J. Eur. Math. Soc. 23, 3279–3321 (2021) DOI 10.4171/JEMS/1085

J.-M. Barbaroux · L. Le Treust · N. Raymond · E. Stockmeyer

On the semiclassical spectrum of the Dirichlet–Pauli operator

Received November 26, 2018

Abstract. This paper is devoted to semiclassical estimates of the eigenvalues of the Pauli operator on a bounded open set with Dirichlet conditions on the boundary. Assuming that the magnetic field is positive and a few generic conditions, we establish the simplicity of the eigenvalues and provide accurate asymptotic estimates involving Segal–Bargmann and Hardy spaces associated with the magnetic field.

Keywords. Pauli operator, magnetic Cauchy–Riemann operators, semiclassical analysis

Contents

J.-M. Barbaroux: Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France; barbarou@univ-tln.fr

L. Le Treust: Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France; loic.le-treust@univ-amu.fr

N. Raymond: LAREMA, Université d'Angers, Faculté des Sciences, Département de Mathématiques, 49045 Angers Cedex 01, France; nicolas.raymond@univ-angers.fr

E. Stockmeyer: Instituto de Física, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Santiago 7820436, Chile; stock@fis.puc.cl

Mathematics Subject Classification (2020): 35P15, 81Q10, 35Jxx

1. Introduction

In this article we consider the magnetic Pauli operator defined on a bounded and simply connected domain $\Omega \subset \mathbb{R}^2$ subject to Dirichlet boundary conditions. This operator is the model Hamiltonian of a non-relativistic spin-$\frac{1}{2}$ particle, constrained to move in Ω, interacting with a magnetic field that is perpendicular to the plane.

Formally the Pauli operator acts on two-dimensional spinors and it is given by

$$\mathscr{P}_h = [\sigma \cdot (-ih\nabla - A)]^2,$$

where $h > 0$ is a semiclassical parameter and σ is a two-dimensional vector whose components are the Pauli matrices σ_1 and σ_2. The magnetic field B enters in the operator through an associated magnetic vector potential $A = (A_1, A_2)$ that satisfies $\partial_1 A_2 - \partial_2 A_1 = B$. Assuming that the magnetic field is positive and a few other mild conditions we provide precise asymptotic estimates for the low energy eigenvalues of $\mathscr{P}_h$ in the semiclassical limit (i.e., as $h \to 0$).

Let us roughly explain our results. Let $\lambda_k(h)$ be the k-th eigenvalue of $\mathscr{P}_h$ counting multiplicity. Assuming that the boundary of Ω is $\mathscr{C}^2$, we show that there exist $\alpha > 0$ and $\theta_0 \in (0, 1]$ such that the following holds: For all $k \in \mathbb{N}^*$, there exists $C_k > 0$ such that, as $h \to 0$,

$$\theta_0 C_k h^{-k+1} e^{-2\alpha/h}(1 + o(1)) \leq \lambda_k(h) \leq C_k h^{-k+1} e^{-2\alpha/h}(1 + o(1)).$$

In particular, this result establishes the simplicity of the eigenvalues in this regime. The constants $\alpha > 0$ and C_k are directly related to the magnetic field, and the geometry of Ω and C_k is expressed in terms of Segal–Bargmann and Hardy norms that are naturally associated to the magnetic field. When Ω is a disk and B is radially symmetric we compute C_k explicitly and find that $\theta_0 = 1$. This substantially improves the known results about the Dirichlet–Pauli operator [6, 11] (for details see Section 1.3.2).

These results may be reformulated in terms of the large magnetic field limit by a simple scaling argument. Indeed,

$$\mu_k(b) = b^2 \lambda_k(1/b),$$

where $\mu_k(b)$ is the k-th eigenvalue of $[\sigma \cdot (-i\nabla - bA)]^2$.

Our results can also be used to describe the spectrum of the magnetic Laplacian with constant magnetic field B_0. For instance, when Ω is bounded, strictly convex with a boundary of class $\mathscr{C}^{1,\gamma}$ ($\gamma > 0$), the k-th eigenvalue of $(-ih\nabla - A)^2$ with Dirichlet boundary conditions, denoted by $\mu_k(h)$, satisfies, for some $c, C > 0$ and h small enough,

$$B_0 h + ch^{-k+1}e^{-2\alpha/h} \leq \mu_k(h) \leq B_0 h + Ch^{-k+1}e^{-2\alpha/h}. \tag{1.1}$$

In particular, the first eigenvalues of the magnetic Laplacian are simple in the semiclassical limit. This asymptotic simplicity was not known before and (1.1) is the most accurate known estimate of the magnetic eigenvalues in the case of the constant magnetic field and Dirichlet boundary conditions (see [10, Section 4] and Section 1.3.2).

Our study presents a new approach that establishes several connections with various aspects of analysis like Cauchy–Riemann operators, uniformization, and, to some extent, Toeplitz operators. We may hope that this work will cast a new light on the magnetic Schrödinger operators.

1.1. Setting and main results

Let $\Omega \subset \mathbb{R}^2$ be an open set. All along the paper Ω will satisfy the following assumption.

Assumption 1.1. Ω is bounded and simply connected.

Consider a magnetic field $B \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$. An associated *vector potential* $A : \overline{\Omega} \to \mathbb{R}^2$ is a function such that

$$B = \partial_1 A_2 - \partial_2 A_1.$$

We will use the following special choice of vector potential.

Definition 1.2. Let ϕ be the unique (smooth) solution of

$$\begin{aligned} \Delta\phi &= B \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

The vector field $A = (-\partial_2\phi, \partial_1\phi)^T := \nabla\phi^\perp$ is a vector potential associated with B.

In this paper, B will be positive (and thus ϕ subharmonic) so that

$$\max_{x\in\overline{\Omega}} \phi = \max_{x\in\partial\Omega} \phi = 0.$$

In particular, the minimum of ϕ will be negative and attained in Ω. Note also that the exterior normal derivative of ϕ, denoted by $\partial_{\mathbf{n}}\phi$, is positive on $\partial\Omega$ if Ω is $\mathscr{C}^2$ [7, Hopf's Lemma, Section 6.4.2].

Notation 1. We denote by $\langle\cdot,\cdot\rangle$ the $\mathbb{C}^n$ ($n \geq 1$) scalar product (antilinear with respect to the left argument), by $\langle\cdot,\cdot\rangle_{L^2(U)}$ the L^2 scalar product on the set U, by $\|\cdot\|_{L^2(U)}$ the L^2-norm on U and by $\|\cdot\|_{L^\infty(U)}$ the L^∞-norm on U. We use o and $\mathscr{O}$ for the standard Landau symbols.

1.2. The Dirichlet–Pauli operator

This paper is devoted to the Dirichlet–Pauli operator $(\mathscr{P}_h, \mathsf{Dom}(\mathscr{P}_h))$ defined for all $h > 0$ on

$$\mathsf{Dom}(\mathscr{P}_h) := H^2(\Omega;\mathbb{C}^2) \cap H_0^1(\Omega;\mathbb{C}^2),$$

and whose action is given by the second order differential operator

$$\begin{aligned}\mathscr{P}_h = [\sigma \cdot (\mathbf{p} - A)]^2 &= \begin{pmatrix} |\mathbf{p} - A|^2 - hB & 0 \\ 0 & |\mathbf{p} - A|^2 + hB \end{pmatrix} \\ &= \begin{pmatrix} \mathscr{L}_h^- & 0 \\ 0 & \mathscr{L}_h^+ \end{pmatrix}. \end{aligned} \tag{1.3}$$

Here $\mathbf{p} = -ih\nabla$, and

$$|\mathbf{p} - A|^2 := (\mathbf{p} - A) \cdot (\mathbf{p} - A) = -h^2\Delta - A \cdot \mathbf{p} - \mathbf{p} \cdot A + |A|^2,$$

and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\sigma \cdot \mathbf{x} = \sigma_1\mathbf{x}_1 + \sigma_2\mathbf{x}_2 + \sigma_3\mathbf{x}_3$ for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and $\sigma \cdot \mathbf{x} = \sigma_1\mathbf{x}_1 + \sigma_2\mathbf{x}_2$ for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$. In terms of quadratic forms, we have by partial integration, for all $\psi \in \mathsf{Dom}(\mathscr{P}_h)$,

$$\begin{aligned}\langle \psi, \mathscr{P}_h\psi \rangle_{L^2(\Omega)} &= \|\sigma \cdot (\mathbf{p} - A)\psi\|_{L^2(\Omega)}^2 \\ &= \|(\mathbf{p} - A)\psi\|_{L^2(\Omega)}^2 - \langle \psi, \sigma_3 hB\psi \rangle_{L^2(\Omega)}. \end{aligned} \tag{1.4}$$

Note that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$,

$$(\sigma \cdot \mathbf{x})(\sigma \cdot \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} 1_2 + i\sigma \cdot (\mathbf{x} \times \mathbf{y}), \tag{1.5}$$

where 1_2 is the 2×2 identity matrix. The operator $\mathscr{P}_h$ is selfadjoint and has compact resolvent. This paper is mainly devoted to the investigation of the lower eigenvalues of $\mathscr{P}_h$.

Notation 2. Let $(\lambda_k(h))_{k\in\mathbb{N}^*}$ $(h > 0)$ denote the increasing sequence of eigenvalues of the operator $\mathscr{P}_h$, each repeated according to its multiplicity. By the min-max theorem,

$$\lambda_k(h) = \inf_{\substack{V \subset \mathsf{Dom}(\mathscr{P}_h) \\ \dim V = k}} \sup_{\psi \in V\setminus\{0\}} \frac{\|\sigma \cdot (\mathbf{p} - A)\psi\|_{L^2(\Omega)}^2}{\|\psi\|_{L^2(\Omega)}^2}. \tag{1.6}$$

Under the assumption that $B > 0$ on $\overline{\Omega}$, the lowest eigenvalues of $\mathscr{P}_h$ are the eigenvalues of $\mathscr{L}_h^-$. More precisely, our main result states that for any fixed $k \in \mathbb{N}^*$ and $h > 0$ small enough, $\lambda_k(h)$ is the k-th eigenvalue of the Schrödinger operator $\mathscr{L}_h^-$.

1.3. Results and relations to the existing literature

1.3.1. Main theorem

Notation 3. Let us denote by $\mathscr{H}(\Omega)$ and $\mathscr{H}(\mathbb{C})$ the sets of holomorphic functions on Ω and $\mathbb{C}$. We consider the (anisotropic) *Segal–Bargmann* space

$$\mathscr{B}^2(\mathbb{C}) = \{u \in \mathscr{H}(\mathbb{C}) : N_{\mathscr{B}}(u) < +\infty\},$$

where

$$N_{\mathscr{B}}(u) = \left(\int_{\mathbb{R}^2} |u(y_1 + iy_2)|^2 e^{-\mathsf{Hess}_{x_{\min}}\phi(y,y)}\,\mathrm{d}y\right)^{1/2}.$$

We also introduce a weighted Hardy space

$$\mathscr{H}^2(\Omega) = \{u \in \mathscr{H}(\Omega) : N_{\mathscr{H}}(u) < +\infty\},$$

where

$$N_{\mathscr{H}}(u) = \left(\int_{\partial\Omega} |u(y_1 + iy_2)|^2 \partial_{\mathbf{n}}\phi\,\mathrm{d}y\right)^{1/2}.$$

Here, $x_{\min} \in \Omega$ and $\mathsf{Hess}_{x_{\min}}\phi \in \mathbb{R}^{2\times 2}$ are defined in Theorem 1.3 below, $\mathbf{n}(s)$ is the outward pointing unit normal to Ω, and $\partial_{\mathbf{n}}\phi(s)$ is the normal derivative of ϕ on $\partial\Omega$ at $s \in \partial\Omega$. We also define for $P \in \mathscr{H}^2(\Omega)$, $A \subset \mathscr{H}^2(\Omega)$,

$$\mathsf{dist}_{\mathscr{H}}(P, A) = \inf\{N_{\mathscr{H}}(P - Q) : Q \in A\},$$

and for $P \in \mathscr{B}^2(\mathbb{C})$, $A \subset \mathscr{B}^2(\mathbb{C})$,

$$\mathsf{dist}_{\mathscr{B}}(P, A) = \inf\{N_{\mathscr{B}}(P - Q) : Q \in A\}.$$

The main results of this paper are gathered in the following theorem.

Theorem 1.3. *Define*

$$\phi_{\min} = \min_{x\in\overline{\Omega}} \phi.$$

Assume that Ω is $\mathscr{C}^2$, satisfies Assumption 1.1, and

(a) $B_0 := \inf\{B(x) : x \in \Omega\} > 0$,

(b) *the minimum of ϕ is attained at a unique point $x_{\min}$,*

(c) *the minimum is non-degenerate, i.e., the Hessian matrix $\mathsf{Hess}_{x_{\min}}\phi$ at $x_{\min}$ (or $z_{\min}$ if seen as a complex number) is positive definite.*

Then there exists $\theta_0 \in (0, 1]$ such that for all fixed $k \in \mathbb{N}^*$,

(i) $\lambda_k(h) \leq C_{\sup}(k) h^{-k+1} e^{2\phi_{\min}/h}(1 + o_{h\to 0}(1))$ *with*

$$C_{\sup}(k) = 2\left(\frac{\mathsf{dist}_{\mathscr{H}}((z - z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega))}{\mathsf{dist}_{\mathscr{B}}(z^{k-1}, \mathscr{P}_{k-2})}\right)^2,$$

where $\mathscr{P}_{k-2} = \mathrm{span}(1, \dots, z^{k-2}) \subset \mathscr{B}^2(\mathbb{C})$, $\mathscr{P}_{-1} = \{0\}$ and

$$\mathscr{H}^2_k(\Omega) = \{u \in \mathscr{H}^2(\Omega) : u^{(n)}(z_{\min}) = 0 \text{ for } n \in \{0, \dots, k-1\}\}. \tag{1.7}$$

(ii) $\lambda_k(h) \geq C_{\inf}(k) h^{-k+1} e^{2\phi_{\min}/h}(1 + o_{h\to 0}(1))$ *with*

$$C_{\inf}(k) = C_{\sup}(k)\theta_0.$$

A precise definition of θ_0 *is given in Remark* 1.10.

Assuming that Ω *is the disk of radius* 1 *centered at* 0, *and that* B *is radial, we have*

$$C_{\sup}(k) = C_{\inf}(k) = C^{\mathrm{rad}}(k) = \frac{B(0)^k \Phi}{2^{k-2}(k-1)!} \qquad (\theta_0 = 1),$$

$$\Phi = \frac{1}{2\pi}\int_\Omega B(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{\partial\Omega} \partial_{\mathbf{n}}\phi\,\mathrm{d}s.$$

Remark 1.4. Assume that $B = B_0 > 0$ and that Ω is strictly convex. Then ϕ has a unique and non-degenerate minimum (see [13, 14] and also [11, Proposition 7.1 and below]). Thus, our assumptions are satisfied in this case.

Remark 1.5. The main properties of the space $\mathscr{H}^2(\Omega)$ can be found in [4, Chapter 10]. Note that whenever $\partial\Omega$ is supposed to be Dini-continuous (in particular $\mathscr{C}^{1,\alpha}$ boundaries, with $\alpha > 0$, are allowed), the set $W^{1,\infty}(\Omega) \cap \mathscr{H}^2(\Omega)$ is dense in $\mathscr{H}^2(\Omega)$ (see Lemma C.1). This assumption is in particular needed in the proof of Theorem 1.3(i) (see Remark 3.4).[1] The definition of Dini-continuous functions is recalled in the context of the boundary behavior of conformal maps in [15, Section 3.3]. It is essentially an integrability property of the derivative of a parametrization of $\partial\Omega$.

Remark 1.6. The Cauchy formula [4, Theorem 10.4] and the Cauchy–Schwarz inequality ensure that

$$|u^{(n)}(z_{\min})| \leq \frac{n!}{2\pi\sqrt{\min_{\partial\Omega}\partial_{\mathbf{n}}\phi}} N_{\mathcal{H}}(u)\left(\int_{\partial\Omega}\frac{|\mathrm{d}z|}{|z - z_{\min}|^{2(n+1)}}\right)^{1/2},$$

for $n \in \mathbb{N}$ and $u \in \mathscr{H}^2(\Omega)$ (see also the proof of Lemma 3.5). This ensures that $\mathscr{H}^2_k(\Omega)$ defined in (1.7) is a closed vector subspace of $\mathscr{H}^2(\Omega)$ and that $\mathrm{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega)) > 0$ (see [3, Corollary 5.4]) since $(z - z_{\min})^{k-1} \notin \mathscr{H}^2_k(\Omega)$.

Remark 1.7. When B is radial on the unit disk $\Omega = D(0, 1)$, we find, using Fourier series, that $(z^n)_{n\geq 0}$ is an orthogonal basis for $N_{\mathcal{B}}$ and $N_{\mathcal{H}}$ which are up to normalization factors, the Szegö polynomials [4, Theorem 10.8]. In particular, $\mathscr{H}^2_k(\Omega)$ is $N_{\mathcal{H}}$-orthogonal to z^{k-1} so that

$$\mathrm{dist}_{\mathcal{H}}(z^{k-1}, \mathscr{H}^2_k(\Omega))^2 = N_{\mathcal{H}}(z^{k-1})^2 = \int_{\partial\Omega}\partial_{\mathbf{n}}\phi = 2\pi\Phi.$$

[1] Note also that we do not use here the stronger notion of Smirnov domain in which the set of polynomials in the complex variable is dense in $\mathscr{H}^2(\Omega)$ (see [4, Theorem 10.6]). Starlike domains and domains with analytic boundary are Smirnov domains.

In addition, $\mathcal{P}_{k-2}$ is $N_{\mathcal{B}}$-orthogonal to z^{k-1} so that

$$\mathsf{dist}_{\mathcal{B}}(z^{k-1}, \mathcal{P}_{k-2})^2 = N_{\mathcal{B}}(z^{k-1})^2 = 2\pi \frac{2^{k-1}(k-1)!}{B(0)^k},$$

and the radial part of Theorem 1.3 follows.

Remark 1.8. The proof of the upper bound can easily be extended to the case where Ω is not necessarily simply connected (see Remark 3.4).

Remark 1.9. Theorem 1.3 is concerned with the asymptotics of each eigenvalue $\lambda_k(h)$ of the operator $\mathscr{P}_h$ ($k \in \mathbb{N}^*$) as $h \to 0$. In particular, $\lambda_k(h)$ tends to 0 exponentially. Of course, this does not mean that all the eigenvalues go to 0 uniformly with respect to k. For $h > 0$, consider for example

$$\nu_1(h) = \inf_{v \in H_0^1(\Omega;\mathbb{C})\setminus\{0\}} \frac{\langle u, \mathscr{L}_h^+ u\rangle}{\|v\|^2_{L^2(\Omega)}},$$

the lowest eigenvalue of the operator $\mathscr{L}_h^+$. For fixed $h > 0$, there exists $k(h) \in \mathbb{N}^*$ such that $\nu_1(h) = \lambda_{k(h)}(h)$. By (1.3), we have $\nu_1(h) \geq 2B_0 h$ and thus $\nu_1(h)$ does not converge to 0 with exponential speed. Actually, Theorem 1.3 ensures that

$$\lim_{h\to 0} \mathsf{card}\{j \in \mathbb{N}^* : \lambda_j(h) \leq \nu_1(h)\} = +\infty, \qquad \lim_{h \to 0} k(h) = +\infty.$$

This accumulation of eigenvalues near 0 in the semiclassical limit is related to the fact that the corresponding eigenfunctions are close to be functions in the Segal–Bargmann space $\mathscr{B}^2(\mathbb{C})$ which is of infinite dimension.

Remark 1.10. The constant θ_0 introduced in Theorem 1.3 does not depend on $k \in \mathbb{N}^*$ and is equal to 1 in the radial case. We conjecture that the upper bounds in Theorem 1.3 (i) are optimal, that is, $\theta_0 = 1$ in the general case.

More precisely, let Ω be a $\mathscr{C}^2$ set satisfying Assumption 1.1. We introduce

$$\mathcal{M}_\Omega := \{G : \overline{\Omega} \to \overline{D(0,1)} \text{ biholomorphic} : c_1 \leq |G'(\cdot)| \leq c_2 \text{ for some } c_1, c_2 > 0\}.$$

Note that $\mathcal{M}_\Omega$ is non-empty by the Riemann mapping theorem. Then the constant θ_0 can be defined by

$$\theta_0 := \frac{\min_{\partial D(0,1)} |(G^{-1})'(y)| \partial_{\mathbf{n}}\phi(G^{-1}(y))}{\max_{\partial D(0,1)} |(G^{-1})'(y)| \partial_{\mathbf{n}}\phi(G^{-1}(y))} \in (0,1],$$

for some $G \in \mathcal{M}_\Omega$ (see Lemma 5.6).[2]

[2]We can even choose

$$\tilde{\theta}_0 := \sup_{G \in \mathcal{M}_\Omega} \frac{\min_{\partial D(0,1)} |(G^{-1})'(y)| \partial_{\mathbf{n}}\phi(G^{-1}(y))}{\max_{\partial D(0,1)} |(G^{-1})'(y)| \partial_{\mathbf{n}}\phi(G^{-1}(y))}.$$

Actually, we can even see from our analysis that there is a class of magnetic fields for which $\theta_0 = 1$. We introduce

$$\mathscr{B} := \{\check{B} \in \mathscr{C}^\infty(\overline{D(0,1)}; \mathbb{R}_+^*) : \\ \exists \check{\phi} \in H_0^1(D(0,1); \mathbb{R}),\ \Delta\check{\phi} = \check{B} \text{ on } D(0,1),\ \partial^2_{\mathbf{n}s}\check{\phi} = 0 \text{ on } \partial D(0,1)\}. \quad (1.8)$$

Here, ∂_s denotes the tangential derivative. Then, for any $\check{B} \in \mathscr{B}$ and $G \in \mathscr{M}_\Omega$, we get $\theta_0 = 1$ and

$$\liminf_{h\to 0} e^{-2\phi_{\min}/h} h^{k-1} \lambda_k(h) \geq C_{\sup}(k)$$

for the magnetic field $B = |G'(z)|^2 \check{B} \circ G(z)$. This follows from the fact that the function

$$\partial D(0,1) \ni y \mapsto |(G^{-1})'(y)| \partial_{\mathbf{n}} \phi(G^{-1}(y))$$

is constant. Here, ϕ is defined in (1.2).

Using the Riemann mapping theorem, we can deduce the following lower bound for Ω with Dini-continuous boundary. Its proof can be found in Section 5.4.

Corollary 1.11. *Assume that Ω is bounded, simply connected and that $\partial\Omega$ is Dini-continuous. Assume also (a)–(c) of Theorem 1.3. Let $k \in \mathbb{N}^*$. Then there exist $c_k, C_k > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$c_k h^{-k+1} e^{2\phi_{\min}/h} \leq \lambda_k(h) \leq C_k h^{-k+1} e^{2\phi_{\min}/h}.$$

Remark 1.12. Note also that our proof ensures that the constants C_k, c_k can be chosen so that C_k/c_k does not depend on $k \in \mathbb{N}^*$.

Our results can be used to describe the spectrum of the magnetic Laplacian with constant magnetic field (see Remark 1.4).

Corollary 1.13. *Assume that Ω is bounded, strictly convex and that $\partial\Omega$ is Dini-continuous. Assume also that (a)–(c) of Theorem 1.3 hold and that B is constant. Then the k-th eigenvalue of $(-ih\nabla - A)^2$ with Dirichlet boundary conditions, denoted by $\mu_k(h)$, satisfies, for some $c, C > 0$ and h small enough,*

$$Bh + ch^{-k+1} e^{2\phi_{\min}/h} \leq \mu_k(h) \leq Bh + Ch^{-k+1} e^{2\phi_{\min}/h}. \quad (1.9)$$

In particular, the first eigenvalues of the magnetic Laplacian are simple in the semiclassical limit.

1.3.2. Relations to the literature. Let us compare our result with the existing literature.

(i) When $B = 1$, our results improve the bound obtained by Erdős for $\lambda_1(h)$ [6, Theorem 1.1 & Proposition A.1] and also the bound by Helffer and Morame [10, Propositions 4.1 and 4.4]. Indeed, (1.9) gives us the optimal behavior of the remainder. When $B = 1$ and $\Omega = D(0,1)$, the asymptotic expansion of the next eigenvalues is considered in [11, Theorem 5.1(c)]. Note that, in this case, $\phi = (|x|^2 - 1)/4$ and that Theorem 1.3 allows one to recover [11, Theorem 5.1(c)] by considering radial magnetic fields.

(ii) In [11] (simply connected case) and [12] (general case), Helffer and Persson Sundqvist have proved, under assumption (a), that

$$\lim_{h\to 0} h \ln \lambda_1(h) = 2\phi_{\min}.$$

Moreover, under the assumptions (a), (b) and (c) of Theorem 1.3, their theorem [11, Theorem 4.2] implies the following upper bound for the *first eigenvalue*

$$\lambda_1(h) \leq 4\Phi \det(\mathsf{Hess}_{x_{\min}} \phi)^{1/2}(1+o(1))e^{2\phi_{\min}/h}.$$

Note that Theorem 1.3(i) provides a better upper bound even for $k = 1$.

They also establish the following lower bound by means of rough considerations:

$$\forall h > 0, \quad \lambda_1(h) \geq h^2 \lambda_1^{\mathsf{Dir}}(\Omega) e^{2\phi_{\min}/h},$$

where $\lambda_1^{\mathsf{Dir}}(\Omega)$ is the first eigenvalue of the corresponding magnetic Dirichlet Laplacian. This estimate is itself an improvement of [5, Theorem 2.1].

Corollary 1.11 is an optimal improvement in terms of the order of magnitude of the pre-factor of the exponential. It also improves the existing results by considering the excited eigenvalues. Describing the behavior of the prefactor is not a purely technical question. Indeed, it is directly related to the simplicity of the eigenvalues and even governs the asymptotic behavior of the spectral gaps. This simplicity was not known before, except in the case of constant magnetic field on a disk.

(iii) The problem of estimating the spectrum of the Dirichlet–Pauli operator is closely connected to the spectral analysis of the Witten Laplacian (see for instance [11, Remark 1.6] and the references therein). For example, in this context, the ground state energy is

$$\min_{\substack{v\neq 0\\ v\in H_0^1(\Omega)}} \frac{\int_\Omega |h\nabla v|^2 e^{-2\phi/h}\,\mathrm{d}x}{\int_\Omega e^{-2\phi/h}|v|^2\,\mathrm{d}x}, \tag{1.10}$$

whereas in the present paper we will focus on

$$\min_{\substack{v\neq 0\\ v\in H_0^1(\Omega)}} \frac{\int_\Omega |h(\partial_{x_1}+i\partial_{x_2})v|^2 e^{-2\phi/h}\,\mathrm{d}x}{\int_\Omega e^{-2\phi/h}|v|^2\,\mathrm{d}x} \tag{1.11}$$

(see also Lemma 2.4). Considering *real-valued* functions v in (1.11) reduces to (1.10). In this sense, (1.11) gives rise to a "less elliptic" minimization problem.

1.4. *The intuition and strategy of the proof*

In this subsection we discuss the main lines of our strategy. It is intended to reveal the intuition behind some of our proofs. We will focus mostly on the ground state energy, which is given by (1.6) as

$$\lambda_1(h) = \min_{\psi\in H_0^1(\Omega;\mathbb{C}^2)\setminus\{0\}} \frac{\|\sigma\cdot(\mathbf{p}-A)\psi\|^2_{L^2(\Omega)}}{\|\psi\|^2_{L^2(\Omega)}}. \tag{1.12}$$

It is easy to guess from (1.3) that the ground state energy has to have the form $\psi = (u, 0)^{\mathrm{T}}$. This is consistent with the physical intuition that, for low energies, the spin of the particle should be parallel to the magnetic field.

The variational problem above can be rewritten by means of a suitable transformation as

$$\lambda_1(h) = h^2 \min_{\substack{v \neq 0 \\ v \in H_0^1(\Omega)}} \frac{\int_\Omega |2\partial_{\bar{z}} v|^2 e^{-2\phi/h}\,\mathrm{d}x}{\int_\Omega |v|^2 e^{-2\phi/h}\,\mathrm{d}x} =: h^2 \min_{\substack{v \neq 0 \\ v \in H_0^1(\Omega)}} \frac{F_h(v,\phi)}{G_h(v,\phi)}, \tag{1.13}$$

where $\partial_{\bar{z}} = (\partial_1 + i\partial_2)/2$ and ϕ is the unique solution to $\Delta\phi = B$ in Ω with Dirichlet boundary conditions (see Definition 1.2). This connection between the spectral analysis of the Dirichlet–Pauli and Cauchy–Riemann operators is known in the literature (see e.g. [2, 6, 11] and [17]), and we describe it in Section 2.

In order to study the problem in (1.13) it is helpful to consider the following heuristics concerning $F_h(v, \phi)$.

Observation 1.14. A minimizer v_h *wants to be* an analytic function in the interior of Ω but, due to the boundary conditions, has to have a different behavior close to the boundary. So, if we set $\Omega_\delta := \{x \in \Omega : \mathsf{dist}(x, \partial\Omega) \geq \delta\}$ for $\delta > 0$, we expect that v_h behaves almost as an analytic function on U with $\Omega_\delta \subset U \subset \Omega$. Moreover, this tendency is enhanced in the semiclassical limit when the presence of the magnetic field becomes stronger. Hence, we also expect that $\delta \to 0$ as $h \to 0$ in some way.

We comment below on how we make Observation 1.14 more precise; for the moment let us just mention that throughout this discussion we work with δ such that

$$\delta^2/h \to 0 \quad \text{and} \quad \delta/h \to \infty \qquad \text{as } h \to 0. \tag{1.14}$$

As a consequence of Observation 1.14 we expect that

$$F_h(v_h, \phi) \sim \int_{T_\delta} |2\partial_{\bar{z}} v_h|^2 e^{-2\phi/h}\,\mathrm{d}x, \tag{1.15}$$

where $T_\delta := \Omega \setminus \Omega_\delta$.

An essential ingredient in our method is the analysis of the minimization problem associated with the RHS of (1.15). The main ideas go as follows: Assume first that Ω is the disk $D(0, 1)$. By writing the integrand $|\partial_{\bar{z}} v_h|^2 e^{-2\phi/h}$ in tubular coordinates (see item (i) from the proof of Lemma 3.7) and Taylor expanding ϕ around any point at the boundary $\partial\Omega$ we get, for δ satisfying (1.14),

$$\int_{T_\delta} |2\partial_{\bar{z}} v|^2 e^{-2\phi/h}\,\mathrm{d}x = (1 + o(h)) \int_0^{2\pi} \int_0^\delta e^{2t\partial_{\mathbf{n}}\phi/h} |(\partial_\tau - i\partial_s)v|^2\,\mathrm{d}s\,\mathrm{d}\tau \tag{1.16}$$

$$=: (1 + o(h)) J_h(v) \tag{1.17}$$

(see also the proof of Lemma 5.5), where $\partial_{\mathbf{n}}\phi \equiv \partial_{\mathbf{n}}\phi(s)$ is the normal derivative at the boundary (see Notation 3).

Observe that if $\partial_{\mathbf{n}}\phi$ is a constant along the boundary, then it equals the flux Φ. In this case, as explained in item (iv) of the proof of Lemma 5.5, the problem of finding a non-trivial solution of

$$\inf_{v\in H^1(T_\delta)} J_h(v) \quad \text{with} \quad v\restriction_{\partial\Omega_\delta} = v_\delta,\ v\restriction_{\partial\Omega} = 0, \tag{1.18}$$

can be reduced to a sum (labeled by the Fourier index) of one-dimensional problems that we solve explicitly in Lemma A.1.

For the particular case of v having only the non-negative Fourier modes on $\partial\Omega_\delta$ (i.e., $v_\delta = \sum_{m\geq 0} \hat{v}_{\delta,m} e^{ims}$) we find that (see Lemma 5.5)

$$J_h(v) \geq \frac{\Phi/h}{1-e^{-2\delta\Phi/h}} \|v\|^2_{L^2(\partial D(0,1-\delta))} = (1+o(h))2\Phi/h\|v\|^2_{L^2(\partial D(0,1-\delta))} \tag{1.19}$$

where the last equality is a trivial consequence of (1.14). Moreover, by Lemma A.1, the latter inequality is saturated when $v_\delta = \hat{v}_{\delta,0}$. Concerning the assumption on v, recall that analytic functions on the disk have only Fourier modes for $m \geq 0$.

Notice that if B is rotationally symmetric then $\partial_{\mathbf{n}}\phi$ is constant. If $\partial_{\mathbf{n}}\phi$ is not a constant we can give a suitable estimate using $\min_{\partial\Omega} \partial_{\mathbf{n}}\phi > 0$. We extend the previous analysis to more general geometries by using the Riemann mapping theorem.

There is another important point to take into account, this time concerning $G_h(v,\phi)$.

Observation 1.15. Recall that $\phi \leq 0$ has an absolute, non-degenerate, minimum at $x_{\min}$. Hence, the weighted norm of v_h, $G_h(v,\phi)$, should have a tendency to concentrate around $x_{\min}$. This is made precise in Lemma 5.3 below. Moreover, observe that using Laplace's method, one formally deduces that, as $h \to 0$,

$$G_h(v,\phi) \sim h\pi |v(x_{\min})|^2 e^{-2\phi_{\min}/h} (\det \mathsf{Hess}_{x_{\min}} \phi)^{-1/2}. \tag{1.20}$$

Observations 1.14 and 1.15 reveal the importance of the behavior of a minimizer around the boundary and close to $x_{\min}$, respectively. In addition, this behavior is naturally captured through the norms $N_{\mathscr{H}}$ and $N_{\mathscr{B}}$ given in Definition 3, which, in turn, provide a natural Hilbert space structure to select linear independent test functions which are used to estimate the excited energies.

In order to show our result we give upper and lower bounds for the variational problem (1.13). This is done in Sections 3 and 5, respectively. Concerning the upper bound: In view of the previous discussion it is natural to choose a trial function (at least for the disk, see Remark 3.2) $v = \omega\chi$ where ω is an analytic function in Ω and χ is such that $\chi\restriction_{\Omega_\delta} = 1$ and decays smoothly to zero towards $\partial\Omega$. We pick $\chi\restriction_{T_\delta}$ as an optimizer of the problem (1.18). For $\lambda_k(h)$, we choose ω to be a polynomial of degree $k-1$. In particular, for the ground-state energy, ω is constant and in view of (1.22) and (1.20) we readily see how the claimed upper bound (at least for the disk with radial magnetic field) is obtained.

As for the lower bound, as a preliminary step, we discuss in Section 4 some ellipticity properties related to the magnetic Cauchy–Riemann operators. Our main result there is Theorem 4.6. It provides elliptic estimates for the magnetic Cauchy–Riemann operators on the orthogonal complement of the kernel which consists, up to an exponential weight, of holomorphic functions. The findings of Section 4 are crucial to proving Proposition 5.4, which gives estimates on the behavior described in Observation 1.14. Indeed, Proposition 5.4, together with the upper bound, roughly states that the non-analytic part of v_h on any open set contained in Ω is, in the semiclassical limit, exponentially small in a sufficiently strong norm. At least for the disk with radial magnetic field, we can argue on how to get the lower bound if we assume that v_h is analytic on an open set U with $\overline{D(0,1-\delta)} \subset U \subset D(0,1)$. Notice that (1.22) holds. Moreover, by Cauchy's Theorem we have $2\pi|v_h(x_{\min})|^2 = 2\pi|v_h(0)|^2 \leq (1+o(h))\|v_h\|^2_{\partial D(0,1-\delta)}$. In this way we see that the lower bound appears by combining (1.22) and (1.20).

Let us finally remark that actually, since the function v in (1.20) depends on h, Laplace's method cannot be applied so easily. Instead, after the change of scale $y = \frac{x-x_{\min}}{h^{1/2}}$, one has formally the Bargmann norm appearing:

$$G_h(v,\phi) \sim he^{-2\phi_{\min}/h} \int |v(x_{\min} + h^{1/2}y)|^2 e^{-\mathsf{Hess}_{x_{\min}}\phi(y,y)}\,\mathrm{d}y. \tag{1.21}$$

Ultimately, in the case of the disk with radial magnetic field, problem (1.13) reduces formally to

$$\lambda_1(h) \gtrsim e^{2\phi_{\min}/h} \inf_{\substack{v\neq 0\\ v\in\mathscr{H}(\Omega)}} 2\left(\frac{N_{\mathscr{H}}(v)}{N_{\mathscr{B}}(v(x_{\min}+h^{1/2}\cdot))}\right)^2, \tag{1.22}$$

which can be computed easily due to the orthogonality of the polynomials $(z^n)_{n\geq 0}$ in the Hilbert spaces $\mathscr{H}^2(\Omega)$ and $\mathscr{B}^2(\mathbb{C})$ (see Remark 1.7). Of course, special attention has to be paid to the domains of integration and the sets where the holomorphic test functions live. In the non-radial case, however, we strongly use the multi-scale structure of (1.22) to get the result of Theorem 1.3 (see Section 5.3). Note that the constant θ_0 of Theorem 1.3 which appears in the computation of (1.22) somehow measures a symmetry breaking rate (see Remark 1.10 and Lemma 5.6).

2. Change of gauge

The following result allows us to remove the magnetic field up to sandwiching the Dirac operator with a suitable matrix.

Proposition 2.1. *We have*

$$e^{\sigma_3\phi/h}\sigma\cdot\mathbf{p}e^{\sigma_3\phi/h} = \sigma\cdot(\mathbf{p}-A), \tag{2.1}$$

as operators acting on $H^1(\Omega;\mathbb{C}^2)$ functions.

This follows from the next two lemmas and Definition 1.2 (see also [17, Theorem 7.3]).

Lemma 2.2. *Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function and A, B be square matrices such that $AB=-BA$. Then*

$$Af(B) = f(-B)A.$$

Lemma 2.3 (Change of gauge for the Dirac operator). *Let $\Phi : \Omega \to \mathbb{R}$ be a regular function. Then*

$$e^{\sigma_3\Phi}(\sigma\cdot\mathbf{p})e^{\sigma_3\Phi} = \sigma\cdot(\mathbf{p} - h\nabla\Phi^{\perp})$$

as operators acting on $H^1(\Omega;\mathbb{C}^2)$ functions and where $\nabla\Phi^{\perp}$ is defined in Definition 1.2.

Proof. By Lemma 2.2, for $k = 1, 2$ we have

$$e^{\sigma_3\Phi}\sigma_k = \sigma_k e^{-\sigma_3\Phi}.$$

Thus, by the Leibniz rule,

$$e^{\sigma_3\Phi}(\sigma\cdot\mathbf{p})e^{\sigma_3\Phi} = (\sigma e^{-\sigma_3\Phi}\cdot\mathbf{p})e^{\sigma_3\Phi} = \sigma\cdot(\mathbf{p} - ih\sigma_3\nabla\Phi).$$

It remains to notice that $-i\sigma\sigma_3 = \sigma^{\perp} := (-\sigma_2, \sigma_1)$ so that

$$e^{\sigma_3\Phi}(\sigma\cdot\mathbf{p})e^{\sigma_3\Phi} = \sigma\cdot\mathbf{p} + h\sigma^{\perp}\cdot\nabla\Phi = \sigma\cdot\mathbf{p} - h\sigma\cdot\nabla\Phi^{\perp}. \qquad \blacksquare$$

We let

$$\partial_z := \frac{\partial_x - i\partial_y}{2}, \qquad \partial_{\overline{z}} := \frac{\partial_x + i\partial_y}{2}.$$

We then obtain the following result.

Lemma 2.4. *Let $k \in \mathbb{N}^*$ be such that $\lambda_k(h) < 2B_0h$. Then*

$$\lambda_k(h) = \inf_{\substack{V\subset H_0^1(\Omega;\mathbb{C})\\ \dim V = k}} \sup_{v\in V\setminus\{0\}} \frac{4\int_\Omega e^{-2\phi/h}|h\partial_{\overline{z}}v|^2\,\mathrm{d}x}{\int_\Omega |v|^2 e^{-2\phi/h}\,\mathrm{d}x}. \tag{2.2}$$

Recall that $\lambda_k(h)$ is defined in (1.6).

Proof. By (1.3) and (1.6), since $\mathscr{L}_h^+ \geq 2B_0h$ we get

$$\lambda_k(h) = \inf_{\substack{V\subset H_0^1(\Omega;\mathbb{C})\\ \dim V = k}} \sup_{u\in V\setminus\{0\}} \frac{\|\sigma\cdot(\mathbf{p} - A)\binom{u}{0}\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

Let $u \in H_0^1(\Omega;\mathbb{C})$ and $h > 0$. Letting $u = e^{-\phi/h}v$ we have, by Proposition 2.1,

$$\begin{aligned}\left\|\sigma\cdot(\mathbf{p} - A)\begin{pmatrix}u\\0\end{pmatrix}\right\|^2_{L^2(\Omega)} &= \left\|e^{\sigma_3\phi/h}\sigma\cdot\mathbf{p}\begin{pmatrix}v\\0\end{pmatrix}\right\|^2_{L^2(\Omega)}\\ &= 4\int_\Omega e^{-2\phi/h}|h\partial_{\overline{z}}v|^2\,\mathrm{d}x,\end{aligned}$$

and

$$\|u\|^2_{L^2(\Omega)} = \int_\Omega |v|^2 e^{-2\phi/h}\,\mathrm{d}x. \qquad \blacksquare$$

3. Upper bounds

This section is devoted to the proof of the following upper bounds.

Proposition 3.1. *Assume that Ω is $\mathscr{C}^2$ and satisfies Assumption 1.1. For all $k \in \mathbb{N}^*$,*

$$\lambda_k(h) \leq C_{\sup}(k) h^{-k+1} e^{2\phi_{\min}/h}(1 + o(1)), \tag{3.1}$$

where $\lambda_k(h)$ and $C_{\sup}(k)$ are defined in (1.6) and in Theorem 1.3 respectively.

3.1. Choice of test functions

Let $k \in \mathbb{N}^*$ and $m \in \mathbb{N}$. By (2.2), we look for a k-dimensional subspace V_h of $H_0^1(\Omega; \mathbb{C})$ such that

$$\sup_{v \in V_h \setminus \{0\}} \frac{4h^2 \int_\Omega |\partial_{\overline{z}} v|^2 e^{-2\phi/h} \, \mathrm{d}x}{\int_\Omega |v|^2 e^{-2(\phi - \phi_{\min})/h} \, \mathrm{d}x} \leq C_{\sup}(k) h^{-k+1}(1 + o(1)).$$

By the min-max principle, this would give (3.1). Formula (2.2) suggests taking functions of the form

$$v(x) = \chi(x) w(x),$$

where

(i) w is holomorphic on a neighborhood on Ω,

(ii) $\chi : \overline{\Omega} \to [0, 1]$ is a Lipschitzian function satisfying the Dirichlet boundary condition and equal to 1 away from a fixed neighborhood of the boundary.

In particular, there exists $\ell_0 \in (0, \mathsf{d}(x_{\min}, \partial\Omega))$ such that

$$\chi(x) = 1 \qquad \text{for all } x \in \Omega \text{ such that } \mathsf{d}(x, \partial\Omega) > \ell_0, \tag{3.2}$$

where d is the usual Euclidean distance.

Remark 3.2. The most naive test functions set could be

$$V_h = \operatorname{span}(\chi_h(z), \dots, \chi_h(x)(z - z_{\min})^{k-1}),$$

where $(\chi_h)_{h \in (0,1]}$ satisfy (3.2). With this choice, one would get

$$\sup_{v \in V_h \setminus \{0\}} \frac{4h^2 \int_\Omega |\partial_{\overline{z}} v|^2 e^{-2\phi/h} \, \mathrm{d}x}{\int_\Omega |v|^2 e^{-2(\phi - \phi_{\min})/h} \, \mathrm{d}x} \leq \widetilde{C_{\sup}}(k) h^{-k+1}(1 + o(1)),$$

where

$$\widetilde{C_{\sup}}(k) = 2\left(\frac{N_{\mathscr{H}}((z - z_{\min})^{k-1})}{\mathsf{dist}_{\mathscr{B}}(z^{k-1}, \mathscr{P}_{k-2})} \right)^2 \geq C_{\sup}(k).$$

Note however that in the radial case $\widetilde{C_{\sup}}(k) = C_{\sup}(k)$. We will rather use functions compatible with the Hardy space structure to get the bound of Proposition 3.1, as explained below.

Notation 4. Let $(P_n)_{n\in\mathbb{N}}$ denote the $N_{\mathcal{B}}$-orthogonal family such that $P_n(Z) = Z^n + \sum_{j=0}^{n-1} b_{n,j} Z^j$ obtained from a Gram–Schmidt process applied to $(1, Z, \ldots, Z^n, \ldots)$. Since P_n is $N_{\mathcal{B}}$-orthogonal to $\mathcal{P}_{n-1}$, we have

$$\begin{aligned}\mathsf{dist}_{\mathcal{B}}(Z^n, \mathcal{P}_{n-1}) &= \mathsf{dist}_{\mathcal{B}}(P_n, \mathcal{P}_{n-1}) = \inf\{N_{\mathcal{B}}(P_n - Q) : Q \in \mathcal{P}_{n-1}\} \\ &= \inf\{\sqrt{N_{\mathcal{B}}(P_n)^2 + N_{\mathcal{B}}(Q)^2} : Q \in \mathcal{P}_{n-1}\} = N_{\mathcal{B}}(P_n) \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{3.3}$$

Let $Q_n \in \mathscr{H}_k^2(\Omega)$ be the unique function such that

$$\mathsf{dist}_{\mathcal{H}}((z - z_{\min})^n, \mathscr{H}_k^2(\Omega)) = N_{\mathcal{H}}((z - z_{\min})^n - Q_n(z))$$

for $n \in \{0, \ldots, k-1\}$ (see Remark 1.6). We recall that $N_{\mathcal{B}}$, $N_{\mathcal{H}}$, $\mathcal{P}_{n-1}$, and $\mathscr{H}_k^2(\Omega)$ are defined in Section 1.3.1.

Lemma 3.3. *For all $n \in \{0, \ldots, k-1\}$, there exists a sequence $(Q_{n,m})_{m\in\mathbb{N}} \subset \mathscr{H}_k^2(\Omega) \cap W^{1,\infty}(\Omega)$ that converges to Q_n in $\mathscr{H}^2(\Omega)$.*

Proof. We can write $Q_n(z) = (z - z_{\min})^{k-1}\widetilde{Q}_n(z)$. Here, $\widetilde{Q}_n$ is a holomorphic function on Ω. Since $z \mapsto (z - z_{\min})^{1-k} \in L^\infty(\partial\Omega)$, we get $\widetilde{Q}_n \in \mathscr{H}^2(\Omega)$. By Lemma C.1, there exists a sequence $(\widetilde{Q}_{n,m})_{m\in\mathbb{N}} \subset \mathscr{H}^2(\Omega) \cap W^{1,\infty}(\Omega)$ converging to $\widetilde{Q}_n$ in $\mathscr{H}^2(\Omega)$. We have

$$N_{\mathcal{H}}((z - z_{\min})^{k-1}(\widetilde{Q}_{n,m} - \widetilde{Q}_n)) \le \|(z - z_{\min})^{k-1}\|_{L^\infty(\partial\Omega)} N_{\mathcal{H}}(\widetilde{Q}_{n,m} - \widetilde{Q}_n),$$

so that the sequence $(Q_{n,m})_{m\in\mathbb{N}} = ((z - z_{\min})^{k-1}\widetilde{Q}_{n,m})_{m\in\mathbb{N}} \subset \mathscr{H}_k^2(\Omega)$ converges to Q_n in $\mathscr{H}^2(\Omega)$. Since $z \mapsto (z - z_{\min})^{k-1} \in L^\infty(\partial\Omega)$, we have $Q_{n,m} \in \mathscr{H}_k^2(\Omega)$. ∎

Let us now define the k-dimensional vector space $V_{h,k,\sup}$ by

$$\begin{aligned} V_{h,k,\sup} &= \operatorname{span}(w_{0,h}, \ldots, w_{k-1,h}), \\ w_{n,h}(z) &= h^{-1/2} P_n\left(\frac{z - z_{\min}}{h^{1/2}}\right) - h^{-(1+n)/2} Q_{n,m}(z) \quad \text{for } n \in \{0, \ldots, k-1\}. \end{aligned} \tag{3.4}$$

At the end of the proof, m will be sent to $+\infty$. Note that we will not need the uniformity of the semiclassical estimates with respect to m. That is why the parameter m does not appear in our notations. Note that $w_{n,h}$, being a non-trivial holomorphic function, does not vanish identically at the boundary. To fulfill the Dirichlet condition, we have to add a cutoff function (see below).

Remark 3.4. Consider

$$\widetilde{\omega}_{n,h}(z) = h^{-1/2} P_n\left(\frac{z - z_{\min}}{h^{1/2}}\right) - h^{-(1+n)/2} Q_n(z).$$

Since Q_n belongs to $\mathscr{H}^2(\Omega) \not\subset H^1(\Omega;\mathbb{C})$, the functions $\widetilde{w}_{n,h} : x \mapsto \widetilde{\omega}_{n,h}(x_1 + ix_2)$ and $\chi\widetilde{w}_{n,h}$ do not belong necessarily to $H^1(\Omega;\mathbb{C})$ and $H_0^1(\Omega;\mathbb{C})$ respectively. That is why we have introduced $Q_{n,m}$. Note that to get $H_0^1(\Omega;\mathbb{C})$ test functions, it suffices to require that χ be compactly supported in Ω. With this strategy, our proof can be adapted to the case where Ω is not necessarily simply connected.

3.2. *Estimate of the L^2-norm*

The aim of this section is to prove the following estimate.

Lemma 3.5. *Let $h \in (0,1]$, $v_h = \chi \sum_{j=0}^{k-1} c_j w_{j,h}$ with $c_0, \dots, c_{k-1} \in \mathbb{C}$, χ satisfying (3.2) and $(w_{j,h})_{j\in\{0,\dots,k-1\}}$ defined in (3.4). Then*

$$\int_\Omega |v_h|^2 e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x = (1+o(1))\sum_{j=0}^{k-1} |c_j|^2 N_{\mathcal{B}}(P_j)^2, \tag{3.5}$$

where $N_{\mathcal{B}}$ is defined in Notation 3 and $o(1)$ does not depend on $c = (c_0, \dots, c_{k-1})$ or χ.

Proof. Let $\alpha \in (1/3, 1/2)$ and $n, n' \in \{0, \dots, k-1\}$.

In the proof, three types of terms will appear after a change of scale around $x_{\min}$: $\langle P_n, P_{n'}\rangle_{\mathcal{B}}$, $\langle P_n, Q_{n',m}\rangle_{\mathcal{B}}$ and $\langle Q_{n,m}, Q_{n',m}\rangle_{\mathcal{B}}$ where $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ is the scalar product associated with $N_{\mathcal{B}}$. Since the polynomials $(P_n)_{n\in\mathbb{N}}$ are $N_{\mathcal{B}}$-orthogonal, we have $\langle P_n, P_{n'}\rangle_{\mathcal{B}} = 0$ if $n \neq n'$ and we will prove that $\langle Q_{n,m}, Q_{n',m}\rangle_{\mathcal{B}} = \mathcal{O}(h)$ and by the Cauchy–Schwarz inequality $\langle P_n, Q_{n',m}\rangle_{\mathcal{B}} = \mathcal{O}(h^{1/2})$. More precisely, we proceed as follows:

(i) Let us estimate the weighted scalar products related to P_n for the weighted L^2-norm. Using the Taylor expansion of ϕ at $x_{\min}$, we get, for all $x \in D(x_{\min}, h^\alpha)$,

$$\frac{\phi(x)-\phi_{\min}}{h} = \frac{1}{2h}\operatorname{Hess}_{x_{\min}}\phi(x-x_{\min}, x-x_{\min}) + \mathcal{O}(h^{3\alpha-1}). \tag{3.6}$$

By using the change of coordinates

$$\Lambda_h : x \mapsto \frac{x-x_{\min}}{h^{1/2}}, \tag{3.7}$$

we find

$$\begin{aligned}
&\int_{D(x_{\min},h^\alpha)} h^{-1} P_n P_{n'}\left(\frac{x_1+ix_2-z_{\min}}{h^{1/2}}\right) e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x \\
&= (1+\mathcal{O}(h^{3\alpha-1}))\int_{D(x_{\min},h^\alpha)} h^{-1} P_n P_{n'}\left(\frac{x_1+ix_2-z_{\min}}{h^{1/2}}\right) e^{-\frac{1}{h}\operatorname{Hess}_{x_{\min}}\phi(x-x_{\min},x-x_{\min})}\,\mathrm{d}x \\
&= (1+\mathcal{O}(h^{3\alpha-1}))\int_{D(0,h^{\alpha-1/2})} P_n P_{n'}(y) e^{-\operatorname{Hess}_{x_{\min}}\phi(y,y)}\,\mathrm{d}y \\
&= (1+\mathcal{O}(h^{3\alpha-1}))\left(\langle P_n, P_{n'}\rangle_{\mathcal{B}} - \int_{\mathbb{C}\setminus D(0,h^{\alpha-1/2})} P_n P_{n'}(y) e^{-\operatorname{Hess}_{x_{\min}}\phi(y,y)}\,\mathrm{d}y\right) \\
&= (1+\mathcal{O}(h^{3\alpha-1}))\langle P_n, P_{n'}\rangle_{\mathcal{B}} + \mathcal{O}(h^\infty),
\end{aligned} \tag{3.8}$$

where the last equality follows from Assumption (c) in Theorem 1.3.

We recall assumptions (b) and (c) of Theorem 1.3. Then, by the Taylor expansion of ϕ at $x_{\min}$, we deduce that

$$\inf_{\Omega\setminus D(x_{\min},\, h^\alpha)} \phi \geq \phi_{\min} + \frac{\lambda_{\min}}{2} h^{2\alpha}(1+\mathcal{O}(h^\alpha)), \tag{3.9}$$

where $\lambda_{\min} > 0$ is the lowest eigenvalue of $\mathsf{Hess}_{x_{\min}} \phi$. Since P_n is of degree n, there exists $C > 0$ such that

$$\sup_{x\in\Omega} \left| h^{-1/2} P_n\left(\frac{x_1 + ix_2 - z_{\min}}{h^{1/2}}\right)\right| \le Ch^{-(n+1)/2}.$$

Using this with (3.9), we get

$$\begin{aligned}\left|\int_{\Omega\setminus D(x_{\min},h^\alpha)} h^{-1}\chi^2 P_n P_{n'}\left(\frac{x_1 + ix_2 - z_{\min}}{h^{1/2}}\right) e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x\right| \\ \le Ch^{-(n+1)/2} h^{-(n'+1)/2} e^{-\lambda_{\min} h^{2\alpha-1}(1+\mathscr{O}(h^\alpha))} = \mathscr{O}(h^\infty). \qquad (3.10)\end{aligned}$$

From (3.8) and (3.10), we find

$$\begin{aligned}\int_\Omega h^{-1}\chi^2 P_n P_{n'}\left(\frac{x_1 + ix_2 - z_{\min}}{h^{1/2}}\right) e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x \\ = (1 + \mathscr{O}(h^{3\alpha-1}))\langle P_n, P_{n'}\rangle_{\mathscr{B}} + \mathscr{O}(h^\infty). \qquad (3.11)\end{aligned}$$

(ii) Let us now deal with the weighted scalar products related to the $Q_{n,m}$. Let $u \in \mathscr{H}^2(\Omega)$ and $z_0 \in D(z_{\min}, h^\alpha)$. By the Cauchy formula (see [4, Theorem 10.4]) and the Cauchy–Schwarz inequality,

$$\begin{aligned}|u^{(k)}(z_0)| &= \frac{k!}{2\pi}\left|\int_{\partial\Omega} \frac{u(z)}{(z-z_0)^{k+1}}\,\mathrm{d}z\right| \\ &\le \frac{k!}{2\pi\sqrt{\min_{\partial\Omega}\partial_{\mathbf{n}}\phi}} N_{\mathscr{H}}(u)\left(\int_{\partial\Omega}\frac{|\mathrm{d}z|}{|z-z_0|^{2(k+1)}}\right)^{1/2} \\ &\le \frac{k!}{2\pi\sqrt{\min_{\partial\Omega}\partial_{\mathbf{n}}\phi}} N_{\mathscr{H}}(u)\left(\int_{\partial\Omega}\frac{|\mathrm{d}z|}{(|z-z_{\min}|-h^\alpha)^{2(k+1)}}\right)^{1/2} \\ &\le CN_{\mathscr{H}}(u). \qquad (3.12)\end{aligned}$$

With the Taylor formula for $u = Q_{n,m}$ at $z_{\min}$, this gives

$$|Q_{n,m}(z_0)| \le C|z_0 - z_{\min}|^k N_{\mathscr{H}}(Q_{n,m}).$$

Using (3.6), this implies

$$\begin{aligned}&\int_{D(x_{\min},h^\alpha)} |h^{-(1+n)/2} Q_{n,m}(x_1 + ix_2)|^2 e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x \\ &\quad\le Ch^{-(1+n)}\int_{D(x_{\min},h^\alpha)} |(x_1 + ix_2) - z_{\min}|^{2k} e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x \\ &\quad\le Ch^{k-n} N_{\mathscr{B}}(z^k)^2 \le Ch. \qquad (3.13)\end{aligned}$$

Using (3.9) and $Q_{n,m} \in W^{1,\infty}(\Omega) \subset L^2(\Omega)$, we get

$$\int_{\Omega\setminus D(x_{\min},h^\alpha)} |h^{-(1+n)/2}\chi Q_{n,m}(x_1+ix_2)|^2 e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x \\ \leq Ch^{-(n+1)}\|Q_{n,m}\|^2_{L^2(\Omega)} e^{-\lambda_{\min}h^{2\alpha-1}(1+\mathscr{O}(h^\alpha))} = \mathscr{O}(h^\infty). \quad (3.14)$$

With (3.13) and (3.14), we deduce

$$\int_\Omega |h^{-(1+n)/2}\chi Q_{n,m}(x_1+ix_2)|^2 e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x = \mathscr{O}(h). \quad (3.15)$$

Applying the Cauchy–Schwarz inequality and (3.15), we obtain

$$\int_\Omega \chi^2 h^{-(1+n)/2} Q_{n,m}(x_1+ix_2))h^{-(1+n')/2}\overline{Q_{n',m}(x_1+ix_2)}e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x = \mathscr{O}(h). \quad (3.16)$$

(iii) Let us now consider the scalar products involving the P_n and the $Q_{n',m}$. Using (3.11), (3.16), and the Cauchy–Schwarz inequality, we get

$$\int_\Omega \chi^2 h^{-1/2} P_n\left(\frac{x_1+ix_2-z_{\min}}{h^{1/2}}\right)h^{-(1+n')/2}\overline{Q_{n',m}(x_1+ix_2)}e^{-2(\phi(x)-\phi_{\min})/h}\,\mathrm{d}x \\ = \mathscr{O}(h^{1/2}). \quad (3.17)$$

The conclusion follows by expanding the square in the left-hand side of (3.5) and by using (3.11), (3.16), (3.17) . ■

Remark 3.6. From Lemma 3.5, we deduce that the vectors $\{\chi w_{j,h} : 0 \leq j \leq k-1\}$ are linearly independent for h small enough.

3.3. Estimate of the energy

The aim of this section is to bound from above the energy on an appropriate subspace.

Lemma 3.7. *There exists a family $(\chi_h)_{h\in(0,1]}$ of functions which satisfy (3.2) and such that, for all $w_h = \sum_{j=0}^{k-1} c_j w_{j,h} \in V_{h,k,\sup}$ with $c_0,\ldots,c_{k-1} \in \mathbb{C}$,*

$$4\int_\Omega h^2 e^{-2\phi/h}|\partial_{\overline{z}}(\chi_h w_h)|^2\,\mathrm{d}x \\ \leq 2h^{1-k}|c_{k-1}|^2 N_{\mathscr{H}}\big((z-z_{\min})^{k-1} - Q_{k-1,m}\big) + o(1)h^{1-k}\|c\|^2_{\ell^2}.$$

Here, $o(1)$ does not depend on $c_0,\ldots,c_{k-1}$.

Proof. Let χ be any function satisfying (3.2). We have

$$\begin{aligned} 4\int_\Omega h^2 e^{-2\phi/h}|\partial_{\overline{z}}\chi w_h|^2\,\mathrm{d}x &= h^2\int_\Omega |w_h|^2 e^{-2\phi/h}|\nabla\chi|^2\,\mathrm{d}x \\ &= h^2\int_{\mathrm{supp}\nabla\chi} |w_h|^2 e^{-2\phi/h}|\nabla\chi|^2\,\mathrm{d}x, \end{aligned}$$

where we have used $|\nabla\chi|^2 = 4|\partial_{\overline{z}}\chi|^2$ since χ is real and $\partial_{\overline{z}} w_h = 0$.

The proof is now divided into three steps. First, we introduce tubular coordinates near the boundary, then we make an explicit choice of χ, and finally we control the remainders.

(i) We only need to define χ in a neighborhood of $\Gamma = \partial\Omega$. To do this, we use the tubular coordinates given by the map

$$\eta : \mathbb{R}/(|\Gamma|\mathbb{Z}) \times (0, t_0) \to \Omega, \qquad (s,t) \mapsto \gamma(s) - t\mathbf{n}(s),$$

for t_0 small enough, γ being a parametrization of Γ with $|\gamma'(s)| = 1$ for all s, and $\mathbf{n}(s)$ the unit outward pointing normal at $\gamma(s)$ (see e.g. [8, §F]). We let

$$\eta^{-1}(x) = (s(x), t(x)) \quad \text{for all } x \in \eta(\mathbb{R}/(|\Gamma|\mathbb{Z}) \times (0, t_0)),$$

the inverse map to η. We let, for all $x \in \Omega$,

$$\chi(x) = \begin{cases} \rho(s(x), \mathsf{d}(x, \partial\Omega)) & \text{if } \mathsf{d}(x, \partial\Omega) \le \varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

The parameter $\varepsilon > 0$ and the function ρ are to be determined. We assume that $\rho(s, 0) = 0$ and $\rho(s,t) = 1$ when $t \ge \varepsilon$. We will choose $\varepsilon = o(h^{1/2})$.

Since the metric induced by the change of variable is the Euclidean metric modulo $\mathscr{O}(\varepsilon)$, we get

$$\begin{aligned} &h^2\int_{\mathrm{supp}\nabla\chi} |w_h|^2 e^{-2\phi/h}|\nabla\chi|^2\,\mathrm{d}x \\ &\qquad \le (1+\mathscr{O}(\varepsilon))h^2\int_\Gamma\int_0^\varepsilon |\tilde{w}_h|^2 e^{-2\tilde{\phi}(s,t)/h}(|\partial_t\rho|^2 + |\partial_s\rho|^2)\,\mathrm{d}s\,\mathrm{d}t, \end{aligned}$$

where $\tilde{w}_h = w_h \circ \eta$ and $\tilde{\phi} = \phi \circ \eta$. Thus, by using the Taylor expansion of $\tilde{\phi}$ at $t = 0$ we get, uniformly in $s \in \Gamma$,

$$\tilde{\phi}(s,t) = t\partial_t\tilde{\phi}(s,0) + \mathscr{O}(t^2) = -t\partial_{\mathbf{n}}\phi(s,0) + \mathscr{O}(\varepsilon^2),$$

and

$$\begin{aligned} &h^2\int_{\mathrm{supp}\nabla\chi} |w_h|^2 e^{-2\phi/h}|\nabla\chi|^2\,\mathrm{d}x \\ &\qquad \le (1+\mathscr{O}(\varepsilon+\varepsilon^2/h))h^2\int_\Gamma\int_0^\varepsilon |\tilde{w}_h|^2 e^{2t\partial_{\mathbf{n}}\phi(s)/h}(|\partial_t\rho|^2 + |\partial_s\rho|^2)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

Since $Q_{n,m} \in W^{1,\infty}(\Omega)$, we have $\partial_t \tilde{Q}_{n,m} \circ \eta \in L^\infty(\Gamma \times (0,\varepsilon))$ and by using the Taylor expansion of $\tilde{w}$ near $t = 0$, we get

$$\begin{aligned}\tilde{w}_h(s,t) &= \Big(\sum_{j=0}^{k-1} c_j w_{j,h}\Big) \circ \eta(s,t) = \tilde{w}_h(s,0) + \int_0^t \partial_t \tilde{w}_h(s,t')\,\mathrm{d}t' \\ &= \tilde{w}_h(s,0) + \mathscr{O}(\varepsilon)\|c_h\|_{\ell^2},\end{aligned}$$

where

$$c_h = (h^{-1/2}c_0, \dots, h^{-k/2}c_{k-1}), \tag{3.18}$$

and $\|\cdot\|_{\ell^2}$ is the canonical Euclidean norm on $\mathbb{C}^k$. Then

$$\begin{aligned}&h^2 \int_{\mathrm{supp}\nabla\chi} |w_h|^2 e^{-2\phi/h}|\nabla\chi|^2\,\mathrm{d}x \\ &\quad\le (1+\mathscr{O}(\varepsilon+\varepsilon^2/h))h^2 \int_\Gamma |\tilde{w}_h(s,0)|^2 \int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi(s)/h}(|\partial_t\rho|^2 + |\partial_s\rho|^2)\,\mathrm{d}s\,\mathrm{d}t \\ &\qquad + Ch^2\varepsilon\|c_h\|^2_{\ell^2} \int_\Gamma\int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi(s)/h}(|\partial_t\rho|^2 + |\partial_s\rho|^2)\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{3.19}$$

(ii) For the right-hand side of (3.19) to be small, we choose ρ to minimize $\partial_t\rho$ far from the boundary. The optimization of

$$\rho \mapsto \int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi/h}\,|\partial_t\rho|^2\,\mathrm{d}t$$

gives us the weight $\partial_{\mathbf{n}}\phi$. More precisely, Lemma A.1 with $\alpha = 2\partial_{\mathbf{n}}\phi/h > 0$ suggests considering the trial state defined, for $t \le \varepsilon$, by

$$\rho(s,t) = \frac{1 - e^{-2t\partial_{\mathbf{n}}\phi(s)/h}}{1 - e^{-2\varepsilon\partial_{\mathbf{n}}\phi(s)/h}},$$

and by 1 otherwise. By Lemma A.1, we get

$$\int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi/h}\,|\partial_t\rho|^2\,\mathrm{d}t = \frac{2\partial_{\mathbf{n}}\phi/h}{1 - e^{-\varepsilon 2\partial_{\mathbf{n}}\phi/h}},$$

and

$$\begin{aligned}\int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi/h}\,|\partial_s\rho|^2\,\mathrm{d}t = |\partial_s\alpha|^2 \int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi/h}|\partial_\alpha\rho_{\alpha,\varepsilon}|^2\,\mathrm{d}t &\le Ch^{-2}(\alpha^{-3} + e^{-\alpha\varepsilon}\varepsilon^2\alpha^{-1}) \\ &\le C(h + e^{-\varepsilon 2\partial_{\mathbf{n}}\phi/h}\varepsilon^2 h^{-1}).\end{aligned}$$

We can choose $\varepsilon = h|\ln h|$ so that

$$\int_\Gamma\int_0^\varepsilon |\tilde{w}_h(s,0)|^2 e^{2t\partial_{\mathbf{n}}\phi(s)/h}\,|\partial_t\rho|^2\,\mathrm{d}s\,\mathrm{d}t = (1+o(1))h^{-1}\int_\Gamma 2\partial_{\mathbf{n}}\phi|\tilde{w}_h(s,0)|^2\,\mathrm{d}s,$$

and (3.19) becomes

$$h^2 \int_{\operatorname{supp}\nabla\chi} |w_h|^2 e^{-2\phi/h} |\nabla\chi|^2 \, dx \\ \leq (1+o(1))h\Big(\int_\Gamma 2\partial_{\mathbf{n}}\phi |\tilde{w}_h(s,0)|^2 \, ds + C\varepsilon \|c_h\|_{\ell^2}^2\Big). \tag{3.20}$$

(iii) Let us consider, for all $h \geq 0$,

$$N_h : \mathbb{C}^k \ni c \mapsto \Big(\int_\Gamma \partial_{\mathbf{n}}\phi \Big|\sum_{j=0}^{k-1} c_j h^{(1+j)/2} \tilde{w}_{j,h}(s,0)\Big|^2 \, ds\Big)^{1/2},$$

where we recall that

$$w_{j,h}(z) = h^{-1/2} P_j\Big(\frac{z - z_{\min}}{h^{1/2}}\Big) - h^{-(j+1)/2} Q_{j,m}(z).$$

The map $\mathbb{C}^k \times [0,1] \ni (c,h) \mapsto N_h(c)$ is well defined and continuous (since the degree of P_j is j). Note in particular that

$$N_0(c) = \Big(\int_\Gamma \partial_{\mathbf{n}}\phi \Big|\sum_{j=0}^{k-1} c_j [(z - z_{\min})^j - Q_{j,m}(z)]\Big|^2 \, ds\Big)^{1/2}.$$

Notice that

$$N_h(c_h)^2 = \int_\Gamma \partial_{\mathbf{n}}\phi |\tilde{w}_h(s,0)|^2 \, ds = N_{\mathscr{H}}(w_h)^2, \tag{3.21}$$

where c_h is defined in (3.18). Since $N_{\mathscr{H}}$ is a norm, and recalling Remark 3.6, we see that the map N_h is a norm when $h \in (0, h_0]$; N_0 is also a norm (as we can see by using the Hardy norm and $Q_{j,m} \in \mathscr{H}_k^2(\Omega)$).

Let us define

$$C_0 = \min_{\substack{h\in[0,h_0] \\ \|c\|_{\ell^2}=1}} N_h(c) > 0.$$

so that, for all $h \in [0, h_0]$ and all $c \in \mathbb{C}^k$,

$$C_0 \|c\|_{\ell^2} \leq N_h(c). \tag{3.22}$$

Using (3.20), (3.21), and replacing c by c_h in (3.22), we conclude that

$$h^2 \int_{\operatorname{supp}\nabla\chi} |w_h|^2 e^{-2\phi/h} |\nabla\chi|^2 \, dx \leq 2(1+o(1)) h N_{\mathscr{H}}(w_h)^2.$$

Let us now estimate $N_{\mathscr{H}}(w_h)$. From the triangle inequality, we get

$$N_{\mathscr{H}}(w_h) \leq |c_{k-1}| N_{\mathscr{H}}(w_{k-1,h}) + \sum_{j=0}^{k-2} |c_j| N_{\mathscr{H}}(w_{j,h}).$$

Then, from degree considerations and the triangle inequality, we get, for $1 \leq j \leq k-2$,

$$N_{\mathcal{H}}(w_{j,h}) = \mathcal{O}(h^{(1-k)/2}),$$

and

$$N_{\mathcal{H}}(w_{k-1,h}) = (1+o(1))h^{-k/2} N_{\mathcal{H}}\big((z-z_{\min})^{k-1} - Q_{k-1,m}\big).$$

Then

$$N_{\mathcal{H}}(w_h)^2 \leq |c_{k-1}|^2 h^{-k} N_{\mathcal{H}}\big((z-z_{\min})^{k-1} - Q_{k-1,m}\big)^2 + o(h^{-k})\|c\|_{\ell^2}^2.$$

This ends the proof. ∎

3.4. *Proof of Proposition* 3.1

Let us define $\widetilde{V}_{h,k,\mathrm{sup}} = \{\chi_h w_h : w_h \in V_{h,k,\mathrm{sup}}\}$, where $V_{h,k,\mathrm{sup}}$ is defined in (3.4) and χ_h in Lemma 3.7. By Lemmas 3.5 and 3.7, we get

$$\frac{4\int_\Omega h^2 e^{-2\phi/h} |\partial_{\overline{z}}(w_h\chi_h)|^2 \,\mathrm{d}x}{\int_\Omega |w_h\chi_h|^2 e^{-2(\phi-\phi_{\min})/h}\,\mathrm{d}x} \leq 2h^{1-k} \frac{|c_{k-1}|^2 N_{\mathcal{H}}((z-z_{\min})^{k-1} - Q_{k-1,m})^2}{\sum_{j=0}^{k-1} |c_j|^2 N_{\mathcal{B}}(P_j)^2} + o(h^{1-k})$$

for all $w_h = \sum_{j=0}^{k-1} c_j w_{j,h} \in V_{h,k,\mathrm{sup}}$ with $c \in \mathbb{C}^k \setminus \{0\}$. From the min-max principle,[3] it follows that

$$\lambda_k(h) \leq 2h^{1-k} N_{\mathcal{H}}\big((z-z_{\min})^{k-1} - Q_{k-1,m}\big)^2 \times \sup_{c\in\mathbb{C}^k\setminus\{0\}} \frac{|c_{k-1}|^2}{\sum_{j=0}^{k-1} |c_j|^2 N_{\mathcal{B}}(P_j)^2} e^{2\phi_{\min}/h} + o(h^{1-k}).$$

Since

$$\sup_{c\in\mathbb{C}^k\setminus\{0\}} \frac{|c_{k-1}|^2}{\sum_{j=0}^{k-1} |c_j|^2 N_{\mathcal{B}}(P_j)^2} = N_{\mathcal{B}}(P_{k-1})^{-2},$$

we deduce

$$\limsup_{h\to 0} h^{k-1} e^{-2\phi_{\min}/h} \lambda_k(h) \leq 2\left(\frac{N_{\mathcal{H}}((z-z_{\min})^{k-1} - Q_{k-1,m})}{\mathrm{dist}_{\mathcal{B}}(z^{k-1}, \mathcal{P}_{k-2})}\right)^2.$$

Taking the limit as $m \to +\infty$ we get

$$\limsup_{h\to 0} h^{k-1} e^{-2\phi_{\min}/h} \lambda_k(h) \leq C_{\mathrm{sup}}(k).$$

[3] By Remark 3.6, $\dim \widetilde{V}_{h,k,\mathrm{sup}} = k$ for h small enough.

3.5. *Computation of $C_{\sup}(k)$ in the radial case*

Let $k \in \mathbb{N}^*$. Assume that Ω is the disk of radius R centered at 0, and that B is radial. In this case $x_{\min} = 0$, $\partial_{\mathbf{n}}\phi$ is constant and $\mathsf{Hess}_{x_{\min}}\phi = B(0)\,\mathrm{Id}/2$. Thus,

$$\mathsf{dist}_{\mathscr{H}}((z - z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega)) = \mathsf{dist}_{\mathscr{H}}(z^{k-1}, \mathscr{H}^2_k(\Omega)) = N_{\mathscr{H}}(z^{k-1})^2 = 2\pi\,\partial_{\mathbf{n}}\phi R^{2k-1},$$

and we notice that $P_n(z) = z^n$ (see Notation 4) so that

$$\begin{aligned}\mathsf{dist}_{\mathscr{B}}(z^{k-1}, \mathscr{P}_{k-2}) &= N_{\mathscr{B}}(P_{k-1})^2\\ &= \int_{\mathbb{R}^2} |y|^{2(k-1)}\, e^{-\mathsf{Hess}_{x_{\min}}\phi(y,y)}\,\mathrm{d}y = 2\pi \int_0^{+\infty} \rho^{2k-1} e^{-B(0)\rho^2/2}\,\mathrm{d}\rho\\ &= \frac{2\pi 2^k}{B(0)^k}\int_0^{+\infty} \rho^{2k-1}e^{-\rho^2}\,\mathrm{d}\rho = \frac{2\pi 2^{k-1}\Gamma(k)}{B(0)^k} = \frac{2\pi 2^{k-1}(k-1)!}{B(0)^k},\end{aligned}$$

We get

$$C_{\sup}(k) = \frac{B(0)^k\,\Phi R^{2k-2}}{2^{k-2}(k-1)!}.$$

Note that this formula extends the upper bound obtained in [11] for constant magnetic fields on the disk.

4. On the magnetic Cauchy–Riemann operators

In this section, U will denote an open bounded subset of $\mathbb{R}^2$. It will be either Ω itself, or a smaller open set.

As we already observed (see (1.3)), the Dirichlet–Pauli operator, considered only as a differential operator, is the square of the magnetic Dirac operator $\sigma\cdot(\mathbf{p} - \mathbf{A})$. It can be written as

$$\sigma\cdot(\mathbf{p} - A) = \begin{pmatrix} 0 & d_{h,A} \\ d^{\times}_{h,A} & 0 \end{pmatrix} \tag{4.1}$$

where $d_{h,A}$ and $d^{\times}_{h,A}$ are the magnetic Cauchy–Riemann operators:

$$d_{h,A} = -2ih\partial_z - A_1 + iA_2, \qquad d^{\times}_{h,A} = -2ih\partial_{\overline{z}} - A_1 - iA_2.$$

Let $(d_{h,A}, \mathsf{Dom}(d_{h,A}))$ be the operator on $L^2(U;\mathbb{C})$ acting as $d_{h,A}$ on $\mathsf{Dom}(d_{h,A}) = H^1_0(U;\mathbb{C})$.

4.1. *Properties of $d_{1,0}$ and $d^*_{1,0}$*

In this part, we study the operators $d_{h,A}$ and $d^*_{h,A}$ in the non-magnetic case $B = 0$ with $h = 1$ in order to describe their properties in this simplified setting in which $-\Delta = d^*_{1,0}d_{1,0}$. Various aspects of this section can be related to the spectral analysis of the "zig-zag" operator (see [16]). The next section will be related to the magnetic case that is needed in our study.

Lemma 4.1. *Assume that U is of class $\mathscr{C}^2$. The following properties hold.*

(a) *The operator $(d_{1,0}, \mathsf{Dom}(d_{1,0}))$ is closed with closed range.*

(b) *The domain of $d_{1,0}^*$ is given by*

$$\mathsf{Dom}(d_{1,0}^*) = \{u \in L^2(U;\mathbb{C}) : \partial_{\overline{z}} u \in L^2(U;\mathbb{C})\}$$
$$= \{u \in L^2(U;\mathbb{C}) : \partial_{\overline{z}} u = 0\} + H^1(U;\mathbb{C}),$$

and $d_{1,0}^*$ acts as $d_{1,0}^\times$. In particular,

$$\ker(d_{1,0}^*) = \{u \in L^2(U;\mathbb{C}) : \partial_{\overline{z}} u = 0\}.$$

(c) *We have*

$$\ker(d_{1,0}^*)^\perp \cap \mathsf{Dom}(d_{1,0}^*) = \{d_{1,0} w : w \in H_0^1(U;\mathbb{C}) \cap H^2(U;\mathbb{C})\} \subset H^1(U;\mathbb{C}),$$

and there exists $C > 0$ such that, for all $v \in \ker(d_{1,0}^*)^\perp \cap \mathsf{Dom}(d_{1,0}^*)$,

$$\|v\|_{H^1(U)} \leq C \|d_{1,0}^* v\|_{L^2(U)}.$$

Proof. Let $u \in \mathsf{Dom}(d_{1,0}) = H_0^1(U;\mathbb{C})$. One easily checks that

$$\|d_{1,0} u\|_{L^2(U)}^2 = \|\nabla u\|_{L^2(U)}^2.$$

Hence, the Poincaré inequality ensures that $(d_{1,0}, \mathsf{Dom}(d_{1,0}))$ is a closed operator with closed range. Then, by definition of the domain of the adjoint,

$$\mathsf{Dom}(d_{1,0}^*) \subset \{u \in L^2(U;\mathbb{C}) : \partial_{\overline{z}} u \in L^2(U;\mathbb{C})\}.$$

Conversely, if $v \in \{u \in L^2(U;\mathbb{C}) : \partial_{\overline{z}} u \in L^2(U;\mathbb{C})\}$, then, for all $w \in \mathscr{C}_0^\infty(U)$,

$$\langle v, -2i\,\partial_z w\rangle_{L^2(U)} = \langle -2i\,\partial_{\overline{z}} v, w\rangle_{L^2(U)}\,.$$

By density, this equality can be extended to $w \in H_0^1(U;\mathbb{C})$. This shows, by definition, that $v \in \mathsf{Dom}(d_{1,0}^*)$ and $d_{1,0}^* v = -2i\,\partial_{\overline{z}} v$.

Moreover, we have

$$\begin{aligned}\ker(d_{1,0}^*)^\perp \cap \mathsf{Dom}(d_{1,0}^*) &= \operatorname{ran}(d_{1,0}) \cap \mathsf{Dom}(d_{1,0}^*)\\ &= \{d_{1,0} w : w \in H_0^1(U;\mathbb{C}) \text{ and } -2i\,\partial_{\overline{z}}(d_{1,0} w) = -\Delta w \in L^2(U;\mathbb{C})\}\\ &= \{d_{1,0} w : w \in H_0^1(U;\mathbb{C}) \cap H^2(U;\mathbb{C})\} \subset H^1(U;\mathbb{C}),\end{aligned}$$

where the last equality follows from the elliptic regularity of the Laplacian. In particular, for all $w \in H_0^1(U;\mathbb{C}) \cap H^2(U;\mathbb{C})$,

$$\|w\|_{H^2(U)} \leq C \|\Delta w\|_{L^2(U)}.$$

Now, take $v \in \ker(d_{1,0}^*)^\perp \cap \mathsf{Dom}(d_{1,0}^*)$. We can write $v = d_{1,0} w$ with $w \in H^2(U;\mathbb{C}) \cap H_0^1(U;\mathbb{C})$. We have $d_{1,0}^* v = -\Delta w$ so that

$$\|v\|_{H^1(U)} \leq C \|d_{1,0}^* v\|_{L^2(U)}.$$ ■

4.2. Properties of $d_{h,A}$ and $d^*_{h,A}$

Let us introduce some notations related to the Riemann mapping theorem.

In the following, we gather some standard properties related to $d_{h,A}$ and $d^*_{h,A}$. We will use the following lemma.

Lemma 4.2. *For all $u \in \mathscr{C}_0^\infty(U;\mathbb{C})$, we have*

$$\|d_{h,A}u\|^2_{L^2(U)} = \|(\mathbf{p}-A)u\|^2_{L^2(U)} + h\int_U B|u|^2\,\mathrm{d}x,$$

$$\|d^\times_{h,A}u\|^2_{L^2(U)} = \|(\mathbf{p}-A)u\|^2_{L^2(U)} - h\int_U B|u|^2\,\mathrm{d}x.$$

These formulas can be extended to $u \in H^1_0(U;\mathbb{C})$.

Proof. This follows by integration by parts and the fact that $d_{h,A}d^\times_{h,A} = |\mathbf{p}-A|^2 - hB$ and $d^\times_{h,A}d_{h,A} = |\mathbf{p}-A|^2 + hB$. The extension to $u \in H^1_0(U;\mathbb{C})$ follows by density. ∎

Remark 4.3. From Lemma 4.2, we deduce[4] that for all $u \in H^1_0(U;\mathbb{C})$,

$$\|(\mathbf{p}-A)u\|^2_{L^2(U)} \geq \int_U hB|u|^2\,\mathrm{d}x.$$

Proposition 4.4. *Assume that U is of class $\mathscr{C}^2$.*

(a) *The operator $(d_{h,A}, \mathsf{Dom}(d_{h,A}))$ is closed with closed range.*

(b) *The adjoint $(d^*_{h,A}, \mathsf{Dom}(d^*_{h,A}))$ acts as $d^\times_{h,A}$ on*

$$\mathsf{Dom}(d^*_{h,A}) = \{u \in L^2(U) : \partial_{\overline{z}}u \in L^2(U)\} = \ker(d^*_{h,A}) + H^1(U;\mathbb{C})$$

and

$$\ker(d^*_{h,A}) = \{e^{-\phi/h}v : v \in L^2(U),\ \partial_{\overline{z}}v = 0\}.$$

(c) *We have* $\ker(d^*_{h,A})^\perp \cap \mathsf{Dom}(d^*_{h,A}) = \{d_{h,A}w : w \in H^1_0(U;\mathbb{C}) \cap H^2(U;\mathbb{C})\}$.

Notation 5. The notation $d_{h,A,U}$ for $d_{h,A}$ emphasizes the dependence on U. We denote by $\Pi_{h,A,U}$ (or simply $\Pi_{h,A}$ if there is no ambiguity) the orthogonal projection on $\ker(d^*_{h,A})$.

Proof of Proposition 4.4. (a) By Lemma 4.2, the graph norm of $d_{h,A}$ and the usual H^1-norm are equivalent. Thus, the graph of $d_{h,A}$ is a closed subspace of $L^2(U) \times L^2(U)$. From Lemma 4.2 and Remark 4.3, we get, for all $u \in H^1_0(U)$,

$$\|d_{h,A}u\|^2_{L^2(U)} \geq h\int_U 2B|u|^2\,\mathrm{d}x.$$

[4] This may also be found in [8, Lemma 1.4.1].

By assumption (a) of Theorem 1.3 and the fact that the operator is closed, the range is closed.

(b) We have $\mathsf{Dom}(d_{h,A}^*) = \mathsf{Dom}(d_{1,0}^*)$, and $d_{h,A}^*$ acts as $d_{h,A}^\times$. By Proposition 2.1 and Lemma 4.1, we deduce

$$\ker(d_{h,A}^*) = \{e^{-\phi/h}v : v \in L^2(U),\ \partial_{\overline{z}}v = 0\}.$$

(c) As in the proof of Lemma 4.1, we get

$$\begin{aligned}\ker(d_{h,A}^*)^\perp \cap \mathsf{Dom}(d_{h,A}^*) &= \mathrm{ran}(d_{h,A}) \cap \mathsf{Dom}(d_{h,A}^*)\\ &= \{d_{h,A}w : w \in H_0^1(U;\mathbb{C}) \text{ and } d_{h,A}^* d_{h,A} w = (|\mathbf{p}-A|^2 + hB)w \in L^2(U;\mathbb{C})\}\\ &= \{d_{h,A}w : w \in H_0^1(U;\mathbb{C}) \text{ and } -\Delta w \in L^2(U;\mathbb{C})\}\\ &= \{d_{h,A}w : w \in H_0^1(U;\mathbb{C}) \cap H^2(U;\mathbb{C})\}.\end{aligned}$$

■

Definition 4.5. We define the self-adjoint operators $(\mathscr{L}_h^\pm, \mathsf{Dom}(\mathscr{L}_h^\pm))$ to act as

$$\mathscr{L}_h^- = d_{h,A} d_{h,A}^\times = |\mathbf{p}-A|^2 - hB, \qquad \mathscr{L}_h^+ = d_{h,A}^\times d_{h,A} = |\mathbf{p}-A|^2 + hB, \tag{4.2}$$

on the respective domains

$$\begin{aligned}\mathsf{Dom}(\mathscr{L}_h^-) &= \{u \in \mathsf{Dom}(d_{h,A}^*) : d_{h,A}^* u \in \mathsf{Dom}(d_{h,A})\},\\ \mathsf{Dom}(\mathscr{L}_h^+) &= \{u \in \mathsf{Dom}(d_{h,A}) : d_{h,A} u \in \mathsf{Dom}(d_{h,A}^*)\}\\ &= H_0^1(U;\mathbb{C}) \cap H^2(U;\mathbb{C}).\end{aligned}$$

4.3. *Semiclassical elliptic estimates for the magnetic Cauchy–Riemann operator*

Notation 6. By the Riemann mapping theorem, and since $\partial\Omega$ is assumed to be $\mathscr{C}^2$, it is Dini-continuous (see [15, Theorem 2.1, and Section 3.3]) and we can consider a biholomorphic function F between $D(0,1)$ and Ω such that $F(\partial D(0,1)) = \partial\Omega$. We write $x = F(y)$. We notice that

$$\partial_{y_1} + i\partial_{y_2} = \overline{F'(y)}(\partial_{x_1} + i\partial_{x_2}) \quad \text{and} \quad \mathrm{d}x = |F'(y)|^2\,\mathrm{d}y.$$

By [15, Theorem 3.5], this biholomorphism can be continuously extended to $\overline{D(0,1)}$, and there exist $c_1, c_2 > 0$ such that, for all $y \in \overline{D(0,1)}$,

$$c_1 \leq |F'(y)| \leq c_2.$$

For $\delta \in (0,1)$, we also let

$$\Omega_\delta = F(D(0,1-\delta)).$$

Note that Ω_δ is actually an analytic manifold.

The following theorem is a crucial ingredient in the proof of the lower bound of $\lambda_k(h)$. It is intimately related to the spectral supersymmetry of Dirac operators [17, Theorem 5.5 and Corollary 5.6].

Theorem 4.6. *There exist* $\delta_0, h_0, c > 0$ *such that, for all* $\delta \in [0, \delta_0)$*, all* $h \in (0, h_0)$*, and all* $u \in \mathsf{Dom}(d^*_{h,A,\Omega_\delta}) \cap \ker(d^*_{h,A,\Omega_\delta})^\perp$,

$$\|d^*_{h,A,\Omega_\delta} u\|_{L^2(\Omega_\delta)} \geq \sqrt{2hB_0}\, \|u\|_{L^2(\Omega_\delta)},$$
$$\|d^*_{h,A,\Omega_\delta} u\|_{L^2(\Omega_\delta)} \geq ch^2(\|\nabla u\|_{L^2(\Omega_\delta)} + \|u\|_{L^2(\partial\Omega_\delta)}),$$

where we use Notation 6.

Theorem 4.6 follows from the next two lemmas.

Lemma 4.7. *For all* $u \in \mathsf{Dom}(d^*_{h,A,\Omega_\delta}) \cap \ker(d^*_{h,A,\Omega_\delta})^\perp$*, we have*

$$\|d^*_{h,A,\Omega_\delta} u\|_{L^2(U)} \geq \sqrt{2hB_0}\, \|u\|_{L^2(U)}.$$

Proof. For notational simplicity, we let $U = \Omega_\delta$ and we write $d_{h,A}$ for $d_{h,A,U}$. Let $u \in \mathsf{Dom}(d^*_{h,A}) \cap \ker(d^*_{h,A})^\perp$. By Proposition 4.4, there exists $w \in H^1_0(U;\mathbb{C}) \cap H^2(U;\mathbb{C})$ such that $u = d_{h,A} w$ and $d^*_{h,A} u = \mathscr{L}^+_h w$. The spectrum of $\mathscr{L}^+_h$ is a subset of $[2hB_0, +\infty)$ (see Remark 4.3). Thus, we get

$$\|\mathscr{L}^+_h w\|_{L^2(U)} \geq 2hB_0 \|w\|_{L^2(U)}.$$

By integration by parts and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} 2hB_0 \|d_{h,A} w\|^2_{L^2(U)} &\leq 2hB_0\, \langle w, \mathscr{L}^+_h w\rangle_{L^2(U)} \leq 2hB_0 \|w\|_{L^2(U)} \|\mathscr{L}^+_h w\|_{L^2(U)} \\ &\leq \|\mathscr{L}^+_h w\|^2_{L^2(U)}. \end{aligned}$$

This ensures that

$$\sqrt{2hB_0}\, \|d_{h,A} w\|_{L^2(U)} \leq \|\mathscr{L}^+_h w\|_{L^2(U)} = \|d^*_{h,A}(d_{h,A} w)\|_{L^2(U)}$$

and the conclusion follows. ∎

Lemma 4.8. *There exist* $\delta_0, h_0, c > 0$ *such that, for all* $\delta \in [0, \delta_0)$*, all* $h \in (0, h_0)$*, and all* $u \in \mathsf{Dom}(d^*_{h,A,\Omega_\delta}) \cap \ker(d^*_{h,A,\Omega_\delta})^\perp$,

$$\|d^*_{h,A,\Omega_\delta} u\|_{L^2(\Omega_\delta)} \geq ch^2 \|\nabla u\|_{L^2(\Omega_\delta)} + ch^2 \|u\|_{L^2(\partial\Omega_\delta)}.$$

Proof. For notational simplicity, we let $U = \Omega_\delta$ and we write $d_{h,A}$ for $d_{h,A,U}$.

With the same notations as in the proof of Lemma 4.7 ($u = d_{h,A} w$), we have

$$d^*_{h,A} u = d^*_{h,A} d_{h,A} w = \mathscr{L}^+_h w, \qquad w \in H^1_0(U) \cap H^2(U).$$

(i) From Lemma 4.2,

$$\begin{aligned} \|d_{h,A} w\|^2_{L^2(U)} &= \|(\mathbf{p} - A) w\|^2_{L^2(U)} + h \int_U B|w|^2 \,\mathrm{d}x \\ &= \langle d^*_{h,A} u, w\rangle_{L^2(U)} \leq \|d^*_{h,A} u\|_{L^2(U)} \|w\|_{L^2(U)}. \end{aligned}$$

Using assumption (a), we get

$$B_0 h\|w\|^2_{L^2(U)} \leq h\int_U B|w|^2\,\mathrm{d}x \leq \|d^*_{h,A}u\|_{L^2(U)}\|w\|_{L^2(U)},$$

and

$$h\|w\|_{L^2(U)} \leq B_0^{-1}\|d^*_{h,A}u\|_{L^2(U)}. \tag{4.3}$$

Since

$$h\int_U B|w|^2\,\mathrm{d}x \leq \|(\mathbf{p}-A)w\|^2_{L^2(U)},$$

we deduce that

$$B_0^{1/2}h^{1/2}\|w\|_{L^2(U)} \leq \|(\mathbf{p}-A)w\|_{L^2(U)},$$

and

$$\begin{aligned}\|(\mathbf{p}-A)w\|^2_{L^2(U)} &\leq \|d^*_{h,A}u\|_{L^2(U)}\|w\|_{L^2(U)}\\ &\leq \|d^*_{h,A}u\|_{L^2(U)}B_0^{-1/2}h^{-1/2}\|(\mathbf{p}-A)w\|_{L^2(U)},\end{aligned} \tag{4.4}$$

so that in view of (4.3) and (4.4), there exists $C>0$ such that

$$h^{1/2}\|(\mathbf{p}-A)w\|_{L^2(U)} + h\|w\|_{L^2(U)} \leq C\|d^*_{h,A}u\|_{L^2(U)}.$$

Since A is bounded,

$$\begin{aligned}h^{3/2}\|\nabla w\|_{L^2(U)} &\leq C\|d^*_{h,A}u\|_{L^2(U)} + Ch^{1/2}\|w\|_{L^2(U)}\\ &\leq Ch^{-1/2}\|d^*_{h,A}u\|_{L^2(U)}.\end{aligned}$$

Thus,

$$h^2\|\nabla w\|_{L^2(U)} + h\|w\|_{L^2(U)} \leq C\|d^*_{h,A}u\|_{L^2(U)}. \tag{4.5}$$

(ii) Let us now deal with the derivatives of order two. From the explicit expression of $\mathscr{L}_h^+ w$, we get

$$-h^2\Delta w = d^*_{h,A}u - 2ihA\cdot\nabla w - |A|^2 w + hBw.$$

Taking the L^2-norm and using (4.5), we get

$$\begin{aligned}&h^2\|\Delta w\|_{L^2(U)}\\ &\quad\leq \|d^*_{h,A}u\|_{L^2(U)} + \|-2ihA\cdot\nabla w\|_{L^2(U)} + \||A|^2 w\|_{L^2(U)} + \|hBw\|_{L^2(U)}\\ &\quad\leq C(1+h^{-1})\|d^*_{h,A}u\|_{L^2(U)}.\end{aligned}$$

Using a standard ellipticity result for the Dirichlet Laplacian, we find

$$h^3\|w\|_{H^2(U)} + h^2\|\nabla w\|_{L^2(U)} + h\|w\|_{L^2(U)} \leq C\|d^*_{h,A}u\|_{L^2(U)}. \tag{4.6}$$

The uniformity of the constant with respect to $\delta\in(0,\delta_0)$ can be checked as in the classical proof of elliptic regularity. Alternatively, using the Riemann mapping theorem, we map Ω onto the unit disk. Then we perform a change of scale for each δ to send $D(0, 1-\delta)$ onto $D(0,1)$ and use a standard ellipticity result on $D(0,1)$. Here, δ appears as a regular parameter in the coefficients of the elliptic operator. Note that $d_{h,A} = L_1 - iL_2$ where $L_j = -ih\partial_j - A_j$. Using (4.6), we deduce that

$$\begin{aligned}\|\nabla d_{h,A} w\|_{L^2(U)} &\leq Ch\|w\|_{H^2(U)} + C\|w\|_{L^2(U)} + C\|\nabla w\|_{L^2(U)}\\ &\leq Ch^{-2}\|d_{h,A}^* u\|_{L^2(U)},\end{aligned}$$

and since $u = d_{h,A} w$,

$$h^2\|\nabla u\|_{L^2(U)} \leq C\|d_{h,A}^* u\|_{L^2(U)}. \tag{4.7}$$

(iii) A classical trace result combined with (4.7) and Lemma 4.7 gives

$$\|u\|_{L^2(\partial U)} \leq C\|u\|_{H^1(U)} \leq Ch^{-2}\|d_{h,A}^* u\|_{L^2(U)},$$

where it can again be checked using the same techniques that C does not depend on $\delta \in (0, \delta_0)$. ■

5. Lower bounds

The aim of this section is to establish the following proposition.

Proposition 5.1. *Assume that Ω is $\mathscr{C}^2$ and satisfies Assumption 1.1. There exists a constant $\theta_0 \in (0,1]$ such that for all $k \in \mathbb{N}^*$,*

$$\liminf_{h\to 0} e^{-2\phi_{\min}/h} h^{k-1}\lambda_k(h) \geq C_{\sup}(k)\theta_0 = C_{\inf}(k).$$

If $\Omega = D(0,1)$ and B is radial, we have

$$\liminf_{h\to 0} e^{-2\phi_{\min}/h} h^{k-1}\lambda_k(h) \geq \frac{4\Phi}{(k-1)!}\det(\mathsf{Hess}_{x_{\min}}\phi)^{k/2}.$$

5.1. Inside approximation by the zero-modes

Let $k \in \mathbb{N}^*$. Let us consider an orthonormal family $(v_{j,h})_{1\leq j\leq k}$ (for the scalar product of $L^2(e^{-2\phi/h}\,\mathrm{d}x)$) associated with the eigenvalues $(\lambda_j(h))_{1\leq j\leq k}$. We define

$$\mathscr{E}_h = \operatorname*{span}_{1\leq j\leq k} v_{j,h}.$$

In this section, we will see that the general upper bound proved in the last section implies that all $v_h \in \mathscr{E}_h$ want to be holomorphic inside Ω.

5.1.1. Concentration of the ground state

Lemma 5.2. *There exist $C, h_0 > 0$ such that for all $v_h \in \mathscr{E}_h$ and $h \in (0, h_0)$,*

$$\|v_h\|^2_{L^2(\Omega)} \leq Ch^{-(1+k)} e^{2\phi_{\min}/h} \int_\Omega e^{-2\phi/h} |v_h|^2 \, dx.$$

This result will be used in the proof of Lemma 5.3 to compute the weighted L^2-norm of v_h on Ω in term of its weighted L^2-norm on a shrinking neighborhood of $x_{\min}$.

Proof of Lemma 5.2. We have $\lambda_k(h) = h^{-k+1}\mathscr{O}(e^{2\phi_{\min}/h})$ (see Proposition 3.1). By using the orthogonality of the $v_{j,h}$, one gets

$$\begin{aligned} \int_\Omega e^{-2\phi/h} |2h\partial_{\overline{z}} v_h|^2 \, dx &\leq \lambda_k(h) \int_\Omega e^{-2\phi/h} |v_h|^2 \, dx \\ &\leq Ch^{-k+1} e^{2\phi_{\min}/h} \int_\Omega e^{-2\phi/h} |v_h|^2 \, dx. \end{aligned} \tag{5.1}$$

Now, we use $\phi \leq 0$ to get

$$\int_\Omega |2\partial_{\overline{z}} v_h|^2 \, dx \leq Ch^{-(1+k)} e^{2\phi_{\min}/h} \int_\Omega e^{-2\phi/h} |v_h|^2 \, dx.$$

Since v_h satisfies the Dirichlet boundary condition and by integration by parts, we find

$$\int_\Omega |\nabla v_h|^2 \, dx \leq Ch^{-(1+k)} e^{2\phi_{\min}/h} \int_\Omega e^{-2\phi/h} |v_h|^2 \, dx.$$

It remains to use the Poincaré inequality. ■

We can now prove a concentration lemma.

Lemma 5.3. *Let $\alpha \in (0, 1/2)$. Then*

$$\lim_{h\to 0} \sup_{v_h \in \mathscr{E}_h \setminus\{0\}} \left| \frac{\int_{D(x_{\min},\, h^\alpha)} e^{-2\phi/h} |v_h(x)|^2 \, dx}{\int_\Omega e^{-2\phi/h} |v_h(x)|^2 \, dx} - 1 \right| = 0,$$

and

$$\lim_{h\to 0} \sup_{\delta\in(0,\delta_0]} \sup_{v_h \in \mathscr{E}_h \setminus\{0\}} \left| \frac{\int_{\Omega_\delta} e^{-2\phi/h} |v_h(x)|^2 \, dx}{\int_\Omega e^{-2\phi/h} |v_h(x)|^2 \, dx} - 1 \right| = 0,$$

where δ_0 is defined in Proposition 4.4.

Proof. Let us remark that the second limit is a consequence of the first one. We have

$$\frac{\int_{D(x_{\min}, h^\alpha)} e^{-2\phi/h} |v_h(x)|^2 \, dx}{\int_\Omega e^{-2\phi/h} |v_h(x)|^2 \, dx} = 1 - \frac{\int_{\Omega\setminus D(x_{\min},\, h^\alpha)} e^{-2\phi/h} |v_h(x)|^2 \, dx}{\int_\Omega e^{-2\phi/h} |v_h(x)|^2 \, dx}.$$

By (3.9) and Lemma 5.2, we deduce that

$$\int_{\Omega\setminus D(x_{\min},\, h^\alpha)} e^{-2\phi/h}|v_h(x)|^2\,\mathrm{d}x \le e^{-2\phi_{\min}/h-\lambda_{\min}h^{2\alpha-1}(1+\mathscr{O}(h^\alpha))}\int_\Omega |v_h(x)|^2\,\mathrm{d}x$$
$$\le Ch^{-(1+k)}e^{-\lambda_{\min}h^{2\alpha-1}(1+\mathscr{O}(h^\alpha))}\int_\Omega e^{-2\phi/h}|v_h|^2\,\mathrm{d}x = \mathscr{O}(h^\infty)\int_\Omega e^{-2\phi/h}|v_h|^2\,\mathrm{d}x,$$

and the conclusion follows. ■

5.1.2. Interior approximation. Now that we know that v_h is localized inside Ω, let us explain why it is close to be a holomorphic function.

Notation 7. Let $\widetilde{\Pi}_{h,\delta}$ denote the orthogonal projection on the kernel of $-i\partial_{\overline{z}}$ (i.e. the Segal–Bargmann functions on Ω_δ, which is defined in Notation 6) for the L^2-scalar product $\langle\cdot, e^{-2\phi/h}\cdot\rangle_{L^2(\Omega_\delta)}$.

We notice that if $u = e^{-\phi/h}v$, we have

$$\Pi_{h,A,\Omega_\delta}u = e^{-\phi/h}\widetilde{\Pi}_{h,\delta}v,$$

where Π_{h,A,Ω_δ} was defined in Notation 5.

Proposition 5.4. *There exist $C, h_0 > 0$ such that for all $\delta \in (0,\delta_0]$ and $h \in (0,h_0)$, and all $v_h \in \mathscr{E}_h$, we have*

(a) $\|e^{-\phi/h}(\mathrm{Id}-\tilde{\Pi}_{h,\delta})v_h\|_{L^2(\Omega_\delta)} \le Ch^{-1/2}\sqrt{\lambda_k(h)}\,\|e^{-\phi/h}v_h\|_{L^2(\Omega_\delta)}$,

(b) $\|e^{-\phi/h}(\mathrm{Id}-\widetilde{\Pi}_{h,\delta})v_h\|_{L^2(\partial\Omega_\delta)} \le Ch^{-2}\sqrt{\lambda_k(h)}\,\|e^{-\phi/h}v_h\|_{L^2(\Omega_\delta)}$,

(c) $\dim \widetilde{\Pi}_{h,\delta}\mathscr{E}_h = k$.

Here, δ_0 is defined in Theorem 4.6.

Proof. For all $v_h \in \mathscr{E}_h$, we have

$$4\|e^{-\phi/h}h\partial_{\overline{z}}v_h\|^2_{L^2(\Omega_\delta)} \le 4\|e^{-\phi/h}h\partial_{\overline{z}}v_h\|^2_{L^2(\Omega)}$$
$$\le \lambda_k(h)\|e^{-\phi/h}v_h\|^2_{L^2(\Omega)} \le (1+o(1))\lambda_k(h)\|e^{-\phi/h}v_h\|^2_{L^2(\Omega_\delta)},$$

where we use Lemma 5.3 to get the last inequality. With $u_h = e^{-\phi/h}v_h$, we have

$$4\|e^{-\phi/h}h\partial_{\overline{z}}v_h\|^2_{L^2(\Omega_\delta)} = 4\|e^{-\phi/h}h\partial_{\overline{z}}(\mathrm{Id}-\tilde{\Pi}_{h,\delta})v_h\|^2_{L^2(\Omega_\delta)}$$
$$= \|d^*_{h,A,\Omega_\delta}(\mathrm{Id}-\Pi_{h,A,\Omega_\delta})u_h\|^2_{L^2(\Omega_\delta)}.$$

Applying Theorem 4.6, we get (a) and (b).

Let $v_h \in \mathscr{E}_h$ be such that $\widetilde{\Pi}_{h,\delta}v_h = 0$. Recalling Proposition 3.1, we have

$$\|e^{-\phi/h}v_h\|_{L^2(\Omega_\delta)} \le \|e^{-\phi/h}(\mathrm{Id}-\widetilde{\Pi}_{h,\delta})v_h\|_{L^2(\Omega_\delta)} + \|e^{-\phi/h}\widetilde{\Pi}_{h,\delta}v_h\|_{L^2(\Omega_\delta)}$$
$$\le Ch^{-k/2}e^{\phi_{min}/h}\|e^{-\phi/h}v_h\|_{L^2(\Omega_\delta)},$$

so that
$$\|e^{-\phi/h}v_h\|_{L^2(\Omega_\delta)}(1 - Ch^{-k/2}e^{\phi_{min}/h}) \leq 0,$$
and $v_h = 0$ on Ω_δ so that $\widetilde{\Pi}_{h,\delta}$ is injective on $\mathscr{E}_h$ and (c) follows. ∎

5.2. A reduction to a holomorphic subspace

In the following, we assume that $\delta \in (0, \delta_0)$ and $h \in (0, h_0)$.

Notation 8. We will use the *Szegö projection*
$$\Pi_+ : L^2(D(0,1)) \ni \sum_{\mathbf{n}\in\mathbb{Z}} a_n(r)e^{ins} \mapsto \sum_{\mathbf{n}\in\mathbb{N}} a_n(r)e^{ins} \in L^2(D(0,1)).$$
Note that the Szegö projection preserves the L^2 holomorphic functions.

Notation 9. We let
$$E := \min_{\partial D(0,1)} |F'(y)|\partial_{\mathbf{n}}\phi(F(y)) \geq c_1 \min_{\Gamma}(\nabla\phi \cdot \mathbf{n}),$$
where F, c_1 are defined in Notation 6.

Lemma 5.5. *Assume that $\delta/h \to +\infty$ and $\delta \to 0$. Then, for all $v_h \in \mathscr{E}_h$,*
$$2hE\|\Pi_+(v_h \circ F)\|^2_{L^2(\partial D(0,1-\delta))}(1 + o(1)) \leq 4h^2 \int_\Omega e^{-2\phi/h}|\partial_{\overline{z}}v_h|^2 \,\mathrm{d}x.$$

Proof. (i) For all $v \in H^1_0(\Omega)$, we let $\check{v} = v \circ F \in H^1_0(D(0,1))$ and $\check{\phi} = \phi \circ F$. We get
$$4h^2 \int_\Omega e^{-2\phi/h}|\partial_{\overline{z}}v|^2 \,\mathrm{d}x = 4h^2 \int_{D(0,1)} e^{-2\check{\phi}/h}|\partial_{\overline{y}}\check{v}|^2 \,\mathrm{d}y.$$

(ii) In polar coordinates, the Cauchy–Riemann operator is
$$-\frac{i}{2}(\partial_1 + i\partial_2) = \frac{-i\gamma'}{2}\left(\frac{\partial_s}{1-t} + i\partial_t\right).$$
We write $\tilde{\psi}(s,t) = \check{\psi}(\eta(s,t))$ for any function $\check{\psi}$ defined on $D(0,1)$. For all $\check{v}$ in $H^1_0(D(0,1))$, we have
$$\begin{aligned} 4h^2 \int_{D(0,1)} e^{-2\check{\phi}/h}|\partial_{\overline{z}}\check{v}|^2 \,\mathrm{d}x &\geq 4h^2 \int_{D(0,1)\setminus D(0,1-\delta)} e^{-2\check{\phi}/h}|\partial_{\overline{z}}\check{v}|^2 \,\mathrm{d}x \\ &= h^2 \int_0^{2\pi}\int_0^\delta (1-t)^{-1}\,|((1-t)\partial_t - i\partial_s)\tilde{v}|^2\, e^{-2\tilde{\phi}/h} \,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{5.2}$$

(iii) Let us notice that
$$\partial_{\check{\mathbf{n}}}\check{\phi}(y) = |F'(y)|\partial_{\mathbf{n}}\phi(F(y)), \tag{5.3}$$

where $x \in \partial\Omega$, $\mathbf{n}(x)$ is the outward unit normal to Ω at x, and $\check{\mathbf{n}}(y)$ is the outward unit normal to $D(0,1)$ at y.

By using the Taylor expansion of $\tilde{\phi}$, there exists $C > 0$ such that, for all (s,t) in $\Gamma \times (0,\delta)$,

$$-\tilde{\phi}(s,t) = -t\partial_t\tilde{\phi}(s,0) + \mathscr{O}(t^2) \geq (1 - C\delta)Et,$$

where $0 < E = \min_{\partial D(0,1)} \partial_{\check{\mathbf{n}}}\check{\phi}$. We have

$$4h^2 \int_{D(0,1)} e^{-2\check{\phi}/h} |\partial_{\overline{z}}\check{v}|^2 \,\mathrm{d}x \geq h^2 \int_0^{2\pi}\int_0^{\delta} (1-t)^{-1} \,|((1-t)\partial_t - i\partial_s)\tilde{v}|^2\, e^{2(1-C\delta)Et/h} \,\mathrm{d}s\,\mathrm{d}t.$$

Consider the new variable $\tau = -\ln|1-t|$. Then we get

$$4h^2 \int_{D(0,1)} e^{-2\check{\phi}/h} |\partial_{\overline{z}}\check{v}|^2 \,\mathrm{d}x \geq h^2 \int_0^{2\pi}\int_0^{-\ln|1-\delta|} |(\partial_\tau - i\partial_s)\mathsf{v}(s,\tau)|^2\, e^{2(1-C\delta)E(1-e^{-\tau})/h} \,\mathrm{d}s\,\mathrm{d}\tau,$$

where $\mathsf{v}(s,\tau) = \tilde{v}(s, 1 - e^{-\tau})$. Since $1 - e^{-\tau} = \tau + \mathscr{O}(\tau^2) = \tau + \mathscr{O}(\delta\tau)$, there exists $\tilde{C} > 0$ such that

$$e^{2(1-C\delta)E(1-e^{-\tau})/h} \geq e^{2E(1-\tilde{C}\delta)\tau/h}.$$

Let $\tilde{E} = E(1 - \tilde{C}\delta)$ and $\tilde{\delta} = -\ln|1-\delta|$ so that

$$4h^2 \int_{D(0,1)} e^{-2\check{\phi}/h} |\partial_{\overline{z}}\check{v}|^2 \,\mathrm{d}x \geq h^2 \int_0^{2\pi}\int_0^{\tilde{\delta}} |(\partial_\tau - i\partial_s)\mathsf{v}(s,\tau)|^2\, e^{2\tilde{E}\tau/h} \,\mathrm{d}s\,\mathrm{d}\tau.$$

(iv) Using Fourier series and the Parseval formula, we get

$$\int_0^{2\pi}\int_0^{\tilde{\delta}} |(\partial_\tau - i\partial_s)\mathsf{v}(s,\tau)|^2\, e^{2\tilde{E}\tau/h} \,\mathrm{d}s\,\mathrm{d}\tau = 2\pi \sum_{m\in\mathbb{Z}} \int_0^{\tilde{\delta}} \left|(\partial_\tau + m)\hat{\mathsf{v}}_m(\tau)\right|^2 e^{2\tilde{E}\tau/h} \,\mathrm{d}\tau,$$

where

$$\hat{\mathsf{v}}_m(\tau) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ims}\mathsf{v}(s,\tau)\,\mathrm{d}s.$$

Let us consider the quadratic form

$$Q_m(w) = \int_0^{\tilde{\delta}} |(\partial_\tau + m)w|^2\, e^{2\tilde{E}\tau/h} \,\mathrm{d}\tau$$

with boundary conditions $w(0) = 0$ and $w(\tilde{\delta}) = 1$.

Notice that

$$Q_m(w) = \tilde{Q}_m(\rho) = \int_0^{\tilde{\delta}} |\partial_\tau \rho|^2 \, e^{2\tau \tilde{E}/h + 2(\tilde{\delta}-\tau)m} \, \mathrm{d}\tau,$$

where $w(\tau) = e^{m(\tilde{\delta}-\tau)}\rho(\tau)$ for all $\tau \in (0, \tilde{\delta})$, $\rho(0) = 0$ and $\rho(\tilde{\delta}) = 1$.

Since $m \mapsto \tilde{Q}_m(\rho)$ is an increasing function, we get $\tilde{Q}_m(\rho) \geq \tilde{Q}_0(\rho)$ for all $m \geq 0$ and, by Lemma A.1,

$$Q_m(w) \geq \Lambda_0(h),$$

where

$$\Lambda_0(h) = \frac{2\tilde{E}/h}{1 - e^{-2\tilde{\delta}\tilde{E}/h}} \geq 0.$$

By forgetting the negative m, we find

$$\begin{aligned} 4h^2 \int_\Omega e^{-2\phi/h} |\partial_{\bar{z}} v|^2 \, \mathrm{d}x &\geq 2\pi h^2 \Lambda_0(h) \sum_{m \geq 0} |\mathsf{v}_m(\delta)|^2 \\ &= h^2 \Lambda_0(h) \| \Pi_+(v \circ F) \|^2_{L^2(\partial D(0,1-\delta))}. \end{aligned}$$ ■

In the following, we choose $\delta = h^{3/4}$.

Using Proposition 5.4, we show in the following lemma that we can replace v_h by $\widetilde{\Pi}_{h,\delta} v_h$ in Lemma 5.5.

Lemma 5.6. *Assume that $\delta = h^{3/4}$ and that $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Then*

$$\begin{aligned} &2\theta_0 h e^{2\phi_{\min}/h} \| \widetilde{\Pi}_{h,\delta} v_h \|^2_{\mathscr{H}^2(\Omega_\delta)} (1 + o(1)) \\ &\qquad \leq \lambda_k(h) \| e^{-\frac{1}{2h} \mathsf{Hess}_{x_{\min}} \phi(x - x_{\min}, x - x_{\min})} \widetilde{\Pi}_{h,\delta} v_h \|^2_{L^2(D(x_{\min}, h^\alpha))}, \end{aligned}$$

where

$$\theta_0 = \frac{\min_{\partial D(0,1)} |F'(y)| \partial_{\mathbf{n}} \phi(F(y))}{\max_{\partial D(0,1)} |F'(y)| \partial_{\mathbf{n}} \phi(F(y))} \in (0, 1],$$

and where we use the notation

$$\|w\|^2_{\mathscr{H}^2(\Omega_\delta)} := \int_{\partial D(0,1-\delta)} |w \circ F|^2 (\partial_{\mathbf{n}} \phi \circ F) |F'| \, \mathrm{d}s.$$

Remark 5.7. Taking $\delta = 0$ in the definition of $\|w\|^2_{\mathscr{H}^2(\Omega_\delta)}$ above gives

$$\|w\|^2_{\mathscr{H}^2(\Omega_0)} = \int_{\partial D(0,1)} |w \circ F|^2 (\partial_{\mathbf{n}} \phi \circ F) |F'| \, \mathrm{d}s = \int_{\partial \Omega} |w|^2 \partial_{\mathbf{n}} \phi \, \mathrm{d}y = N_{\mathscr{H}}(w)^2$$

for $w \in \mathscr{H}^2(\Omega)$.

Proof of Lemma 5.6. (i) From Lemma 5.5 and the definition of v_h, we have

$$2Eh\|\Pi_+(v_h\circ F)\|^2_{L^2(\partial D(0,1-\delta))}(1+o(1))\le h^2\int_\Omega e^{-2\phi/h}|\partial_{\overline{z}}v_h|^2\,\mathrm{d}x \le \lambda_k(h)\int_\Omega e^{-2\phi/h}|v_h|^2\,\mathrm{d}x.$$

Thus, by Lemma 5.3,

$$2Eh\|\Pi_+(v_h\circ F)\|^2_{L^2(\partial D(0,1-\delta))}(1+o(1))\le \lambda_k(h)\int_{D(x_{\min},h^\alpha)} e^{-2\phi/h}|v_h|^2\,\mathrm{d}x. \tag{5.4}$$

(ii) Let $\check{\Pi}_{h,\delta}$ be the orthogonal projection on $\mathscr{H}(D(0,1-\delta))$ for the $L^2(e^{-2\check{\phi}/h}\,\mathrm{d}y)$ scalar product. Note that $\check{\Pi}_{h,\delta}\Pi_+=\Pi_+\check{\Pi}_{h,\delta}=\check{\Pi}_{h,\delta}$ (see Notation 8). Let us now replace Π_+ by $\check{\Pi}_{h,\delta}$. Proposition 5.4 ensures that

$$\|e^{-\check{\phi}/h}(\mathrm{Id}-\check{\Pi}_{h,\delta})v_h\circ F\|_{L^2(\partial D(0,1-\delta))}\le Ch^{-2}\sqrt{\lambda_k(h)}\,\|e^{-\phi/h}v_h\|_{L^2(\Omega_\delta)},$$

Using the Taylor expansion of $\check{\phi}$ near the boundary and (5.3), we have, on $\partial D(0,1-\delta)$,

$$e^{-\check{\phi}/h}\ge(1+o(1))e^{Eh^{-1/4}},$$

so that

$$\|(\mathrm{Id}-\check{\Pi}_{h,\delta})v_h\circ F\|_{L^2(\partial D(0,1-\delta))}\le Ch^{-2}\sqrt{\lambda_k(h)}\,e^{-Eh^{-1/4}}\|e^{-\phi/h}v_h\|_{L^2(\Omega_\delta)}. \tag{5.5}$$

Since Π_+ is a projection and $\check{\Pi}_{h,\delta}$ is valued in the holomorphic functions,

$$\begin{aligned}\|(\mathrm{Id}-\check{\Pi}_{h,\delta})v_h\circ F\|_{L^2(\partial D(0,1-\delta))} &\ge \|\Pi_+(\mathrm{Id}-\check{\Pi}_{h,\delta})v_h\circ F\|_{L^2(\partial D(0,1-\delta))}\\ &\ge \|\Pi_+v_h\circ F-\check{\Pi}_{h,\delta}v_h\circ F\|_{L^2(\partial D(0,1-\delta))}\\ &\ge \big|\|\Pi_+v_h\circ F\|_{L^2(\partial D(0,1-\delta))}-\|\check{\Pi}_{h,\delta}v_h\circ F\|_{L^2(\partial D(0,1-\delta))}\big|.\end{aligned}$$

Then, with (5.5),

$$\|\Pi_+v_h\circ F\|_{L^2(\partial D(0,1-\delta))} \ge \|\check{\Pi}_{h,\delta}v_h\circ F\|_{L^2(\partial D(0,1-\delta))}-\mathscr{O}(h^\infty)\sqrt{\lambda_k(h)}\,\|e^{-\phi/h}v_h\|_{L^2(\Omega_\delta)}.$$

By (5.4) and Lemma 5.3,

$$\sqrt{2Eh}\,\|\check{\Pi}_{h,\delta}v_h\circ F\|_{L^2(\partial D(0,1-\delta))}(1+o(1)) \le \sqrt{\lambda_k(h)}\,\|e^{-\phi/h}v_h\|_{L^2(D(x_{\min},h^\alpha))}. \tag{5.6}$$

Thus, coming back to Ω_δ (without forgetting the Jacobian of F),

$$\sqrt{2Eh}\,\||F'(F(\cdot))|^{-1/2}\widetilde{\Pi}_{h,\delta}v_h\|_{L^2(\partial\Omega_\delta)}(1+o(1)) \leq \sqrt{\lambda_k(h)}\,\|e^{-\phi/h}v_h\|_{L^2(D(x_{\min},h^\alpha))}.$$

Then, by using the (weighted) Hardy norm, we have

$$\sqrt{2\theta_0 h}\,\|\widetilde{\Pi}_{h,\delta}v_h\|_{\mathscr{H}^2(\Omega_\delta)}(1+o(1)) \leq \sqrt{\lambda_k(h)}\,\|e^{-\phi/h}v_h\|_{L^2(D(x_{\min},h^\alpha))}. \tag{5.7}$$

(iii) Using Proposition 5.4 and Lemma 5.3, we get

$$\begin{aligned}\|e^{-\phi/h}v_h\|_{L^2(D(x_{\min},h^\alpha))} &\leq \|e^{-\phi/h}\widetilde{\Pi}_{h,\delta}v_h\|_{L^2(D(x_{\min},h^\alpha))} + \|e^{-\phi/h}(\mathrm{Id}-\widetilde{\Pi}_{h,\delta})v_h\|_{L^2(D(x_{\min},h^\alpha))}\\ &\leq \|e^{-\phi/h}\widetilde{\Pi}_{h,\delta}v_h\|_{L^2(D(x_{\min},h^\alpha))} + \|e^{-\phi/h}(\mathrm{Id}-\widetilde{\Pi}_{h,\delta})v_h\|_{L^2(\Omega_\delta)}\\ &\leq \|e^{-\phi/h}\widetilde{\Pi}_{h,\delta}v_h\|_{L^2(D(x_{\min},h^\alpha))}\\ &\quad + Ch^{-1/2}\sqrt{\lambda_k(h)}\,\|e^{-\phi/h}v_h\|_{L^2(D(x_{\min},h^\alpha))}.\end{aligned}$$

Combing this with (5.7) and Proposition 3.1, we find

$$2\theta_0 h\|\widetilde{\Pi}_{h,\delta}v_h\|^2_{\mathscr{H}^2(\Omega_\delta)}(1+o(1)) \leq \lambda_k(h)\|e^{-\phi/h}\widetilde{\Pi}_{h,\delta}v_h\|^2_{L^2(D(x_{\min},h^\alpha))}.$$

(iv) Using the Taylor expansion of ϕ at $x_{\min}$, we get, for all $x \in D(x_{\min}, h^\alpha)$,

$$\frac{\phi(x)-\phi_{\min}}{h} = \frac{1}{2h}\,\mathsf{Hess}_{x_{\min}}\,\phi(x-x_{\min}, x-x_{\min}) + \mathscr{O}(h^{3\alpha-1}),$$

and the conclusion follows. ∎

Remark 5.8. Lemma 5.6 shows in particular that

$$2(1+o(1))\theta_0 h\tilde{\lambda}_k(h) \leq \lambda_k(h),$$

where

$$\tilde{\lambda}_k(h) = \inf_{\substack{V\subset\mathscr{H}^2(\Omega_\delta)\\ \dim V=k}}\ \sup_{v\in V\setminus\{0\}} \frac{\|v\|^2_{\mathscr{H}^2(\Omega_\delta)}}{\|e^{-\frac{1}{2h}\,\mathsf{Hess}_{x_{\min}}\,\phi(x-x_{\min},x-x_{\min})}v\|^2_{L^2(D(x_{\min},h^\alpha))}}.$$

In the next section, we will essentially provide a lower bound of $\tilde{\lambda}_k(h)$. Note that if we could replace $\mathscr{H}^2(\Omega_\delta)$ by the set of polynomials, then we would get the bound presented in Remark 3.2. However, there is no hope to do so, since in general

$$\mathsf{dist}_{\mathscr{H}}((z-z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega)) < N_{\mathscr{B}}((z-z_{\min})^{k-1}),$$

(this inequality is an equality in the radial case). We still have to work to get the lower bound of Theorem 1.3.

5.3. Reduction to a polynomial subspace: Proof of Proposition 5.1

We can now prove Proposition 5.1.

(i) By (3.12), there exist $C, h_0 > 0$ such that, for all $h \in (0, h_0)$, all $w \in \mathscr{H}^2(\Omega_\delta)$, all $z_0 \in D(x_{\min}, h^\alpha)$, and all $n \in \{0, \dots, k\}$,

$$|w^{(n)}(z_0)| \le C\|w\|_{\mathscr{H}^2(\Omega_\delta)}. \tag{5.8}$$

Define, for all $w \in \mathscr{H}^2(\Omega_\delta)$,

$$N_h(w) = \|e^{-\frac{1}{2h}\mathsf{Hess}_{x_{\min}}\phi(x-x_{\min},x-x_{\min})}w\|_{L^2(D(x_{\min},h^\alpha))}.$$

Let $w_h = \widetilde{\Pi}_{h,\delta}v_h$. By the Taylor formula, we can write

$$w_h = \mathrm{Tayl}_{k-1}\, w_h + R_{k-1}(w_h),$$

where

$$\mathrm{Tayl}_{k-1}\, w_h = \sum_{n=0}^{k-1}\frac{w_h^{(n)}(z_{\min})}{n!}(z-z_{\min})^n,$$

and, for all $z \in D(z_{\min}, h^\alpha)$,

$$|R_{k-1}(w_h)(z)| \le C|z-z_{\min}|^k \sup_{D(z_{\min},h^\alpha)}|w_h^{(k)}|.$$

With (5.8) and a rescaling, the Taylor remainder satisfies

$$N_h(R_{k-1}(w_h)) \le Ch^{k/2}h^{1/2}\|w_h\|_{\mathscr{H}^2(\Omega_\delta)}.$$

Thus, by the triangle inequality,

$$N_h(w_h) \le N_h(\mathrm{Tayl}_{k-1}\, w_h) + Ch^{k/2}h^{1/2}\|w_h\|_{\mathscr{H}^2(\Omega_\delta)}.$$

Thus, with Lemma 5.6, we get

$$\begin{aligned}(1+o(1))e^{\phi_{\min}/h}\sqrt{2\theta_0 h}\,\|w_h\|_{\mathscr{H}^2(\Omega_\delta)}&\\ \le \sqrt{\lambda_k(h)}\,N_h(\mathrm{Tayl}_{k-1}\, w_h) &+ C\sqrt{\lambda_k(h)}\,h^{(1+k)/2}\|w_h\|_{\mathscr{H}^2(\Omega_\delta)},\end{aligned}$$

so that, thanks to Proposition 3.1,

$$\begin{aligned}(1+o(1))e^{\phi_{\min}/h}\sqrt{2\theta_0 h}\,\|w_h\|_{\mathscr{H}^2(\Omega_\delta)} &\le \sqrt{\lambda_k(h)}\,N_h(\mathrm{Tayl}_{k-1}\, w_h)\\ &\le \sqrt{\lambda_k(h)}\,\hat{N}_h(\mathrm{Tayl}_{k-1}\, w_h),\end{aligned} \tag{5.9}$$

with

$$\hat{N}_h(w) = \|e^{-\frac{1}{2h}\mathsf{Hess}_{x_{\min}}\phi(x-x_{\min},x-x_{\min})}w\|_{L^2(\mathbb{R}^2)}.$$

This inequality shows in particular that $\mathrm{Tayl}_{k-1}\,\widetilde{\Pi}_{\delta,h}$ is injective on $\mathscr{E}_h$ and

$$\dim \mathrm{Tayl}_{k-1}(\widetilde{\Pi}_{\delta,h}\mathscr{E}_h) = k. \tag{5.10}$$

(ii) Let us recall that

$$\mathscr{H}^2_k(\Omega_\delta) = \{\psi \in \mathscr{H}^2(\Omega_\delta) : \forall n \in \{0,\dots,k-1\},\ \psi^{(n)}(x_{\min}) = 0\}.$$

Since $w_h - \mathrm{Tayl}_{k-1}\, w_h \in \mathscr{H}^2_k(\Omega_\delta)$, we have, by the triangle inequality,

$$\begin{aligned}\|w_h\|_{\mathscr{H}^2(\Omega_\delta)} &\geq \left\| \frac{w_h^{(k-1)}(z_{\min})}{(k-1)!}(z-z_{\min})^{k-1} + (w_h - \mathrm{Tayl}_{k-1}\, w_h)\right\|_{\mathscr{H}^2(\Omega_\delta)} \\ &\quad - \|\mathrm{Tayl}_{k-2}\, w_h\|_{\mathscr{H}^2(\Omega_\delta)} \\ &\geq \frac{|w_h^{(k-1)}(z_{\min})|}{(k-1)!}\mathsf{dist}_{\mathscr{H},\delta}((z-z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega_\delta)) \\ &\quad - \|\mathrm{Tayl}_{k-2}\, w_h\|_{\mathscr{H}^2(\Omega_\delta)},\end{aligned}$$

where

$$\begin{aligned}&\mathsf{dist}_{\mathscr{H},\delta}((z-z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega_\delta)) \\ &\qquad = \inf\{\|(z-z_{\min})^{k-1} - Q(z)\|_{\mathscr{H}^2(\Omega_\delta)} : Q \in \mathscr{H}^2_k(\Omega_\delta)\}.\end{aligned}$$

Using again the triangle inequality, we get

$$\|\mathrm{Tayl}_{k-2}\, w_h\|_{\mathscr{H}^2(\Omega_\delta)} \leq C \sum_{n=0}^{k-2} |w_h^{(n)}(z_{\min})|.$$

Moreover,

$$\begin{aligned}\sum_{n=0}^{k-2} |w_h^{(n)}(z_{\min})| &\leq h^{-(k-2)/2} \sum_{n=0}^{k-2} h^{n/2}|w_h^{(n)}(z_{\min})| \leq h^{-(k-2)/2} \sum_{n=0}^{k-1} h^{n/2}|w_h^{(n)}(z_{\min})| \\ &\leq C h^{-(k-2)/2} h^{-1/2} \hat{N}_h(\mathrm{Tayl}_{k-1}\, w_h),\end{aligned}$$

where we use the rescaling property

$$\hat{N}_h\Big(\sum_{n=0}^{k-1} c_n (z-z_{\min})^n\Big) = h^{1/2}\hat{N}_1\Big(\sum_{n=0}^{k-1} c_n h^{n/2}(z-z_{\min})^n\Big), \tag{5.11}$$

and the equivalence of norms in finite dimension:

$$\exists C > 0, \forall d \in \mathbb{C}^k, \quad C^{-1}\sum_{n=0}^{k-1}|d_n| \leq \hat{N}_1\Big(\sum_{n=0}^{k-1} d_n (z-z_{\min})^n\Big) \leq C \sum_{n=0}^{k-1}|d_n|.$$

We find

$$\begin{aligned}\|w_h\|_{\mathscr{H}^2(\Omega_\delta)} &\geq \frac{|w_h^{(k-1)}(z_{\min})|}{(k-1)!}\mathsf{dist}_{\mathscr{H},\delta}((z-z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega_\delta)) \\ &\quad - C h^{-(k-2)/2} h^{-1/2}\hat{N}_h(\mathrm{Tayl}_{k-1}\, w_h),\end{aligned}$$

and thus, by (5.9),

$$(1+o(1))e^{\phi_{\min}/h}\sqrt{2\theta_0 h}\,\frac{|w_h^{(k-1)}(z_{\min})|}{(k-1)!}\mathsf{dist}_{\mathscr{H},\delta}((z-z_{\min})^{k-1},\mathscr{H}_k^2(\Omega_\delta))$$
$$\leq (\sqrt{\lambda_k(h)}+Ch^{(2-k)/2}e^{\phi_{\min}/h})\hat{N}_h(\mathrm{Tayl}_{k-1}\,w_h). \quad (5.12)$$

(iii) Since we have (5.10), we deduce that

$$(1+o(1))e^{\phi_{\min}/h}\sqrt{2\theta_0 h}\,\mathsf{dist}_{\mathscr{H},\delta}((z-z_{\min})^{k-1},\mathscr{H}_k^2(\Omega_\delta))\sup_{c\in\mathbb{C}^k}\frac{|c_{k-1}|}{\hat{N}_h(\sum_{n=0}^{k-1}c_n(z-z_{\min})^n)}$$
$$\leq \sqrt{\lambda_k(h)}+Ch^{(2-k)/2}e^{\phi_{\min}/h}. \quad (5.13)$$

By (5.11), we infer that

$$h^{1/2}\sup_{c\in\mathbb{C}^k}\frac{|c_{k-1}|}{\hat{N}_h(\sum_{n=0}^{k-1}c_n(z-z_{\min})^n)}=\sup_{c\in\mathbb{C}^k}\frac{h^{(1-k)/2}|c_{k-1}|}{\hat{N}_1(\sum_{n=0}^{k-1}c_n(z-z_{\min})^n)}.$$

Since $\hat{N}_1$ is related to the Segal–Bargmann norm $N_{\mathscr{B}}$ via a translation, and recalling Notation 4, we get

$$\sup_{c\in\mathbb{C}^k}\frac{|c_{k-1}|}{\hat{N}_1(\sum_{n=0}^{k-1}c_n(z-z_{\min})^n)}=\sup_{c\in\mathbb{C}^k}\frac{|c_{k-1}|}{N_{\mathscr{B}}(\sum_{n=0}^{k-1}c_nz^n)}=\frac{1}{N_{\mathscr{B}}(P_{k-1})}.$$

Thus,

$$(1+o(1))h^{(1-k)/2}e^{\phi_{\min}/h}\sqrt{2\theta_0}\,\frac{\mathsf{dist}_{\mathscr{H},\delta}((z-z_{\min})^{k-1},\mathscr{H}_k^2(\Omega_\delta))}{N_{\mathscr{B}}(P_{k-1})}\leq\sqrt{\lambda_k(h)}. \quad (5.14)$$

(iv) Since Ω is regular enough, the Riemann mapping theorem ensures that

$$\lim_{h\to 0}\mathsf{dist}_{\mathscr{H},\delta}((z-z_{\min})^{k-1},\mathscr{H}_k^2(\Omega_\delta))=\mathsf{dist}_{\mathscr{H}}((z-z_{\min})^{k-1},\mathscr{H}_k^2(\Omega)).$$

The conclusion follows.

5.4. *Proof of Corollary* 1.11

We recall Notation 6 where F, c_1 and c_2 are defined. Let us notice that we can choose F such that $F(0)=x_{\min}$.

For all $v\in H_0^1(\Omega)$, we let $\check{v}=v\circ F\in H_0^1(D(0,1))$, and we get

$$\frac{1}{c_2}\,\frac{\int_{D(0,1)}e^{-2\check{\phi}/h}|\partial_{\overline{y}}\check{v}|^2\,\mathrm{d}y}{\int_{D(0,1)}e^{-2\check{\phi}/h}|\check{v}|^2\,\mathrm{d}y}\leq\frac{\int_{D(0,1)}e^{-2\check{\phi}/h}|\partial_{\overline{y}}\check{v}|^2\,\mathrm{d}y}{\int_{D(0,1)}e^{-2\check{\phi}/h}|\check{v}|^2|F'(y)|^2\,\mathrm{d}y}=\frac{\int_\Omega e^{-2\phi/h}|\partial_{\overline{z}}v|^2\,\mathrm{d}x}{\int_\Omega e^{-2\phi/h}|v|^2\,\mathrm{d}x},$$

where $\check{\phi}=\phi\circ F$ has a unique and non-degenerate minimum at $y=0$ and $\check{\phi}(0)=\phi_{\min}$.

In the same way, we get

$$\frac{\int_\Omega e^{-2\phi/h}|\partial_{\overline{z}} v|^2\,\mathrm{d}x}{\int_\Omega e^{-2\phi/h}|v|^2\,\mathrm{d}x} \leq \frac{1}{c_1}\,\frac{\int_{D(0,1)} e^{-2\check{\phi}/h}|\partial_{\overline{y}}\check{v}|^2\,\mathrm{d}y}{\int_{D(0,1)} e^{-2\check{\phi}/h}|\check{v}|^2\,\mathrm{d}y}.$$

These inequalities, the min-max principle, and Theorem 1.3 imply Corollary 1.11.

Appendix A. A unidimensional optimization problem

The goal of this section is to minimize, for each fixed s, the quantity

$$\int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi(s)/h}\,|\partial_t\rho|^2\;\mathrm{d}t.$$

This leads to the following lemma.

Lemma A.1. *For $\alpha, \varepsilon > 0$, let $I = (0, \varepsilon)$ and*

$$\mathscr{V} = \{\rho \in H^1(I) : \rho(0) = 0,\ \rho(\varepsilon) = 1\},$$

and for all $\rho \in \mathscr{V}$, consider

$$F_{\alpha,\varepsilon}(\rho) := \int_0^\varepsilon e^{\alpha\ell}|\rho'(\ell)|^2\,\mathrm{d}\ell.$$

(a) *The minimization problem*

$$\inf\{F_{\alpha,\varepsilon}(\rho) : \rho \in \mathscr{V}\}$$

has a unique minimizer

$$\rho_{\alpha,\varepsilon}(\ell) = \frac{1-e^{-\alpha\ell}}{1-e^{-\alpha\varepsilon}}.$$

(b) *We have*

$$\inf\{F_{\alpha,\varepsilon}(\rho) : \rho \in \mathscr{V}\} = \frac{\alpha}{1-e^{-\varepsilon\alpha}},$$

(c) *Let $c_0 > 0$. Assume $1 - e^{-\alpha\varepsilon} \geq c_0$. Then there exists $C > 0$ such that*

$$\int_0^\varepsilon e^{\alpha\ell}|\partial_\alpha\rho_{\alpha,\varepsilon}|^2\,\mathrm{d}\ell \leq C(\alpha^{-3} + e^{-\alpha\varepsilon}\varepsilon^2\alpha^{-1}).$$

Proof. (i) Since $\alpha > 0$, we have $F_{\alpha,\varepsilon}(\rho) \geq \int_0^\varepsilon |\rho'(\ell)|^2\,\mathrm{d}\ell$ for all $\rho \in \mathscr{V}$. There exists $C > 0$ such that, for all $\rho \in \mathscr{V}$,

$$\int_0^\varepsilon |\rho'(\ell)|^2\,\mathrm{d}\ell \geq C\int_0^\varepsilon |\rho(\ell)|^2\,\mathrm{d}\ell.$$

This ensures that any minimizing sequence $(\rho_n)_{n\in\mathbb{N}} \subset \mathscr{V}$ is bounded in $H^1(I)$ and any H^1-weak limit is a minimizer of $\inf\{F_{\alpha,\varepsilon}(\rho) : \rho \in \mathscr{V}\}$.

(ii) $F_{\alpha,\varepsilon}^{1/2}$ is a Euclidean norm on $\mathscr{V}$ so that $F_{\alpha,\varepsilon}$ is strictly convex and the minimizer is unique.

(iii) At a minimum ρ, the Euler–Lagrange equation is

$$(e^{\alpha\ell}\rho')' = 0.$$

Thus, there exists $(c,d) \in \mathbb{R}^2$ such that, for all $\ell \in I$,

$$\rho(\ell) = d - c\alpha^{-1}e^{-\alpha\ell},$$

so that from the boundary conditions we find the function $\rho_{\alpha,\varepsilon}$.

(iv) We have

$$\int_0^\varepsilon e^{\alpha\ell}|\rho'(\ell)|^2 \,\mathrm{d}\ell = \alpha^2(1-e^{-\varepsilon\alpha})^{-2}\int_0^\varepsilon e^{-\alpha\ell}\,\mathrm{d}\ell = \frac{\alpha}{1-e^{-\varepsilon\alpha}}.$$

(v) We also have

$$\partial_\alpha\rho_{\alpha,\varepsilon}(\ell) = \frac{1}{(1-e^{-\alpha\varepsilon})^2}\big(\ell e^{-\alpha\ell}(1-e^{-\alpha\varepsilon}) - (1-e^{-\alpha\ell})\varepsilon e^{-\alpha\varepsilon}\big),$$

for $\ell \in (0,\varepsilon)$ and

$$\begin{aligned}\int_0^\varepsilon e^{\alpha\ell}|\partial_\alpha\rho_{\alpha,\varepsilon}|^2\,\mathrm{d}\ell &\le \frac{1}{(1-e^{-\alpha\varepsilon})^4}\bigg(\int_0^\varepsilon \ell^2 e^{-\alpha\ell}\,\mathrm{d}\ell(1-e^{-\alpha\varepsilon})^2 + \int_0^\varepsilon e^{-\alpha\ell}\,\mathrm{d}\ell(\varepsilon e^{-\alpha\varepsilon})^2 \\ &\qquad + \int_0^\varepsilon e^{\alpha\ell}\,\mathrm{d}\ell(\varepsilon e^{-\alpha\varepsilon})^2\bigg) \\ &\le C(\alpha^{-3} + e^{-\alpha\varepsilon}\varepsilon^2\alpha^{-1}).\end{aligned}$$

■

Appendix B. Hopf's lemma with Dini-regularity

In the following lemma, we present a simple proof of an extension of Hopf's lemma to the case when Ω is Dini-regular. The standard version of Hopf's lemma given for instance in [7, Hopf's Lemma, Section 6.4.2] requires essentially $\mathscr{C}^2$ regularity. However, the regularity can be lowered down to Dini (see [1] and the references therein).

Lemma B.1. *Let Ω be a simply connected, Dini-regular, bounded open set. If ϕ is the solution of* (1.2)*, then the function $\partial\Omega \ni s \mapsto \partial_{\mathbf{n}}\phi(s)$ is continuous and*

$$\partial_{\mathbf{n}}\phi > 0 \quad \text{on } \partial\Omega.$$

Proof. Let ϕ be the solution of (1.2). By the Riemann mapping theorem [15], there exists a bi-holomorphic map $F: D(0,1) \to \Omega$ such that F' is continuous on $\overline{D(0,1)}$. The function $\check{\phi} = \phi \circ F$ is the solution of (1.2) on $D(0,1)$ for $\check{B} = |F'|^2 B \circ F$. By [9, Corollary 8.36], $\check{\phi}$ is $\mathscr{C}^{1,1^-}$ on $D(0,1)$ and Hopf's lemma [7, Hopf's Lemma, Section 6.4.2] ensures that $\partial_{\mathbf{n}}\check{\phi} > 0$. The result follows from the fact that

$$\partial_{\mathbf{n}}\phi = |(F^{-1})'|\partial_{\mathbf{n}}\check{\phi} \circ F^{-1}.$$

■

Appendix C. A density result

Lemma C.1. *Assume that Ω is bounded, simply connected and $\partial\Omega$ is Dini-continuous. Then the set $\mathscr{H}^2(\Omega) \cap W^{1,\infty}(\Omega)$ is dense in $\mathscr{H}^2(\Omega)$.*

Proof. We recall Notation 6. Let $u \in \mathscr{H}^2(\Omega)$. Then $u \circ F = \sum_{k\geq 0} a_k z^k$ is holomorphic on $D(0,1)$ and $(a_k)_{k\geq 0} \in \ell^2(\mathbb{N})$. Let $\varepsilon \in (0,1)$. The function

$$\widetilde{u}_\varepsilon : D(0,1) \ni z \mapsto u \circ F((1-\varepsilon)z) \in \mathbb{C}$$

is holomorphic on $D(0, 1/(1-\varepsilon))$. We denote $u_\varepsilon = \widetilde{u}_\varepsilon \circ F^{-1}$. We have

$$\begin{aligned}
\|u - u_\varepsilon\|^2_{\mathscr{H}^2(\Omega)} &:= \int_{\partial\Omega} |u(x) - u_\varepsilon(x)|^2 \partial_{\mathbf{n}}\phi \, \mathrm{d}x \\
&= \int_{\partial D(0,1)} |u \circ F(y) - u \circ F((1-\varepsilon)y)|^2 |F'(y)| \partial_{\mathbf{n}}\phi \circ F(y) \, \mathrm{d}y \\
&\leq c_2 \|\partial_{\mathbf{n}}\phi\|_{L^\infty} \int_{\partial D(0,1)} |u \circ F(y) - u \circ F((1-\varepsilon)y)|^2 \, \mathrm{d}y \\
&\leq c_2 \|\partial_{\mathbf{n}}\phi\|_{L^\infty} \sum_{k\geq 1} |a_k|^2 |1 - (1-\varepsilon)^k|^2.
\end{aligned}$$

Note that $\partial_{\mathbf{n}}\phi$ is bounded by Lemma B.1. By Lebesgue's theorem, $(u_\varepsilon)_{\varepsilon\in(0,1)}$ converges to u in $\mathscr{H}^2(\Omega)$. Since also $(u_\varepsilon)_{\varepsilon\in(0,1)} \subset W^{1,\infty}(\Omega)$, the result follows. ∎

Acknowledgments. N.R. is grateful to T. Hmidi for sharing the reference [15]. L. L.T. is grateful to S. Monniaux, J. Olivier, S. Rigat and E. H. Youssfi for useful discussions. The authors are grateful to Bernard Helffer for many useful comments.

Funding. L. L.T. is supported by ANR DYRAQ ANR-17-CE40-0016-01. E.S. has been partially funded by Fondecyt (Chile) project # 118–0355.

References

[1] Apushkinskaya, D. E., Nazarov, A. I.: A counterexample to the Hopf–Oleinik lemma (elliptic case). Anal. PDE **9**, 439–458 (2016) Zbl 1354.35024 MR 3513140

[2] Bonthonneau, Y., Raymond, N.: WKB constructions in bidimensional magnetic wells. Math. Res. Lett. **27**, 647–664 (2020) Zbl 07317388 MR 4216562

[3] Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York (2011) Zbl 1220.46002 MR 2759829

[4] Duren, P. L.: Theory of H^p Spaces. Dover Publ. (2000) Zbl 0215.20203 MR 0268655

[5] Ekholm, T., Kovařík, H., Portmann, F.: Estimates for the lowest eigenvalue of magnetic Laplacians. J. Math. Anal. Appl. **439**, 330–346 (2016) Zbl 1386.35295 MR 3474366

[6] Erdős, L.: Rayleigh-type isoperimetric inequality with a homogeneous magnetic field. Calc. Var. Partial Differential Equations **4**, 283–292 (1996) Zbl 0846.35094 MR 1386738

[7] Evans, L. C.: Partial Differential Equations. Grad. Stud. Math. 19, Amer. Math. Soc., Providence, RI (1998) Zbl 0902.35002 MR 1625845

[8] Fournais, S. R., Helffer, B.: Spectral Methods in Surface Superconductivity. Progr. Nonlinear Differential Equations Appl. 77, Birkhäuser Boston, Boston, MA (2010) Zbl 1256.35001 MR 2662319

[9] Gilbarg, D., Trudinger, N. S.: Elliptic Partial Differential Equations of Second Order. 2nd ed., Grundlehren Math. Wiss. 224, Springer, Berlin (1983) Zbl 0562.35001 MR 737190

[10] Helffer, B., Morame, A.: Magnetic bottles in connection with superconductivity. J. Funct. Anal. **185**, 604–680 (2001) Zbl 1078.81023 MR 1856278

[11] Helffer, B., Persson Sundqvist, M.: On the semi-classical analysis of the ground state energy of the Dirichlet Pauli operator. J. Math. Anal. Appl. **449**, 138–153 (2017) Zbl 1356.81132 MR 3595196

[12] Helffer, B., Persson Sundqvist, M.: On the semiclassical analysis of the ground state energy of the Dirichlet Pauli operator in non-simply connected domains. J. Math. Sci. (N.Y.) **226**, 531–544 (2017) Zbl 1414.35134 MR 3705157

[13] Kawohl, B.: Rearrangements and Convexity of Level Sets in PDE. Lecture Notes in Math. 1150, Springer, Berlin (1985) Zbl 0593.35002 MR 810619

[14] Kawohl, B.: When are solutions to nonlinear elliptic boundary value problems convex? Comm. Partial Differential Equations **10**, 1213–1225 (1985) Zbl 0587.35026 MR 806439

[15] Pommerenke, C.: Boundary Behaviour of Conformal Maps. Grundlehren Math. Wiss. 299, Springer, Berlin (1992) Zbl 0762.30001 MR 1217706

[16] Schmidt, K. M.: A remark on boundary value problems for the Dirac operator. Quart. J. Math. Oxford Ser. (2) **46**, 509–516 (1995) Zbl 0864.35092 MR 1366620

[17] Thaller, B.: The Dirac Equation. Springer, Berlin (1992) Zbl 0765.47023 MR 1219537