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Galois self-dual cuspidal types and Asai local factors

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Abstract. Let $\mathrm{F}/\mathrm{F}_\mathrm{o}$ be a quadratic extension of non-archimedean locally compact fields of odd residual characteristic and σ be its non-trivial automorphism. We show that any σ-self-dual cuspidal representation of $\mathrm{GL}_n(\mathrm{F})$ contains a σ-self-dual Bushnell–Kutzko type. Using such a type, we construct an explicit test vector for Flicker's local Asai L-function of a $\mathrm{GL}_n(\mathrm{F}_\mathrm{o})$-distinguished cuspidal representation and compute the associated Asai root number. Finally, by using global methods, we compare this root number to Langlands–Shahidi's local Asai root number, and more generally we compare the corresponding epsilon factors for any cuspidal representation.

Keywords. Asai local factor, distinction, root number, test vector, type theory

1. Introduction

1.1. Let $\mathrm{F}/\mathrm{F}_\mathrm{o}$ be a quadratic extension of locally compact non-archimedean fields of odd residual characteristic p and let σ denote the non-trivial element of $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_\mathrm{o})$. Let G denote the general linear group $\mathrm{GL}_n(\mathrm{F})$, make σ act on G componentwise and let G^σ be the σ-fixed points subgroup $\mathrm{GL}_n(\mathrm{F}_\mathrm{o})$. In [19], using the Rankin–Selberg method, Flicker has associated Asai local factors to any generic irreducible (smooth, complex) representation of G. Let N denote the subgroup of upper triangular unipotent matrices in G, and ψ be a non-degenerate character of N trivial on $\mathrm{N}^\sigma = \mathrm{N} \cap \mathrm{G}^\sigma$. Given a generic irreducible representation of G, let $\mathcal{W}(\pi, \psi)$ denote its Whittaker model with respect to the Whittaker datum (N, ψ), that is, the unique subrepresentation of the smooth induced representation

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$\mathrm{Ind}_{\mathrm{N}}^{\mathrm{G}}(\psi)$ which is isomorphic to π. Given a function $\mathrm{W} \in \mathcal{W}(\pi, \psi)$, a smooth compactly supported complex function Φ on $\mathrm{F}_{\mathrm{o}}^{n}$ and a complex number $s \in \mathbf{C}$, the associated local Asai integral is

$$\mathrm{I}_{\mathrm{As}}(s, \Phi, \mathrm{W}) = \int_{\mathrm{N}^{\sigma}\backslash\mathrm{G}^{\sigma}} \mathrm{W}(g)\Phi((0 \ \dots \ 0 \ 1)g)|\det(g)|_{\mathrm{o}}^{s}\, dg,$$

where $|\cdot|_{\mathrm{o}}$ is the normalized absolute value of F_{o} and dg is a right invariant measure on $\mathrm{N}^{\sigma}\backslash\mathrm{G}^{\sigma}$. This integral is convergent when the real part of s is large enough, and it is a rational function in q_{o}^{-s}, where q_{o} is the cardinality of the residual field of F_{o}. When one varies the functions W and Φ, these integrals generate a fractional ideal of $\mathbf{C}[q_{\mathrm{o}}^{s}, q_{\mathrm{o}}^{-s}]$. The Asai L-function $\mathrm{L}_{\mathrm{As}}(s, \pi)$ of π is defined as a generator, suitably normalized, of this fractional ideal. It does not depend on the choice of the non-degenerate character ψ.

1.2. Now consider a cuspidal (irreducible, smooth, complex) representation π of G, and suppose that its Asai L-function $\mathrm{L}_{\mathrm{As}}(s, \pi)$ is non-trivial. By [35], this happens if and only if π has a *distinguished* unramified twist, that is, an unramified twist carrying a non-zero G^{σ}-invariant linear form. In this case, the Asai L-function can be described explicitly (see Proposition 7.5). We prove that it can be realized as a single Asai integral:

Theorem 1.1 (Theorem 7.14 and Corollary 7.15). *Let π be a cuspidal representation of* G *having a distinguished unramified twist. Then there is an explicit function* W_0 *in* $\mathcal{W}(\pi, \psi)$ *such that*

$$\mathrm{I}_{\mathrm{As}}(s, \Phi_0, \mathrm{W}_0) = \mathrm{L}_{\mathrm{As}}(s, \pi)$$

where Φ_0 is the characteristic function of the lattice $\mathcal{O}_{\mathrm{o}}^{n}$ in $\mathrm{F}_{\mathrm{o}}^{n}$ and $\mathcal{O}_{\mathrm{o}}$ is the ring of integers of F_{o}.

This provides an integral formula for the Asai L-function of π. As an application of this theorem, we compute the associated root number: the Asai L-functions of π and its contragredient $\pi^{\vee}$ are related by a functional equation (8.3), which features a local Asai epsilon factor $\epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathrm{o}}, \delta)$ depending on a non-trivial character ψ_{o} of F_{o} and a non-zero scalar $\delta \in \mathrm{F}^{\times}$ such that $\mathrm{tr}_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(\delta) = 0$. We prove the following theorem conjectured in [1, Remark 4.4]. It can be seen as the twisted tensor analogue of [11, Theorem 2] in the cuspidal case and also as the Rankin–Selberg counterpart of [1, Theorem 1.1] in the cuspidal distinguished case.

Theorem 1.2 (Theorem 8.4). *Let π be a distinguished cuspidal representation of* G. *Then*

$$\epsilon_{\mathrm{As}}(1/2, \pi, \psi_{\mathrm{o}}, \delta) = 1.$$

Our proof of this theorem is purely local and relies on Theorem 1.1.

Independently, using a global argument, we compare the Asai epsilon factor $\epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathrm{o}}, \delta)$ with the local Asai epsilon factor $\epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \pi, \psi_{\mathrm{o}})$ defined via the Langlands–Shahidi method.

Theorem 1.3 (Theorem 9.29). *Let π be a cuspidal representation of* G *with central character ω_π. For any non-trivial character ψ_o of F_o and non-zero scalar $\delta \in F$ such that* $\mathrm{tr}_{F/F_o}(\delta) = 0$, *we have*

$$\epsilon_{\mathrm{As}}(s, \pi, \psi_o, \delta) = \omega_\pi(\delta)^{n-1} \cdot |\delta|^{n(n-1)(s-1/2)/2} \cdot \lambda(F/F_o, \psi_o)^{-n(n-1)/2} \cdot \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \pi, \psi_o)$$

where $|\delta|$ is the normalized absolute value of δ and $\lambda(F/F_o, \psi_o)$ is the local Langlands constant.

When in addition π is distinguished, Theorem 1.3 and [1, Theorem 1.1] together give another proof, using a global argument, of Theorem 1.2.

1.3. Let us now explain our strategy to prove Theorem 1.1. The basic idea is to use Bushnell–Kutzko's theory of types [14], which provides an explicit model for a cuspidal representation π of G as a compactly induced representation from an extended maximal simple type: a pair $(\mathbf{J}, \boldsymbol{\lambda})$ consisting of an irreducible representation $\boldsymbol{\lambda}$ of a compact-mod-centre, open subgroup $\mathbf{J}$ of G, constructed via a precise recipe, such that

$$\pi \simeq \mathrm{ind}_{\mathbf{J}}^{G}(\boldsymbol{\lambda}). \tag{1.4}$$

Such an extended maximal simple type $(\mathbf{J}, \boldsymbol{\lambda})$ is unique up to G-conjugacy, and when (1.4) holds we say that π contains the type $(\mathbf{J}, \boldsymbol{\lambda})$.

Suppose now π is distinguished. By a result of Prasad and Flicker [43, 19], it is then σ*-self-dual*, that is, its contragredient representation $\pi^\vee$ is isomorphic to $\pi^\sigma = \pi \circ \sigma$. In order to compute a test vector for π, that is, the explicit function W_0 of Theorem 1.1, our first task is to isolate a type, among those contained in π, which behaves well with respect to σ.

Our first main result is the following. For further use in another context (see [46]), we state and prove it for cuspidal representations of G with coefficients not necessarily in **C**, but more generally in an algebraically closed field R of characteristic different from p. For Bushnell–Kutzko's theory in this more general context, in particular the description of cuspidal R-representations by compact induction of extended maximal simple types, see [54, 38].

Theorem 1.5 (Theorem 4.1). *Let π be a cuspidal representation of* G *with coefficients in* R. *Then π is σ-self-dual if and only if it contains a σ-self-dual type, that is, a type $(\mathbf{J}, \boldsymbol{\lambda})$ such that $\sigma(\mathbf{J}) = \mathbf{J}$ and $\lambda^\sigma \simeq \lambda^\vee$.*

Theorem 1.5 generalizes [39, Lemma 2.1], which deals with the case where π is essentially tame, that is, the number of unramified characters χ of G such that $\pi\chi \simeq \pi$ is prime to p.

The proof relies on Bushnell–Henniart's theory of endo-classes and tame lifting [9, 13]; although they are technical in nature, it is both natural and necessary to consider endo-classes since, in the case of representations that are not essentially tame, there are no simpler canonical parameters to construct types. The assumption that $p \neq 2$ is crucial here, since at various places we use the fact that the first cohomology set of $\mathrm{Gal}(F/F_0)$ in a pro-p-group is trivial.

1.4. In general, a σ-self-dual type as in Theorem 1.5 is not unique up to G^σ-conjugacy: see Proposition 4.31. To construct explicit Whittaker functions, we need to go further and isolate those σ-self-dual types which are compatible with the Whittaker model of π.

Recall that we have fixed a Whittaker datum (N, ψ) in §1.1. A type $(\mathbf{J}, \boldsymbol{\lambda})$ contained in a cuspidal representation of G is said to be *generic* (with respect to ψ) if $\mathrm{Hom}_{\mathbf{J}\cap N}(\boldsymbol{\lambda}, \psi)$ is non-zero.

Proposition 1.6 (Proposition 5.5). *Any σ-self-dual cuspidal representation of* G *with coefficients in* R *contains a generic σ-self-dual type. Such a type is uniquely determined up to* N^σ*-conjugacy.*

We then prove the following result.

Proposition 1.7 (Corollary 6.6). *A σ-self-dual cuspidal representation π of* G *with coefficients in* R *is distinguished if and only if any generic σ-self-dual type* $(\mathbf{J}, \boldsymbol{\lambda})$ *contained in π is distinguished, that is, the space* $\mathrm{Hom}_{\mathbf{J}\cap G^\sigma}(\boldsymbol{\lambda}, \mathbf{1})$ *is non-zero.*

Our proof of Proposition 1.7 is based on a result of Ok [40] proved for any irreducible complex representation of G, and which we prove for any cuspidal representation of G with coefficients in R in Appendix B. Note that Proposition 1.7 is proved in another way in [46], without using Ok's result (see Remark 6.7).

1.5. For the remainder of the introduction we go back to *complex* representations. Given a generic type $(\mathbf{J}, \boldsymbol{\lambda})$ in a cuspidal representation π of G, a construction of Paskunas–Stevens [41] defines an explicit Whittaker function $W_{\boldsymbol{\lambda}} \in \mathcal{W}(\pi, \psi)$. The key point for this paper is that if $(\mathbf{J}, \boldsymbol{\lambda})$ is both generic and σ-self-dual, then $W_{\boldsymbol{\lambda}}$ is well suited to computing the local Asai integral. We make Theorem 1.1 more precise.

Theorem 1.8 (Theorem 7.14). *Let π be a distinguished cuspidal representation of* G, *and* $(\mathbf{J}, \boldsymbol{\lambda})$ *be a generic σ-self-dual type contained in π. Then there is a unique right invariant measure on* $N^\sigma\backslash G^\sigma$ *such that*

$$I_{As}(s, \Phi_0, W_{\boldsymbol{\lambda}}) = L_{As}(s, \pi)$$

where Φ_0 is the characteristic function of the lattice $\mathcal{O}_\circ^n$ in $F_\circ^n$.

Let us briefly explain how we prove this theorem. Following the method of [30], we compute the local Asai integral and get

$$I_{As}(s, \Phi_0, W_{\boldsymbol{\lambda}}) = \frac{1}{1 - q_\circ^{-sn/e_\circ}}$$

where $e_\circ$ a positive integer attached to the generic σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$ in §5.4. On the other hand, starting from Proposition 7.5 giving a formula for $L_{As}(s, \pi)$, and using the dichotomy theorem (§7.1) together with Proposition 7.2, we find the same expression for $L_{As}(s, \pi)$.

1.6. We now explain how we prove Theorem 1.2. First, thanks to the functional equation (8.3) together with the fact that π is distinguished, the Asai root number $\epsilon_{\mathrm{As}}(1/2,\pi,\psi_{\mathrm{o}},\delta)$ must be equal to either 1 or -1. Then, by using the explicit integral expression for $\mathrm{L}_{\mathrm{As}}(s,\pi)$ provided by our test vectors, we prove that $\epsilon_{\mathrm{As}}(1/2,\pi,\psi_{\mathrm{o}},\delta)$ is positive.

1.7. We now explain our global argument for proving Theorem 1.3. We first prove that, for any generic irreducible representation π of G and any $\delta\in\mathrm{F}$ such that $\mathrm{tr}_{\mathrm{F}/\mathrm{F}_0}(\delta)=0$, the quantity

$$\omega_\pi(\delta)^{1-n}\cdot|\delta|^{-n(n-1)(s-1/2)/2}\cdot\lambda(\mathrm{F}/\mathrm{F}_{\mathrm{o}},\psi_{\mathrm{o}})^{n(n-1)/2}\cdot\epsilon_{\mathrm{As}}(s,\pi,\psi_{\mathrm{o}},\delta)$$

does not depend on δ. When π is in addition unramified, we also prove that it is equal to the local Asai epsilon factor $\epsilon^{\mathrm{LS}}_{\mathrm{As}}(s,\pi,\psi_{\mathrm{o}})$ obtained via the Langlands–Shahidi method. This leads us to:

Definition 1.9 (Definition 9.10). For any generic irreducible representation π of G, we set

$$\epsilon^{\mathrm{RS}}_{\mathrm{As}}(s,\pi,\psi_{\mathrm{o}})=\omega_\pi(\delta)^{1-n}\cdot|\delta|^{-n(n-1)(s-1/2)/2}\cdot\lambda(\mathrm{F}/\mathrm{F}_{\mathrm{o}},\psi_{\mathrm{o}})^{n(n-1)/2}\cdot\epsilon_{\mathrm{As}}(s,\pi,\psi_{\mathrm{o}},\delta).$$

We note that Beuzart-Plessis also has the same normalization in [7].

Now consider a quadratic extension k/k_{o} of global fields of characteristic different from 2 such that any place of k_{o} dividing 2, as well as any archimedean place in the number field case, splits in k.

Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A})$, where $\mathbf{A}$ is the ring of adeles of k, with local component Π_v for each place v of k_{o}. We also fix a non-trivial character ψ_{o} of $\mathbf{A}_{\mathrm{o}}$ trivial on k_{o}, where $\mathbf{A}_{\mathrm{o}}$ is the ring of adeles of k_{o}, and denote by $\psi_{\mathrm{o},v}$ its local component at v. We then set

$$\epsilon^{\mathrm{RS}}_{\mathrm{As}}(s,\Pi)=\prod_v\epsilon^{\mathrm{RS}}_{\mathrm{As}}(s,\Pi_v,\psi_{\mathrm{o},v})$$

where the product is taken over all places v of k_{o}, where $\epsilon^{\mathrm{RS}}_{\mathrm{As}}(s,\Pi_v,\psi_{\mathrm{o},v})$ is defined by Definition 1.9 when v is inert, and is the Jacquet–Piatetski-Shapiro–Shalika epsilon factor $\epsilon^{\mathrm{RS}}_{\mathrm{As}}(s,\pi_1,\pi_2,\psi_{\mathrm{o},v})$ when v is split and Π_v identifies with $\pi_1\otimes\pi_2$ as representations of $\mathrm{GL}_n(k_v)\simeq\mathrm{GL}_n(k_{\mathrm{o},v})\times\mathrm{GL}_n(k_{\mathrm{o},v})$.

We define the global factor $\epsilon^{\mathrm{LS}}_{\mathrm{As}}(s,\Pi)$ similarly. Using the equality (which we prove when F has characteristic p in Appendix A) of the Flicker and Langlands–Shahidi Asai L-functions of Π_v for all v, the comparison of the global functional equations gives:

Theorem 1.10 (Theorem 9.26). *Let* Π *be a cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbf{A})$. *Then*

$$\epsilon^{\mathrm{RS}}_{\mathrm{As}}(s,\Pi)=\epsilon^{\mathrm{LS}}_{\mathrm{As}}(s,\Pi).$$

Realizing any cuspidal representation of G as a local component of some cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A})$ with prescribed ramification at other places, and combining with Theorems 1.10 and 9.13, we get Theorem 1.3.

1.8. Finally, we must explain the interconnections between [46] and the present paper. The starting point of both papers is the σ-self-dual type theorem for cuspidal R-representations, namely Theorem 1.5, which is proved in Section 4 below. Starting from this theorem, and independently from the rest of this paper, one gives in [46] a necessary and sufficient condition of distinction for σ-self-dual *supercuspidal* R-representations. In particular, for complex representations, in which case all cuspidal representations are supercuspidal, this implies the two results stated in §7.1 (i.e., Theorem 7.1 and Proposition 7.2) which we use in the proof of Theorem 1.1. We also use Proposition 5.8, which is also proved in [46], for any σ-self-dual supercuspidal R-representation of G.

2. Notation

Let $\mathrm{F}/\mathrm{F}_\mathrm{o}$ be a quadratic extension of locally compact non-archimedean fields of residual characteristic $p \neq 2$. Write σ for the non-trivial F_o-automorphism of F.

For any finite extension E of F_o, we denote by $\mathcal{O}_\mathrm{E}$ its ring of integers, by $\mathfrak{p}_\mathrm{E}$ the unique maximal ideal of $\mathcal{O}_\mathrm{E}$ and by $\boldsymbol{k}_\mathrm{E}$ its residue field. We abbreviate $\mathcal{O}_\mathrm{F}$ to $\mathcal{O}$ and $\mathcal{O}_{\mathrm{F}_\mathrm{o}}$ to $\mathcal{O}_\mathrm{o}$, and define similarly $\mathfrak{p}$, $\mathfrak{p}_\mathrm{o}$, $\boldsymbol{k}$, $\boldsymbol{k}_\mathrm{o}$. The involution σ induces a $\boldsymbol{k}_\mathrm{o}$-automorphism of $\boldsymbol{k}$, still denoted σ. It is a generator of the Galois group $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{k}_\mathrm{o})$. We write q_o for the cardinality of $\boldsymbol{k}_\mathrm{o}$ and $|\cdot|_\mathrm{o}$ for the normalized absolute value on F_o.

Let R be an algebraically closed field of characteristic ℓ different from p; note that ℓ can be 0. We will say we are in the "modular case" when we consider the case where $\ell > 0$.

We also denote by $\omega_{\mathrm{F}/\mathrm{F}_\mathrm{o}}$ the character of $\mathrm{F}_\mathrm{o}^\times$ whose kernel contains the subgroup of $\mathrm{F}/\mathrm{F}_\mathrm{o}$-norms and is non-trivial if $\ell \neq 2$.

Let G denote the locally profinite group $\mathrm{GL}_n(\mathrm{F})$, with $n \geq 1$, equipped with the involution σ acting componentwise. Its σ-fixed points form the closed subgroup $\mathrm{G}^\sigma = \mathrm{GL}_n(\mathrm{F}_\mathrm{o})$. We will identify the centre of G with $\mathrm{F}^\times$, and that of G^σ with $\mathrm{F}_\mathrm{o}^\times$.

By a *representation* of a locally profinite group, we mean a smooth representation on an R-module. Given a representation π of a closed subgroup H of G, we write $\pi^\vee$ for the smooth contragredient of π and π^σ for the representation $\pi \circ \sigma$ of $\sigma(\mathrm{H})$. We also write $\mathrm{Ind}_\mathrm{H}^\mathrm{G}(\pi)$ for the smooth induction of π to G, and $\mathrm{ind}_\mathrm{H}^\mathrm{G}(\pi)$ for the compact induction of π to G. If χ is a character of H, we write $\pi\chi$ for the representation $g \mapsto \chi(g)\pi(g)$.

A pair (K, π), consisting of an open subgroup K of G and a smooth irreducible representation π of K, is called *σ-self-dual* if K is σ-stable and π^σ is isomorphic to $\pi^\vee$. When $\mathrm{K} = \mathrm{G}$, we will just talk about π being σ-self-dual.

Let χ be a character of $\mathrm{F}_\mathrm{o}^\times$. A pair (K, π), consisting of a σ-stable open subgroup K of G and an irreducible representation π of K, is called *χ-distinguished* if

$$\mathrm{Hom}_{\mathrm{K}^\sigma}(\pi, \chi \circ \det) \neq \{0\}$$

where det denotes the determinant on G and $\mathrm{K}^\sigma = \mathrm{K} \cap \mathrm{G}^\sigma$. We say that (K, π) is *distinguished* if it is **1**-distinguished, that is, distinguished by the trivial character of $\mathrm{F}_\mathrm{o}^\times$. When $\mathrm{K} = \mathrm{G}$, we will just talk about π being χ-distinguished.

Given $g \in \mathrm{G}$ and a subset $\mathrm{X} \subseteq \mathrm{G}$, we set $\mathrm{X}^g = \{g^{-1}xg \mid x \in \mathrm{X}\}$. If f is a function on X, we write f^g for the function $x \mapsto f(gxg^{-1})$ on X^g.

For any finite extension E of $\mathrm{F_o}$ and any integer $n \geq 1$, we write $\mathrm{N}_n(\mathrm{E})$ for the subgroup of $\mathrm{GL}_n(\mathrm{E})$ made up of all upper triangular unipotent matrices, and $\mathrm{P}_n(\mathrm{E})$ for the standard mirabolic subgroup of all matrices in $\mathrm{GL}_n(\mathrm{E})$ with final row $(0 \cdots 0\ 1)$.

Throughout the paper, by a *cuspidal* representation of G, we mean a cuspidal irreducible (smooth) representation of G.

3. Preliminaries on simple types

We recall the main results on simple strata, characters and types [14, 9, 13, 38] that we will need.

3.1. Simple strata

Let $[\mathfrak{a}, \beta]$ be a simple stratum in the F-algebra $\mathrm{M}_n(\mathrm{F})$ of $n \times n$ matrices with entries in F for some $n \geq 1$. Recall that $\mathfrak{a}$ is a hereditary $\mathcal{O}$-order in $\mathrm{M}_n(\mathrm{F})$ and β is a matrix in $\mathrm{M}_n(\mathrm{F})$ such that:

(i) the F-algebra $\mathrm{E} = \mathrm{F}[\beta]$ is a field, whose degree over F is denoted d;
(ii) the multiplicative group $\mathrm{E}^\times$ normalizes $\mathfrak{a}$;

plus an additional technical condition (see [14, (1.5.5)]). The centralizer of E in $\mathrm{M}_n(\mathrm{F})$, denoted B, is an E-algebra isomorphic to $\mathrm{M}_m(\mathrm{E})$, where $n = md$. The intersection $\mathfrak{a} \cap \mathrm{B}$, denoted $\mathfrak{b}$, is a hereditary $\mathcal{O}_\mathrm{E}$-order in B. We write $\mathfrak{p}_\mathfrak{a}$ for the Jacobson radical of $\mathfrak{a}$ and $\mathrm{U}^1(\mathfrak{a})$ for the compact open pro-p-subgroup $1 + \mathfrak{p}_\mathfrak{a}$ of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$, and define $\mathrm{U}^1(\mathfrak{b})$ similarly. Note that $\mathrm{U}^1(\mathfrak{b}) = \mathrm{U}^1(\mathfrak{a}) \cap \mathrm{B}^\times$.

Note that we use the simplified notation of [13] for simple strata: what we denote by $[\mathfrak{a}, \beta]$ would be denoted $[\mathfrak{a}, v, 0, \beta]$ in [14, 9], where v is the non-negative integer defined by $\beta\mathfrak{a} = \mathfrak{p}_\mathfrak{a}^{-v}$.

Associated with $[\mathfrak{a}, \beta]$, there are compact open subgroups

$$\mathrm{H}^1(\mathfrak{a}, \beta) \subseteq \mathrm{J}^1(\mathfrak{a}, \beta) \subseteq \mathrm{J}(\mathfrak{a}, \beta)$$

of $\mathfrak{a}^\times$ and a finite set $\mathcal{C}(\mathfrak{a}, \beta)$ of characters of $\mathrm{H}^1(\mathfrak{a}, \beta)$ called *simple characters*. This set depends on the choice of a character of F, trivial on $\mathfrak{p}$ but not on $\mathcal{O}$, which we assume to be σ-stable and which is fixed from now on. Such a choice is possible since $p \neq 2$. Write $\mathbf{J}(\mathfrak{a}, \beta)$ for the compact-mod-centre subgroup of G generated by $\mathrm{J}(\mathfrak{a}, \beta)$ and the normalizer of $\mathfrak{b}$ in $\mathrm{B}^\times$.

Proposition 3.1 ([13, 2.1]). *We have the following properties:*

(i) *The group* $\mathrm{J}(\mathfrak{a}, \beta)$ *is the unique maximal compact subgroup of* $\mathbf{J}(\mathfrak{a}, \beta)$.
(ii) *The group* $\mathrm{J}^1(\mathfrak{a}, \beta)$ *is the unique maximal normal pro-p-subgroup of* $\mathrm{J}(\mathfrak{a}, \beta)$.
(iii) *The group* $\mathrm{J}(\mathfrak{a}, \beta)$ *is generated by* $\mathrm{J}^1(\mathfrak{a}, \beta)$ *and* $\mathfrak{b}^\times$, *and we have*

$$\mathrm{J}(\mathfrak{a}, \beta) \cap \mathrm{B}^\times = \mathfrak{b}^\times, \quad \mathrm{J}^1(\mathfrak{a}, \beta) \cap \mathrm{B}^\times = \mathrm{U}^1(\mathfrak{b}).$$

(iv) *The normalizer of any simple character* $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ *in* G *is equal to* $\mathbf{J}(\mathfrak{a}, \beta)$.

3.2. Simple characters and endo-classes

Simple characters have remarkable intertwining and transfer properties. Let $[\mathfrak{a}', \beta']$ be another simple stratum in $M_{n'}(F)$ for some integer $n' \geq 1$, and suppose that we have an isomorphism of F-algebras $\varphi : F[\beta] \to F[\beta']$ such that $\varphi(\beta) = \beta'$. Then there is a canonical bijective map

$$\mathcal{C}(\mathfrak{a}, \beta) \to \mathcal{C}(\mathfrak{a}', \beta'),$$

called the *transfer map* [14, Theorem 3.6.14].

Now let $[\mathfrak{a}, \beta_1]$ and $[\mathfrak{a}, \beta_2]$ be simple strata in $M_n(F)$, and assume that we have two simple characters $\theta_1 \in \mathcal{C}(\mathfrak{a}, \beta_1)$ and $\theta_2 \in \mathcal{C}(\mathfrak{a}, \beta_2)$ that intertwine; that is, there is a $g \in GL_n(F)$ such that

$$\theta_2(x) = \theta_1(gxg^{-1}) \quad \text{for all } x \in H^1(\mathfrak{a}, \beta_2) \cap g^{-1}H^1(\mathfrak{a}, \beta_1)g.$$

For $i = 1, 2$, let $[\mathfrak{a}', \beta'_i]$ be a simple stratum in $M_{n'}(F)$ for some $n' \geq 1$, such that θ_i transfers to a simple character $\theta'_i \in \mathcal{C}(\mathfrak{a}', \beta'_i)$. Then the simple characters θ'_1, θ'_2 are conjugate under $GL_{n'}(F)$ (see [9, Theorem 8.7] and [14, Theorem 3.5.11]).

Now let $[\mathfrak{a}_1, \beta_1]$ and $[\mathfrak{a}_2, \beta_2]$ be simple strata in $M_{n_1}(F)$ and $M_{n_2}(F)$, respectively, for $n_1, n_2 \geq 1$. We say that two simple characters $\theta_1 \in \mathcal{C}(\mathfrak{a}_1, \beta_1)$ and $\theta_2 \in \mathcal{C}(\mathfrak{a}_2, \beta_2)$ are *endo-equivalent* if there are simple strata $[\mathfrak{a}', \beta'_1]$ and $[\mathfrak{a}', \beta'_2]$ in $M_{n'}(F)$, for some $n' \geq 1$, such that θ_1 and θ_2 transfer to simple characters $\theta'_1 \in \mathcal{C}(\mathfrak{a}', \beta'_1)$ and $\theta'_2 \in \mathcal{C}(\mathfrak{a}', \beta'_2)$ which intertwine (or equivalently which are $GL_{n'}(F)$-conjugate). This defines an equivalence relation on the set

$$\bigcup_{[\mathfrak{a}, \beta]} \mathcal{C}(\mathfrak{a}, \beta)$$

where the union is taken over all simple strata of $M_n(F)$ for all $n \geq 1$ [9, Section 8]. An equivalence class for this relation is called an *endo-class*.

Given a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, the degree of E/F, its ramification order and its residue class degree depend only on the endo-class of θ. These integers are called the degree, ramification order and residue class degree of this endo-class. The field extension E/F is not uniquely determined, but its maximal tamely ramified subextension is uniquely determined, up to F-isomorphism, by the endo-class of θ. This tamely ramified subextension is called the *tame parameter field* of the endo-class [13, 2.2, 2.4].

Let $\mathcal{E}(F)$ denote the set of all endo-classes of simple characters in all general linear groups over F. Given a finite tamely ramified extension T of F, there is a surjective map

$$\mathcal{E}(T) \to \mathcal{E}(F)$$

with finite fibres, called the *restriction map* [13, 2.3]. Given $\boldsymbol{\Theta} \in \mathcal{E}(F)$, the endo-classes $\boldsymbol{\Psi} \in \mathcal{E}(T)$ which restrict to $\boldsymbol{\Theta}$ are called the T/F-*lifts* of $\boldsymbol{\Theta}$. If $\boldsymbol{\Theta}$ has tame parameter field T, then $\mathrm{Aut}_F(T)$ acts transitively and faithfully on the set of T/F-lifts of $\boldsymbol{\Theta}$ [13, 2.3, 2.4].

3.3. Simple types and cuspidal representations

Let us write $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ for some $n \geq 1$. A family of pairs (J, λ) called *simple types*, formed by a compact open subgroup J of G and an irreducible representation λ of J, has been constructed in [14] (see also [38] for the modular case).

By construction, given a simple type (J, λ) in G, there are a simple stratum $[\mathfrak{a}, \beta]$ and a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ such that $\mathrm{J}(\mathfrak{a}, \beta) = \mathrm{J}$ and θ is contained in the restriction of λ to $\mathrm{H}^1(\mathfrak{a}, \beta)$. Such a simple character is said to be *attached to* λ.

Definition 3.2. When the hereditary order $\mathfrak{b} = \mathfrak{a} \cap \mathrm{B}$ is a maximal order in B, we say that the simple stratum $[\mathfrak{a}, \beta]$ and the simple characters in $\mathcal{C}(\mathfrak{a}, \beta)$ are *maximal*. A simple type with a maximal attached simple character is called a *maximal simple type*.

When the simple stratum $[\mathfrak{a}, \beta]$ is maximal, and given a homomorphism $\mathrm{B} \simeq \mathrm{M}_m(\mathrm{E})$ of E-algebras identifying $\mathfrak{b}$ with the standard maximal order, one has group isomorphisms

$$\mathrm{J}(\mathfrak{a}, \beta)/\mathrm{J}^1(\mathfrak{a}, \beta) \simeq \mathfrak{b}^\times/\mathrm{U}^1(\mathfrak{b}) \simeq \mathrm{GL}_m(\boldsymbol{k}_\mathrm{E}). \tag{3.3}$$

The following proposition gives a description of cuspidal (irreducible) representations of G in terms of maximal simple types.

Proposition 3.4. *Let π be a cuspidal representation of* G.

(i) *There is a maximal simple type* (J, λ) *such that λ occurs as a subrepresentation of the restriction of π to* J. *This simple type is uniquely determined up to* G*-conjugacy.*
(ii) *The simple character θ attached to λ is uniquely determined up to* G*-conjugacy. Its endo-class $\boldsymbol{\Theta}$ is called the* endo-class *of π.*
(iii) *If $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$ is a simple character in* G*, then the restriction of π to $\mathrm{H}^1(\mathfrak{a}', \beta')$ contains θ' if and only if θ' is maximal and has endo-class $\boldsymbol{\Theta}$, that is, if and only if θ, θ' are* G*-conjugate.*
(iv) *Let $[\mathfrak{a}, \beta]$ be a maximal simple stratum such that $\mathrm{J} = \mathrm{J}(\mathfrak{a}, \beta)$ and $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$. The simple type λ extends uniquely to a representation $\boldsymbol{\lambda}$ of the normalizer $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ of θ in* G *such that the compact induction of $\boldsymbol{\lambda}$ to* G *is isomorphic to π.*

Proof. This follows from [14, 6.2, 8.4]. See [38, Section 3] in the case that R has positive characteristic. □

A pair $(\mathbf{J}, \boldsymbol{\lambda})$ constructed in this way is called an *extended maximal simple type* in G. Compact induction induces a bijection between G-conjugacy classes of extended maximal simple types and isomorphism classes of cuspidal representations of G ([14, 6.2] and [38, Theorem 3.11]).

4. The σ-self-dual type theorem

We state our first main theorem. We fix an integer $n \geq 1$ and write $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$.

Theorem 4.1. *Let π be a cuspidal representation of* G. *Then $\pi^\sigma \simeq \pi^\vee$ if and only if π contains an extended maximal simple type $(\mathbf{J}, \boldsymbol{\lambda})$ such that $\mathbf{J}$ is σ-stable and $\boldsymbol{\lambda}^\sigma \simeq \boldsymbol{\lambda}^\vee$.*

In other words, a cuspidal representation of G is σ-self-dual if and only if it contains a σ-self-dual extended maximal simple type.

If $(\mathbf{J}, \boldsymbol{\lambda})$ is an extended maximal simple type for the cuspidal representation π, then $(\sigma(\mathbf{J}), \boldsymbol{\lambda}^{\sigma})$ is an extended maximal simple type for π^{σ} and $(\mathbf{J}, \boldsymbol{\lambda}^{\vee})$ is an extended maximal simple type for $\pi^{\vee}$. Thus, if π contains an extended maximal simple type $(\mathbf{J}, \boldsymbol{\lambda})$ such that $\mathbf{J}$ is σ-stable and $\boldsymbol{\lambda}^{\sigma}, \boldsymbol{\lambda}^{\vee}$ are isomorphic, then $\pi^{\sigma}, \pi^{\vee}$ are isomorphic. The rest of Section 4 is devoted to the proof of the converse statement.

4.1. The endo-class

Start with a cuspidal representation π of G, and suppose that $\pi^{\sigma} \simeq \pi^{\vee}$. Let $\boldsymbol{\Theta}$ be its endo-class over F. Associated with it, there are its degree $d = \deg(\boldsymbol{\Theta})$ and its tame parameter field T: this is a tamely ramified finite extension of F, unique up to F-isomorphism (see §3.2).

If $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ is a maximal simple character contained in π, then $\theta^{-1} \in \mathcal{C}(\mathfrak{a}, -\beta)$ is contained in $\pi^{\vee}$ and $\theta \circ \sigma \in \mathcal{C}(\sigma(\mathfrak{a}), \sigma(\beta))$ is contained in π^{σ}. Note that we use the fact that the character of F fixed in §3.1 is σ-stable in order to have $\theta \circ \sigma \in \mathcal{C}(\sigma(\mathfrak{a}), \sigma(\beta))$. We write $\boldsymbol{\Theta}^{\vee}$ for the endo-class of θ^{-1}, and $\boldsymbol{\Theta}^{\sigma}$ for that of $\theta \circ \sigma$. The assumption on π implies that $\boldsymbol{\Theta}^{\sigma} = \boldsymbol{\Theta}^{\vee}$. We will prove the following theorem.

Theorem 4.2. *Let $\boldsymbol{\Theta} \in \mathcal{E}(\mathrm{F})$ be an endo-class of degree dividing n such that $\boldsymbol{\Theta}^{\sigma}$ is equal to $\boldsymbol{\Theta}^{\vee}$, and let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a simple character in* G *of endo-class $\boldsymbol{\Theta}$. There are a simple stratum $[\mathfrak{a}', \beta']$ and a simple character $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$ such that:*

(i) *the character θ' is* G*-conjugate to θ;*
(ii) *the group $\mathrm{H}^1(\mathfrak{a}', \beta')$ is σ-stable and $\theta' \circ \sigma = \theta'^{-1}$;*
(iii) *the order $\mathfrak{a}'$ is σ-stable and $\sigma(\beta') = -\beta'$.*

Before proving Theorem 4.2, we show how it implies Theorem 4.1. By applying Theorem 4.2 to any simple character ϑ contained in π, which is maximal by Proposition 3.4(iii), we get a maximal simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, conjugate to ϑ, such that $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$ and

$$\theta \circ \sigma = \theta^{-1}.$$

Thus θ is contained in π and its normalizer $\mathbf{J}$ in G is σ-stable. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be an extended maximal simple type for π with attached simple character θ. Since π is σ-self-dual, it contains both $(\mathbf{J}, \boldsymbol{\lambda})$ and $(\mathbf{J}, \boldsymbol{\lambda}^{\vee\sigma})$. By Proposition 3.4, this implies that they are conjugate by an element $g \in \mathrm{G}$, that is, g normalizes $\mathbf{J}$ and $\boldsymbol{\lambda}^{\vee\sigma}$ is isomorphic to $\boldsymbol{\lambda}^{g}$. Now consider the simple characters $\theta^{-1} \circ \sigma = \theta$ and θ^{g}. Both are contained in $\boldsymbol{\lambda}^{g}$. Restricting $\boldsymbol{\lambda}^{g}$ to the intersection

$$\mathrm{H}^1(\mathfrak{a}, \beta) \cap \mathrm{H}^1(\mathfrak{a}, \beta)^{g} \tag{4.3}$$

we get a direct sum of copies of θ containing the restriction of θ^{g} to (4.3). It follows that g intertwines θ. By [14, Theorem 3.3.2], which describes the intertwining set of a simple character, we have $g \in \mathbf{J}\mathrm{B}^{\times}\mathbf{J}$. We may thus assume that $g \in \mathrm{B}^{\times}$. By uniqueness of the maximal compact subgroup in $\mathbf{J}$, the identity $\mathbf{J}^{g} = \mathbf{J}$ gives us $\mathrm{J}^{g} = \mathrm{J}$. Intersecting with $\mathrm{B}^{\times}$ gives $\mathfrak{b}^{\times g} = \mathfrak{b}^{\times}$. It follows that g normalizes the order $\mathfrak{b}$. We thus have $g \in \mathbf{J}$, thus $\boldsymbol{\lambda}^{\sigma} \simeq \boldsymbol{\lambda}^{\vee}$. Theorem 4.1 is proved.

Remark 4.4. Assuming that Theorem 4.2 holds, and using Intertwining Implies Conjugacy [14, Theorem 5.7.1], the same argument shows that if π is a σ-self-dual irreducible representation of G that contains a simple type, then π contains a σ-self-dual simple type. In particular, any σ-self-dual discrete series representation of G contains a σ-self-dual simple type.

Remark 4.5. However, an arbitrary σ-self-dual irreducible representation of G may not contain a σ-self-dual semisimple type. See [16, 38] for the notion of semisimple type and §4.9 for a counterexample.

It thus remains to prove Theorem 4.2. For this, one can forget about the representation π.

4.2. *A prelude*

We first show how to deal with the (second part of the) third condition of Theorem 4.2. Recall (see [14]) that a stratum $[\mathfrak{a}, v, r, \beta]$ in $M_n(F)$ is *pure* if $F[\beta]$ is a field, $F[\beta]^\times$ normalizes $\mathfrak{a}$ and $\beta\mathfrak{a} = \mathfrak{p}_\mathfrak{a}^{-v}$. (See §3.1 for the comment on the notation.)

Here again (see §4.1), we use the fact that the character of F fixed in §3.1 is σ-stable.

Lemma 4.6. *Let $[\mathfrak{a}, v, r, \beta]$ be a pure stratum in $M_n(F)$ with $\sigma(\mathfrak{a}) = \mathfrak{a}$ and $\sigma(\beta) + \beta \in \mathfrak{p}_\mathfrak{a}^{-r}$. There is a simple stratum $[\mathfrak{a}, v, r, \gamma]$ such that $\beta - \gamma \in \mathfrak{p}_\mathfrak{a}^{-r}$ and $\sigma(\gamma) + \gamma = 0$.*

Proof. The proof is exactly as in [52, Proposition 1.10], using the involution σ instead of the adjoint involution $x \mapsto \overline{x}$ used in [52]. □

Proposition 4.7. *Let $[\mathfrak{a}, \beta]$ be a simple stratum in $M_n(F)$ with $\sigma(\mathfrak{a}) = \mathfrak{a}$. Suppose that there is a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ such that $H^1(\mathfrak{a}, \beta)$ is σ-stable and $\theta \circ \sigma = \theta^{-1}$. Then there is a simple stratum $[\mathfrak{a}, \gamma]$ such that $\theta \in \mathcal{C}(\mathfrak{a}, \gamma)$ and $\sigma(\gamma) + \gamma = 0$.*

Proof. The proof is exactly the same as in [52, Theorem 6.3], using the involution σ instead of the adjoint involution used in [52], and replacing [52, Proposition 1.10] by Lemma 4.6. □

4.3. *The tame parameter field*

From now on, and until the end of this section, $\boldsymbol{\Theta} \in \mathcal{E}(F)$ is an endo-class, with degree d dividing n, which is σ-self-dual—that is, such that $\boldsymbol{\Theta}^\sigma = \boldsymbol{\Theta}^\vee$. In this subsection, we will see that this symmetry condition on $\boldsymbol{\Theta}$ implies that its tame parameter field T/F inherits certain properties.

Note that we *do not* assume that $\boldsymbol{\Theta}$ is the endo-class of some σ-self-dual cuspidal representation π of G. For the notion of a T/F-lift of $\boldsymbol{\Theta}$, we refer to §3.2.

Lemma 4.8. *Let $\boldsymbol{\Theta}$ be a σ-self-dual endo-class and T/F be its tame parameter field.*

(i) *Given a T/F-lift $\boldsymbol{\Psi}$ of $\boldsymbol{\Theta}$, there is a unique involutive F_o-automorphism α of T extending σ such that $\boldsymbol{\Psi}^\vee = \boldsymbol{\Psi}^\alpha$.*
(ii) *For any F-automorphism γ of T, the F_o-involution of T associated with $\boldsymbol{\Psi}^\gamma$ is $\gamma^{-1}\alpha\gamma$.*

Proof. The tame parameter field of $\boldsymbol{\Theta}^\vee$ is T, and that of $\boldsymbol{\Theta}^\sigma$ is the field T endowed with the map $x \mapsto \sigma(x)$ from F to T. The assumption on $\boldsymbol{\Theta}$ implies that these tame parameter fields are F-isomorphic. Thus there exists an F_o-automorphism of T whose restriction to F is σ.

Let $\boldsymbol{\Psi}$ be a T/F-lift of $\boldsymbol{\Theta}$ (see §3.2). Then $\boldsymbol{\Psi}^\vee$ is a T/F-lift of $\boldsymbol{\Theta}^\vee$, and the bijection $\alpha \mapsto \boldsymbol{\Psi}^\alpha$ between automorphisms of T/F_o and T/F_o-lifts of $\boldsymbol{\Theta}$ induces a bijection between F_o-automorphisms of T extending σ and T/F-lifts of $\boldsymbol{\Theta}^\sigma$. Thus there is a unique F_o-automorphism α of T extending σ such that $\boldsymbol{\Psi}^\vee = \boldsymbol{\Psi}^\alpha$. Since $\boldsymbol{\Psi}^{\vee\vee} = \boldsymbol{\Psi}$, we deduce that $\boldsymbol{\Psi}^{\alpha^2} = \boldsymbol{\Psi}$. That α^2 is trivial follows from the fact that α^2 is in $\mathrm{Aut}_F(T)$, which acts faithfully on the set of T/F-lifts of $\boldsymbol{\Theta}$. □

Remark 4.9. It is *not* in general true that every involutive F_o-automorphism α of T extending the F_o-automorphism σ of F has the additional property required by Lemma 4.8(i). For example, if F/F_o is unramified and T/F is ramified quadratic, then T/F_o is a biquadratic extension and the two automorphisms fixing the ramified quadratic subextensions of F_o in T are both involutions extending σ; however, they are not conjugate so, by the uniqueness statement in Lemma 4.8, cannot both have the additional property.

Let α be an F_o-involution of T given by Lemma 4.8, and let T_o be the fixed points of α in T. Thus $T_o \cap F = F_o$.

Lemma 4.10. *The canonical homomorphism* $T_o \otimes_{F_o} F \to T$ *of* $T_o \otimes_{F_o} F$*-modules is an isomorphism.*

Proof. The canonical homomorphism is an isomorphism if and only if F does not embed in T_o as an F_o-algebra. Assume that there is such an embedding. Since F is Galois over F_o, its image is F. Thus F is contained in T_o, which contradicts $T_o \cap F = F_o$. □

Write t for the degree of T over F.

Corollary 4.11. *There is an embedding of* F*-algebras* $\iota : T \hookrightarrow M_t(F)$ *such that*

$$\iota(\alpha(x)) = \sigma(\iota(x))$$

for all $x \in T$*. In particular, the image of* ι *in* $M_t(F)$ *is* σ*-stable.*

Proof. Fix an F_o-embedding ι_o of T_o in $M_t(F_o)$. Then $\iota = \iota_o \otimes F$ has the required property, thanks to Lemma 4.10. □

Remark 4.12. The natural group homomorphism

$$\mathrm{Aut}_{F_o}(T) \to \mathrm{Aut}_{F_o}(T_o) \rtimes \mathrm{Gal}(F/F_o)$$

(where the semidirect product is defined with respect to α) is an isomorphism.

4.4. The maximal and totally wild case

In this subsection, we will assume that $d = n$ and $\mathrm{T} = \mathrm{F}$.

Proposition 4.13. *Let θ be a simple character in* G *with endo-class $\boldsymbol{\Theta}$. There is a simple character $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$ which is* G*-conjugate to θ, such that $\mathfrak{a}'$ is σ-stable and $\theta' \circ \sigma = \theta'^{-1}$.*

Let $[\mathfrak{a}, \beta]$ be a simple stratum such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$. We may and will assume that the principal order $\mathfrak{a}$ is standard (that is, $\mathfrak{a}$ is made up of matrices with coefficients in $\mathcal{O}$ and its reduction mod $\mathfrak{p}$ is made up of upper block triangular matrices), thus σ-stable. The extension $\mathrm{F}[\beta]$ is totally wildly ramified over F. In particular, $\mathfrak{a}$ is a minimal order in $\mathrm{M}_n(\mathrm{F})$.

Write $\mathrm{U} = \mathfrak{a}^\times$, which is the standard Iwahori subgroup of G. For all $i \geq 1$, write $\mathrm{U}^i = 1 + \mathfrak{p}_{\mathfrak{a}}^i$, which is a normal subgroup of U. Then $\mathrm{U}/\mathrm{U}^1 \simeq \boldsymbol{k}^{\times n}$ is abelian, of order prime to p, and $\mathrm{U}^i/\mathrm{U}^{i+1}$ is an abelian p-group for all $i \geq 1$.

Since $\boldsymbol{\Theta}^\sigma = \boldsymbol{\Theta}^\vee$ and $\mathfrak{a}$ is σ-stable, the characters $\theta \circ \sigma \in \mathcal{C}(\mathfrak{a}, \sigma(\beta))$ and $\theta^{-1} \in \mathcal{C}(\mathfrak{a}, -\beta)$ intertwine. By Intertwining Implies Conjugacy for simple characters [14, Theorem 3.5.11], there is a $u \in \mathrm{U}$ such that $\mathrm{H}^1(\mathfrak{a}, \sigma(\beta)) = u^{-1}\mathrm{H}^1(\mathfrak{a}, -\beta)u$ and $\theta \circ \sigma = (\theta^{-1})^u$. Since σ is involutive and the G-normalizer of θ is $\mathbf{J}$, this gives

$$u\sigma(u) \in \mathbf{J} \cap \mathrm{U} = \mathrm{J}. \tag{4.14}$$

We search for an $x \in \mathrm{G}$ such that the character $\theta' = \theta^x \in \mathcal{C}(\mathfrak{a}^x, \beta^x)$ has the desired property. This amounts to the condition $u\sigma(x)x^{-1} \in \mathbf{J}$.

Note that $\mathrm{J} = \mathcal{O}^\times \mathrm{J}^1$ since $\mathrm{F}[\beta]$ is totally ramified over F. Thus the image of J in $\mathrm{U}/\mathrm{U}^1 \simeq \boldsymbol{k}^{\times n}$ is the image of the diagonal embedding of $\boldsymbol{k}^\times$ in $\boldsymbol{k}^{\times n}$. Let M be the torus made up of all diagonal matrices of G.

Lemma 4.15. *There is a $y \in \mathrm{M}$ such that $u\sigma(y)y^{-1} \in \mathrm{JU}^1 = \mathcal{O}^\times\mathrm{U}^1$.*

Proof. There are $u_1, \dots, u_n \in \boldsymbol{k}^\times$ such that u mod U^1 is equal to $(u_1, \dots, u_n)$ in $\mathrm{U}/\mathrm{U}^1 \simeq \boldsymbol{k}^{\times n}$. Changing u in the equivalence class $\mathcal{O}^\times u$, we may assume that $u_1 = 1$.

The condition (4.14) says that $u\sigma(u)$ mod U^1 is in the image of the diagonal embedding of $\boldsymbol{k}^\times$ in $\boldsymbol{k}^{\times n}$. Since $u_1 = 1$, this gives us $u_i\sigma(u_i) = 1$ for all $i \in \{1, \dots, n\}$.

Assume first that F is unramified over F_o. Then $\boldsymbol{k}$ is quadratic over $\boldsymbol{k}_\mathsf{o}$ and σ induces the non-trivial $\boldsymbol{k}_\mathsf{o}$-automorphism of $\boldsymbol{k}$. We search for $y = (y_1, \dots, y_n) \in \boldsymbol{k}^{\times n}$ such that $u\sigma(y)y^{-1} = 1$ in $\boldsymbol{k}^{\times n}$. This is possible by Hilbert's Theorem 90, since $u_i\sigma(u_i) = 1$ for all i.

Assume now F is ramified over F_o. Then σ is trivial on $\boldsymbol{k} = \boldsymbol{k}_\mathsf{o}$. We thus have $u_i^2 = 1$ which implies $u_i \in \{-1, 1\}$. Let ϖ be a uniformizer of F such that $\sigma(\varpi) = -\varpi$. Such a choice is possible since $p \neq 2$. We are searching for a $y = (y_1, \dots, y_n) \in \mathrm{F}^{\times n}$ such that $\sigma(y)y^{-1} \in \mathrm{U}$ and $u\sigma(y)y^{-1} = 1$ in $\boldsymbol{k}^{\times n}$. Let $y_i = 1$ if $u_i = 1$, and let $y_i = \varpi$ otherwise. This gives a $y \in \mathrm{M}$ satisfying the required condition. □

Let us write $zu\sigma(y)y^{-1} \in \mathrm{U}^1$ for some $y \in \mathrm{M}$ and $z \in \mathcal{O}^\times$ given by Lemma 4.15. By replacing the stratum $[\mathfrak{a}, \beta]$ by $[\mathfrak{a}^y, \beta^y]$, the simple character θ by $\theta^y \in \mathcal{C}(\mathfrak{a}^y, \beta^y)$ and u by $y^{-1}zu\sigma(y)$, which does not affect the fact that the order is σ-stable, we may and will assume that $u \in \mathrm{U}^1$. We write $\mathrm{J}^0 = \mathrm{J}$ and $\mathrm{J}^i = \mathrm{J} \cap \mathrm{U}^i$ for $i \geq 1$.

Lemma 4.16. *Let $v \in \mathrm{U}^i$ for some $i \geq 1$, and assume that $v\sigma(v) \in \mathrm{J}^i$. Then there are $j \in \mathrm{J}^i$ and $x \in \mathrm{U}^i$ such that $jv\sigma(x)x^{-1} \in \mathrm{U}^{i+1}$.*

Proof. Recall that $\mathrm{U}^i/\mathrm{U}^{i+1}$ is abelian, and write $h = v\sigma(v)$. We have $\sigma(h) \equiv h \bmod \mathrm{U}^{i+1}$. This implies that $h \in \mathrm{V} = \mathrm{J}^i\mathrm{U}^{i+1} \cap \sigma(\mathrm{J}^i\mathrm{U}^{i+1}) \supseteq \mathrm{U}^{i+1}$. We thus have $v\sigma(v) \equiv 1 \bmod \mathrm{V}$. The quotient $\mathrm{W} = \mathrm{U}^i/\mathrm{V}$ is an abelian, finite and σ-stable p-group, and the first cohomology group of $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_\mathrm{o})$ in W is trivial since $p \neq 2$. We thus have $v \equiv x\sigma(x)^{-1} \bmod \mathrm{V}$ for some element $x \in \mathrm{U}^i$. This gives us $v\sigma(x)x^{-1} \in \mathrm{V} \subseteq \mathrm{J}^i\mathrm{U}^{i+1}$ as required. □

Lemma 4.17. *There is a sequence of triples $(x_i, j_i, v_i) \in \mathrm{U}^i \times \mathrm{J}^i \times \mathrm{U}^{i+1}$, for $i \geq 0$, satisfying the following conditions:*

(i) $(x_0, j_0, v_0) = (1, 1, u)$;
(ii) *for all $i \geq 0$, if we set $y_i = x_0x_1 \dots x_i \in \mathrm{U}^1$, then the simple character $\theta_i = \theta^{y_i} \in \mathcal{C}(\mathfrak{a}, \beta^{y_i})$ satisfies $\theta_i \circ \sigma = ({\theta_i}^{-1})^{v_i}$;*
(iii) *for all $i \geq 1$, we have $y_iv_i = j_iy_{i-1}v_{i-1}\sigma(x_i)$.*

Proof. Assume the triples (x_k, j_k, v_k) have been defined for all $k < i$, for some $i \geq 1$. Applying Lemma 4.16 to $v_{i-1} \in \mathrm{U}^i$, which satisfies

$$v_{i-1}\sigma(v_{i-1}) \in \mathbf{J}^{y_{i-1}} \cap \mathrm{U}^i = \mathrm{J}^i(\mathfrak{a}, \beta^{y_{i-1}})$$

thanks to condition (ii), we obtain $h_i \in \mathrm{J}^i(\mathfrak{a}, \beta^{y_{i-1}})$ and $x_i \in \mathrm{U}^i$ such that $h_iv_{i-1}\sigma(x_i){x_i}^{-1} \in \mathrm{U}^{i+1}$. Now define $j_i \in \mathrm{J}^i$ and $v_i \in \mathrm{U}^{i+1}$ by $j_iy_{i-1} = y_{i-1}h_i$ and $x_iv_i = h_iv_{i-1}\sigma(x_i)$. Setting $y_i = y_{i-1}x_i$ and $\theta_i = \theta^{y_i}$, we get

$$\theta_i \circ \sigma = (\theta_{i-1} \circ \sigma)^{\sigma(x_i)} = (\theta_{i-1}^{-1})^{v_{i-1}\sigma(x_i)} = (\theta_{i-1}^{-1})^{x_iv_i}$$

since $h_i \in \mathrm{J}^i(\mathfrak{a}, \beta^{y_{i-1}})$ normalizes θ_{i-1}. Since $\theta_{i-1}^{x_i}$ is equal to θ_i, we get the expected result. □

Let $x \in \mathrm{U}^1$ be the limit of $y_i = x_0x_1 \dots x_i$ and $h \in \mathrm{J}^1$ that of $j_i \dots j_1j_0$ when i tends to infinity. We have

$$y_iv_i{y_i}^{-1} = (j_i \dots j_1j_0)u\sigma(y_i){y_i}^{-1} \in \mathrm{U}^i.$$

Passing to the limit, we get $u\sigma(x)x^{-1} = h^{-1} \in \mathrm{J}$, as expected.

4.5. The maximal case

In this subsection, we assume that $d = n$ only. We generalize Proposition 4.13 to this situation.

Proposition 4.18. *Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a simple character in G of endo-class $\boldsymbol{\Theta}$. There is a simple character $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$ which is G-conjugate to θ and such that $\mathfrak{a}'$ is σ-stable and $\theta' \circ \sigma = \theta'^{-1}$.*

Proof. Let E be the field extension $F[\beta]$, and let T be the maximal tamely ramified extension of F in E. It is the tame parameter field for the endo-class $\boldsymbol{\Theta}$. The simple character θ determines a T/F-lift $\boldsymbol{\Psi}$ of $\boldsymbol{\Theta}$ as in [9, Section 9]. Namely, let C denote the centralizer of T in $M_n(F)$. The intersection $\mathfrak{c} = \mathfrak{a} \cap C$ is a minimal order in C, giving rise to a simple stratum $[\mathfrak{c}, \beta]$ in C. The restriction of θ to $H^1(\mathfrak{c}, \beta)$, denoted θ_T, is a simple character associated to this simple stratum, called the interior T/F-lift of θ in [9]. Its endo-class, denoted $\boldsymbol{\Psi}$, is a T/F-lift of $\boldsymbol{\Theta}$.

Lemma 4.8 gives us a unique F_o-involution α of T such that $\alpha|_F = \sigma$ and $\boldsymbol{\Psi}^\vee = \boldsymbol{\Psi}^\alpha$. Let us fix an F-embedding ι of T in $M_t(F)$ as in Corollary 4.11. Composing with the diagonal embedding of $M_t(F)$ in $M_n(F)$ gives us an F-embedding of T in $M_n(F)$ such that

$$\iota(\alpha(x)) = \sigma(\iota(x)), \quad x \in T.$$

By the Skolem–Noether theorem, this embedding is implemented by conjugating by some $g \in G$. Thus, conjugating $[\mathfrak{a}, \beta]$ and θ by g, we may assume that T is σ-stable and that the F_o-involution σ of $M_n(F)$ induces α on T. Note that C is σ-stable and is canonically isomorphic to the T-algebra $M_{n/t}(T)$. The restriction of σ to C identifies with the involution α acting componentwise. From now on, we will abuse the notation and write σ instead of α.

We now apply Proposition 4.13 to the simple character θ_T whose endo-class $\boldsymbol{\Psi}$ satisfies $\boldsymbol{\Psi}^\vee = \boldsymbol{\Psi}^\sigma$. We thus get a $y \in C^\times$ such that $\mathfrak{c}^y$ is σ-stable and the simple character $\vartheta = \theta_T^y$ satisfies $\vartheta \circ \sigma = \vartheta^{-1}$. Since the map $\mathfrak{a} \mapsto \mathfrak{a}^\times \cap C^\times$ is injective on hereditary orders of $M_n(F)$ normalized by $T^\times$ (see for instance [9, Section 2]), we deduce that the order $\mathfrak{a}' = \mathfrak{a}^y$ is σ-stable. Since interior T/F-lifting is injective from $\mathcal{C}(\mathfrak{a}^y, \beta^y)$ to $\mathcal{C}(\mathfrak{c}^y, \beta^y)$ by [9, Theorem 7.10], the simple character $\theta' = \theta^y$ has the expected property $\theta' \circ \sigma = \theta'^{-1}$. □

4.6. The general case

In this subsection, we prove Theorem 4.2 in the general case. Write $n = md$ with $m \geq 1$.

Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a simple character of endo-class $\boldsymbol{\Theta}$. By conjugating in G, we may assume that $\mathfrak{a}$ is σ-stable.

Fix an F-algebra homomorphism $\iota : F[\beta] \to M_d(F)$. Let $\mathfrak{a}_0$ denote the unique hereditary order in $M_d(F)$ normalized by $F[\iota\beta]^\times$ and $\theta_0 \in \mathcal{C}(\mathfrak{a}_0, \iota\beta)$ denote the transfer of θ. By Proposition 4.18, there are a maximal simple stratum $[\mathfrak{a}'_0, \beta'_0]$ and a simple character $\theta'_0 \in \mathcal{C}(\mathfrak{a}'_0, \beta'_0)$ such that:

(i) the character θ'_0 is conjugate to θ_0 under $GL_d(F)$;
(ii) the group $H^1(\mathfrak{a}'_0, \beta'_0)$ is σ-stable and $\theta'_0 \circ \sigma = \theta'^{-1}_0$;
(iii) the order $\mathfrak{a}_0$ is σ-stable.

Proposition 4.7 implies that, without changing $\mathfrak{a}_0$, we may assume that $\sigma(\beta'_0) = -\beta'_0$.

Let us now embed $M_d(F)$ diagonally in the F-algebra $M_n(F)$. This gives us an F-algebra homomorphism $\iota' : F[\beta'_0] \to M_n(F)$. Write $\beta' = \iota'\beta'_0$ and $E' = F[\beta']$. Since $\sigma(\beta') = -\beta'$, the field E′ is stable by σ. The centralizer B′ of E′ in $M_n(F)$ naturally identifies with $M_m(E')$.

Let $\mathfrak{b}'$ be a standard hereditary order in B', and let $\mathfrak{a}'$ be the unique hereditary order in $\mathrm{M}_n(\mathrm{F})$ normalized by $\mathrm{E}'^{\times}$ such that $\mathfrak{a}' \cap \mathrm{B}' = \mathfrak{b}'$. Then we have a simple stratum $[\mathfrak{a}', \beta']$ in $\mathrm{M}_n(\mathrm{F})$. Let $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$ be the transfer of θ. Since $\mathfrak{a}'$ is σ-stable and $\sigma(\beta') = -\beta'$, we have

$$\sigma(\mathrm{H}^1(\mathfrak{a}', \beta')) = \mathrm{H}^1(\sigma(\mathfrak{a}'), \sigma(\beta')) = \mathrm{H}^1(\mathfrak{a}', -\beta') = \mathrm{H}^1(\mathfrak{a}', \beta').$$

Let M be the standard Levi subgroup of G isomorphic to $\mathrm{GL}_d(\mathrm{F}) \times \cdots \times \mathrm{GL}_d(\mathrm{F})$. Write P for the standard parabolic subgroup of G generated by M and upper triangular matrices, and N for its unipotent radical. Let N^- be the unipotent radical of the parabolic subgroup opposite to P with respect to M. By [47, Theorem 2.17], we have

$$\left.\begin{aligned} \mathrm{H}^1(\mathfrak{a}', \beta') &= (\mathrm{H}^1(\mathfrak{a}', \beta') \cap \mathrm{N}^-) \cdot (\mathrm{H}^1(\mathfrak{a}', \beta') \cap \mathrm{M}) \cdot (\mathrm{H}^1(\mathfrak{a}', \beta') \cap \mathrm{N}), \\ \mathrm{H}^1(\mathfrak{a}', \beta') \cap \mathrm{M} &= \mathrm{H}^1(\mathfrak{a}'_0, \beta'_0) \times \cdots \times \mathrm{H}^1(\mathfrak{a}'_0, \beta'_0). \end{aligned}\right\} \tag{4.19}$$

Moreover, the character θ' is trivial on $\mathrm{H}^1(\mathfrak{a}', \beta') \cap \mathrm{N}$ and $\mathrm{H}^1(\mathfrak{a}', \beta') \cap \mathrm{N}^-$, and the restriction of θ' to $\mathrm{H}^1(\mathfrak{a}', \beta') \cap \mathrm{M}$ is equal to $\theta'_0 \otimes \cdots \otimes \theta'_0$. As M, N, N^- and $\mathrm{H}^1(\mathfrak{a}', \beta')$ are σ-stable, and by uniqueness of the Iwahori decomposition (4.19), we get $\theta' \circ \sigma = \theta'^{-1}$. Finally, as $\mathrm{F}[\beta]$ and E' have the same ramification index over F (see §3.2) we may choose the order $\mathfrak{b}'$ such that $\mathfrak{a}$ and $\mathfrak{a}'$ are conjugate. The transfer map from $\mathcal{C}(\mathfrak{a}, \beta)$ to $\mathcal{C}(\mathfrak{a}', \beta')$ is thus implemented by conjugacy by an element of G. It follows that θ and θ' are G-conjugate.

Definition 4.20. A maximal simple stratum $[\mathfrak{a}, \beta]$ in $\mathrm{M}_n(\mathrm{F})$ is said to be σ-*standard* if:

(i) the hereditary order $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$;
(ii) the element β has the block diagonal form

$$\begin{pmatrix} \beta_0 & & \\ & \ddots & \\ & & \beta_0 \end{pmatrix} = \beta_0 \otimes 1 \in \mathrm{M}_d(\mathrm{F}) \otimes_{\mathrm{F}} \mathrm{M}_m(\mathrm{F}) = \mathrm{M}_n(\mathrm{F})$$

for some $\beta_0 \in \mathrm{M}_d(\mathrm{F})$, where $d = \deg_{\mathrm{F}}(\beta)$ and $n = md$; the centralizer B of $\mathrm{E} = \mathrm{F}[\beta]$ in $\mathrm{M}_n(\mathrm{F})$ is thus equal to $\mathrm{M}_m(\mathrm{E})$, equipped with the involution σ acting componentwise;
(iii) the order $\mathfrak{b} = \mathfrak{a} \cap \mathrm{B}$ is the standard maximal order of $\mathrm{M}_m(\mathrm{E})$.

In conclusion, the following corollary refines Theorem 4.1.

Corollary 4.21. *Let π be a σ-self-dual cuspidal representation of* G. *Then π contains a σ-self-dual type attached to a σ-standard stratum.*

Remark 4.22. Let π be a σ-self-dual cuspidal representation of G, and $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a simple character in π such that $\theta \circ \sigma = \theta^{-1}$ and $\sigma(\beta) = -\beta$. Let E denote the field extension $\mathrm{F}[\beta]$ and write $\mathrm{E}_\mathrm{o} = \mathrm{E}^\sigma$. Let T denote the maximal tamely ramified subextension of E/F, that is, the tame parameter field of the endo-class of π, and write $\mathrm{T}_\mathrm{o} = \mathrm{T}^\sigma$. Then:

(i) The canonical homomorphism $\mathrm{E}_\mathrm{o} \otimes_{\mathrm{F}_\mathrm{o}} \mathrm{F} \to \mathrm{E}$ of $\mathrm{E}_\mathrm{o} \otimes_{\mathrm{F}_\mathrm{o}} \mathrm{F}$-modules is an isomorphism.
(ii) The extensions $\mathrm{E}/\mathrm{E}_\mathrm{o}$ and $\mathrm{T}/\mathrm{T}_\mathrm{o}$ have the same ramification index.

For the first property, see Lemma 4.10 and its proof. The second one follows from the fact that E is totally wildly ramified over T and p is odd, thus [E : T] is odd.

4.7. *Classification of σ-self-dual types*

From now on, we will abbreviate *σ-self-dual extended maximal simple type* to *σ-self-dual type*. In this paragraph, we determine the G^{σ}-orbits of σ-self-dual types in a σ-self-dual cuspidal representation of G.

Lemma 4.23. *Let π be a cuspidal representation of* G *containing a σ-self-dual type* $(\mathbf{J}, \boldsymbol{\lambda})$. *The σ-self-dual types in π are the* $(\mathbf{J}^g, \boldsymbol{\lambda}^g)$ *for* $g \in \mathrm{G}$ *such that* $\sigma(g)g^{-1} \in \mathbf{J}$.

Proof. By Proposition 3.4, any (extended maximal simple) type contained in π is G-conjugate to $(\mathbf{J}, \boldsymbol{\lambda})$. Given $g \in \mathrm{G}$, we have $(\boldsymbol{\lambda}^g)^{\sigma} = (\boldsymbol{\lambda}^{\sigma})^{\sigma(g)}$ and $(\boldsymbol{\lambda}^g)^{\vee} = (\boldsymbol{\lambda}^{\vee})^g$. Thus $(\mathbf{J}^g, \boldsymbol{\lambda}^g)$ is σ-self-dual if and only if $\sigma(g)g^{-1}$ normalizes $\boldsymbol{\lambda}$, that is, $\sigma(g)g^{-1} \in \mathbf{J}$. □

Corollary 4.24. *Let* $(\mathbf{J}, \boldsymbol{\lambda})$ *be a σ-self-dual type in* G. *There is a maximal simple stratum* $[\mathfrak{a}, \beta]$ *in* $\mathrm{M}_n(\mathrm{F})$ *such that:*

(i) $\mathfrak{a}$ *is σ-stable and* $\sigma(\beta) = -\beta$,
(ii) $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ *and the simple character θ associated to* $\boldsymbol{\lambda}$ *belongs to* $\mathcal{C}(\mathfrak{a}, \beta)$.

Proof. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-self-dual type in G. It induces to a σ-self-dual cuspidal representation π of G. Let $(\mathbf{J}_0, \boldsymbol{\lambda}_0)$ be a σ-self-dual type in π defined with respect to a simple stratum $[\mathfrak{a}_0, \beta_0]$ such that $\mathfrak{a}_0$ is σ-stable and $\sigma(\beta_0) = -\beta_0$. Then $(\mathbf{J}, \boldsymbol{\lambda}) = (\mathbf{J}_0^g, \boldsymbol{\lambda}_0^g)$ for some $g \in \mathrm{G}$ such that $\gamma = \sigma(g)g^{-1} \in \mathbf{J}_0$. We may thus assume that $(\mathbf{J}, \boldsymbol{\lambda})$ is defined with respect to the maximal simple stratum $[\mathfrak{a}_0^g, \beta_0^g]$. We have $\sigma(\mathfrak{a}_0^g) = (\mathfrak{a}_0^{\gamma})^g$, which is equal to $\mathfrak{a}_0^g$ since $\mathbf{J}_0$ is contained in the normalizer of $\mathfrak{a}_0$. The result now follows from Proposition 4.7. □

Lemma 4.25. *Let* $[\mathfrak{a}, \beta]$ *be a σ-standard maximal simple stratum in* $\mathrm{M}_n(\mathrm{F})$ *in the sense of Definition* 4.20. *Write* $\mathrm{E} = \mathrm{F}[\beta]$ *and* $\mathrm{E}_{\mathsf{o}} = \mathrm{E}^{\sigma}$. *Let* $g \in \mathrm{G}$ *and suppose that* $\sigma(g)g^{-1}$ *is in* $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$.

(i) *If* E *is unramified over* E_{o}, *then* $g \in \mathbf{J}\mathrm{G}^{\sigma}$;
(ii) *If* E *is ramified over* E_{o}, *and* ϖ_{E} *is a uniformizer of* E, *then:*
   (a) *there is a unique integer i such that* $0 \leq 2i \leq m$ *and* $g \in \mathbf{J}t_i\mathrm{G}^{\sigma}$, *where*
$$t_i = \mathrm{diag}(\varpi_{\mathrm{E}}, \dots, \varpi_{\mathrm{E}}, 1, \dots, 1) \in \mathrm{B}^{\times} = \mathrm{GL}_m(\mathrm{E}) \tag{4.26}$$
   with ϖ_{E} occurring i times;
   (b) *the double cosets* $\mathbf{J}t_i\mathrm{G}^{\sigma}$, $0 \leq i \leq \lfloor m/2 \rfloor$, *are all distinct.*

Proof. For any group Γ equipped with an action of σ, we will write $\mathsf{H}^1(\sigma, \Gamma)$ for the first cohomology set of $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_{\mathsf{o}})$ in Γ. Write $\gamma = \sigma(g)g^{-1}$. The identity $\sigma(\gamma) = \gamma^{-1}$ implies that γ has valuation 0 in $\mathbf{J}$. We thus have $\gamma \in \mathrm{J} = \mathrm{J}(\mathfrak{a}, \beta)$. Write $\mathrm{J}^1 = \mathrm{J}^1(\mathfrak{a}, \beta)$ and identify J/J^1 with $\mathrm{GL}_m(\boldsymbol{k}_{\mathrm{E}})$, denoted $\mathcal{G}$, as in (3.3). Let x denote the image of γ in $\mathcal{G}$. It satisfies $x\sigma(x) = 1$.

If E is unramified over E_o, then $x = \sigma(y)y^{-1}$ for some $y \in \mathcal{G}$, thus

$$\sigma(a^{-1})\gamma a = \sigma(a^{-1}g)g^{-1}a \in J^1 \tag{4.27}$$

for some $a \in J$. Since J^1 is a pro-p-group and $p \neq 2$, the first cohomology set $H^1(\sigma, J^1)$ is trivial. The left hand side of (4.27) can thus be written $\sigma(j)j^{-1}$ for some $j \in J^1$, thus we have $g \in JG^\sigma$.

Suppose now that E is ramified over E_o, so that σ acts trivially on $\boldsymbol{k}_E$. We may and will assume that ϖ_E has been chosen such that $\sigma(\varpi_E) = -\varpi_E$. Then x is conjugate in $\mathcal{G}$ to a class δJ^1 where

$$\delta = \delta_i = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1) \in \mathfrak{b}^\times \subseteq \mathrm{GL}_m(E)$$

with -1 occurring i times for some $i \in \{0, \dots, m\}$. We thus have $\sigma(a)\gamma a^{-1} \in \delta J^1$ for some $a \in J$. Notice that $\delta t_i = \sigma(t_i)$. If we write $h = t_i^{-1}xg$, we get $\sigma(h)h^{-1} \in J^{1t_i}$. Since J^{1t_i} is a σ-stable pro-p-group, the set $H^1(\sigma, J^{1t_i})$ is trivial, thus $h \in J^{1t_i}G^\sigma$, which implies that $g \in Jt_iG^\sigma$.

Now suppose that $\mathbf{J}t_iG^\sigma = \mathbf{J}t_kG^\sigma$ for some integers $0 \leq i, k \leq m$. Then $\delta_k = \sigma(a)\delta_i a^{-1}$ for some $a \in \mathbf{J}$. If we write $a = ut^r$ for some $r \in \mathbf{Z}$ and $u \in J$, then the images of δ_k and $(-1)^r\delta_i$ in $\mathcal{G}$ are conjugate, thus either r is even and $k = i$, or r is odd and $k = m - i$.

Finally, we have $\mathbf{J}t_iG^\sigma = \mathbf{J}t_{m-i}G^\sigma$ since $t_m \in \mathbf{J}$, $t_i^2 \in G^\sigma$ and the group of permutation matrices in $B^\times = \mathrm{GL}_m(E)$ is contained in $\mathbf{J} \cap G^\sigma$. □

Remark 4.28. If E is ramified over E_o, then the pairs

$$(\mathbf{J}^{t_i}, \boldsymbol{\lambda}^{t_i}), \quad i \in \{0, \dots, \lfloor m/2 \rfloor\},$$

where t_i is defined by (4.26), form a set of representatives of the G^σ-conjugacy classes of σ-self-dual types in π. The integer i is called the *index* of the G^σ-conjugacy class. If one identifies the quotient $J(\mathfrak{a}, \beta)^{t_i}/J^1(\mathfrak{a}, \beta)^{t_i}$ with $\mathrm{GL}_m(\boldsymbol{k}_E)$ via

$$J(\mathfrak{a}, \beta)^{t_i}/J^1(\mathfrak{a}, \beta)^{t_i} \simeq J(\mathfrak{a}, \beta)/J^1(\mathfrak{a}, \beta) \simeq U(\mathfrak{b})/U^1(\mathfrak{b}) \simeq \mathrm{GL}_m(\boldsymbol{k}_E),$$

then σ acts on $\mathrm{GL}_m(\boldsymbol{k}_E)$ by conjugacy by the diagonal element

$$\delta_i = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1),$$

where -1 occurs i times, and the group $(J(\mathfrak{a}, \beta)^{t_i} \cap G^\sigma)/(J^1(\mathfrak{a}, \beta)^{t_i} \cap G^\sigma)$ of σ-fixed points identifies with the Levi subgroup $(\mathrm{GL}_i \times \mathrm{GL}_{m-i})(\boldsymbol{k}_E)$ of $\mathrm{GL}_m(\boldsymbol{k}_E)$.

The inconvenience of the extension E/E_o is that it is not canonically determined by π. We remedy this in the next subsection.

4.8. The quadratic extension T/T_o

Let $\boldsymbol{\Theta} \in \mathcal{E}(F)$ be an endo-class of degree d such that $\boldsymbol{\Theta}^{\sigma} = \boldsymbol{\Theta}^{\vee}$. By Theorem 4.2, given any multiple n of d, there are a maximal simple stratum $[\mathfrak{a}, \beta]$ in $M_n(F)$ and a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ of endo-class $\boldsymbol{\Theta}$ such that $\theta \circ \sigma = \theta^{-1}$, the order $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$. Thus $E = F[\beta]$, its centralizer B and the maximal order $\mathfrak{b} = \mathfrak{a} \cap B$ are stable by σ.

Denote by E_o the field of σ-fixed points in E, by T the maximal tamely ramified subextension of E over F, and set $T_o = T \cap E_o$. Note that T is the tame parameter field of $\boldsymbol{\Theta}$, and that d is the degree $[E : F]$. We also write $n = md$.

Lemma 4.29. *The* F_o*-isomorphism class of the extension* T/T_o *only depends on* $\boldsymbol{\Theta}$*. Namely, if* T'/T'_o *is another extension obtained from* $\boldsymbol{\Theta}$ *as above, then there is an isomorphism* $\phi : T \to T'$ *of* F_o*-algebras such that* $\phi(T_o) = T'_o$.

Proof. Let $[\mathfrak{a}', \beta']$ be a maximal simple stratum in $M_{n'}(F)$ for some multiple n' of d, and let θ' be a simple character in $\mathcal{C}(\mathfrak{a}', \beta')$ of endo-class $\boldsymbol{\Theta}$ such that $\theta' \circ \sigma = \theta'^{-1}$, the order $\mathfrak{a}'$ is σ-stable and $\sigma(\beta') = -\beta'$. Associated with this, there are a tamely ramified extension T' of F and its σ-fixed points T'_0.

Suppose first that $\theta' = \theta$. Write J^1 for the maximal normal compact open pro-p-subgroup of the G-normalizer of θ. By [13, Proposition 2.6], one has $T' = T^x$ for some $x \in J^1$. Since T' is stable by σ, the element $y = \sigma(x)x^{-1} \in J^1$ normalizes T, thus centralizes it by [13, Proposition 2.6]. Applying Hilbert's Theorem 90 to the element y in the centralizer G_T of T in G implies that $x \in G_T G^{\sigma}$. It follows that T' is G^{σ}-conjugate to T. The F_o-isomorphism class of T/T_o thus only depends on θ, not on the simple stratum $[\mathfrak{a}, \beta]$ such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$.

Suppose now that $n' = n$. Since θ, θ' have the same endo-class, we have $\theta' = \theta^g$ for some $g \in G$. Since they are both σ-self-dual, we have $\sigma(g)g^{-1} \in \mathbf{J}$, where $\mathbf{J}$ is the G-normalizer of θ. By Lemma 4.25, we may even assume, up to G^{σ}-conjugacy, that $g \in B^{\times}$, thus $\sigma(g)g^{-1} \in B^{\times}$ centralizes T. Thanks to the first case, we may also assume that $\mathfrak{a}' = \mathfrak{a}^g$ and $\beta' = \beta^g$. We thus have $T' = T^g$ with $\sigma(g)g^{-1} \in G_T$. By the same cohomological argument as above, we deduce that T' is G^{σ}-conjugate to T.

We now consider the general case. By the first two cases and Corollary 4.21, we may assume, replacing θ, θ' by G-conjugate characters if necessary, that $[\mathfrak{a}, \beta]$ and $[\mathfrak{a}', \beta']$ are σ-standard. Hence we may transfer θ and θ' to $GL_d(F)$ without changing the F_o-isomorphism classes of T/T_o and T'/T'_o. We are thus reduced to the previous case. □

Now let π be a σ-self-dual cuspidal representation of G. Its endo-class, denoted $\boldsymbol{\Theta}$, has degree dividing n and satisfies $\boldsymbol{\Theta}^{\sigma} = \boldsymbol{\Theta}^{\vee}$. Associated with it, there is thus a quadratic extension T/T_o, uniquely determined up to F_o-isomorphism. Let us record this fact for future reference.

Proposition 4.30. *The* F_o*-isomorphism class of* T/T_o *depends only on the endo-class of* π.

Unlike E/E_o, the quadratic extension T/T_o is canonically attached to π. By applying Lemmas 4.23 and 4.25 together with Remarks 4.22 and 4.28, we get the following proposition.

Proposition 4.31. *Let π be a σ-self-dual cuspidal representation of* G, *and* T/T_o *be the quadratic extension canonically attached to it.*

(i) *If* T *is unramified over* T_o, *the σ-self-dual types contained in π form a single* G^σ*-conjugacy class.*
(ii) *If* T *is ramified over* T_o, *the σ-self-dual types contained in π form exactly* $\lfloor m/2 \rfloor + 1$ *different* G^σ*-conjugacy classes, characterized by their index.*

4.9. A counterexample in the semisimple case

We end this section by looking at a natural question which lies slightly outside the main thrust of this paper but which we find intriguing: namely, is there, for *any* σ-self-dual irreducible representation π, a σ-self-dual type contained in π. If one requires the type to be *semisimple* (in the sense of [16, 38]) then the answer is *no*, as the following example shows.

Let χ be a tamely ramified character of $F^\times$ such that the character $\chi(\chi\circ\sigma)$ is ramified. We consider the representation π of $GL_2(F)$ obtained by applying the functor or normalized parabolic induction to the character $\chi \otimes (\chi^{-1} \circ \sigma)$ of the Levi subgroup $F^\times \times F^\times$. This is an irreducible and σ-self-dual representation of level 0. By looking at its cuspidal support, one deduces that any semisimple type in π is conjugate to one of the following:

(i) the pair (I, λ) where I is the standard Iwahori subgroup (the one whose reduction mod $\mathfrak{p}_F$ is made up of upper triangular matrices) and λ is the character

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi(a)\chi(\sigma(d))^{-1};$$

(ii) the pair (I, λ') where λ' is the character

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi(\sigma(a))^{-1}\chi(d).$$

Note that the latter is conjugate to the former by the element

$$h = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \in GL_2(F)$$

where ϖ is a uniformizer of F. Thus any semisimple type in π is conjugate to (I, λ).

Now assume that π contains a σ-self-dual semisimple type. There is then a $g \in GL_2(F)$ such that

$$\sigma(I^g) = I^g, \quad \lambda^g \circ \sigma = (\lambda^g)^{-1}.$$

The first condition says that $\gamma = \sigma(g)g^{-1}$ normalizes I. The second one gives us $(\lambda\circ\sigma)^\gamma = \lambda^{-1}$. But $(\lambda\circ\sigma)^{-1} = \lambda' = \lambda^h$, thus $h\gamma$ normalizes λ. Let us write N for the normalizer of I in $\mathrm{GL}_2(\mathrm{F})$. It is generated by I and h, and carries a valuation homomorphism $v : \mathrm{N} \to \mathbf{Z}$ with kernel I. Since I is σ-stable, we have $v \circ \sigma = v$. Since $\sigma(\gamma) = \gamma^{-1}$ we have $v(\gamma) = 0$, thus $\gamma \in \mathrm{I}$. Since $h\gamma$ and I normalize λ, this implies that h normalizes λ: a contradiction.

Remark 4.32. The example above shows that there is no σ-self-dual semisimple type for π. This also implies that there is no σ-self-dual type for π which is a cover of type for its cuspidal support (in the sense of [15]). However, writing $\mathrm{K} = \mathrm{GL}_2(\mathcal{O})$ for the standard maximal compact subgroup of $\mathrm{GL}_2(\mathrm{F})$ and using the other notation above, the pair $(\mathrm{K}, \mathrm{ind}_{\mathrm{I}}^{\mathrm{K}} \lambda)$ is a type for π, which is σ-self-dual. Thus the question of whether or not all irreducible σ-self-dual representations of G possess a σ-self-dual type remains an interesting open question.

5. Generic σ-self-dual types

For this section, we place ourselves in a slightly more general setting. We again take $\mathrm{F_o}$ to be a non-archimedean local field of odd residual characteristic p, but we allow G to be the group of rational points of any connected reductive group defined over $\mathrm{F_o}$ equipped with a non-trivial involution σ defined over $\mathrm{F_o}$.

5.1. Let N be a σ-stable unipotent subgroup of G.

Lemma 5.1. *The group* N *is a union of* σ*-stable pro-*p*-subgroups.*

Proof. We write $\mathrm{N} = \bigcup_{i \geqslant 0} \mathrm{N}_i$ as a nested union of compact subgroups N_i which are open in N, so that $\mathrm{N}_i \subseteq \mathrm{N}_j$ for $0 \leqslant i \leqslant j$. For any $u \in \mathrm{N}$, there exist $i, j \geqslant 0$ such that $u \in \mathrm{N}_i$ and $\sigma(u) \in \mathrm{N}_j$. Then, taking $k = \max\{i, j\}$, we have $u \in \mathrm{N}_k \cap \sigma(\mathrm{N}_k)$. Thus $\mathrm{N} = \bigcup_{k \geqslant 0}(\mathrm{N}_k \cap \sigma(\mathrm{N}_k))$, as required. □

Lemma 5.2 (cf. [51, Lemma 2.1]). *Let* K *be a* σ*-stable open subgroup of* G*, and let* $g \in \mathrm{G}$.

(i) *If the double coset* $\mathrm{N}g\mathrm{K}$ *is* σ*-stable then it contains a* σ*-stable left* K*-coset.*
(ii) *If* $g\mathrm{K}$ *is* σ*-stable then every* σ*-stable left* K*-coset in* $\mathrm{N}g\mathrm{K}$ *lies in* $\mathrm{N}^\sigma g\mathrm{K}$.
(iii) $(\mathrm{NK})^\sigma = \mathrm{N}^\sigma \mathrm{K}^\sigma$.

Proof. (i) Suppose $\mathrm{N}g\mathrm{K}$ is σ-stable, so that $\sigma(g) = ugk$ for some $u \in \mathrm{N}$ and $k \in \mathrm{K}$. By Lemma 5.1, there is a σ-stable pro-p-subgroup N_0 of N containing u, so that $\sigma(g)$ is in $\mathrm{N}_0 g\mathrm{K}$. In particular, the double coset $\mathrm{N}_0 g\mathrm{K}$ is σ-stable.

Now we decompose $\mathrm{N}_0 g\mathrm{K}$ as a union of K-cosets. Since $\mathrm{N}_0 g\mathrm{K}/\mathrm{K}$ is in bijection with the quotient $\mathrm{N}_0/(\mathrm{N}_0 \cap g\mathrm{K}g^{-1})$, which is finite of order a power of p (odd), there is some coset $h\mathrm{K} \subset \mathrm{N}_0 g\mathrm{K}$ which is σ-stable.

(ii) Suppose $g\mathrm{K}$ is σ-stable so that $g^{-1}\sigma(g) = k \in \mathrm{K}$. If $ug\mathrm{K}$ is σ-stable, then

$$u^{-1}\sigma(u) = u^{-1}\sigma(ug)\sigma(g^{-1}) = gk_1k^{-1}g^{-1}$$

for some $k_1 \in \mathrm{K}$. Thus $\tau \mapsto u^{-1}\tau(u)$ defines a 1-cocycle in $\mathsf{H}^1(\langle\sigma\rangle, g\mathrm{K}g^{-1} \cap \mathrm{N})$, which is trivial, so there exists $v \in g\mathrm{K}g^{-1} \cap \mathrm{N}$ such that $u^{-1}\sigma(u) = v\sigma(v^{-1})$. Then $ug\mathrm{K} = uvg\mathrm{K}$ and $uv \in \mathrm{N}^\sigma$.

(iii) Suppose $h \in (\mathrm{NK})^\sigma$. Then certainly $h\mathrm{K}$ is σ-stable. On the other hand, $\mathrm{N}h\mathrm{K} = \mathrm{NK}$ and K itself is also σ-stable, so applying (ii) with $g = 1$ we find that every σ-stable left coset in NK lies in $\mathrm{N}^\sigma\mathrm{K}$; thus h is in $\mathrm{N}^\sigma\mathrm{K}$. Writing $h = uk$ with $u \in \mathrm{N}^\sigma$ and $k \in \mathrm{K}$, the fact that h is σ-invariant implies $k \in \mathrm{K}^\sigma$, so $h \in \mathrm{N}^\sigma\mathrm{K}^\sigma$. □

5.2. We suppose from now on that G is quasi-split. As before, a pair (K, τ), consisting of an open subgroup K of G and an irreducible representation τ of K, is called *σ-self-dual* if $\sigma(\mathrm{K}) = \mathrm{K}$ and $\tau^\sigma \simeq \tau^\vee$.

A *Whittaker datum* for G is a pair (N, ψ) consisting of (the $\mathrm{F_o}$-points of) the unipotent radical N of an $\mathrm{F_o}$-Borel subgroup of G and a character ψ of N such that the stabilizer of ψ in G is ZN, where Z denotes the $\mathrm{F_o}$-points of the centre of G. If a Whittaker datum (N, ψ) is σ-self-dual then, since $\mathrm{F_o}$ is not of characteristic two, ψ is trivial on N^σ.

Proposition 5.3 (cf. [10, Proposition 1.6]). *Suppose that* G *is quasi-split. Let* (N, ψ) *be a σ-self-dual Whittaker datum in* G *and let π be an irreducible σ-self-dual cuspidal representation of* G *such that the space* $\mathrm{Hom_N}(\pi, \psi)$ *is one-dimensional. Suppose that* $(\mathbf{J}, \rho)$ *is a σ-self-dual pair, with* $\mathbf{J}$ *a compact-mod-centre open subgroup of* G, *such that* $\pi \simeq \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}\,\rho$.

(i) *There exists a σ-self-dual pair* $(\mathbf{J}', \rho')$ *conjugate to* $(\mathbf{J}, \rho)$ *such that*

$$\mathrm{Hom}_{\mathbf{J}'\cap\mathrm{N}}(\rho', \psi) \neq 0.$$

(ii) *The pair* $(\mathbf{J}', \rho')$ *as in* (i) *is uniquely determined up to conjugacy by* N^σ.

(iii) *For any pair* $(\mathbf{J}', \rho')$ *as in* (i), *the space* $\mathrm{Hom}_{\mathbf{J}'\cap\mathrm{N}}(\rho', \psi)$ *is one-dimensional.*

Proof. We follow the proof of [10, Proposition 1.6] which, although written only for $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$, is valid more generally. Let us write $\mathscr{V}_\rho$ for the space of ρ, and $\mathscr{H}(\mathrm{G}, \rho, \psi)$ for the space of functions $\varphi : \mathrm{G} \to \mathrm{Hom_R}(\mathscr{V}_\rho, \mathrm{R})$ such that $\varphi(ugk) = \psi(u)\varphi(g) \circ \rho(k)$ for all $u \in \mathrm{N}$, $g \in \mathrm{G}$ and $k \in \mathbf{J}$. By the main Theorem of [31] (which is valid also for R-representations), we have a natural G-isomorphism

$$\mathscr{H}(\mathrm{G}, \rho, \psi) \simeq \mathrm{Hom_G}(\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}\,\rho, \mathrm{Ind}_{\mathrm{N}}^{\mathrm{G}}\,\psi).$$

In particular, we see that $\dim_\mathrm{R} \mathscr{H}(\mathrm{G}, \rho, \psi) = 1$, whence (cf. [10, (1.8)]) there is a unique double coset $\mathrm{N}g\mathbf{J}$ which supports a non-zero element of $\mathscr{H}(\mathrm{G}, \rho, \psi)$ (that is, intertwines ψ with ρ), and moreover the space of $\varphi \in \mathscr{H}(\mathrm{G}, \rho, \psi)$ supported on $\mathrm{N}g\mathbf{J}$ is one-dimensional—that is, $\mathrm{Hom}_{\mathrm{N}^g\cap\mathbf{J}}(\rho, \psi^g)$ is one-dimensional. Note that $\mathrm{N}^g \cap \mathbf{J}$ is a compact subgroup of N^g, so is pro-p; in particular, the restriction of ρ to $\mathrm{N}^g \cap \mathbf{J}$ is semisimple.

Applying σ and taking contragredients, we see that $\mathrm{Hom}_{\mathrm{N}^{\sigma(g)}\cap\mathbf{J}}(\psi^{\sigma(g)}, \rho)$ is also non-zero; by semisimplicity, the same is true of $\mathrm{Hom}_{\mathrm{N}^{\sigma(g)}\cap\mathbf{J}}(\rho, \psi^{\sigma(g)})$, so, by uniqueness, $\sigma(g)$ lies in $\mathrm{N}g\mathbf{J}$. Since the double coset $\mathrm{N}g\mathbf{J}$ is then σ-stable, Lemma 5.2 implies that it contains a σ-stable coset $h\mathbf{J}$, and that any σ-stable $\mathbf{J}$-coset in $\mathrm{N}g\mathbf{J}$ lies in $\mathrm{N}^{\sigma}h\mathbf{J}$. Then the pair $({}^{h}\mathbf{J}, {}^{h}\rho)$ satisfies the hypotheses of (i), while the uniqueness statements in (ii) and (iii) also follow. □

5.3. Finally in this subsection, we specialize to the case $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$, where $\mathrm{F}/\mathrm{F}_{\mathsf{o}}$ is a quadratic extension and σ the Galois involution as in the rest of the paper. By the σ-self-dual type Theorem 4.1 together with [23, Corollary 1] (or [54, III.5.10] in the modular case), the hypotheses of Proposition 5.3 are satisfied for any irreducible σ-self-dual cuspidal representation π of $\mathrm{GL}_n(\mathrm{F})$.

Remark 5.4. Note that [54, III.5.10] is for cuspidal representations with coefficients in an algebraic closure $\overline{\mathbf{F}}_\ell$ of a finite field of characteristic $\ell \neq p$ only, but one can easily extend it to representations with coefficients in a general R of characteristic ℓ. Indeed, if π is a cuspidal R-representation, then, by twisting it by a character, we may assume that its central character has values in $\overline{\mathbf{F}}_\ell \subseteq \mathrm{R}$. Then by [54, II.4] there is a cuspidal $\overline{\mathbf{F}}_\ell$-representation π_1 such that π is isomorphic to $\pi_1 \otimes_{\overline{\mathbf{F}}_\ell} \mathrm{R}$. It now follows that the hypotheses of Proposition 5.3 are satisfied by π, since they are satisfied by π_1.

Proposition 5.5. *Let π be a σ-self-dual cuspidal representation of $\mathrm{GL}_n(\mathrm{F})$, and let $\mathrm{T}/\mathrm{T}_{\mathsf{o}}$ be the quadratic extension associated with it by Proposition 4.30. Let d be the degree of the endo-class of π, and write $n = md$.*

(i) *Let (N, ψ) be a σ-self-dual Whittaker datum in $\mathrm{GL}_n(\mathrm{F})$. Then the representation π contains a σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$ such that*

$$\mathrm{Hom}_{\mathbf{J}\cap\mathrm{N}}(\boldsymbol{\lambda}, \psi) \neq 0. \tag{5.6}$$

The pair $(\mathbf{J}, \boldsymbol{\lambda})$ is uniquely determined up to conjugacy by N^{σ} and $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{N}}(\boldsymbol{\lambda}, \psi)$ has dimension 1.

(ii) *The set of all σ-self-dual types contained in π and satisfying (5.6) for some σ-self-dual Whittaker datum (N, ψ) is a single $\mathrm{GL}_n(\mathrm{F}_{\mathsf{o}})$-conjugacy class.*

(iii) *If T is unramified over T_{o}, the conjugacy class in (ii) is the unique $\mathrm{GL}_n(\mathrm{F}_{\mathsf{o}})$-conjugacy class of σ-self-dual types in π.*

(iv) *If T is ramified over T_{o}, the conjugacy class in (ii) is the unique $\mathrm{GL}_n(\mathrm{F}_{\mathsf{o}})$-conjugacy class of σ-self-dual types in π of index $\lfloor m/2 \rfloor$ (see Remark 4.28).*

Proof. Assertion (i) follows from Proposition 5.3, and (ii) follows from (i) together with the fact that any two σ-self-dual Whittaker data in $\mathrm{GL}_n(\mathrm{F})$ are $\mathrm{GL}_n(\mathrm{F}_{\mathsf{o}})$-conjugate. Indeed, if (N', ψ') is a σ-self-dual Whittaker datum, it can be written $(\mathrm{N}^{g}, \psi^{g})$ for some $g \in \mathrm{GL}_n(\mathrm{F})$ such that $\sigma(g)g^{-1}$ is in ZN. Writing $\sigma(g)g^{-1} = zu$ with $z \in \mathrm{Z} \simeq \mathrm{F}^{\times}$ and $u \in \mathrm{N}$, we get $z\sigma(z) = u\sigma(u) = 1$. The result now follows from a simple cohomological argument.

Assertion (iii) follows from Proposition 4.31.

We now prove (iv). By Proposition 4.31, there are $\lfloor m/2 \rfloor + 1$ conjugacy class of σ-self-dual types in π and each conjugacy class has an *index* i as in Remark 4.28. If $(\mathbf{J}, \boldsymbol{\lambda})$ is a σ-self-dual type with index i then, identifying J/J^1 with $\mathrm{GL}_m(\boldsymbol{k}_{\mathrm{E}})$, the involution σ acts via conjugation by the diagonal element

$$\delta = \delta_i = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1)$$

with -1 occurring i times.

If $(\mathbf{J}, \boldsymbol{\lambda})$ is as in (ii), then the image $\mathcal{U}$ of $\mathrm{J} \cap \mathrm{N}$ in $\mathrm{GL}_m(\boldsymbol{k}_\mathrm{E})$ is a σ-stable maximal unipotent subgroup on which ψ induces a σ-self-dual character $\overline{\psi}$. By [41, Remark 4.15 and Theorem 3.3], the character $\overline{\psi}$ is non-degenerate.

Now there is a $g \in \mathrm{GL}_m(\boldsymbol{k}_\mathrm{E})$ such that $\mathcal{U}^g$ is equal to $\mathcal{N}$, the standard maximal unipotent subgroup. Since $\mathcal{U}$ and $\mathcal{N}$ are σ-stable, the element $\gamma = \sigma(g)g^{-1}$ normalizes $\mathcal{N}$. It can thus be written $\gamma = n_0 t$ with $n_0 \in \mathcal{N}$ and t diagonal. Since $\gamma^{-1} = \sigma(\gamma) = \delta\gamma\delta^{-1}$, we have $t^{-1} = t$. Write $\delta' = t\delta$ and let σ' be the involution of $\mathrm{GL}_m(\boldsymbol{k}_\mathrm{E})$ given by conjugacy by δ'. Then

$$n_0\sigma'(n_0) = n_0 t\delta n_0 \delta^{-1} t^{-1} = \gamma\delta\gamma\delta^{-1} = 1.$$

Since $\mathcal{N}$ is a p-group with p odd, there is $n_1 \in \mathcal{N}$ such that $n_0 = \delta' n_1 \delta'^{-1} n_1^{-1}$. Write $h = n_1^{-1} g$. Then $\mathcal{U}^h = \mathcal{N}$ and $\sigma(h)h^{-1} = t$. Thus, replacing g by h, we may assume that $n_0 = 1$. Moreover, if we identify $\mathcal{U}$ with $\mathcal{N}$, then σ is replaced by σ', that is, conjugacy by the diagonal matrix δ'.

Now consider the σ'-self-dual non-degenerate character $\psi' = (\overline{\psi})^g$ of $\mathcal{N}$. There are $a_1, \dots, a_{m-1} \in \boldsymbol{k}_\mathrm{E}^\times$ such that

$$\psi'(n) = \varphi(a_1 n_{1,2} + \dots + a_{m-1} n_{m-1,m})$$

for all $n \in \mathcal{N}$, where φ is a fixed non-trivial character of $\boldsymbol{k}_\mathrm{E}$. The fact that ψ' is σ'-self-dual implies that $\delta'_{k+1} = -\delta'_k$ for all $k = 1, \dots, m-1$. Since the number of -1s and the number of 1s differ by at most 1, and since $i \leq \lfloor m/2 \rfloor$ by definition, it follows that $i = \lfloor m/2 \rfloor$. □

Definition 5.7. We call a type in the conjugacy class of Proposition 5.5(ii) a *generic σ-self-dual type* for π.

Proposition 5.5 thus says that when T is unramified over T_o, any σ-self-dual type contained in π is generic, and when T is ramified over T_o, a σ-self-dual type contained in π is generic if and only if its index is $\lfloor m/2 \rfloor$.

5.4. We continue with the notation of §5.3. The main result of this subsection is Lemma 5.10, which will be useful in Sections 6 and 7.

We assume, in this subsection, that π is a σ-self-dual *supercuspidal* representation of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$. Recall that a cuspidal representation of G is supercuspidal if it does not occur as a subquotient of the parabolic induction of an irreducible representation of a proper Levi subgroup of G.

By Proposition 5.5, this representation contains a generic σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$, uniquely determined up to G^σ-conjugacy. Fix a maximal simple stratum $[\mathfrak{a}, \beta]$ such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ with $\mathfrak{a}$ a σ-stable hereditary order and $\sigma(\beta) = -\beta$. Let E denote the F-extension $\mathrm{F}[\beta]$. Let T be the maximal tamely ramified subextension of E over F, and let T_0 denote its σ-fixed points. We also write $m = n/\mathrm{deg}_\mathrm{F}(\beta)$.

Proposition 5.8 ([46, Proposition 8.1]). *Let π be a σ-self-dual supercuspidal representation of the group* $\mathrm{GL}_n(\mathrm{F})$. *If* $\mathrm{T}/\mathrm{T}_\circ$ *is ramified, then either $m = 1$ or m is even.*

Remark 5.9. (i) Note that Proposition 5.8 does not hold if π is only assumed to be σ-self-dual cuspidal: see [46, Remark 7.5].
(ii) In the situation of Proposition 5.8, but with $\mathrm{T}/\mathrm{T}_\circ$ unramified instead of ramified, it is proved in [46, Proposition 8.14] that m is odd, but we will not need this result.

The parahoric subgroup $\mathfrak{a}^\times$ of G is σ-stable; thus $\mathfrak{a}^\times \cap \mathrm{G}^\sigma$ is a parahoric subgroup of G^σ and has the form $\mathfrak{a}_\circ^\times$ for some $\mathcal{O}_\circ$-hereditary order $\mathfrak{a}_\circ$ in $\mathrm{M}_n(\mathrm{F}_\circ)$. Let $e_\circ$ denote the $\mathcal{O}_\circ$-period of $\mathfrak{a}_\circ$. As usual, we also write B for the centralizer of E in $\mathrm{M}_n(\mathrm{F})$. Then $\mathfrak{b} = \mathfrak{a} \cap \mathrm{B}$ is an $\mathcal{O}_\mathrm{E}$-hereditary order in B, and $\mathfrak{b}_\circ = \mathfrak{b} \cap \mathfrak{a}_\circ$ is an $\mathcal{O}_{\mathrm{E}_\circ}$-hereditary order in $\mathrm{B}^\sigma \simeq \mathrm{M}_m(\mathrm{E}_\circ)$.

Lemma 5.10. *Let π be a σ-self-dual supercuspidal representation of* $\mathrm{GL}_n(\mathrm{F})$. *The orders $\mathfrak{a}_\circ$ and $\mathfrak{b}_\circ$ defined as above are principal. Moreover:*

(i) *If* $\mathrm{T}/\mathrm{T}_\circ$ *is ramified and $m \neq 1$, then $e_\circ = 2e(\mathrm{E}_\circ/\mathrm{F}_\circ)$ and $\mathfrak{b}_\circ$ has $\mathcal{O}_{\mathrm{E}_\circ}$-period* 2.
(ii) *Otherwise, $e_\circ = e(\mathrm{E}_\circ/\mathrm{F}_\circ)$ and $\mathfrak{b}_\circ$ is maximal.*

Proof. Note that $\mathfrak{a}_\circ$ is a hereditary order of $\mathrm{M}_n(\mathrm{F}_\circ)$ normalized by $\mathrm{E}_\circ^\times$. Suppose first that $\mathrm{T}/\mathrm{T}_\circ$ is unramified. Then one may assume that $(\mathbf{J}, \boldsymbol{\lambda})$ is attached to a σ-standard stratum (Definition 4.20), thus $\mathfrak{b}$ is the standard maximal order of $\mathrm{M}_m(\mathrm{E})$. It follows that $\mathfrak{b}_\circ$ is the standard maximal order of $\mathrm{M}_m(\mathrm{E}_\circ)$, thus $\mathfrak{a}_\circ$ is the unique hereditary order of $\mathrm{M}_n(\mathrm{F}_\circ)$ normalized by $\mathrm{E}_\circ^\times$. It is thus principal, and its period is equal to $e(\mathrm{E}_\circ/\mathrm{F}_\circ)$.

Suppose now that $\mathrm{T}/\mathrm{T}_\circ$ is ramified. By Proposition 5.8, the integer m is either 1 or even, and Proposition 5.5 says that the index of $(\mathbf{J}, \boldsymbol{\lambda})$ is $\lfloor m/2 \rfloor$. Suppose first that $m = 1$. Then $\mathfrak{b}_\circ$ identifies with $\mathcal{O}_{\mathrm{E}_\circ}$. It is thus maximal and $\mathfrak{a}_\circ$ is principal of period $e(\mathrm{E}_\circ/\mathrm{F}_\circ)$, as in the unramified case.

Suppose now that $m = 2r$ for some $r \geq 1$. Since $(\mathbf{J}, \boldsymbol{\lambda})$ has index r, we may assume

$$\mathfrak{b} = \begin{pmatrix} \mathcal{O}_\mathrm{E} & \cdots & \mathcal{O}_\mathrm{E} & \mathfrak{p}_\mathrm{E}^{-1} & \cdots & \mathfrak{p}_\mathrm{E}^{-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{O}_\mathrm{E} & \cdots & \mathcal{O}_\mathrm{E} & \mathfrak{p}_\mathrm{E}^{-1} & \cdots & \mathfrak{p}_\mathrm{E}^{-1} \\ \mathfrak{p}_\mathrm{E} & \cdots & \mathfrak{p}_\mathrm{E} & \mathcal{O}_\mathrm{E} & \cdots & \mathcal{O}_\mathrm{E} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathfrak{p}_\mathrm{E} & \cdots & \mathfrak{p}_\mathrm{E} & \mathcal{O}_\mathrm{E} & \cdots & \mathcal{O}_\mathrm{E} \end{pmatrix}$$

where each block has size $r \times r$. Since E is ramified over $\mathrm{E}_\circ$, a simple calculation based on the fact that $\mathfrak{p}_\mathrm{E}^{-1} \cap \mathrm{E}_\circ = \mathcal{O}_{\mathrm{E}_\circ}$ shows that $\mathfrak{b}_\circ$ is principal of period 2. We have a similar description of $\mathfrak{a}$: if $\mathfrak{a}_{\mathrm{E}/\mathrm{F}}$ denotes the unique $\mathcal{O}$-order of $\mathrm{End}_\mathrm{F}(\mathrm{E})$ normalized by $\mathrm{E}^\times$, then

$$\mathfrak{a} = \begin{pmatrix} \mathfrak{a}_{\mathrm{E}/\mathrm{F}} & \cdots & \mathfrak{a}_{\mathrm{E}/\mathrm{F}} & \mathfrak{p}_{\mathrm{E}/\mathrm{F}}^{-1} & \cdots & \mathfrak{p}_{\mathrm{E}/\mathrm{F}}^{-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathfrak{a}_{\mathrm{E}/\mathrm{F}} & \cdots & \mathfrak{a}_{\mathrm{E}/\mathrm{F}} & \mathfrak{p}_{\mathrm{E}/\mathrm{F}}^{-1} & \cdots & \mathfrak{p}_{\mathrm{E}/\mathrm{F}}^{-1} \\ \mathfrak{p}_{\mathrm{E}/\mathrm{F}} & \cdots & \mathfrak{p}_{\mathrm{E}/\mathrm{F}} & \mathfrak{a}_{\mathrm{E}/\mathrm{F}} & \cdots & \mathfrak{a}_{\mathrm{E}/\mathrm{F}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathfrak{p}_{\mathrm{E}/\mathrm{F}} & \cdots & \mathfrak{p}_{\mathrm{E}/\mathrm{F}} & \mathfrak{a}_{\mathrm{E}/\mathrm{F}} & \cdots & \mathfrak{a}_{\mathrm{E}/\mathrm{F}} \end{pmatrix}$$

where $\mathfrak{p}_{E/F} = \mathfrak{p}_E \mathfrak{a}_{E/F}$ is the Jacobson radical of $\mathfrak{a}_{E/F}$. As in the $m = 1$ case, $(\mathfrak{a}_{E/F})_\mathrm{o}$ is principal of period $e(E_\mathrm{o}/F_\mathrm{o})$. A simple calculation shows that $\mathfrak{a}_\mathrm{o}$ is principal of period $2e(E_\mathrm{o}/F_\mathrm{o}) = e(E/F_\mathrm{o})$. □

Remark 5.11. Let ϖ_λ be an element of the principal order $\mathfrak{b}_\mathrm{o}$ generating its Jacobson radical. Then $\mathbf{J} \cap G^\sigma$ is generated by ϖ_λ and $J \cap G^\sigma$.

6. Distinction and Whittaker functions

We return to the notation of the rest of the paper, so that F/F_o is a quadratic extension, $G = \mathrm{GL}_n(F)$ for some $n \geq 1$ and σ is the involution on G induced by the Galois involution.

6.1. Distinguished linear forms and Whittaker functions

In this subsection we begin to look at the question of distinction. Recalling that $P = P_n(F)$ denotes the standard mirabolic subgroup of G, we will prove the following analogue of a result of Ok [40].

Proposition 6.1. *Let* $(\mathbf{J}, \boldsymbol{\lambda})$ *be a* σ*-self-dual type such that the compactly induced representation* $\pi = \mathrm{ind}_{\mathbf{J}}^{G} \boldsymbol{\lambda}$ *is distinguished. Then*

$$\mathrm{Hom}_{\mathbf{J}^\sigma \cap P}(\boldsymbol{\lambda}, \mathbf{1}) = \mathrm{Hom}_{\mathbf{J}^\sigma}(\boldsymbol{\lambda}, \mathbf{1}).$$

Recall that saying that $(\mathbf{J}, \boldsymbol{\lambda})$ is distinguished means that the space on the right hand side is non-zero. The condition in the proposition that the σ-self-dual cuspidal representation π is distinguished is *a priori* weaker than this; however, see Remark 6.7.

In order to prove this proposition, we need a small lemma which again applies in a more general setting. Let G be a locally profinite group, let K be an open subgroup of G and let $\mathsf{H}' \subseteq \mathsf{H}$ be closed subgroups of G. Let ρ be a smooth representation of K and let τ be a smooth representation of H. For $g \in \mathsf{G}$, we write $\mathrm{ind}_{\mathsf{K}}^{\mathsf{K}g\mathsf{H}} \rho$ for the subspace of $\mathrm{ind}_{\mathsf{K}}^{\mathsf{G}} \rho$ consisting of functions with support contained in $\mathsf{K}g\mathsf{H}$. Then the Mackey decomposition gives

$$\mathrm{ind}_{\mathsf{K}\cap\mathsf{H}}^{\mathsf{H}} \mathrm{Res}_{\mathsf{K}\cap\mathsf{H}}^{\mathsf{K}} \rho \simeq \mathrm{ind}_{\mathsf{K}}^{\mathsf{KH}} \rho \subseteq \bigoplus_{\mathsf{K}\backslash\mathsf{G}/\mathsf{H}} \mathrm{ind}_{\mathsf{K}}^{\mathsf{K}g\mathsf{H}} \rho = \mathrm{Res}_{\mathsf{H}}^{\mathsf{G}} \mathrm{ind}_{\mathsf{K}}^{\mathsf{G}} \rho$$

and, by Frobenius reciprocity applied to the natural projection in the opposite direction, we get natural maps

$$\mathrm{Hom}_{\mathsf{K}\cap\mathsf{H}}(\rho, \tau) \simeq \mathrm{Hom}_{\mathsf{H}}(\mathrm{ind}_{\mathsf{K}}^{\mathsf{KH}} \rho, \tau) \hookrightarrow \mathrm{Hom}_{\mathsf{H}}(\mathrm{ind}_{\mathsf{K}}^{\mathsf{G}} \rho, \tau).$$

We get similar maps with H replaced by H', and the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathsf{K}\cap\mathsf{H}}(\rho, \tau) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{H}}(\mathrm{ind}_{\mathsf{K}}^{\mathsf{KH}} \rho, \tau) & \longrightarrow & \mathrm{Hom}_{\mathsf{H}}(\mathrm{ind}_{\mathsf{K}}^{\mathsf{G}} \rho, \tau) \\
\downarrow{\scriptstyle \iota_0} & & \downarrow & & \downarrow{\scriptstyle \iota_1} \\
\mathrm{Hom}_{\mathsf{K}\cap\mathsf{H}'}(\rho, \tau) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{H}'}(\mathrm{ind}_{\mathsf{K}}^{\mathsf{KH}'} \rho, \tau) & \longrightarrow & \mathrm{Hom}_{\mathsf{H}'}(\mathrm{ind}_{\mathsf{K}}^{\mathsf{G}} \rho, \tau)
\end{array}$$

where the vertical maps are given by natural inclusion.

Lemma 6.2. *Suppose, in the situation above, that the inclusion* ι_1 *is an equality. Then the inclusion* ι_0 *is also an equality.*

Proof. Certainly the inclusion ι_0 is an injection. Conversely, any $\varphi \in \mathrm{Hom}_{\mathsf{K}\cap\mathsf{H}'}(\rho,\tau)$ corresponds to a map $\Phi \in \mathrm{Hom}_{\mathsf{H}'}(\mathrm{ind}_{\mathsf{K}}^{\mathsf{G}}\rho,\tau)$ which is trivial on all the summands $\mathrm{ind}_{\mathsf{K}}^{\mathsf{K}g\mathsf{H}'}\rho$ with $g \notin \mathsf{KH}'$. Then, since ι_1 is an equality, $\Phi \in \mathrm{Hom}_{\mathsf{H}}(\mathrm{ind}_{\mathsf{K}}^{\mathsf{G}}\rho,\tau)$; moreover, it is trivial on all H-submodules of $\mathrm{ind}_{\mathsf{K}}^{\mathsf{G}}\rho$ which do not contain $\mathrm{ind}_{\mathsf{K}}^{\mathsf{KH}'}\rho$, whence trivial on all summands $\mathrm{ind}_{\mathsf{K}}^{\mathsf{K}g\mathsf{H}}\rho$ with $g \notin \mathsf{KH}$. In particular, we see that $\Phi \in \mathrm{Hom}_{\mathsf{H}}(\mathrm{ind}_{\mathsf{K}}^{\mathsf{KH}}\rho,\tau)$ so that $\phi \in \mathrm{Hom}_{\mathsf{K}\cap\mathsf{H}}(\rho,\tau)$, as required. □

Proof of Proposition 6.1. We apply the lemma to our situation, where we recall that $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$, $\mathrm{P} = \mathrm{P}_n(\mathrm{F})$ is a σ-stable mirabolic subgroup, and $(\mathbf{J},\boldsymbol{\lambda})$ is a σ-self-dual type—by which, we recall, we mean a σ-self-dual extended maximal simple type—with $\pi = \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}\boldsymbol{\lambda}$ an irreducible distinguished σ-self-dual cuspidal representation of G. The result of Ok [40, Theorem 3.1.2] (see also [35, Proposition 2.1]), proved for any *irreducible complex* representation of G and which we generalize to any *cuspidal* representation of G *with coefficients in* R in Appendix B (see Proposition B.23), says that, in this situation, we have an equality

$$\mathrm{Hom}_{\mathrm{P}^\sigma}(\pi,\mathbf{1}) = \mathrm{Hom}_{\mathrm{G}^\sigma}(\pi,\mathbf{1}).$$

We set $\mathsf{G} = \mathrm{G}$, $\mathsf{H} = \mathrm{G}^\sigma$ and $\mathsf{H}' = \mathrm{P}^\sigma$, with $\tau = \mathbf{1}$ the trivial representation of H, and $(\mathsf{K},\rho) = (\mathbf{J},\boldsymbol{\lambda})$. Then the result follows at once from Lemma 6.2. □

We turn now to Whittaker functions. Let $\mathrm{N} = \mathrm{N}_n(\mathrm{F})$ denote the standard maximal unipotent subgroup (consisting of upper triangular unipotent matrices) and let ψ be a σ-self-dual non-degenerate character of N. If π is any generic irreducible representation of G, recall also that its *Whittaker model* (*with respect to* ψ) is the subspace $\mathcal{W}(\pi,\psi)$ of $\mathrm{Ind}_{\mathrm{N}}^{\mathrm{G}}\psi$ which is the image of π under any non-zero map in the one-dimensional space $\mathrm{Hom}_{\mathrm{G}}(\pi,\mathrm{Ind}_{\mathrm{N}}^{\mathrm{G}}\psi)$.

Now let π be an irreducible σ-self-dual cuspidal representation of G. By Theorem 4.1 and Proposition 5.5, it contains a σ-self-dual type $(\mathbf{J},\boldsymbol{\lambda})$ such that $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{N}}(\boldsymbol{\lambda},\psi) \neq 0$. We use the usual notation for data associated to this type; in particular, we have the unique maximal simple character θ contained in $\boldsymbol{\lambda}$ and normalized by $\mathbf{J}$, defined on the normal subgroup H^1 of $\mathbf{J}$, as well as the normal subgroups $\mathrm{J} \supseteq \mathrm{J}^1$ of $\mathbf{J}$.

Let $\mathcal{U} = (\mathrm{N}\cap\mathrm{J})\mathrm{H}^1$ and extend ψ to a character $\psi_{\boldsymbol{\lambda}}$ of $\mathcal{U}$ as in [41, Definition 4.2]:

$$\psi_{\boldsymbol{\lambda}}(uh) = \psi(u)\theta(h) \qquad \text{for } u \in \mathrm{N}\cap\mathrm{J},\ h \in \mathrm{H}^1.$$

We fix a normal compact open subgroup $\mathcal{N}$ of $\mathcal{U}$ contained in $\ker(\psi_{\boldsymbol{\lambda}})$ and define the *Bessel function* $\mathcal{J}_{\boldsymbol{\lambda}} : \mathbf{J} \to \mathrm{R}$ of $\boldsymbol{\lambda}$ by

$$\mathcal{J}_{\boldsymbol{\lambda}}(j) = \frac{1}{(\mathcal{U}:\mathcal{N})}\sum_{u\in\mathcal{U}/\mathcal{N}}\psi_{\boldsymbol{\lambda}}(u)^{-1}\,\mathrm{tr}\,\boldsymbol{\lambda}(ju) \qquad \text{for } j \in \mathbf{J},$$

where $\mathrm{tr}\,\boldsymbol{\lambda}$ is the trace character of $\boldsymbol{\lambda}$. This is independent of the choice of $\mathcal{N}$. Note that this definition makes sense over R, since $\mathcal{U}$ is a pro-p-group.

We then define a function $\mathrm{W}_{\boldsymbol{\lambda}} : \mathrm{G} \to \mathrm{R}$ supported in $\mathrm{N}\mathbf{J}$ by

$$\mathrm{W}_{\boldsymbol{\lambda}}(nj) = \psi(n)\mathcal{J}_{\boldsymbol{\lambda}}(j) \qquad \text{for } n \in \mathrm{N},\ j \in \mathbf{J}. \tag{6.3}$$

One checks that the function $\mathrm{W}_{\boldsymbol{\lambda}}$ is well defined, and that $\mathrm{W}_{\boldsymbol{\lambda}}(ng) = \psi(n)\mathrm{W}_{\boldsymbol{\lambda}}(g)$ for all $n \in \mathrm{N}$ and $g \in \mathrm{G}$.

We set $\mathcal{M} = (\mathrm{P} \cap \mathrm{J})\mathrm{J}^1$ and note that, by [41, Corollary 4.8], the subgroup $\mathrm{P} \cap \mathbf{J} = \mathrm{P} \cap \mathrm{J}$ is contained in $\mathcal{M}$. Let $\mathcal{S}_{\boldsymbol{\lambda}}$ denote the space of functions $f : \mathcal{M} \to \mathrm{R}$ such that $f(um) = \psi_{\boldsymbol{\lambda}}(u)f(m)$ for all $u \in \mathcal{U}$ and $m \in \mathcal{M}$. For each $j \in \mathbf{J}$, we define an operator $\mathrm{L}(j)$ on $\mathcal{S}_{\boldsymbol{\lambda}}$ by

$$\mathrm{L}(j)f : m \mapsto \sum_{x \in \mathcal{M}/\mathcal{U}} \mathcal{J}_{\boldsymbol{\lambda}}(mjx)f(x^{-1}) \tag{6.4}$$

for all $f \in \mathcal{S}_{\boldsymbol{\lambda}}$ and $m \in \mathcal{M}$. This defines a representation L of $\mathbf{J}$ on $\mathcal{S}_{\boldsymbol{\lambda}}$. We claim that this representation is isomorphic to $\boldsymbol{\lambda}$. When R is the field of complex numbers, or more generally when R has characteristic 0, this is [41, Theorem 5.4]. Let us explain briefly how to deduce the modular case from the characteristic 0 case. Assume that R has characteristic $\ell > 0$. First, by the same argument as in Remark 5.4, it is enough to prove the result when R is the field $\overline{\mathbf{F}}_\ell$. Then fix an extended maximal simple type $\widetilde{\boldsymbol{\lambda}}$ with coefficients in $\overline{\mathbf{Q}}_\ell$ whose reduction mod ℓ is isomorphic to $\boldsymbol{\lambda}$ (which is possible by [38, Proposition 2.39]). We thus have an isomorphism between $\widetilde{\boldsymbol{\lambda}}$ and the representation on $\mathcal{S}_{\widetilde{\boldsymbol{\lambda}}}$ defined as in (6.4). Reducing mod ℓ, we get the claimed result. In what follows, we will identify the space of $\boldsymbol{\lambda}$ with $\mathcal{S}_{\boldsymbol{\lambda}}$. It follows as in [41, Section 5.2] that the function $\mathrm{W}_{\boldsymbol{\lambda}}$ defined by (6.3) belongs to the Whittaker model $\mathcal{W}(\pi, \psi)$ of π. Note also (see [41, Proposition 5.3(iii)]) that the restriction of $\mathcal{J}_{\boldsymbol{\lambda}}$ to $\mathcal{M}$ lies in $\mathcal{S}_{\boldsymbol{\lambda}}$.

Proposition 6.5. *Let π be a σ-self-dual cuspidal representation of* G*, and let* $(\mathbf{J}, \boldsymbol{\lambda})$ *be a generic σ-self-dual type contained in π.*

(i) *Let dm be a right invariant measure on* $(\mathrm{J} \cap \mathrm{N}^\sigma)\backslash(\mathrm{J} \cap \mathrm{P}^\sigma)$*. The linear form on $\boldsymbol{\lambda}$ defined by*

$$\mathcal{L}_{\boldsymbol{\lambda}}(f) = \int_{(\mathrm{J}\cap\mathrm{N}^\sigma)\backslash(\mathrm{J}\cap\mathrm{P}^\sigma)} f(m)\,dm$$

for any $f \in \mathcal{S}_{\boldsymbol{\lambda}}$ is $\mathrm{J} \cap \mathrm{P}^\sigma$*-invariant, and $\mathcal{L}_{\boldsymbol{\lambda}}(\mathcal{J}_{\boldsymbol{\lambda}})$ is non-zero.*

(ii) *Moreover, if π is distinguished, then $\mathcal{L}_{\boldsymbol{\lambda}}$ is $\mathbf{J}^\sigma$-invariant.*

Proof. The form $\mathcal{L}_{\boldsymbol{\lambda}}$ is clearly $\mathrm{J} \cap \mathrm{P}^\sigma$-invariant by its definition. By [41, Proposition 5.3(iv)], the proof of which is written for complex representations but still works in the modular case, the function $\mathcal{J}_{\boldsymbol{\lambda}}$ is identically zero on the complement of $\mathcal{U}^\sigma$ in $\mathcal{M}^\sigma$. On the other hand, for $u \in \mathcal{U}^\sigma$, we have $\mathcal{J}_{\boldsymbol{\lambda}}(u) = \psi_{\boldsymbol{\lambda}}(u) = 1$, since $\psi_{\boldsymbol{\lambda}}$ is a σ-self-dual character of a pro-p-group $\mathcal{U}$ and p is odd. Consequently, the value $\mathcal{L}_{\boldsymbol{\lambda}}(\mathcal{J}_{\boldsymbol{\lambda}}) = dm((\mathrm{J} \cap \mathrm{N}^\sigma)\backslash(\mathrm{J} \cap \mathrm{N}^\sigma)(\mathrm{H}^1 \cap \mathrm{P}^\sigma))$ is non-zero, since H^1 is pro-p. The final statement follows immediately from the fact that $\mathbf{J} \cap \mathrm{P} = \mathrm{J} \cap \mathrm{P}$ together with Proposition 6.1. □

We deduce the following corollary from Proposition 6.5.

Corollary 6.6. *Let π be a σ-self-dual cuspidal representation of* G. *Then π is distinguished if and only if any of its generic σ-self-dual types is distinguished.*

Remark 6.7. Putting Corollary 6.6 and Proposition 5.5 together, we obtain a different proof of a result of [46] saying that a σ-self-dual cuspidal representation π of G is distinguished if and only if it contains a distinguished σ-self-dual type, and that, if the quadratic extension $\mathrm{T}/\mathrm{T}_\circ$ associated with π by Proposition 4.30 is ramified, any distinguished σ-self-dual type contained in π has index $\lfloor m/2 \rfloor$, where $n = md$ and d is the degree of the endo-class of π.

6.2. Explicit Whittaker functions and restriction to $\mathrm{GL}_n(\mathrm{F}_\circ)$

We continue with the same notation, and write $\mathrm{K} = \mathrm{GL}_n(\mathcal{O})$ and $\mathrm{K}^\sigma = \mathrm{GL}_n(\mathcal{O}_\circ)$. In order to make computations, we need to be somewhat more careful with our choice of non-degenerate character ψ to ensure that the corresponding generic σ-self-dual type is well-positioned with respect to the standard maximal compact subgroup K^σ of G^σ.

Let $\mathfrak{S}_n$ denote the group of permutation matrices in G^σ. The Bruhat decomposition in the finite quotient of K^σ by its pro-p unipotent radical, together with the Iwasawa decomposition of G^σ, yields the *Bruwasawa decomposition* $\mathrm{G}^\sigma = \mathrm{B}^\sigma \mathfrak{S}_n \mathrm{I}_\circ$, where B is the standard Borel subgroup of G, and $\mathrm{I}_\circ$ is the standard Iwahori subgroup of G^σ. In particular, this decomposition implies that any parahoric subgroup of G^σ is conjugate by N^σ to a parahoric subgroup in the standard apartment, where N is the unipotent radical of B.

If $(\mathbf{J}, \boldsymbol{\lambda})$ is a σ-self-dual type in G then we can write $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$, with $\mathfrak{a}$ a σ-stable hereditary order and $\sigma(\beta) = -\beta$. As in §5.4, we have $\mathfrak{a}^\times \cap \mathrm{G}^\sigma = \mathfrak{a}_\circ^\times$ for some $\mathcal{O}_\circ$-hereditary order $\mathfrak{a}_\circ$ in $\mathrm{M}_n(\mathrm{F}_\circ)$. Write $e_\circ$ for the $\mathcal{O}_\circ$-period of $\mathfrak{a}_\circ$, and $\Lambda_\circ$ for the $\mathcal{O}_\circ$-lattice chain in the vector space $\mathrm{F}_\circ^n$ consisting of $\mathfrak{a}_\circ$-lattices. These depend only on the pair $(\mathbf{J}, \boldsymbol{\lambda})$.

Writing $\mathbf{e}_1, \dots, \mathbf{e}_n$ for the standard basis of F^n, and using the notation above, we get the following.

Lemma 6.8. *Let π be a σ-self-dual cuspidal representation of* G. *There are a σ-self-dual Whittaker datum* (N, ψ) *and a generic σ-self-dual type* $(\mathbf{J}, \boldsymbol{\lambda})$ *in π such that:*

(i) *the space* $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{N}}(\boldsymbol{\lambda}, \psi)$ *is non-zero;*
(ii) *there is a numbering on the $\mathcal{O}_\circ$-lattice chain $\Lambda_\circ$ associated to* $(\mathbf{J}, \boldsymbol{\lambda})$ *such that*

$$\Lambda_\circ(k) = \bigoplus_{i=1}^{n} \mathfrak{p}_\circ^{a_i(k)} \mathbf{e}_i \quad \textit{for } k \in \mathbf{Z},$$

where the $a_i : \mathbf{Z} \to \mathbf{Z}$ are non-decreasing functions satisfying:

(a) $a_i(k + e_\circ) = a_i(k) + 1$ *for all* $k \in \mathbf{Z}$ *and* $a_i(0) = 0$ *for* $i = 1, \dots, n$,
(b) $a_n(0) = \cdots = a_n(e_\circ - 1) = 0$.

Note that condition (ii) implies in particular that $J^\sigma \subseteq K^\sigma$ (though it is not equivalent to this). It is also worth noting that it is *not* in general possible to find $(\mathbf{J}, \boldsymbol{\lambda})$ satisfying condition (i) and the stronger condition $J \subseteq K$ (see Remark 6.9).

Proof of Lemma 6.8. We pick a σ-self-dual Whittaker datum (N, ψ) where $N = N_n(F)$ is the standard maximal unipotent subgroup. By Proposition 5.5, we have a σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$ satisfying (i). Fix a maximal simple stratum $[\mathfrak{a}, \beta]$ as above, denote by $\mathfrak{a}_\circ$ the $\mathcal{O}_\circ$-hereditary order associated to it and by $e_\circ$ its period. There is an element $u \in N^\sigma$ which sends $\mathfrak{a}_\circ$ to a point in the standard apartment. Conjugating by u, we assume $\mathfrak{a}_\circ$ is itself in the standard apartment.

Writing $\Lambda_\circ$ for the $\mathcal{O}_\circ$-lattice chain in $F_\circ^n$ consisting of $\mathfrak{a}_\circ$-lattices, we can number the lattices so that

$$\Lambda_\circ(0) \cap F_\circ \mathbf{e}_n = \mathcal{O}_\circ \mathbf{e}_n, \qquad \Lambda_\circ(-1) \cap F_\circ \mathbf{e}_n = \mathfrak{p}_\circ^{-1} \mathbf{e}_n.$$

Since $\mathfrak{a}_\circ$ lies in the standard apartment, we can find $t_1, \dots, t_{n-1} \in F_\circ^\times$ such that

$$\Lambda_\circ(0) = \mathcal{O}_\circ t_1 \mathbf{e}_1 \oplus \cdots \oplus \mathcal{O}_\circ t_{n-1} \mathbf{e}_{n-1} \oplus \mathcal{O}_\circ \mathbf{e}_n.$$

Conjugating both $(\mathbf{J}, \boldsymbol{\lambda})$ and the Whittaker datum (N, ψ) by $t = \mathrm{diag}(t_1, \dots, t_{n-1}, 1)$ (which is in the diagonal torus of G^σ), we obtain the result. □

Remark 6.9. Suppose that $F/F_\circ$ is ramified, $n = 2$ and π is a σ-self-dual depth zero cuspidal representation of $GL_2(F)$. Then any generic σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$ in π has index 1 so $\mathbf{J}$ is $GL_2(F_\circ)$-conjugate to $t_1 K t_1^{-1}$ where $t_1 = \mathrm{diag}(\varpi, 1)$ and ϖ is a uniformizer of F. In particular, the group $\mathbf{J}$ is not $GL_2(F_\circ)$-conjugate to (any subgroup of) K.

6.3. Suppose now that π is a σ-self-dual supercuspidal representation (see §5.4) and choose our non-degenerate character ψ and generic σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$ as in Lemma 6.8. We have an order $\mathfrak{a}_\circ$ as above. By Lemma 5.10, it is a principal order. We choose $\varpi_{\boldsymbol{\lambda}} \in \mathbf{J}^\sigma$ as in Remark 5.11, so that $\mathbf{J}^\sigma$ is generated by $\varpi_{\boldsymbol{\lambda}}$ and J^σ.

The following lemma shows a useful property of the Iwasawa decomposition of $\varpi_{\boldsymbol{\lambda}}$ in G^σ, which will be key to our computation to come.

Lemma 6.10. *Let $i \in \mathbf{Z}$. We have $\varpi_{\boldsymbol{\lambda}}^i \in P^\sigma K^\sigma$ if and only if $i \in \{0, \dots, e_\circ - 1\}$. In that case, if we choose $p_i \in P^\sigma$ and $k_i \in K^\sigma$ such that $\varpi_{\boldsymbol{\lambda}}^i = p_i k_i$, then $|\det(p_i)|_\circ = |\det(\varpi_{\boldsymbol{\lambda}}^i)|_\circ = q_\circ^{-in/e_\circ}$.*

Proof. Note first that $P^\sigma K^\sigma$ consists precisely of those matrices whose last row lies in $(\mathcal{O}_\circ, \dots, \mathcal{O}_\circ)$ but not in $(\mathfrak{p}_\circ, \dots, \mathfrak{p}_\circ)$. Considering the action of $\varpi_{\boldsymbol{\lambda}}$ on the lattice chain $\Lambda_\circ$, it follows at once from the previous lemma that the last row of $\varpi_{\boldsymbol{\lambda}}^i$ belongs to $(\mathfrak{p}_\circ^{\lfloor i/e_\circ \rfloor}, \dots, \mathfrak{p}_\circ^{\lfloor i/e_\circ \rfloor})$—that is, the entries of this row all have valuation $\geq \lfloor i/e_\circ \rfloor$—but does not belong to $(\mathfrak{p}_\circ^{\lceil i/e_\circ \rceil}, \dots, \mathfrak{p}_\circ^{\lceil i/e_\circ \rceil})$, which implies the first statement. The second is immediate, since $|\det(k)|_\circ = 1$ for all $k \in K^\sigma$. □

For $i \in \{0, \dots, e_\circ - 1\}$, we fix from now $p_i \in P^\sigma$ and $k_i \in K^\sigma$ such that $\varpi_{\boldsymbol{\lambda}}^i = p_i k_i$, as in Lemma 6.10.

We recall from the previous section that we have an explicit Whittaker function $\mathrm{W}_\lambda \in \mathcal{W}(\pi, \psi)$ with support $\mathrm{N}\mathbf{J}$.

Proposition 6.11. *For each $l \in \mathbf{Z}$, let W_λ^l denote the function from G^σ to R supported on the subset $\{g \in \mathrm{G}^\sigma \cap \mathrm{N}\mathbf{J} \mid |\det(g)|_\circ = q_\circ^{-l}\}$ and coinciding with W_λ on it.*

(i) *The function $\mathrm{W}_\lambda^l|_{\mathrm{P}^\sigma\mathrm{K}^\sigma}$ is zero unless $l = in/e_\circ$ with $i \in \{0, \dots, e_\circ - 1\}$, in which case*

$$\mathrm{supp}(\mathrm{W}_\lambda^l|_{\mathrm{P}^\sigma\mathrm{K}^\sigma}) \subseteq \mathrm{N}^\sigma \varpi_\lambda^i \mathrm{J}^\sigma.$$

(ii) *If $\mathrm{W}_\lambda^{in/e_\circ}(pk) \neq 0$ with $p \in \mathrm{P}^\sigma$, $k \in \mathrm{K}^\sigma$ and $i \in \{0, \dots, e_\circ - 1\}$, then k is in $(\mathrm{P}^\sigma \cap \mathrm{K}^\sigma)k_i\mathrm{J}^\sigma$.*

(iii) *If $\mathrm{W}_\lambda^{in/e_\circ}(p\varpi_\lambda^i j) \neq 0$ with $p \in \mathrm{P}^\sigma$, $j \in \mathrm{J}^\sigma$ and $i \in \{0, \dots, e_\circ - 1\}$, then p is in $\mathrm{N}^\sigma(\mathrm{P}^\sigma \cap \mathrm{J}^\sigma)$.*

Proof. Note that $\mathrm{W}_\lambda|_{\mathrm{G}^\sigma}$ is supported in $\mathrm{G}^\sigma \cap \mathrm{N}\mathbf{J}$, equal to $\mathrm{N}^\sigma\mathbf{J}^\sigma$ by Lemma 5.2(iii). By definition of ϖ_λ, the set $\mathrm{N}^\sigma\mathbf{J}^\sigma$ is the disjoint union of the $\mathrm{N}^\sigma\varpi_\lambda^i\mathrm{J}^\sigma$ for $i \in \mathbf{Z}$, and thus (i) follows from Lemma 6.10. The remaining parts follow exactly as in the proof of [30, Proposition 8.4]. □

Finally, using that $\mathrm{J}^\sigma \subseteq \mathrm{K}^\sigma$ thanks to our choice of basis, as in [30, Lemma 7.2], we have the following lemma, which we will use in Section 7.

Lemma 6.12. *There is a unique right invariant complex valued measure dk on $(\mathrm{P}^\sigma \cap \mathrm{K}^\sigma)\backslash\mathrm{K}^\sigma$ such that*

$$dk((\mathrm{P}^\sigma \cap \mathrm{K}^\sigma)\backslash(\mathrm{P}^\sigma \cap \mathrm{K}^\sigma)k_i\mathrm{J}^\sigma) = q_\circ^{-in/e_\circ}$$

for all $i \in \{0, \dots, e_\circ - 1\}$.

Proof. Let dk be any right invariant measure on $(\mathrm{P}^\sigma \cap \mathrm{K}^\sigma)\backslash\mathrm{K}^\sigma$. Following the first part of the proof of [30, Lemma 7.2], and thanks to Lemma 6.10, we have

$$dk((\mathrm{P}^\sigma \cap \mathrm{K}^\sigma)\backslash(\mathrm{P}^\sigma \cap \mathrm{K}^\sigma)k_i\mathrm{J}^\sigma) = q_\circ^{-in/e_\circ} \cdot dk((\mathrm{P}^\sigma \cap \mathrm{K}^\sigma)\backslash(\mathrm{P}^\sigma \cap \mathrm{K}^\sigma)\mathrm{J}^\sigma)$$

for all $i \in \{0, \dots, e_\circ - 1\}$. Thus the required measure is that for which $\mathrm{K}^\sigma \cap \mathrm{P}^\sigma)\backslash(\mathrm{K}^\sigma \cap \mathrm{P}^\sigma)\mathrm{J}^\sigma$ has volume 1. □

7. Asai L-functions and test vectors

From now on, until the end of the paper, all representations are complex, that is, R is now the field $\mathbf{C}$ of complex numbers.

7.1. Distinction and dichotomy

We will need two further key results on distinction of σ-self-dual cuspidal complex representations, which we recall from [46]. Recall that $\omega_{\mathrm{F}/\mathrm{F}_\circ}$ denotes the non-trivial character of $\mathrm{F}_\circ^\times$ which is trivial on $\mathrm{N}_{\mathrm{F}/\mathrm{F}_\circ}(\mathrm{F}^\times)$. The first result is *dichotomy*. It is proved for discrete series representations when F has characteristic 0 by Flicker [19], Kable [29] and Anandavardhanan, Kable and Tandon [2], and we prove in Appendix A (Theorem A.2) that the global arguments of [29] and [2] remain valid when F has characteristic p. This result is also proved by Sécherre [46] for cuspidal representations, in a purely local way, with no assumption on the characteristic of F (see also Remark 7.3).

Theorem 7.1 ([46, Theorem 10.8]). *Let π be a cuspidal (complex) representation of* $\mathrm{GL}_n(\mathrm{F})$, $n \geq 1$.

(i) *π is σ-self-dual if and only if it is either distinguished or $\omega_{\mathrm{F}/\mathrm{F}_\mathrm{o}}$-distinguished.*
(ii) *π cannot be both distinguished and $\omega_{\mathrm{F}/\mathrm{F}_\mathrm{o}}$-distinguished.*

Given a σ-self-dual cuspidal representation π of $\mathrm{GL}_n(\mathrm{F})$, we denote by $\mathrm{T}/\mathrm{T}_\mathrm{o}$ the quadratic extension associated to π by Proposition 4.30. Let d denote the degree of the endo-class of π. It is a divisor of n, and we write $n = md$.

Proposition 7.2 ([46, Proposition 10.12]). *Let π be a distinguished cuspidal (complex) representation of* $\mathrm{GL}_n(\mathrm{F})$. *Then π has an $\omega_{\mathrm{F}/\mathrm{F}_\mathrm{o}}$-distinguished unramified twist if and only if either* $\mathrm{T}/\mathrm{T}_\mathrm{o}$ *is unramified or* $m > 1$.

Remark 7.3. These two results are proved in [46] in a more general setting: π is a *supercuspidal* representation of $\mathrm{GL}_n(\mathrm{F})$ with coefficients in R, where R has characteristic different from p. Note that when R has characteristic 0, any cuspidal representation is supercuspidal.

7.2. Definition of the integrals

As before, we suppose that ψ is a σ-self-dual non-degenerate character of N. Let π be a generic irreducible representation of G. For W a function in the Whittaker model $\mathcal{W}(\pi, \psi)$ of π and Φ in the space $\mathcal{C}_c^\infty(\mathrm{F}_\mathrm{o}^n)$ of locally constant functions on F_o^n with compact support, define the local Asai integral

$$\mathrm{I}_{\mathrm{As}}(s, \Phi, \mathrm{W}) = \int_{\mathrm{N}^\sigma \backslash \mathrm{G}^\sigma} \mathrm{W}(g)\Phi(\tau g)|\det(g)|_\mathrm{o}^s \, dg, \tag{7.4}$$

where τ is the row vector $\begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}$ and dg is a right invariant measure on $\mathrm{N}^\sigma \backslash \mathrm{G}^\sigma$ which will be fixed later in §7.4. It turns out (see [29, Theorem 2]) that, for $s \in \mathbf{C}$ with sufficiently large real part, the integral (7.4) is a rational function in q_o^{-s}; moreover, as W varies in $\mathcal{W}(\pi, \psi)$ and Φ varies in $\mathcal{C}_c^\infty(\mathrm{F}_\mathrm{o}^n)$, these functions generate a fractional ideal of $\mathbf{C}[q_\mathrm{o}^s, q_\mathrm{o}^{-s}]$ which has a unique generator $\mathrm{L}_{\mathrm{As}}(s, \pi)$ which is an Euler factor (i.e., of the form $1/\mathrm{P}(q_\mathrm{o}^{-s})$ where P is a polynomial with constant term 1).

Now let π be a cuspidal representation of G and let $\mathrm{X}(\pi)$ denote the set of unramified characters χ of G^σ such that π is χ-distinguished. We recall the following description of the Asai L-function of a cuspidal representation, the proof of which is valid (as the rest of [35]) when F has positive characteristic as well:

Proposition 7.5 ([35, Proposition 3.6]). *Let π be a cuspidal representation of* G. *Then*

$$\mathrm{L}_{\mathrm{As}}(s,\pi)=\prod_{\chi\in\mathrm{X}(\pi)}(1-\chi(\varpi_{\mathrm{o}})q_{\mathrm{o}}^{-s})^{-1}$$

where ϖ_{o} is a fixed uniformizer of F_{o}.

Let $t(\pi)$ denote the *torsion number* of π, that is, the number of unramified characters of $\mathrm{F}^{\times}$ such that $\pi(\chi\circ\det)$ is isomorphic to π. Thanks to Theorem 7.1, we deduce the following formula.

Corollary 7.6. *Let π be a distinguished cuspidal representation of* G. *Then*

$$\mathrm{L}_{\mathrm{As}}(s,\pi)=\begin{cases}\frac{1}{1-q_{\mathrm{o}}^{-st(\pi)}} & \text{if } \mathrm{F}/\mathrm{F}_{\mathrm{o}} \text{ is unramified,}\\ \frac{1}{1-q_{\mathrm{o}}^{-st(\pi)}} & \text{if } \mathrm{F}/\mathrm{F}_{\mathrm{o}} \text{ is ramified and no unramified twist of } \pi \text{ is } \omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}\text{-distinguished,}\\ \frac{1}{1-q_{\mathrm{o}}^{-st(\pi)/2}} & \text{if } \mathrm{F}/\mathrm{F}_{\mathrm{o}} \text{ is ramified and an unramified twist of } \pi \text{ is } \omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}\text{-distinguished.}\end{cases}$$

As the Rankin–Selberg and Langlands–Shahidi Asai local L-functions agree (see Theorem A.1), one can deduce Corollary 7.6 from [4, Theorem 1.1]. We give another proof, based on Proposition 7.5 and Theorem 7.1.

Proof of Corollary 7.6. Let $\mathrm{R}(\pi)$ denote the group of unramified characters of $\mathrm{F}^{\times}$ such that $\pi(\chi\circ\det)$ is isomorphic to π. It is cyclic and has order $t(\pi)$. Let us fix uniformizers ϖ and ϖ_{o} of F and F_{o}, respectively. Since π is distinguished, it is σ-self-dual. Let $\mathrm{U}(\pi)$ denote the subgroup of unramified characters χ of $\mathrm{F}_{\mathrm{o}}^{\times}$ such that $\pi(\widetilde{\chi}\circ\det)$ is σ-self-dual for any unramified character $\widetilde{\chi}$ of $\mathrm{F}^{\times}$ extending χ. An unramified character χ belongs to $\mathrm{U}(\pi)$ if and only if

$$\chi(\mathrm{N}_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(\varpi))^{t(\pi)}=1.$$

Note that $\omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}\in\mathrm{U}(\pi)$.

Let $\mathrm{Y}(\pi)$ denote the set of unramified characters χ of $\mathrm{F}_{\mathrm{o}}^{\times}$ such that π is $\omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}\chi$-distinguished. Then Theorem 7.1 says that $\mathrm{U}(\pi)$ decomposes as the disjoint union of $\mathrm{X}(\pi)$ and $\mathrm{Y}(\pi)$.

We first treat the case where $\mathrm{F}/\mathrm{F}_{\mathrm{o}}$ unramified. If χ is an unramified character of $\mathrm{F}_{\mathrm{o}}^{\times}$, then $\chi\in\mathrm{U}(\pi)$ if and only if $\chi(\varpi_{\mathrm{o}})^{2t(\pi)}=1$, hence $\mathrm{U}(\pi)$ is cyclic of order $2t(\pi)$. But we have $\omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}\in\mathrm{U}(\pi)$, hence $\mathrm{Y}(\pi)=\omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}\mathrm{X}(\pi)$, and $\mathrm{X}(\pi)$ is of order $t(\pi)$; this proves the expected equality in the first case.

We now suppose that $\mathrm{F}/\mathrm{F}_{\mathrm{o}}$ is ramified, hence for an unramified character χ of $\mathrm{F}_{\mathrm{o}}^{\times}$, one has $\chi\in\mathrm{U}(\pi)$ if and only if $\chi(\varpi_{\mathrm{o}})^{t(\pi)}=1$ so $\mathrm{U}(\pi)$ is cyclic of order $t(\pi)$. If no unramified twist of π is $\omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}$-distinguished, then $\mathrm{Y}(\pi)$ is empty, hence $\mathrm{X}(\pi)=\mathrm{U}(\pi)$ and $\mathrm{X}(\pi)$ is of order $t(\pi)$, whereas if an unramified twist $\pi\mu$ of π is $\omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}$-distinguished, then setting $\chi=\mu|_{\mathrm{F}_{\mathrm{o}}^{\times}}$, one has $\mathrm{Y}(\pi)=\chi\mathrm{X}(\pi)$, thus $\mathrm{X}(\pi)$ is of order $t(\pi)/2$. The last two equalities follow immediately. □

Remark 7.7. By [14, 6.2.5], the torsion number $t(\pi)$ is equal to n/e, where e is a divisor of n equal to the ramification index of the endo-class of π (see §3.2), which is $e(\mathrm{E}/\mathrm{F})$ with the notation of §5.4. Using the invariant e_o introduced in §5.4 and computed in Lemma 5.10, together with Proposition 7.2, we deduce that Corollary 7.6 is equivalent to the equality

$$\mathrm{L}_{\mathrm{As}}(s,\pi) = \frac{1}{1-q_\mathrm{o}^{-sn/e_\mathrm{o}}}. \tag{7.8}$$

7.3. A decomposition of the integral

We continue with π a cuspidal (complex) representation of G. For computational convenience, we introduce a second integral: for W in the Whittaker model $\mathcal{W}(\pi,\psi)$ of π, we put

$$\mathrm{I}^{(0)}_{\mathrm{As}}(s,\mathrm{W}) = \int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}(p)|\det(p)|_\mathrm{o}^{s-1}\,dp \tag{7.9}$$

where dp is a right invariant measure on $\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma$ which will be fixed later in Proposition 7.13. Again, if $s\in\mathbf{C}$ has sufficiently large real part, $\mathrm{I}^{(0)}_{\mathrm{As}}(s,\mathrm{W})$ is a rational function in q_o^{-s}.

Now let dk be the measure on $(\mathrm{P}^\sigma\cap\mathrm{K}^\sigma)\backslash\mathrm{K}^\sigma$ given by Lemma 6.12 and $d^\times a$ be the Haar measure on $\mathrm{F}_\mathrm{o}^\times$ giving measure 1 to $\mathcal{O}_\mathrm{o}^\times$. Then, as noticed in [19, Section 4], if s has a sufficiently large real part and if the function $\Phi\in\mathcal{C}_c^\infty(\mathrm{F}_\mathrm{o}^n)$ is chosen to be K^σ-invariant, there is a unique right invariant measure dg on $\mathrm{N}^\sigma\backslash\mathrm{G}^\sigma$, depending only on the choice of dp, such that

$$\mathrm{I}_{\mathrm{As}}(s,\Phi,\mathrm{W}) = \int_{\mathrm{F}_\mathrm{o}^\times}\Phi(\tau a)\omega_\pi(a)|a|_\mathrm{o}^{ns}\,d^\times a\int_{(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)\backslash\mathrm{K}^\sigma}\mathrm{I}^{(0)}_{\mathrm{As}}(s,k\cdot\mathrm{W})\,dk \tag{7.10}$$

where ω_π denotes the central character of π and $g\cdot\mathrm{W}$ denotes the action of $g\in\mathrm{G}$ on $\mathcal{W}(\pi,\psi)$, that is, $(g\cdot\mathrm{W})(x)=\mathrm{W}(xg)$ for $x\in\mathrm{G}$. From now on, we will assume that dg is chosen with respect to dp so that (7.10) holds.

Suppose that ω_π is trivial when restricted to $\mathrm{F}_\mathrm{o}^\times$, which is the case when π is distinguished. If Φ is the characteristic function $\mathbf{1}_{\mathcal{O}_\mathrm{o}^n}$ of $\mathcal{O}_\mathrm{o}^n$, then

$$\int_{\mathrm{F}_\mathrm{o}^\times}\mathbf{1}_{\mathcal{O}_\mathrm{o}^n}(\tau a)\omega_\pi(a)|a|_\mathrm{o}^{ns}\,d^\times a = \int_{\mathcal{O}_\mathrm{o}\backslash\{0\}}|a|_\mathrm{o}^{ns}\,d^\times a = \frac{1}{1-q_\mathrm{o}^{-ns}},$$

by Tate's thesis [17]. Therefore, we have the following decomposition which we record as a lemma:

Lemma 7.11. *Let π be a distinguished cuspidal complex representation of* G. *Then, for all functions* $\mathrm{W}\in\mathcal{W}(\pi,\psi)$*, we have*

$$\mathrm{I}_{\mathrm{As}}(s,\mathbf{1}_{\mathcal{O}_\mathrm{o}^n},\mathrm{W}) = \frac{1}{1-q_\mathrm{o}^{-ns}}\int_{(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)\backslash\mathrm{K}^\sigma}\mathrm{I}^{(0)}_{\mathrm{As}}(s,k\cdot\mathrm{W})\,dk.$$

For $\mathrm{W}\in\mathcal{W}(\pi,\psi)$ and $l\in\mathbf{Z}$, we write W_o^l for the function from G^σ to $\mathbf{C}$ supported on the subset $\{g\in\mathrm{G}^\sigma \mid |\det(g)|_\mathrm{o}=q_\mathrm{o}^{-l}\}$ and coinciding with W on it. Finally we decompose the integral given in Lemma 7.11 by the absolute value:

$$\int_{(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)\backslash\mathrm{K}^\sigma} \mathrm{I}^{(0)}_{\mathrm{As}}(s,k\cdot\mathrm{W})\,dk = \sum_{l\in\mathbf{Z}}\int_{(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)\backslash\mathrm{K}^\sigma}\int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}^l_{\mathsf{o}}(pk)|\det(pk)|_{\mathsf{o}}^{s-1}\,dp\,dk$$
$$= \sum_{l\in\mathbf{Z}} q_{\mathsf{o}}^{-l(s-1)}\int_{(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)\backslash\mathrm{K}^\sigma}\int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}^l_{\mathsf{o}}(pk)\,dp\,dk.$$

Since π is cuspidal, the right hand sum is finite [6]. We call

$$\mathsf{c}_l(\mathrm{W}) = \int_{(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)\backslash\mathrm{K}^\sigma}\int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}^l_{\mathsf{o}}(pk)\,dp\,dk$$

the l^{th} *coefficient* of the integral, and we record:

Lemma 7.12. *Let π be a distinguished cuspidal complex representation of* G*. Then, for all functions* $\mathrm{W}\in\mathcal{W}(\pi,\psi)$*, we have*

$$\mathrm{I}_{\mathrm{As}}(s,\mathbf{1}_{\mathcal{O}^n_{\mathsf{o}}},\mathrm{W}) = \frac{1}{1-q_{\mathsf{o}}^{-ns}}\Big(\sum_{l\in\mathbf{Z}}\mathsf{c}_l(\mathrm{W})q_{\mathsf{o}}^{-l(s-1)}\Big).$$

7.4. Test vectors

Until the end of this section, π is a distinguished cuspidal representation of G and $(\mathbf{J},\boldsymbol{\lambda})$ is a generic σ-self-dual type as in Lemma 6.8. Now we compute the Asai integral of the explicit Whittaker vector $\mathrm{W}_{\boldsymbol{\lambda}}$ showing it is a test vector, making use of the decomposition of Lemma 7.12.

Proposition 7.13. *Let $l\in\mathbf{Z}$.*

(i) *The l^{th} coefficient $\mathsf{c}_l(\mathrm{W}_{\boldsymbol{\lambda}})$ is zero unless $l=in/e_{\mathsf{o}}$ for some $i\in\{0,\dots,e_{\mathsf{o}}-1\}$.*
(ii) *There is a unique right invariant measure dp on $\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma$ such that $\mathsf{c}_{in/e_{\mathsf{o}}}(\mathrm{W}_{\boldsymbol{\lambda}}) = q_{\mathsf{o}}^{-in/e_{\mathsf{o}}}$ for $i\in\{0,\dots,e_{\mathsf{o}}-1\}$.*

Proof. By definition, we have

$$\mathsf{c}_l(\mathrm{W}_{\boldsymbol{\lambda}}) = \int_{(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)\backslash\mathrm{K}^\sigma}\int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}^l_{\boldsymbol{\lambda}}(pk)\,dp\,dk$$

where dk is the measure given by Lemma 6.12 and dp is a right invariant measure on $\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma$.

Part (i) follows from Proposition 6.11(i).

Now assume that $l=in/e_{\mathsf{o}}$ for some $i\in\{0,\dots,e_{\mathsf{o}}-1\}$. We recall that we have our fixed decompositions $\varpi^i_{\boldsymbol{\lambda}} = p_ik_i$ with $p_i\in\mathrm{P}^\sigma$, $k_i\in\mathrm{K}^\sigma$. Then it follows from Proposition 6.11(ii) that

$$\mathsf{c}_{in/e_{\mathsf{o}}}(\mathrm{W}_{\boldsymbol{\lambda}}) = \int_{(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)\backslash(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)k_i\mathrm{J}^\sigma}\int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}^{in/e_{\mathsf{o}}}_{\boldsymbol{\lambda}}(pk)\,dp\,dk$$
$$= \int_{(\mathrm{J}^\sigma\cap(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)^{k_i})\backslash\mathrm{J}^\sigma}\int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}^{in/e_{\mathsf{o}}}_{\boldsymbol{\lambda}}(pk_ij)\,dp\,dj$$

where dj is the right invariant measure on $(\mathrm{J}^\sigma\cap(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)^{k_i})\backslash\mathrm{J}^\sigma$ corresponding to dk.

Let us compute the inner integral. By applying the change of variable $p \mapsto pp_i^{-1}$ and then Proposition 6.11(iii), we get

$$\begin{aligned}\int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}_{\boldsymbol{\lambda}}^{in/e_\circ}(pk_i j)\,dp &= \int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}_{\boldsymbol{\lambda}}^{in/e_\circ}(p\varpi_{\boldsymbol{\lambda}}^i j)\,dp \\ &= \int_{\mathrm{N}^\sigma\backslash\mathrm{N}^\sigma(\mathrm{P}^\sigma\cap\mathrm{J}^\sigma)} \mathrm{W}_{\boldsymbol{\lambda}}^{in/e_\circ}(p\varpi_{\boldsymbol{\lambda}}^i j)\,dp \\ &= \int_{(\mathrm{N}^\sigma\cap\mathrm{J}^\sigma)\backslash(\mathrm{P}^\sigma\cap\mathrm{J}^\sigma)} \mathrm{W}_{\boldsymbol{\lambda}}^{in/e_\circ}(m\varpi_{\boldsymbol{\lambda}}^i j)\,dm\end{aligned}$$

where dm is the right invariant measure on $(\mathrm{N}^\sigma\cap\mathrm{J}^\sigma)\backslash(\mathrm{P}^\sigma\cap\mathrm{J}^\sigma)$ corresponding to dp. Since $\varpi_{\boldsymbol{\lambda}}^i j \in \mathbf{J}^\sigma$ by Lemma 6.10, and thanks to Proposition 6.5(ii), this is equal to $\mathscr{L}_{\boldsymbol{\lambda}}(\mathcal{J}_{\boldsymbol{\lambda}})$.

Now let us fix dm so that $\mathscr{L}_{\boldsymbol{\lambda}}(\mathcal{J}_{\boldsymbol{\lambda}}) = 1$, which is possible thanks to Proposition 6.5(i). This defines dp uniquely. Then our choice of dk gives us

$$\mathsf{c}_{in/e_\circ}(\mathrm{W}_{\boldsymbol{\lambda}}) = dk((\mathrm{P}^\sigma\cap\mathrm{K}^\sigma)\backslash(\mathrm{P}^\sigma\cap\mathrm{K}^\sigma)k_i\mathrm{J}^\sigma) = q_\circ^{-in/e_\circ}$$

as expected. □

We now prove our main result on test vectors for Asai L-functions.

Theorem 7.14. *Suppose π is a distinguished cuspidal representation of* G. *Then*

$$\mathrm{I}_{\mathrm{As}}(s, \mathbf{1}_{\mathcal{O}_\circ^n}, \mathrm{W}_{\boldsymbol{\lambda}}) = \frac{1}{1-q_\circ^{-sn/e_\circ}} = \mathrm{L}_{\mathrm{As}}(s,\pi)$$

where the right invariant measure dg defining the left hand side is chosen so that (7.10) *holds and the measure dp defining* (7.9) *is the one given by Proposition* 7.13.

Proof. By Lemma 7.11, we have

$$\mathrm{I}_{\mathrm{As}}(s, \mathbf{1}_{\mathcal{O}_\circ^n}, \mathrm{W}_{\boldsymbol{\lambda}}) = \frac{1}{1-q_\circ^{-sn}} \int_{(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma)\backslash\mathrm{K}^\sigma} \mathrm{I}_{\mathrm{As}}^{(0)}(s, k\cdot\mathrm{W}_{\boldsymbol{\lambda}})\,dk.$$

By Proposition 7.13, we have

$$\mathrm{I}_{\mathrm{As}}(s, \mathbf{1}_{\mathcal{O}_\circ^n}, \mathrm{W}_{\boldsymbol{\lambda}}) = \frac{1}{1-q_\circ^{-sn}} \sum_{i=0}^{e_\circ-1} q_\circ^{-in/e_\circ} q_\circ^{-\frac{in}{e_\circ}(s-1)} = \frac{1}{1-q_\circ^{-sn/e_\circ}}.$$

The result then follows immediately from (7.8). □

Corollary 7.15. *Let π be a cuspidal representation of* G *such that* $\mathrm{L}_{\mathrm{As}}(s,\pi)$ *is not* 1. *Let χ be an unramified character of* $\mathrm{F}_\circ^\times$ *such that π is χ-distinguished. Then*

$$\mathrm{I}_{\mathrm{As}}(s, \mathbf{1}_{\mathcal{O}_\circ^n}, (\widetilde{\chi}\circ\det)\mathrm{W}_{\boldsymbol{\lambda}}) = \mathrm{L}_{\mathrm{As}}(s,\pi)$$

for any unramified character $\widetilde{\chi}$ of $\mathrm{F}^\times$ *extending χ.*

8. Flicker–Kable root numbers for cuspidal representations

In this section, using Theorem 7.14, we compute the local Asai root number, as defined by Flicker and Kable, of a cuspidal distinguished representation of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$. Our methods here are purely local.

Let us fix once and for all a non-trivial complex character $\psi_\circ$ of $\mathrm{F}_\circ$, and a non-zero element $\delta \in \mathrm{F}^\times$ such that $\mathrm{tr}_{\mathrm{F}/\mathrm{F}_\circ}(\delta) = 0$. We consider the character

$$\psi_\circ^{\mathrm{F},\delta} : x \mapsto \psi_\circ(\mathrm{tr}_{\mathrm{F}/\mathrm{F}_\circ}(\delta x)) \tag{8.1}$$

of F. As the characteristic of $\mathrm{F}_\circ$ is not 2, this is a non-trivial character of F trivial on $\mathrm{F}_\circ$. Conversely, each non-trivial character of F trivial on $\mathrm{F}_\circ$ is of the form $\psi_\circ^{\mathrm{F},t\delta}$ for a unique $t \in \mathrm{F}_\circ^\times$.

We denote by $\psi = \psi^\delta$ the standard σ-self-dual non-degenerate character of N attached to (8.1), namely

$$\psi = \psi^\delta : u \mapsto \psi_\circ(\mathrm{tr}_{\mathrm{F}/\mathrm{F}_\circ}(\delta(u_{1,2} + \cdots + u_{n-1,n}))). \tag{8.2}$$

Given a generic irreducible complex representation π of G, its Asai integrals satisfy a local functional equation (see [20, appendix] and [29, Theorem 3]): there is a unique element $\epsilon_{\mathrm{As}}(s, \pi, \psi_\circ, \delta)$ in the units of $\mathbf{C}[q_\circ^s, q_\circ^{-s}]$, called the *local Asai epsilon factor*, such that, for all functions $\mathrm{W} \in \mathcal{W}(\pi, \psi^\delta)$ and $\Phi \in \mathcal{C}_c^\infty(\mathrm{F}_\circ^n)$, we have

$$\frac{\mathrm{I}_{\mathrm{As}}(1-s, \widehat{\Phi}, \widetilde{\mathrm{W}})}{\mathrm{L}_{\mathrm{As}}(1-s, \pi^\vee)} = \epsilon_{\mathrm{As}}(s, \pi, \psi_\circ, \delta) \cdot \frac{\mathrm{I}_{\mathrm{As}}(s, \Phi, \mathrm{W})}{\mathrm{L}_{\mathrm{As}}(s, \pi)} \tag{8.3}$$

where:

(i) $\widehat{\Phi} = \widehat{\Phi}^{\psi_\circ}$ denotes the Fourier transform of Φ with respect to the character $\psi_\circ \otimes \cdots \otimes \psi_\circ$ of $\mathrm{F}_\circ^n$ and its associated self-dual Haar measure;
(ii) $\widetilde{\mathrm{W}}$ is the function in $\mathcal{W}(\pi^\vee, \psi^{-\delta})$ defined by

$$\widetilde{\mathrm{W}}(g) = \mathrm{W}(w_0 g^*), \quad g \in \mathrm{G},$$

where w_0 is the antidiagonal permutation matrix of maximal length and g^* is the transpose of g^{-1}.

Notice that the epsilon factor defined above is the one used in [29]; it differs in sign from the one defined in [20]. In the next section we will address the question of proper normalization.

Before stating the main result of this section, let us make one observation on Asai root numbers of distinguished generic representations of G. If π is such a representation, then applying the functional equation for $\mathrm{I}_{\mathrm{As}}(s, \Psi, \mathrm{W})$ and $\mathrm{I}_{\mathrm{As}}(1-s, \widehat{\Phi}, \widetilde{\mathrm{W}})$ gives us

$$\epsilon_{\mathrm{As}}(s, \pi, \psi_\circ, \delta) \cdot \epsilon_{\mathrm{As}}(1-s, \pi^\vee, \psi_\circ, -\delta) = \omega_\pi(-1)$$

as in [29, Theorem 3]. Since π is distinguished, its central character is trivial on $F_o^\times$ and $\pi^\vee \simeq \pi^\sigma$. Since π and π^σ have the same local Asai L-factor and $\epsilon_{As}(s, \pi, \psi_o, \delta) = \epsilon_{As}(s, \pi^\sigma, \psi_o, -\delta)$, we get

$$\epsilon_{As}(1/2, \pi, \psi_o, \delta) \in \{-1, 1\}.$$

It is expected that this number is 1 (cf. [1, Remark 4.4]). Here we prove it when π is a distinguished cuspidal representation.

Theorem 8.4. *Let π be a distinguished cuspidal representation of* G. *Then*

$$\epsilon_{As}(1/2, \pi, \psi_o, \delta) = 1.$$

Proof. Since we have already observed that the possible values for this epsilon factor are -1 and 1, we just need to show that $\epsilon_{As}(1/2, \pi, \psi_0, \delta)$ is positive. It is sufficient to show that $\epsilon_{As}(0, \pi, \psi_o, \delta)$ is positive since

$$\epsilon_{As}(s, \pi, \psi_o, \delta) = q_o^{m(s-1/2)} \cdot \epsilon_{As}(1/2, \pi, \psi_o, \delta)$$

for some $m \in \mathbf{Z}$ as $\epsilon_{As}(s, \pi, \psi_o, \delta)$ is just a unit in $\mathbf{C}[q_o^s, q_o^{-s}]$.

Fix a Whittaker datum (N, ψ_1) and a σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$ as in Lemma 6.8. The symbol $\sim$ will stand for equality up to a positive constant. By Theorem 7.14, there is $W_{\boldsymbol{\lambda}} \in \mathcal{W}(\pi, \psi_1)$ such that

$$I_{As}(s, \Phi_0, W_{\boldsymbol{\lambda}}) \sim L_{As}(s, \pi)$$

where Φ_0 is the characteristic function of $\mathcal{O}_o^n$ in F_o^n. As $\psi_1(u) = \psi(tut^{-1})$ for some diagonal matrix t with coefficients in $F_o^\times$, the function

$$W_0 : g \mapsto W_{\boldsymbol{\lambda}}(t^{-1}g)$$

belongs to $\mathcal{W}(\pi, \psi)$. We may (and will) even assume that the bottom coefficient on the diagonal of t is 1. Applying the change of variable $g \mapsto t^{-1}g$, we check that

$$I_{As}(s, \Phi_0, W_0) \sim I_{As}(s, \Phi_0, W_{\boldsymbol{\lambda}}).$$

Applying the functional equation, we get

$$\frac{I_{As}(1, \widehat{\Phi}_0, \widetilde{W}_0)}{L_{As}(1, \pi^\vee)} \sim \epsilon_{As}(0, \pi, \psi_o, \delta).$$

Let l and l' denote the linear forms on $\mathcal{W}(\pi, \psi)$ defined by

$$l : W \mapsto \int_{N^\sigma \backslash P^\sigma} W(h)\, dh \quad \text{and} \quad l' : W \mapsto \int_{N^\sigma \backslash P^\sigma} \widetilde{W}(h)\, dh.$$

Both these linear forms are defined by convergent integrals: by [6, Corollary 5.19] the supports in G^σ of the integrands are compact mod N^σ on G^σ. They are G^σ-invariant by

[40, Theorem 3.1.2]; by multiplicity 1, they thus differ by a scalar, which is positive by [3, Theorem 7.2]. By the proof of [2, Theorem 1.4], we have

$$\mathrm{I}_{\mathrm{As}}(1, \widehat{\Phi}_0, \widetilde{\mathrm{W}}_0) \sim \Phi_0(0) l'(\mathrm{W}_0).$$

On the other hand,

$$\mathrm{L}_{\mathrm{As}}(1, \pi^\vee) = \mathrm{L}_{\mathrm{As}}(1, \pi^\sigma) = \mathrm{L}_{\mathrm{As}}(1, \pi) \sim \mathrm{I}_{\mathrm{As}}(1, \Phi_0, \mathrm{W}_0) \sim \widehat{\Phi}_0(0) l(\mathrm{W}_0),$$

the last equality by [2] again. In particular

$$\frac{\mathrm{I}_{\mathrm{As}}(1, \widehat{\Phi}_0, \widetilde{\mathrm{W}}_0)}{\mathrm{L}_{\mathrm{As}}(1, \pi^\vee)} \sim \Phi_0(0)\widehat{\Phi}_0(0)^{-1}$$

and the right hand side is positive thanks to our choice of Φ_0. Hence $\epsilon_{\mathrm{As}}(0, \pi, \psi_{\mathrm{o}}, \delta) > 0$, which implies that $\epsilon_{\mathrm{As}}(1/2, \pi, \psi_{\mathrm{o}}, \delta) = 1$. □

Remark 8.5. In the proof above, we used results written in characteristic 0 only. Let us explain why they are valid in characteristic p as well. First notice that as $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, \mathbf{1})$ and $\mathrm{Hom}_{\mathrm{P}^\sigma}(\pi, \mathbf{1})$ are equal by Ok [40, Theorem 3.1.2], the computation borrowed from the proof of [2, Theorem 1.4] holds for F of arbitrary characteristic. In [3, Theorem 7.2], and more generally in [3], the field F is assumed to have characteristic 0. In fact, appealing to [3, Theorem 6.3] is enough in the cuspidal case, since a distinguished cuspidal representation of G is always unitary (as its central character is). Now the only ingredient in the proof of [3, Theorem 6.3] which uses this restriction on the characteristic of F is that the Godement–Jacquet epsilon factor $\epsilon(1/2, \pi, \psi)$ is equal to 1, for which [3] refers to [37], but the cuspidal case of this result is already in [40] and this reference does not assume that the characteristic of F is 0.

9. Comparing Asai epsilon factors

In this section, we compare, for π a generic unramified representation of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ (not necessarily distinguished), the Flicker–Kable Asai epsilon factor to the Asai epsilon factor of π defined via the Langlands–Shahidi method. This naturally leads to the normalization we give in Definition 9.10. Beuzart-Plessis has the same normalization in [7]. Then, we show by a global argument that, for cuspidal representations, all these definitions of the Asai epsilon factor coincide. In particular we answer some questions posed in [1, Remark 4.4].

9.1. Changing the additive character

We denote by $\mathcal{W}_{\mathrm{F}}$ the Weil group of F with repect to a given separable closure $\overline{\mathrm{F}}$ of F, and by $\mathcal{W}'_{\mathrm{F}}$ the corresponding Weil–Deligne group, that is, its direct product by $\mathrm{SL}(2, \mathbf{C})$. We use a similar notation for F_{o}. We will write $\mathrm{Ind}'_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}$ and $\mathrm{M}'_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}$ for induction and multiplicative induction (defined for instance in [42, Section 7]) from $\mathcal{W}'_{\mathrm{F}}$ to $\mathcal{W}'_{\mathrm{F}_{\mathrm{o}}}$. We will also write $\mathrm{Ind}_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}$ and $\mathrm{M}_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}$ for induction and multiplicative induction from $\mathcal{W}_{\mathrm{F}}$ to $\mathcal{W}_{\mathrm{F}_{\mathrm{o}}}$.

Given an irreducible representation π of G, we denote by $\rho(\pi)$ its Langlands parameter, which is a finite-dimensional representation of $\mathcal{W}'_{\mathrm{F}}$. Then, using local class field theory to identify characters of $\mathcal{W}'_{\mathrm{F}}$ and of $\mathrm{F}^{\times}$, we have

$$\det(\mathrm{M}'_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(\rho(\pi))) = \omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}^{n(n-1)/2} \cdot \omega_{\pi}^{n}|_{\mathrm{F}_{\mathrm{o}}^{\times}}. \tag{9.1}$$

When $\pi = \chi$ is a character of $\mathrm{F}^{\times}$, which we identify with $\rho(\chi)$, this tells us that $\mathrm{M}'_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(\chi)$ is the restriction of χ to $\mathrm{F}_{\mathrm{o}}^{\times}$.

Given a generic irreducible representation π of G, we denote:

(i) by $\epsilon_{\mathrm{As}}^{\mathrm{LS}}(s,\pi,\psi_{\mathrm{o}})$ and $\mathrm{L}_{\mathrm{As}}^{\mathrm{LS}}(s,\pi)$ the Asai local factors attached to π via the Langlands–Shahidi method (see [50] when F has characteristic 0 and [32] when F has characteristic p);

(ii) by $\epsilon_{\mathrm{As}}^{\mathrm{Gal}}(s,\pi,\psi_{\mathrm{o}})$ and $\mathrm{L}_{\mathrm{As}}^{\mathrm{Gal}}(s,\pi)$ the Langlands–Deligne local constants of the local Asai transfer $\mathrm{M}'_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(\rho(\pi))$ of the Langlands parameter of π (see [42, Section 7]).

When F has characteristic 0, the Asai local L-functions $\mathrm{L}_{\mathrm{As}}(s,\pi)$, $\mathrm{L}_{\mathrm{As}}^{\mathrm{LS}}(s,\pi)$ and $\mathrm{L}_{\mathrm{As}}^{\mathrm{Gal}}(s,\pi)$ are known to be all equal. We will see in the appendix (Theorem A.1) that this still holds in characteristic p.

By [50] when F has characteristic 0 and by [26] when F has characteristic p, when π is unramified and generic we have

$$\epsilon_{\mathrm{As}}^{\mathrm{LS}}(s,\pi,\psi_{\mathrm{o}}) = \epsilon_{\mathrm{As}}^{\mathrm{Gal}}(s,\pi,\psi_{\mathrm{o}})$$

whereas by [25] when F has characteristic 0 and [26] when F has characteristic p, when π is generic we have

$$\epsilon_{\mathrm{As}}^{\mathrm{LS}}(s,\pi,\psi_{\mathrm{o}}) = \zeta \cdot \epsilon_{\mathrm{As}}^{\mathrm{Gal}}(s,\pi,\psi_{\mathrm{o}})$$

where ζ is a root of unity independent of ψ_{o}, which is expected to be 1, and known to be 1 when F has characteristic p.

We first describe how all these epsilon factors depend on ψ_{o}. Given $t \in \mathrm{F}_{\mathrm{o}}^{\times}$, we write $\psi_{\mathrm{o},t}$ for the character $x \mapsto \psi_{\mathrm{o}}(tx)$ of F_{o}.

Lemma 9.2. *Let π be generic irreducible representation of* G *and $t \in \mathrm{F}_{\mathrm{o}}^{\times}$. Then:*

(i) $\epsilon_{\mathrm{As}}(s,\pi,\psi_{\mathrm{o},t},\delta) = \omega_{\pi}(t)^{n} \cdot |t|_{\mathrm{o}}^{n^{2}(s-1/2)} \cdot \epsilon_{\mathrm{As}}(s,\pi,\psi_{\mathrm{o}},\delta)$;

(ii) $\epsilon_{\mathrm{As}}^{\mathrm{LS}}(s,\pi,\psi_{\mathrm{o},t}) = \omega_{\pi}(t)^{n} \cdot |t|_{\mathrm{o}}^{n^{2}(s-1/2)} \cdot \omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(t)^{n(n-1)/2} \cdot \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s,\pi,\psi_{\mathrm{o}})$;

(iii) $\epsilon_{\mathrm{As}}^{\mathrm{Gal}}(s,\pi,\psi_{\mathrm{o},t}) = \omega_{\pi}(t)^{n} \cdot |t|_{\mathrm{o}}^{n^{2}(s-1/2)} \cdot \omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(t)^{n(n-1)/2} \cdot \epsilon_{\mathrm{As}}^{\mathrm{Gal}}(s,\pi,\psi_{\mathrm{o}})$.

Proof. We first give the proof of (i) for convenience of the reader; it follows verbatim the analogue for Rankin–Selberg L-factors in [27, p. 7]. As before we set $\psi = \psi^{\delta}$. We introduce the matrix

$$a = \mathrm{diag}(t^{n-1},\ldots,t,1).$$

Thus $\mathrm{W} \in \mathcal{W}(\pi,\psi^{\delta})$ if and only if the function $\mathrm{W}_{a} : g \mapsto \mathrm{W}(ag)$ is in $\mathcal{W}(\pi,\psi^{t\delta})$. Now take $\mathrm{W} \in \mathcal{W}(\pi,\psi)$ and $\Phi \in \mathcal{C}_{c}^{\infty}(\mathrm{F}_{\mathrm{o}}^{n})$, and notice that $\Phi(\tau a^{-1}h) = \Phi(\tau h)$ for all

$h \in \mathrm{G}^{\sigma}$. Then

$$\begin{aligned}\mathrm{I}_{\mathrm{As}}(s, \Phi, \mathrm{W}_a) &= \int_{\mathrm{N}^{\sigma}\backslash \mathrm{G}^{\sigma}} \mathrm{W}(ah)\Phi(\tau h)|\det(h)|_{\mathrm{o}}^{s}\, dh \\ &= \mu(t) \cdot \int_{\mathrm{N}^{\sigma}\backslash \mathrm{G}^{\sigma}} \mathrm{W}(h)\Phi(\tau h)|\det(a^{-1}h)|_{\mathrm{o}}^{s}\, dh \\ &= \mu(t) \cdot |t|_{\mathrm{o}}^{-n(n-1)s/2} \cdot \mathrm{I}_{\mathrm{As}}(s, \Phi, \mathrm{W})\end{aligned}$$

for some positive character μ of $\mathrm{F}_{\mathrm{o}}^{\times}$. On the other hand, for all $h \in \mathrm{G}^{\sigma}$, we have

$$\mathrm{W}(aw_0h^*) = \mathrm{W}(w_0(a^{*w_0}h)^*) = \widetilde{\mathrm{W}}(a^{*w_0}h) = \widetilde{\mathrm{W}}(t^{1-n}ah).$$

It follows that

$$\begin{aligned}\mathrm{I}_{\mathrm{As}}(1-s, \widehat{\Phi}^{\psi_{\mathrm{o},t}}, \widetilde{\mathrm{W}}_a) &= \int_{\mathrm{N}^{\sigma}\backslash \mathrm{G}^{\sigma}} \widetilde{\mathrm{W}}(t^{1-n}ah)\widehat{\Phi}^{\psi_{\mathrm{o},t}}(\tau h)|\det(h)|_{\mathrm{o}}^{1-s}\, dh \\ &= \omega_{\pi}(t)^{n-1} \cdot \int_{\mathrm{N}^{\sigma}\backslash \mathrm{G}^{\sigma}} \widetilde{\mathrm{W}}(ah)\widehat{\Phi}^{\psi_{\mathrm{o},t}}(\tau h)|\det(h)|_{\mathrm{o}}^{1-s}\, dh \\ &= \omega_{\pi}(t)^{n-1} \cdot \mu(t) \cdot \int_{\mathrm{N}^{\sigma}\backslash \mathrm{G}^{\sigma}} \widetilde{\mathrm{W}}(h)\widehat{\Phi}^{\psi_{\mathrm{o},t}}(\tau h)|\det(a^{-1}h)|_{\mathrm{o}}^{1-s}\, dh \\ &= \omega_{\pi}(t)^{n-1} \cdot \mu(t) \cdot |t|_{\mathrm{o}}^{n(n-1)(s-1)/2} \cdot \int_{\mathrm{N}^{\sigma}\backslash \mathrm{G}^{\sigma}} \widetilde{\mathrm{W}}(h)\widehat{\Phi}^{\psi_{\mathrm{o},t}}(\tau h)|\det(h)|_{\mathrm{o}}^{1-s}\, dh.\end{aligned}$$

Now we use the relation

$$\widehat{\Phi}^{\psi_{\mathrm{o},t}}(x) = |t|_{\mathrm{o}}^{n/2} \cdot \widehat{\Phi}(tx), \quad x \in \mathrm{F}_{\mathrm{o}}^{n}, \tag{9.3}$$

to get

$$\begin{aligned}\int_{\mathrm{N}^{\sigma}\backslash \mathrm{G}^{\sigma}} \widetilde{\mathrm{W}}(h)\widehat{\Phi}^{\psi_{\mathrm{o},t}}(\tau h)|\det(h)|_{\mathrm{o}}^{1-s}\, dh &= |t|_{\mathrm{o}}^{n/2} \cdot \int_{\mathrm{N}^{\sigma}\backslash \mathrm{G}^{\sigma}} \widetilde{\mathrm{W}}(h)\widehat{\Phi}(\tau th)|\det(h)|_{\mathrm{o}}^{1-s}\, dh \\ &= |t|_{\mathrm{o}}^{n/2} \cdot \int_{\mathrm{N}^{\sigma}\backslash \mathrm{G}^{\sigma}} \widetilde{\mathrm{W}}(t^{-1}h)\widehat{\Phi}(\tau h)|\det(t^{-1}h)|_{\mathrm{o}}^{1-s}\, dh \\ &= |t|_{\mathrm{o}}^{n/2} \cdot |t|_{\mathrm{o}}^{n(s-1)} \cdot \omega_{\pi}(t) \cdot \mathrm{I}_{\mathrm{As}}(1-s, \widehat{\Phi}, \widetilde{\mathrm{W}}).\end{aligned}$$

We thus get the relation

$$\epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathrm{o},t}, \delta) = \frac{\omega_{\pi}(t)^{n} \cdot \mu(t) \cdot |t|_{\mathrm{o}}^{n(n-1)(s-1)/2+n(s-1/2)}}{\mu(t) \cdot |t|_{\mathrm{o}}^{-n(n-1)s/2}} \cdot \epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathrm{o}}, \delta),$$

which gives us the expected result.

Now, as we noticed that $\epsilon_{\mathrm{As}}^{\mathrm{Gal}}(s, \pi, \psi_{\mathrm{o}})$ and $\epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \pi, \psi_{\mathrm{o}})$ are equal up to a non-zero constant which does not depend on ψ_{o}, it is enough to prove (iii). Then by the properties of the Langlands–Deligne constants in [53], one has

$$\epsilon_{\mathrm{As}}^{\mathrm{Gal}}(s, \pi, \psi_{0,t}) = |t|_{\mathrm{o}}^{n^2(s-1/2)} \cdot \det(\mathrm{M}'_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(\rho(\pi))) \cdot \epsilon_{\mathrm{As}}^{\mathrm{Gal}}(s, \pi, \psi_{\mathrm{o}}),$$

which, together with (9.1), gives the expected result. ◻

We will also need the following relation satisfied by $\epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathsf{o}}, \delta)$. Note that though $\psi_{\mathsf{o},t}^{\mathrm{F},\delta} = \psi_{\mathsf{o}}^{\mathrm{F},t\delta}$, it is not true that $\epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathsf{o},t}, \delta) = \epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathsf{o}}, t\delta)$ since changing the character ψ_{o} changes the Fourier transform in the functional equation. Here is what happens when one changes δ.

Lemma 9.4. *Let π be a generic irreducible representation of* G *and $t \in \mathrm{F}_{\mathsf{o}}^{\times}$. Then*

$$\epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathsf{o}}, t\delta) = \omega_\pi(t)^{n-1} \cdot |t|_{\mathsf{o}}^{n(n-1)(s-1/2)} \cdot \epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathsf{o}}, \delta).$$

Proof. Going through the exact same computations as in the proof of Lemma 9.2, but taking the Fourier transform of Φ with respect to ψ_{o} rather than $\psi_{\mathsf{o},t}$, we arrive at

$$\mathrm{I}_{\mathrm{As}}(1-s, \widehat{\Phi}, \widetilde{\mathrm{W}}_a) = \omega_\pi(t)^{n-1} \cdot \mu(t) \cdot |t|_{\mathsf{o}}^{n(n-1)(s-1)/2} \cdot \mathrm{I}_{\mathrm{As}}(1-s, \widehat{\Phi}, \widetilde{\mathrm{W}})$$

whereas the relation

$$\mathrm{I}_{\mathrm{As}}(s, \Phi, \mathrm{W}_a) = \mu(t) \cdot |t|_{\mathsf{o}}^{-n(n-1)s/2} \cdot \mathrm{I}_{\mathrm{As}}(s, \Phi, \mathrm{W})$$

does not change. From this we obtain

$$\begin{aligned}
&\epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathsf{o}}, \delta)\\
&\quad = (|t|_{\mathsf{o}}^{-n(n-1)(1-s)/2} \cdot \mu(t) \cdot \omega_\pi(t)^{n-1})^{-1} \cdot \mu(t) \cdot |t|_{\mathsf{o}}^{-n(n-1)s/2} \cdot \epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathsf{o}}, t\delta)\\
&\quad = \omega_\pi(t)^{1-n} \cdot |t|_{\mathsf{o}}^{n(n-1)(1-2s)/2} \cdot \epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathsf{o}}, t\delta),
\end{aligned}$$

which gives us the expected result. □

9.2. *Unramified representations*

We are going to compute $\epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathsf{o}}, \delta)$ and $\epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \pi, \psi_{\mathsf{o}}, \delta)$ when π is generic and unramified. From now on, Haar measures on any closed subgroup H of G will be normalized so that they give volume 1 to H ∩ K. This also normalizes all right invariant measures on quotients of the type U\H whenever U is a unimodular closed subgroup of H.

First, we perform a test vector computation similar to that done by Flicker when $\mathrm{F}/\mathrm{F}_{\mathsf{o}}$ is unramified. We suppose that π is a generic unramified representation of G; we denote by W_0 the normalized spherical vector in $\mathcal{W}(\pi, \psi)$ and by Φ_0 the characteristic function of $\mathcal{O}_{\mathsf{o}}^n$.

Recall that the *conductor* of an additive character of a finite extension E of F_{o} is the largest integer i such that it is trivial on $\mathfrak{p}_{\mathrm{E}}^{-i}$.

Proposition 9.5. *Let π be a generic unramified representation of* G *and suppose that the character $\psi_{\mathsf{o}}^{\mathrm{F},\delta}$ defined by* (8.1) *has conductor* 0. *Then*

$$\mathrm{I}_{\mathrm{As}}(s, \Phi_0, \mathrm{W}_0) = \mathrm{L}_{\mathrm{As}}(s, \pi).$$

Proof. When $\mathrm{F}/\mathrm{F_o}$ is unramified, this is proved in [19, Section 3] where the unitarity assumption is unnecessary. In the ramified case, we have $q = q_{\mathrm{o}}$. We write $\pi = \mu_1 \times \cdots \times \mu_n$ where the product notation stands for parabolic induction, and the characters $\mu_1, \dots, \mu_n$ of $\mathrm{F}^\times$ are unramified. Let us fix a uniformizer ϖ of F such that $\varpi_{\mathrm{o}} = \varpi^2$ is a uniformizer of $\mathrm{F_o}$. For $i = 1, \dots, n$, set $z_i = \mu_i(\varpi)$. With notations [19, p. 306], as $\varpi_{\mathrm{o}}^\lambda = \varpi^{2\lambda}$, we find

$$\mathrm{I_{As}}(s, \Phi_0, \mathrm{W}_0) = \sum_\lambda q^{-s\cdot \mathrm{tr}(\lambda)} s_{2\lambda}(z_1, \dots, z_n) = \sum_\lambda s_{2\lambda}(z_1 q^{-s/2}, \dots, z_n q^{-s/2})$$

where the sum ranges over all partitions of length $\leq n$ and $s_{2\lambda}$ is the Schur function (see [33, (3.1) p. 40]) associated to the partition 2λ obtained by multiplying the entries of λ by 2. By [33, Example 5a, p. 77], the sum above is equal to

$$\prod_{1\leq i\leq n} (1 - z_i^2 q^{-s}) \cdot \prod_{1\leq k<l\leq n} (1 - z_k z_l q^{-s}) \tag{9.6}$$

Now the Langlands parameter $\rho(\pi)$ is the direct sum $\mu_1 \oplus \cdots \oplus \mu_n$. Since it is trivial on $\mathrm{SL}_2(\mathbf{C})$ we consider it as a representation of $\mathcal{W}_\mathrm{F}$ only. Since $\mu_i \circ \sigma = \mu_i$ for all i, we have

$$\mathrm{M_{F/F_o}}(\mu_1 \oplus \cdots \oplus \mu_n) = \bigoplus_{1\leq i\leq n} \mu_i|_{\mathrm{F_o^\times}} \oplus \bigoplus_{1\leq k<l\leq n} \mathrm{Ind_{F/F_o}}(\mu_k\mu_l)$$

by [42, Lemma 7.1]. Thus $\mathrm{L_{As}^{Gal}}(s, \pi)$ is equal to (9.6). The result follows from Theorem A.1. □

Remark 9.7. At this point, we note that the authors of [3] appeal to Flicker's unramified computation even when $\mathrm{F}/\mathrm{F_o}$ is ramified, but Proposition 9.5 shows that there is no harm in doing that.

When $\mathrm{F}/\mathrm{F_o}$ is unramified, one can choose δ to be a unit, whereas when $\mathrm{F}/\mathrm{F_o}$ is ramified, one can choose δ to have valuation -1. In both cases, the character $\psi_{\mathrm{o}}^{\mathrm{F},\delta}$ has conductor 0 if ψ_{o} has conductor 0. In this case, the functional equation, together with Proposition 9.5, the fact that $\widehat{\Phi}_0 = \Phi_0$ and that $\widetilde{\mathrm{W}}_0$ is the normalized spherical vector in $\mathcal{W}(\pi^\vee, \psi^{-1})$, tells us that:

Corollary 9.8. *Suppose that π is a generic unramified representation of* G, *that ψ_{o} has conductor* 0 *and δ has valuation $1 - e(\mathrm{F}/\mathrm{F_o})$. Then $\epsilon_{\mathrm{As}}(s, \pi, \psi_{\mathrm{o}}, \delta) = 1$.*

Let us compare this with the unramified situation for the Asai constant defined via the Langlands–Shahidi method. To do this we introduce the local Langlands constant $\lambda(\mathrm{F}/\mathrm{F_o}, \psi_{\mathrm{o}})$ (see for instance [12, (30.4.1)] for a definition). We note that $\lambda(\mathrm{F}/\mathrm{F_o}, \psi_{\mathrm{o}})$ is equal to $\epsilon(1/2, \omega_{\mathrm{F/F_o}}, \psi_{\mathrm{o}})$, the Tate root number of the quadratic character $\omega_{\mathrm{F/F_o}}$. We will freely use the relation [12, (30.4.2)]

$$\epsilon(s, \mathrm{Ind_{F/F_o}}\, \rho, \psi_{\mathrm{o}}) = \lambda(\mathrm{F}/\mathrm{F_o}, \psi_{\mathrm{o}})^{\dim(\rho)} \cdot \epsilon(s, \rho, \psi_{\mathrm{o}} \circ \mathrm{tr_{F/F0}})$$

where ρ a semisimple representation of $\mathcal{W}_\mathrm{F}$ and $\mathrm{Ind_{F/F_o}}$ denotes induction from $\mathcal{W}_\mathrm{F}$ to $\mathcal{W}_{\mathrm{F_o}}$. We will also use the fact that if χ is an unramified character of $\mathrm{E}^\times$ for any finite extension E of F_o and ψ_E is a character of E of conductor 0, then $\epsilon(s, \chi, \psi_E) = 1$ (see [53, remark after (3.2.6.1)]). More generally, we refer to [53] for the basic facts and relations concerning epsilon factors of characters that we will use in this section without necessarily recalling.

Proposition 9.9. *Suppose that π is a generic unramified representation of* G*, and that ψ_o has conductor* 0 *and δ has valuation $1 - e(F/F_o)$. Then*

$$\epsilon_{As}^{LS}(s, \pi, \psi_o) = \epsilon_{As}^{Gal}(s, \pi, \psi_0) = \omega_\pi(\delta)^{1-n} \cdot |\delta|^{-n(n-1)(s-1/2)/2} \cdot \lambda(F/F_o, \psi_o)^{n(n-1)/2}$$

where $|\delta|$ denotes the normalized absolute value of δ.

Proof. We use the notation of the proof of Proposition 9.5. We thus have

$$\begin{aligned}
\epsilon_{As}^{Gal}(1/2, \pi, \psi_o) &= \prod_{1\leq k<l\leq n} \epsilon(1/2, \mathrm{Ind}_{F/F_o}(\mu_k\mu_l), \psi_o) \\
&= \lambda(F/F_o, \psi_o)^{n(n-1)/2} \cdot \prod_{1\leq k<l\leq n} \epsilon(1/2, \mu_k\mu_l, \psi_o \circ \mathrm{tr}_{F/F_o}) \\
&= \lambda(F/F_o, \psi_o)^{n(n-1)/2} \cdot \prod_{1\leq k<l\leq n} (\mu_k\mu_l)(\delta)^{-1}\epsilon(1/2, \mu_k\mu_l, \psi_o^{F,\delta}) \\
&= \lambda(F/F_o, \psi_o)^{n(n-1)/2} \cdot \prod_{1\leq k<l\leq n} (\mu_k\mu_l)(\delta)^{-1} \\
&= \lambda(F/F_o, \psi_o)^{n(n-1)/2} \cdot \omega_\pi(\delta)^{1-n}.
\end{aligned}$$

where we ignore the epsilon factors equal to 1. However

$$\epsilon_{As}^{Gal}(s, \pi, \psi_o) = \epsilon_{As}^{Gal}(1/2, |\cdot|^{(s-1/2)/2}\pi, \psi_o)$$

hence the previous equality gives the result. □

9.3. *Rankin–Selberg epsilon factors*

Proposition 9.9, together with Corollary 9.8, suggests introducing the following definition.

Definition 9.10. For any generic irreducible representation π, we set

$$\epsilon_{As}^{RS}(s, \pi, \psi_o, \delta) = \omega_\pi(\delta)^{1-n} \cdot |\delta|^{-n(n-1)(s-1/2)/2} \cdot \lambda(F/F_o, \psi_o)^{n(n-1)/2} \cdot \epsilon_{As}(s, \pi, \psi_o, \delta).$$

The following result, which is an immediate consequence of Lemma 9.4, was brought to our attention by Beuzart-Plessis.

Lemma 9.11. *For any generic irreducible representation π of* G *and $t \in F_o^\times$, we have*

$$\epsilon_{As}^{RS}(s, \pi, \psi_o, t\delta) = \epsilon_{As}^{RS}(s, \pi, \psi_o, \delta).$$

In particular we can remove δ from the notations, and set

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi,\psi_{\mathrm{o}}) = \epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi,\psi_{\mathrm{o}},\delta)$$

for any generic irreducible representation π, any non-trivial character ψ_{o} and any element $\delta \in \mathrm{F}^{\times}$ of trace 0. By Lemma 9.2 and the relation

$$\lambda(\mathrm{F}/\mathrm{F}_{\mathrm{o}},\psi_{\mathrm{o},t}) = \omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(t)\cdot\lambda(\mathrm{F}/\mathrm{F}_{\mathrm{o}},\psi_{\mathrm{o}}),$$

we have:

Lemma 9.12. *For any generic irreducible representation π of* G *and $t \in \mathrm{F}_{\mathrm{o}}^{\times}$, we have*

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi,\psi_{\mathrm{o},t}) = \omega_{\mathrm{F}/\mathrm{F}_{\mathrm{o}}}(t)^{n(n-1)/2}\cdot\omega_{\pi}(t)^{n}\cdot|t|_{\mathrm{o}}^{n^2(s-1/2)}\cdot\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi,\psi_{\mathrm{o}}).$$

Combining Proposition 9.9, Corollary 9.8 and Lemmas 9.11 and 9.12, we get the following result.

Theorem 9.13. *For any generic unramified irreducible representation π of* G*, we have*

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi,\psi_{\mathrm{o}}) = \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s,\pi,\psi_{\mathrm{o}}). \tag{9.14}$$

At the end of this section (see Theorem 9.29), we will show that (9.14) holds for cuspidal representations as well.

9.4. The local factors at split places

Let k/k_{o} be a quadratic extension of global fields of characteristic different from 2. Given a place v of k_{o}, we write $k_{\mathrm{o},v}$ for the completion of k_{o} at v and $k_v = k\otimes_{k_{\mathrm{o}}} k_{\mathrm{o},v}$. In this subsection, we consider a place v which splits in k, that is, k_v is a split $k_{\mathrm{o},v}$-algebra. There are thus two isomorphisms of $k_{\mathrm{o},v}$-algebras between k_v and $k_{\mathrm{o},v}\oplus k_{\mathrm{o},v}$, and one passes from one to the other by applying the automorphism $(x,y)\mapsto(y,x)$.

Let π_v be a generic irreducible representation of $\mathrm{G}_v = \mathrm{GL}_n(k_v)$ and set $\mathrm{N}_v = \mathrm{N}_n(k_v)$. We fix a non-trivial character $\psi_{\mathrm{o},v}$ of $k_{\mathrm{o},v}$ and an element $\delta_v \in k_v^{\times}$ such that $\mathrm{tr}_{k_v/k_{\mathrm{o},v}}(\delta_v) = 0$, and set

$$\psi_{\mathrm{o},v}^{k_v,\delta_v} : x \mapsto \psi_{\mathrm{o},v}(\mathrm{tr}_{k_v/k_{\mathrm{o},v}}(\delta_v x)),$$

which is a non-trivial character of k_v trivial on $k_{\mathrm{o},v}$. We denote by $|\cdot|_{\mathrm{o},v}$ the normalized absolute value on $k_{\mathrm{o},v}$.

We first suppose that v is finite, and write $q_{\mathrm{o},v}$ for the cardinality of $k_{\mathrm{o},v}$. Take $\mathrm{W}_v \in \mathcal{W}(\pi_v,\psi_v)$ and $\Phi_v \in \mathcal{C}_c^{\infty}(k_{\mathrm{o},v}^n)$. By [28, Theorem 2.7], the integral

$$\mathrm{I}_{\mathrm{As}}(s,\Phi_v,\mathrm{W}_v) = \int_{\mathrm{N}(k_{\mathrm{o},v})\backslash\mathrm{G}(k_{\mathrm{o},v})} \mathrm{W}_v(h)\Phi_v(\tau h)|\det(h)|^s\,dh$$

is absolutely convergent when the real part of s is larger than a real number r_v depending only on π_v, it extends to an element of $\mathbf{C}[q_{\mathrm{o},v}^{s},q_{\mathrm{o},v}^{-s}]$, and these integrals span a fractional ideal of $\mathbf{C}[q_{\mathsf{o},v}^{s}, q_{\mathsf{o},v}^{-s}]$ generated by a unique Euler factor denoted $\mathrm{L}_{\mathrm{As}}(s,\pi_v)$. Also, there is a unit in $\mathbf{C}[q_{\mathsf{o},v}^{s}, q_{\mathsf{o},v}^{-s}]$, which we denote by $\epsilon_{\mathrm{As}}(s,\pi_v,\psi_{\mathsf{o},v},\delta_v)$ for the sake of coherent notations, such that

$$\frac{\mathrm{I}_{\mathrm{As}}(1-s,\widehat{\Phi}_v,\widetilde{\mathrm{W}}_v)}{\mathrm{L}_{\mathrm{As}}(1-s,\pi_v^{\vee})} = \epsilon_{\mathrm{As}}(s,\pi_v,\psi_{\mathsf{o},v},\delta_v)\cdot\frac{\mathrm{I}_{\mathrm{As}}(s,\Phi_v,\mathrm{W}_v)}{\mathrm{L}_{\mathrm{As}}(s,\pi_v)}$$

where the Fourier transform of Φ_v is defined with repect to the character $\psi_{\mathsf{o},v}$.

Definition 9.15. We set

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi_v,\psi_{\mathsf{o},v},\delta_v) = \omega_\pi(\delta_v)^{1-n}\cdot|\delta_v|^{-n(n-1)(s-1/2)/2}\cdot\epsilon_{\mathrm{As}}(s,\pi_v,\psi_{\mathsf{o},v},\delta_v).$$

Remark 9.16. Compared with Definition 9.10 in the inert case, there is no Langlands constant appearing in Definition 9.15. However, note that the character $\omega_{k_v/k_{\mathsf{o},v}}$ of $k_{\mathsf{o},v}^{\times}$ trivial on $k_v/k_{\mathsf{o},v}$-norms is trivial. In analogy with the inert case, we may set the Langlands constant $\lambda(k_v/k_{\mathsf{o},v},\psi_{\mathsf{o},v})$ to be equal to $\epsilon(1/2,\omega_{k_v/k_{\mathsf{o},v}},\psi_{\mathsf{o},v})$, but this root number is equal to 1 by the classical properties of Tate epsilon factors.

A computation similar to the one carried out in the proof of Lemma 9.4 shows that this local factor $\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi_v,\psi_{\mathsf{o},v},\delta_v)$ is independent of δ_v hence we write

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi_v,\psi_{\mathsf{o},v}) = \epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi_v,\psi_{\mathsf{o},v},\delta_v).$$

When v is archimedean, the discussion above remains true up to the appropriate modifications (the L-factor is meromorphic rather than an Euler factor, and the epsilon factor is entire rather than a Laurent monomial) appealing to [27, Theorem 2.1] instead of [28, Theorem 2.7], and we define the local factor $\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi_v,\psi_{\mathsf{o},v},\delta_v)$ as in Definition 9.15.

Now we compare these epsilon factors to the epsilon factors of pairs defined by the authors of [28] and [27].

Lemma 9.17. *Let ϕ be an isomorphism of $k_{\mathsf{o},v}$-algebras between k_v and $k_{\mathsf{o},v}\oplus k_{\mathsf{o},v}$. It induces an isomorphism of groups between $\mathrm{GL}_n(k_v)$ and $\mathrm{GL}_n(k_{\mathsf{o},v})\times\mathrm{GL}_n(k_{\mathsf{o},v})$, still denoted ϕ. Write $\pi_v\circ\phi$ as a tensor product $\pi_{1,v}\otimes\pi_{2,v}$ of two generic irreducible representations of $\mathrm{GL}_n(k_{\mathsf{o},v})$. Then*

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi_v,\psi_{\mathsf{o},v}) = \epsilon^{\mathrm{RS}}(s,\pi_{1,v},\pi_{2,v},\psi_{\mathsf{o},v}) = \epsilon^{\mathrm{RS}}(s,\pi_{2,v},\pi_{1,v},\psi_{\mathsf{o},v})$$

where $\epsilon^{\mathrm{RS}}(s,\pi_{1,v},\pi_{2,v},\psi_{\mathsf{o},v})$ is the epsilon factor denoted $\epsilon(s,\pi_{1,v},\pi_{2,v},\psi_{\mathsf{o},v})$ in [28, Theorem 2.7] if v is finite, and is the one canonically associated to the gamma factor of [27, Theorem 2.1] if v is archimedean.

Proof. Since $\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s,\pi_v,\psi_{\mathsf{o},v})$ does not depend on δ_v, we can choose $\delta_v=\phi(1,-1)$. Then $\psi_v\circ\phi$ can be written $\psi_{\mathsf{o},v}\otimes\psi_{\mathsf{o},v}^{-1}$ and we have

$$\mathcal{W}(\pi_v\circ\phi,\psi_v\circ\phi) = \mathcal{W}(\pi_{1,v},\psi_{\mathsf{o},v})\otimes\mathcal{W}(\pi_{2,v},\psi_{\mathsf{o},v}^{-1}).$$

Moreover, $\omega_{\pi_v}(\delta_v)^{n-1} = \omega_{\pi_{2,v}}(-1)^{n-1}$, hence

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \pi_v, \psi_{\mathrm{o},v}) = \epsilon^{\mathrm{RS}}(s, \pi_{1,v}, \pi_{2,v}, \psi_{\mathrm{o},v}).$$

Now replace ϕ by the other isomorphism ϕ' of $k_{\mathrm{o},v}$-algebras such that $\phi' \circ \phi^{-1} : (x, y) \mapsto (y, x)$ and replace δ_v by $-\delta_v = \phi'(1, -1)$. We then get

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \pi_v, \psi_{\mathrm{o},v}) = \epsilon^{\mathrm{RS}}(s, \pi_{2,v}, \pi_{1,v}, \psi_{\mathrm{o},v}),$$

which proves the expected result. □

We give another reason for the lemma above to be true for the possibly surprised reader.

Remark 9.18. It is in fact well known as a part of the local Langlands correspondence for $\mathrm{GL}_n(k_{\mathrm{o},v})$ that

$$\epsilon^{\mathrm{RS}}(s, \pi_{1,v}, \pi_{2,v}, \psi_{\mathrm{o},v}) = \epsilon^{\mathrm{RS}}(s, \pi_{2,v}, \pi_{1,v}, \psi_{\mathrm{o},v}) \tag{9.19}$$

as it is equal to the Langlands–Deligne constant

$$\epsilon(s, \rho(\pi_{1,v}) \otimes \rho(\pi_{2,v}), \psi_{\mathrm{o},v}) = \epsilon(s, \rho(\pi_{2,v}) \otimes \rho(\pi_{1,v}), \psi_{\mathrm{o},v}).$$

Equality (9.19) can also be checked as follows. Using the notation of [28, Theorem 2.7], one has

$$\frac{\Psi(1-s, \widetilde{\mathrm{W}}_{1,v}, \widetilde{\mathrm{W}}_{2,v}, \widehat{\Phi})}{\mathrm{L}^{\mathrm{RS}}(1-s, \pi_{1,v}^{\vee}, \pi_{2,v}^{\vee})} = \omega_{\pi_{2,v}}(-1)^{n-1} \cdot \epsilon^{\mathrm{RS}}(s, \pi_{1,v}, \pi_{2,v}, \psi_{\mathrm{o},v}) \cdot \frac{\Psi(1-s, \mathrm{W}_{1,v}, \mathrm{W}_{2,v}, \Phi)}{\mathrm{L}^{\mathrm{RS}}(s, \pi_{1,v}, \pi_{2,v})}$$

for $\mathrm{W}_{1,v} \in \mathcal{W}(\pi_{1,v}, \psi_{\mathrm{o},v})$, $\mathrm{W}_{2,v} \in \mathcal{W}(\pi_{2,v}, \psi_{\mathrm{o},v}^{-1})$ and $\Phi \in \mathcal{C}_c^{\infty}(k_{\mathrm{o},v}^n)$. Similarly,

$$\frac{\Psi(1-s, \widetilde{\mathrm{W}}_{2,v}, \widetilde{\mathrm{W}}_{1,v}, \widehat{\Phi}^{\psi_{\mathrm{o},v}^{-1}})}{\mathrm{L}^{\mathrm{RS}}(1-s, \pi_{2,v}^{\vee}, \pi_{1,v}^{\vee})} = \omega_{\pi_{1,v}}(-1)^{n-1} \cdot \epsilon^{\mathrm{RS}}(s, \pi_{2,v}, \pi_{1,v}, \psi_{\mathrm{o},v}^{-1}) \cdot \frac{\Psi(1-s, \mathrm{W}_{2,v}, \mathrm{W}_{1,v}, \Phi)}{\mathrm{L}^{\mathrm{RS}}(s, \pi_{2,v}, \pi_{1,v})}$$

for $\mathrm{W}_{1,v} \in \mathcal{W}(\pi_1, \psi_{\mathrm{o},v})$, $\mathrm{W}_{2,v} \in \mathcal{W}(\pi_2, \psi_{\mathrm{o},v}^{-1})$ and $\Phi \in \mathcal{C}_c^{\infty}(k_{\mathrm{o},v}^n)$. The L-factors do not depend on the ordering of the representations, and a simple change of variable using the relation (9.3) gives

$$\Psi(1-s, \widetilde{\mathrm{W}}_{2,v}, \widetilde{\mathrm{W}}_{1,v}, \widehat{\Phi}^{\psi_{\mathrm{o},v}^{-1}}) = \omega_{\pi_1}(-1) \cdot \omega_{\pi_2}(-1) \cdot \Psi(1-s, \widetilde{\mathrm{W}}_{2,v}, \widetilde{\mathrm{W}}_{1,v}, \widehat{\Phi}),$$

whereas

$$\epsilon^{\mathrm{RS}}(s, \pi_{2,v}, \pi_{1,v}, \psi_{\mathrm{o},v}^{-1}) = \omega_{\pi_1}(-1)^n \cdot \omega_{\pi_2}(-1)^n \cdot \epsilon^{\mathrm{RS}}(s, \pi_{2,v}, \pi_{1,v}, \psi_{\mathrm{o},v})$$

by [27, p. 7]. Putting all together yields the equality we were looking for.

Remark 9.20. In [29, p. 811], the author notices a sign ambiguity in the identification of the Asai epsilon factor with $\epsilon^{\mathrm{RS}}(s, \pi_{1,v}, \pi_{2,v}, \psi_{\mathrm{o},v})$ due to the ordering of $\pi_{1,v}$ and $\pi_{2,v}$. Lemma 9.17 or Remark 9.18 show that there is in fact no such ambiguity.

Remark 9.21. With the same assumptions as in Lemma 9.17, we also have the equalities

$$\mathrm{L}_{\mathrm{As}}(s, \pi_v) = \mathrm{L}^{\mathrm{RS}}(s, \pi_{1,v}, \pi_{2,v}) = \mathrm{L}^{\mathrm{RS}}(s, \pi_{2,v}, \pi_{1,v})$$

between local L-factors. Note that

$$\mathrm{L}^{\mathrm{RS}}(s, \pi_{1,v}, \pi_{2,v}) = \mathrm{L}^{\mathrm{LS}}(s, \pi_{1,v}, \pi_{2,v}),$$
$$\epsilon^{\mathrm{RS}}(s, \pi_{1,v}, \pi_{2,v}, \psi_{\mathrm{o},v}) = \epsilon^{\mathrm{LS}}(s, \pi_{1,v}, \pi_{2,v}, \psi_{\mathrm{o},v})$$

where the factors on the right hand side are the Langlands–Shahidi factors of [48]. This is known by [49] in the non-archimedean case, and by [48] in the archimedean case.

9.5. Global factors

As in the previous subsection, k/k_{o} is a quadratic extension of global fields of characteristic different from 2. We denote by $\mathbf{A}$ the ring of adeles of k and by $\mathbf{A}_{\mathrm{o}}$ that of k_{o}. We suppose that all places of k_{o} dividing 2, as well as all archimedean places in the number field case, are split in k.

We fix once and for all a non-trivial character ψ_{o} of $\mathbf{A}_{\mathrm{o}}/k_{\mathrm{o}}$ and a non-zero element $\delta \in k$ such that $\mathrm{tr}_{k/k_0}(\delta) = 0$. Thus

$$\psi_{\mathrm{o}}^{k,\delta} : x \mapsto \psi_{\mathrm{o}}(\mathrm{tr}_{k/k_0}(\delta x)) \tag{9.22}$$

is a non-trivial character of $\mathbf{A}$ trivial on $k + \mathbf{A}_{\mathrm{o}}$. Given a place v of k_{o}, we denote by $\psi_{\mathrm{o},v}$ the local component of ψ_{o} at v.

Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A})$ as in [8]. It decomposes as a restricted tensor product

$$\Pi = \bigotimes_v{}' \, \Pi_v$$

where v ranges over the set of all places of k_{o}. If v is inert in k, then Π_v is the local component of Π at the place of k above v. When v is split in k, and given an isomorphism ϕ_v of $k_{\mathrm{o},v}$-algebras between k_v and $k_{\mathrm{o},v} \oplus k_{\mathrm{o},v}$, the representation $\Pi_v \circ \phi_v$ decomposes as $\Pi_{1,v} \otimes \Pi_{2,v}$.

Note that

$$\mathrm{L}_{\mathrm{As}}(s, \Pi_v) = \mathrm{L}_{\mathrm{As}}^{\mathrm{LS}}(s, \Pi_v) = \mathrm{L}_{\mathrm{As}}^{\mathrm{Gal}}(s, \Pi_v)$$

for any place v of k_{o}. See Theorem A.1 when v is inert, and Remark 9.21 when v is split.

The factors $\mathrm{L}_{\mathrm{As}}(s, \Pi_v)$ and $\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi_v, \psi_{\mathrm{o},v})$ have now been defined at all places of k_{o}. We set

$$\mathrm{L}_{\mathrm{As}}(s, \Pi) = \prod_v \mathrm{L}_{\mathrm{As}}(s, \Pi_v),$$
$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi) = \prod_v \epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi_v, \psi_{\mathrm{o},v}), \qquad \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \Pi) = \prod_v \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \Pi_v, \psi_{\mathrm{o},v})$$

where the products are taken over all places v of k_{o}.

Note that by [50] and [32], the factor $\epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \Pi)$ is indeed independent of the character (9.22) and one has the functional equation

$$\mathrm{L}_{\mathrm{As}}(s, \Pi) = \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \Pi) \cdot \mathrm{L}_{\mathrm{As}}(1 - s, \Pi^{\vee}). \tag{9.23}$$

In fact, we claim that with our normalization (see Definitions 9.10 and 9.15),

$$\mathrm{L}_{\mathrm{As}}(s, \Pi) = \epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi) \cdot \mathrm{L}_{\mathrm{As}}(1 - s, \Pi^{\vee}). \tag{9.24}$$

Let us prove this claim. Whatever the place v of k_{o} is, the local functional equation is of the form

$$\begin{aligned} \frac{\mathrm{I}_{\mathrm{As}}(1 - s, \widehat{\Phi}_v, \widetilde{\mathrm{W}}_v)}{\mathrm{L}_{\mathrm{As}}(1 - s, \Pi_v^{\vee})} &= \omega_{\Pi_v}(\delta_v)^{n-1} \cdot |\delta|_v^{n(n-1)(s-1/2)/2} \\ &\quad \cdot \lambda(k_v/k_{\mathsf{o},v}, \psi_{\mathsf{o},v})^{n(n-1)/2} \cdot \epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi_v, \psi_{\mathsf{o},v}) \cdot \frac{\mathrm{I}_{\mathrm{As}}(s, \Phi_v, \mathrm{W}_v)}{\mathrm{L}_{\mathrm{As}}(s, \Pi_v)}. \end{aligned} \tag{9.25}$$

By [29, Proposition 5] and [19, Section 2, Proposition], together with local multiplicity 1 for Whittaker functionals, if we take a decomposable global Schwartz function Φ and a decomposable global Whittaker function W in the global Whittaker model of Π, we have

$$\prod_v \mathrm{I}_{\mathrm{As}}(1 - s, \widehat{\Phi}_v, \widetilde{\mathrm{W}}_v) = \prod_v \mathrm{I}_{\mathrm{As}}(s, \Phi_v, \mathrm{W}_v).$$

To be more precise, the left hand side makes sense when the real part of $-s$ is large enough, whereas the right hand side makes sense when the real part of s is large enough, and both terms admit meromorphic continuations to $\mathbf{C}$. It is these meromorphic continuations that are equal.

Now let T be a finite set of places of k_{o}, containing the set of archimedean places, such that for all $v \notin \mathrm{T}$ one has

$$\begin{aligned} \mathrm{I}_{\mathrm{As}}(s, \Phi_v, \mathrm{W}_v) &= \mathrm{L}_{\mathrm{As}}(s, \Phi_v, \mathrm{W}_v), \\ \omega_{\Pi_v}(\delta_v) &= 1, \\ |\delta|_v &= 1, \\ \lambda(k_v/k_{\mathsf{o},v}, \psi_{\mathsf{o}}) &= 1, \end{aligned}$$

hence $\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi_v, \psi_{\mathsf{o},v}) = 1$. Taking the product of the equalities (9.25) for all v, we get

$$\begin{aligned} \mathrm{L}_{\mathrm{As}}(s, \Pi) = \Big(\prod_{v \in \mathrm{T}} & \omega_{\Pi_v}(\delta_v)^{n-1} \cdot |\delta|_v^{n(n-1)(s-1/2)/2} \\ & \cdot \lambda(k_v/k_{\mathsf{o},v}, \psi_{\mathsf{o}})^{n(n-1)/2} \cdot \epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi_v, \psi_{\mathsf{o},v})\Big) \cdot \mathrm{L}_{\mathrm{As}}(1 - s, \Pi^{\vee}). \end{aligned}$$

Since $\delta \in k^{\times}$, we have

$$\prod_v \omega_{\Pi_v}(\delta_v) = \omega_{\Pi}(\delta) = 1 \quad \text{and} \quad \prod_v |\delta|_v = 1.$$

On the other hand, we have (see Remark 9.16)

$$\prod_v \lambda(k_v/k_{\mathsf{o},v}, \psi_{\mathsf{o},v}) = \prod_v \epsilon(1/2, \omega_{k_v/k_{\mathsf{o},v}}, \psi_{\mathsf{o},v}) = \epsilon(1/2, \omega_{k/k_{\mathsf{o}}}).$$

However, the global root number $\epsilon(1/2, \omega_{k/k_{\mathrm{o}}})$ is equal to 1 by the dimension 1 case of the main result of [21]. By the assumption on T we get

$$\prod_{v \in \mathrm{T}} \epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi_v, \psi_{\mathrm{o},v}) = \epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi)$$

and (9.24) follows. In particular (9.23) and (9.24) imply:

Theorem 9.26. *Let* Π *be a cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbf{A})$. *Then*

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi) = \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \Pi).$$

Note that the functional equation of [29, Theorem 5] has a different epsilon factor and moreover is up to sign. The presence of this sign is due to the fact that at an inert place v of k_{o}, Kable takes the local factor $\epsilon_{\mathrm{As}}(s, \Pi_v, \psi_{\mathrm{o},v}, \delta_v)$ whereas we take $\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi_v, \psi_{\mathrm{o},v})$.

9.6. Cuspidal representations

Let $\mathrm{F}/\mathrm{F}_{\mathrm{o}}$ be our usual quadratic extension of non-archimedean local fields of residual characteristic different from 2. Fix a global field k_{o} such that $k_{\mathrm{o},w} \simeq \mathrm{F}_{\mathrm{o}}$ at some place w. Write $\mathrm{F} \simeq \mathrm{F}_{\mathrm{o}}[\mathrm{X}]/(\mathrm{P}_w)$ for P_w a polynomial of degree 2 with coefficients in F_{o}. We also fix, whenever v is a place of k_{o} in the set S made up of all archimedean places and all finite places dividing 2, a polynomial $\mathrm{P}_v \in k_{\mathrm{o},v}[\mathrm{X}]$ of degree 2 with simple roots. By the weak approximation lemma, there is a $\mathrm{P} \in k_{\mathrm{o}}[\mathrm{X}]$ of degree 2, as close as we want to P_v in $k_{\mathrm{o},v}[\mathrm{X}]$ for each $v \in \mathrm{S} \cup \{w\}$. Thanks to Krasner's lemma, we take P close enough such that the extension spanned by its roots in the separable closure $\overline{\mathrm{F}}$ of F is equal to F, and such that $k_{\mathrm{o},v}[\mathrm{X}]/(\mathrm{P})$ is split for $v \in \mathrm{S}$. Setting $k = k_{\mathrm{o}}[\mathrm{X}]/(\mathrm{P})$, we have:

- (i) k is split at all archimedean places (when k is a number field) and at all places dividing 2;
- (ii) $k_{\mathrm{o},w} \simeq \mathrm{F}_{\mathrm{o}}$ and $k_w \simeq \mathrm{F}$.

We explain below how to realize a cuspidal representation π of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ as the local component at w of some suitable cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A})$. First, we realize its central character ω_π as a local component of some character Ω of $\mathbf{A}^\times/k^\times$.

Lemma 9.27. *Let* ω *be a unitary character of* $\mathrm{F}^\times$, *and* u *be a finite place of* k_{o} *different from* w. *Then there exists a unitary automorphic character* $\Omega : \mathbf{A}^\times/k^\times \to \mathbf{C}^\times$ *such that:*

- (i) *the local component of* Ω *at* w *is* ω;
- (ii) *for all* $v \neq u, w$, *the local component* Ω_v *is unramified.*

Proof. The subgroup

$$\mathrm{U} = \prod_{v \neq u,w} k_v^{\times 0},$$

where v ranges over all places of k_{o} different from u, w and where $k_v^{\times 0}$ is the maximal compact subgroup of $k_v^\times$, is compact in $\mathbf{A}^\times$, thus $k^\times \mathrm{U}$ is closed in $\mathbf{A}^\times$. The intersection $k^\times \mathrm{U} \cap k_w^\times$ is trivial, thus $k_w^\times$ identifies with a locally compact subgroup of $\mathbf{A}^\times/k^\times \mathrm{U}$. By Pontryagin duality, ω extends to a unitary character Ω of $\mathbf{A}^\times/k^\times \mathrm{U}$, which satisfies the required conditions. □

Lemma 9.28. *Let π be a unitary cuspidal representation of* G*, and u be a finite place of k_{o} which is split in k. Then there is a cuspidal automorphic representation Π of* $\mathrm{GL}_n(\mathbf{A})$ *such that:*

(i) *the local component of Π at w is isomorphic to π;*
(ii) *for all $v \neq u$, w, the local component Π_v is unramified.*

Proof. The proof follows that of [24, Appendice 1], adapted to our context, the reductive group of interest here being the restriction of GL_n from k to k_{o}. Let Ω be a unitary character as in Lemma 9.27 extending the central character ω_π.

For each finite place $v \neq u, w$ of k_{o}, we let f_v denote the complex function on $\mathrm{GL}_n(k_v)$ supported on $k_v^\times \mathrm{GL}_n(\mathcal{O}_{k_v})$ such that $f_v(zk) = \Omega_v(z)$ for all $z \in k_v^\times$ and all $k \in \mathrm{GL}_n(\mathcal{O}_{k_v})$.

If k_{o} is a number field and v is archimedean, we choose a smooth complex function f_v on $\mathrm{GL}_n(k_v)$, compactly supported mod the centre $k_v^\times$, such that $f_v(1) = 1$ and $f_v(zg) = \Omega_v(z) f_v(g)$ for all elements $z \in k_v^\times$ and $g \in \mathrm{GL}_n(k_v)$.

We let f_w be a coefficient of π such that $f_w(1) = 1$.

Finally we choose a smooth complex function f_u on $\mathrm{GL}_n(k_u)$, compactly supported mod the centre, such that $f_u(1) = 1$ and $f_u(zg) = \Omega_u(z) f_u(g)$ for all $z \in k_u^\times$ and $g \in \mathrm{GL}_n(k_u)$, and of support small enough such that

$$f(g^{-1}) f(\gamma g) = 0 \quad \text{for all } g, \gamma \in \mathrm{GL}_n(k) \text{ such that } \gamma \notin k^\times,$$

where f is the product of all the f_v, as in [24, Appendice 1, top of p. 148].

We may also assume that $f_v(g^{-1}) = \overline{f_v(g)}$ for all v and all $g \in \mathrm{GL}_n(k_v)$.

Then there is a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A})$ such that f_v acts non-trivially on Π_v for each place v of k_{o}. In particular $\Pi_w \simeq \pi$ and Π_v is unramified at every place different from w and u. □

Now let us consider a cuspidal representation π of $\mathrm{G} \simeq \mathrm{GL}_n(k_w)$. The character $\omega_\pi |\cdot|_w^{-s}$ is unitary for some $s \in \mathbf{C}$, thus $\pi_1 = \pi |\det|_w^{-s}$ is unitary. Lemma 9.27 gives us a cuspidal automorphic representation Π_1 of $\mathrm{GL}_n(\mathbf{A})$. Denoting by $|\cdot|$ the idelic norm on $\mathbf{A}^\times / k^\times$, the cuspidal automorphic representation $\Pi = \Pi_1 |\det|^s$ has a local component at w isomorphic to π, and all its local compotents at $v \neq w, u$ are unramified.

We recalled in Remark 9.21 that

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi_v, \psi_{\mathsf{o},v}) = \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \Pi_v, \psi_{\mathsf{o},v})$$

when v is split (in particular when $v = u$), hence from Theorems 9.13 and 9.26, we get

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \Pi_w, \psi_{\mathsf{o},w}) = \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \Pi_w, \psi_{\mathsf{o},w}).$$

Thus we have proved:

Theorem 9.29. *Let π be a cuspidal representation of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ and ψ_{o} be a non-trivial character of F_{o}. Then*

$$\epsilon_{\mathrm{As}}^{\mathrm{RS}}(s, \pi, \psi_{\mathsf{o}}) = \epsilon_{\mathrm{As}}^{\mathrm{LS}}(s, \pi, \psi_{\mathsf{o}}).$$

When π is cuspidal, Theorem 9.29 tells us that

$$\epsilon_{\mathrm{As}}(1/2, \pi, \psi_{\mathrm{o}}, \delta) = \omega_\pi(\delta)^{n-1} \cdot \lambda(\mathrm{F}/\mathrm{F_o}, \psi_{\mathrm{o}})^{-n(n-1)/2} \cdot \epsilon_{\mathrm{As}}^{\mathrm{LS}}(1/2, \pi, \psi_{\mathrm{o}}). \tag{9.30}$$

If in addition π is distinguished, then combining this equality with Theorem 8.4 and since $\omega_\pi(\delta^{-1}) = \omega_\pi(\delta)$ as π is distinguished and $\delta^2 \in \mathrm{F}_{\mathrm{o}}^\times$, we recover [1, Theorem 1.1] for distinguished cuspidal representations.

Remark 9.31. When π is cuspidal and $\omega_{\mathrm{F}/\mathrm{F}_0}$-distinguished, we may go in the opposite direction: applying [1, Theorem 1.1] together with (9.30) gives us the value of $\epsilon_{\mathrm{As}}(1/2, \pi, \psi_{\mathrm{o}}, \delta)$ when π is cuspidal and $\omega_{\mathrm{F}/\mathrm{F}_0}$-distinguished.

Remark 9.32. It is shown in [1, Theorem 1.2] that the global Asai root number of a σ-self-dual cuspidal automorphic representation is 1. Hence, by Theorem 9.26, the same holds for the Asai factor defined via the Rankin–Selberg method. Globalizing local distinguished cuspidal representations as local components of distinguished cuspidal automorphic representations as in [44] or [22] and following the methods of [1], it is possible to prove that $\epsilon_{\mathrm{As}}(1/2, \pi, \psi_{\mathrm{o}}, \delta) = 1$ by global methods as well. However, our proof in this paper has the advantage of being purely local.

Remark 9.33. In [7], Beuzart-Plessis extends Theorem 9.29 to all generic representations, using a global method as well.

Appendix A. Some remarks in positive characteristic

We use the notation of Section 2 and §9.1. In particular, G denotes the group $\mathrm{GL}_n(\mathrm{F})$ for some $n \geq 1$, and we have defined Asai local L-factors $\mathrm{L}_{\mathrm{As}}(s, \pi)$, $\mathrm{L}_{\mathrm{As}}^{\mathrm{LS}}(s, \pi)$ and $\mathrm{L}_{\mathrm{As}}^{\mathrm{Gal}}(s, \pi)$ for all generic irreducible representations of G. We will first prove that these factors are all equal.

Theorem A.1. *For any generic irreducible complex representation π of* G, *we have*

$$\mathrm{L}_{\mathrm{As}}(s, \pi) = \mathrm{L}_{\mathrm{As}}^{\mathrm{Gal}}(s, \pi) = \mathrm{L}_{\mathrm{As}}^{\mathrm{LS}}(s, \pi).$$

Proof. When F has characteristic 0, this follows from [34, 35, 36] and [5], [25].

Now we notice that the local results in [29, Section 3] hold in positive characteristic, and the global results of [29, Section 4] also hold in positive characteristic though written in characteristic 0 only. Indeed they refer to [19] which is for any global field. The main point is that [29, Theorem 5] is true for function fields, and its proof slightly simplifies because of the absence of archimedean places. This implies that when F has characteristic $p \neq 2$, the equality

$$\mathrm{L}_{\mathrm{As}}(s, \pi) = \mathrm{L}_{\mathrm{As}}^{\mathrm{LS}}(s, \pi)$$

holds for any discrete series representation: the ingredients which make the proof of [5, Theorem 1.6] work are then all available. Once again, notice that its proof simplifies in positive characteristic as there are no archimedean places to worry about.

Now notice that [35, Theorem 3.1] holds when F has characteristic p. Indeed, its proof relies on [40, Theorem 3.1.2] which is for any non-archimedean local field of odd residual characteristic. Then the classification of generic distinguished representations in [36] relies only on the geometric lemma of Bernstein–Zelevinsky, the Bernstein–Zelevinsky explicit description of discrete series representations and their Jacquet modules, and the fact that a distinguished irreducible representation of G is σ-self-dual. All the aforementioned results are true in positive characteristic (different from 2 for the latter), hence the classification of [36] still holds when F has characteristic p.

Finally, the Cogdell–Piatetski-Shapiro method of derivatives to analyze the exceptional poles used in [34] works in positive characteristic as well (for example the original paper [18] is written in arbitrary characteristic), hence the inductivity relation of $\mathrm{L}_{\mathrm{As}}(s,\pi)$ for any generic irreducible representation (see [34, Proposition 4.22]) follows. All in all, when F has characteristic p, we have

$$\mathrm{L}_{\mathrm{As}}(s,\pi) = \mathrm{L}_{\mathrm{As}}^{\mathrm{Gal}}(s,\pi)$$

for any generic irreducible representation. □

We now prove that the dichotomy theorem of [29] and [2] holds when F has characteristic p.

Theorem A.2. *Let π be a σ-self-dual discrete series representation of* G. *Then π is either distinguished or $\omega_{\mathrm{F}/\mathrm{F}_\mathrm{o}}$-distinguished, but not both.*

Proof. When F has characteristic 0, this is [29, Theorem 4] and [2, Corollary 1.6].

Assume that F has characteristic p. If π is a discrete series representation of G and ω is a character of $\mathrm{F}^\times$ extending $\omega_{\mathrm{F}/\mathrm{F}_0}$, the equality

$$\mathrm{L}(s,\pi,\pi^\sigma) = \mathrm{L}_{\mathrm{As}}(s,\pi)\cdot\mathrm{L}_{\mathrm{As}}(s,\omega\otimes\pi)$$

becomes a consequence of the relation

$$\mathrm{Ind}'_{\mathrm{F}/\mathrm{F}_\mathrm{o}}(\rho(\pi)\otimes\rho(\pi)^\sigma) = \mathrm{M}'_{\mathrm{F}/\mathrm{F}_\mathrm{o}}(\rho(\pi))\oplus\omega_{\mathrm{F}/\mathrm{F}_\mathrm{o}}\mathrm{M}'_{\mathrm{F}/\mathrm{F}_\mathrm{o}}(\rho(\pi)).$$

Then, if π is σ-self-dual, the Rankin–Selberg local L-factor $\mathrm{L}^{\mathrm{RS}}(s,\pi,\pi^\sigma)$ has a simple pole at $s=0$ according to [28, Proposition 8.1 and Theorem 8.2], hence either $\mathrm{L}_{\mathrm{As}}(s,\pi)$ or $\mathrm{L}_{\mathrm{As}}(s,\omega\otimes\pi)$ has a pole at $s=0$ but not both. Finally one concludes appealing to [35, Proposition 3.4] (the paper [35] is valid for F of characteristic p as it only relies on the paper [40] which is true in this setting). □

Notice that [46] gives a purely local proof of Theorem A.2 when π is cuspidal.

Appendix B. Modular versions of results by Bruhat, Kable and Ok

In this appendix, which culminates in §B.3, we generalize three results which were known for complex representations only.

In §B.1, we generalize a result of Bruhat on equivariant distributions to the case of smooth representations of a locally profinite group with coefficients in an (almost) arbitrary commutative ring. For complex representations, a formal proof can be found in an unpublished version of Rodier's paper [45] on Whittaker models. The result is also stated in [45] as Theorem 4 and invokes Bruhat's thesis as a reference.

B.1. A modular version of a result of Bruhat on equivariant distributions

In this subsection, G is a locally profinite group, H is a closed subgroup of G, and R is a commutative ring with unit. We assume that there is a right invariant R-valued measure dh on H giving measure 1 to some compact open subgroup of H. According to [54, I.2.4], this is equivalent to assuming that H has a compact open subgroup whose pro-order is invertible in R.

Let ρ be a smooth representation of H on an R-module V. Write $\mathcal{C}_c^\infty(\mathsf{G},\mathsf{V})$ for the space of locally constant, compactly supported functions on G with values in V, which canonically identifies with $\mathcal{C}_c^\infty(\mathsf{G},\mathsf{R})\otimes\mathsf{V}$, and write $\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\rho)$ for the compact induction of ρ to G. Both are equipped with an action of G by right translations, denoted $g\cdot f : x\mapsto f(xg)$ for all $g,x\in\mathsf{G}$.

We denote by $\delta=\delta_{\mathsf{H}}$ the character of H such that $d(xh)=\delta(x)dh$ for all $x\in\mathsf{H}$, that is,

$$\int_{\mathsf{H}} f(xh)\,dh = \delta(x)^{-1}\cdot\int_{\mathsf{H}} f(h)\,dh$$

for all $f\in\mathcal{C}_c^\infty(\mathsf{H},\mathsf{R})$ and $x\in\mathsf{H}$. We will use the fact that

$$\int_{\mathsf{H}} f(h^{-1})\,dh = \int_{\mathsf{H}} \delta(h)^{-1} f(h)\,dh$$

as well as the fact that the restriction of δ to any compact open subgroup of H is trivial.

We start with the following lemma, proved by Rodier in [45, Proposition 1]. Unlike Rodier, we use unnormalized induction.

Lemma B.1. (i) *For all* $f\in\mathcal{C}_c^\infty(\mathsf{G},\mathsf{V})$, *the function*

$$\mathrm{p}(f) : g\mapsto \int_{\mathsf{H}} \tau(h^{-1})f(hg)\,dh$$

is in $\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\tau)$.

(ii) *The map* $\mathrm{p}:\mathcal{C}_c^\infty(\mathsf{G},\mathsf{V})\to\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\tau)$ *defined in* (i) *is surjective.*

(iii) *The map* p *is* G*-equivariant and, for all* $f\in\mathcal{C}_c^\infty(\mathsf{G},\mathsf{V})$ *and* $x\in\mathsf{H}$, *one has*

$$\mathrm{p}(f_x)=\mathrm{p}(\delta(x)^{-1}\tau(x)f)$$

where f_x *is the function* $g\mapsto f(xg)$.

Proof. First, we prove (i). For all $g \in \mathsf{G}$, the integral

$$\int_{\mathsf{H}} \tau(h^{-1}) f(hg)\, dh$$

is a finite sum since f is smooth and compactly supported. For any $x \in \mathsf{H}$ and $g \in \mathsf{G}$,

$$\mathrm{p}(f)(xg) = \int_{\mathsf{H}} \tau(h^{-1}) f(hxg)\, dh = \int_{\mathsf{H}} \tau(xh^{-1}) f(hg)\, dh = \tau(x)(\mathrm{p}(f)(g))$$

since dh is right invariant. It follows that $\mathrm{p}(f)$ is in $\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\tau)$.

Let us prove (ii). Given $v \in \mathrm{V}$ and $g \in \mathsf{G}$, there is an open subgroup J of G such that $\mathsf{H} \cap g\mathrm{J}g^{-1}$ leaves v invariant and its measure is invertible in R. Let $\phi : \mathsf{G} \to \mathrm{V}$ be the function supported in $\mathsf{H}g\mathrm{J}$ and defined by $\phi(hgj) = \tau(h)v$ for all $h \in \mathsf{H}$ and $j \in \mathrm{J}$. It belongs to $\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\tau)$, and the linear span of all such maps is the full induced representation, hence it suffices to show that such a ϕ is in the image of p to prove that p is surjective. Let $f : \mathsf{G} \to \mathrm{V}$ be the function supported in $g\mathrm{J}$ and defined for all $x \in g\mathrm{J}$ by

$$f(x) = \frac{1}{dh(\mathsf{H} \cap g\mathrm{J}g^{-1})} \cdot v.$$

One checks immediately that $f \in \mathcal{C}_c^\infty(\mathsf{G}, \mathrm{V})$ and $\mathrm{p}(f) = \phi$.

Finally, let us prove (iii). One has

$$\mathrm{p}(f_x) = \int_{\mathsf{H}} \tau(h^{-1}) f(xhg)\, dh = \int_{\mathsf{H}} \tau(h^{-1}x) f(hg) \delta(x)^{-1}\, dh$$
$$= \int_{\mathsf{H}} \tau(h^{-1})(\delta(x)^{-1}\tau(x) f(hg))\, dh,$$

which is indeed equal to $\mathrm{p}(\delta(x)^{-1}\tau(x) f)$. □

Remark B.2. One can reformulate Lemma B.1(iii) as follows. The space $\mathcal{C}_c^\infty(\mathsf{G}, \mathrm{V})$ has an action of G by right translations, as well as an action of H defined by

$$x \odot f : g \mapsto \delta(x)^{-1}\tau(x) f(x^{-1}g) \tag{B.3}$$

for all $x \in \mathsf{H}$ and $f \in \mathcal{C}_c^\infty(\mathsf{G}, \mathrm{V})$. Then the map p is G-equivariant and H-invariant.

Recall that, given $f \in \mathcal{C}_c^\infty(\mathsf{G}, \mathrm{V})$ and $x \in \mathsf{H}$, we write f_x for the function $g \mapsto f(xg)$.

Lemma B.4. (i) *Let $\mathscr{L}$ be a linear form on $\mathcal{C}_c^\infty(\mathsf{G}, \mathrm{V})$ such that*

$$\mathscr{L}(f_x) = \mathscr{L}(\delta(x)^{-1}\tau(x) f) \tag{B.5}$$

for all $f \in \mathcal{C}_c^\infty(\mathsf{G}, \mathrm{V})$ and $x \in \mathsf{H}$. Then there is a unique linear form $\mathscr{L}'$ on $\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\tau)$ such that $\mathscr{L} = \mathscr{L}' \circ \mathrm{p}$.

(ii) *The map $\mathscr{L}' \mapsto \mathscr{L}' \circ \mathrm{p}$ is an isomorphism of R-modules:*

$$\mathrm{Hom}_{\mathsf{R}}(\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\tau), \mathsf{R}) \simeq \mathrm{Hom}_{\mathsf{H}}(\mathcal{C}_c^\infty(\mathsf{G}, \mathrm{V}), \mathsf{R})$$

where the right hand side is made up of all linear forms on $\mathcal{C}_c^\infty(\mathsf{G}, \mathrm{V})$ satisfying (B.5).

Proof. First, we notice that if $\mathscr{L}'$ is a linear form on $\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\tau)$, then $\mathscr{L} = \mathscr{L}' \circ \mathrm{p}$ is a linear form on $\mathcal{C}_c^{\infty}(\mathsf{G}, \mathrm{V})$ satisfying (B.5), by Lemma B.1.

Now let $\mathscr{L}$ be a linear form on $\mathcal{C}_c^{\infty}(\mathsf{G}, \mathrm{V})$ satisfying (B.5). We denote by p' the natural projection from $\mathcal{C}_c^{\infty}(\mathsf{G}, \mathrm{R})$ onto $\mathcal{C}_c^{\infty}(\mathsf{H}\backslash\mathsf{G}, \mathrm{R})$ defined by

$$\mathrm{p}'(\phi)(g) = \int_{\mathsf{H}} \phi(hg)\, dh.$$

Note that $\mathcal{C}_c^{\infty}(\mathsf{H}\backslash\mathsf{G}, \mathrm{R})$ is the compact induction $\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\mathbf{1})$ of the trivial R-character of H, thus p' is a particular case of the projection given by Lemma B.1 when one chooses for ρ the trivial character. Fix $f \in \mathcal{C}_c^{\infty}(\mathsf{G}, \mathrm{V})$ and $\phi \in \mathcal{C}_c^{\infty}(\mathsf{G}, \mathrm{R})$, and notice that both $\phi\mathrm{p}(f)$ and $\mathrm{p}'(\phi) f$ considered as functions on G are in $\mathcal{C}_c^{\infty}(\mathsf{G}, \mathrm{V})$. We are going to prove that

$$\mathscr{L}(\phi\mathrm{p}(f)) = \mathscr{L}(\mathrm{p}'(\phi) f).$$

Let K be a compact open subgroup of H leaving f and ϕ fixed under left translations, and acting trivially on all vectors in the image of f (which is possible because the linear span of the image of f in V is finite-dimensional). One defines the compact subset

$$\mathrm{C} = \mathrm{K}\big[\big(\mathrm{supp}(f)\,\mathrm{supp}(\phi)^{-1} \cup \mathrm{supp}(\phi)\,\mathrm{supp}(f)^{-1}\big) \cap \mathsf{H}\big]\mathrm{K}$$

of H. It is stable under $x \mapsto x^{-1}$ and K-bi-invariant. We claim that there is a compact open subgroup U of K such that:

(i) $x\mathrm{U}x^{-1} \subseteq \mathrm{K}$ for all $x \in \mathrm{C}$;
(ii) the pro-order of U is invertible in R.

Indeed, consider the continuous function $\mu : \mathsf{H} \times \mathsf{H} \to \mathsf{H}$ defined by $(x, y) \mapsto xyx^{-1}$. The preimage $\mu^{-1}(\mathrm{K})$ is an open subset of $\mathsf{H} \times \mathsf{H}$ containing $\mathrm{C} \times \{1\}$. For all $x \in \mathrm{C}$, there are an open neighbourhood B_x of x and a compact open subgroup U_x of H such that $\mathrm{B}_x \times \mathrm{U}_x$ is contained in $\mu^{-1}(\mathrm{K})$. As C is compact, $\mathrm{C} \times \{1\}$ is contained in the union of finitely many $\mathrm{B}_x \times \mathrm{U}_x$. The intersection U of these finitely many U_x satisfies (i). To get (ii), one chooses a small enough open subgroup of U.

We are now in a position to prove the sought equality. First notice that, for all $g \in \mathsf{G}$, the function $h \mapsto \phi(g)\tau(h)^{-1} f(hg)$ is constant on any U-double coset in C. Let A be a set of representatives of $\mathrm{U}\backslash\mathrm{C}/\mathrm{U}$. Writing $k(a) = dh(\mathrm{U}a\mathrm{U})$ for all $a \in \mathrm{A}$, and noticing that $\phi(g)\tau(h)^{-1} f(hg)$ vanishes when $h \notin \mathrm{supp}(f)\,\mathrm{supp}(\phi)^{-1}$, we get

$$\phi\mathrm{p}(f)(g) = \int_{\mathrm{C}} \phi(g)\tau(h)^{-1} f(hg)\, dh = \sum_a k(a)\phi(g)\tau(a^{-1}) f_a(g) \tag{B.6}$$

for all $g \in \mathsf{G}$, where a ranges over A.

Similarly, using the formula

$$\mathrm{p}'(\phi)(g) = \int_{\mathsf{H}} \phi(hg)\, dh = \int_{\mathsf{H}} \delta(h)^{-1}\phi(h^{-1}g)\, dh$$

for all $g \in \mathsf{G}$, one has

$$\mathrm{p}'(\phi) f = \sum_a k(a)\delta(a)^{-1}\phi_{a^{-1}} f.$$

Hence

$$\mathscr{L}(\mathrm{p}'(\phi)f) = \sum_a k(a)\mathscr{L}(\delta(a)^{-1}\phi_{a^{-1}}f) = \sum_a k(a)\mathscr{L}(\delta(a)^{-1}(\phi f_a)_{a^{-1}})$$
$$= \sum_a k(a)\mathscr{L}(\tau(a)^{-1}\phi f_a),$$

which is equal to $\mathscr{L}(\phi\mathrm{p}(f))$ by (B.6).

Now, by Lemma B.1(ii), there is $\phi \in \mathcal{C}_c^\infty(\mathsf{G}, \mathsf{R})$ such that $\mathrm{p}'(\phi)$ is equal to 1 on $\mathrm{supp}(f)$. For such a ϕ, one has $\mathscr{L}(f) = \mathscr{L}(\phi\mathrm{p}(f))$. Hence the kernel of p is contained in that of $\mathscr{L}$, which proves Lemma B.4. □

Now let H' be another closed subgroup of G and χ be an R-character of H'. Let $\mathscr{L}$ be a linear form as in Lemma B.4 and suppose that

$$\mathscr{L}(y \cdot f) = \chi(y)\mathscr{L}(f), \qquad f \in \mathcal{C}_c^\infty(\mathsf{G}, \mathsf{V}),\ y \in \mathsf{H}'. \tag{B.7}$$

Then, by uniqueness of the linear form $\mathscr{L}'$ corresponding to $\mathscr{L}$, one has

$$\mathscr{L}'(y \cdot \phi) = \chi(y)\mathscr{L}'(\phi), \qquad \phi \in \mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\tau). \tag{B.8}$$

We arrive at the following result which we shall use many times.

Corollary B.9. *The map $\mathscr{L}' \mapsto \mathscr{L}' \circ \mathrm{p}$ is an isomorphism of* R*-modules between:*

(i) *linear forms $\mathscr{L}'$ on $\mathrm{ind}_{\mathsf{H}}^{\mathsf{G}}(\tau)$ satisfying* (B.8)*, and*
(ii) *linear forms $\mathscr{L}$ on $\mathcal{C}_c^\infty(\mathsf{G}, \mathsf{V})$ satisfying* (B.5) *and* (B.7)*.*

B.2. A modular version of a result of Kable

In this subsection, we generalize a result of Kable [29, Proposition 1] to the case of smooth representations of $\mathrm{GL}_n(\mathrm{F})$ with coefficients in a commutative ring with sufficiently many roots of unity of p-power order and in which p is invertible. In fact, we expand and simplify Kable's proof, appealing to Lemma B.4 when he appeals to Warner [55].

We go back to the main notation of the paper: G is the group $\mathrm{GL}_n(\mathrm{F})$ where $\mathrm{F}/\mathrm{F}_\mathsf{o}$ is a quadratic extension, σ is the Galois involution and P is the mirabolic subgroup of G. We also write G' for the group $\mathrm{GL}_{n-1}(\mathrm{F})$ considered as a subgroup of G in the usual way, and P' for the mirabolic subgroup of G'. Denoting by U the unipotent radical of P, one has the semidirect product decomposition $\mathrm{P} = \mathrm{G}'\mathrm{U}$.

We also assume that R is a commutative ring with unit such that p is invertible in R and there is a non-trivial R-character ψ_o of F_o.

Let ψ_U be the restriction to U of the standard σ-self-dual non-degenerate character ψ of N defined by (8.2) for some non-zero $\delta \in \mathrm{F}^\times$ of trace 0.

Since p is invertible in R and G is locally pro-p, there is a non-zero right invariant measure dh on $\mathrm{P}'\mathrm{U}$ with values in R, giving measure 1 to some compact open subgroup.

Given any smooth representation τ of P' on an R-module V, we denote by $\tau \otimes \psi_{\mathrm{U}}$ the representation of $\mathrm{P}'\mathrm{U}$ defined by

$$\tau \otimes \psi_{\mathrm{U}} : xu \mapsto \psi_{\mathrm{U}}(u)\tau(x)$$

for $x \in \mathrm{P}'$ and $u \in \mathrm{U}$. Following [6], we set

$$\Phi^{+}(\tau) = \mathrm{ind}_{\mathrm{P}'\mathrm{U}}^{\mathrm{P}}(\tau \otimes \psi_{\mathrm{U}}).$$

This defines a functor from smooth R-representations of P' to smooth R-representations of P. Note that, since we use the unnormalized version of the functor Φ^{+} as in [6], we do not have to worry about the existence of a square root of q in R.

We will write ν and ν_{o} for the unramified characters $g \mapsto |\det(g)|$ and $g \mapsto |\det(g)|_{\mathrm{o}}$, respectively.

Proposition B.10. *For any smooth* R*-representation* τ *of* P' *and any character* χ *of* P^{σ}*, one has an isomorphism*

$$\mathrm{Hom}_{\mathrm{P}^{\sigma}}(\Phi^{+}(\tau), \chi) \simeq \mathrm{Hom}_{\mathrm{P}'^{\sigma}}(\tau, \chi\nu_{\mathrm{o}})$$

of R*-modules.*

Proof. First, we apply Corollary B.9 with $\mathsf{G} = \mathrm{P}$, $\mathsf{H} = \mathrm{P}'\mathrm{U}$ and $\rho = \tau \otimes \psi_{\mathrm{U}}$. Since the character $\delta_{\mathrm{P}'\mathrm{U}}$ associated with $\mathrm{P}'\mathrm{U}$ is equal to ν^{2}, we get an isomorphism of R-modules from $\mathrm{Hom}_{\mathrm{P}^{\sigma}}(\Phi^{+}(\tau), \chi)$ to the space of all linear forms T on $\mathcal{C}_c^{\infty}(\mathrm{P}, \mathrm{V})$ such that

$$\mathrm{T}(g_{\mathrm{o}} \cdot f) = \chi(g'_{\mathrm{o}}) \cdot \mathrm{T}(f), \tag{B.11}$$
$$\mathrm{T}(u_{\mathrm{o}} \cdot f) = \mathrm{T}(f), \tag{B.12}$$
$$\mathrm{T}(f_g) = \mathrm{T}(\nu(g)^{-2}\tau(g)f), \tag{B.13}$$
$$\mathrm{T}(f_u) = \psi_{\mathrm{U}}(u)\mathrm{T}(f), \tag{B.14}$$

for all $g_{\mathrm{o}} \in \mathrm{G}'^{\sigma}$, $u_{\mathrm{o}} \in \mathrm{U}^{\sigma}$, $g \in \mathrm{P}'$, $u \in \mathrm{U}$ and $f \in \mathcal{C}_c^{\infty}(\mathrm{P}, \mathrm{V})$. We now consider the R-linear map $\mathscr{A}$ from $\mathcal{C}_c^{\infty}(\mathrm{P}, \mathrm{V})$ to $\mathcal{C}_c^{\infty}(\mathrm{P}'\mathrm{G}'^{\sigma}, \mathrm{V})$ defined by

$$\mathscr{A}(f) : x \mapsto \int_{\mathrm{U}} \psi_{\mathrm{U}}^{-1}(u) f(ux)\, du$$

for all $f \in \mathcal{C}_c^{\infty}(\mathrm{P}, \mathrm{V})$ and $x \in \mathrm{P}'\mathrm{G}'^{\sigma}$, where du is some right invariant measure on U. It is obtained by composing the map

$$f \mapsto \left(x \mapsto \int_{\mathrm{U}} \psi_{\mathrm{U}}^{-1}(u) f(ux)\, du\right),$$

with $x \in \mathrm{G}'$, with the restriction map from $\mathcal{C}_c^{\infty}(\mathrm{G}', \mathrm{V})$ to $\mathcal{C}_c^{\infty}(\mathrm{P}'\mathrm{G}'^{\sigma}, \mathrm{V})$. The former is surjective since $\mathcal{C}_c^{\infty}(\mathrm{P}, \mathrm{V})$ canonically identifies with $\mathcal{C}_c^{\infty}(\mathrm{U}, \mathrm{R}) \otimes \mathcal{C}_c^{\infty}(\mathrm{G}', \mathrm{V})$, and so is the latter since $\mathrm{P}'\mathrm{G}'^{\sigma}$ is a closed subset of G' (being made up of all matrices in G' with the last row fixed by σ). Thus the adjoint map

$$\mathscr{A}^{*} : \mathrm{Hom}_{\mathrm{R}}(\mathcal{C}_c^{\infty}(\mathrm{P}'\mathrm{G}'^{\sigma}, \mathrm{V}), \mathrm{R}) \to \mathrm{Hom}_{\mathrm{R}}(\mathcal{C}_c^{\infty}(\mathrm{P}, \mathrm{V}), \mathrm{R}) \tag{B.15}$$

is injective. We claim that its image is the space of all linear forms T satisfying (B.12) and (B.14).

First, let us check that the image of $\mathscr{A}^*$ is contained in that space. Indeed, given a linear form S in the left hand side of (B.15) and $f \in \mathcal{C}_c^\infty(\mathrm{P}, \mathrm{V})$, one has $\mathscr{A}(f_u) = \psi_{\mathrm{U}}(u)\mathscr{A}(f)$ for all $u \in \mathrm{U}$ and

$$\mathscr{A}(u_\circ \cdot f)(x) = \int_{\mathrm{U}} \psi_{\mathrm{U}}^{-1}(u) f(uxu_\circ)\, du = \psi_{\mathrm{U}}^{-1}(xu_\circ x^{-1}) \int_{\mathrm{U}} \psi_{\mathrm{U}}^{-1}(u) f(ux)\, du$$

for all $x \in \mathrm{P}'\mathrm{G}'^\sigma$ and $u_\circ \in \mathrm{U}^\sigma$. Since $\psi_{\mathrm{U}}(xu_\circ x^{-1}) = 1$ for all $x \in \mathrm{P}'\mathrm{G}'^\sigma$, this is equal to $\mathscr{A}(f)(x)$ as expected. (Note that we have used the fact that ψ_{U} is trivial on U^σ.) To prove surjectivity, we follow [29, second paragraph of the proof of Proposition 1, p. 797].

Now consider a linear form S on $\mathcal{C}_c^\infty(\mathrm{P}'\mathrm{G}'^\sigma, \mathrm{V})$. We check immediately that $\mathscr{A}^*(\mathrm{S}) = \mathrm{S} \circ \mathscr{A}$ satisfies (B.11) if and only if

$$\mathrm{S}(g'_\circ \cdot f) = \chi(g'_\circ) \cdot \mathrm{S}(f) \tag{B.16}$$

for all $f \in \mathcal{C}_c^\infty(\mathrm{P}'\mathrm{G}'^\sigma, \mathrm{V})$ and $g'_\circ \in \mathrm{G}'^\sigma$. On the other hand, $\mathrm{S} \circ \mathscr{A}$ satisfies (B.13) if and only if

$$\mathrm{S}(f_{p'}) = \mathrm{S}(\nu^{-1}(p')\tau(p')f) \tag{B.17}$$

for all $f \in \mathcal{C}_c^\infty(\mathrm{P}'\mathrm{G}'^\sigma, \mathrm{V})$ and $p' \in \mathrm{P}'$. Indeed, notice that

$$\begin{aligned}\mathscr{A}(f_{p'})(x) &= \int_{\mathrm{U}} \psi_{\mathrm{U}}^{-1}(u) f(p'ux)\, du = \int_{\mathrm{U}} \psi_{\mathrm{U}}^{-1}(p'up'^{-1}) f(up'x)\nu^{-1}(p')\, du \\ &= \int_{\mathrm{U}} \psi_{\mathrm{U}}^{-1}(u) f(up'x)\nu^{-1}(p')\, du\end{aligned}$$

for all $f \in \mathcal{C}_c^\infty(\mathrm{P}, \mathrm{V})$, $x \in \mathrm{P}'\mathrm{G}'^\sigma$ and $p' \in \mathrm{P}'$, where the second equality follows from the fact that the character $\delta_{\mathrm{P}'}$ associated with P' is ν, and the third one from the fact that P' normalizes ψ_{U}. It follows that $\mathscr{A}^*$ induces an isomorphism of R-modules between:

(i) the space of linear forms T on $\mathcal{C}_c^\infty(\mathrm{P}, \mathrm{V})$ satisfying (B.11)–(B.14), and
(ii) the space of linear forms S on $\mathcal{C}_c^\infty(\mathrm{P}'\mathrm{G}'^\sigma, \mathrm{V})$ satisfying (B.16) and (B.17).

Now consider the map $(x, y) \mapsto x^{-1}y$ from $\mathrm{P}' \times \mathrm{G}'^\sigma$ onto $\mathrm{P}'\mathrm{G}'^\sigma$. It identifies $\mathrm{P}'\mathrm{G}'^\sigma$ with the homogeneous space $\mathrm{P}'^\sigma \backslash (\mathrm{P}' \times \mathrm{G}'^\sigma)$ where $\mathrm{P}'^\sigma = \mathrm{P}' \cap \mathrm{G}'^\sigma$ is diagonally embedded in $\mathrm{P}' \times \mathrm{G}'^\sigma$. This identifies the space $\mathcal{C}_c^\infty(\mathrm{P}'\mathrm{G}'^\sigma, \mathrm{V})$ with the compact induction $\mathrm{ind}_{\mathrm{P}'^\sigma}^{\mathrm{P}' \cap \mathrm{G}'^\sigma}(\mathbf{1} \otimes \mathrm{V})$ where $\mathbf{1} \otimes \mathrm{V}$ denotes the trivial representation of P'^σ on V. Namely, $f \in \mathcal{C}_c^\infty(\mathrm{P}'\mathrm{G}'^\sigma, \mathrm{V})$ identifies with the function ϕ on $\mathrm{P}' \cap \mathrm{G}'^\sigma$ defined by $\phi(x, y) = f(x^{-1}y)$ for $(x, y) \in \mathrm{P}' \cap \mathrm{G}'^\sigma$. This gives us an isomorphism of R-modules between:

(i) the space of linear forms S on $\mathcal{C}_c^\infty(\mathrm{P}'\mathrm{G}'^\sigma, \mathrm{V})$ satisfying (B.16) and (B.17), and
(ii) the space of linear forms Q on $\mathrm{ind}_{\mathrm{P}'^\sigma}^{\mathrm{P}' \cap \mathrm{G}'^\sigma}(\mathbf{1} \otimes \mathrm{V})$ such that

$$\mathrm{Q}((p', g'_\circ) \cdot \phi) = \chi(g'_\circ) \cdot \mathrm{Q}(\nu(p')\tau(p'^{-1})\phi) \tag{B.18}$$

for all $\phi \in \mathrm{ind}_{\mathrm{P}'^\sigma}^{\mathrm{P}' \cap \mathrm{G}'^\sigma}(\mathbf{1} \otimes \mathrm{V})$ and $(p', g'_\circ) \in \mathrm{P}' \times \mathrm{G}'^\sigma$.

We now apply Corollary B.9 again, with $G = P' \times G'^{\sigma}$, $H = P'^{\sigma}$ and $\rho = \mathbf{1} \otimes V$. Since the character $\delta_{P'^{\sigma}}$ associated with P'^{σ} is equal to ν_{o}, we get an isomorphism of R-modules between:

(i) the space of linear forms Q on $\mathrm{ind}_{P'^{\sigma}}^{P' \cap G'^{\sigma}}(\mathbf{1} \otimes V)$ satisfying (B.18), and
(ii) the space of linear forms L on $\mathcal{C}_c^{\infty}(P' \times G'^{\sigma}, V)$ such that

$$L((p', g'_{o}) \cdot \phi) = \chi(g'_{o}) \cdot L(\nu(p')\tau(p'^{-1})\phi), \tag{B.19}$$
$$L(\phi_{p'_{o}}) = \nu_{o}^{-1}(p'_{o}) \cdot L(\phi), \tag{B.20}$$

for all $\phi \in \mathcal{C}_c^{\infty}(P' \times G'^{\sigma}, V)$, $p'_{o} \in P'^{\sigma}$ and $(p', g'_{o}) \in P' \times G'^{\sigma}$.

For ϕ and L as above, we define $\phi^{\vee} : (x, y) \mapsto \phi(x^{-1}, y^{-1})$ and $M(\phi) = L(\phi^{\vee})$. This defines an isomorphism of R-modules between:

(i) the space of linear forms L on $\mathcal{C}_c^{\infty}(P' \times G'^{\sigma}, V)$ satisfying (B.19) and (B.20), and
(ii) the space of linear forms M on $\mathcal{C}_c^{\infty}(P' \times G'^{\sigma}, V)$ such that

$$M(p'_{o} \cdot \phi) = \nu_{o}(p'_{o}) \cdot M(\phi), \tag{B.21}$$
$$M(\phi_{(p', g'_{o})}) = M(\chi^{-1}(g'_{o})\nu^{-1}(p')\tau(p')\phi), \tag{B.22}$$

for all $\phi \in \mathcal{C}_c^{\infty}(P' \times G'^{\sigma}, V)$, $p'_{o} \in P'^{\sigma}$ and $(p', g'_{o}) \in P' \times G'^{\sigma}$.

We now apply Corollary B.9 again, with $G = H = P' \times G'^{\sigma}$ and $\rho = \tau \otimes \chi^{-1}$. Since the character $\delta_{P' \times G'^{\sigma}}$ associated with $P' \times G'^{\sigma}$ is equal to $\nu^{-1} \otimes 1$, we get an isomorphism of R-modules between:

(i) the space of linear forms M on $\mathcal{C}_c^{\infty}(P' \times G'^{\sigma}, V)$ satisfying (B.21) and (B.22), and
(ii) the space of linear forms t on $\mathrm{ind}_{P' \times G'^{\sigma}}^{P' \times G'^{\sigma}}(\tau \otimes \chi^{-1})$ such that

$$t(p'_{o} \cdot \varphi) = \nu_{o}(p'_{o}) \cdot \varphi$$

for all $\varphi \in \mathrm{ind}_{P' \times G'^{\sigma}}^{P' \times G'^{\sigma}}(\tau \otimes \chi^{-1})$ and $p'_{o} \in P'^{\sigma}$.

Finally, one verifies that the map $\varphi \mapsto \varphi(1, 1)$ from $\mathrm{ind}_{P' \times G'^{\sigma}}^{P' \times G'^{\sigma}}(\tau \otimes \chi^{-1})$ to V induces an isomorphism of R-modules between the space of linear forms t as above and $\mathrm{Hom}_{P'^{\sigma}}(\tau, \chi\nu_{o})$, which ends the proof of the proposition. □

B.3. A modular version of a result of Ok for cuspidal representations

In this subsection, we generalize a result of Ok [40, Theorem 3.1.2] on irreducible complex representations of $G = GL_n(F)$. More precisely, using Proposition B.10, we prove it for any *cuspidal* representation of G with coefficients in an algebraically closed field of characteristic different from p.

In this subsection, R is an algebraically closed field of characteristic different from p.

Proposition B.23. *Let π be a cuspidal representation of* G *with coefficients in* R. *Then the space* $\mathrm{Hom}_{P^{\sigma}}(\pi, \mathbf{1})$ *has dimension* 1. *If in addition π is* H*-distinguished, then*

$$\mathrm{Hom}_{P^{\sigma}}(\pi, \mathbf{1}) = \mathrm{Hom}_{G^{\sigma}}(\pi, \mathbf{1}).$$

Proof. By [6] and [54, III.1], the restriction of π to P is isomorphic to $\mathrm{ind}_{\mathrm{N}}^{\mathrm{P}}(\psi)$, where ψ is the standard σ-self-dual non-degenerate character of N which has been fixed at the beginning of B.2. This induced representation can be written $(\Phi^+)^{n-1}\Psi^+(\mathbf{1})$, where $\mathbf{1}$ denotes the trivial character of the trivial group, $\Psi^+(\mathbf{1})$ is the trivial character of the (trivial) mirabolic subgroup $\mathrm{P}_1(\mathrm{F})$ and Φ^+ is the functor which has been defined in B.2. Applying Proposition B.10 $n-1$ times, we get the expected result. □

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