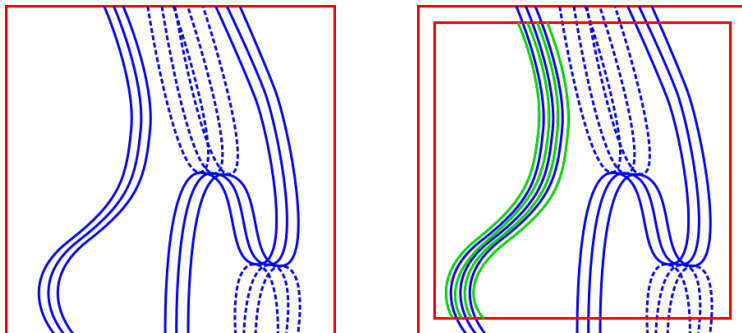
**Lemma 5.2.** *Let  $A \subset \hat{\mathbb{C}}$ . Any holomorphic motion  $f : \mathbb{D} \times A \rightarrow \hat{\mathbb{C}}$  of  $A$  over  $\mathbb{D}$  extends to a holomorphic motion  $\mathbb{D} \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  of  $\hat{\mathbb{C}}$  over  $\mathbb{D}$ .*

Let  $L_R(a)$  be a leaf of the dynamical vertical lamination, with a linearization map  $\varphi_a : \mathbb{C} \rightarrow L(a)$ . Recall that by the assumption that  $f$  is substantially dissipative we have  $L_R(a) = \varphi_a(D)$  for some bounded simply connected set  $D \subset \mathbb{C}$ . Let  $D$  be compactly contained in a slightly larger simply connected set  $D'$ , and let  $\xi : \mathbb{D} \rightarrow D'$  be the Riemann mapping. Define  $\psi_a = \varphi_a \circ \xi : \mathbb{D} \rightarrow L_R(a)$ . Then there exists a biholomorphic map  $\Psi : \Delta_\epsilon \times \mathbb{D} \rightarrow \mathbb{C}^2$  with  $\Psi(\zeta, 0) = \psi_a(\zeta)$ , mapping to a tubular neighborhood of the “core”  $L_R(a)$ . Consider all leaves of the dynamical vertical lamination that intersect a small neighborhood  $\Psi(\Delta_\delta \times \mathbb{D})$ , where  $\delta$  is chosen sufficiently small so that these dynamical leaves are completely contained in  $\Psi(\Delta_\epsilon \times \mathbb{D})$ . If  $\epsilon$  is sufficiently small then the inverse images under  $\Psi$  of these leaves form a collection of pairwise disjoint “dynamical” graphs over  $\mathbb{D}$  in  $\Delta_\epsilon \times \mathbb{D}$ , thus giving a holomorphic motion of a set  $A \subset \Delta_\epsilon$ .

By the  $\lambda$ -lemma the motion extends to a holomorphic motion over  $\Delta_\epsilon$ . The graphs over  $\mathbb{D}$  that are completely contained in  $\Delta_\epsilon \times \mathbb{D}$  can be mapped back by  $\Psi$ . By restricting to a slightly smaller vertical disk  $D_{1-\eta}$ , we can guarantee that all graphs that intersect a sufficiently small neighborhood of the core  $\{0\} \times D_{1-\eta}$  are completely contained in  $\Delta_\delta \times D_{1-\eta}$ , and can therefore be mapped back to  $\mathbb{C}^2$  by  $\Psi$ . We obtain a collection of pairwise disjoint “graphs” over  $L_R(a)$ , filling a neighborhood and all remaining in the neighborhood sufficiently close to  $L_R(a)$ . Moreover, by construction the newly constructed graphs cannot intersect any dynamical graphs.

We will refer to such an extension as a *flow box*, and to the leaves as *vertical*. Note that the leaves of the dynamical vertical lamination were globally defined, while the new leaves in the flow boxes are only defined in  $\Delta_R^2$ .

By compactness the Euclidean radii of the tubular neighborhoods of the leaves of the dynamical vertical lamination can be chosen uniformly, and hence the dynamical



**Fig. 2.** Local extension of the lamination on a slightly smaller bidisk.

vertical lamination is contained in a finite number of flow boxes. One could apply the  $\lambda$ -lemma to each of these, but a priori there is no reason why new leaves coming from different flow boxes should not intersect transversely. The main result in this section is Proposition 5.8, where a single extension to a neighborhood of the dynamical vertical lamination is constructed. Let us give an outline of the argument before going into details.

The extension of the lamination will be constructed by applying the  $\lambda$ -lemma to a finite number of tubular neighborhoods, each time taking into account the leaves that have been considered in previous steps.

A difficulty is that the leaves that we construct in a local extension are not global, they only are defined in some tubular neighborhood. In particular, even if we can guarantee that all new leaves are graphs over the core of other tubular neighborhoods they intersect, they may not be graphs over the entire core (see Figure 3a). We can deal with this by starting with a strictly larger bidisk  $\Delta_{R'}^2$ , and reducing the radius  $R'$  after each local extension by twice the radius of the tubular neighborhood. The goal is therefore to reduce the radius by at most the difference  $R' - R$  we start with.

Starting with an even larger constant  $R'$  is not of help, as that would affect the size and geometry of the flow boxes. Just reducing the radii of the flow boxes seems useless as well, as that would increase the number of flow boxes needed. The solution is to carry out the  $\lambda$ -lemma on large numbers of pairwise disjoint flow boxes simultaneously. By a covering lemma à la Besicovitch it follows that we can finish the process in a number of steps  $N$  that is independent of the radii of the flow boxes. By starting with a slightly larger bidisk  $\Delta_{R'}^2$  and choosing the radii  $\epsilon$  so that  $N \cdot 2 \times \epsilon < R' - R$  we will obtain the desired extension.

The version of Besicovitch Covering Theorem that we will use relies on the fact that the dynamical vertical lamination is Lipschitz, which follows from the following:

**Lemma 5.3.** *The holonomy maps of the dynamical vertical lamination are  $C^{1+\epsilon}$  smooth.*

*Proof.* The analogous statement in the uniformly hyperbolic case was proved in [Lyu99], with a similar proof. It is sufficient to prove that under the holonomy maps induced by

the dynamical vertical lamination, the dilatation of images of small disks of radius  $r > 0$  is  $1 + r^\epsilon$  for some  $\epsilon > 0$ .

We consider holonomy between horizontal transversals through two points  $z, w$  with  $w \in W_{\text{loc}}^s(z)$ . Let  $\eta > 0$  be such that any horizontal disk through any point  $f^n(z)$  or  $f^n(w)$  of radius at most  $\eta$  is a graph over the horizontal tangent line.

Let  $D(z)$  and  $D(w)$  be horizontal transversal disks, let  $0 < r < \eta$ , and let  $\Delta_r(z) \subset D(z)$  be a graph over the disk of radius  $r$ . We choose  $n \in \mathbb{N}$  so that

$$\|df^n|_{E_z^c}\| \sim r/\eta.$$

Note that

$$\|df^n|_{E_z^c}\| \cdot \|df^n|_{E_z^s}\| \sim |\text{Jac } f|^n,$$

and thus decreases exponentially fast. It follows that for  $r$  sufficiently small, the disks  $\Delta_r^n(z) = f^n \Delta_r(z)$  have size  $< \eta$ , so they are graphs over the horizontal tangent line at  $f^n(z)$ . Hence the composition of  $f^n : \Delta_r(z) \rightarrow \Delta_r^n(z)$  with the respective projections to and from the respective horizontal tangent lines at  $z$  and  $f^n(z)$  produces a univalent function.

Consider the disks  $\Delta_{r^2}(z) \subset \Delta_r(z)$ . By the Koebe Distortion Theorem, the dilatation of  $\Delta_{r^2}^n(z)$  is bounded by  $1 + O(r)$ . By our choice of  $n$  it follows that  $\|z^n - w^n\|$  is of order  $r$ . Hence the holonomy from  $\Delta_r^n(z)$  to its image  $\Delta_r^n(w)$  is quasiconformal of order  $r$ , and the dilatation of  $\Delta_{r^2}^n(w)$  is bounded by  $1 + O(r)$ .

The modulus  $-\log(r)$  of the annulus  $\Delta_r(z) \setminus \Delta_{r^2}(z)$  is preserved under conformal maps, hence  $\text{Mod}(\Delta_r^n(z) \setminus \Delta_{r^2}^n(z)) = -\log(r)$ . Since the distance between the disks  $\Delta_r(z)$  and  $\Delta_r(w)$  is of order  $r$ , the holonomy map from one to the other changes the modulus of the respective annuli by at most a factor of order  $1 - r$ , hence

$$\text{Mod}(\Delta_r^n(w) \setminus \Delta_{r^2}^n(w)) \geq (1 - O(r)) \cdot \log(1/r) \geq \log(1/r^{1-O(r)}).$$

Therefore, we can again apply the Koebe Distortion Theorem to the conformal map  $f^{-n} : \Delta_r^n(w) \rightarrow \Delta_r(w)$ , and it follows that the dilatation of  $\Delta_{r^2}^n(w)$  is bounded by  $1 + r^{1-O(r)}$ .

Note that restricted to the dynamical vertical lamination, the holonomy maps that we considered by mapping back and forth by  $f^n$  are all equal, hence the dilatation bound of  $1 + r^{1-O(r)}$  applies to the holonomy in the original flow box as well. Letting  $r^2 = \rho$ , it follows that the dilatation on a disk of radius  $\rho$  is  $1 + \rho^\epsilon$  for  $\epsilon > 0$  that can in fact be chosen arbitrarily close to  $1/2$ . This completes the proof.  $\square$

In the real differentiable setting there have been a number of results regarding the smoothness of holonomy maps in the partially hyperbolic setting: see for example [PSW97, PSW00]. Often smoothness holds when the center eigenvalues are sufficiently close to each other, i.e. satisfy some ‘‘center-bunching condition’’. Such a condition is trivially satisfied when the center direction is one-dimensional, or as here, in the conformal setting.

We note that the above proof shows that the holonomy is  $C^{1+\epsilon}$  on the dynamical vertical lamination for any  $\epsilon < 1/2$ . We will not use this estimate. In fact, we will only use the fact that the dynamical vertical lamination is Lipschitz.

Note that in later steps of the procedure, after having already found a partial extension of the dynamical vertical lamination by a number of applications of the  $\lambda$ -lemma, the lamination under consideration may no longer be Lipschitz. However, we will see that this does not present difficulties when only the leaves in sufficiently small neighborhoods of leaves of the dynamical vertical lamination are kept.

**Definition 5.4.** Since the holonomy maps of the dynamical vertical lamination are Lipschitz, there exists a constant  $k > 0$ , independent of  $\epsilon > 0$  for  $\epsilon$  sufficiently small, such that any leaf of the dynamical vertical lamination that intersects an  $\epsilon$ -tubular neighborhood of a leaf of the dynamical vertical lamination must be contained in the corresponding  $(k \cdot \epsilon)$ -tubular neighborhood. Note that  $\epsilon$  and  $k \cdot \epsilon$  refer to the *Euclidean* radius of the tubular neighborhoods in  $\mathbb{C}^2$ .

For given  $\epsilon > 0$  we will consider tubular neighborhoods of three different radii:  $\epsilon$ ,  $k \cdot \epsilon$  and  $k^2\epsilon$ . Given a collection  $\{L_R(a_i)\}$  of leaves, we will denote the tubular neighborhood of radius  $r_i$  centered at  $L_R(a_i)$  as  $T_i(r_i)$ . In what follows we consider tubular neighborhoods in different bidisks  $\Delta_{R'}^2$ , where  $R' > R$  decreases in each step. Without loss of generality we may assume that the constant  $k > 0$  defined above will be sufficiently large for the maximal bidisk  $\Delta_{R'}^2$ , as well. We will write  $T_i(r_i, R')$  to indicate the radius  $R'$  of the bidisk we consider.

Let us fix a straight horizontal line

$$\mathbb{L}_0 = \{(x, y) : y = y_0\}$$

for some  $|y_0| < R$ .

**Lemma 5.5.** *The dynamical vertical lamination has only finitely many horizontal tangencies in  $\mathbb{L}_0$ .*

*Proof.* Apply the  $\lambda$ -lemma to a given tubular neighborhood of a leaf of the dynamical vertical lamination, and consider the holonomy map from  $\mathbb{L}_0$  to a small disk transverse to the lamination. Such holonomy maps are quasi-regular, hence critical points are isolated. The critical points are exactly given by the tangencies to  $\mathbb{L}_0$ , thus finiteness follows from compactness of  $J_R^+$ . □

We may assume, by either increasing  $R$  or by changing  $y_0$ , that if a global leaf has more than one tangency with  $\mathbb{L}_0$ , then those tangencies are all contained in a single semilocal leaf. This is not necessary for what follows but makes the statement and proof of the Tubular Covering Lemma below more convenient.

We first consider a planar covering lemma. Let  $K \subset \mathbb{C}$  be compact, let  $\gamma > 1$  and  $s \in \mathbb{N}$ . For each  $\alpha \in K$  let  $\Lambda(\alpha) \subset K$  be a set of cardinality at most  $s$ , defining an equivalence relation, i.e.  $\alpha \in \Lambda(\beta)$  if and only if  $\beta = \Lambda(\alpha)$ .

We assume that the sets  $\Lambda(\alpha)$  vary lower semicontinuously with  $\alpha$ , i.e.

$$\Lambda(\alpha) \subset \liminf_{\alpha_j \rightarrow \alpha} \Lambda(\alpha_j).$$

We assume moreover that

$$\widehat{\Lambda(\alpha)} = \limsup_{\beta \rightarrow \alpha} \Lambda(\beta)$$

is also finite and of cardinality at most  $s$ . Finally, we assume that there exists a Lipschitz constant  $C > 1$  for the equivalence relation. That is, for  $\delta > 0$  sufficiently small and  $\alpha \in K$ , the set

$$\bigcup_{\beta \in D_\delta(\alpha)} \Lambda(\beta)$$

is contained in at most  $s$  disks of radius  $C \cdot \delta$ . Here we write  $D_\delta(\alpha)$  as usual for the disk centered at  $\alpha$  of radius  $\epsilon$ .

We define

$$E(\alpha, \epsilon) = \bigcup_{\beta \in \Lambda(\alpha)} D_{\gamma \cdot \epsilon}(\beta).$$

**Lemma 5.6** (Planar Covering Lemma). *There exists  $N$  such that for every sufficiently small  $\epsilon > 0$  the set  $K$  can be covered by a finite collection  $\{D_\epsilon(\alpha)\}$ , whose set of centers  $A$  can be partitioned into subcollections  $A_1, \dots, A_N$  such that for every  $j = 1, \dots, N$  and any  $\alpha, \beta \in A_j$ , the sets  $E_\epsilon(\alpha)$  and  $E_\epsilon(\beta)$  are disjoint.*

*Proof.* Recursively construct a finite collection  $A = \{\alpha\}$  for which the disks  $D_\epsilon(\alpha)$  cover  $K$ , by at each step selecting a center  $\alpha$  that is not yet contained in the previous disks. It follows that there is an upper bound, independent of  $\epsilon$ , on the number of centers  $\alpha \in A$  contained in any disk of radius  $\epsilon$ . For  $\gamma \cdot \epsilon < \delta$  it follows from the Lipschitz bound  $C$  that given  $\alpha \in K$ , the set of points  $\beta \in K$  for which  $E_\epsilon(\beta)$  and  $E_\epsilon(\alpha)$  intersect is contained in at most  $s^2$  disks of radius  $2C \cdot \gamma\epsilon$ . Therefore there is also an upper bound, again independent of  $\epsilon$ , on the number of centers in  $A$  contained in those larger disks.

It follows that for any  $\alpha \in A$  the number of centers  $\beta \in A$  for which  $E_\epsilon(\beta) \cap E_\epsilon(\alpha) \neq \emptyset$  is bounded by a constant  $M$  that does not depend on  $\epsilon$ . Recursively partition  $A$  into sets  $A_1, \dots, A_N$ , at each step taking a maximal number of the remaining centers  $\alpha \in A$  for which the sets  $E_\epsilon(\alpha)$  are pairwise disjoint. This process must end in at most  $N \leq M + 1$  steps.  $\square$

We stress that the bound  $N$  is allowed to depend on  $C$ ,  $\gamma$  and  $s$ .

**Lemma 5.7** (Tubular Covering Lemma). *Let  $R' > R$ . Given  $\epsilon_1 > 0$  sufficiently small, there exists an  $N \in \mathbb{N}$  such that for any sufficiently small  $\epsilon_2 > 0$  we can cover the dynamical vertical lamination in  $\Delta_R^2$  with finitely many tubular neighborhoods  $\{T_i(r_i)\}$  centered at leaves of the dynamical vertical lamination, satisfying the following:*

- (i) *The finite set of tubular neighborhoods can be partitioned into  $N$  collections  $A_1, \dots, A_N$ .*
- (ii) *The tubular neighborhoods in  $A_1$  have radius  $r_i = \epsilon_1$ , and all other tubular neighborhoods have radius  $r_i = \epsilon_2$ .*
- (iii) *For  $\alpha = 1, \dots, N$  and  $T_i(r_i), T_j(r_j) \in A_\alpha$  one has*

$$T_i(k^2 \cdot r_i, R') \cap T_j(k^2 \cdot r_j, R') = \emptyset.$$

*Proof.* Note that each leaf of the dynamical vertical lamination must pass through the line  $\mathbb{L}_0$ , so it is sufficient to consider tubular neighborhoods that cover the intersection of the dynamical vertical lamination with  $\mathbb{L}_0$ .



By Lemma 5.5, there are only finitely many semilocal leaves with horizontal tangencies in  $\mathbb{L}_0$ . We may assume that  $\epsilon_1 > 0$  is sufficiently small such that the corresponding tubular neighborhoods  $T_i(k^2 \cdot \epsilon_1, R')$  do not intersect. Let  $A_1$  be the set of corresponding tubular neighborhoods  $T_i(\epsilon_1)$ . *From now on we consider tubular neighborhoods centered at leaves not contained in these finitely many tubular neighborhoods.*

We claim that we are left with the situation of the Planar Covering Lemma. The set  $K$  is the intersection of the remaining dynamical vertical lamination with  $\mathbb{L}_0$ . The equivalence classes  $L(\alpha)$  are given by the intersection points of semilocal leaves of the dynamical vertical lamination in the bigger bidisk  $\Delta_{R'}^2$ .

Recall from Lemma 5.1 that the semilocal leaves are branched covers with uniformly bounded degrees. The lower semicontinuity, and upper bound on  $\widehat{L(\alpha)}$  follow as in the proof of Lemma 5.1.

We can choose  $\epsilon_2 > 0$  sufficiently small so that for any tubular neighborhood  $T_i(\epsilon_2, R)$  not contained in one of the tubular neighborhoods in  $A_1$ , the intersection  $T_i(k^2 \cdot \epsilon_2, R') \cap \mathbb{L}_0$  consists of a finite number of connected components, each containing an intersection point of the core leaf.

Each connected component of the intersection closely resembles an ellipse, whose direction and eccentricity is determined by the tangent vector of the leaf at the corresponding intersection point with  $\mathbb{L}_0$ . Since we consider only sufficiently small tubular neighborhoods of points bounded away from the tangencies in  $\mathbb{L}_0$ , the eccentricity of these ellipses is bounded. In other words, there exists  $\ell > 1$  independent of  $\epsilon_2$  sufficiently small such that each component of each  $T_i(k^2 \cdot \epsilon_2, R') \cap \mathbb{L}_0$  is contained in a disk of radius  $\ell \cdot \epsilon_2$ , and contains the concentric disk of radius  $(1/\ell)\epsilon_2$ .

Thus, each intersection  $T_i(\epsilon_2) \cap \mathbb{L}_0$  contains a disk of radius  $(1/\ell)\epsilon_2$  centered at a point  $\alpha \in K$ , while the intersection  $T_i(k^2 \cdot \epsilon_2, R') \cap \mathbb{L}_0$  is contained in a bounded number of disks of radius  $\ell k^2 \epsilon_2$ , thus we are in the situation of the Planar Covering Lemma for  $\gamma = \ell^2 k^2$ . The existence of the Lipschitz constant  $C$  follows from the fact that the holonomy maps are Lipschitz and from the bound from below on the angle between the transversal  $\mathbb{L}_0$  and the remaining dynamical vertical lamination.

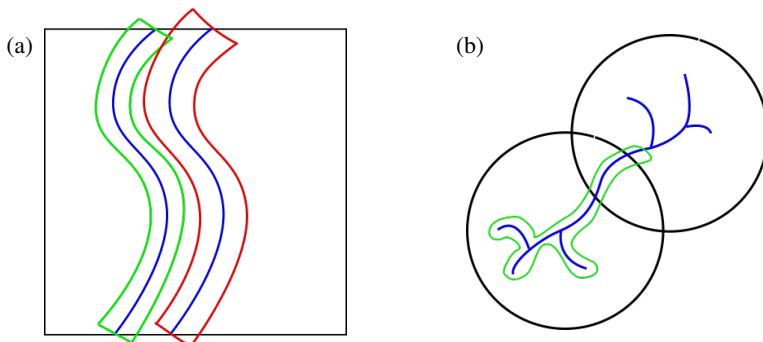
The existence of the partition  $A_1, \dots, A_N$  therefore follows from the Planar Covering Lemma. The constants  $C$  and  $\gamma$  depend on  $\epsilon_1$ , hence so does  $N$ , but  $N$  is independent of  $\epsilon_2$ . □

**Proposition 5.8.** *The dynamical vertical lamination in  $\Delta_R^2$  can be extended to an open neighborhood.*

*Proof.* We consider  $R' > R$  and apply the previous lemma to the dynamical vertical lamination in  $\Delta_{R'}^2$ . Let  $2k^2 \cdot \epsilon_1 < (R' - R)/2$ , let  $N$  be as in the previous lemma, and let  $\epsilon_2 > 0$  be sufficiently small such that

$$2k^2 \cdot \epsilon_2 \cdot N < (R' - R)/2,$$

where as before  $k$  is an upper bound for the Lipschitz constant of the holonomy maps. We can cover the dynamical vertical lamination in  $\Delta_R^2$  by tubular neighborhoods as in the previous lemma, and write  $A_1, \dots, A_N$  for the partition into pairwise disjoint tubular



**Fig. 3.** (a) Two tubular neighborhoods. (b) From above: thin neighborhood.

neighborhoods. We may assume that  $\epsilon_1$  and  $\epsilon_2$  are chosen sufficiently small such that the  $\lambda$ -lemma can be applied to each tubular neighborhood of radius  $k \cdot \epsilon_i$  for  $i = 1, 2$ .

We first apply the  $\lambda$ -lemma to each of the tubular neighborhoods  $T_i(k^2 \cdot \epsilon_1, R')$  in  $A_1$ , keeping only the leaves that intersect the tubular neighborhood of radius  $k \cdot \epsilon_1$ . Since the tubular neighborhoods of radius  $k^2 \cdot \epsilon_1$  are pairwise disjoint, it follows that the new leaves are all pairwise disjoint as well.

Note that while the leaves of the dynamical vertical lamination were global leaves, the new leaves are semilocal, and contained in the tubular neighborhoods in  $A_1$ . In order to guarantee that the new leaves are still graphs over the entire cores of the tubular neighborhoods in  $A_2, \dots, A_N$  that they intersect, we decrease the radius of the bidisk  $\Delta_{R'}^2$  by  $2k^2 \cdot \epsilon_1$ .

Note that the laminations constructed using the  $\lambda$ -lemma may not be Lipschitz. In order to guarantee the preservation of the modulus of continuity  $k$  for each of the selected tubular neighborhoods that will be used in later steps, we keep only the newly constructed leaves that intersect a sufficiently thin neighborhood of the dynamical vertical lamination; see the two tubular neighborhoods illustrated in Figure 3b, where the leaves of the dynamical vertical lamination are represented by the blue continuum, and only the new leaves in the small green neighborhood are kept. Since the holonomy maps will be continuous, choosing a thin enough neighborhood of the dynamical vertical lamination will guarantee that the leaves that intersect the tubular neighborhoods of radii  $\epsilon_2$  and  $k \cdot \epsilon_2$  will still be contained in the corresponding tubular neighborhoods of radii  $k \cdot \epsilon_2$  and  $k^2 \cdot \epsilon_2$  respectively.

We continue with the tubular neighborhoods in  $A_2, A_3, \dots, A_{N-1}$ , and finally to those in  $A_N$ , each time following the same procedure as above: first apply the  $\lambda$ -lemma to each tubular neighborhood of radius  $k^2 \cdot \epsilon_2$  in the current bidisk, keeping only those leaves that intersect the tubular neighborhood of radius  $k \cdot \epsilon_2$ , then decrease the radius of the bidisk by  $k^2 \cdot \epsilon_2$ , and finally only keeping the newly constructed leaves in a very thin neighborhood of the dynamical vertical lamination in order to maintain the modulus of continuity.

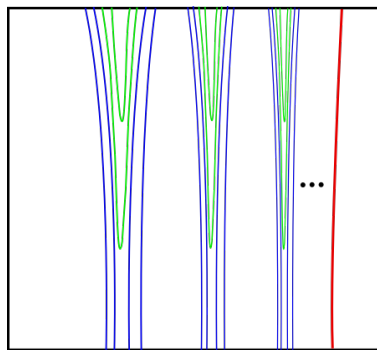
By our choice of  $\epsilon_2$  we end up with the required extended lamination on a bidisk of radius at least  $R$ .  $\square$

We will refer to this extension of the dynamical vertical lamination as the *artificial vertical lamination*, and denote it by  $\mathcal{L}$ . By slight abuse of terminology, we will also write  $\mathcal{L}$  for the union of the (artificial) vertical leaves, which gives a neighborhood of  $J_R^+$ . We may assume that the dynamical vertical lamination is extended to a thin enough neighborhood such that  $\mathcal{L}$  is contained in the region where the dominated splitting is defined, and by continuity we may assume that its tangent bundle lies in the vertical cone field. *From now on we write  $\mathcal{N}(J_R^+)$  for the region where both the cone field and the artificial vertical lamination are defined.*

### 5.3. Adjusting the artificial vertical lamination on wandering domains

Note that there is no reason for the artificial vertical lamination  $\mathcal{L}$  to be invariant under  $f$ . Here we discuss how to define and modify the lamination on the (hypothetical) wandering Fatou components.

Recall the set  $\mathcal{V}^+$  (2) foliated by stable manifolds  $W^s(z)$ . Let  $U$  be a wandering Fatou component of  $f$ . As  $f^n z \rightarrow J$  for any  $z \in U$ , we have  $U \subset \mathcal{V}^+$ . So,  $U$  is foliated by strong stable manifolds; we call it the *dynamical foliation*  $\mathcal{F}_U$  of  $U$ . Putting these foliations together, we obtain the invariant dynamical foliation on the union  $\mathcal{U}$  of all wandering components.

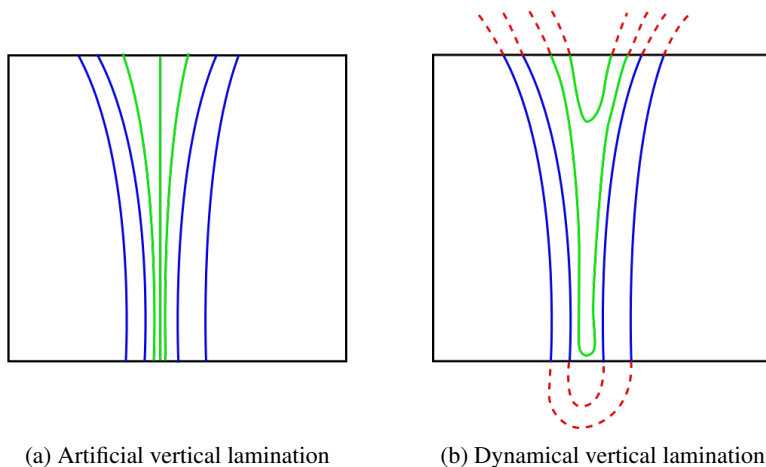


**Fig. 4.** The pullback is not continuous in the limit.

However, in general, this foliation cannot be extended to the closure of this union (see Figure 4). To deal with this problem, we combine this dynamical foliation on some “semilocal” wandering components with the non-dynamical extension on the others, to obtain a lamination of  $\mathcal{U} \cap \Delta_R^2$  which is invariant everywhere except on finitely many semilocal wandering components.

A *semilocal wandering component*  $V$  is a connected component of  $U \cap \Delta_R^2$ . Note that there are at most finitely many semilocal wandering components  $V \subset U \cap \Delta_R^2$  not contained in  $\mathcal{N}(J_R^+)$ . We refer to such a component  $V$  as a *component with hole*.

Let  $V$  be a semilocal component with hole, let  $V^{-1}$  be a connected component of  $f^{-1}(V) \cap \Delta_R^2$ , and assume that neither  $V^{-1}$  nor any connected component of



**Fig. 5.** Conflicting laminations in a wandering component.

$f^{-n}(V^{-1}) \cap \Delta_R^2$  has a hole. Then  $V^{-1}$  is contained in  $\mathcal{L}$  and hence foliated by vertical leaves. Note that vertical leaves in  $\mathcal{L}$  sufficiently close to the (vertical) boundary of  $V^{-1}$  are necessarily *dynamical* leaves contained in the dynamical vertical lamination, since the forward orbits of such points remain sufficiently close to  $J_R^+$ . Recall that the dynamical vertical lamination is invariant under  $f$ . We modify the artificial vertical lamination  $\mathcal{L}$  by pulling back the leaves in  $V^{-1}$  to all components of  $f^{-n}(V^{-1}) \cap \Delta_R^2$  for all  $n \geq 2$ .

If  $V^n$  lies in  $\mathcal{N}(J_R^+)$  for all  $n \geq n_0$ , then the artificial vertical lamination on  $V^{n_0}$  is dynamical, and we can pull back the *dynamical* lamination on  $V^{n_0}$  to all components  $V^j$  with  $0 \leq j \leq n_0$ . See Figure 5 for a sketch of the two conflicting laminations that one obtains by pulling back the dynamical vertical lamination to  $V^{-1}$ . Note that the artificial vertical lamination near the boundary of the component is dynamical, and is therefore identical in both pictures.

By following this procedure for all grand orbits of Fatou components with holes, we obtain a lamination that is invariant on all but finitely many components, and for each bi-infinite orbit of components there is at most one step in which the lamination is not invariant. To be more precise, if  $(V^n)_{n \in \mathbb{Z}}$  is a sequence of semilocal wandering components with  $f(V^n) \subset V^{n+1}$ , then there is at most one  $n \in \mathbb{Z}$  for which the leaves of the artificial vertical lamination in  $V^n$  are not mapped into leaves of the lamination in  $V^{n+1}$ , and this can only occur when  $V^j$  lies in the region of dominated splitting for  $j \leq n$  but not for  $j = n + 1$ .

#### 5.4. Choice of horizontal line

In the one-dimensional argument we considered iterated inverse images of a given disk  $D_r(z)$ . This is problematic in the Hénon case. The reason is that such preimages are very likely to land at least partially outside of the bidisk  $\Delta_R^2$ . Instead, we will start

with a flat horizontal complex line  $\mathbb{L}_0$ , map it forward by  $f^n$ , consider a small disk  $V^0$  inside  $f^n(\mathbb{L}_0) \cap \Delta_R^2$ , and consider the pullbacks  $V^1, \dots, V^n$  of this disk. Of course instead of working with a full horizontal line  $\mathbb{L}_0$  it is equivalent to work with a disk of radius  $R$ .

We will now make a suitable choice for the horizontal line:

**Lemma 5.9.** *There exists  $y_0 \in \mathbb{C}$  with  $|y_0| < R$  such that the vertical lamination of  $J^+$  is transverse to  $\{y = y_0\}$ .*

*Proof.* For each horizontal complex plane, the tangencies of this plane with the artificial vertical lamination are isolated, thus, by restricting the neighborhood of  $J^+$  if necessary, there are at most finitely many tangencies. We can remove the tangencies one by one by making arbitrarily small perturbations in  $y_0$  for which the tangencies are transferred to nearby leaves in the Fatou set. More precisely, if a leaf is tangent to a horizontal plane, then each nearby vertical leaf is tangent to some nearby horizontal plane. Locally the number of tangencies, counted with multiplicities, is constant. Thus, we can take any nearby leaf in the Fatou set, take the horizontal plane for which that leaf is tangent, and reduce the number of tangencies in  $J^+$  by at least one. After a finite number of perturbations we obtain a desired horizontal line  $\{y = y_0\}$ . □

**Definition 5.10** (Choice of  $\mathbb{L}_0$ ). From now on we fix  $y_0$  so that the dynamical lamination of  $J^+$  is transverse to  $\mathbb{L}_0$ , and so that  $\mathbb{L}_0$  does not contain any parabolic periodic points.

It follows that the line  $\mathbb{L}_0$  is transverse to the artificial vertical lamination in a sufficiently small neighborhood of  $J^+$ .

### 6. Uniformization of wandering components

In this section we will show that any wandering component  $U$  can be uniformized by the straight cylinder  $\mathbb{D} \times \mathbb{C}$  in such a way that the dynamical foliation of  $U$  becomes vertical. It will imply a bound for the (appropriately understood) degrees of the maps  $f^n|_U$ .

#### 6.1. Contraction

**Lemma 6.1.** *For any wandering component  $U$ , the derivatives  $\|df^n\|$  converge to 0 uniformly on compact subsets of  $U$ .*

*Proof.* Replacing  $U$  with its iterated image, if needed, we can ensure that all the images  $U^n = f^n(U)$  lie in the domain of dominate splitting. Hence  $U$  is filled with global strong stable manifolds  $W^s(z)$ .

Arguing by contradiction, we can find a sequence of unit vectors  $v^m \in E^c(z^m)$  converging to a vector  $v \in E^c(z)$ ,  $z \in U^0$ , and a sequence of times  $n_m \rightarrow \infty$  such that

$$\|df^{n_m}(v^m)\| \geq \varepsilon > 0. \tag{4}$$

Take a small horizontal disk  $D \ni z$  tangent to  $v$ , and find a sequence of horizontal disks  $D^m \ni z^m$  tangent to the  $v^m$  and converging to  $D$ . Since the family of iterates is normal

near  $z$ , the derivatives  $d(f^n|_{D_m})$  have a uniformly bounded distortion. Together with (4), this implies

$$\|df^{n_m}(w)\| \geq \varepsilon' > 0, \quad \forall w \in TD^m,$$

so the images  $f^{n_m}(D^m)$  are horizontal disks of definite size. Hence the local stable manifolds through each of them fill a ball of definite radius. This contradicts the fact that these infinitely many balls must be disjoint yet bounded.  $\square$

Recall the set  $\mathcal{V}^+$  (2) foliated by the strong stable manifolds  $W^s(z)$ . For  $z \in \mathcal{V}^+$ , let  $\phi_z : \mathbb{C} \rightarrow W^s(z)$  be a uniformization of  $W^s(z)$ , normalized so that  $\phi_z(0) = z$  and  $\|d\phi_z(0)\| = 1$ . We note that  $\phi_z$  is unique up to multiplication in  $\mathbb{C}$  by a constant  $e^{i\theta}$ . Hence for  $\zeta = \phi_z(u) \in W^s(z)$  we can define the (asymmetric) *intrinsic distance* as

$$\text{dist}^i(z, \zeta) = |u|,$$

which is independent of the choice of  $\phi_z$ .

- Lemma 6.2.** (i) For  $\zeta \in W_{\text{loc}}^s(z)$ , we have  $\text{dist}^i(z, \zeta) \asymp \|z - \zeta\|$ .  
 (ii) There exists an  $\varepsilon > 0$  with the following property: For any  $\zeta \in W^s(z) \setminus W_{\text{loc}}^s(z)$ , there exists  $n \asymp 1 + \log^+(\varepsilon^{-1} \text{dist}^i(z, \zeta))$  such that  $\|f^n z - f^n \zeta\| \geq \varepsilon$ .

*Proof.* (i) The linearizing maps  $\phi_z$  are locally bi-Lipschitz with a constant continuously depending on  $z$ , which implies the first assertion.

(ii) Let us select  $\varepsilon > 0$  so that each  $\phi_z$  maps the disk  $\Delta_\varepsilon$  into  $W_{\text{loc}}^s(z)$ . Since

$$f^n \circ \phi_z(u) = \phi_{z_n}(\lambda_n u) \quad \text{with } \lambda_n = \|df_z^n|_{E^s}\|, \tag{5}$$

the intrinsic distance is contracted at an exponential rate. Hence the number of iterates it takes for it to become of order  $\varepsilon$  depends logarithmically on  $\text{dist}^i(z, \zeta)$ . Application of (i) concludes the proof.  $\square$

### 6.2. Global transversals

Recall from §5.3 that given a wandering component  $U$ ,  $\mathcal{F}_U$  stands for the dynamical vertical lamination of  $U$  by the global stable manifolds  $W^s(z) \approx \mathbb{C}$ .

Let us say that  $D$  is a *global transversal* to a wandering component  $U$  if

- (T1)  $D$  is a non-singular holomorphic disk properly embedded into  $U$ ;
- (T2)  $D$  is relatively compactly contained in a non-singular holomorphic curve  $D'$ ;
- (T3) for any  $z \in D$ , the curve  $D'$  is transverse to the stable line  $E_z^s$  (see §4.0.3).

**Lemma 6.3.** *If  $D$  is a global transversal to a wandering component  $U$ , then for  $n$  large enough the images  $f^n(D)$  are horizontal with respect to the cone field.*

*Proof.* By assumption (T3), the angle between the tangent line  $T_z D$  and the stable line  $E_z^s$  is bounded below by some  $\alpha > 0$  independent of  $z \in D$ . Hence it takes a bounded number of iterates to bring  $T_z D$  to a horizontal cone.  $\square$

**Lemma 6.4.** *Let  $U$  be a wandering component, and let  $z \in U \cap \Delta_R^2$ . Then  $W^s(z) \subset U$  intersects a global transversal.*

In particular, any wandering domain contains a global transversal.

*Proof.* By Lemma 5.9, there exists a horizontal line  $\mathbb{L}_0$  transverse to the dynamical vertical lamination on  $J^+$ . For large  $n \in \mathbb{N}$  connected components of  $f^n(U) \cap \Delta_R^2$  will be contained in an arbitrarily small neighborhood of  $J^+$ , and hence the dynamical vertical lamination in those components is transverse to  $\mathbb{L}_0$ . Since  $f$  is substantially dissipative, the semilocal strong stable manifold  $W_R^s(f^n(z))$  intersects  $\mathbb{L}_0$  in a connected component  $D_n \subset \mathbb{L}_0 \cap f^n(U)$ .

Since  $f^n(U)$  is a Fatou component, the maximum principle implies that  $D_n$  is simply connected. Let  $D'_n \subset \mathbb{L}_0$  be a slightly larger domain where the dynamical vertical lamination is still transverse to  $\mathbb{L}_0$ . Pulling back by  $f^n$  gives the required  $D \subset D'$ . □

Select a continuous unit vector field  $v(z) \in E_z^s$  on  $D'$ , and for  $z \in D$ , let  $\phi_z : \mathbb{C} \rightarrow W^s(z)$  be the uniformization of  $W^s(z)$  normalized so that  $\phi_z(0) = z$  and  $\phi'_z(0) = v(z)$ . Then we obtain a continuous map

$$\Phi : D \times \mathbb{C} \rightarrow U, \quad (z, u) \mapsto \phi_z(u). \tag{6}$$

Our goal is to prove that  $\Phi$  is a homeomorphism.

### 6.3. Holonomy group

Let  $D$  and  $\Delta$  be global transversals to  $U$ , let  $z \in D$ , and suppose that the global stable manifold  $W^s(z)$  intersects  $\Delta$  in a point  $w \in \Delta$ . Then holonomy induces a map  $h$  from a neighborhood of  $z$  in  $D$  to a neighborhood of  $w$  in  $\Delta$ .

**Lemma 6.5.** *The map  $h$  admits a unique extension along any path  $\gamma$  in  $D$ .*

*Proof.* Assume there exists a path  $\gamma : [0, 1] \rightarrow D$ ,  $z^0 = \gamma(0)$ , such that  $h$  extends along  $\gamma : [0, 1) \rightarrow D$  but does not extend to  $z^1 = \gamma(1) \in D$ . We let

$$z^t = \gamma(t), \quad h(z^t) = \Phi(z^t, u^t), \quad t \in [0, 1).$$

Since  $h$  does not extend to  $z^1$  it follows that

$$\text{dist}^t(z^t, h(z^t)) = |u^t| \rightarrow \infty \quad \text{as } t \rightarrow 1.$$

Otherwise we would have a subsequence  $t_k \rightarrow 1$  with bounded  $u^{t_k}$ . Then we could take a limit point  $u^1$  of the  $u^{t_k}$  and obtain a local holonomy from  $z^1$  to  $h(z^1) = \Phi(z^1, u^1)$ . This local holonomy must then agree with the holonomy along  $\gamma$  for  $t$  close to 1, giving a contradiction.

For  $t \in [0, 1)$ , let  $\delta^t$  be the ‘‘intrinsically straight’’ path in  $W^s(z^t)$  connecting  $z^t$  to  $h(z^t)$ :

$$\delta^t : [0, u^t] \rightarrow W^s(z^t), \quad \delta^t(\cdot) = \Phi(z^t, \cdot).$$

Let  $D_n = f^n(D)$ ,  $\gamma_n = f^n(\gamma)$ ,  $z_n^t = f^n(z^t)$ ,  $\delta_n^t = f^n(\delta^t)$ , and let  $h_n = f^n \circ h \circ f^{-n}$  be the pushforward holonomy defined on  $\gamma_n : [0, 1) \rightarrow D_n$ . By Lemma 6.1,

$$\text{length } \gamma_n \rightarrow 0 \quad \text{and} \quad \text{length } \delta_n^0 \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where length stands for the Euclidean length. In particular, the path  $\delta_n^0$  lies in  $W^s(z_n^0)$  for  $n$  sufficiently large. On the other hand, there exists an  $\varepsilon > 0$  such that for any given  $n$  and  $t \in [1 - \eta_n, 1)$  with  $\eta_n \rightarrow 0$ ,  $\text{length } \delta_n^t \geq \varepsilon$ . Let us select the smallest  $t = t_n$  with this property.

Let us use a local coordinate system near  $z_n^0$  with axes  $E^c$  and  $E^s$  at that point. Then the holonomy  $h_n$  on the short path  $\gamma_n(t)$ ,  $0 \leq t \leq t_n$ , quickly goes from a small height (equal to  $\text{length } \delta_n^0$ ) to a definite height (of order  $\varepsilon$ ), so it has a large average slope. It follows that somewhere either  $\Delta_n$  or  $D_n$  must have a small angle with the stable direction, contradicting the property that for large  $n$  they are horizontal with respect to the cone field. □

**Corollary 6.6.** *Any local holonomy map  $h$  extends to a homeomorphism  $D \rightarrow \Delta$ .*

*Proof.* Since  $D$  is simply connected and any  $h$  extends uniquely along all paths, the usual argument of the Monodromy Theorem implies that  $h$  extends uniquely to a global continuous map  $D \rightarrow \Delta$ . Since the same is true for  $h^{-1}$ , it is a homeomorphism. □

In particular, we note that when  $\Delta = D$ , the holonomy maps form a group  $\mathcal{H}_D$  of homeomorphisms of  $D$ .

Denote by  $W^s(D)$  the union of the strong stable manifolds through  $D$ .

**Lemma 6.7.** *Let  $D$  be a global transversal to a wandering component  $U$ . Then  $U = W^s(D)$ .*

*Proof.* It is immediate that  $W^s(D)$  is an open subset of  $U$ . Let  $z \in U \cap \overline{W^s(D)}$ . Let  $n \in \mathbb{N}$  be such that  $f^n(U) \cap \Delta_R^2$  is contained in  $\mathcal{V}^+$ . By Lemma 6.4 there exists a global transversal  $\Delta \subset L$  intersecting  $W_R^s(z_n)$ , or equivalently  $z_n$  is contained in the semilocal strong stable manifold of some  $w \in \Delta$ . Then  $W^s(\Delta)$  contains a neighborhood of  $z_n$ , and hence intersects  $f^n(D)$ . Thus, we obtain a local holonomy map from  $\Delta$  to  $f^n(D)$ , which by Corollary 6.6 extends to a homeomorphism  $h : \Delta \rightarrow D_n$ . It follows that  $W^s(D_n) = W^s(\Delta)$ , implying  $z_n \in W^s(D_n)$ . We conclude that  $z \in W^s(D)$ , completing the proof. □

### 6.4. Uniformization

**Proposition 6.8.** *Let  $D$  be a global transversal to a wandering component  $U$ . Then any stable manifold intersects  $D$  in at most one point.*

*Proof.* Suppose for a contradiction that a strong stable manifold intersects  $D$  in two distinct points, and denote the induced holonomy homeomorphism by  $h : D \rightarrow D$ . Let us consider pushforward holonomies

$$h_n = f^n \circ h \circ f^{-n} : D_n \rightarrow D_n, \quad \text{where } D_n = f^n(D).$$



For any point  $z \in D$  and  $n$  large enough (depending on  $z$ ), it is the holonomy along the local stable foliation in some flow box  $B_i$  containing  $z_n = f^n z$ . By the  $\lambda$ -lemma [MnSS83],  $h_n$  is locally quasiconformal (“qc”) near  $z_n$ . Since a biholomorphic map  $f^n$  does not change the dilatation,  $h$  is locally qc near  $z$ , with the same local dilatation (depending only on  $B_i$  but not on  $h, z$  or  $n$ ). Since there are only finitely many flow boxes  $B_i$  covering the whole domain of the dominated splitting, these local dilatations are uniformly bounded for all  $h \in \mathcal{H} \equiv \mathcal{H}_D$  and  $z \in D$ . Hence each  $h \in \mathcal{H}$  is globally qc on  $D$  with uniformly bounded dilatation, so the holonomy group  $\mathcal{H}$  acts uniformly qc on  $D$ .

Furthermore,  $\mathcal{H}$  acts freely on  $D$  since fixed points of the action would be tangencies between the stable foliation and  $D$ .

Moreover, for any point  $z \in D$ , the intersection  $W^s(z) \cap D$  is discrete in the intrinsic topology of  $W^s(z)$ . Indeed, otherwise there would exist distinct points  $w^m = \phi_z(u^m) \in D$  with bounded  $u^m$ . Then we could select a subsequence converging to a point  $w = \phi_z(u) \in \overline{D} \subset D'$ , which would be a non-isolated point of the intersection  $W^s(z) \cap D'$ .

In fact, this discreteness is uniform in the following sense: For any  $M$  and any  $z \in \overline{D}$ ,  $\text{dist}^i(z, hz) > M$  for all but finitely many holonomy homeomorphisms  $h \in \mathcal{H}$ . Indeed, if there is a sequence  $w^m = h_m(z^m) = \Phi(z^m, u^m)$  with bounded  $u^m$ , then we can select a converging subsequence  $u^m \rightarrow u, z^m \rightarrow z \in \overline{D}, w^m \rightarrow w \in \overline{D}$  so that  $w = \Phi(z, u) = h(z)$  for some  $h \in \mathcal{H}$ . Then  $h(z^m) = \Phi(z^m, t^m)$  with  $t^m \rightarrow u$ , and hence  $|t^m - u^m| \rightarrow 0$ . But for  $m$  large enough, both  $\Phi(z^m, t^m)$  and  $w^m = \Phi(z^m, u^m)$  lie in the same flow box  $B$  around  $w$ . Hence they lie in the same local leaf of  $B$ . Since the latter intersects  $\overline{D}$  in a single point, we conclude that  $w^m = h(z^m)$ , and so  $h_m = h$  (for all large enough  $m$ ).

Let us now show that  $\mathcal{H}$  acts properly discontinuously on  $D$ , i.e., for any two neighborhoods  $Z$  and  $W$  compactly contained in  $D$ , we have  $h(Z) \cap W = \emptyset$  for all but finitely many  $h \in \mathcal{H}$ . Indeed, assume there is a sequence of distinct  $h^m \in \mathcal{H}$  and of points  $z^m \in Z, w^m = h^m(z^m) \in W$ . As we have just shown,  $\text{dist}^i(z^m, w^m) \rightarrow \infty$ . Now we can apply our usual argument to arrive at a contradiction. Namely, there exist times  $n_m \rightarrow \infty$  that bring the points  $z^m$  and  $w^m$  to the same local stable manifold, implying that  $\|f^{n_m}(z^m) - f^{n_m}(w^m)\| \asymp 1$ , which contradicts Lemma 6.1.

Hence the quotient  $S = D/\mathcal{H}_D$  is a qc surface (i.e., a surface endowed with qc local charts with uniformly bounded dilatation). Taking any conformal structure (a Beltrami differential)  $\mu$  on  $S$  and pulling it back to  $D$ , we obtain an  $\mathcal{H}$ -invariant conformal structure on  $D$ . By the Measurable Riemann Mapping Theorem, there exists a qc map  $\psi : D \rightarrow \mathbb{D}$  such that  $g = \psi \circ h \circ \psi^{-1}$  is Möbius for any  $h \in \mathcal{H}_D$ .

Let  $\zeta \in \mathbb{D}$  and denote its orbit by  $\zeta_n = g^n(\zeta)$ . Then the hyperbolic distance between  $\zeta_n$  and  $\zeta_{n+1}$  is independent of  $n$  since it is preserved under holomorphic automorphisms. Let  $z_n = \psi^{-1}(\zeta_n) \in D$ . Since  $\psi$  is quasiconformal, it is a quasi-isometry, that is,  $\psi$  expands the hyperbolic distance by a bounded factor for scales bounded away from zero. It follows that the hyperbolic distance between  $z_n$  and  $z_{n+1}$  is bounded for all  $n$ .

Since  $g$  does not have fixed points in  $\mathbb{D}$ , the  $\zeta_n$  converge to a Denjoy–Wolff point in  $\partial\mathbb{D}$ . Hence the sequence  $(z_n)_{n \in \mathbb{N}}$  escapes to the boundary  $\partial D$ . Since near the boundary the hyperbolic metric of  $D$  explodes relative to the Euclidean metric of  $D'$ , we conclude that

$$\|z_n - z_{n+1}\| \rightarrow 0. \tag{7}$$

Note that all the points  $z_n$  lie in the same stable manifold  $W^s(z)$ , so we can measure the intrinsic distance between them. Property (7), together with Lemma 6.2, implies that  $\text{dist}^i(z_n, z_{n+1}) \rightarrow 0$ . It follows that any limit point  $q \in \overline{D} \subset D'$  for  $(z_n)$  is a tangency between  $D'$  and  $W^s(q)$ , and this contradiction completes the proof.  $\square$

**Remark 6.9.** The above application of the Measurable Riemann Mapping Theorem is a special case of Sullivan's Theorem concerning uniformly qc group actions [Sul81], [Tuk86].

Since stable manifolds in  $U^0$  each intersect  $D$  in a unique point, we conclude:

**Corollary 6.10.** *Let  $D$  be a global transversal to a wandering component  $U$ . Then the uniformization  $\Phi : D \times \mathbb{C} \rightarrow U$  is a vertically holomorphic homeomorphism.*

### 6.5. Degree bound

Let  $U^0$  be a semilocal wandering component of a wandering component  $U$ , and let  $U^n$  be the component of  $f^n(U) \cap \Delta_R^2$  containing  $f^n(U^0)$ . We define the degree of  $f^n : U^0 \rightarrow U^n$  as the maximal number of semilocal dynamical leaves in  $U^0$  that are mapped into a single semilocal dynamical leaf in  $U^n$ .

**Lemma 6.11.** *For any semilocal wandering component  $U^0$ , the degree of  $f^n : U^0 \rightarrow U^n$  is uniformly bounded over all  $n$  (with a bound depending on  $U^0$ ).*

*Proof.* By replacing  $U^0$  with an appropriate  $U^m$ , we can ensure that all the domains  $U^n$ ,  $n \geq 0$ , are contained in the neighborhood  $\mathcal{N}(J^+)$ , so the dynamical and the artificial vertical laminations coincide on these domains. Note that degrees of compositions are submultiplicative, so a bound on the degrees of  $f^n : U^m \rightarrow U^{m+n}$  implies a bound on the degrees of  $f^{m+n} : U^0 \rightarrow U^{m+n}$ .

Let us consider a horizontal line  $L = \mathbb{L}_0$ , and let  $t$  be the number of tangencies between  $\mathbb{L}_0$  and the artificial vertical lamination. Let  $L^0 := U^0 \cap L$ , and let  $L^n := f^n(L^0)$ .

As each semilocal dynamical leaf of  $U^0$  intersects  $L^0$ , the degree of  $f^n : U^0 \rightarrow U^n$  is bounded by the maximal number of intersections between  $L^n$  and the vertical leaves of  $U^n$ . Since the artificial vertical lamination of  $U^n$  coincides with the (invariant) dynamical vertical lamination, the degree of  $f^n : U^0 \rightarrow U^n$  is bounded by the maximal number of intersections between  $L^n$  and dynamical leaves of  $f^n(U)$ .

Let  $D^n$  be a global transversal to  $f^n(U)$  and let  $\Phi_n : D^n \times \mathbb{C} \rightarrow f^n(U)$  be the corresponding uniformization. Then the maximal number of intersections between  $L^n$  and dynamical leaves of  $f^n(U)$  is equal to the degree  $d$  of the horizontal projection  $\Phi_n^{-1}(L^n) \rightarrow D^n$ . This projection is a branched covering since  $L_n$  is properly embedded into  $U^n$ . By the Riemann–Hurwitz formula,  $d$  equals at least 1 plus the number of tangencies (counted with multiplicities) between  $L_n$  and the dynamical foliation. But the latter is preserved by the dynamics, so it equals the number of tangencies between  $L^0$  and the dynamical foliation of  $U^0$ , which is bounded by  $t$ . The conclusion follows.  $\square$

### 7. Horizontal lamination and $\text{deg}_{\text{crit}}$

We let  $U^0$  be a semilocal wandering Fatou component, and for  $n \in \mathbb{N}$  we write  $U^{-n}$  for a semilocal wandering component satisfying  $f^n(U^{-n}) \subset U^0$ .

**Assumption A.** *Let us say that a semilocal wandering component  $U^0$  satisfies Assumption A if all possible choices of the domains  $U^{-n}$ ,  $n \geq 0$ , are contained in  $\mathcal{N}(J_R^+)$  (defined at the end of §5.2), and thus in the domain of dominated splitting.*

In what follows, through Corollary 7.5, we will assume that  $U^0$  satisfies Assumption A.

We write  $W^{-n}$  for a connected component of  $U^{-n} \cap \mathbb{L}_0$ , and consider holomorphic disks  $f^n(W^{-n}) \subset U^0$ , ranging over all  $n \in \mathbb{N}$  and all choices of  $U^{-n}$  and  $W^{-n}$ . Since the horizontal line  $\mathbb{L}_0$  is chosen so that it is transverse to the artificial vertical lamination near  $J^+$ , for  $n$  sufficiently large the tangent spaces to the holomorphic disks  $f^n(W^{-n})$  are contained in the horizontal cone field. In fact, by taking  $n$  sufficiently large we may assume that the horizontal cone field is arbitrarily thin.

Let  $z \in U^0$ , and consider small bidisks  $\Delta_\rho^2(z) \subset \Delta_r^2(z) \subset U^0$ , with respect to affine coordinates contained in the horizontal respectively vertical cone field, and with boundary bounded away from  $J^+$ . For  $\rho \ll r \ll 1$  any connected component of  $f^n(W^{-n}) \cap \Delta_r^2(z)$  intersecting  $\Delta_\rho^2(z)$  is a horizontal graph. The collection of these graphs forms a normal family, hence any sequence has a subsequence that converges locally uniformly. We consider the Riemann surfaces  $\mathcal{S}_v$  that are locally given as uniform limits of these horizontal graphs.

**Lemma 7.1.** *If two limits  $\mathcal{S}_1$  and  $\mathcal{S}_2$  intersect, then they are equal.*

*Proof.* Assume towards a contradiction that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  intersect at a point  $z_0$ , but locally they do not coincide. Assume that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are locally given as limits of graphs in respectively  $f^{n_j}(W^{-n_j})$  and  $f^{m_l}(W^{-m_l})$ . In local coordinates  $\Delta_r^2(z_0)$  we can write  $\mathcal{S}_1 = \{y = \varphi(x)\}$  and  $\mathcal{S}_2 = \{y = \psi(x)\}$ .

It follows that for large enough  $j$  and  $l$  the horizontal graphs in  $f^{n_j}(W^{-n_j})$  and  $f^{m_l}(W^{-m_l})$  intersect at a point near  $z_0$ .

Let  $j$  and  $l$  be large with, say,  $n_j > m_l$ . Then there exists a point

$$\zeta_{m_l} \in f^{n_j - m_l}(W^{-n_j}) \cap W^{-m_l}$$

for which  $f^{m_l}(\zeta_{m_l})$  lies near  $z_0$ . As discussed above, for some large fixed  $N$  independent of  $l$  and  $j$  the local graphs in  $f^N \circ f^{n_j - m_l}(W^{-n_j})$  and  $f^N(W^{-m_l})$  are both horizontal. Given that  $f$  acts as an exponential contraction in the vertical direction, while being at most subexponentially contracting in the horizontal direction (by Lemma 4.3), it follows that near  $f^{m_l}(\zeta_{m_l})$  the distance between the horizontal graphs in  $f^{n_j}(W^{-n_j})$  and  $f^{m_l}(W^{-m_l})$  shrinks exponentially fast as  $l \rightarrow \infty$ . Therefore, the limits  $\varphi(\mathbb{D}_r)$  and  $\psi(\mathbb{D}_{r'})$  coincide locally. Since they are both proper holomorphic disks, they must coincide globally, which gives a contradiction.  $\square$

We will refer to the collection of Riemann surfaces  $\mathcal{S}_v$  as the *horizontal lamination* in  $U^0$ . It is clear that this lamination is contained in the backward Julia set  $J^-$ . Since  $U^0$  is

Kobayashi hyperbolic, each leaf is a hyperbolic Riemann surface. The leaves are locally given as limits of horizontal graphs, hence are themselves also horizontal.

**Lemma 7.2.** *The horizontal and the vertical laminations do not share leaves.*

*Proof.* The horizontal leaves are locally given as uniform limits  $f^n(W^{-n}) \cap \Delta_r^2(z)$ . Recall that  $V^-$ , one of the sets defining the filtration of the Hénon map, is strictly backward invariant, i.e. the interior of  $f^{-1}(V^-)$  is contained in  $V^-$ . It follows that the forward images  $f^n(W^{-n})$  are bounded away from  $V^{-1}$ , hence horizontal leaves cannot intersect the horizontal boundary of  $\Delta^2(R)$ . But all vertical leaves do intersect this boundary.  $\square$

**Corollary 7.3.** *For any semilocal component  $U^0$  satisfying Assumption A the order of tangency between horizontal and vertical leaves in  $U^0$  is bounded.*

*Proof.* This follows from two observations:

- The order of tangency between the leaves depends upper semicontinuously on the intersection point.
- Near  $J^+$  the laminations are transverse.  $\square$

**Corollary 7.4.** *For any semilocal component  $U^0$  satisfying Assumption A, any preimage  $U^{-j}$ , and any component  $W^{-j}$  of the horizontal slice  $\mathbb{L}_0 \cap U^{-j}$ , the order of tangency of the holomorphic disk  $f^j(W^{-j})$  with the dynamical vertical foliation  $\mathcal{F}_{U^0}$  is bounded.*

*Proof.* This is obvious for small  $j$ . For a large  $j$ , the disk  $f^j(W^{-j})$  is a small perturbation of some horizontal leaf  $L$  in  $U^0$ , so the order of tangency between  $f^j(W^{-j})$  and  $\mathcal{F}_{U^0}$  is bounded by the order of tangency between  $L$  and  $\mathcal{F}_{U^0}$ .  $\square$

For a component  $U^0$  satisfying Assumption A, let us define  $\deg_{\text{crit}}(U^0)$  as the maximum order of tangency between the above holomorphic disks  $f^j(W^{-j})$  and the dynamical vertical foliation  $\mathcal{F}_{U^0}$ .

**Corollary 7.5.**  *$\deg_{\text{crit}}(U^0)$  is bounded over all components  $U^0$  satisfying Assumption A.*

*Proof.* If all forward components  $U^n$  are contained in  $\mathcal{N}(J^+)$  then the dynamical vertical lamination  $\mathcal{F}_{U^0}$  is tangent to the vertical line field  $E^v$ , which is transverse to the horizontal lamination. Hence  $\deg_{\text{crit}} U^0 = 1$  in this case.

Therefore we only need to consider the case where some forward component  $U^n$  is not contained in  $\mathcal{N}(J^+)$ . Since  $\deg_{\text{crit}}(U^0)$  is defined by means of two dynamical laminations, both invariant under  $f$ , it remains the same for all semilocal preimages  $U^{-j}$  of  $U^0$ . Thus, it suffices to consider only those semilocal components  $U^n$  for which  $U^{n+1}$  does not satisfy Assumption A. Since there are only finitely many such components, the conclusion follows.  $\square$

Finally, let us get rid of Assumption A:

**Lemma 7.6.** *For an arbitrary semilocal component  $U$ , any component  $W$  of the horizontal slice  $\mathbb{L}_0 \cap U$ , and any integer  $n \geq 0$ , the order of tangency of the holomorphic disk  $f^n(W)$  with the dynamical vertical foliation  $\mathcal{F}_{U^n}$  is bounded (where  $U^n$  is the semilocal component containing  $f^n(U)$ ).*

*Proof.* We already know this for components satisfying Assumption A, so let us deal with other components.

Assume  $U^n \subset \mathcal{N}(J^+)$ ,  $n = 0, 1, \dots$ . Then the vertical dynamical foliations on the  $U^n$  are tangent to the vertical line field  $E^v$ . On the other hand, by the choice of  $\mathbb{L}_0$ , the slice  $W$  is transverse to this line field. Thus,  $W$  is transverse to  $\mathcal{F}_U$ . By invariance of the dynamical foliation, the forward iterates  $f^n(W)$  are transverse to  $\mathcal{F}_{U^n}$ : no tangencies in play.

This leaves us with finitely many components  $U$ . For each of them,  $W$  has finitely many tangencies with  $\mathcal{F}_U$  counted with multiplicities (by construction of  $\mathbb{L}$ ). By invariance of the dynamical foliations,  $f^n(W)$  has the same number of tangencies with  $\mathcal{F}_{U^n}$  for any integer  $n \geq 0$ , and with the same orders. The conclusion follows.  $\square$

**Definition 7.7** ( $\text{deg}_{\text{crit}}$ ). Let  $\text{deg}_{\text{crit}}$  be the maximum of the orders of tangency that appear in the above lemma.

**Corollary 7.8.** *For any semilocal component  $U^0$ , any preimage  $U^{-j}$ , and any component  $W^{-j}$  of the horizontal slice  $\mathbb{L}_0 \cap U^{-j}$ , the order of tangency of the holomorphic disk  $f^j(W^{-j})$  with the dynamical vertical foliation  $\mathcal{F}_{U^0}$  is bounded by  $\text{deg}_{\text{crit}}$ .*

### 8. Final preparations

The following is a rephrasing of Corollary 3.3.

**Lemma 8.1.** *Let  $\epsilon > 0$ . Then there exists a domain  $\Omega \subset \Delta_R^2$  for which*

$$f^{-1}(\Omega) \cap \Delta_R^2 \subset \Omega$$

and

$$(J^+ \cap \Delta_R^2) \cup \bigcup (\text{semilocal wandering components}) \setminus \{\text{parabolic cycles}\} \subset \Omega,$$

and which satisfies two conditions:

- (i) *For any  $z \in J^+$  there exist  $w \in \Delta_R^2$  with  $|z - w| < \epsilon$  such that the semilocal leaf through  $w$  does not intersect  $\Omega$ .*
- (ii) *Any  $z \in \Omega$  that does not lie in a wandering Fatou component lies  $\epsilon$ -close to  $J^+$ .*

Note that property (2) follows since  $J^+$  is the boundary of the basin of the point  $[1 : 0 : 0]$ , and  $\Omega$  can be chosen not to intersect any compact subset of this basin. Alternatively note that in the proof of Corollary 3.3 the domain  $\Omega$  is chosen as a subset of  $\Delta_R^2 \cap \{G^+ < \epsilon\}$  for  $\epsilon > 0$  arbitrarily small; this constant  $\epsilon$  is directly related to the  $\epsilon$  in Lemma 8.1, but typically they are not equal.

In several instances of the proofs that follow, the value of the constant  $\epsilon > 0$  must be sufficiently small, which we will refer to by saying that  $\Omega$  should be *sufficiently thin*. The domain  $\Omega$  will only be fixed after all bounds on degrees and diameters are determined. To avoid a circular argument we should take care that the constants that appear in those bounds can be defined independently of the exact choice of  $\Omega$ . At this time we only guarantee that  $\Omega$  is chosen sufficiently thin so that every point  $z \in \Omega$  lies on a semilocal leaf of the artificial vertical lamination.

**Lemma 8.2.** *Given constants  $d \in \mathbb{N}$ ,  $r < 1$  and  $\mu > 0$  there exists a constant  $C = C(r, d, \mu)$  with the following property. For any hyperbolic Riemann surface  $V$ , and any proper quasiregular map  $f : V \rightarrow \mathbb{D}$  of degree at most  $d$  whose dilatation is bounded by  $\mu$ , the hyperbolic diameter of any connected component of  $f^{-1}D_r(0)$  is bounded by  $C$ . For fixed  $d, \mu$  the constant  $C(r, d, \mu)$  converges to 0 as  $r \rightarrow 0$ .*

*Proof.* The quasiregular map can be written as the composition of a proper quasiconformal homeomorphism with dilatation bounded by  $\mu$ , with a proper holomorphic map of degree at most  $d$ . The statement holds for both of these maps, and hence also for the composition.  $\square$

**Definition 8.3** (Protected lifts). Let  $S \subset \Delta_R^2$  be a properly embedded holomorphic disk, bounded away from the horizontal boundary  $\{|y| = R\}$ . For a disk  $D \subset \mathbb{L}_0$  consider all artificial vertical leaves through points in  $D$ . Write  $V \subset S$  for a connected component of the set of intersection points of these leaves with  $S$ .

We say that  $V$  is a *lift* of  $D$  if the holonomy correspondence is *proper*, i.e., for any compact  $E \subset D$  the set of intersection points of the leaves through  $E$  with  $V$  is compact.

We will consider lifts in  $f^j \mathbb{L}_0 \cap \Omega$ . Let  $t > 1$  and consider two concentric disks  $D_r(z) \subset D_{t \cdot r}(z) \subset \mathbb{L}_0$ . If the disks  $D_r(z)$  and  $D_{t \cdot r}(z)$  can be lifted to  $V_r(z) \subset V_{t \cdot r}(z) \subset f^j \mathbb{L}_0 \cap \Omega$ , then we say that  $V_r(z)$  is a *protected lift* of  $D_r(z)$ . We define the *degree* of  $V_r(z)$  as the maximal number of intersections between  $V_r(z)$  and artificial vertical leaves through  $D_r(z)$ .

Recall that the artificial vertical leaves intersect  $\mathbb{L}_0$  transversely near  $J^+$ . Thus, for sufficiently small disks  $D_r(z) \subset \mathbb{L}_0$  sufficiently close to  $J^+$  the lift is traditional: a pullback under the holonomy map. However, if  $D_r(z)$  intersects an artificial leaf non-transversely then the holonomy from  $V_r(z)$  to  $D_r(z)$  cannot be single valued, so we talk about the holonomy correspondence.

The properness of lifts is not automatic, and may be violated when an artificial vertical leaf through a disk  $D$  is tangent to the boundary of  $\Delta_R^2$ . It is therefore possible that a disk  $D_r(z) \subset \mathbb{L}_0$  cannot be lifted, even when all artificial vertical leaves through  $D$  are contained in  $\Omega$ . However, for every point  $z \in \mathbb{L}_0$  for which the vertical leaf through  $z$  intersects  $f^j \mathbb{L}_0 \cap \Omega$  there is a sufficiently small disk that can be lifted. Conversely, every point in  $f^j \mathbb{L}_0 \cap \Omega$  is contained in some lift  $V_r(z)$ .

**Definition 8.4.** Let  $V$  be a Riemann surface, embedded in a bounded open set  $U \subset \mathbb{C}^2$ . Let  $a, b \in V$ . By the *intrinsic Kobayashi distance* between  $a$  and  $b$  in  $U$  we mean the infimum of the Kobayashi lengths in  $U$  of paths contained in  $V$  from  $a$  to  $b$ . The intrinsic

Kobayashi diameter of  $V$  in  $U$ , denoted by  $\text{diam}_U V$ , is the supremum of distances between points  $a, b \in V$ . This definition will not cause problems, as we will never consider the actual Kobayashi diameter of  $V$  in  $U$ , i.e. we will never consider paths that travel outside of  $V$ .

**Lemma 8.5.** *Let  $V_r(z) \subset V_{t,r}(z)$  be a protected lift of degree  $d$ . Then there exists a constant  $C_1(1/t, d) > 0$  such that*

$$\text{diam}_\Omega V_r(z) \leq C_1(1/t, d),$$

and given  $d$  the constant  $C_1(1/t, d)$  converges to 0 as  $t \rightarrow \infty$ .

*Proof.* If  $D_{t,r}$  intersects artificial vertical leaves at most once, then the holonomy defines a proper quasi-regular map from  $V_{t,r}$  to  $D_{t,r}$ . The degree of the holonomy map is then exactly the maximal number of intersections of  $V_{t,r}$  with leaves of the artificial lamination, which is  $d$ . It follows from Lemma 8.2 that the Poincaré diameter

$$\text{diam}_{V_{t,r}(z)} V_r(z)$$

is then bounded by  $C(1/t, d, \mu)$ , for a bound  $\mu$  on the order of qc-dilatation of the holonomy maps induced by the artificial vertical lamination. By definition of protected lifts we have  $V_{t,r}(z) \subset \Omega$ , which implies the same bound on the intrinsic Kobayashi diameter in  $\Omega$ .

Recall that the vertical lamination of  $J^+$  is transverse to  $\mathbb{L}_0$ , hence the above discussion applies to sufficiently small disks in a sufficiently small neighborhood of  $J^+$ . By choosing  $\Omega$  sufficiently thin, it follows that  $D_{t,r}(z) \subset \mathbb{L}_0$  intersects each vertical leaf in a unique point, unless  $D_{t,r}(z)$  intersects one of finitely many wandering Fatou components.

It is clear that for each given  $D_r(z) \subset D_{t,r}(z) \subset \mathbb{L}_0$  there does exist a bound on

$$\text{diam}_{V_{t,r}(z)} V_r(z),$$

depending only on the degree of  $V_{t,r}(z)$ . Hence by compactness we obtain a bound when the radius  $r$  is bounded away from zero. But when  $r$  is sufficiently small, either the disk  $D_{t,r}(z)$  lies in a neighborhood of  $J^+$  that guarantees that  $D_{t,r}(z)$  intersects each artificial vertical leaf at most once, or  $D_{t,r}(z)$  lies well inside a semilocal wandering Fatou component  $U$ , of which there are at most finitely many. It follows that

$$\text{diam}_U V_r(z) \rightarrow 0$$

as  $r \rightarrow 0$ . Since  $U \subset \Omega$ , this completes the proof. □

**Lemma 8.6.** *There exists a constant  $\epsilon > 0$  such that the following holds. Let  $V \subset \mathbb{L}_0$  be a holomorphic disk, and write  $V^j = f^j(V)$ . Suppose that for  $j = 0, \dots, n$  we have  $V^j \in \Delta_R^2$  and*

$$\sup_{z \in V^j} d(z, J^+) < \epsilon,$$

where  $d(\cdot, \cdot)$  refers to the Euclidean distance in  $\mathbb{C}^2$ . Then each  $V^j$  is transverse to the artificial lamination. If the Euclidean diameter of each  $V^j$  is sufficiently small then it follows moreover that each  $V^j$  has degree 1.

*Proof.* Recall that  $\mathbb{L}_0$  is transverse to the artificial vertical lamination in a small neighborhood of  $J^+$ . Since the disks  $V^j$  remain in the region of dominated splitting, their tangent spaces lie in some large horizontal cone field, while in a small neighborhood of  $J^+$  the tangent spaces to the vertical leaves do not intersect those horizontal cones. Thus transversality follows, in fact with uniform bounds on the angles between the tangent spaces of the disks  $V^j$  and the leaves of the artificial vertical lamination. It follows that sufficiently small disks will have degree 1.  $\square$

**Lemma 8.7.** *Let  $t > 1$ . There exists an  $N_0 = N_0(t) \in \mathbb{N}$  such that the following holds. For each protected lift  $V_r(z) \subset V_{t,r}(z)$  of disks  $D_r(z) \subset D_{t,r}(z) \subset \mathbb{L}_0$ , the preimage  $V_r^{-1}(z)$  can be covered by protected lifts of at most  $N_0$  disks  $D_{r_k}(z_k) \subset D_{2t,r_k}(z_k)$ .*

*Moreover, in the particular case where the disk  $D_{t,r}(z)$  lies in a semilocal wandering component  $U$ , the lifts  $V_{2t,r_k}(z_k)$  are all contained in the component  $U^{-1}$ . In all other cases, the lifts  $V_{2t,r_k}(z_k)$  are all contained in  $V_{tr}^{-1}(z)$ .*

*Proof.* It is clear that for every  $r > 0$  and  $z \in \mathbb{L}_0$  and every choice of protected lift  $V_r(z) \subset V_{t,r}(z)$ , the disk  $V_r^{-1}(z)$  can be covered with a finite number of protected lifts  $V_{r_k}(z_k)$  for which  $V_{2tr_k}(z_k) \subset V_{tr}^{-1}(z)$ . Suppose for a contradiction that there exists a sequence  $(r, z, V_r(z))$  for which the minimal number of lifts needed converges to infinity.

By restricting to a subsequence we may assume that the lifts  $V_{t,r}(z)$  are either contained in wandering Fatou components, in periodic Fatou components, or all intersect  $J^+$ . We consider these cases separately.

First suppose that the lifts  $V_{t,r}(z)$  are contained in wandering Fatou components. Suppose first that  $V_{t,r}(z)$  is contained a wandering domain  $U$  where the artificial lamination is backward invariant. By invariance the image of  $V_{t,r}^{-1}(z)$  in  $\mathbb{L}_0$  under holonomy is independent of the choice of lift, and the argument is the same as in the one-dimensional setting: For disks of radius bounded away from zero the bound on  $N_0$  follows from compactness. But sufficiently small disks are either contained in a small neighborhood of  $J^+$ , where the lamination is transverse to  $\mathbb{L}_0$ , or well inside wandering domains. In the latter case it is clear that  $V_r^{-1}(z)$  can in fact be covered by a single lift. In the former case holonomy induces a quasiconformal map of bounded dilatation, which gives a bound on the distortion and thus on  $N_0$ .

Since the vertical lamination is invariant except in finitely many components, we may therefore assume that all  $V_{t,r}(z)$  are in one of the wandering components for which the vertical lamination is not backward invariant. Recall that in this wandering Fatou component the vertical lamination is still invariant in a neighborhood of the boundary  $J^+$ . The bound on  $N_0$  follows again when  $r$  remains bounded away from zero, hence we may assume that  $r \rightarrow 0$ . But in that case either the lifts are very close to  $J^+$ , where the lamination is invariant and the bound on  $N_0$  follows as above, or the lifts are bounded away from  $J^+$ . But then for sufficiently small  $r$  the preimage  $V_r^{-1}(z)$  can again be covered by a single lift  $V_\rho(w)$  for which  $V_{2t,\rho}(w) \subset U^{-1}$ .

Now suppose that the lifts  $V_{t,r}(z)$  are contained in periodic Fatou components. Recall that  $\mathbb{L}_0$  was chosen to be transverse to the dynamical lamination near  $J^+$ . Thus, by making  $\Omega$  sufficiently thin, we may assume that both  $V_{t,r}(z)$  and  $V_{t,r}^{-1}(z)$  are transverse to the



vertical lamination, with angles bounded from below (see Lemma 8.6). Again it follows that holonomy from  $\mathbb{L}_0$  to  $V_{t,r}^{-1}(z)$  and from  $V_{t,r}(z)$  back to  $\mathbb{L}_0$  induces a quasiconformal map of bounded dilatation, which implies a bound on  $N_0$ .

The last case to be considered is when the lift  $V_{t,r}(z)$  intersects  $J^+$ . Suppose first that there exists a subsequence for which  $r$  converges to zero. In that case  $n$  must converge to infinity, since otherwise the preimages  $V_{t,r}^{-i}(z)$  are all contained in the domain of dominated splitting, which implies that the lifts  $V_{t,r}$  are horizontal, giving a bound on  $N_0$ . In fact, by Lemma 4.3 the contraction in the horizontal direction is subexponential, and hence backwards in time the expansion is subexponential, therefore we may assume that

$$n \geq \log_\alpha(r)$$

for any  $\alpha > 1$  and  $r$  sufficiently small.

Since  $n$  is large the disks  $V_{t,r}(z)$  must intersect  $J^+$  in a point where  $G^-$  is very close to zero, which implies that there is a nearby point  $x \in J$  on the same local stable manifold. For each  $x \in J$  there exists a small closed loop  $\gamma$  around  $x$  in  $W^s(p)$  where  $G^-$  is strictly positive. Consider small disks through each point in  $\gamma$ , normal to  $W^s(p)$ , and therefore in particular horizontal, and denote their union by  $\Gamma$ . By making these horizontal disks sufficiently small, we can guarantee that  $G^-$  is strictly positive on  $\Gamma$ . It follows that there exists an  $N$  such that

$$f^{-N}(\Gamma) \cap \Delta_R^2 = \emptyset.$$

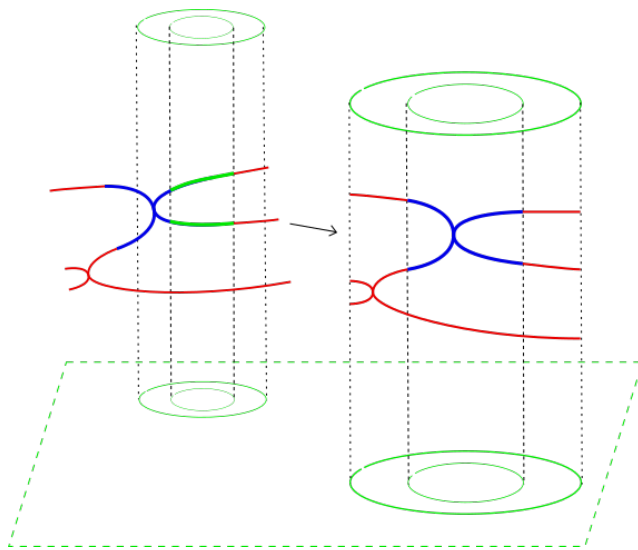
By compactness of  $J$ , we can find a uniform bound from above on the diameter of the  $\gamma$ , a uniform bound from below on the size of the horizontal disks through  $\gamma$ , and a uniform bound on  $N$ .

By the exponential contraction in the vertical direction and the fact that the inverse images  $f^{-j}(V_{t,r}(z))$  are all contained in the bidisk  $\Delta_R^2$ , it follows that  $V_{t,r}(z)$  is contained in a tubular neighborhood with radius of order  $r^\alpha$  around the forward image of a horizontal disk  $D$  (with radius of order  $t \cdot r$ ) in an  $f^{n-N}\Gamma(p)$ , for some point  $p \in J$ , with similar estimates for  $V_{t,r}^{-1}(z)$  and  $f^{-1}(D)$ . Therefore it is sufficient to consider the respective lifts in  $D$  and  $f^{-1}(D)$ , and the fact that these disks are horizontal implies a bound on  $N_0$  by the same argument as above.

Thus the remaining lifts  $V_{t,r}(z)$  we need to consider have radius bounded away from zero and intersect  $J^+$ . By making  $\Omega$  sufficiently thin we can therefore guarantee that for some  $1 < t_2 < t$  the lift  $V_{t_2,r}(z)$  is contained in a wandering domain  $U$ . The existence of the bound  $N_0$  follows by the same argument as when  $V_{t,r}(z) \subset U$ . □

Figure 6 illustrates the covering of  $V_r^{-1}(z)$  by lifts of disks  $D_{r_k}(z_k) \subset D_{t \cdot r_k}(z_k)$ . The lift  $V_r(z)$  and its inverse image  $V_r^{-1}(z)$  are depicted in blue; the rest of the larger  $V_{t,r}(z)$  and  $V_{t,r}^{-1}(z)$  in red. In the sketch the vertical lamination is given by straight vertical lines. Two distinct lifts of a single disk  $V_{r_k}(z_k)$  are depicted in green.

**Remark 8.8.** By construction the bound  $N_0$  is independent of  $\Omega$ , that is,  $N_0$  does not need to be changed when  $\Omega$  is made smaller.



**Fig. 6.** The set  $V_r^{-1}$  is covered by lifts of disks  $D_{r_k}(z_k) \subset D_{t-r_k}(z_k)$ .

Let  $U$  be a semilocal wandering domain on which the artificial lamination is not equal to the dynamical lamination. We will compare lifts of disks  $D_r(z) \subset \mathbb{L} \cap U$  with respect to both laminations.

**Proposition 8.9.** *There exists  $\delta = \delta(U) > 0$  such that the following holds. Let  $D \subset \mathbb{L}_0 \cap U$  be a holomorphic disk of hyperbolic diameter (in  $\mathbb{L}_0 \cap U$ ) at most  $\delta$ , and let  $V \subset f^j \mathbb{L}_0$  be a lift with respect to the dynamical lamination. Then  $V$  is contained in a lift with respect to the artificial lamination of a protected disk  $D_\rho(w) \subset D_{2,\rho}(w) \subset \mathbb{L}_0 \cap U$ .*

*Proof.* Note that any connected component of  $\mathbb{L}_0 \cap U$  is simply connected. Therefore, as in the proof of Lemma 2.10, a subset  $E \subset \mathbb{L}_0 \cap U$  of sufficiently small hyperbolic diameter is contained in a disk  $D_r(w)$  satisfying  $D_r(w) \subset D_{2,r}(w) \subset U$ . Hence the proof is completed by showing that  $V$  is a lift (with respect to the artificial lamination) of a set  $E \subset \mathbb{L}_0 \cap U$  of sufficiently small hyperbolic diameter.

Since the dynamical and artificial laminations coincide near the boundary of  $U$ , the statement holds trivially when  $V$  is sufficiently close to  $\partial U$ . We will therefore consider lifts of disks  $D$  that are bounded away from  $\partial U$ .

Recall from the previous section that the holomorphic disks  $f^n(W^{-n})$  converge to the horizontal lamination. Both the horizontal leaves and the disks  $f^n(W^{-n})$  cannot coincide with vertical dynamical leaves, hence for a sufficiently small tubular neighborhood  $\mathcal{N}$  of a dynamical leaf, the intersections  $f^n(W^{-n}) \cap \mathcal{N}$  have arbitrarily small Euclidean diameters. Since we consider leaves that are bounded away from the boundary, small Euclidean diameters imply small intrinsic Kobayashi diameters in  $U$ , and thus  $V$  can be assumed to be a lift of some  $E \subset \mathbb{L}_0 \cap U$  of arbitrarily small hyperbolic diameter, which completes the proof.  $\square$

**Definition 8.10** ( $\beta$ ). Let  $\Lambda_R(a)$  be a vertical leaf that is either contained in  $J^+$  or in a semilocal wandering domain, and assume that  $\Lambda_R(a)$  is not one of the finitely many parabolic leaves in  $J^+$  that were removed from  $\Omega$ . Then the leaf  $\Lambda_R(a)$  is contained in  $\Omega$ . Note that the sets  $f^j \mathbb{L} \cap \Delta_R^2$  stay bounded away from the horizontal boundary  $\{|w| = R\}$ . It follows that there exists an upper bound  $\beta$  such that for any  $j \in \mathbb{N}$  the intrinsic Kobayashi distance in  $\Omega$  between points in  $\Lambda_R(a) \cap f^j \mathbb{L}_0$  is bounded by  $\beta$ . Note that the choice of  $\beta$  holds when  $\Omega$  is made thinner.

**Definition 8.11** ( $\text{diam}_{\max}$ ). For  $t > 1$  and an integer  $d \geq 2$  we define

$$\text{diam}_{\max}(t, d) := 2N_0(t) \cdot C_1(1/2, d) + \beta.$$

In what follows  $t$  will equal either 2 or  $2K$ , where the constant  $K$  will be introduced in Proposition 9.2. The integer  $d$  will be either 1 or  $\text{deg}_{\text{crit}}$ .

**Definition 8.12** (Big wandering domain). We say that a semilocal wandering domain  $U$  is a *big wandering domain* if the vertical lamination on  $U$  is not backward invariant, or if there exist  $j \in \mathbb{N}$  and  $S \subset U \cap f^j \mathbb{L}_0$  of intrinsic Kobayashi diameter

$$\text{diam}_U S \leq \text{diam}_{\max}(2, 1)$$

intersecting a vertical leaf  $\Lambda_R(a)$  for which  $f^{-1} \Lambda_R(a) \cap \Delta_R^2$  has more than one component intersecting  $f^{-1} S$ .

**Lemma 8.13.** *There are at most finitely many big semilocal wandering domains.*

*Proof.* The fact that there are only finitely many semilocal wandering domains for which the vertical lamination is not backward invariant follows immediately from the construction of the artificial vertical lamination. Indeed, there are only finitely many semilocal wandering domains that are large enough not to be entirely contained in the domain of dominated splitting, hence only finitely many grand orbits of semilocal wandering domains where the artificial vertical lamination is not dynamical. By the adjustment of the vertical lamination on wandering components in Subsection 5.3, it follows that in these finitely many grand orbits of semilocal wandering domains there are only finitely many domains where the vertical lamination is not invariant.

Let us from now on consider the second condition regarding the distance between different points in vertical leaves. First observe that if a vertical leaf  $\Lambda_R(a)$  intersects  $S$  in a second point  $b \neq a$ , such that  $f^{-1}(a)$  and  $f^{-1}(b)$  are contained in different components of  $f^{-1} \Lambda_R(a) \cap \Delta_R^2$ , then the Euclidean distance between  $a$  and  $b$  is bounded from below. This follows from the fact that the intersections  $f^{j-1} \mathbb{L}_0 \cap \Delta^2(R)$  are uniformly bounded away from the horizontal boundary of  $\Delta^2(R)$ .

Suppose for of a contradiction that there are infinitely many big semilocal wandering domains, with corresponding points  $a, b$  in the intersection with disks in  $f^j \mathbb{L}_0$ . Then we may consider either a sequence for which  $j$  is constant, or a sequence for which  $j \rightarrow \infty$ .

We first consider the case where  $j$  is constant. By hypothesis the diameter bound is violated on infinitely many domains  $U$ , hence we can restrict to components  $U$  for which

the maps  $f^j : U_{-j} \rightarrow U$  preserve the artificial vertical lamination, as there are only finitely many semilocal components for which this is not the case. Since  $\mathbb{L}_0$  is transverse to the artificial vertical lamination, it follows that  $f^j \mathbb{L}_0$  is also transverse to the artificial vertical lamination, with uniform bounds on the angles.

It follows that as the inner radius of  $U$  gets thinner and thinner, the intersections  $f^j \mathbb{L}_0 \cap U$  must also become thinner, i.e. the Kobayashi metric of  $U$  blows up along tangent directions of  $f^j \mathbb{L}_0$ , which implies that the intrinsic Kobayashi distance between  $a$  and  $b$  converges to infinity. This contradicts the hypothesis.

Therefore we may assume that  $j \rightarrow \infty$ . Considering the orbits from  $\mathbb{L}_0$  to  $S \cap f^j \mathbb{L}_0$  we can distinguish three possibilities: (i) orbits pass only through semilocal components where the artificial vertical lamination is invariant, (ii) after a bounded number of steps orbits pass through a component where the lamination is not invariant, and (iii) after a larger and larger number of steps orbits pass through components where the lamination is not invariant. In case (i) the argument used for constant  $j$  holds as well, hence we only need to consider cases (ii) and (iii). In both of these cases the number of tangencies between  $f^j \mathbb{L}_0$  and the artificial vertical lamination is bounded, in case (iii) by the properties of the horizontal lamination discussed in Section 7. Due to the exponential contraction in the vertical direction in the domain of dominated splitting, away from arbitrarily small neighborhoods of those tangencies the disks  $S \subset f^j \mathbb{L} \cap U$  become horizontal, i.e. the tangent space lies in the horizontal cone field.

Again it follows that the Kobayashi metric blows up along these tangent directions, and it follows from the fact that the Euclidean distance between  $a$  and  $b$  is bounded from below that the intrinsic Kobayashi distance must blow up, giving a contradiction. This completes the proof in all cases.  $\square$

**Definition 8.14** (Regular wandering domains). A semilocal wandering domain  $U$  is said to be *regular* if none of the components  $U^{-n}$  for  $n \geq 0$  are big wandering domains.

There exist only finitely many bi-infinite orbits of semilocal wandering domains that are not regular. A component that is not regular is called *post-critical*. Note that there are at most finitely many grand orbits of semilocal wandering Fatou components that contain post-critical domains. If  $U^n$  is regular but  $U^{n+1}$  is post-critical then we say that  $U^n$  is *critical*.

**Definition 8.15** ( $\deg_{\max}$ ). Recall from Lemma 6.11 that the degree of the maps  $f^n : U^0 \rightarrow U^n$  is bounded from above by a constant independent of  $n$ . Thus, such a forward degree bound exists for each critical wandering component. Since there are only finitely many critical components, there exists a uniform bound, which we will denote by  $\deg_{\max}$ , analogously to the one-dimensional setting.

## 9. Diameter and degree bounds: proof for Hénon maps

Let us recall the constants and objects that play a role in the upcoming proofs, listed in the order of their dependency.

$\deg_{\text{crit}}$ : The maximal local degree (Def. 7.7).

$\mathbb{L}_0$ : Convenient choice of horizontal line (Def. 5.10).

- $\beta$ : Upper bound on the intrinsic Kobayashi distance between points in artificial vertical leaves (Def. 8.10).
- $N_0(t)$ : Maximal number of  $t$ -protected disks whose lifts cover  $V_r^{-1}(z)$  (Lemma 8.7).
- $C_1(1/t, d)$ : Upper bound on the intrinsic Kobayashi diameter for lifts in  $\Omega$  (Lemma 8.5).
- $\text{deg}_{\max}$ : Bound on global degrees of iterates on critical wandering domains (Def. 8.15).
- $K$ : Defined in Prop. 9.2 below. We will consider protected lifts  $V_r(z) \subset V_{2K \cdot r}(z)$ .
- $\text{diam}_{\max}(t, d) := 2N_0(t) \cdot C_1(1/2, d) + \beta$  (Def. 8.11).
- $\Omega$ : Domain under consideration, chosen sufficiently thin (Lemma 8.1).

In this section we prove the main estimates on the diameters and degree of lifts  $V_r(z)$  and their preimages. Just as in the one-dimensional argument, we distinguish between three different kinds of lifts. First we consider lifts that are deeply contained in wandering components, i.e. lifts of disks  $D_r(z)$  for which  $D_{K \cdot r}(z)$  is contained in the same component for a sufficiently large constant  $K$ . Afterwards we consider the remaining two cases, namely lifts of disks that are not contained in wandering components, and lifts of disks  $D_r(z)$  that are contained in wandering components but for which the protecting disks  $D_{K \cdot r}(z)$  are not.

9.1. Lifts deeply contained in wandering domains.

In what follows we let  $(U^n)_{n \in \mathbb{Z}}$  be a bi-infinite orbit of semilocal wandering Fatou components, and consider protected lifts  $V_r \subset V_{t \cdot r}$  for which  $V_{t \cdot r}$  is contained in one of the domains  $U^n$ . As post-critical components are harder to deal with than regular components, we start with the latter.

**Lemma 9.1.** *Let  $U^n$  be a regular wandering semilocal Fatou component and let  $V_r(z) \subset V_{2r}(z) \subset f^j \mathbb{L}_0 \cap U^n$  be protected lifts of disks  $D_r(z) \subset D_{2r}(z) \subset \mathbb{L}_0$ . Then for  $i = 0, \dots, j$  we have*

$$\text{diam}_{\Omega} V_r^{-i}(z) \leq \text{diam}_{\max}(2, 1) \quad \text{and} \quad \text{deg } V_r^{-i}(z) = 1.$$

*Proof.* Assume the statement holds for a given  $j \in \mathbb{N}$ . By Lemma 8.7 the holomorphic disk  $V_r^{-1}(z)$  can be covered by lifts  $V_{r_k}(z_k)$  of at most  $N_0$  disks  $D_{r_k}(z_k) \subset D_{4 \cdot r_k}(z_k) \subset \mathbb{L}_0 \cap U^{n-1}$ . By the induction assumption each  $V_{2 \cdot r_k}^{-i}(z_k)$  has degree 1, and thus

$$\text{diam}_{U^{n-i-1}} V_{r_k}^{-i}(z_k) \leq C_1(1/2, 1).$$

Note that a priori we do not have a bound on the number of lifts  $V_{r_k}(z_k)$ . Since  $U^n$  is regular, the vertical lamination on the components  $(U^{n-i})_{i \geq 0}$  is invariant. Let  $x \in V_r(z)$ , and write  $x_{-i} = f^{-i}(x)$ . Each point in a lift  $V_{r_k}(z_k)$  can be connected to the semilocal leaf  $\Lambda(x_{-1})$  by a path that travels through at most  $N_0$  other lifts  $V_{r_l}(z_l)$ . Hence

$$\text{diam}_{U^{n-1}} V_r^{-1}(z) \leq \text{diam}_{\max}(2, 1).$$

Recall that by Definitions 8.12 and 8.14 of respectively *big* and *regular* Fatou components, this bound implies that  $\Lambda(x_{-2})$  is the unique semilocal leaf in  $f^{-1}\Lambda(x_{-1}) \cap \Delta_R^2$  that intersects  $V_r^{-2}(z)$ . The same argument gives the same bound on the hyperbolic diameter of  $V_r^{-2}(z)$  in  $U^{n-2}$ . Continuing by induction on  $i$  gives

$$\text{diam}_{U^{n-i}} V_r^{-i}(z) \leq \text{diam}_{\max}(2, 1)$$

for all  $i \leq j$ . We obtain the required bounds on the hyperbolic diameters in  $\Omega$ , as  $\Omega$  contains all semilocal wandering components. Since the semilocal wandering component  $U$  is regular it follows that

$$\text{deg } V_r^{-i}(z) = 1$$

for  $i = 0, \dots, j$ . □

**Proposition 9.2.** *There exists a constant  $K > 0$  such that the following holds. Let  $U^n$  be a semilocal wandering component, and let  $V_r(z) \subset V_{K \cdot r}(z) \subset f^j \mathbb{L}_0 \cap U^n$  be protected lifts of disks  $D_r(z) \subset D_{K \cdot r}(z) \subset \mathbb{L}_0$ . Then for all  $i \leq j$  we have*

$$\text{diam}_\Omega V_r^{-i}(z) \leq \text{diam}_{\max}(2, 1) \quad \text{and} \quad \text{deg } V_r^{-i}(z) \leq \text{deg}_{\text{crit}}.$$

*Proof.* Write  $U^{n-i}$  for the semilocal wandering Fatou component in  $f^{-i}U^n \cap \Delta_R^2$  that contains the lift  $V_r^{-i}(z)$ . By renumbering the orbit  $U^j$  we may assume that  $U^0$  is critical. It follows from the previous lemma that  $K \geq 2$  suffices when  $n \leq 0$ , so suppose  $n > 0$ .

We first consider the hyperbolic diameters in components  $U^{n-i}$  for  $i \leq n$ . We write  $D_r(z) \subset D_{k \cdot r}(z) \subset U^n \cap \mathbb{L}_0$  for the disks giving the lifts  $V_r(z) \subset V_{K \cdot r}(z)$ . We also consider the lifts of these disks to  $f^i \mathbb{L}_0$ .

Recall from the proof of Lemma 6.4 that for  $n$  sufficiently large any connected component  $D$  of  $\mathbb{L}_0 \cap U^n$  is a global transversal. The composition of  $f^i$  with holonomy from  $f^i(\mathbb{L}_0 \cap U^{n-i}) \rightarrow D$  induces quasi-regular maps, whose degrees are bounded by  $\text{deg}_{\max}$  and have bounded dilatation. It follows that by making  $K$  sufficiently large, the hyperbolic diameter of the preimage in  $\mathbb{L}_0 \cap U^{n-i}$  can be made arbitrarily small. Note that  $V_r^{-i}(z)$  is a lift of this preimage with respect to the *dynamical lamination*. It follows that the intrinsic Kobayashi diameter of  $V_r^{-i}(z)$  in  $U^{-i}$ , and thus also in  $\Omega$ , can be made arbitrarily small by choosing  $K$  large.

Let us consider the finitely many values of  $n$  for which we do not know that  $D$  is a global transversal. For any fixed  $n$ , making  $K$  sufficiently large implies that  $V_r^{-i}(z)$  is contained in an arbitrarily small neighborhood of a dynamical leaf, and thus we immediately obtain the same bounds on the hyperbolic diameter in  $\mathbb{L}_0 \cap U^{n-i}$ . Therefore we can drop the assumption that  $n$  is sufficiently large.

We now consider the artificial vertical lamination on  $U^0$ . We have seen that  $V_r^{-i}(z)$  is the lift with respect to the dynamical lamination of a set  $E \subset \mathbb{L}_0 \cap U^0$ , whose hyperbolic diameter can be made arbitrarily small by choosing  $K$  sufficiently large. Lemma 8.9 implies that  $V_r^{-i}(z)$  is contained in the lift of a disk  $D_\rho(w) \subset D_{2 \cdot \rho}(w) \subset \mathbb{L}_0 \cap U^0$  with respect to the artificial lamination when the hyperbolic diameter of  $E$  is less than  $\delta = \delta(U^0)$ . Since there are only finitely many critical components  $U^0$ , the constant  $\delta$  can

be chosen independent of the critical component, and thus  $K$  can be chosen independent as well.

By applying Lemma 9.1 to the lift of the disk  $D_\rho(w)$  we obtain the required diameter bounds on  $V_r^{-i}(z)$  for  $n \leq i \leq j$ , as well as

$$\deg V_r^{-i}(z) = 1$$

for  $n \leq i \leq n$ . Hence if  $K$  is chosen sufficiently large it follows that

$$\deg V_r^{-i}(z) \leq \deg_{\text{crit}}$$

for all  $i \leq j$ . □

9.2. Lifts that are not deeply contained.

We are ready to prove the main technical result:

**Proposition 9.3.** *Let  $z \in \Omega$  and let  $r > 0$  be such that  $D_r(z) \subset D_{2K \cdot r}(z) \subset \mathbb{L}_0 \cap \Omega$ . Then for every  $j \in \mathbb{N}$  and any lift  $V_r(z) \subset f^n \mathbb{L}_0$  one has*

$$\text{diam}_\Omega V_r^{-j}(z) \leq \text{diam}_{\max}(2K, \deg_{\text{crit}}) \quad \text{for } j < n,$$

and

$$\deg V_r(z) \leq \deg_{\text{crit}}.$$

*Proof.* Having previously dealt with lifts that are sufficiently deeply contained in wandering domains, we only need to consider the other two cases: of lifts not contained in a wandering component, or those that are contained in a wandering component, but are lifts of disks  $D_r(z)$  for which  $D_{K \cdot r}(z)$  intersects  $J^+$ .

We again assume the induction hypothesis that both bounds hold for all  $i \leq j$ . We will first prove the induction step under the assumption that  $D_r(z)$  is not contained in a wandering domain, and afterwards deal with the case where  $D_r(z)$  is contained in a wandering domain. The result in the former case will be used to prove the latter.

*Case 0.* Let us first consider the situation where  $V_r(z)$  is contained in a periodic Fatou component. By choosing  $\Omega$  sufficiently thin we can guarantee that  $V_r(z)$  is horizontal and that the disks  $D_r(z)$  and  $D_{2K \cdot r}(z)$  are arbitrarily small. It follows that  $V_r(z)$  must have degree 1. Hence we can cover  $V_r^{-1}(z)$  with at most  $N_0(2K)$  protected lifts  $V_{r_k}(z_k) \subset V_{4K \cdot r_k}(z_k) \subset V_{2K \cdot r}(z)$ . It follows by the induction assumption that the disks  $V_{2 \cdot r_k}(z_k)$  all have degree at most  $\deg_{\text{crit}}$ . Recall from the proof of Lemma 8.5 that the Poincaré diameter of  $V_{r_k}(z_k)_v$  with respect to  $V_{2 \cdot r_k}(z_k)_v$  is bounded by  $C_1(1/2, \deg_{\text{crit}})$ . Since  $f$  is an automorphism, we obtain the same bounds for the pairs  $V_{r_k}^{-i}(z_k)_v \subset V_{2 \cdot r_k}^{-i}(z_k)_v$ , and since each  $V_{2 \cdot r_k}^{-i}(z_k)_v$  lies in  $\Omega$ , we have the bounds

$$\text{diam}_\Omega V_{r_k}^{-i}(z_k) \leq C_1(1/2, \deg_{\text{crit}}).$$

This implies that

$$\text{diam}_\Omega V_r^{-i}(z) \leq N_0(2K) \cdot C_1(1/2, \deg_{\text{crit}}),$$

which in turn gives the necessary degree bounds.

*Case 1.* Now assume that  $V_r(z)$  is not contained in a Fatou component. Then there exists  $y_0 \in D_r(z) \cap J^+$ , and we let  $x_0 \in V_r(z)$  lie in the semilocal vertical leaf through  $y_0$ , which we denote by  $\Lambda_R(x_0)$ . We similarly write  $\Lambda_R^{-i}(x_0)$  for the semilocal leaf through  $f^{-i}(x_0)$ , which by invariance of the vertical lamination on  $J^+$  equals the connected component of  $f^{-i}(\Lambda_R(x_0))$  that contains  $f^{-i}(x_0)$ .

Cover  $V_r^{-1}(z)$  by protected lifts  $V_{r_k}(z_k)_v \subset V_{4K \cdot r_k}(z_k)_v \subset \Omega$  of at most  $N_0(2K)$  disks  $D_{r_k}(z_k) \subset D_{4K \cdot r_k}(z_k) \subset \mathbb{L}_0$ . The induction hypothesis implies that the degree of each lift  $V_{2 \cdot r_k}(z_k)_v$  is bounded by  $\text{deg}_{\text{crit}}$ , hence as above we obtain

$$\text{diam}_\Omega V_{r_k}^{-i}(z_k)_v \leq C_1(1/2, \text{deg}_{\text{crit}}).$$

As before we do not have an a priori estimate on the number of lifts  $V_{r_k}(z_k)_v$ . We apply the same induction argument as in Proposition 9.2. By choosing  $\Omega$  sufficiently thin we can guarantee that the Euclidean diameter of  $V_r(z)$  is sufficiently small, so that for each vertical leaf  $\Lambda_R(a)$  intersecting  $V_r(z)$  there is a unique component of  $f^{-1}\Lambda_R(a) \cap \Delta_R^2$  intersecting  $V_r^{-1}(z)$ .

Any point in  $V_r^{-1}(z)$  can then be connected to a point in  $\Lambda_R^{-1}(x_0)$  by a path that passes through at most  $N_0(2K)$  lifts  $V_{r_k}(z_k)_v$ . By the definition of  $\beta$  it follows that

$$\text{diam}_\Omega V_r^{-1}(z) \leq \text{diam}_{\max}(2K, \text{deg}_{\text{crit}}).$$

Hence we can continue the induction procedure, and obtain

$$\text{diam}_\Omega V_r^{-i}(z) \leq \text{diam}_{\max}(2K, \text{deg}_{\text{crit}})$$

for all  $i \leq j$ .

By choosing  $\Omega$  sufficiently thin it follows from Lemma 8.6 that each  $V_r^{-i}(z)$  is transverse to the artificial vertical lamination and that

$$\text{deg } V_r^{-i}(z) = 1$$

for all  $i \leq j$ , completing the proof for disks  $D_r(z)$  that are not contained in a wandering component.

*Case 2.* In the remainder of this proof we assume that  $D_r(z)$  is contained in a semilocal wandering component  $U$ , but  $D_{K \cdot r}(z)$  is not contained in  $U$ . We again cover  $V_r^{-1}(z)$  with protected lifts  $V_{r_k}(z_k)_v \subset V_{4K \cdot r}(z_k)_v \subset f^{j-1}\mathbb{L}_0 \cap \Omega$  of at most  $N_0(2K)$  disks  $D_{r_k}(z_k) \subset D_{4K \cdot r}(z_k) \subset \mathbb{L}_0$ , and by induction obtain the diameter bounds

$$\text{diam}_\Omega V_{r_k}^{-i}(z_k)_v \leq \text{diam}_{\max}(2K, \text{deg}_{\text{crit}}).$$

Again the difficulty lies in the fact that a priori we do not have a bound on the number of lifts  $V_{r_k}(z_k)_v$ . As before we will obtain the diameter bounds for each inverse image  $V_r^{-i}(z)$  by connecting the disks  $V_{r_k}^{-i+1}(z_k)_v$  to a single semilocal vertical leaf.

Write  $W$  for the component of  $U \cap \mathbb{L}_0$  that contains  $D_r(z)$ , and let  $w \subset \partial W$  be such that  $|z - w|$  is minimal, and write  $[z, w] \subset \mathbb{L}_0$  for the closed interval.



Consider the disk  $D_{r/2}(w)$ . Since  $D_{K \cdot r}(w) \subset D_{2K \cdot r}(z)$  it follows that  $D_{K \cdot r}(w) \subset \Omega$ . Hence  $D_{r/2}(w)$  satisfies the conditions of the previously discussed case, and we obtain the estimates

$$\text{diam}_\Omega V_{r/2}^{-i}(w) \leq \text{diam}_{\max}(2K, \text{deg}_{\text{crit}})$$

for any protected lifts  $V_{r/2}(w) \subset V_{K \cdot r}(w) \subset f^n \mathbb{L}_0 \cap \Omega$  of the disks  $D_{r/2}(w) \subset D_{K \cdot r}(w)$ .

The interval  $[z, w]$  can be covered by  $D_{r/2}(w)$  plus a bounded number of disks

$$D_{s_1}(w_1), \dots, D_{s_{N_1}}(w_{N_1}),$$

each satisfying  $D_{K \cdot s_v}(w_v) \subset U^n$ . The maximal number of disks needed is bounded by a constant  $N_1 \in \mathbb{N}$  that only depends on  $K$  (see Figure 1).

Write

$$V_{s_v}(w_v)_\xi \subset V_{K \cdot s_v}(w_v)_\xi \subset f^n \mathbb{L}_0 \cap \Omega$$

for protected lifts of the disks  $D_{s_v}(w_v) \subset D_{K \cdot s_v}(w_v)$ . It follows that these lifts satisfy the conditions of Proposition 9.2, and hence their preimages satisfy

$$\text{diam}_\Omega V_{s_v}^{-i}(w_v)_\xi \leq \text{diam}_{\max}(2K, \text{deg}_{\text{crit}}).$$

As in Case 1, we obtain a bound on the hyperbolic distance from each point  $f^{-i}(z)$  to  $\partial U^{-i} \subset J^+$ . By choosing  $\Omega$  sufficiently thin, the points  $f^{-i}(z)$  can all be assumed to lie arbitrarily close to  $J^+$ . In particular we may assume that the vertical leaf through each  $f^{-i}(z)$  is dynamical, and these semilocal leaves  $\Lambda_R^{-i}(z)$  are in particular invariant under  $f$ . Recall from Definition 8.10 that

$$\text{diam}_\Omega f^{n-i} \mathbb{L}_0 \cap \Lambda_R^{-i}(z)$$

is bounded by  $\beta$ . Hence we can use the same argument as in Case 1, using  $\Lambda_R^{-i}(z)$  instead of a semilocal leaf in  $J^+$ , to obtain the required diameter and degree bounds.  $\square$

### 10. Consequences

As before we will assume that the Hénon map  $f$  is substantially dissipative and admits a dominated splitting on  $J$ .

**Lemma 10.1.** *There are no wandering Fatou components.*

*Proof.* Suppose for a contradiction that there does exist a wandering Fatou component  $U$ . Without loss of generality we may assume that  $U$  intersects  $\Delta_R^2$ . Since  $U \cap \Delta_R^2$  is foliated by semilocal strong stable manifolds, the intersection  $U \cap \mathbb{L}_0$  is a non-empty relatively open set. Let  $D$  be a holomorphic disk relatively compactly contained in  $U \cap \mathbb{L}_0$ . By construction, we may assume that  $\Omega$  is sufficiently thin and  $D \subset U \cap \mathbb{L}_0$  is sufficiently large such that the hyperbolic diameter of  $D$  in  $\Omega$  is larger than  $\text{diam}_{\max}$ .

Since  $U$  lies in the Fatou set there exists a sequence  $(n_j)$  for which  $f^{n_j}$  converges uniformly on compact subsets of  $U$  to a map  $h : U \rightarrow J$ . By the dominated splitting the image  $h(U)$  is a point  $p \in J$ , and without loss of generality we may assume that  $p$  is not

contained in a parabolic cycle. Let  $D_r(p) \subset D_{2r}(p) \subset \mathbb{L}_0 \cap \Omega$  be a transverse disk. Since  $D$  is relatively compact in  $U$ , it follows that for sufficiently large  $j$  the image  $f^{nj}(D)$  is contained in a lift of  $D_r(p)$  in  $f^{nj}(\mathbb{L}_0)$ . But then it follows from Proposition 9.3 that the hyperbolic diameter of  $D$  in  $\Omega'$  is bounded by  $\text{diam}_{\max}$ , leading to a contradiction.  $\square$

In the absence of wandering domains the proof of Proposition 9.3 becomes considerably simpler. An immediate consequence is a better degree bound.

**Corollary 10.2.** *The constant  $\text{deg}_{\text{crit}}$  can be taken equal to 1.*

*Proof.* Since there are no wandering Fatou components, it follows from the proof of Proposition 9.3 that the degree of the disk  $V_r(z)$  is bounded by the degree of  $D_r(z)$ , which is a small straight disk contained in  $\mathbb{L}_0$ . Recall that  $y_0$  was chosen so that there are no tangent intersections between  $\mathbb{L}_0$  and the dynamical vertical lamination on  $J^+$ . Now that we know that there are no wandering Fatou components, the set  $\Omega$  can be constructed as an arbitrarily thin neighborhood of  $J_R^+$ . Hence we may assume that there are no tangencies in  $\mathbb{L}_0$  with the artificial vertical lamination in  $\Omega$ . Since the disks  $D_r(z)$  may be assumed to have arbitrarily small Euclidean diameter, it follows that  $D_r(z)$  intersects each vertical leaf in at most one point.  $\square$

**Proposition 10.3.** *Let  $x \in \Omega \cap J$ . Then there exists a local unstable manifold through  $x$ .*

*Proof.* For each  $n \in \mathbb{N}$ , the semilocal strong stable manifold  $W_R^s(f^{-n}(x))$  must intersect the horizontal disk  $\mathbb{L}_0$ . Let  $y_n$  be such an intersection point. Since the degree of the semilocal strong stable manifolds is uniformly bounded and  $f$  is uniformly contracting on the family of strong stable manifolds, it follows that  $f^n(y_n)$  converges to  $x$  exponentially fast. Moreover, by the exponential contraction in the vertical direction, it follows that locally the disks  $D_n \subset f^n(\mathbb{L}_0)$  passing through  $f^n(y_n)$  converge to a holomorphic disk through  $x$ , which we denote by  $D$ .

We claim that  $D$  must be an unstable manifold, i.e. the diameter of  $f^{-n}(D)$  converges to 0. By the exponential contraction in the vertical direction it suffices to show that the diameter of  $f^{-n}(D_n)$  converges to zero. Suppose that this is not the case. Recall that the diameters of the disks  $f^{-n}(D_n)$  are uniformly bounded, hence are given by images of a normal family of holomorphic maps. Hence if the diameters do not converge to zero, then there is a subsequence  $(n_j)$  for which  $f^{-n_j}(D_{n_j}) \subset \mathbb{L}_0$  converges to a holomorphic disk  $E$ , necessarily intersecting  $J^+$ . By construction  $\mathbb{L}_0$  is transverse to  $J^+$ , hence  $E$  must intersect vertical leaves through points in the basin of infinity, say in a point  $t$ . Since those vertical leaves must be contained in the basin of infinity, so must  $t$ . Note that since  $t \in E$ , it follows that  $t \in f^{-n_j}(D_{n_j})$  for  $j$  large enough. This however leads to a contradiction, as the forward orbit  $t$  must escape the bidisk in finite time, and hence  $f^{n_j}(t)$  cannot be contained in  $D_{n_j}$  for  $j$  large. The contradiction finishes the proof.  $\square$

Since  $D$  is a uniform limit of holomorphic disks with horizontal tangent bundles, it follows that the tangent bundles to  $D$  must also lie in the horizontal cone field. The size of the local unstable manifolds is uniform on any subset of  $J$  that is bounded away from the parabolic cycles.

**Corollary 10.4.** *The map  $f$  does not have any attracting-rotating Fatou components.*

*Proof.* Suppose there does exist an attracting rotating Fatou component, and let  $U$  be the connected component of the intersection with  $\Delta_R^2$  that contains the rotating disk  $\Sigma$ . Let  $x \in \partial\Sigma$ , which implies that  $x \in J$ . The local unstable manifold  $D$  through  $x$  is horizontal and thus intersects all nearby strong stable manifolds. Since  $U$  is foliated by strong stable manifolds,  $D$  must intersect  $U$  in a point  $y$  local strong stable manifold through  $\Sigma$ . The sets  $f^{-n}(D)$  may all be assumed to have arbitrarily small diameters, so that  $f^{-n}(y)$  remains arbitrarily close to  $\Sigma$ . It follows that  $y \in \Sigma$ . But while the backward orbits of points on this local unstable manifold converge to  $J$ , backward orbits in  $\Sigma$  do not, giving a contradiction. □

**Proposition 10.5.** *If  $f$  does not have any parabolic cycles then  $f$  is hyperbolic.*

*Proof.* If  $f$  does not have parabolic cycles, then there exist local unstable manifolds through any point in  $J$ , and the family of these unstable manifolds is invariant under  $f$ . By compactness of  $J$  there are uniform estimates on the rate at which these unstable manifolds are contracted, hence the center direction of the dominated splitting is actually unstable, proving that the map is hyperbolic. □

**Proposition 10.6.** *The Julia set  $J^+$  has zero Lebesgue measure.*

*Proof.* The one-dimensional counterpart of this result was proven in [Lyu82, DH85a]. The 1D argument can be adapted to our setting as follows. Take any point  $x \in J^+$  that does not belong to strong stable manifolds of the parabolic points. Then there is a sequence of times  $n_k \rightarrow \infty$  for which  $f^{n_k}x$  stays away from the parabolic points. Let  $L$  be a complex horizontal line through  $x$ . Proposition 10.3 implies that there exists a shrinking nest of ovals  $V_k \subset L$  of bounded shape around  $x$  (i.e. with bounded ratio of the inner and outer radii centered at  $x$ ) such that:

- (i) Each  $D_k := f^{n_k}(V_k)$  is a horizontal-like oval of a definite size and bounded shape around  $f^{n_k}x$ .
- (ii) The maps  $f^{n_k} : V_k \rightarrow D_k$  have a bounded distortion.

It follows from (i) that the ovals  $D_k$  contain gaps in  $J^+$  of definite relative size. Then (ii) implies that so do the ovals  $V_k$ . Hence  $x$  is not a Lebesgue density point for  $L \cap J^+$ .

By the Lebesgue Density Points Theorem, the horizontal slices of  $J^+$  have zero area. Since the dynamical vertical lamination of  $J^+$  is smooth (Lemma 5.3),  $J^+$  has zero volume. □

On the complex line the fact that a rational function has only finitely many attracting or parabolic cycles is a direct consequence of there only being finitely many critical points. For complex Hénon maps this kind of argument cannot be used. Indeed, there do exist Hénon maps with infinitely many attracting cycles: see for example [Buz97]. It was shown in [BS91a] that a hyperbolic Hénon map has only finitely many Fatou components, each contained in the basin of an attracting cycle.

**Corollary 10.7.** *There are at most finitely many attracting cycles.*

*Proof.* This follows from the fact that an attracting cycle cannot be completely contained in the neighborhood of  $J^+$  where the dominated splitting exists. But the complement of this neighborhood is contained in only finitely many Fatou components.  $\square$

**Corollary 10.8.** *There are at most finitely many parabolic cycles.*

*Proof.* By the existence of unstable manifolds through any point in  $J \cap \Omega$  it follows that there can be no parabolic cycles in  $\Omega$ , which implies that the only parabolic cycles can be those finitely many that were removed from  $J$  in the construction of  $\Omega$ .  $\square$

**Remark 10.9.** An interesting question is whether Corollary 2.16 can be extended to the 2D situation, asserting that any Hénon map with dominated splitting lies on the boundary of some hyperbolic component. There are two issues to address for such an extension.

The first issue is to perturb each parabolic cycle so that it bifurcates into multiple cycles, including one attracting cycle whose basin of attraction contains an arbitrarily large compact subset of the parabolic basin. Such a result for *non-degenerate* parabolic cycles was recently announced by Bedford and Ueda (as a follow-up of [BSU17]). However, for our purposes perturbations of arbitrarily degenerate parabolic cycles are needed.

The other issue is whether such a perturbation can be performed so that *all* parabolic points will be simultaneously turned into attracting ones. To the best of our knowledge, all 1D proofs of such an assertion (see Douady and Hubbard [DH85b] for polynomials and Shishikura [Shi87] for rational functions) are based on the machinery of quasiconformal surgery, which is not available in 2D. However, it is easy to make a weaker statement: to perform such a perturbation in the infinite-dimensional family of Hénon-like maps, kicking the map in question into a hyperbolic component in this big space.

Modulo these two issues, the proof of Corollary 2.16 works for Hénon maps as well. Indeed, one obtains a one-parameter family of perturbations  $f_t$  converging to  $f$  with small backward invariant neighborhoods  $\Omega_t$  of the Julia set  $J_R^+(f_t)$ , which in turn is contained in a small neighborhood of  $J_R^+(f)$ . One can therefore guarantee that  $\Omega_t$  is contained in the domain of dominated splitting of  $f$ , which is stable under perturbations. Since  $\Omega_t$  is a full neighborhood of  $J_R^+(f_t)$ , every  $f_t$  is horizontally expanding, implying hyperbolicity.

**Corollary 10.10.** *The two Julia sets  $J^*$  and  $J$  are equal.*

We note that in the more recent paper [GP18] the assertion  $J = J^*$  is proved using different methods, under the weaker assumption that the substantially dissipative Hénon map  $f$  admits a dominated splitting on the potentially smaller set  $J^*$ .

*Proof of Corollary 10.10.* Let  $x \in J$  be a point that is not contained in one of the parabolic cycles. Let  $p$  be a saddle periodic point. By [BS91b] the stable manifold  $W^s(p)$  accumulates on all of  $J^+$ , and hence also on  $x$ , and similarly  $W^u(p)$  must accumulate on  $x$ . Note that the tangent space to  $W^s(p)$  must be vertical at all points.

A sufficiently small local unstable manifold  $W_{\text{loc}}^u(p)$  is contained in  $\Omega$ . Since  $\Omega$  is relatively backward invariant, it follows that if  $y \in W_{\text{loc}}^u(p)$  is such that  $f^n(y)$  contained in  $\Omega$  for some  $n \in \mathbb{N}$ , then  $f^j(y) \in \Omega$  for  $j = 0, \dots, n$ . Since  $\Omega$  is contained in the region of dominated splitting and the tangent bundle to  $W_{\text{loc}}^u(p)$  is horizontal, and the horizontal

cone field is forward invariant, it follows that the tangent space to  $W_{\text{loc}}^u(f^n(p))$  at  $f^n(y)$  is horizontal. Thus  $W^u(p)$  is horizontal in a small neighborhood of  $x$ .

Since both  $W^s(p)$  and  $W^u(p)$  accumulate at  $x$  and are respectively vertical and horizontal near  $p$ , it follows that there exist intersection points of  $W^s(p)$  and  $W^u(p)$  arbitrarily close to  $x$ . By a standard construction there are saddle periodic points arbitrarily close to homoclinic intersection points, and hence also arbitrarily close to  $x$ . Since  $J^*$  is the closure of the set of saddle points, it follows that  $x \in J^*$ .

Since  $J^*$  is closed and we have proved that all but finitely many points of  $J$  are contained in  $J^*$ , the fact that  $J$  does not have isolated points implies that  $J \subset J^*$ .  $\square$

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