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# Strongly minimal groups in o-minimal structures

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**Abstract.** We prove Zilber's Trichotomy Conjecture for strongly minimal expansions of 2-dimensional groups, definable in o-minimal structures:

*Theorem.* Let  $\mathcal{M}$  be an o-minimal expansion of a real closed field,  $(G; +)$  a 2-dimensional group definable in  $\mathcal{M}$ , and  $\mathcal{D} = \langle G; +, \dots \rangle$  a strongly minimal structure, all of whose atomic relations are definable in  $\mathcal{M}$ . If  $\mathcal{D}$  is not locally modular, then an algebraically closed field  $K$  is interpretable in  $\mathcal{D}$ , and the group  $G$ , with all its induced  $\mathcal{D}$ -structure, is definably isomorphic in  $\mathcal{D}$  to an algebraic  $K$ -group with all its induced  $K$ -structure.

**Keywords.** O-minimality, strongly minimal groups, Zilber's conjecture

## 1. Introduction

### 1.1. Zilber's Conjecture (ZC)

In [41], Boris Zilber formulated the following conjecture.

**Zilber's Trichotomy Conjecture.** *The geometry of every strongly minimal structure  $\mathcal{D}$  is either (i) trivial, (ii) non-trivial and locally modular, or (iii) isomorphic to the geometry of an algebraically closed field  $K$  definable in  $\mathcal{D}$ . Moreover, in (iii) the structure induced on  $K$  from  $\mathcal{D}$  is already definable in  $K$  (that is, the field  $K$  is "pure" in  $\mathcal{D}$ ).*

The conjecture reduces by [8] to: *if a strongly minimal structure  $\mathcal{D}$  is not locally modular, then it interprets a field  $K$ , and the field  $K$  is pure in  $\mathcal{D}$ .*

In the early 1990s, Hrushovski refuted both parts of the conjecture. Using his amalgamation method he showed the existence of a strongly minimal structure which is not locally modular and yet does not interpret any group (so certainly not a field) [10]. In addition he showed the existence of a *proper* strongly minimal expansion of a field [9],

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thus disproving also the purity of the field. Nevertheless, Zilber's Conjecture stayed alive since it turned out to be true in various restricted settings, and moreover its verification in those settings gave rise to important applications (such as Hrushovski's proof of the function field Mordell–Lang conjecture in all characteristics [11]).

A common feature to many cases where the conjecture is true is the presence of an underlying geometry putting strong restrictions on the definable sets in the strongly minimal structure  $\mathcal{D}$ . This is for example the case when  $\mathcal{D}$  is definable in an algebraically closed field [7, 19, 35], in a differentially closed field [17], in a separably closed field [11], or in an algebraically closed valued field [16]. This is also the case when  $\mathcal{D}$  is endowed with a Zariski geometry [13].

Thus, it is interesting to examine the conjecture in various geometric settings. In this paper, we consider Zilber's Conjecture in the o-minimal geometric setting, introduced in the 1980s [15, 33, 37]. O-minimality imposes strong conditions on definable complex analytic objects, forcing them in many cases to be algebraic (see [28] for a survey, and [1] for a recent application). The results of this paper can be seen as another manifestation of the same phenomenon.

### 1.2. The connection to o-minimality

The complex field is an example of a strongly minimal structure definable in the o-minimal structure  $\langle \mathbb{R}; +, \cdot, < \rangle$ , and indeed the underlying Euclidean geometry is an important component in understanding complex algebraic varieties. This leads to examining in greater generality those strongly minimal structures definable in o-minimal ones, and to the following restricted variant of Zilber's Conjecture, formulated by the third author in a model theory conference at East Anglia in 2005.

**The o-minimal ZC.** *Let  $\mathcal{M}$  be an o-minimal structure and  $\mathcal{D}$  a strongly minimal structure whose underlying set and atomic relations are definable in  $\mathcal{M}$ . If  $\mathcal{D}$  is not locally modular, then an algebraically closed field  $K$  is interpretable in  $\mathcal{D}$ , and moreover  $K$  is a pure field in  $\mathcal{D}$ .*

- Remark 1.1.** (1) Because every algebraically closed field of characteristic zero ( $\text{ACF}_0$ ) is definable in an o-minimal real closed field, Zilber's Conjecture for reducts of algebraically closed fields of characteristic zero is a special case of the o-minimal ZC. This variant of the conjecture is still open for reducts whose universe is not an algebraic curve.
- (2) The purity of the field in the o-minimal setting was already proven in [26], thus the o-minimal ZC reduces to proving the interpretability of a field in  $\mathcal{D}$ .
- (3) Since every definable algebraically closed field in an o-minimal structure has dimension 2 (see [30]), it is not hard to see that the above conjecture implies that the underlying universe of  $\mathcal{D}$  must be 2-dimensional in  $\mathcal{M}$ . Therefore, it is natural to consider the o-minimal ZC under the 2-dimensionality assumption on  $\mathcal{D}$ , which is the case of our Theorem 1.3 below.

- (4) By [6], if  $\mathcal{D}$  is strongly minimal, interpretable in an o-minimal structure and in addition  $\dim_{\mathcal{M}} D = 1$ , then  $\mathcal{D}$  must be locally modular, thus trivially implying the o-minimal ZC in the case when  $\dim_{\mathcal{M}} D = 1$ .
- (5) The theory of compact complex manifolds, denoted by CCM (see [42]), is the multi-sorted theory of the structure whose sorts are all compact complex manifolds, endowed with all analytic subsets and analytic maps. It is known [42, Theorems 3.4.3 and 3.2.8] that each sort in this structure has finite Morley rank, and also that the structure is interpretable in the o-minimal structure  $\mathbb{R}_{\text{an}}$ . Hence, every sufficiently saturated structure elementarily equivalent to a CCM is interpretable in an o-minimal structure.

By [20], every set of Morley rank 1 in any model of CCM is definably isomorphic to an algebraic curve. Thus, Zilber’s conjecture for reducts of CCM whose universe is analytically 1-dimensional reduces to the work in [7]. The higher-dimensional cases may also reduce to the conjecture for  $\text{ACF}_0$  but this is still open.

In [5] the following case of the o-minimal ZC was proven.

**Theorem 1.2.** *Let  $\mathcal{R} := \langle R, +, \cdot, <, \dots \rangle$  be an o-minimal expansion of a real closed field,  $K := R[i]$  its algebraic closure. Let  $f : K \rightarrow K$  be an  $\mathcal{R}$ -definable function. If  $\mathcal{D} = \langle K; +, f \rangle$  is strongly minimal and is not locally modular (equivalently,  $f$  is not an affine map), then up to conjugation by an invertible  $2 \times 2$   $R$ -matrix and finitely many corrections,  $f$  is a  $K$ -rational function. In addition, a function  $\odot : K^2 \rightarrow K$  is definable in  $\mathcal{D}$ , making  $\langle K; +, \odot \rangle$  an algebraically closed field.*

In our current result below we replace the additive group of  $K$  above by an arbitrary  $\mathcal{R}$ -definable 2-dimensional group  $G$ . Moreover, we let  $\mathcal{D}$  be an arbitrary expansion of  $G$  and not only by a map  $f : G \rightarrow G$ . Since strongly minimal groups are abelian [34, Corollary 3.1], we write the group below additively. Here is the main theorem of our article.

**Theorem 1.3.** *Let  $\mathcal{M}$  be an o-minimal expansion of a real closed field  $R$ , and let  $\langle G; \oplus \rangle$  be a 2-dimensional group definable in  $\mathcal{M}$ . Let  $\mathcal{D} = \langle G; \oplus, \dots \rangle$  be a strongly minimal structure expanding  $G$ , all of whose atomic relations are definable in  $\mathcal{M}$ .*

*Then there are in  $\mathcal{D}$  an interpretable algebraically closed field  $K$ , a  $K$ -algebraic group  $H$  with  $\dim_K H = 1$ , and a definable isomorphism  $\varphi : G \rightarrow H$ , such that the definable sets in  $\mathcal{D}$  are precisely those of the form  $\varphi^{-1}(X)$  for  $X$  a  $K$ -constructible subset of  $H^n$ .*

*In fact, the structure  $\mathcal{D}$  and the field  $K$  are bi-interpretable.*

Note that the theorem implies in particular that  $G$  is definably isomorphic in  $\mathcal{D}$  to either  $\langle K; + \rangle$ ,  $\langle K^\times; \cdot \rangle$  or an elliptic curve over  $K$ .

### 1.3. The general strategy: from real geometry and strong minimality to complex algebraic geometry

Let  $\mathcal{M}$ ,  $G$  and  $\mathcal{D}$  be as in Theorem 1.3. Since  $G$  is a group definable in an o-minimal expansion of a real closed field  $R$ , it admits a differentiable structure which makes it into

a Lie group with respect to  $R$  (see [31]). We let  $\mathfrak{F}$  be the collection of all differentiable (with respect to that Lie structure) partial functions  $f : G \rightarrow G$ , with  $f(0_G) = 0_G$ , such that for some  $\mathcal{D}$ -definable strongly minimal  $S_f \subseteq G^2$ , we have  $\text{graph}(f) \subseteq S_f$ . We let  $J_0 f$  denote the Jacobian matrix of  $f$  at 0. The following is easy to verify, using the chain rule for differentiable functions:

$$J_0(f \oplus g) = J_0 f + J_0 g, \quad J_0(f \circ g) = J_0 f \cdot J_0 g,$$

where on the left hand side of each equation we use the group operation and functional composition, and on the right hand side the usual matrix operations in  $M_2(R)$ . Let also

$$\mathfrak{R} = \{J_0 f \in M_2(R) : f \in \mathfrak{F}\}.$$

The key observation, going back to Zilber, is that via the above equations we can recover a ring structure on  $\mathfrak{R}$  by performing addition and composition of curves in  $\mathcal{D}$ . Most importantly, for the ring structure to be  $\mathcal{D}$ -definable, one needs to recognize tangency of curves at a point  $\mathcal{D}$ -definably. The geometric idea for that goes back to Rabinovich’s work [35], and requires us to develop a sufficient amount of intersection theory for  $\mathcal{D}$ -definable sets, so as to recognize “combinatorially” when two curves are tangent.

This paper establishes in several distinct steps the necessary ingredients for the proof. In each of these steps we prove an additional property of  $\mathcal{D}$ -definable sets which shows their resemblance to complex algebraic sets. We briefly describe these steps.

We call  $S \subseteq G^2$  a *plane curve* if it is  $\mathcal{D}$ -definable and  $\text{RM}(S) = 1$  (we recall the definition of Morley rank in Section 2.1). In Section 4 we investigate the frontier of plane curves, where the frontier of a set  $S$  is  $\text{cl}(S) \setminus S$ . We prove that every plane curve has finite frontier in the group topology on  $G$ .

In Section 5 we consider the poles of plane curves, where a pole of  $S \subseteq G^2$  is a point  $a \in G$  such that for every neighborhood  $U \ni a$ , the set  $(U \times G) \cap S$  is “unbounded”. We prove that every plane curve has at most finitely many poles.

As a corollary of the above two results we establish in Section 6 another geometric property which is typically true for complex analytic curves. Namely, we show that every plane curve  $S$  whose projection on both coordinates is finite-to-one, is locally, outside finitely many points, the graph of a homeomorphism.

Next, we discuss the differential properties of plane curves, and consider in Section 7 the collection,  $\mathfrak{R}$ , of all Jacobian matrices at 0 of local smooth maps from  $G$  to  $G$  whose graph is contained in a plane curve. Using our previous results we prove that this collection forms an algebraically closed subfield  $K$  of  $M_2(R)$ , and thus up to conjugation by a fixed invertible matrix, every such Jacobian matrix at 0 satisfies the Cauchy–Riemann equations.

In Section 8 we establish elements of complex intersection theory, showing that if two plane curves  $E$  and  $X$  are tangent at some point, then by varying  $E$  within a sufficiently well-behaved family, we gain additional intersection points with  $X$ . This allows us to identify tangency of curves in  $\mathcal{D}$  by counting intersection points.

Finally, in Section 9 we use the above results in order to interpret an algebraically closed field in  $\mathcal{D}$  and prove our main theorem.

## 2. Preliminaries

We briefly review the basic model-theoretic notions appearing in the text. We refer to any standard textbook in model theory (such as [18, §6, §7]) for more details. Standard facts on o-minimality can be found in [38] whose Sections 1.1 and 1.2 provide most of the basic background needed on structures and definability.

### 2.1. Strong minimality and related notions

Throughout the text, given a structure  $\mathcal{N}$ , by  $\mathcal{N}$ -definable we mean definable in  $\mathcal{N}$  with parameters, unless stated otherwise. We drop the prefix ‘ $\mathcal{N}$ -’ if it is clear from the context. In the next subsection, we will adopt a global convention about this prefix to be enforced in Sections 4–9.

Let  $\mathcal{N} = \langle N, \dots \rangle$  be an  $\omega$ -saturated structure. A definable set  $S$  is *strongly minimal* if every definable subset of  $S$  is finite or co-finite. We call  $\mathcal{N}$  *strongly minimal* if  $N$  is a strongly minimal set.

Let  $\mathcal{N} = \langle N, <, \dots \rangle$  be an expansion of a dense linear order without endpoints. We call  $\mathcal{N}$  *o-minimal* if every definable subset of  $N$  is a finite union of points from  $N$  and open intervals whose endpoints lie in  $N \cup \{\pm\infty\}$ . The standard topology in  $\mathcal{N}$  is the order topology on  $N$  and the product topology on  $N^n$ .

Now let  $\mathcal{N}$  be a strongly minimal structure or an o-minimal structure. The algebraic closure operator  $\text{acl}$  in both cases is known to give rise to a pregeometry. We refer to [18, §6.2] and [31, §1] for all details, and recall here only some. Given  $A \subseteq N$  and  $a \in N^n$ , we let  $\dim(a/A)$  be the size of a maximal  $\text{acl}$ -independent subtuple of  $a$  over  $A$ . Given a set  $C \subseteq N^n$  definable over  $A$  we let

$$\dim C = \max \{ \dim(a/A) : a \in C \},$$

and we call an element  $a \in C$  *generic in  $C$  over  $A$  in  $\mathcal{N}$*  if  $\dim(a/A) = \dim C$ . We also note that  $\mathcal{N}$  eliminates the  $\exists^\infty$  quantifier. Namely, if  $\varphi(x, y)$  is a formula, then the set of all  $x$  for which there are infinitely many  $y$  such that  $\varphi(x, y)$  holds is a definable set. We say that  $\exists^\infty y \varphi(x, y)$  defines that set.

If  $\mathcal{N}$  is a strongly minimal structure, then  $\dim(C)$  coincides with the Morley rank of  $C$ , and we denote  $\dim(a/A)$  and  $\dim C$  by  $\text{RM}(a/A)$  and  $\text{RM}(C)$ , respectively. We denote the Morley degree of  $C$  by  $\text{MD}(C)$ . In the o-minimal case,  $\dim C$  coincides with topological dimension of  $C$ , and we keep the notation  $\dim(a/A)$  and  $\dim C$ .

Let  $\mathcal{N}$  be any structure. Given a definable set  $X$ , a *canonical parameter for  $X$*  is an element in  $\mathcal{N}^{\text{eq}}$  which is interdefinable with the set  $X$ , namely  $\bar{a}$  is a canonical parameter for  $X$  if  $\varphi(\bar{x}, \bar{a})$  defines  $X$  and  $\varphi(\bar{x}, \bar{a}') \neq X$  for all  $\bar{a}' \neq \bar{a}$ . Any two canonical parameters are interdefinable over  $\emptyset$ , and so we use  $[X]$  to denote any such parameter. Note that if  $X = X_{t_0}$  for some definable family of sets over  $\emptyset$ ,  $\{X_t : t \in T\}$ , then  $[X] \in \text{dcl}(t_0)$ , but  $t_0$  need not be a canonical parameter for  $X$ .

A structure  $\mathcal{N} = \langle N, \dots \rangle$  is *interpretable* in  $\mathcal{M}$  if there is an isomorphism of structures  $\alpha : \mathcal{N} \rightarrow \mathcal{N}'$ , where the universe of  $\mathcal{N}'$  and all  $\mathcal{N}'$ -atomic relations are interpretable in  $\mathcal{M}$ .

If  $\mathcal{N}$  is interpretable in  $\mathcal{M}$  via  $\alpha$  and  $\mathcal{M}$  is interpretable in  $\mathcal{N}$  via  $\beta$ , and if in addition  $\beta \circ \alpha$  is definable in  $\mathcal{N}$  and  $\alpha \circ \beta$  is definable in  $\mathcal{M}$ , then we say that  $\mathcal{M}$  and  $\mathcal{N}$  are *bi-interpretable*.

Note that if  $\mathcal{M}$  is an o-minimal expansion of an ordered group, then by definable choice, every interpretable structure in  $\mathcal{M}$  is also definable in  $\mathcal{M}$ .

### 2.2. The setting

Throughout Sections 4–9, we fix a sufficiently saturated o-minimal expansion  $\mathcal{M} = \langle R; +, \cdot, <, \dots \rangle$  of a real closed field. As described in [38, Chapters 6–7], definable sets in  $\mathcal{M}$  admit various topological properties with respect to the underlying order topology on  $R$  and the product topology on  $R^n$ . In addition, a theory of differentiability with respect to  $R$  is developed there, allowing notions which are analogous to classical ones, such as manifolds, differentials of definable maps, Jacobian matrices, etc. We are going to exploit this theory heavily, similarly to the way  $\mathbb{R}$ -differentiability is often used when developing complex algebraic geometry.

Throughout the same sections, we also fix a 2-dimensional  $\mathcal{M}$ -definable group  $G$ . By [31], the group  $G$  admits a definable  $C^1$ -manifold structure with respect to the field  $R$ , such that the group operation and inverse function are  $C^1$  maps with respect to it. The topology and differentiable structure which we refer to below are always those of this smooth group structure on  $G$ . Note that the group  $G$  is definably isomorphic, as a topological group, to a definable group whose domain is a closed subset of some  $R^r$ , endowed with the  $R^r$ -topology (see, for example, [29, Claim 3.1]). Thus, we assume that  $G$  is a closed subset of  $R^r$  and its topology is the subspace topology.

Finally, throughout Sections 4–9, we fix a strongly minimal non-locally-modular structure  $\mathcal{D} = \langle G; \dots \rangle$  definable in  $\mathcal{M}$ . We treat  $\mathcal{M}$  as the default structure and thus use “definable” to mean “definable in  $\mathcal{M}$ ”, and use “ $\mathcal{D}$ -definable” to mean “definable in  $\mathcal{D}$ ”. Similarly, we use  $\text{acl}$ ,  $\text{dim}$  and “generic” to denote the corresponding notions in  $\mathcal{M}$ , and let  $\text{acl}_{\mathcal{D}}$ ,  $\text{RM}$ , “ $\mathcal{D}$ -generic” and “ $\mathcal{D}$ -canonical parameter” denote the corresponding notions in  $\mathcal{D}$ .

Since the underlying universe of the strongly minimal structure  $\mathcal{D}$  is the 2-dimensional set  $G$ , it follows that for every  $\mathcal{D}$ -definable set  $X \subseteq G^n$ , we have

$$\dim X = 2 \text{RM}(X).$$

Also, for  $a \in G^n$  and  $A \subseteq G$ , we have

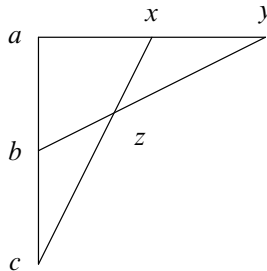
$$\dim(a/A) \leq 2 \text{RM}(a/A),$$

and in particular, if  $X \subseteq G^n$  is definable in  $\mathcal{D}$  and  $a \in X$  is generic in  $X$  over  $A$ , then it is also  $\mathcal{D}$ -generic in  $X$  over  $A$ . The converse fails: indeed, let  $\mathcal{M}$  be the real field and  $\mathcal{D}$  the complex field, interpretable in the real field  $\mathcal{M}$ . The element  $\pi \in \mathbb{C}$  is  $\mathcal{D}$ -generic in  $\mathbb{C}$  over  $\emptyset$  but it is not generic in  $\mathbb{C}$  over  $\emptyset$  because it is contained in the definable, 1-dimensional set  $\mathbb{R}$ .

2.3. The field configuration

Recall the following definition.

**Definition 2.1.** Let  $\mathcal{N}$  be a strongly minimal structure. A set  $\{a, b, c, x, y, z\}$  of tuples is called a *field configuration* in  $\mathcal{N}$  if:



- (1) All elements of the diagram are pairwise independent and  $RM(a, b, c, x, y, z) = 5$ .
- (2)  $RM(a) = RM(b) = RM(c) = 2, RM(x) = RM(y) = RM(z) = 1$ .
- (3) All triples of tuples lying on the same line are dependent, and moreover  $RM(a, b, c) = 4, RM(a, x, y) = RM(b, z, y) = RM(c, x, z) = 3$ .
- (4)  $RM(Cb(x, y)/a) = RM(a), RM(Cb(y, z)/c) = RM(b)$  and  $RM(Cb(x, z)/c) = RM(c)$ .

(For the notion  $Cb$  of a canonical base, see [32, p. 19].)

**Remark 2.2.** Consider the following minimality condition on a set  $\{a, b, c, x, y, z\}$  of tuples in  $\mathcal{N}$ :

- (4)' There are no  $a' \in acl(a), b' \in acl(b)$  and  $c' \in acl(c)$  with  $RM(a') = RM(b') = RM(c') = 1$  such that (1)–(3) above hold with  $a', b', c'$  replacing  $a, b, c$ .

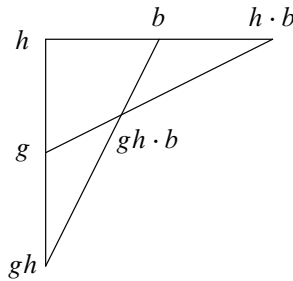
Standard Morley rank calculations show that the above conditions (1)–(4) are equivalent to (1)–(3) and (4)'.

For a proof of the following theorem, see [4, Main Theorem, Proposition 2] and the discussion following Proposition 2 there.

**Fact 2.3** (Hrushovski). *If a strongly minimal structure  $\mathcal{N}$  admits a field configuration, then  $\mathcal{N}$  interprets an algebraically closed field.*

Let  $\mathbb{G}_m$  and  $\mathbb{G}_a$  denote the multiplicative and additive groups of an algebraically closed field  $K$ . The action of  $\mathbb{G}_m \times \mathbb{G}_a$  on  $\mathbb{G}_a$  (defined by  $(a, c) \cdot b = ab + c$ ) gives rise, naturally, to a field configuration on the structure  $(K, +, \cdot)$  as follows: take  $g, h \in \mathbb{G}_m \times \mathbb{G}_a$  independent generics (in  $K$ ), and  $b \in \mathbb{G}_a$  generic over  $g, h$ . Then

$$\mathcal{F} := \{h, g, gh, b, h \cdot b, gh \cdot b\}$$



where  $\cdot$  denotes the action of  $\mathbb{G}_m \times \mathbb{G}_a$  on  $\mathbb{G}_a$ , is readily verified to be a field configuration in the field  $K$  (we will prove a slightly more general statement in Lemma 3.20 allowing us to construct field configurations from certain families of plane curves).

When constructing a field configuration in Section 9, we will need the lemma below. Given an algebraically closed field  $K$ , denote by  $\text{AGL}_1(K)$  the group of its affine transformations. Let  $\mathcal{M}$  be an o-minimal expansion of a real closed field, and  $\mathcal{D}$  a 2-dimensional definable strongly minimal structure. Here and below, we follow the conventions mentioned in Section 2.2. Namely, notions such as definability, genericity,  $\dim$  and  $\text{acl}$  refer to  $\mathcal{M}$ , unless indexed otherwise.

**Lemma 2.4.** *Let  $K$  be a definable algebraically closed field and  $h, g \in \text{AGL}_1(K)$  independent generics. Let  $b \in K$  be a generic independent from  $g, h$ . Let  $\mathcal{Y} = \{h', g', k', b', c', d'\} \subseteq \mathcal{D}^n$  be such that:*

- $h', g', b'$  are interalgebraic over  $\emptyset$  with  $h, g, b$  respectively.
- $k', c', d'$  are interalgebraic over  $\emptyset$  with  $gh, h \cdot b, gh \cdot b$  respectively.

Then  $\mathcal{Y}$  is a field configuration in  $\mathcal{D}$  if and only if it satisfies (3) of Definition 2.1.

*Proof.* Because  $\mathcal{D}$  is 2-dimensional, if  $S$  is a  $\mathcal{D}$ -definable set, then by what we have already explained in Section 2.2,  $\dim S = 2 \text{RM}(S)$ . Since o-minimal dimension is preserved under interalgebraicity, it will suffice to show that (1), (2) and (4) of Definition 2.1 hold with  $\text{RM}$  replaced by  $\frac{1}{2} \dim$ .

Because  $\dim K = 2$  we get  $\dim \text{AGL}_1(K) = 4$ . By exchange (in  $\mathcal{M}$ ), we see that (1) and (2) above hold. So it remains to verify (4). Note that, by genericity of  $b$ , for example, the point  $(b, h \cdot b)$  is generic on the affine  $K$ -line  $(x, h_1x + h_2)$  where  $h = (h_1, h_2)$ . Since any two distinct affine lines intersect in at most one point, any automorphism fixing the affine line  $(x, h_1x + h_2)$  setwise must also fix  $h$  (pointwise). So  $h$  is a canonical base for  $\text{tp}_K(b, h \cdot b/h)$ . Using the interalgebraicity it follows that  $h'$  is  $\mathcal{D}$ -interalgebraic with  $\text{Cb}(b', c'/h')$ . Similarly, the rest of clause (4) carries over from  $K$  to  $\mathcal{D}$ . ■

#### 2.4. Notation

If  $S$  is a set in a topological space, its closure, interior, boundary and frontier are denoted by  $\text{cl}(S)$ ,  $\text{int}(S)$ ,  $\text{bd}(S) := \text{cl}(S) \setminus \text{int}(S)$  and  $\text{fr}(S) := \text{cl}(S) \setminus S$ , respectively. Given



a group  $\langle G, + \rangle$  and sets  $A, B \subseteq G$ , we denote by  $A - B$  the Minkowski difference of the two sets,  $A - B = \{x - y : x \in A, y \in B\}$ . Given a set  $X$  and  $S \subseteq X^2$ , we denote  $S^{\text{op}} = \{(y, x) \in X^2 : (x, y) \in S\}$ . The graph of a function  $f$  is denoted by  $\Gamma_f$ . If  $\gamma : (a, b) \rightarrow \mathbb{R}^n$  is a definable curve we will let  $\gamma$  also denote the image of  $\gamma$  in  $\mathbb{R}^n$ . Thus, if for some definable function  $f$  we have  $\gamma(t) \in \text{dom}(f)$  for all  $t$ , we may write  $f(\gamma)$  instead of  $f(\text{Im}(\gamma))$ . For  $\mathcal{M} = \langle \mathbb{R}; +, \cdot, <, \dots \rangle$  as above and  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , we write  $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ .

### 3. Plane curves

In this section, we work in a strongly minimal structure  $\mathcal{D}$  and prove some lemmas about the central objects of our study, plane curves. When  $\mathcal{D}$  expands a group  $G$  and is not locally-modular, we construct in Sections 3.3 and 3.4 two special definable families of plane curves which will be used in the subsequent sections.

#### 3.1. Some basic definitions and notations

Let  $\mathcal{D}$  be a strongly minimal structure.

**Definition 3.1.** A  $\mathcal{D}$ -plane curve (or just plane curve) is a  $\mathcal{D}$ -definable subset of  $G^2$  of Morley rank 1.

**Definition 3.2.** For two plane curves  $C_1, C_2$ , we write  $C_1 \sim C_2$  if  $|C_1 \Delta C_2| < \infty$ . Note that this gives a  $\mathcal{D}$ -definable equivalence relation on any  $\mathcal{D}$ -definable family of plane curves. A  $\mathcal{D}$ -definable family of plane curves,  $\mathcal{F} = \{C_t : t \in T\}$ , is *faithful* if for  $t_1 \neq t_2$  in  $T$ ,  $C_{t_1} \Delta C_{t_2}$  is finite (i.e.,  $C_1 \sim C_2$ ). It is *almost faithful* if all  $\sim$ -equivalence classes are finite.

Note that if  $\mathcal{F} = \{C_t : t \in T\}$  is a faithful family of plane curves, then  $t$  is a canonical parameter for  $C_t$ . If  $\mathcal{F}$  is almost faithful, then  $t$  is interalgebraic with a canonical parameter of  $C_t$ .

Given a  $\mathcal{D}$ -definable family  $\mathcal{F}$  of plane curves, there exists a  $\mathcal{D}$ -definable almost faithful family  $\mathcal{F}' = \{C'_t : t \in T'\}$  of plane curves (possibly over additional parameters) such that every curve in  $\mathcal{F}$  has an equivalent curve in  $\mathcal{F}'$  and vice versa (see for example [9, p. 137]).<sup>1</sup> It is not hard to see that  $\text{RM}(T')$  is independent of the choice of  $\mathcal{F}'$ . Thus, we can make the following definition.

**Definition 3.3.** A  $\mathcal{D}$ -definable family of plane curves  $\mathcal{F}$  as above is said to be *n-dimensional*, written  $\text{RM}(\mathcal{F}) = n$ , if in the corresponding almost faithful family  $\mathcal{F}'$  as above, we have  $\text{RM}(T') = n$ . We call  $\mathcal{F}$  *stationary* if  $\text{MD}(T') = 1$ .

We call  $\mathcal{D}$  a *non-locally-modular structure* if there exists a  $\mathcal{D}$ -definable family of plane curves  $\mathcal{F}$  with  $\text{RM}(\mathcal{F}) \geq 2$ .

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<sup>1</sup>Allowing imaginary elements, we can always obtain faithful families of plane curves. The point here is to work in the real sort only.

In fact, by [32, Proposition 5.3.2], if  $\mathcal{D}$  is not locally modular, then for every  $n$  there exists an  $n$ -dimensional  $\mathcal{D}$ -definable family of plane curves. We will sketch a proof of a slightly stronger result in Proposition 3.21 below.

The following terminology is inspired by [13].

**Definition 3.4.** Let  $\mathcal{F} = \{C_t : t \in T\}$  be a  $\mathcal{D}$ -definable family of plane curves. For every  $p \in G^2$ , denote

$$T(p) = \{t \in T : p \in C_t\}, \quad \mathcal{F}(p) = \{C_t : p \in C_t\}.$$

We say that  $\mathcal{F}$  is (*generically*) *very ample* if for any  $p \neq q \in G^2$  (each  $\mathcal{D}$ -generic over the parameters defining  $\mathcal{F}$ ),

$$\text{RM}(T(p) \cap T(q)) < \text{RM}(T(p)).$$

In the rest of this section,  $\mathcal{D} = \langle G; +, \dots \rangle$  denotes a strongly minimal expansion of a group  $G$ .

### 3.2. Local modularity

Here we recall some basic facts about local modularity.

**Definition 3.5.** A  $\mathcal{D}$ -definable set is *G-affine* if it is a finite boolean combination of cosets of  $\mathcal{D}$ -definable subgroups of  $G$ .

We use the following simple observation without further reference (see, for example, [18, Lemma 7.2.5, Corollary 7.1.6 and Corollary 7.2.4] for details).

**Remark 3.6.** If  $S \subseteq G^2$  is  $\mathcal{D}$ -definable and strongly minimal, then  $S$  is *G-affine* if and only if  $S \sim H + a$  for some  $\mathcal{D}$ -definable strongly minimal subgroup  $H \subseteq G^2$ .

**Definition 3.7.** Given a strongly minimal plane curve  $C$ , the *stabilizer* of  $C$  is the set

$$\text{Stab}^*(C) = \{g \in G^2 : C \sim C + g\}.$$

The stabilizer of  $C$  is easily seen to be a  $\mathcal{D}$ -definable subgroup of  $G^2$ . The next properties are easy to verify.

**Lemma 3.8.** For  $C$  a strongly minimal  $\mathcal{D}$ -plane curve, and  $p, q \in G^2$ :

- (1)  $C + p \sim C + q$  if and only if  $p - q \in \text{Stab}^*(C)$ .
- (2)  $\text{Stab}^*(C)$  is trivial if and only if  $\{C + p : p \in G^2\}$  is a faithful family.
- (3)  $\text{Stab}^*(C)$  is finite if and only if  $\{C + p : p \in G^2\}$  is almost faithful.
- (4)  $\text{Stab}^*(C)$  is infinite if and only if  $C$  is *G-affine*.

We only use here the following characterization of non-local-modularity in expansions of groups, which follows from [12].

**Fact 3.9.** If  $\mathcal{D}$  is a strongly minimal expansion of a group  $G$ , then  $\mathcal{D}$  is not locally modular if and only if there exists a  $\mathcal{D}$ -plane curve which is not *G-affine*.

Note that if  $\mathcal{F} = \{C + p : p \in T\}$  is a  $\mathcal{D}$ -definable family of plane curves, with  $C$  strongly minimal and  $T = G^2$ , then for every  $p \in G^2$ ,

$$T(p) = \{q \in T : p \in C + q\} = p - C.$$

In particular  $T(p)$  is strongly minimal so that, in fact, if  $\text{RM}(T(p) \cap T(q)) = \text{RM}(T(p))$ , then  $T(p) \sim T(q)$ . We thus have the following lemma.

**Lemma 3.10.** *If  $C$  is a strongly minimal plane curve and  $\mathcal{F} = \{C + p : p \in G^2\}$ , then the following are equivalent:*

- (1)  $\mathcal{F}$  is very ample.
- (2)  $\mathcal{F}$  is faithful.
- (3)  $\text{Stab}^*(C)$  is trivial.

Finally, we will need the following definition.

**Definition 3.11.** Let  $\mathcal{F} = \{C_t : t \in T\}$  be a  $\mathcal{D}$ -definable family of plane curves. We call  $C_t$  a  $\mathcal{D}$ -generic curve in  $\mathcal{F}$  over  $A$  if  $t$  is  $\mathcal{D}$ -generic in  $T$  over  $A$ . We say that  $\mathcal{F}$  is generically strongly minimal if every  $\mathcal{D}$ -generic curve in  $\mathcal{F}$  is strongly minimal.

### 3.3. Dividing by a finite subgroup of $G$

The main goal of this subsection is to prove Lemma 3.12 below, which will be used in the proof of Theorem 1.3 in Section 9. It will also allow us to assume, without loss of generality, the existence of a  $\mathcal{D}$ -definable faithful, very ample family of strongly minimal plane curves of Morley rank 2 (Proposition 3.14 below).

Given a strongly minimal plane curve  $C$  which is not  $G$ -affine, we plan to work with the family  $\mathcal{F} = \{C + p : p \in G^2\}$ . We know that  $\text{Stab}^*(C)$  cannot be infinite but it can be a finite, non-trivial group, in which case  $\mathcal{F}$  is neither faithful nor very ample. We prove below that dividing the structure  $\mathcal{D}$  by a finite group is harmless.

Given a finite subgroup  $F \subseteq G$ ,  $\mathcal{D}$ -definable over  $\emptyset$ , we consider the map  $\pi_F : G \rightarrow G/F$ , and still use  $\pi_F : G^n \rightarrow (G/F)^n$  to denote the map  $\pi_F(g_1, \dots, g_n) = (\pi_F(g_1), \dots, \pi_F(g_n))$ .

We let  $\mathcal{D}_F$  be the structure whose universe is  $G/F$  and whose atomic relations are all sets of the form  $\pi_F(S)$  for  $S \subseteq G^n$  a  $\emptyset$ -definable set in  $\mathcal{D}$ . The structure  $\mathcal{D}_F$  is again an expansion of a group.

The following result implies that for the purpose of our main theorem we may work with  $\mathcal{D}_F$  instead of  $\mathcal{D}$ .

**Lemma 3.12.** *Assume that the group  $G$  has unbounded exponent. Then the structures  $\mathcal{D}$  and  $\mathcal{D}_F$  are bi-interpretable, without parameters. In particular,  $\mathcal{D}$  is bi-interpretable with an algebraically closed field if and only if  $\mathcal{D}_F$  is.*

*Proof.* Because  $F$  and  $\pi_F$  are  $\emptyset$ -definable in  $\mathcal{D}$ , the structure  $\mathcal{D}_F$  is interpretable, with no additional parameters, in  $\mathcal{D}$ , via the identity interpretation  $\alpha(g + F) = g + F$ .

Next, let us see how we interpret  $\mathcal{D}$  in  $\mathcal{D}_F$ . Let  $n = |F|$ , and let  $\pi_F^* : G/F \rightarrow G$  be the map defined as follows: given  $y \in G/F$ , and  $x \in G$  for which  $\pi_F(x) = y$ , let

$$\pi_F^*(y) = nx.$$

Because  $G$  is commutative, if  $\pi(x) = \pi(x') = y$ , then  $nx = nx' + ng$  for some  $g \in F$ . Since  $ng = 0$ , this proves that  $\pi_F^*$  is a well-defined group homomorphism with kernel  $\pi_F(G[n])$ , where  $G[n] = \{x \in G : nx = 0\}$ .

Since  $G$  is strongly minimal and has unbounded exponent, the group  $G[n]$  is finite and hence  $\ker(\pi_F^*)$  is finite, so  $\dim \text{Im}(\pi_F^*) = \dim G/F = \dim G$ . Because  $G$  is definably connected,  $\pi_F^*$  is surjective. Thus the homomorphism  $\pi_F^*$  induces an isomorphism of  $(G/F)/\pi_F(G[n])$  with  $G$ . Its inverse  $\beta : G \rightarrow (G/F)/\pi_F(G[n])$  is given by

$$\beta(g) = (g/n + F) + \pi_F(G[n]),$$

where  $g/n$  is any element  $h \in G$  such that  $nh = g$  (note that a strongly minimal group of unbounded exponent is divisible [34, §3.3]).

By our assumptions,  $\pi_F(G[n])$  is  $\emptyset$ -definable in  $\mathcal{D}_F$ , and hence  $(G/F)/\pi_F(G[n])$  is  $\emptyset$ -definable in  $\mathcal{D}_F$ . Now, given any  $\emptyset$ -definable  $X \subseteq G^k$  in  $\mathcal{D}$ , the set  $\{(g_i/n)_{i=1}^k \in G^k : g \in X\}$  is also  $\emptyset$ -definable in  $\mathcal{D}$ , and hence its image in  $(G/F)^k/\pi_F(G[n])^k$  is  $\emptyset$ -definable in  $\mathcal{D}_F$ . We have thus shown that  $\mathcal{D}$  is interpretable, without parameters, in  $\mathcal{D}_F$  via  $\beta$ .

To see that this is indeed bi-interpretation, we first note that the isomorphism between  $\mathcal{D}$  and its interpretation in  $\mathcal{D}_F$  is  $\alpha \circ \beta$ , which equals  $\beta$ . It is clearly definable in  $\mathcal{D}$ .

Let us examine the map induced on  $G/F$  by  $\beta \circ \alpha$  and prove that it is definable in  $\mathcal{D}_F$ . We denote by  $F/n$  the preimage of  $F$  in  $G$  under the map  $g \mapsto ng$ . It is not hard to see that the image of  $F$  inside  $\beta(G)$  is the group  $\pi_F(F/n) + \pi_F(G[n]) = \pi_F(F/n)$ , and hence the isomorphism which  $\beta \circ \alpha$  induces on  $G/F$  is

$$g + F \mapsto g/n + \pi_F(F/n).$$

This map is definable in the group  $G/F$  by sending  $g + F$  to the unique coset  $h + F/n$  such that  $nh + F = g + F$ .

This completes the proof that  $\mathcal{D}$  and  $\mathcal{D}_F$  are bi-interpretable over  $\emptyset$ . ■

Note that in our case, when the group  $G$  is abelian and definable in an o-minimal structure, by [36] the group  $G$  has unbounded exponent, so the above result holds.

For the rest of this subsection, assume that  $\mathcal{D}$  is not locally modular, and fix (after possibly absorbing into the language a finite set of parameters) a strongly minimal plane curve  $C \subseteq G^2$  which is  $\mathcal{D}$ -definable over  $\emptyset$  and not  $G$ -affine. By Lemma 3.8(4),  $F' = \text{Stab}^*(C) \subseteq G^2$  is a finite subgroup and let  $F \subseteq G$  be a  $\mathcal{D}$ - $\emptyset$ -definable subgroup such that  $F' \subseteq F \times F$ . Consider the structure  $\mathcal{D}_F$  expanding  $\langle G/F, + \rangle$  as above.

**Claim 3.13.**  $\pi_F(C)$  is strongly minimal in  $\mathcal{D}_F$  and  $\text{Stab}^*(\pi_F(C))$  in  $(G/F)^2$  is trivial.

*Proof.* The strong minimality of  $\pi_F(C)$  is immediate from the strong minimality of  $C$  in  $\mathcal{D}$ .

Assume that  $q \in \text{Stab}^*(\pi_F(C)) \subseteq (G/F)^2$ , namely  $q + \pi_F(C) \sim \pi_F(C)$ . Let  $\tilde{F} = F \times F \subseteq G^2$  and fix  $p \in G^2$  such that  $\pi_F(p) = q$ . Then  $p + C + \tilde{F} \cap C + \tilde{F}$  is infinite and since  $\tilde{F}$  is finite, there exist  $g, h \in \tilde{F}$  such that  $C + p + g \cap C + h$  is infinite. But then  $p + g - h \in \text{Stab}^*(C) \subseteq \tilde{F}$ , implying that  $p \in \tilde{F}$ , and hence  $0 = \pi_F(p) = q$ .

We have thus shown that  $\text{Stab}^*(\pi_F(C))$  is trivial. ■

Combining Lemmas 3.10, 3.12 and Claim 3.13, we can deduce the following statement.

**Proposition 3.14.** *Assume  $\mathcal{D}$  is not locally modular, expanding a group  $G$  of unbounded exponent. Then there exists a finite group  $F \subseteq G$ , possibly trivial, and in the structure  $\mathcal{D}_F$  defined above there exists a definable family  $\mathcal{L} = \{l_t : t \in Q\}$  of strongly minimal plane curves, which is faithful, very ample, and  $\text{RM}(Q) = 2$ .*

*The structures  $\mathcal{D}$  and  $\mathcal{D}_F$  are bi-interpretable, over the parameters defining  $F$ .*

**Assumption:** *for the rest of the article, we replace the structure  $\mathcal{D}$  with the structure  $\mathcal{D}_F$ , and thus assume that a family  $\mathcal{L}$  as above is definable in  $\mathcal{D}$ .*

### 3.4. Very ample families of high dimension

The goal of this subsection is to construct a larger family  $\mathcal{L}'$  of plane curves which still has the geometric properties of the family  $\mathcal{L}$  from Proposition 3.14. The main method is to use composition of binary relations and families of plane curves. Recall the notion of a composition of binary relations, extending composition of functions: given  $S_1, S_2 \subseteq G^2$ , we let

$$S_1 \circ S_2 = \{(x, z) \in G^2 : \exists y (x, y) \in S_2 \text{ and } (y, z) \in S_1\}.$$

Clearly, if  $S_1, S_2$  are  $\mathcal{D}$ -definable, then so is  $S_1 \circ S_2$ . We will be mostly interested in the composition of plane curves, and even more so, in the composition of families of plane curves: if  $\mathcal{L}_1, \mathcal{L}_2$  are  $\mathcal{D}$ -definable families of plane curves, we let  $\mathcal{L}_1 \circ \mathcal{L}_2 := \{C_1 \circ C_2 : C_1 \in \mathcal{L}_1, C_2 \in \mathcal{L}_2\}$ .

**Definition 3.15.** A plane curve  $S \subseteq G^2$  is a *straight line* if there exists  $a \in G^2$  such that either  $S \sim \{a\} \times G$  or  $S \sim G \times \{a\}$ .

As a rule, geometric properties are not preserved under compositions of (families of) curves. The composition of two strongly minimal curves which are not both straight lines has, indeed, Morley rank 1, but it need not be strongly minimal. More generally, a  $\mathcal{D}$ -generic curve of  $\mathcal{L}_1 \circ \mathcal{L}_2$  need not be strongly minimal, and even if it were,  $\mathcal{L}_1 \circ \mathcal{L}_2$  need not be faithful. In fact, although the dimension of  $\mathcal{L}_1 \circ \mathcal{L}_2$  cannot decrease, it need not be greater than that of  $\mathcal{L}_1$  or  $\mathcal{L}_2$ . For example, if both families are the family of affine lines in  $\mathbb{A}^2$ , then  $\mathcal{L}_1 \circ \mathcal{L}_2 = \mathcal{L}_1$ .

We will need a series of lemmas to address these issues. We start with the following easy observation.

**Lemma 3.16.** *Assume that  $\mathcal{L}_1 = \{C_t : t \in T\}$  and  $\mathcal{L}_2 = \{D_r : r \in R\}$  are two  $\mathcal{D}$ -definable almost faithful families of plane curves, none of which is a straight line, and let  $\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_2$ .*

- (1) *For every  $\mathcal{D}$ -generic  $p$  in  $G^2$ , we have  $\text{RM}(\mathcal{L}(p)) = \text{RM}(R) + \text{RM}(T) - 1$ .*
- (2) *If  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are generically very ample, then so is  $\mathcal{L}$ .*

*Proof.* (1) Let  $\mathcal{L} := \mathcal{L}_1 \circ \mathcal{L}_2$ ,  $C \in \mathcal{L}$  a  $\mathcal{D}$ -generic curve and  $(a, b) \in C$  a  $\mathcal{D}$ -generic point. So  $[C]$  forks over  $(a, b)$  and therefore  $\text{RM}([C]/(a, b)) \leq \text{RM}(R) + \text{RM}(T) - 1$ . Let us see that equality holds, or equivalently  $\text{RM}(\mathcal{L}(a, b)) = \text{RM}(R) + \text{RM}(T) - 1$ . Fix some  $\mathcal{D}$ -generic  $e \in G$ . Then  $(e, b)$  is  $\mathcal{D}$ -generic in  $G^2$ . So  $\mathcal{L}_1(e, b)$  has Morley rank  $\text{RM}(T) - 1$ . Similarly  $\mathcal{L}_2(a, e)$  has Morley rank  $\text{RM}(R) - 1$ . So the set

$$\mathcal{L}(a, b)_e := \{(t, r) \in T \times R : (a, e) \in D_r \wedge (e, b) \in C_t\}$$

has rank  $\text{RM}(R) + \text{RM}(T) - 2$ . But because for  $\mathcal{D}$ -independent generics  $e, e'$ , the sets  $\mathcal{L}(a, b)_e$  and  $\mathcal{L}(a, b)_{e'}$  are disjoint up to a set of lower rank,  $\mathcal{L}(a, b)$  has rank  $\text{RM}(R) + \text{RM}(T) - 1$ .

(2) In order to show that  $\mathcal{L}$  is generically very ample it will suffice to show that  $\mathcal{L}(a, b) \cap \mathcal{L}(c, d)$  has rank at most  $\text{RM}(R) + \text{RM}(T) - 2$  for  $(a, b), (c, d)$  distinct  $\mathcal{D}$ -generics. Fix some  $r \in R$ . Then for  $t \in T$  we find that  $(t, r) \in \mathcal{L}(a, b)$  only if for some  $e$  such that  $(a, e) \in D_r$  we also have  $(e, b) \in C_t$ . If in addition  $(t, r) \in \mathcal{L}(c, d)$ , then there exists  $e'$  such that  $(a, e') \in D_r$  and  $(e', b) \in C_t$ . But as there are only finitely many  $e$  such that  $(a, e) \in D_r$  and only finitely many  $e'$  such that  $(e', b) \in C_t$  it follows that there is a  $\mathcal{D}$ -generic (over  $a, b, c, d$ ) element of  $\mathcal{L}_1(e, b)$  that is also an element of  $\mathcal{L}_1(e', d)$ . Unless  $b = d$ , this contradicts generic very ampleness of  $\mathcal{L}_1$ . So we are reduced to the case where  $b = d$ , in which case a symmetric argument will show that unless also  $a = c$  we get a similar contradiction. But since  $(a, b) \neq (c, d)$ , we are done. ■

**Definition 3.17.** Given two  $\mathcal{D}$ -definable families of plane curves,  $\mathcal{L}$  and  $\mathcal{L}'$ , we say that  $\mathcal{L}$  extends  $\mathcal{L}'$  if for every  $C' \in \mathcal{L}'$  there exists  $C \in \mathcal{L}$  such that  $C' \subseteq C$ .

In the next couple of lemmas we show that although the composition of two families of curves need not preserve the properties of the original families (as already discussed), it extends a family of curves that does.

**Lemma 3.18.** *Let  $\mathcal{L}$  be a  $k$ -dimensional almost faithful  $\mathcal{D}$ -definable family of plane curves. Let  $E$  be a plane curve. Assume neither  $E$  nor any  $\mathcal{D}$ -generic plane curve is a straight line. Then  $E \circ \mathcal{L}$  extends a  $k$ -dimensional almost faithful  $\mathcal{D}$ -definable family of plane curves whose  $\mathcal{D}$ -generic members are strongly minimal. In fact, if  $C \in \mathcal{L}$  is  $\mathcal{D}$ -generic over  $[E]$ , then for any strongly minimal  $C_E \subseteq E \circ C$  we have  $\text{RM}([C_E]/[E]) = k$ .*

*Proof.* Fix some  $C \in \mathcal{L}$  which is  $\mathcal{D}$ -generic over  $[E]$  and  $C_E \subseteq E \circ C$  strongly minimal. Note that  $(E^{-1} \circ C_E) \cap C$  is infinite, and since  $C$  is strongly minimal,  $E^{-1} \circ C_E$  is a set of Morley rank 1, containing the set  $C$ , up to a finite set. It follows that  $[C] \in \text{acl}_{\mathcal{D}}([C_E][E])$ . Since  $\text{RM}([C]/[E]) = \text{RM}([C])$ , we see, by exchange, that  $\text{RM}([C]) = \text{RM}([C]/[E]) = \text{RM}([C_E]/[E])$ .

Absorbing  $[E]$  into the language, we can find  $\bar{c} \in \text{acl}_{\mathcal{D}}([C])$  and a formula  $\varphi(x, \bar{c})$  defining  $C_E$ . By compactness, there is a formula  $\theta \in \text{tp}(\bar{c})$  such that whenever  $\bar{c}' \models \theta$  there is some  $C' \in \mathcal{L}$  such that  $\varphi(x, \bar{c}') \subseteq E \circ C'$ , and for all  $\mathcal{D}$ -generic  $\bar{c}' \models \theta$  the formula  $\varphi(x, \bar{c}')$  is strongly minimal. We may further require – by compactness again – that if  $\varphi(x, \bar{c}') \wedge \varphi(x, \bar{c}'')$  is infinite, then the symmetric difference  $\varphi(x, \bar{c}') \Delta \varphi(x, \bar{c}'')$  is finite for all  $\bar{c}', \bar{c}'' \models \theta$ . By rank considerations, the family  $\{\varphi(G^2, \bar{c}') : \theta(\bar{c}')\}$  is almost faithful of rank  $k$ . ■

As an immediate application (since the only families of straight lines are 1-dimensional), we get the following statement.

**Corollary 3.19.** *Let  $\mathcal{L}_1, \mathcal{L}_2$  be almost faithful  $k$ -dimensional  $\mathcal{D}$ -definable families of plane curves, with  $k > 1$ . Then  $\mathcal{L}_1 \circ \mathcal{L}_2$  extends an almost faithful, stationary, generically strongly minimal family of plane curves of dimension at least  $k$ .*

We can now show that a 2-dimensional family of plane curves closed under composition (such as the family of affine lines in a field) gives rise to a field configuration.

**Lemma 3.20.** *Let  $\mathcal{L}_1, \mathcal{L}_2$  be almost faithful 2-dimensional families of plane curves. Assume that  $\mathcal{L}_1, \mathcal{L}_2$  are  $\mathcal{D}$ -definable over  $\emptyset$ . Let  $X \in \mathcal{L}_1$  and  $Y \in \mathcal{L}_2$  be  $\mathcal{D}$ -independent generic curves, and  $E \subseteq X \circ Y$  strongly minimal.*

- (1) *If  $\text{RM}([E]/\emptyset) = 2$ , then  $\mathcal{D}$  interprets an infinite field.*
- (2) *If  $\text{RM}([E]/\emptyset) = k > 2$  and  $\mathcal{L}_1, \mathcal{L}_2$  are generically very ample, then  $\mathcal{L}_1 \circ \mathcal{L}_2$  extends a  $k$ -dimensional almost faithful, generically strongly minimal, stationary and generically very ample family of curves.*

*Proof.* (1) As we note at the beginning of the proof of Lemma 3.18, each of  $[X], [Y], [E]$  is in the algebraic closure of the other two. Since  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are almost faithful and 2-dimensional, we have  $\text{RM}([X]/\emptyset) = \text{RM}([Y]/\emptyset) = \text{RM}([E]/\emptyset) = 2$ , and the Morley rank of any two of  $[X], [Y], [E]$  is 4.

Now choose a  $\mathcal{D}$ -generic  $(x, y) \in X$ , and  $z$  so that  $(y, z) \in Y$ , and hence  $(x, z) \in E$ . We claim that  $\{[X], [Y], [E], x, y, z\}$  is a field configuration as in Definition 2.1. We have

$$\text{RM}([X], x, y/\emptyset) = \text{RM}([Y], y, z/\emptyset) = \text{RM}([E], x, z/\emptyset) = 3$$

and

$$\text{RM}(x/\emptyset) = \text{RM}(y/\emptyset) = \text{RM}(z/\emptyset) = 1.$$

Also, the Morley rank of the whole configuration over  $\emptyset$  is 5. It thus remains to verify (4) of Definition 2.1.

Because  $(x, y) \in X$  and  $\text{RM}([X]/\emptyset) = 2$ , we have  $\text{RM}(\text{Cb}(x, y/[X])) = 2$ . We similarly verify the other conditions and therefore  $\{[X], [Y], [E], x, y, z\}$  is indeed a field configuration. By Fact 2.3, an infinite field is interpretable in  $\mathcal{D}$ .

(2) Let  $\mathcal{L} := \mathcal{L}_1 \circ \mathcal{L}_2$ . Let  $\mathcal{F}$  be an  $\text{acl}_{\mathcal{D}}(\emptyset)$ -definable, generically strongly minimal, almost faithful and stationary family of plane curves, so that  $E$  is contained, up to finitely many points, in a  $\mathcal{D}$ -generic member of  $\mathcal{L}$  (such a family always exists). The family  $\mathcal{L}$

extends  $\mathcal{F}$  and by Lemma 3.16(2) is generically very ample. We need to show that so is  $\mathcal{F}$ .

Since  $\mathcal{F}$  is  $k$ -dimensional and almost faithful, for every  $\mathcal{D}$ -generic  $p \in G^2$  we have  $\text{RM}(\mathcal{F}(p)) = k - 1$ . It is thus sufficient to prove that for  $p, q$ , each  $\mathcal{D}$ -generic in  $G^2$ , we have  $\text{RM}(\mathcal{F}(p) \cap \mathcal{F}(q)) < k - 1$ .

We write  $\mathcal{L}_1 = \{C_t : t \in T\}$  and  $\mathcal{L}_2 = \{D_r : r \in R\}$ . By our assumption on  $E$ , it is a strongly minimal subset of  $C_t \circ D_r$  for some  $(t, r)$   $\mathcal{D}$ -generic in  $T \times R$ . That is,  $\text{RM}(t, r/\emptyset) = 4$ . It follows that  $\text{RM}(t, r/[E]) = 4 - k$  and thus for every  $\mathcal{D}$ -generic  $(t', r') \in T \times R$  there exists a strongly minimal  $E' \subseteq^* C_{t'} \circ D_{r'}$  in  $\mathcal{F}$  with  $\text{RM}(t', r'/[E']) = 4 - k$ . Here  $\subseteq^*$  means “contained up to finitely many points”.

Now let  $p, q$  be  $\mathcal{D}$ -generic in  $G^2$  and assume towards a contradiction that  $\text{RM}(\mathcal{F}(p) \cap \mathcal{F}(q)) = k - 1$ . Take  $E'$   $\mathcal{D}$ -generic in  $\mathcal{F}(p) \cap \mathcal{F}(q)$  over  $p, q$ . Consider the set

$$P = \{(t_1, r_1) \in T \times R : E' \subseteq^* C_{t_1} \circ D_{r_1}\}.$$

Since  $\text{RM}(t', r'/[E']) = 4 - k$  and  $(t', r') \in P$ , we have  $\text{RM}(P) \geq 4 - k$ . Fix  $(t_0, r_0)$   $\mathcal{D}$ -generic in  $P$  over  $[E']$ ,  $p$  and  $q$ . We have

$$\text{RM}(t_0, r_0, [E']/p, q) = \text{RM}(t_0, r_0/[E'], p, q) + \text{RM}([E']/p, q) \geq (4 - k) + (k - 1) = 3.$$

Finally, since  $[E'] \in \text{acl}_{\mathcal{D}}(t_0, r_0)$ , we have  $\text{RM}(t_0, r_0/p, q) \geq 3$  and in addition  $(t_0, r_0) \in (\mathcal{L}_1 \circ \mathcal{L}_2)(p) \cap (\mathcal{L}_1 \circ \mathcal{L}_2)(q)$  (because  $E' \subseteq C_{t_0} \circ D_{r_0}$ ). However, by Lemma 3.16(1, 2) we have  $\text{RM}(\mathcal{L}_1 \circ \mathcal{L}_2)(p) \cap (\mathcal{L}_1 \circ \mathcal{L}_2)(q) < 3$ , a contradiction. Thus  $\mathcal{F}$  is indeed generically very ample. ■

Under our standing assumptions at the end of Sections 3.1 and 3.3, we can finally deduce the last result of this section.

**Proposition 3.21.** *There exists a  $\mathcal{D}$ -definable almost faithful, stationary family of generically strongly minimal plane curves,  $\mathcal{F} = \{C_t : t \in T\}$ , which is generically very ample, and  $\text{RM}(T) \geq 3$ .*

*Proof.* Let  $\mathcal{L}$  be an almost faithful family of rank 2 as in Proposition 3.14 and consider the family  $\mathcal{L} \circ \mathcal{L}$ . Let  $C \in \mathcal{L} \circ \mathcal{L}$  be  $\mathcal{D}$ -generic. By Lemma 3.18, there exists a strongly minimal  $E \subseteq C$  with  $\text{RM}([E]) \geq 2$ . Either  $\text{RM}([E]) = 2$  and by Lemma 3.20(1) there is an infinite field interpretable in  $\mathcal{D}$ , in which case a family as required exists (take the family of graphs of polynomials of degree  $d > 1$  over  $K$ ); or  $\text{RM}([E]) > 2$ , in which case Lemma 3.20(2) gives a  $\mathcal{D}$ -definable family of curves as required. ■

*From now on, until the end of the paper, we fix a sufficiently saturated  $o$ -minimal expansion of a real closed field  $\mathcal{M} = \langle R; +, \cdot, <, \dots \rangle$ , and a 2-dimensional group  $G = \langle G; \oplus \rangle$  definable in  $\mathcal{M}$ . We also fix a strongly minimal non-locally-modular structure  $\mathcal{D} = \langle G; \oplus, \dots \rangle$  definable in  $\mathcal{M}$ . As discussed in Section 2.2, we include the index  $\mathcal{D}$  when referring to definability, genericity etc. in the structure  $\mathcal{D}$ , and omit the index when referring to  $\mathcal{M}$ . We also assume the existence of a  $\mathcal{D}$ -definable, very ample stationary family of plane curves  $\mathcal{L}$ , as noted after Proposition 3.14. In Sections 4–6, we denote  $\oplus$  by  $+$ , for simplicity.*



### 4. Frontiers of plane curves

#### 4.1. Strategy

Our goal is to show (Theorem 4.9) that if  $S \subseteq G^2$  is a plane curve, then its frontier  $\text{fr}(S)$  is finite and in fact contained in  $\text{acl}_{\mathcal{D}}([S])$ . The geometric idea originates in [26] and it is implemented in Lemma 4.7 below, as follows. We consider the family  $\mathcal{L}$  from the assumption following Proposition 3.14. We also fix  $b \in \text{fr}(S)$  and consider a curve  $l_q \in \mathcal{L}$  going through  $b$  with  $q$  generic over  $[S]$ . If  $l_q$  meets  $S$  transversely at every point of intersection and  $b$  is sufficiently generic in  $G^2$ , then by moving  $l_q$  to an appropriate  $l_{q'}$  close to  $l_q$ , the curve  $l_{q'}$  will intersect  $S$  near all points of  $l_q \cap S$ , and in addition at a new point near  $b$ . Since  $b$  itself was not in  $S$ , it follows that a generic  $l_q$  through  $b$  intersects  $S$  at fewer points than a generic curve in  $\mathcal{L}$ . Thus  $b$  is  $\mathcal{D}$ -algebraic over  $[S]$  and in particular  $\text{fr}(S)$  is finite.

While this strategy works well when the curves in  $\mathcal{L}$  are complex lines in  $\mathbb{C}^2$ , the problem becomes more difficult when they are arbitrary plane curves and  $b$  is not necessarily generic in  $G^2$ . To get around this problem, the idea in [5] was to replace  $S$  by its image under composition with a “generic enough” curve from a new “large” family  $\mathcal{L}'$  (Proposition 3.21). We carry out this replacement in Lemma 4.8 below. An additional complication of this strategy in the current setting is that instead of the functional language in [5] we need to work with arbitrary curves, and control their composition.

#### 4.2. Two technical lemmas about 2-dimensional sets in $G^2$

The following lemmas will be used in what follows.

**Lemma 4.1.** *Assume that  $\{Y_e : e \in E\}$  is a definable family of 2-dimensional subsets of  $G^2$  with  $\dim E = k \geq 2$ . Assume that for all  $e \in E$  there are at most finitely many  $e' \in E$  such that  $|Y_e \cap Y_{e'}| = \infty$ . Then  $\dim \bigcup_{e \in E} Y_e = 4$ .*

*Proof.* The set

$$\{(e, s) : e \in E, s \in Y_e\}$$

has dimension  $k + 2$ . Therefore, if the union of the  $Y_e$  had dimension smaller than 4, then for a generic  $s$  in this union, the dimension of  $E(s) = \{e \in E : s \in Y_e\}$  is at least  $k - 1 \geq 1$ , and in particular, is infinite. Hence, there are  $e_1, e_2 \in E(s)$ , independent and generic over  $s$ . Therefore,  $\dim(e_1, e_2/s) = 2k - 2$  and hence

$$\dim(e_1, e_2, s) = 2k - 2 + 3 = 2k + 1.$$

But this is impossible since  $\dim(e_1, e_2/\emptyset) \leq 2k$  and, by our assumption on the family, the set  $Y_{e_1} \cap Y_{e_2}$  is finite, so  $s \in \text{acl}(e_1, e_2)$ . ■

**Definition 4.2.** We say that two 2-dimensional sets  $C_1$  and  $C_2$  intersect transversely at  $p \in C_1 \cap C_2$  if  $C_1$  and  $C_2$  are both smooth at  $p$ , and their tangent spaces at  $p$  generate the full tangent space of  $G^2$  at  $p$ , namely  $T_p C_1 + T_p C_2 = T_p G^2$ .

**Lemma 4.3.** *Let  $\mathcal{L} = \{l_q : q \in Q\}$  be a  $\emptyset$ -definable family of 2-dimensional subsets of  $G^2$ , and  $S \subseteq G^2$  a  $\emptyset$ -definable 2-dimensional set. Let  $q$  be generic in  $Q$  over  $\emptyset$  and assume that  $l_q$  and  $S$  intersect transversely at  $s$ . Then for every neighborhood  $U \subseteq G^2$  of  $s$ , there is a neighborhood  $V \subseteq Q$  of  $q$  such that for every  $q' \in V$ , we have  $l_{q'} \cap S \cap U \neq \emptyset$ .*

*Proof.* Without loss of generality,  $U$  is definable over  $\emptyset$  and  $l_q \cap U$  is smooth (we can shrink it so that  $q$  is generic in  $Q$  over the parameters defining it). Reducing  $U$  further, if needed, we may – by cell decomposition, and the assumption that  $l_q$  is smooth at  $s$  – write  $l_q \cap U$  as the zero set of a definable  $C^1$ -map  $F_q : U \rightarrow R^2$ , and similarly write  $S$  as the zero set of a  $C^1$ -map  $G : U \rightarrow R^2$ . The transverse intersection of  $l_q$  and  $S$  implies that the joint map  $(F_q, G) : U \rightarrow R^4$  is a diffeomorphism at  $s$ , so in particular there is  $U_0 \subseteq U$  such that  $(F_q, G)$  is a diffeomorphism on  $U_0$  and  $\bar{0} \in R^4$  is in its open image. We may choose  $U_0$  so  $q$  is still generic over the parameters defining  $U_0$ . It follows that there is a neighborhood  $V \subseteq Q$  of  $q$  such that for every  $q' \in V$ ,  $l_{q'} \cap U = F_{q'}^{-1}(0)$  for some definable  $F_{q'} : U_0 \rightarrow R^2$ , and the map  $(F_{q'}, G)$  is still a diffeomorphism on  $U_0 \subseteq U$ , with  $\bar{0}$  in its image. But now, if  $(F_{q'}, G)(s') = \bar{0}$ , then  $s' \in U_0 \cap l_{q'} \cap S$ . ■

### 4.3. Bad points

Recall that  $\mathcal{L} = \{l_q : q \in Q\}$  is a faithful and generically very ample  $\mathcal{D}$ -definable family of strongly minimal plane curves, with  $\text{RM}(Q) = 2$ . Notice that for  $b \in G^2$  generic, the set  $Q(b) = \{q \in Q : b \in l_q\}$  has Morley rank 1. As in Section 3, we let  $\mathcal{L}(b) = \{l_q : q \in Q(b)\}$ .

**Definition 4.4.** Let  $U \subseteq G^2$  be an open set and  $b \in G^2$ . We say that  $\mathcal{L}(b)$  fibers  $U$  if for every  $s \in U$  there exists a unique  $q \in Q(b)$  such that  $s \in l_q$ , the set  $Q(b)$  is smooth at  $q$  and furthermore the function  $s \mapsto q : U \rightarrow Q(b)$  is a submersion at  $s$  (that is, the differential map between the tangent spaces is surjective).

**Definition 4.5.** For  $b \in G^2$ , we say that a point  $s = (s_1, s_2) \in G^2$  is  $b$ -good if:

- (1) There exists an open neighborhood  $U \subseteq G^2$  of  $s$  such that the family  $\mathcal{L}(b)$  fibers  $U$ .
- (2) For all  $q \in Q(b)$  such that  $s \in l_q$ , the curve  $l_q$  is smooth at  $s$ .

Otherwise, we say that  $s$  is a  $b$ -bad. We denote by  $\text{Bad}(b)$  the set of all  $b$ -bad points.

Clearly, the set  $\text{Bad}(b)$  is definable over  $b$ .

**Lemma 4.6.** *For every  $b \in G^2$ , the set  $\text{Bad}(b)$  has dimension at most 3.*

*Proof.* Note that since  $\mathcal{L}(b)$  is faithful, it follows that  $\text{RM}(G^2 \setminus \bigcup_{q \in Q(b)} l_q) \leq 1$ .

By cell decomposition, for a fixed generic  $q \in Q$ , the set of points  $s \in l_q$  failing (2) is at most 1-dimensional. So the set of all points  $s$  failing (2) is at most 3-dimensional.

We now fix  $s \in G^2$  generic over  $b$ , and show that it satisfies (1). The set of singular points  $q$  on  $Q(b)$  has dimension 1, and for every such  $q$ ,  $l_q$  has dimension 2. Thus, the union of all such  $l_q$  has dimension at most 3, and does not contain  $s$ . So if  $s \in l_q$  for some  $q \in Q(b)$ , then  $q$  is a smooth point on  $Q(b)$ .

Since  $s$  is generic in  $G^2$ , there are at most finitely many curves in  $\mathcal{L}(b)$  containing  $s$ . Hence, there is an open neighborhood  $W \subseteq Q(b)$  such that  $W \cap Q(b) \cap Q(s) = \{q\}$ .

We may choose  $W$  to be definable over generic parameters. Hence the first order property over  $b$ , “ $\varphi(s') := |W \cap Q(s') \cap Q(b)| = 1$ ”, must hold for all  $s'$  in a neighborhood  $U \subseteq G^2$  of  $s$ . Let  $g : U \rightarrow Q(b)$  be the map sending  $s'$  to the unique  $q' \in W \cap Q(b)$  with  $s' \in l_{q'}$ . Note that for every  $q' \in g(U)$ ,  $g^{-1}(q') = l_{q'} \cap U$ . Since the family  $\mathcal{L}(b)$  is faithful, we have  $\dim g(U) = 2 = \dim Q(b)$ , and by the genericity of  $s$  in  $\text{dom}(g)$ , the function  $g$  is a submersion at  $s$ , thus  $s$  is a  $b$ -good point. ■

4.4. Finiteness of the frontier

The heart of the geometric argument is contained in the following lemma showing that in a generic enough setting the frontier of  $S$  is indeed contained in  $\text{acl}_{\mathcal{D}}([S])$ .

**Lemma 4.7.** *Let  $\mathcal{F} = \{S_t : t \in T\}$  be a  $\mathcal{D}$ -definable stationary almost faithful family of plane curves with  $\text{RM}(T) \geq 3$ . Assume that  $b \in G^2$  with  $\dim(b/\emptyset) = 4$ , and  $t_0 \in T$  generic over  $\emptyset$ . If  $b \in \text{fr}(S_{t_0})$ , then  $b \in \text{acl}_{\mathcal{D}}(t_0)$ .*

*Proof.* We may assume first that  $S_{t_0}$  is strongly minimal. Indeed,  $S_{t_0}$  is a finite union of strongly minimal sets, each definable over  $\text{acl}_{\mathcal{D}}(t_0)$ , and  $b$  is in the frontier of one of them, so we may replace  $S_{t_0}$  by this strongly minimal set, and modify the family  $\mathcal{F}$  accordingly.

Denote  $S = S_{t_0}$  and  $B = \text{Bad}(b)$ .

**Claim 1.**  $\dim(S \cap B) \leq 1$ .

*Proof of Claim 1.* Since  $\dim(t_0/\emptyset) \geq 6$  and  $\dim(b/\emptyset) \leq 4$ , we obtain  $\dim(t_0/b) \geq 2$ . Assume towards a contradiction that  $\dim(S \cap B) = 2$ . Let

$$I = \{t \in T : \dim(S_t \cap B) = 2\}.$$

Notice that  $I$  is defined over  $b$  and  $t_0 \in I$ , so  $\dim I \geq 2$ . Because  $\mathcal{F}$  is almost faithful,  $\{S_t \cap B : t \in I\}$  is a definable family of 2-dimensional subsets of  $G^2$  satisfying the assumptions of Lemma 4.1. It follows that  $\dim \bigcup_{t \in T} (S_t \cap B) = 4$ . But  $\bigcup (S_t \cap B) \subseteq B$ , contradicting Lemma 4.6. ■

**Claim 2.** For every  $q' \in Q$ ,  $S \cap l_{q'}$  is finite.

*Proof of Claim 2.* If not, then by strong minimality of  $S$ , we would have  $S \sim l_{q'}$  for some  $q' \in Q$ , implying – since  $S = S_{t_0}$  and  $\mathcal{F}$  is almost faithful – that  $t_0 \in \text{acl}(q')$ . However, we have assumed that  $\dim(t_0/\emptyset) \geq 6$ , while  $\dim(q'/\emptyset) \leq 4$ , a contradiction. ■

We fix an element  $q \in Q(b)$  generic over  $t_0$  and  $b$ . Since  $\dim(b/\emptyset) = 4$ ,  $q$  is generic in  $Q$  over  $\emptyset$ , hence we have  $\dim(q/\emptyset) = 4$ .

Since  $\mathcal{L}$  is very ample, no two points in  $G^2$  belong to infinitely many curves in  $\mathcal{L}$ , and hence each  $s \in S \cap l_q$  is interalgebraic with  $q$  over  $t_0$  and  $b$ . Thus such an  $s$  is generic in  $S$  over  $t_0$  and  $b$ . So in particular  $S$  is smooth at  $s$ . It is not hard to see now (using the fact that  $\mathcal{F}$  is almost faithful) that  $\dim(s/b) = 4$ .

For the rest of this proof, we fix an element  $s \in S \cap l_q$ .

**Claim 3.** *The curve  $l_q$  is smooth at  $s$ , and  $S$  and  $l_q$  intersect transversely at  $s$ .*

*Proof of Claim 3.* Because  $\dim(s/b) = 4$ , it follows from Claim 1 that  $s$  is  $b$ -good, so in particular  $l_q$  is smooth at  $s$  and there exist neighborhoods  $U \subseteq G^2$  of  $s$  and  $W \subseteq Q(b)$  of  $q$ , and a  $\mathcal{D}$ -definable parameter choice function  $g_b : U \rightarrow W$  such that  $g(s')$  is the unique  $q' \in W$  with  $s' \in l_q \cap U$ . Restricting  $U, W$  if needed we may assume that  $l_q \cap U$  (which equals  $g_b^{-1}(q)$ ) is a  $C^1$ -submanifold of  $G^2$ . Thus, the tangent space to  $l_q$  at  $s$ ,  $T_s(l_q)$ , equals  $\ker(d_s(g_b))$ , where  $d_s(g_b)$  is the differential of  $g_b$  at  $s$  viewed as a linear map between the tangent spaces (see Definition 4.4). If the intersection is not transverse, then  $\dim(T_s(l_q) \cap T_s(S)) \geq 1$ . It follows that  $\dim(d_s(g_b)(T_s(S))) \leq 1$ , and by genericity of  $s$  in  $S$  over  $t_0, b$ , the same is true of any  $s' \in S$  in some open neighborhood  $U' \ni s$ . Thus, the image of  $g_b(S \cap U)$  is a 1-dimensional manifold (or finite), and it follows that for some  $q'$  in this image,  $l_{q'} \cap S$  is infinite. This contradicts Claim 2. ■

**Claim 4.** *For every neighborhood  $V \subseteq Q$  of  $q$ , there exists a neighborhood  $U \subseteq G^2$  of  $b$  such that for every  $b' \in U$  there are infinitely many  $q' \in V$  with  $b' \in l_{q'}$ .*

*Proof of Claim 4.* By assumptions,  $b \in l_q$  is generic in  $G^2$  over  $\emptyset$ . Thus, by shrinking  $V$  if needed, we may assume  $b$  is still generic in  $G^2$  over the parameters defining  $V$ . Since  $Q(b) \cap V$  is infinite, the first order statement

$$\varphi(b') := (\exists^\infty q' \in V)(b' \in l_{q'})$$

holds for  $b$  and therefore there is a neighborhood  $U \ni b$  for which it holds. ■

Let  $N$  be the number of intersection points of a curve from  $\mathcal{L}$ , generic over  $t_0$ , with  $S$  (recall that  $\text{MD}(\mathcal{L}) = 1$ , so  $\mathcal{L}$  has a unique generic type).

**Claim 5.** *The curve  $l_q$  intersects  $S$  in less than  $N$  points.*

*Proof of Claim 5.* We write  $l_q \cap S = \{s_1, \dots, s_n\}$  (note that  $b$  is not among them). We first fix some open disjoint neighborhoods  $U_1, \dots, U_n \subseteq G^2$ , of  $s_1, \dots, s_n$ , respectively. By Claim 3 and Lemma 4.3, applied to each of the  $s_i$ , there is a neighborhood  $V \subseteq Q$  of  $q$  such that for every  $q' \in V$ , the curve  $l_{q'}$  intersects  $S$  at least  $n$  times – at least once in each  $U_i, i = 1, \dots, n$ . Next, we apply Claim 4 to  $V$  and find  $U_0 \ni b$ , which we may assume is disjoint from all the  $U_i$ , as in Claim 4.

Because  $b$  is in  $\text{cl}(S) \setminus S$ , we can find in  $S \cap U_0$  some  $s'$ , an element  $\mathcal{D}$ -generic over  $t_0$ , and by Claim 4, we can find in  $V$  some  $q' \in Q(s')$  generic over  $s'$  and  $t_0$ . But now  $l_{q'}$  intersects  $S$  at least  $n + 1$  times: at  $s'$  and in each of  $U_1, \dots, U_n$ . Since  $S \cap l_{q'}$  is finite, the curve  $l_{q'}$  is generic in  $\mathcal{L}$  over  $t_0$ . So we have  $N \geq n + 1 > n = |l_q \cap S|$ . ■

Finally, let us see that  $b \in \text{acl}_{\mathcal{D}}(t_0)$ . The set  $Y$  of all  $q_1 \in Q$  such that  $|l_{q_1} \cap S| < N$  is  $\mathcal{D}$ -definable over  $t_0$  and has Morley rank at most 1. Since  $q$  is generic in  $Q(b)$  over  $t_0$  and  $b$ , it follows from Claim 5 that  $\text{RM}(Q(b) \cap Y) = 1$ . Also, there are at most finitely many  $b$ 's such that  $\text{RM}(Q(b) \cap Y) = 1$ , for otherwise there would be  $b_1 \neq b_2$  such that  $Q(b_1) \cap Q(b_2)$  is infinite, contradicting the very ampleness of  $\mathcal{L}$ . Thus  $b \in \text{acl}_{\mathcal{D}}(t_0)$ . This ends the proof of Lemma 4.7. ■

In our next step we show that the assumptions of Lemma 4.7 can be met for a  $\mathcal{D}$ -definable set  $S$  with  $\text{RM}(S) = 1$ , after replacing  $S$  by its composition with a generic enough curve in a family  $\mathcal{L}'$  as in Proposition 3.21.

**Lemma 4.8.** *Let  $S \subseteq G^2$  be a  $\mathcal{D}$ -definable strongly minimal set which is not a straight line, and assume that  $c$  is generic in  $G^2$  over  $\emptyset$  and belongs to  $\text{fr}(S)$ . Then there are:*

- (1) *An almost faithful stationary family of plane curves  $S' = \{S'_t : t \in T\}$ ,  $\mathcal{D}$ -definable over  $[S]$ , with  $\text{RM}(T) \geq 3$ .*
- (2)  *$t_0$  generic in  $T$  over  $c \cup \text{acl}_{\mathcal{D}}([S])$ .*
- (3)  *$b$  which is  $\mathcal{D}$ -interalgebraic with  $c$  over  $t_0 \cup [S]$ .*
- (4)  *$b \in \text{fr}(S'_{t_0})$  with  $\dim(b/\emptyset) = 4$ .*

*Proof.* Let  $\mathcal{L}' = \{C_t : t \in T\}$  be a  $\mathcal{D}$ -definable family of plane curves as in Proposition 3.21. Recall that, for every  $(a, b) \in G^2$ ,

$$T(a, b) := \{t \in T : (a, b) \in C_t\}.$$

If we write  $c = (c_1, c_2)$ , then by assumption,  $c_2$  is generic in  $G$  over  $\emptyset$ . Fix  $b_2 \in G$  generic over  $c_2 \cup \text{acl}_{\mathcal{D}}([S])$  (abusing notation, in the present proof we will write  $[S]$  for  $\text{acl}_{\mathcal{D}}([S])$ ), and let  $t_0$  be generic in  $T(c_2, b_2)$  over  $c_1, c_2, b_2$  and  $[S]$ . Note that  $(c_2, b_2) \in G^2$  is generic and  $C_{t_0}$  is generic through the point  $(c_2, b_2)$ . So  $\dim(t_0 c_2 b_2) = \dim T + 2$ , whereas  $\dim(t_0/c_2 b_2) = \dim T - 2$ . Since  $b_2 \in \text{acl}_{\mathcal{D}}(t_0 c_2)$ , we find that  $\dim(t_0/c_2) = \dim T$ . Because  $t_0$  was chosen generic over  $c_1, [S]$  too, we get  $\dim(t_0/c_1 c_2 [S]) = \dim T$ .

We set  $b := (c_1, b_2)$ . Since  $(c_2, b_2) \in C_{t_0}$  and  $\text{RM}(C_{t_0}) = 1$ ,  $b_2$  and  $c_2$  are interalgebraic in  $\mathcal{D}$  over  $t_0$  and  $[S]$ , and hence so are  $(c_1, b_2)$  and  $(c_1, c_2)$ .

**Claim.**  $b \in \text{fr}(C_{t_0} \circ S)$ .

*Proof of Claim.* Since  $c_2$  is generic in  $G$  over  $\emptyset$ ,  $(c_2, b_2)$  is generic in  $G^2$  over  $\emptyset$  and therefore, by our choice of  $t_0$ , the point  $(c_2, b_2)$  is also generic in  $C_{t_0}$  over  $t_0$ . Hence, the curve  $C_{t_0}$  is a homeomorphism at  $(c_2, b_2)$ . Denote this local map by  $f_0$ . It follows that the map  $(x, y) \mapsto (x, f_0(y))$  is a local homeomorphism on a neighborhood  $W$  of  $(c_1, c_2)$ , sending  $(c_1, c_2)$  to  $(c_1, b_2)$ . It is easy to verify that it sends every point in  $S \cap W$  to a point in  $C_{t_0} \circ S$ , and therefore sends every point in  $\text{cl}(S) \cap W$  to a point in  $\text{cl}(C_{t_0} \circ S)$ . We conclude that  $(c_1, b_2) \in \text{cl}(C_{t_0} \circ S)$ .

It remains to see that  $(c_1, b_2) \notin C_{t_0} \circ S$ . Let

$$S_{c_1} = \{y \in G : (c_1, y) \in S\} = \{d_1, \dots, d_k\}.$$

Note that since  $(c_1, c_2) \notin S$ , we have  $c_2 \notin S_{c_1}$ . Also,  $(c_1, b_2) \in C_{t_0} \circ S$  if and only if there is some  $i = 1, \dots, k$  for which  $(d_i, b_2) \in C_{t_0}$ .

Since  $\mathcal{L}'$  is very ample, for every  $i = 1, \dots, k$ ,  $\dim(T(c_2, b_2) \cap T(d_i, b_2)) < \dim T$ . But  $t_0$  is generic in  $T(c_2, b_2)$  over  $\{c_1, c_2, d_1, \dots, d_k, b_2, [S]\}$  and therefore  $t_0 \notin T(c_2, b_2) \cap T(d_i, b_2)$ . That is, none of the points  $(d_i, b_2)$  are in  $C_{t_0}$ . It follows that  $(c_1, b_2) \notin C_{t_0} \circ S$ , so we may conclude that  $b = (c_1, b_2) \in \text{fr}(C_{t_0} \circ S)$ . ■

Since  $S$  is not a straight line, we have  $\text{RM}(C_{t_0} \circ S) = 1$ , and hence there is a strongly minimal  $C \subseteq C_{t_0} \circ S$  such that  $b \in \text{fr}(C)$ . By Lemma 3.18,  $\text{RM}[C] = \text{RM}[C_{t_0}] = \text{RM}(T)$  and is therefore contained in an almost faithful family  $S'$  of the same rank. This gives condition (1) of the lemma; (2) is by the choice of  $t_0$ ; (3) is the line before the above claim, and (4) is what we have just shown. So the lemma is proved. ■

We can now conclude the main result of this section.

**Theorem 4.9.** *Let  $S \subseteq G^2$  be a  $\mathcal{D}$ -definable set with  $\text{RM}(S) = 1$ . Then  $\text{fr}(S) \subseteq \text{acl}_{\mathcal{D}}([S])$  and hence  $\text{fr}(S)$  is finite. In particular,  $S$  is locally closed, namely every  $p \in S$  has a neighborhood  $U \ni p$  in  $G^2$  such that  $S \cap U$  is closed in  $U$ .*

*Proof.* Since  $\text{RM}(S) = 1$ ,  $S$  can be written as  $\bigcup_{i=1}^k S_i$  for some strongly minimal sets  $\mathcal{D}$ -definable over  $\text{acl}_{\mathcal{D}}([S])$ . Since  $\text{fr}(\bigcup_{i=1}^k S_i) \subseteq \bigcup_{i=1}^k \text{fr}(S_i)$ , it suffices to prove the theorem for  $S$  strongly minimal. Moreover, if  $S$  is a straight line, then clearly its frontier is contained in finitely many points, which are in  $\text{acl}_{\mathcal{D}}([S])$ . So we may assume that  $S$  is strongly minimal not coinciding with any straight line.

Fix  $c \in \text{fr}(S)$ . Replacing  $S$  by  $S + p$  for  $p$  generic in  $G^2$  over  $c$  and  $[S]$ , we may assume that  $\dim(c/\emptyset) = 4$ . We can now apply Lemma 4.8 and obtain  $t_0, S'_{t_0}$  and  $b$  in  $\text{fr}(S'_{t_0})$  as in the lemma. Working first in a richer language where  $[S]$  is  $\emptyset$ -definable, we may apply Lemma 4.7, and then conclude that  $b \in \text{acl}_{\mathcal{D}}(t_0, [S])$ .

By Lemma 4.8,  $c$  is interalgebraic with  $b$  over  $t_0$  and  $[S]$ , hence  $c \in \text{acl}_{\mathcal{D}}(t_0, [S])$ . Since  $\dim(t_0/c, [S]) = \dim(t_0/c)$ , we find that  $c \in \text{acl}_{\mathcal{D}}([S])$ .

For  $p \in S$ , let  $U \ni p$  be any neighborhood such that  $U \cap \text{fr}(S) = \emptyset$ , and so  $S \cap U$  is closed in  $U$ . ■

#### 4.5. Two structural corollaries on plane curves

The first corollary will be used in the next subsection.

**Corollary 4.10.** *Let  $\mathcal{L}$  be a family of plane curves. Assume  $\mathcal{L}$  is  $\mathcal{D}$ -definable over  $\emptyset$ . Then there exists a family of plane curves  $\mathcal{L}'$ , also  $\mathcal{D}$ -definable over  $\emptyset$ , such that:*

- (1) Every curve in  $\mathcal{L}'$  is closed.
- (2) For every curve  $X_s \in \mathcal{L}$ , there exists a curve  $X'_s$ , defined over the same parameters, such that  $X_s \sim X'_s$ .
- (3) For every  $X'_s \in \mathcal{L}'$ , there exists  $X_s \in \mathcal{L}$ , defined over the same parameters, such that  $X'_s \sim X_s$ .

*Proof.* Let  $\chi(x, y)$  define  $\mathcal{L}$  and  $\psi(y) := (\exists^\infty x) \chi(x, y)$ . We prove the corollary by induction on  $(\text{RM}(\psi), \text{MD}(\psi))$ . For  $\text{RM}(\psi) = 0$ , the corollary is Theorem 4.9.

In the general case, fix  $s \models \psi(y)$  generic. By definition,  $[X_s] \in \text{dcl}_{\mathcal{D}}(s)$ . By Theorem 4.9, there is a finite set  $R_s$ ,  $\mathcal{D}$ -definable over  $[X_s]$ , so a fortiori also over  $s$ , such that  $\text{fr}(X_s) \subseteq R_s$ . Let  $\varphi(x, s)$  define  $R_s$ . By compactness, there is a formula  $\theta(y) \in \text{tp}(s)$  such that for all  $r \models \theta$  the formula  $\chi(x, r)$  is algebraic and, if not empty, its set of realisations contains  $\text{fr}(X_r)$ . So, for all  $r \models \theta$ , the formula  $\varphi(x, r) \vee \chi(x, r)$  defines a closed plane

curve  $\sim$ -equivalent to  $X_r$ . Because  $\theta(y) \in \text{tp}(s)$  and  $X_s$  was generic in  $\mathcal{L}$ , we see that  $\psi(y) \wedge \neg\theta(y)$  has smaller (RM, MD) (in the lexicographic order) than  $\psi(y)$ . So we are done by the induction hypothesis. ■

The second corollary below will be used several times in the rest of the paper.

**Definition 4.11.** Let  $S \subseteq G^2$  and  $a = (a_1, a_2) \in S$ . We say that  $S$  is *injective at  $a$  over  $a_1$*  if there is an open neighborhood  $U_1 \times U_2 \subseteq G \times G$  of  $a$  such that for every  $y \in U_2$  there exists at most one  $x \in U_1$  such that  $(x, y) \in S$ . Namely,  $S \cap (U_1 \times U_2)$  is the graph of a function from a subset of  $U_2$  into  $U_1$ . We say that  $a$  is an *injective point of  $S$*  if  $S$  is injective at  $a$  over  $a_1$  and  $S^{\text{op}}$  is injective at  $(a_2, a_1)$  over  $a_2$ . Otherwise, we say that  $a$  is a *non-injective point of  $S$* .

Let  $S \subseteq G^2$  and  $a_1 \in G$ . We say that  $S$  is *injective over  $a_1$*  if for every  $a = (a_1, a_2) \in S$ , the set  $S$  is injective at  $a$  over  $a_1$ .

Note that  $S$  is injective at every isolated point. Also, we cannot yet rule out the possibility that  $a$  is an injective point of  $S$  belonging to a 1-dimensional component of  $S$ .

**Corollary 4.12.** Let  $S \subseteq G^2$  be a  $\mathcal{D}$ -definable strongly minimal set. If  $S$  is not  $\sim$ -equivalent to any fiber  $G \times \{a\}$ , then the set of  $x \in G$  such that  $S$  is non-injective over  $x$  is finite and contained in  $\text{acl}_{\mathcal{D}}([S])$ . If  $S$  is not a straight line, then the set of non-injective points of  $S$  is finite and contained in  $\text{acl}_{\mathcal{D}}([S])$ .

*Proof.* By Theorem 4.9, we may assume that  $S$  is closed. Let

$$S_1 = \{(x_1, x_2) \in G^2 : x_1 \neq x_2 \ \& \ \exists y ((x_1, y) \in S \wedge (x_2, y) \in S)\} = (S^{\text{op}} \circ S) \setminus \Delta.$$

The set  $S_1$  is  $\mathcal{D}$ -definable over the same parameters as  $S$ . Since  $S$  is not  $\sim$ -equivalent to any fiber  $G \times \{a\}$ , we have  $\text{RM}(S_1) \leq 1$ . Note that  $(x, x) \notin \text{fr}(S_1)$  if and only if there exists an open  $U \ni x$  such that for all  $y \in G$  there exists at most one  $x' \in U$  such that  $(x', y) \in S$ . It follows that  $(x, x) \notin \text{fr}(S_1)$  if and only if  $S$  is injective over  $x$ . By Theorem 4.9,  $\text{fr}(S_1) \subseteq \text{acl}_{\mathcal{D}}([S])$  thus the set of  $x \in G$  such that  $S$  is non-injective over  $x$  is finite and contained in  $\text{acl}_{\mathcal{D}}([S])$ .

The second clause follows immediately by applying the first one also to  $S^{\text{op}}$ . ■

#### 4.6. On $\mathcal{D}$ -functions

Every plane curve  $S \subseteq G^2$  gives rise to a definable partial function from  $G$  into  $G$  around almost every point in  $S$  (except when  $S$  is contained in finitely many straight lines  $\{a\} \times G$ ). The goal of this subsection is to establish the basic theory of such functions.

**Definition 4.13.** Let  $U \subseteq G$  be a definable open set and  $f : U \rightarrow G$  be a definable continuous function.

- (1) We say that  $f$  is a  *$\mathcal{D}$ -function* if there exists a plane curve  $S \subseteq G^2$  such that  $\Gamma_f \subseteq S$ . We say in this case that  $S$  *represents  $f$* .
- (2) We say that  $f$  is  *$\mathcal{D}$ -represented over  $A$*  if there exists  $S$  representing  $f$  which is  $\mathcal{D}$ -definable over  $A$ .

- (3) We say that a plane curve  $S$  represents the germ of  $f$  at  $x_0 \in U$  if there exists an open neighborhood  $W \ni x_0$  with  $W \subseteq \text{dom}(f)$  such that  $\Gamma_{f|_W} \subseteq S$ .

Note that our definition does not require that  $S$  is, locally at  $(x_0, f(x_0))$ , the graph of a function, but only that it contains the graph of  $f$ . Indeed, at least for some of the  $\mathcal{D}$ -functions we need to consider, we do not know whether this stronger property can be achieved as well.

**Lemma 4.14.** *Let  $U \subseteq G$  be a definably connected open set and  $f : U \rightarrow G$  a continuous  $\mathcal{D}$ -function,  $\mathcal{D}$ -represented over  $A$ . Then  $f$  can be  $\mathcal{D}$ -represented over  $\text{acl}_{\mathcal{D}}(A)$  by a strongly minimal set.*

*Proof.* Assume that  $f : U \rightarrow G$  is  $\mathcal{D}$ -represented over  $A$  by  $S$ . We let  $S = S_1 \cup \dots \cup S_r$  be a decomposition of  $S$  into strongly minimal sets, definable in  $\mathcal{D}$  over  $\text{acl}(A)$ . By Theorem 4.9, we may assume, by adding finitely many points in  $\text{acl}_{\mathcal{D}}(A)$ , that each  $S_i$  is closed in  $G^2$ , but now the intersection  $S_i \cap S_j$  for  $i \neq j$  may be non-empty and finite. We claim that one of the  $S_i$  must contain  $\Gamma_f$ . Indeed, for each  $i = 1, \dots, r$ , let  $C_i = \pi(S_i \cap \Gamma_f) \subseteq U$ , where  $\pi : G^2 \rightarrow G$  is the projection on the first coordinate. By the continuity of  $f$ , these are definable, relatively closed subsets of  $U$ , whose pairwise intersection is at most finite.

Let  $U' := U \setminus \bigcup_{i \neq j} (C_i \cap C_j)$ . Because  $U$  is open and definably connected, so is  $U'$ . For  $i = 1, \dots, r$  let  $C'_i = C_i \cap U'$ . The  $C'_i$ 's are pairwise disjoint and still relatively closed in  $U'$ . Hence each  $C'_i$  is clopen (having a closed complement) in  $U'$ , so for some  $j$ ,  $C'_j = U'$ . Because  $C_j$  is closed in  $U$  it follows that  $C_j = U$ . ■

**Proposition 4.15.** *Let  $\{S_t : t \in T\}$  be a family of plane curves. Assume that this family is  $\mathcal{D}$ -definable over  $A$ , and that for every  $t \in T$ ,  $(0, 0) \in S_t$ . Then there exists a family  $\mathcal{F} = \{f_s : s \in T_0\}$ , definable (in  $\mathcal{M}$ ) over  $A$ , of functions in  $\mathfrak{F}$  (defined in Section 1.3), such that:*

- (1) *For every  $t \in T$ , if  $S_t$  represents the germ at 0 of a  $\mathcal{D}$ -function  $f \in \mathfrak{F}$ , then there exists  $s \in T_0$  and an open  $W \ni 0$  such that  $f|_W = f_s|_W$ .*
- (2) *For every  $s \in T_0$  there exists  $t \in T$  such that  $S_t$  represents the germ at 0 of  $f_s$ .*

*Proof.* By Corollary 4.10, there exists a  $\mathcal{D}$ -definable family  $\mathcal{L}'$  of closed plane curves such that each curve in  $\mathcal{L}$  is  $\sim$ -equivalent to one in  $\mathcal{L}'$  and vice versa. Note that whenever  $S_t$  represents the germ of a  $\mathcal{D}$ -function  $f_t$  at 0, if  $S'_t \in \mathcal{L}'$  is  $\sim$ -equivalent to  $S_t$ , then it also represents the germ of  $f$  at 0. Thus, we may replace  $\mathcal{L}$  with  $\mathcal{L}'$  and assume that every curve  $S_t$  is closed.

By fixing a coordinate system near 0, we can identify some neighborhood  $W \ni 0$  in  $G$  with an open subset of  $R^2$ . For each  $r > 0$ , we consider the disc  $B_r$  centered at 0, and let  $S'_t = S_t \cap (B_r \times W)$ . By o-minimality, there exists a uniform cell decomposition of the sets  $\{S'_t : t \in T, r > 0\}$ . In particular, there is a bound  $k \in \mathbb{N}$  such that every such decomposition contains at most  $k$  cells. By allowing cells to be empty, we obtain a definable collection of cells  $\{C^r_{t,i} : t \in T, r > 0, i = 1, \dots, k\}$  such that for every  $t \in T$



and  $r > 0$ ,

$$S_t^r = \bigcup_{i=1}^k C_{t,i}^r.$$

Recall that the notion of a decomposition implies that for  $C_{t,i}^r, C_{t,j}^r$ , if  $\pi : G^2 \rightarrow G$  is the projection onto the first coordinate, then either  $\pi(C_{t,i}^r) = \pi(C_{t,j}^r)$  or  $\pi(C_{t,i}^r) \cap \pi(C_{t,j}^r) = \emptyset$ .

**Claim.** For every  $t \in T$ , and a  $\mathcal{D}$ -function  $f \in \mathfrak{F}$ , the following are equivalent:

- (1)  $S_t$  represents the germ of  $f$  at 0.
- (2) There exist  $r > 0$  and  $A \subseteq \{1, \dots, k\}$  such that

$$\Gamma_{f|_{B_r}} = \bigcup_{i \in A} C_{t,i}^r.$$

*Proof of Claim.* (1) $\Rightarrow$ (2). We assume that  $S_t \cap (B_r \times G)$  contains the graph of  $f|_{B_r}$ , for all  $r > 0$ . To simplify notation we omit  $r$  and consider the cell decomposition  $S_t = C_{t,1} \cup \dots \cup C_{t,k}$ .

We let  $A \subseteq \{1, \dots, k\}$  be all  $i$  such that  $C_i \cap \Gamma_f \neq \emptyset$ . We fix a cell  $C = C_i$  with  $i \in A$  and claim that  $C \subseteq \Gamma_f$ . Without loss of generality  $\dim C > 0$ , and since  $C \subseteq \Gamma_f$ , the projection  $\pi : C \rightarrow G$  is injective. Since  $C$  is definably connected, it is sufficient to prove that  $C \cap \Gamma_f$  is clopen inside  $C$ . Because  $C$  is locally closed and  $f$  is continuous, it follows that  $C \cap \Gamma_f$  is closed in  $C$ , so we need to prove that it is also open in  $C$ .

Fix some  $x_0 \in \pi(C \cap \Gamma_f)$ . Since  $\Gamma_f \subseteq S_t$ , there exists a cell  $C'$  in the decomposition of  $S_t$  containing  $\Gamma_f \cap [(U \cap \pi(C)) \times G]$  for some open set  $U \ni x_0$ . But then  $\pi(C) \cap \pi(C') \neq \emptyset$  and therefore  $\pi(C) = \pi(C')$ . By the continuity of  $f$ , it follows that  $(x_0, f(x_0)) \in C'$ , forcing  $C' = C$ . It follows that  $C \cap \Gamma_f$  is clopen in  $C$ , and therefore  $C \subseteq \Gamma_f$ .

We have shown that  $C_i \subseteq \Gamma_f$  for each  $i \in A$ , and hence  $\Gamma_f = \bigcup_{i \in A} C_i$ .

(2) $\Rightarrow$ (1). This is immediate, since  $\Gamma_{f|_{B_r}} \subseteq S_t$ . ■

We now return to the proof of Proposition 4.15 and consider the uniform decomposition

$$S_t^r = \bigcup_{i=1}^k C_{t,i}^r.$$

For each  $A \subseteq \{1, \dots, k\}$ , we consider

$$G_{t,A}^r = \bigcup_{i \in A} C_{t,i}^r.$$

The family

$$\mathfrak{F} = \{G_{t,A}^r : G_{t,A}^r \text{ is the graph of a continuous function on } B_r\}$$

is definable in  $\mathcal{M}$ , as  $t$  varies in  $T$ ,  $A$  varies among subsets of  $\{1, \dots, k\}$  and  $r > 0$ . By the above claim, this family satisfies our requirements. ■

- Remark 4.16.** (1) Note that in the above family  $\mathcal{F}$  of  $\mathcal{D}$ -functions, each germ of a function appears infinitely often since we allow arbitrarily small  $r$ . One can divide the family, definably in  $\mathcal{M}$ , by the equivalence of germs at 0 and then, using definable choice in o-minimal structures, obtain a unique  $\mathcal{D}$ -function in the family representing each germ. Thus, if  $f \in \mathcal{F}$  is represented by the plane curve  $S_t$ , then there exists  $g \in \mathcal{F}$  which has the same germ as  $f$  at 0 and is definable in  $\mathcal{M}$  over  $t$ .
- (2) It follows from the above that if  $S_t$  represents  $f \in \mathcal{F}$ , then  $J_0(f)$  is in  $\text{dcl}(t)$  (recall from Section 1.3 that  $J_0 f$  is the Jacobian of  $f$  at 0 with respect to some fixed differential structure on  $G$ ).

**Notation.** For a  $\mathcal{D}$ -function  $f$ , we reserve the notation  $S_f$  for a strongly minimal set representing  $f$ . Note that  $S_f$  is unique only up to  $\sim$ -equivalence.

We conclude this section with an open mapping theorem for  $\mathcal{D}$ -functions.

**Theorem 4.17.** *Let  $U \subseteq G$  be an open definably connected set and  $f : U \rightarrow G$  a continuous non-constant  $\mathcal{D}$ -function. Then  $f$  is an open map.*

*Proof.* By Lemma 4.14, there exists a  $\mathcal{D}$ -definable strongly minimal  $S_f \subseteq G^2$  representing  $f$ . Because  $f$  is not constant, the projection of  $S_f$  onto both coordinates is finite-to-one, so it is not a straight line. By Corollary 4.12,  $S_f$  is injective at co-finitely many points, and therefore so is also  $f$ . By the o-minimal version of Brouwer’s invariance of domain [14], it follows that  $f$  is open at every injective point of its domain. So  $f$  is open after possibly removing finitely many points from its domain. It is easy to check (see, for example, the proof of [5, Proposition 4.7] for details) that a function which is continuous on a disc and open on the punctured disc is open on the whole disc. So  $f$  is open.

### 5. Poles of plane curves

Recall that we assume that  $G$  is a definable closed subset of some  $R^n$ , equipped with the subspace topology, making it a topological group.

The goal of this section is to prove that just like affine algebraic curves in  $\mathbb{C}^2$ , every plane curve has at most finitely many poles. We may assume that  $0_G = 0 \in R^n$ . For  $x \in G$  and  $\epsilon > 0$  in  $R$ , we write

$$B(x; \epsilon) = \{g \in G : |x - g| < \epsilon\},$$

and  $B_\epsilon$  for  $B(0; \epsilon)$ . For  $A \subseteq G$  and  $\epsilon > 0$ , we let

$$B(A; \epsilon) = \{y \in G : \exists x \in A \ y \in B(x; \epsilon)\}.$$

In this section, we will also consider definable curves, that is, definable maps  $\gamma : (0, 1) \rightarrow U \subseteq R^n$ , which we will denote, for simplicity, by  $\gamma(t) \in U$ . We recall from [38, §6.1] that if  $x \in \text{cl}(X)$  for some definable  $X \subseteq R^n$ , then by curve selection for  $\mathcal{M}$ , there is a definable path  $\gamma(t) \in X$  with  $\lim_{t \rightarrow 0} \gamma(t) = x$ .

**Definition 5.1.** Let  $S \subseteq G^2$  be a definable set. We call  $a \in G$  a *pole* of  $S$  if for every open  $U \subseteq R^n$  containing  $a$ , the set  $(U \times G) \cap S$  is an unbounded subset of  $R^n$ . We denote the set of poles of  $S$  by  $S_{\text{pol}}$ .

Given  $S \subseteq G^2$  and  $U \subseteq G$ , we define

$$S(U) := \{y \in G : \exists x \in U (x, y) \in S\} \subseteq G.$$

Note that then  $a \in S_{\text{pol}}$  if and only if for every open  $U \subseteq G$  containing  $a$ ,  $S(U)$  is unbounded. Another remark is that if  $S$  is  $G$ -affine, then  $S_{\text{pol}} = \emptyset$ . Indeed, if  $S$  is a subgroup of  $G^2$  or its coset, then its projection onto the first coordinate is a finite-to-one topological covering map, and hence  $S$  has no poles.

The main result of this section is the following.

**Theorem 5.2.** *If  $S \subseteq G^2$  is a  $\mathcal{D}$ -definable set and  $\text{RM}(S) = 1$ , then  $S_{\text{pol}}$  is finite.*

Notice that if  $G$  is a definably compact group (for example, a complex elliptic curve), then  $G$  is a closed and bounded subset of  $R^n$ , and hence  $S_{\text{pol}} = \emptyset$ . So the theorem is of interest for those  $G$  which are not definably compact.

Let us first introduce the key notion of ‘‘approximated points’’ and then discuss the strategy of our proof. Recall that for  $S \subseteq G^2$  and  $x \in G$ , we let  $S_x = \{y \in G : (x, y) \in S\}$ .

**Definition 5.3.** Let  $S \subseteq G^2$ ,  $b, x_1, x_2 \in G$ , and  $I \subseteq G$ . We say that:

- (1) •  $b$  is  $S$ -attained at  $(x_1, x_2)$  if  $b \in S_{x_1} - S_{x_2}$ .
- $b$  is  $S$ -attained in  $I$  if it is  $S$ -attained at some  $(x_1, x_2) \in I$ .
- (2) •  $b$  is  $S$ -attained near  $(x_1, x_2)$  if for every  $\epsilon > 0$ ,  $b$  is  $S$ -attained in  $B((x_1, x_2); \epsilon)$ .
- $b$  is  $S$ -attained near  $I$  if for every  $\epsilon > 0$ ,  $b$  is  $S$ -attained in  $B(I; \epsilon)$ .
- (3) •  $b$  is  $S$ -approximated near  $(x_1, x_2)$  if for every  $\epsilon > 0$ , some  $b' \in B(b; \epsilon)$  is  $S$ -attained at  $(x_1, x_2)$ .
- $b$  is  $S$ -approximated near  $I$  if for every  $\epsilon > 0$ , there are  $x_1, x_2 \in B(I; \epsilon)$  such that  $B(b; \epsilon) \cap (S_{x_1} - S_{x_2}) \neq \emptyset$ . The set of such points  $b$  is denoted by  $A(S, I)$ .

We omit  $S$  from the above notation whenever it is clear from the context.

The following claim is immediate from the definitions.

**Claim 5.4.** *For any  $S \subseteq G^2$  and  $I \subseteq G$ ,*

$$b \text{ attained at } I \implies b \text{ is attained near } I \implies b \text{ is approximated near } I.$$

*If, in addition,  $S$  and  $I$  are closed and bounded, then the above notions are equivalent and  $A(S, I) = S(I) - S(I)$ .*

Here is a simple example.

**Example 5.5.** Let  $G = (\mathbb{C}, +)$  and consider the complex algebraic curve

$$S = \{(z, w) \in \mathbb{C}^2 : zw = 1\}.$$

The following are easy to verify:  $S_{\text{pol}} = \{0\}$ , every  $b \in \mathbb{C}$  is attained near 0, and thus  $A(S, \{0\}) = \mathbb{C}$ .

The strategy of the proof of Theorem 5.2 is as follows. Assume towards a contradiction that the theorem fails. It is easy to see that we may assume that  $S$  is closed, strongly minimal and not  $G$ -affine. Now, for any such  $\mathcal{D}$ -definable set  $S$  and infinite definable  $I \subseteq G$ , we first find an infinite definable set  $I_0 \subseteq I$  and an open bounded ball  $B \subseteq R^n$  such that the set  $A(S, I_0) \setminus B$  is at most 1-dimensional (Proposition 5.6(1)). Then, using further the fact that  $S_{\text{pol}}$  is infinite, we construct (Proposition 5.10) another  $\mathcal{D}$ -definable set  $\hat{S}$ , again closed, strongly minimal and not  $G$ -affine, and an infinite definable  $\hat{I} \subseteq G$ , such that for every infinite definable set  $T \subseteq \hat{I}$  and open bounded ball  $B$ , the set  $A(\hat{S}, T) \setminus B$  is 2-dimensional. This gives the desired contradiction.

### 5.1. An upper bound on the dimension of the set of approximated points

The goal of this subsection is to prove the following proposition.

**Proposition 5.6.** *Assume that  $S \subseteq G^2$  is a  $\mathcal{D}$ -definable strongly minimal closed set which is not  $G$ -affine, and let  $I \subseteq G$  be an infinite definable set. Then there is a definable 1-dimensional  $I_0 \subseteq I$  such that:*

- (1) *There exists a bounded  $B \subseteq G$  such that the set  $A(S, I_0) \setminus B$  is at most 1-dimensional.*
- (2) *For every definable open  $V \ni 0$  in  $G$  there exist  $\epsilon > 0$  and a bounded ball  $B' \ni 0$  such that for all  $x \in G \setminus B'$ ,*

$$x + V \not\subseteq S(B(I_0, \epsilon)).$$

The rest of this subsection is devoted to the proof of Proposition 5.6. Throughout, we fix  $S$  as in its assumptions. Since  $\dim S = 2 \text{RM}(S) = 2$ , it follows easily from cell decomposition for  $\mathcal{M}$  that  $\dim S_{\text{pol}} \leq 1$ . Absorbing  $[S]$  into the language, we assume that  $S$  is  $\mathcal{D}$ -definable over  $\emptyset$ .

We begin with an observation regarding the notions of Definition 5.3.

**Lemma 5.7.** *Let  $I \subseteq G$  be a definable bounded set over  $\emptyset$ .*

- (1) *If  $b \in G$  is attained near  $I$ , then there are  $x_1, x_2 \in \text{cl}(I)$  such that  $b$  is attained near  $(x_1, x_2)$ .*
- (2) *If  $b \in G$  is generic over  $\emptyset$  and  $b$  is approximated near  $I$ , then  $b$  is attained near  $I$ .*

*Proof.* (1) By assumption, and by curve selection in  $\mathcal{M}$ , there are definable curves  $x_1(\epsilon), x_2(\epsilon), y_1(\epsilon), y_2(\epsilon) \in G$  such that for every  $\epsilon$ , we have  $x_1(\epsilon), x_2(\epsilon) \in B(I; \epsilon)$ ,  $(x_i(\epsilon), y_i(\epsilon)) \in S, i = 1, 2$ , and  $b = y_1(\epsilon) - y_2(\epsilon)$ . Since  $I$  is bounded, the curves  $x_i(\epsilon)$  have limits  $x_1, x_2 \in \text{cl}(I)$ , so  $b$  is attained near  $(x_1, x_2)$ .

(2) Fix  $b$  generic in  $G$  over  $\emptyset$ , and assume that it is approximated near  $I$ . It follows from the definition that for every  $\epsilon > 0$ , the element  $b$  is in the closure of

$$Y_\epsilon = \{y_2 - y_1 : \exists x_1, x_2 \in B(I; \epsilon) (x_1, y_1), (x_2, y_2) \in S\}.$$

Notice that the collection of  $Y_\epsilon$  forms a definable chain of definable sets decreasing with  $\epsilon$ . We may now take  $\epsilon$  sufficiently small, so that  $b$  is still generic in  $G$  over  $\epsilon$ , and therefore  $b$  is generic in  $\text{cl}(Y_\epsilon)$  over  $\epsilon$ . Hence,  $b \notin \text{fr}(Y_\epsilon)$ , a set of dimension at most 1. It follows that  $b \in Y_\epsilon$  for all sufficiently small  $\epsilon$ , and so  $b$  is attained near  $I$ . ■

The following technical claim about definable and  $\mathcal{D}$ -definable sets will be used in the subsequent lemma.

**Claim 5.8.** *Let  $P \subseteq G^2 \times G$  be a  $\mathcal{D}$ -definable set of Morley rank 1 whose projection on the  $G^2$ -coordinate is finite-to-one. Then for any definable sets  $I, J \subseteq G$  of dimension at most 1,*

$$\dim(P \cap (I \times J \times G)) \leq 1.$$

*Proof.* Since the projection  $\pi : P \rightarrow G^2$  is finite-to-one,

$$\dim(P \cap (I \times J \times G)) = \dim(\pi(P \cap (I \times J \times G))) \leq \dim(I \times J),$$

so if one of  $I$  and  $J$  is finite, then  $\dim \pi(P) \leq 1$  and we are done.

Suppose now that  $\dim I = \dim J = 1$ . Since  $P$  is  $\mathcal{D}$ -definable and infinite, the projection of  $P$  on one of the coordinates of  $G^2$  has infinite image. Let us assume it is the projection on the first coordinate. Hence, since  $\text{RM}(P) = 1$ , for every  $\mathcal{D}$ -generic  $a \in G$ , the set  $\{(w, z) \in G \times G : (a, w, z) \in P\}$  is finite. Since  $I \subseteq G$  is infinite, every generic of  $I$  is also  $\mathcal{D}$ -generic in  $G$ . But then, for such an  $a \in I$  the set  $\{(w, z) \in J \times G : (a, w, z) \in P\}$  is finite. It follows that  $\dim(P \cap (I \times J \times G)) = \dim I = 1$ . ■

We proceed with a lemma towards the proof of Proposition 5.6.

**Lemma 5.9.** *There exists a finite set  $F \subseteq G$  with  $F \subseteq \text{acl}_{\mathcal{D}}(\emptyset)$  and a definable set  $X \subseteq G$  with  $\dim X \leq 1$  such that for every  $b \in G \setminus X$  and for every  $(x_1, x_2) \in G^2 \setminus F^2$ , if  $b$  is attained near  $(x_1, x_2)$ , then  $b$  is attained at  $(x_1, x_2)$ .*

*Proof.* Consider the  $\mathcal{D}$ -definable set

$$T = \{(x_1, x_2, b) \in G^3 : b \in S_{x_1} - S_{x_2}\}.$$

Since every generic fiber  $S_x$  is finite,  $\text{RM}(T) = 2$ . Also, by fixing  $x_1$  and letting  $x_2$  vary, it is easy to see that the projection of  $T$  on the last coordinate is infinite and hence for every  $\mathcal{D}$ -generic  $b \in G$  the set

$$T^b = \{(x_1, x_2) \in G^2 : (x_1, x_2, b) \in T\}$$

has Morley rank 1. Note that  $(x_1, x_2) \in T^b$  if and only if  $b$  is attained at  $(x_1, x_2)$ , and  $(x_1, x_2) \in \text{cl}(T^b)$  if and only if  $b$  is attained near  $(x_1, x_2)$ . We also note (although we will not use this) that  $(x_1, x_2, b) \in \text{cl}(T)$  if and only if  $b$  is approximated near  $(x_1, x_2)$ .

**Claim 1.** *For  $b \in G$  and  $x_1, x_2 \in G$ , the following are equivalent:*

- (1)  $b$  is attained near  $(x_1, x_2)$  but not attained at  $(x_1, x_2)$ .

- (2)  $(x_1, x_2) \in \text{fr}(T^b)$  and  $x_1, x_2 \in S_{\text{pol}}$ .
- (3)  $(x_1, x_2) \in \text{fr}(T^b)$ .

*Proof of Claim 1.* (1) $\Rightarrow$ (2). The fact  $(x_1, x_2) \in \text{fr}(T^b)$  is immediate from the notes just above the claim. Since  $b$  is attained near  $(x_1, x_2)$ , by curve selection in  $\mathcal{M}$ , we can find definable curves  $(x_1(t), y_1(t)) \in S$  and  $(x_2(t), y_2(t)) \in S$  such that  $x_i(t) \rightarrow x_i$  for  $i = 1, 2$ , and  $y_1(t) - y_2(t) = b$ . Notice that  $y_1(t)$  is bounded if and only if  $y_2(t)$  is bounded, in which case, since  $S$  is closed, their limit points  $y_1, y_2$  satisfy  $(x_1, y_1), (x_2, y_2) \in S$  and  $y_2 - y_1 = b$ , so  $b$  is attained at  $(x_1, x_2)$ . Because we have assumed that this is not the case,  $y_1(t)$  and  $y_2(t)$  are unbounded, hence  $x_1, x_2$  are both in  $S_{\text{pol}}$ .

The other implications are easy, thus ending the proof of Claim 1. ■

By Theorem 4.9, for each  $b \in G$ ,  $\text{fr}(T^b) \subseteq \text{acl}_{\mathcal{D}}(b)$  (recall that  $[S]$  was absorbed into the language). By compactness, we may therefore find a set  $P \subseteq G^2 \times G$ ,  $\mathcal{D}$ -definable over  $\emptyset$ , such that for every  $b \in G$  the set  $P^b$  is finite and contains  $\text{fr}(T^b)$ . It follows that  $\text{RM}(P) = 1$ . Note however that we do not claim that  $P^b = \text{fr}(T^b)$  for every  $b \in G$ . Thus, for example, we allow at this stage the possibility that the set of  $b$  for which  $T^b$  is not closed is 1-dimensional.

Now, by Claim 1, if  $b$  is attained near  $(x_1, x_2)$  but not at  $(x_1, x_2)$  then  $(x_1, x_2) \in P^b$ .

Assume first that the image of  $P$  under the projection onto the  $G^2$ -coordinates, call it  $F_1$ , is finite, and let  $F \subseteq G$  be a finite set,  $\mathcal{D}$ -definable over  $\text{acl}_{\mathcal{D}}(\emptyset)$ , such that  $F_1 \subseteq F^2$ . We may take  $X = \emptyset$  and complete the proof of the lemma in this case. Assume then that  $F_1$  is infinite.

Let  $F_0 \subseteq G^2$  be the set of all  $p \in G^2$  such that  $P_p \subseteq G$  is infinite. This is a finite set,  $\mathcal{D}$ -definable over  $\text{acl}_{\mathcal{D}}(\emptyset)$ , and because we have assumed that  $F_1$  is infinite, the set  $P^* := (G^2 \setminus F_0) \times G$  still has Morley rank 1, and the projection map from  $P^*$  onto the  $G^2$ -coordinate is finite-to-one.

Set

$$X = \{b \in G : \text{fr}(T^b) \setminus F_0 \neq \emptyset\},$$

a definable set in  $\mathcal{M}$ .

**Claim 2.**  $\dim X \leq 1$ .

*Proof of Claim 2.* Assume towards a contradiction that  $\dim X = 2$ . For every  $b \in X$  there exists  $(x_1, x_2) \in \text{fr}(T^b) \setminus F_0$ . By Claim 1 and our choice of  $P$ ,  $(x_1, x_2) \in (P^*)^b \cap (S_{\text{pol}} \times S_{\text{pol}})$ , so since  $\dim X = 2$ , it follows that

$$\dim(P^* \cap (S_{\text{pol}} \times S_{\text{pol}} \times X)) \geq 2.$$

This contradicts Claim 5.8. ■

By Claim 1, for every  $b \in G$  and every  $(x_1, x_2) \in G^2$ , if  $b$  is attained near  $(x_1, x_2)$  but not at  $(x_1, x_2)$ , then  $(x_1, x_2) \in \text{fr}(T^b) \subseteq P^b$ . Now, either  $(x_1, x_2) \in F_0$ , or  $b \in X$ . Thus, we may take any finite set  $F \subseteq \text{acl}_{\mathcal{D}}(\emptyset)$  with  $F_0 \subseteq F^2$  to complete the proof of Lemma 5.9. ■

We now fix a definable 1-dimensional set  $I \subseteq G$ . Fix also a finite set  $F \subseteq G$  as in Lemma 5.9, and a definable 1-dimensional closed set  $I_0 \subseteq I$  such that  $I_0 \cap F = \emptyset$ .

*Proof of Proposition 5.6(1).* Because  $S \cap (I_0 \times G)$  is a 1-dimensional subset of  $G \times G$ , we may shrink  $I_0$  further and assume that the set  $S \cap (I_0 \times G)$  is closed and bounded. Thus, the set

$$B = \{b \in G : b \text{ is attained at } I_0\} = S(I_0) - S(I_0)$$

is a closed and bounded subset of  $G$ . By Lemma 5.9 and the choice of  $I_0$ , there is a definable  $X \subseteq G$  with  $\dim X \leq 1$  such that for every  $b \in G \setminus X$ , if  $b$  is attained near  $(x_1, x_2) \in I_0^2$ , then it is attained at  $(x_1, x_2)$ . Assume towards a contradiction that the set  $A(S, I_0) \setminus B$  has dimension 2. By Lemma 5.7(2), the set  $L$  of all  $b \in G \setminus B$  which are attained near  $I_0$  has dimension 2, and therefore there is some  $b \in L$  which is not in  $X$ . By Lemma 5.7(1),  $b$  is attained near some  $(x_1, x_2) \in \text{cl}(I_0) = I_0$ , and since  $b \notin X$ , it is attained at  $(x_1, x_2)$ . Thus,  $b \in S(I_0) - S(I_0) = B$ , a contradiction. ■

The rest of this subsection is devoted to the proof of Proposition 5.6(2). Fix an open  $V \subseteq G$  containing 0. We may assume that  $V$  is bounded and symmetric, that is,  $-V = V$ . Given  $r > 0$ , let  $P_r = \text{cl}(B_r) \cap G$  and  $S_r = \text{fr}(B_r) \cap G$ , where  $B_r$  is defined at the beginning of this section. Let  $B$  be as in Proposition 5.6(1).

**Claim 1.** *There are  $r_1 > r_0 > 0$  sufficiently large such that  $B \subseteq P_{r_0} \subseteq P_{r_1}$  and  $S_{r_0} + V \subseteq P_{r_1} \setminus B$ .*

*Proof of Claim 1.* Since  $B + V$  is bounded, there exists  $r_0 > 0$  such that  $B \subseteq P_{r_0}$  and  $B + V$  does not intersect  $S_{r_0}$ . Since  $V$  is symmetric, it follows that  $(S_{r_0} + V) \cap B = \emptyset$ . Because  $S_{r_0} + V$  is bounded there exists  $r_1 > r_0$  such that  $S_{r_0} + V \subseteq P_{r_1}$ . It follows that  $S_{r_0} + V \subseteq P_{r_1} \setminus B$ . ■

Fix such  $r_0, r_1$ . For  $\epsilon > 0$  let, as in Lemma 5.7,

$$Y_\epsilon = \{y_2 - y_1 : \exists x_1, x_2 \in B(I_0; \epsilon) (x_1, y_1), (x_2, y_2) \in S\}.$$

**Claim 2.** *There exists  $\epsilon_0 > 0$  such that no translate of  $V$  is contained in  $(P_{r_1} \setminus B) \cap Y_{\epsilon_0}$ .*

*Proof of Claim 2.* The family of  $Y_\epsilon$  decreases with  $\epsilon$ , and it is immediate from the definitions that

$$A(S, I_0) = \bigcap_{\epsilon} \text{cl}(Y_\epsilon).$$

We restrict our attention to the definably compact set  $P_{r_1} \setminus \text{int}(B)$  and let

$$\bar{Y}_\epsilon^{r_1} = \text{cl}(Y_\epsilon) \cap (P_{r_1} \setminus \text{int}(B)) \quad \text{and} \quad A_{r_1}(S, I_0) = A(S, I_0) \cap (P_{r_1} \setminus \text{int}(B)).$$

Thus,  $A_{r_1}(S, I_0) = \bigcap_{\epsilon > 0} \bar{Y}_\epsilon^{r_1}$ . Since each  $\bar{Y}_\epsilon^{r_1}$  is definably compact, so is  $A_{r_1}(S, I_0)$ .

By the choice of  $B$ , Proposition 5.6(1) implies that  $\dim(A(S, I_0) \setminus B) \leq 1$ , and hence, since the boundary of  $B$  is at most 1-dimensional, also  $\dim(A(S, I_0) \setminus \text{int}(B)) \leq 1$ . It follows that  $A_{r_1}(S, I_0)$  is a definably compact set which is at most 1-dimensional. Using

that, it is not hard to see that for a sufficiently small open  $W \ni 0$  the set  $A_{r_1}(S, I_0) + W$  does not contain any translate of our open set  $V$ . Fix such a set  $W$ .

Because  $A_{r_1}(S, I_0) = \bigcap_{\epsilon} \bar{Y}_{\epsilon}^{r_1}$  it is not hard to see that there exists  $\epsilon_0 > 0$  such that  $\bar{Y}_{\epsilon_0}^{r_1} \subseteq A_{r_1}(S, I_0) + W$ . It follows that  $\bar{Y}_{\epsilon_0}^{r_1}$  does not contain any translate of  $V$ , proving Claim 2. ■

It remains to show that setting  $\epsilon := \epsilon_0$  with  $\epsilon_0$  as in Claim 2, the requirements of Proposition 5.6(2) are satisfied.

**Claim 3.** *There exists  $r > 0$  such that  $x + V \not\subseteq S(B(I_0, \epsilon_0))$  for all  $x \in G \setminus P_r$ .*

*Proof of Claim 3.* Assume towards a contradiction that no such  $r$  exists. Then we can find an unbounded, definably connected curve  $\Gamma \subseteq G$  such that  $\Gamma + V \subseteq S(B(I_0, \epsilon_0))$ . It follows from the definition of  $Y_{\epsilon_0}$  that  $(\Gamma + V) - (\Gamma + V) \subseteq Y_{\epsilon_0}$ .

Fix any  $\gamma_0 \in \Gamma$  and let  $\Gamma_0 = \Gamma - \gamma_0$ . The curve  $\Gamma_0$  is unbounded, definably connected, with  $0 \in \Gamma_0$  and in addition  $\Gamma_0 + V \subseteq (\Gamma + V) - (\Gamma + V) \subseteq Y_{\epsilon_0}$ . Note that  $\Gamma_0 \cap S_{r_0} \neq \emptyset$  with  $r_0$  as in Claim 1. Indeed, although  $S_{r_0} = \text{fr}(B_{r_0}) \cap G$  need not be definably connected,  $\Gamma_0 \cap B_{r_0} \neq \emptyset$  because  $\Gamma_0$  is unbounded, definably connected and contains 0. Fix  $x_0 \in \Gamma_0 \cap S_{r_0}$ . This intersection point necessarily lies in  $S_{r_0}$ .

By our choice of  $\Gamma_0$ ,  $x_0 + V \subseteq \Gamma_0 + V \subseteq Y_{\epsilon_0}$ , and by our choice of  $r_0$  in Claim 1,  $x_0 + V \subseteq P_{r_1} \setminus B$ . However, by Claim 2, no translate of  $V$  is contained in  $Y_{\epsilon_0} \cap (P_{r_1} \setminus B)$ , a contradiction. ■

Choose  $r$  as in Claim 3. Setting  $B' = P_r$  and  $\epsilon = \epsilon_0$  finishes the proof of Proposition 5.6(2). ■

### 5.2. A lower bound on the dimension of the set of approximated points

In this subsection, assuming that  $S_{\text{pol}}$  is infinite, we modify the set  $S$  from Proposition 5.6 to a set  $\hat{S}$  as in the next proposition, using an idea from [5, Section 4]. The proof of Theorem 5.2 in the next subsection is by contradiction, and towards that we need this proposition.

**Proposition 5.10.** *Let  $S \subseteq G^2$  be a  $\mathcal{D}$ -definable, strongly minimal, closed set which is not  $G$ -affine, and assume that  $S_{\text{pol}}$  is infinite. Then there is a strongly minimal closed set  $\hat{S} \subseteq G^2$  which is not  $G$ -affine, definable in  $\mathcal{D}$  (over additional parameters), and there exists an infinite definable  $\hat{I} \subseteq G$ , such that for every infinite definable set  $T \subseteq \hat{I}$  and any bounded ball  $B$ , the set  $A(\hat{S}, T) \setminus B$  is 2-dimensional.*

The rest of this subsection is devoted to the proof of Proposition 5.10. We fix the sets  $S$  and  $S_{\text{pol}}$  as in its assumptions. Applying Proposition 5.6 to  $S$  and  $S_{\text{pol}}$  (in the role of  $I$  there) we fix a definable 1-dimensional  $I_0 \subseteq S_{\text{pol}}$  satisfying clauses (1) and (2) of that proposition.

**Lemma 5.11.** *There is a definable smooth 1-dimensional  $I_1 \subseteq I_0$  and*

- *a definably connected bounded open  $U \subseteq G$ ,*



- a definable continuous function  $f : U \rightarrow G$  with  $\Gamma_f \subseteq S$ , and
- a definable family  $\{\gamma_x : x \in I_1\}$  of curves  $\gamma_x : (0, 1) \rightarrow U$

such that

$$\lim_{t \rightarrow 0} \gamma_x(t) = x, \quad \lim_{t \rightarrow 0} f(\gamma_x(t)) = \infty,$$

and for all  $x_1, x_2 \in I_1$ ,

$$\lim_{t \rightarrow 0} (f(\gamma_{x_1}(t)) - f(\gamma_{x_2}(t))) = 0.$$

*Proof.* Using o-minimality and the fact that the projection of  $S$  onto  $G$  is finite-to-one, we may partition  $S$  and  $I_0$  into finitely many cells and reach the following situation. There is a definable, definably connected bounded open set  $U \subseteq G$  and a definable 1-dimensional smooth set  $I_1 \subseteq I_0$ , with  $I_1$  on the boundary of  $U$  and  $U \cup I_1$  a manifold with boundary. We may assume that  $\text{cl}(U) \cap S_{\text{pt}} = \text{cl}(I_1)$ . Furthermore, there is a definable, injective, continuous function  $f : U \rightarrow G$  whose graph is contained in  $S$ , such that for every  $x_0 \in I_1$  and every curve  $\gamma : (0, 1) \rightarrow U$  tending to  $x_0$  at 0, the image of  $\gamma$  under  $f$  is unbounded.

After applying a definable local diffeomorphism, we may assume that  $I_1 = (a, b) \times \{0\} \subseteq \mathbb{R}^2$  and  $U = (a, b) \times (0, 1) \subseteq \mathbb{R}^2$ . By shrinking  $I_1$  if needed we may assume that  $f$  is defined on the box  $[a, b] \times (0, 1]$ . For  $\epsilon \leq 1$ , let

$$U_\epsilon = (a, b) \times (0, \epsilon) \subseteq U$$

and

$$\begin{aligned} C_\epsilon &= f([a, b] \times \{\epsilon\}), \\ \Gamma_{\epsilon,1} &= f(\{a\} \times (0, \epsilon)), \\ \Gamma_{\epsilon,2} &= f(\{b\} \times (0, \epsilon)). \end{aligned}$$

When  $\epsilon = 1$ , we denote  $C_1, \Gamma_{1,1}$  and  $\Gamma_{1,2}$  by  $C, \Gamma_1$  and  $\Gamma_2$ , respectively. For every  $\epsilon \leq 1$ , the set  $C_\epsilon$  is bounded and  $\Gamma_{\epsilon,i}$  are unbounded curves for  $i = 1, 2$ . Recall that  $\partial f(U_\epsilon)$  denotes the boundary of  $f(U_\epsilon)$  (which is contained in  $G$ , since  $G \subseteq \mathbb{R}^n$  is closed). Because  $f : U \rightarrow G$  is continuous and injective, it is in fact a homeomorphism, by [14], hence

$$\partial f(U_\epsilon) = \Gamma_{\epsilon,1} \cup \Gamma_{\epsilon,2} \cup C_\epsilon$$

(we use here the fact that the limit of  $|f(x)|$  as  $x$  tends to any point in  $I_1$  is  $\infty$ ).

The next claim roughly says that for an infinitesimal  $\epsilon$ , the set  $f(U_\epsilon)$  is contained in two infinitesimal tubes around  $\Gamma_1$  and  $\Gamma_2$ .

**Claim 1.** For every  $\epsilon_1 > 0$  there exists  $\epsilon_2 > 0$  such that

$$f(U_{\epsilon_2}) \subseteq \bigcup_{i=1}^2 \Gamma_i + B_{\epsilon_1}.$$

*Proof of Claim 1.* We fix  $\epsilon_1 > 0$ . Using Proposition 5.6(2), we can find  $\epsilon > 0$  and a bounded neighborhood  $B' \subseteq G$  of 0 such that  $y + B_{\epsilon_1} \not\subseteq f(U_\epsilon)$  for every  $y \in G \setminus B'$ . Next, choose  $0 < \epsilon_2 < \min\{\epsilon, \epsilon_1\}$  such that  $f(U_{\epsilon_2})$  does not intersect the bounded sets  $B'$  and  $C_\epsilon + B_{\epsilon_1}$ . This can be done since the limit of  $|f(x)|$  is  $\infty$  as  $x$  tends in  $U$  to any point in  $I_1$ . We claim that this  $\epsilon_2$  satisfies our requirements.

Indeed, given  $x \in U_{\epsilon_2}$ , we have  $f(x) \notin B'$  and hence  $f(x) + B_{\epsilon_1} \not\subseteq f(U_\epsilon)$ . However, clearly  $f(x) \in f(U_\epsilon)$  (since  $\epsilon_2 < \epsilon$ ) and so, because  $f(x) + B_{\epsilon_1}$  is definably connected, we must have  $(f(x) + B_{\epsilon_1}) \cap \partial f(U_\epsilon) \neq \emptyset$ . Since  $f(x) \notin C_\epsilon + B_{\epsilon_1}$ , we have  $(f(x) + B_{\epsilon_1}) \cap C_\epsilon = \emptyset$ , and therefore  $f(x) + B_{\epsilon_1}$  must intersect  $\Gamma_{\epsilon_1} \cup \Gamma_{\epsilon_2}$ , and hence also  $\Gamma_1 \cup \Gamma_2$ . It now follows that for some  $i = 1, 2$ ,  $f(x) \in \Gamma_i + B_{\epsilon_1}$ . ■

**Claim 2.** *There is a definable 1-dimensional subset  $I_2 \subseteq I_1$ , and a definable family  $\{\gamma_x : x \in I_2\}$  of curves  $\gamma_x : (0, 1) \rightarrow U$  with  $\lim_{t \rightarrow 0} \gamma_x(t) = x$ , such that for every  $x_1, x_2 \in I_2$ ,*

$$\lim_{t \rightarrow 0} (f(\gamma_{x_1}(t)) - f(\gamma_{x_2}(t))) = 0.$$

*Proof of Claim 2.* Consider the unbounded curves  $\Gamma_1, \Gamma_2 \subseteq \partial f(U)$ , and for each  $i = 1, 2$  fix a definable parametrization  $\gamma_i(t) : (0, 1) \rightarrow G$  for  $\Gamma_i$  such that  $\lim_{t \rightarrow 0} |\gamma_i(t)| = \infty$ .

Now fix a definable family  $\{\gamma_x : x \in I_1\}$  of curves  $\gamma_x : (0, 1) \rightarrow U$  with  $\lim_{t \rightarrow 0} \gamma_x(t) = x$ . By Claim 1, for each  $x \in I_1$ , the curve  $f(\gamma_x(t))$  approaches one of the  $\Gamma_i$  as  $t$  tends to 0, and therefore, after possibly re-parameterizing  $\gamma_x$ , we can find  $\gamma_i, i = 1, 2$ , such that  $\lim_{t \rightarrow 0} (f(\gamma_x(t)) - \gamma_i(t)) = 0$ . The re-parametrization can be done uniformly in  $x$ . We can now find an infinite subinterval  $I_2 \subseteq I_1$  and  $i \in \{1, 2\}$  such that if  $x \in I_2$ , then  $\lim_{t \rightarrow 0} (f(\gamma_x(t)) - \gamma_i(t)) = 0$ . ■

Replacing  $I_1$  by  $I_2$  finishes the proof of Lemma 5.11. ■

The rest of this subsection is devoted to the proof of Proposition 5.10. We fix  $I_1 \subseteq I_0$ ,  $U, f, \{\gamma_x : x \in I_1\}$  as in Lemma 5.11.

Because  $I_0$  is smooth on the boundary of  $U$ , we can find an infinite subcell  $\hat{I} \subseteq I_0$  and  $c \in G$  generic over  $\emptyset$  such that  $\text{cl}(\hat{I} + c)$  is contained in  $U$ . We fix such  $\hat{I}$  and  $c$ . We say that two definable sets  $X, Y$  have *the same germ at 0* if there is some open neighborhood  $W \ni 0$  such that  $X \cap W = Y \cap W$ . With this in hand, the key initial observation is the following.

**Claim 5.12.** *For any infinite definable set  $T \subseteq \hat{I}$ , the set  $V_c = f(T + c) - f(T + c)$  is a 2-dimensional bounded set.*

*Proof.* Since  $f$  is continuous and  $\text{cl}(\hat{I} + c) \subseteq U$ , it follows that  $V_c$  is bounded. Assume now towards a contradiction that  $\dim V_c = 1$ . By shrinking  $T$  further if needed, we deduce from [21, Lemma 2.7] that  $f(T + c)$  is  $G$ -linear, that is, the sets  $f(T + c) - g$  and  $f(T + c) - h$  have the same germ at 0 for all  $h, g \in f(T + c)$ .

By shrinking  $T$  if needed, we may assume that  $c$  is still generic in  $G$  over the parameters defining  $T$ . It follows that there is an open neighborhood  $W \ni c$  such that for all  $c' \in W$  the set  $f(T + c')$  is  $G$ -linear. By definable choice, there is a definable function  $g : W \rightarrow G$  such that  $g(c') \in f(T + c')$  for all  $c' \in W$ . Denote  $H(c') := f(T + c') - g(c')$  and define

an equivalence relation  $E$  on  $W$  by  $E(c_1, c_2)$  if  $H(c_1)$  and  $H(c_2)$  have the same germ at 0. Since  $f(T + c')$  was  $G$ -linear, we easily get (see [21] for details) that  $H(c')$  is a local subgroup.

We claim that there is a generic  $E$ -equivalence class that is infinite. Since  $W$  is 2-dimensional, it will suffice to show that the class of germs at 0 of the sets  $H(c')$  is at most 1-dimensional as  $c'$  varies on  $W$ . Since the tangent space to  $H(c)$  at 0 is a subspace of the 2-dimensional tangent space to  $G$  at 0, our claim will follow from the fact that  $H(c)$  and  $H(c')$  have the same germ at 0 if and only if they have the same tangent space at 0. This latter fact is [23, Claim 2.20] (note that the argument given there for definable subgroups goes through verbatim for germs of definable local subgroups).

If we now fix generic and independent  $x, y, z \in T$  sufficiently close to each other, then there is  $w \in T$  and there are infinitely many  $E$ -equivalent  $c'$  such that

$$f(x + c') - f(y + c') + f(z + c') = f(w + c').$$

Since  $\Gamma_f \subseteq S$ , it follows readily from the above that  $\text{Stab}^*(S)$  is infinite and therefore, by Lemma 3.8(4), that  $S$  is  $G$ -affine, a contradiction. ■

Consider now the  $\mathcal{D}$ -definable set

$$S' = \{(x, y_1 - y_2) : (x + c, y_1), (x, y_2) \in S\}.$$

and the continuous function  $\hat{f} : U \rightarrow G$ ,

$$\hat{f}(x) = f(x + c) - f(x).$$

Clearly,  $\text{RM}(S') = 1$  and  $\Gamma(\hat{f}) \subseteq S'$ . By Lemma 4.14, there is a  $\mathcal{D}$ -definable strongly minimal set  $\hat{S} \subseteq S'$  containing  $\Gamma(\hat{f})$ . Clearly,  $\Gamma(\hat{f})_{\text{pol}} \subseteq \hat{S}_{\text{pol}}$ . Since  $\text{fr}(\hat{S})$  is finite, we may assume that  $\hat{S}$  is closed.

**Claim 5.13.**  $\hat{I} \subseteq \hat{S}_{\text{pol}}$ .

*Proof.* It suffices to prove  $\hat{I} \subseteq \Gamma(\hat{f})_{\text{pol}}$ . Take  $x \in \hat{I}$ , and denote by  $\gamma$  our fixed  $\gamma_x : (0, 1) \rightarrow U$ . Then  $\lim_{t \rightarrow 0} \gamma(t) = x$ . Also  $\hat{f}(\gamma(t)) = f(\gamma(t) + c) - f(\gamma(t))$ . Since  $\lim_{t \rightarrow 0} \gamma(t) + c = x + c$ , it follows that  $\lim_{t \rightarrow 0} f(\gamma(t) + c) = f(x + c)$ , and because  $\lim_{t \rightarrow 0} |f(\gamma(t))| = \infty$ , also  $\lim_{t \rightarrow 0} |\hat{f}(\gamma(t))| = \infty$ , so  $x$  is a pole of  $\Gamma(\hat{f})$ . ■

Since  $\hat{S}_{\text{pol}} \neq \emptyset$ , it follows that  $\hat{S}$  is not  $G$ -affine.

We can now proceed with the proof of Proposition 5.10. Let  $T$  be any infinite definable subset of  $\hat{I}$ , and  $B$  any open bounded ball. We want to prove that  $A(\hat{S}, T) \setminus B$  has dimension 2.

**Claim 1.** *There is a definable unbounded 1-dimensional subgroup  $H \subseteq G$  such that for every  $x \in T$  and  $h \in H$ , there is a definable  $\pi : (0, 1) \rightarrow (0, 1)$  with  $\pi(0^+) = 0^+$  and*

$$\lim_{t \rightarrow 0} (f(\gamma_x(\pi(t))) - f(\gamma_x(t))) = h.$$

*Proof of Claim 1.* We first recall a theorem from [30]: given a definable curve  $\sigma : (0, 1) \rightarrow G$  with  $\lim_{t \rightarrow 0} |\sigma(t)| = \infty$ , the set of all limit points of  $\sigma(t) - \sigma(s)$ , as  $s$  and  $t$  tend to 0, forms an 1-dimensional torsion-free unbounded subgroup  $H_\sigma \subseteq G$ . In particular, for each  $h \in H_\sigma$  there is a definable function  $\pi_h : (0, 1) \rightarrow (0, 1)$  with  $\pi_h(0^+) = 0^+$  such that  $\lim_{t \rightarrow 0} (\sigma(\pi_h(t)) - \sigma(t)) = h$ . It follows from the definition of  $H_\sigma$  that for any other definable curve  $\sigma' : (0, 1) \rightarrow G$ , if  $\lim_{t \rightarrow 0} (\sigma'(t) - \sigma(t)) = 0$ , then  $H_\sigma = H_{\sigma'}$ . We now apply this result to the unbounded curves  $f(\gamma_x(t))$ ,  $x \in T$ , and obtain the desired  $H$ . ■

**Claim 2.** For every  $b \in V_1 := f(T + c) - f(T + c)$  and  $h \in H$ , we have  $b + h \in A(\hat{S}, T)$ .

*Proof of Claim 2.* Let  $b = f(x_1 + c) - f(x_2 + c) \in V_1$ , where  $x_1, x_2 \in T$ , and let  $\pi$  be as in Claim 1, for  $x = x_2$  and  $h$ . Hence  $h = \lim_{t \rightarrow 0} (f(\gamma_{x_2}(\pi(t))) - f(\gamma_{x_2}(t)))$ . We have

$$\begin{aligned} \hat{f}(\gamma_{x_1}(t)) - \hat{f}(\gamma_{x_2}(\pi(t))) &= \hat{f}(\gamma_{x_1}(t)) - \hat{f}(\gamma_{x_2}(\pi(t))) + f(\gamma_{x_2}(t)) - f(\gamma_{x_2}(t)) \\ &= [f(\gamma_{x_1}(t) + c) - f(\gamma_{x_1}(t))] \\ &\quad - [f(\gamma_{x_2}(\pi(t)) + c) - f(\gamma_{x_2}(\pi(t)))] + f(\gamma_{x_2}(t)) - f(\gamma_{x_2}(t)) \\ &= [f(\gamma_{x_1}(t) + c) - f(\gamma_{x_2}(\pi(t)) + c)] \\ &\quad + [f(\gamma_{x_2}(t)) - f(\gamma_{x_1}(t))] + [f(\gamma_{x_2}(\pi(t))) - f(\gamma_{x_2}(t))]. \end{aligned}$$

As  $t$  tends to 0, for  $i = 1, 2$ , the curve  $\gamma_{x_i}(\pi(t)) + c$  still tends to  $x_i + c$ , since  $\pi(0^+) = 0^+$ , so its image under  $f$  tends to  $f(x_i + c)$ . By Lemma 5.11(3),  $\lim_{t \rightarrow 0} (f(\gamma_{x_2}(t)) - f(\gamma_{x_1}(t))) = 0$ . Thus, by Claim 1, the above expression tends to  $f(x_1 + c) - f(x_2 + c) + h = b + h$ , proving that  $b + h$  can be approximated near  $T$ . ■

We can now conclude the proof of Proposition 5.10, as follows. Because  $V_1$  and  $B$  are bounded, we can find  $r_0 > 0$  such that  $V_1 + h \subseteq G \setminus B$  for every  $h \in G \setminus B_{r_0}$ . In particular,  $b + (H \setminus B_{r_0}) \subseteq G \setminus B$  for every  $b \in V_1$ . Moreover, since  $H$  is unbounded,  $H \setminus B_{r_0}$  has dimension 1. Hence, by Claim 2, the 2-dimensional set  $V_1 + (H \setminus B_{r_0})$  is contained in  $A(\hat{S}, T) \setminus B$ , as needed.

### 5.3. Proof of Theorem 5.2

Assume towards a contradiction that  $S_{\text{pol}}$  is infinite. Since for any  $S_1, S_2 \subseteq G^2$ ,  $(S_1 \cup S_2)_{\text{pol}} = S_{1\text{pol}} \cup S_{2\text{pol}}$ , and  $S_{\text{pol}} = \text{cl}(S)_{\text{pol}}$ , we may assume that  $S$  is strongly minimal and closed. Since  $S_{\text{pol}} \neq \emptyset$ , we infer that  $S$  is not  $G$ -affine. By Proposition 5.10, there is a  $\mathcal{D}$ -definable set  $\hat{S}$  which is closed, strongly minimal and not  $G$ -affine, and an infinite definable  $\hat{I} \subseteq G$  such that for every infinite set  $T \subseteq \hat{I}$  and any open bounded ball  $B$ ,  $A(\hat{S}, T) \setminus B$  is 2-dimensional. This contradicts Proposition 5.6(1) for  $\hat{S}$  and  $\hat{I}$ .

**Example 5.14.** One of the difficulties in the above proof was the need to replace the initial set  $S$  with a set  $\hat{S}$ , in order to reach a situation where  $\dim(A(\hat{S}, T) \setminus B) = 2$ , for every infinite definable  $T \subseteq \hat{I} \subseteq \hat{S}_{\text{pol}}$  and any open bounded ball  $B$ . The following example

shows that the initial  $S$  can indeed have infinitely many poles and yet  $\dim A(S, I_0) = 1$  for some (in fact, any bounded) infinite  $I_0 \subseteq S_{\text{pol}}$ . Consider the graph of the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$f(x, y) = \begin{cases} (x, 0) & \text{if } y = 0, \\ (xy, 1/y) & \text{if } y \neq 0, \end{cases}$$

with  $G = \langle \mathbb{C}, + \rangle$ . The function  $f$  is a bijection of  $\mathbb{C}$  which is its own inverse. Its set of poles is the  $x$ -axis. For every  $x \in \mathbb{R}$ , as  $(x, y) \rightarrow (x, 0)$ ,  $f(x, y)$  approaches the  $y$ -axis with  $|f(x, y)| \rightarrow \infty$ . Thus, for any bounded  $I_0 \subseteq \mathbb{R} \times \{0\}$ ,  $A(S, I_0) = y$ -axis. After moving to  $\hat{S}$  as in the proof of Proposition 5.10, we can see that  $\dim(A(\hat{S}, T) \setminus B) = 2$  for any infinite  $T \subseteq \hat{S}_{\text{pol}}$  and any bounded ball  $B$ .

### 6. Topological corollaries

We establish here several topological properties of plane curves, typically true for complex algebraic plane curves. These properties are used later on in our proof of the main theorem. Our first definition generalizes the notion of a function being open at a point.

**Definition 6.1.** Let  $S \subseteq G^2$  and  $a = (a_1, a_2) \in S$ . We say that  $S$  is *open at  $a$  over  $a_1$*  if for every open neighborhood  $U$  of  $a$ ,  $a_1$  is in the interior of  $\pi_1(U \cap S)$ . We say that  $S$  is *open at  $a$*  if  $S$  is open at  $a$  over  $a_1$  and  $S^{\text{op}}$  is open at  $a$  over  $a_2$ .

Let  $S \subseteq G^2$  and  $a_1 \in G$ . We say that  $S$  is *open over  $a_1 \in \pi_1(S)$*  if for every  $(a_1, a_2) \in S$ ,  $S$  is open at  $a$  over  $a_1$ .

We note that if  $B \ni a = (a_1, a_2)$  is an open box such that  $a_1 \notin \text{int}(\pi_1(B \cap S))$ , then the same remains true for all smaller open boxes.

**Lemma 6.2.** *Assume that  $S \subseteq G^2$  is a plane curve. Then there are at most finitely many  $a_1 \in \pi_1(S)$  such that  $S$  is not open over  $a_1$ . In particular,  $S$  does not contain any 1-dimensional components.*

*If  $S$  does not contain any straight line, then there are at most finitely points  $a \in S$  such that  $S$  is not open at  $a$ .*

*Proof.* First note that if  $S = S_1 \cup S_2$  and  $S$  is not open at  $a$  over  $a_1 \in G$ , then either  $S_1$  or  $S_2$  is not open at  $a$  over  $a_1$ . Thus we may assume that  $S$  is strongly minimal. Without loss of generality,  $S$  is  $\mathcal{D}$ -definable over  $\emptyset$ .

Assume towards a contradiction that the set  $N$  of all  $x$  in  $\pi_1(S)$  over which  $S$  is not open is infinite. Pick  $a_1$  generic in  $N$  over  $\emptyset$ . Because  $\text{RM}(\pi_1(S)) = 1$ , the point  $a_1$  is  $\mathcal{D}$ -generic in  $\pi_1(S)$  over  $\emptyset$ .

Fix  $a = (a_1, a_2) \in S$  and  $B = B_1 \times B_2 \ni a$  such that  $a_1 \notin \text{int}(\pi_1(S \cap B))$ . Let  $B_S = S \cap B$  and write  $\bar{B}_S := \text{cl}(B_S)$ . Note that  $a$  is  $\mathcal{D}$ -generic in  $S$  over  $\emptyset$ .

By Theorem 5.2,  $S$  has finitely many poles and since  $\dim(a_1/\emptyset) \geq 1$ , the point  $a_1$  is not a pole of  $S$ . By Corollary 4.12, there are at most finitely many points in  $\pi_1(S)$  over

which  $S$  is non-injective, and each of them is in  $\text{acl}_{\mathcal{D}}([S]) = \text{acl}_{\mathcal{D}}(\emptyset)$ . Thus  $S$  is injective over  $a_1$ . By Theorem 4.9,  $\text{fr}(S) \subseteq \text{acl}_{\mathcal{D}}(\emptyset)$  and hence  $(\{a_1\} \times G) \cap \text{fr}(S) = \emptyset$ .

Since  $a_1 \notin \text{int}(\pi_1(B_S))$ , there exists a definable curve  $\gamma : (0, 1) \rightarrow B_1 \setminus \pi_1(B_S)$  such that  $\lim_{t \rightarrow 0} \gamma(t) = a_1$ . Notice that for  $t$  small enough  $\gamma(t)$  must be  $\mathcal{D}$ -generic in  $G$ , and therefore, because  $\pi_1(S)$  is co-finite in  $G$ ,  $\gamma(t)$  is  $\mathcal{D}$ -generic in  $\pi_1(S)$  over  $\emptyset$ . So, we may assume that the fiber  $S_{\gamma(t)}$  has constant size  $n \geq 1$ . For each  $t$ , let  $y_1(t), \dots, y_n(t) \in G$  be distinct such that  $(\gamma(t), y_i(t)) \in S$ . Because  $\gamma(t) \notin \pi_1(B_S)$ , none of the  $y_i(t)$  is in  $B_2$ .

Since  $a_1 \notin S_{\text{pol}}$ , each of the curves  $\gamma_i(t)$  is bounded, and hence has a limit  $y_i \in G \setminus B_2$ . Since  $(\{a_1\} \times G) \cap \text{fr}(S) = \emptyset$ , each of the limit points  $(a_1, y_i)$  is in  $S$  and in addition  $(a_1, a_2) \in S$ , with  $a_2 \neq y_i$  for all  $i$ . However, since  $a_1$  is  $\mathcal{D}$ -generic, we must have  $|S_{a_1}| = n$ . This implies that  $y_i = y_j$  for some  $i \neq j$ , so  $S$  is non-injective at  $(a_1, y_i)$ , a contradiction.

Assume now that the intersection of  $S$  with any straight line is finite. We apply the above to both  $S$  and  $S^{\text{op}}$ , and then by removing from  $\pi_1(S)$  and  $\pi_1(S^{\text{op}})$  finitely many points, we remain, by our assumption on  $S$ , with a co-finite subset  $S'$  of  $S$  such that  $S$  is open at each point of  $S'$ . ■

**Corollary 6.3.** *Assume that  $S \subseteq G^2$  is strongly minimal and  $a = (a_1, a_2)$  is a non-isolated point of  $S$ .*

- (1) *If  $S$  is not a straight line, then  $S$  is open at  $a$ .*
- (2) *If there is  $y \in G$  such that  $S \sim G \times \{y\}$ , then  $y = a_2$  and there exists an open  $U \ni a_1$  such that  $U \times \{a_2\} \subseteq S$ . In particular,  $S$  is open at  $a$  over  $a_1$ .*
- (3) *If  $S^{\text{op}}$  is injective at  $(a_2, a_1)$  over  $a_2$ , then either  $S \sim G \times \{a_2\}$  or there exists an open  $B = B_1 \times B_2 \ni a$  such that  $S \cap B$  is the graph of an open continuous map from  $B_1$  into  $B_2$ .*
- (4) *If  $S$  is not a straight line and  $a$  is  $\mathcal{D}$ -generic in  $S$ , then there exists an open  $U \ni a$  such that  $S \cap U$  is the graph of a homeomorphism from  $\pi_1(U)$  onto  $\pi_2(U)$ .*

*Proof.* (1) We assume that  $S$  is not a straight line and show that  $S$  is open at  $a$ . Assume towards a contradiction that  $S$  is not open at  $a$  over  $a_1$ . In order to reach a contradiction it is sufficient, by Lemma 6.2, to deduce that there are infinitely many points in  $\pi_1(S)$  over which  $S$  is not open.

By Theorem 4.9, we can find an open box  $B = B_1 \times B_2$  containing  $a$  such that  $S \cap \text{cl}(B)$  is closed, and  $a_1 \notin \text{int}(\pi_1(B \cap S))$ . Let  $B_S = B \cap S$  and denote  $\bar{B}_S = \text{cl}(B_S)$ . Repeating the argument with a smaller box, we see that also  $a_1 \notin \text{int}(\pi_1(\bar{B}_S)) = \text{int}(\pi_1(S \cap \text{cl}(B)))$ . Because  $S \cap (\{a_1\} \times G)$  is finite, we may also assume that  $S \cap (\{a_1\} \times \text{cl}(B_2)) = \{a\}$ .

The set  $\pi_1(\bar{B}_S)$  is closed in  $G$  and since, by Lemma 6.2,  $S$  has no 1-dimensional components and  $\pi_1$  is finite-to-one, it is 2-dimensional. The point  $a_1$  belongs to the boundary of  $\pi_1(\bar{B}_S)$ , so by o-minimality, there exists a definable curve  $\gamma_1 : (0, 1) \rightarrow \partial(\pi_1(\bar{B}_S))$  with  $a_1 = \lim_{t \rightarrow 0} \gamma_1(t)$ . Since  $\bar{B}_S = S \cap \text{cl}(B)$ , there exists a definable curve  $\gamma_2 : (0, 1) \rightarrow \text{cl}(B_2)$  such that  $(\gamma_1(t), \gamma_2(t)) \in S \cap \text{cl}(B)$  for every  $t$ . Let  $b = \lim_{t \rightarrow 0} \gamma_2(t) \in \text{cl}(B_2)$ .

Since  $S \cap \text{cl}(B)$  is closed, it follows that  $(a_1, b) \in S$ , and therefore by our choice of  $B_2, b = a_2$ . But then the curve  $\gamma(t) = (\gamma_1(t), \gamma_2(t))$  tends to  $a$ , so for  $t$  small enough, it must belong to the open set  $B$ , and its projection is not in  $\text{int}(\pi_1(B_S))$ . Therefore for all  $t$  small enough,  $S$  is not open over  $\gamma_1(t)$ . This contradicts Lemma 6.2 and ends the proof of (1).

(2) Assume that  $S_1 := S \cap G \times \{y\}$  is infinite. Because  $S$  is strongly minimal and  $a$  is non-isolated we must have  $(a_1, a_2) \in S_1$ , so  $y = a_2$ . The set  $S_1$  is strongly minimal, thus its projection on the first coordinate is co-finite and so  $S_1$  (and therefore  $S$ ) is locally near  $a$  the graph of a constant function. In particular,  $S$  is open at  $a$  over  $a_1$ .

(3) Assume now that  $S^{\text{op}}$  is injective at  $(a_2, a_1)$  over  $a_2$  and that we are not under clause (2), so the intersection of  $S$  with any line  $G \times \{y\}$  is finite.

By definition of injectivity, there exists an open box  $B = B_1 \times B_2$  such that  $B \cap S$  is the graph of a function, call it  $f_S$ , from a subset of  $B_1$  into  $B_2$ , so the intersection of each  $\{x\} \times G$  with  $S$  is finite. We may also assume that  $B \cap S$  has no isolated point (by o-minimality, there are only finitely many). By (1), we may shrink  $B$  so that  $B_S$  is open over every point in  $\pi_1(B_S)$  and  $B_S^{-1}$  is open over every point in  $\pi_1(B_S)$ . It follows that the domain of  $f_S$  is the whole of  $B_1$  and in addition  $f_S$  is continuous and open.

(4) By Corollary 4.12,  $S$  is injective at  $a$  over  $a_1$  and  $S^{\text{op}}$  is injective at  $(a_2, a_1)$  over  $a_2$ . The result follows from (3), applied to  $S$  and to  $S^{\text{op}}$ . ■

Notice that even though, by o-minimality, the set of isolated points of any plane curve  $S$  is finite we do not know yet that it is contained in  $\text{acl}_{\mathcal{D}}(\{S\})$ .

## 7. The ring of Jacobian matrices

### 7.1. The ring $\mathfrak{R}$

Our next goal is to show that if  $f$  is a  $\mathcal{D}$ -function, then its Jacobian matrix vanishes at 0 if and only if  $f$  is not locally invertible at 0. This will be done in this and the next section. In the present section we prove that similarly to a complex analytic function, the Jacobian matrix is non-zero if and only if it is an invertible matrix.

Throughout this section we fix a definable local coordinate system for  $G$  near  $0_G$ , identifying  $0_G$  with  $0 \in \mathbb{R}^2$ . From now on we identify  $G$  locally with an open subset of  $\mathbb{R}^2$ . For a differentiable  $\mathcal{D}$ -function  $f$  in a neighborhood of 0, with  $f(0) = 0$ , the Jacobian matrix at  $x$ , denoted by  $J_x f$ , is computed with respect to this fixed coordinate system, and we denote by  $|J_x f|$  its determinant. We use  $d_x f$  to denote the differential of  $f$ , viewed as a map from the tangent space of  $G$  at  $x$ , denoted by  $T_x(G)$ , to  $T_{f(x)}(G)$ . As we soon observe, the collection of all matrices  $J_0 f$  is a subring of  $M_2(\mathbb{R})$ , and the main goal of this section is to show that it is in fact a field (thus every non-zero matrix is invertible).

We first observe the following statement.

**Lemma 7.1.** *Let  $f : U \rightarrow G$  be a non-constant  $\mathcal{D}$ -function. Then:*

- (1) *The set of  $a \in U$  at which  $|J_a f| = 0$  is at most 1-dimensional.*
- (2) *The set of  $a \in U$  at which  $J_a f = 0$  is finite.*

*Proof.* (1) By strong minimality, for every open  $V \subseteq U$ , we have  $\dim f(V) = 2$ , for otherwise the preimage of some point is infinite and co-infinite. By the o-minimal version of Sard’s Theorem [39, Theorem 2.7], the set of singular points of  $f$  is at most 1-dimensional. For (2), note that if  $J_a f = 0$  on a definably connected path then  $f$  must be constant there, which by strong minimality implies that  $f$  is constant on  $U$ . ■

**Definition 7.2.** Recall from Section 1.3 that  $\mathfrak{F}$  is the collection of all  $\mathcal{D}$ -functions  $f$  which are  $C^1$  in a neighborhood of 0, with  $f(0) = 0$ . We let

$$\mathfrak{R} = \{J_0 f \in M_2(R) : f \in \mathfrak{F}\}.$$

It is important here to distinguish between the group operation in  $G$  and the usual ring operations in  $M_2(R)$ . Thus we reserve the additive notation  $\pm$  for matrix addition, and let  $\oplus, \ominus$  denote the group operations in  $G$ .

**Lemma 7.3.** *The set  $\mathfrak{R}$  is a unital subring of  $M_2(R)$  and for every  $A \in \mathfrak{R}$  which is invertible,  $A^{-1} \in \mathfrak{R}$ .*

*Proof.* We first note that the collection of germs of functions in  $\mathfrak{F}$  is closed under  $\oplus$  and functional composition. Indeed, if  $S_f$  and  $S_g$  represent  $\mathcal{D}$ -functions  $f$  and  $g$  in  $\mathfrak{F}$ , then the plane curve  $S_f \circ S_g$  represents  $f \circ g$  and the plane curve

$$S_f \boxplus S_g = \{(x, y_1 \oplus y_2) : (x, y_1) \in S_f, (x, y_2) \in S_g\}$$

represents  $f \oplus g$ .

Using the chain rule it is easy to verify that for  $f, g \in \mathfrak{F}$ ,

$$J_0(f \oplus g) = J_0 f + J_0 g \quad \text{and} \quad J_0(f \circ g) = J_0 f \cdot J_0 g.$$

Since the germs in  $\mathfrak{F}$  are closed under  $\oplus$  and functional composition, it follows that  $\mathfrak{R}$  is a ring. If  $J_0 f$  is invertible, then  $f$  is a locally invertible function, in which case it is clear that  $f^{-1}$  is also in  $\mathfrak{F}$ , and therefore  $(J_0 f)^{-1} \in \mathfrak{R}$ . ■

**Note.** Given  $\mathcal{D}$ -functions  $f, g \in \mathfrak{F}$  it seems possible that every strongly minimal set representing  $f \circ g$  (or every set representing  $f \oplus g$ ) will have a nodal singularity at  $(0, 0)$  and thus will not be locally at  $(0, 0)$  the graph of a function.

### 7.2. Definability and dimension of $\mathfrak{R}$

Our aim is to show that  $\mathfrak{R}$  is a definable field isomorphic to  $R(\sqrt{-1})$ . This is achieved in several steps. We first show (Theorem 7.13) that  $\mathfrak{R}$  is a definable ring of one of two kinds, and then – by eliminating one of these possibilities – we deduce the desired result.

We are going to use the following operation extensively.

**Definition 7.4.** For a  $\mathcal{D}$ -function  $f$  which is  $C^1$  in a neighborhood of some  $a \in G$ , we let

$$\tilde{J}_a f = J_0(f(x \oplus a) \ominus f(a)), \quad \tilde{d}_a f = d_0(f(x \oplus a) \ominus f(a)).$$

Note that  $f(x \oplus a) \ominus f(a)$  is in  $\mathfrak{F}$  and thus  $\tilde{J}_a f \in \mathfrak{R}$ . We let  $\ell_a(x) = x \oplus a$ .



- Lemma 7.5.** (1)  $\tilde{d}_a f = (d_0 \ell_{f(a)})^{-1} \circ d_a f \circ d_0 \ell_a$ .  
 (2) For every  $a \in \text{dom}(f)$ ,  $J_a f$  is invertible if and only if  $\tilde{J}_a f$  is invertible, and  $J_a f = 0 \Leftrightarrow \tilde{J}_a f = 0$ .  
 (3) For any two differentiable  $\mathcal{D}$ -functions  $f, g : U \rightarrow G$  and  $x_0 \in U$ ,  $\tilde{J}_{x_0}(f \oplus g) = \tilde{J}_{x_0} f - \tilde{J}_{x_0} g$ .

*Proof.* (1) is easy to verify and (2) follows easily, so we prove (3). Note that

$$\tilde{J}_{x_0}(f \oplus g) = J_0[(f \oplus g)(x_0 \oplus x) \ominus (f \oplus g)(x_0)],$$

which equals

$$J_0[((f(x_0 \oplus x) \ominus f(x_0)) \ominus (g(x_0 \oplus x) \ominus g(x_0))).$$

As noted in the proof of Lemma 7.3,  $J_0(h_1 \oplus h_2) = J_0 h_1 - J_0 h_2$ , so the above equals

$$J_0(f(x_0 \oplus x) \ominus f(x_0)) - J_0(g(x_0 \oplus x) \ominus g(x_0)) = \tilde{J}_{x_0}(f) - \tilde{J}_{x_0}(g). \quad \blacksquare$$

**Definition 7.6.** We say that a  $\mathcal{D}$ -function  $f : U \rightarrow G$  is  $G$ -affine if there exist non-empty open sets  $V \subseteq U$  and  $W \ni 0$  such that for all  $x_1, x_2 \in V$  and  $x \in W$ ,

$$f(x + x_1) - f(x_1) = f(x + x_2) - f(x_2).$$

We note that a  $\mathcal{D}$ -function  $f$  is  $G$ -affine if and only if  $S_f$  is  $G$ -affine if and only if  $\text{Stab}^*(S_f)$  is infinite. Indeed, by Lemma 3.8(4),  $S_f$  is  $G$ -affine if and only if  $\text{Stab}^*(S_f)$  is infinite, and strong minimality implies that if  $f$  is  $G$ -affine, then so is  $S_f$ . Furthermore, since  $S_f$  is unique up to  $\sim$ -equivalence (as noted in the concluding paragraph of Section 4.6), this does not depend on the choice of  $S_f$ .

**Remark 7.7.** If  $f$  is  $G$ -affine and  $f(0) = 0$ , then  $f$  is a partial group homomorphism in a neighborhood of 0. It follows that for all  $a$  in some open  $V \ni 0$  we have  $\tilde{J}_a f = J_0(f(x \oplus a) \ominus f(a)) = J_0 f$ . Thus, if  $J_0 f = 0$ , then  $f$  vanishes on  $V$ . Since  $f$  is represented by some strongly minimal  $S_f$ , if  $f$  vanishes on some infinite set, it vanishes on its domain.

As we already saw in Fact 3.9, since  $\mathcal{D}$  is not locally modular, there exists at least one  $\mathcal{D}$ -function which is not  $G$ -affine.

We are going to need the following lemma.

**Lemma 7.8.** *There are invertible matrices in  $\mathfrak{R}$  arbitrarily close to the zero matrix.*

*Proof.* We go via the following claim which is also used later in the text.

**Claim 7.9.** *There exists  $g \in \mathfrak{F}$  which is not  $G$ -affine, with  $J_0 g = 0$ .*

*Proof.* For  $y \in G$  and  $n \in \mathbb{N}$  we write  $ny := \overbrace{y \oplus \cdots \oplus y}^{n \text{ times}}$ . Fix  $f : U \rightarrow G$  in  $\mathfrak{F}$  which is not  $G$ -affine, and for  $n \in \mathbb{N}$ , let  $g_n(x) = f(nx) - nf(x)$ . It is easy to see that  $g_n \in \mathfrak{F}$  and  $J_0 g_n = nJ_0 f - nJ_0 f = 0$ . We want to show that for some  $n \in \mathbb{N}$ , the function  $g_n$  is not  $G$ -affine, so gives the desired  $g$ .

Notice that if  $g_n$  is  $G$ -affine, then since  $J_0g_n = 0$ , the function  $g_n$  must vanish on its domain (by Remark 7.7). Assume towards a contradiction that for every  $n \in \mathbb{N}$ , the function  $g_n$  vanishes on its domain, so  $f(nx) = nf(x)$  whenever  $nx \in U$ . Pick a  $\mathcal{D}$ -generic  $x \in U$  sufficiently close to 0 so that for all  $n$ , we have  $nx \in U$  and  $nf(x) \in U$  (we can do it by saturation). For all  $n$  we have

$$f(x + nx) = f((n + 1)x) = (1 + n)f(x) = f(x) + nf(x) = f(x) + f(nx).$$

Thus, since  $x$  is generic, it is not torsion, and hence there are infinitely many  $y \in G$  such that  $f(x + y) = f(x) + f(y)$ . Because  $f$  is a  $\mathcal{D}$ -function it follows that for almost all  $y$  with  $x + y \in U$ ,  $f(x + y) = f(x) + f(y)$ . Since  $x$  is  $\mathcal{D}$ -generic, the function  $f$  must be  $G$ -affine, a contradiction. ■

We return to the proof of Lemma 7.8. Take the function  $g : V \rightarrow G$  from Claim 7.9. By Lemma 7.1, for every  $x \in V$  generic,  $J_xg$  and hence  $\tilde{J}_xg$  is invertible. Because  $g$  is smooth and  $J_0g = 0$ , there are invertible matrices of the form  $\tilde{J}_xg \in \mathfrak{R}$  arbitrarily close to the zero matrix. ■

**Definition 7.10.** Given a set  $W \subseteq \mathfrak{R}$  and a family  $\mathcal{F} = \{f_t : t \in T_0\}$  of  $\mathcal{D}$ -functions, we say that  $W$  is realized by  $\mathcal{F}$  if  $W = \{J_0f_t : t \in T_0\}$ .

**Proposition 7.11.** The ring  $\mathfrak{R}$  is a definable subring of  $M_2(R)$  which is also an  $R$ -vector subspace.

*Proof.* We first show that  $\mathfrak{R}$  is a  $\bigvee$ -definable subring of  $M_2(R)$ , that is,  $\mathfrak{R}$  is a bounded union of definable subsets of  $M_2(R)$ . Let  $M \in \mathfrak{R}$ . By definition, there exists some  $\mathcal{D}$ -function  $f \in \mathfrak{F}$  such that  $J_0f = M$ . Let  $S_f$  represent  $f$ . Let  $\varphi(x, a)$   $\mathcal{D}$ -define  $S_f$  such that  $\varphi(x, y)$  is a family of plane curves all passing through  $(0, 0)$ . By Proposition 4.15, there is a  $\emptyset$ -definable family  $\mathcal{F}$  of  $\mathcal{D}$ -functions in  $\mathfrak{F}$  such that the germ of  $f$  at 0 is represented in  $\mathcal{F}$ . Since  $J_0f$  only depends on the germ of  $f$  at 0, we find that  $M$  is realized as the Jacobian matrix at 0 of some  $\mathcal{D}$ -function in  $\mathcal{F}$ . Let  $T_{\mathcal{F}}$  be the set of all Jacobian matrices of  $\mathcal{D}$ -functions in  $\mathcal{F}$ , where  $\mathcal{F}$  is a  $\emptyset$ -definable family of  $\mathcal{D}$ -functions in  $\mathfrak{F}$ . We have thus seen that  $\mathfrak{R}$  can be covered by all the sets  $T_{\mathcal{F}}$ . There are a bounded number of such sets, where the bound is given by the cardinality of the language of  $\mathcal{D}$ .

It follows that there is a definable open neighborhood  $U \subseteq M_2(R)$  of the zero matrix such that  $U \cap \mathfrak{R}$  is definable (for more on  $\bigvee$ -definable groups and rings see [25]). More precisely, there exists a  $\emptyset$ -definable family of  $\mathcal{D}$ -functions which realizes  $U \cap \mathfrak{R}$ .

We now proceed to show that  $\mathfrak{R}$  is actually a definable subset of  $M_2(R)$ . Let  $U \ni 0$  be a neighborhood of 0 in  $M_2(R)$  such that  $U \cap \mathfrak{R}$  is definable as above. We claim that

$$\mathfrak{R} = \{AB^{-1} : A, B \in U \cap \mathfrak{R}, B \text{ invertible}\}.$$

Indeed, for every  $C \in \mathfrak{R}$  we can find, by Lemma 7.8, an invertible matrix  $B \in U \cap \mathfrak{R}$ , sufficiently close to 0, such that  $CB \in U \cap \mathfrak{R}$ . It follows that  $\mathfrak{R}$  is definable.

Finally, the subring of scalar matrices in  $\mathfrak{R}$  is in particular a subgroup of  $(R, +)$ , and it is non-trivial since it contains 1. By o-minimality, the only non-trivial definable subgroup of  $(R, +)$  is  $(R, +)$  itself. So  $\mathfrak{R}$  contains all diagonal matrices, and is therefore an  $R$ -vector subspace of  $M_2(R)$ . ■

**Proposition 7.12.** *Let  $U \subseteq G$  be an open neighborhood of 0, and assume that  $f : U \rightarrow G$  is a continuously differentiable  $\mathcal{D}$ -function which is not  $G$ -affine. Then the set  $\tilde{J}(U) = \{\tilde{J}_a f \in M_2(R) : a \in U\}$  has dimension 2. In particular,  $\dim \mathfrak{R} \geq 2$ .*

*Proof.* Since  $\dim U = 2$ , we have  $\dim \tilde{J}(U) \leq 2$ . Assume towards a contradiction that  $\dim \tilde{J}(U) \leq 1$ .

**Claim.** *There exists  $g_0 \in G \setminus \text{dcl}(\emptyset)$  and infinitely many  $a \in G$  such that  $\tilde{J}_a f = \tilde{J}_a f(x \oplus g_0)$ .*

*Proof of Claim.* For every matrix  $A \in \tilde{J}(U)$  let  $C_A := \{x \in U : \tilde{J}_x f = A\}$ . By our assumptions, there exists  $A \in \tilde{J}(U)$  such that  $\dim C_A \geq 1$ , and by possibly shrinking  $U$ , we may assume that  $C_A$  is definably connected. Consider  $B_A = C_A \ominus C_A \subseteq G$ . There are two cases to consider:

**Case 1.** There exists  $A \in \tilde{J}(U)$  such that  $\dim B_A = 1$ .

We may apply [21, Lemma 2.7] and conclude that the set  $C_A$  is contained in a coset of a  $\sqrt{\phantom{x}}$ -definable 1-dimensional subgroup  $\mathcal{H}$  of  $G$ . It follows that for  $g_0 \in \mathcal{H}$  sufficiently close to 0, there are infinitely many  $a \in C_A$  such that  $a \oplus g_0 \in C_A$ , and thus  $\tilde{J}_a f = \tilde{J}_a f(x \oplus g_0) = \tilde{J}_{a+g_0} f$ .

**Case 2.** For all  $A \in \tilde{J}(U)$ ,  $\dim B_A = 2$ , so  $B_A$  contains an open subset of  $G$ .

Given  $A$  generic in  $\tilde{J}(U)$  we may find an open set  $W \subseteq G$  in  $B_A$  such that  $A$  is still generic in  $\tilde{J}(U)$  over the parameters defining  $W$ . Thus there are infinitely many  $A \in \tilde{J}(U)$  for which  $W \subseteq B_A$ . Pick  $g_0$  generic in  $W$  and then for each  $A$  such that  $g_0 \in B_A$  there are  $a, b \in C_A$  such that  $a \ominus b = g_0$ , so  $a = b \oplus g_0$ . By definition of  $C_A$ , we know that for every such pair  $(a, b)$  we have  $\tilde{J}_a f = \tilde{J}_b f$ , so  $\tilde{J}_{b \oplus g_0} f = \tilde{J}_b f$ . We get

$$\tilde{J}_b f(x \oplus g_0) = J_0(f(x \oplus b \oplus g_0) \ominus f(b \oplus g_0)) = \tilde{J}_{b \oplus g_0} f = \tilde{J}_b f.$$

Since there are infinitely many such pairs  $b, b \oplus g_0$  as  $A$  varies, we are done. ■

To conclude the proof, fix  $g_0$  as in the claim and infinitely many  $a$  such that  $\tilde{J}_a f = \tilde{J}_a f(x \oplus g_0)$ . By Lemma 7.5(3), for each such  $a$ ,  $\tilde{J}_a(f(x \oplus g_0) \ominus f(x)) = 0$ . But then, by Lemma 7.5(2), for the  $\mathcal{D}$ -function  $k(x) = f(x \oplus g_0) \ominus f(x)$  there are infinitely many  $a$ , such that  $J_a k = 0$ , so  $k(x)$  is constant on its domain, say of value  $d$ . By strong minimality of  $\mathcal{D}$ ,  $(g_0, d)$  is in  $\text{Stab}^*(S_f)$ . Since  $g_0$  is not in  $\text{dcl}(\emptyset)$ , it is not a torsion element so  $\text{Stab}^*(S_f)$  is infinite and therefore  $f$  is  $G$ -affine, a contradiction. ■

### 7.3. The structure of $\mathfrak{R}$

The main result of this section is the following theorem.

**Theorem 7.13.** *There exists a fixed invertible matrix  $M \in M_2(R)$  such that one of the following two holds:*

$$(1) \quad \mathfrak{R} = \left\{ M^{-1} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} M : a, b \in R \right\}.$$

In particular,  $\mathfrak{R}$  is a field which is definably isomorphic to  $R(\sqrt{-1})$ .

$$(2) \quad \mathfrak{R} = \left\{ M^{-1} \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} M : a, b \in R \right\}.$$

We need some preliminaries.

**Lemma 7.14.** *Let  $U \subseteq G$  be a definably connected open neighborhood of 0. Let  $f : U \rightarrow G$  be a non-constant  $\mathcal{D}$ -function. Then  $|J_x f|$  has constant sign at all  $x \in U$  where  $f$  is differentiable and  $J_x f$  is invertible.*

*Proof.* By Corollary 4.12, we may assume – possibly removing finitely many points from  $U$  – that  $f$  is locally injective. The result now follows from [27, Theorem 3.2]. ■

Now, for  $f \in \mathfrak{F}$  non-constant we denote by  $\sigma(f)$  the sign of  $|J_x f|$  for all  $x$  sufficiently close to 0 at which  $J_x f$  is invertible.

**Proposition 7.15.** *Every invertible  $A \in \mathfrak{R}$  has positive determinant.*

*Proof.* Fix  $A_0 \in \mathfrak{R}$  generic over  $\emptyset$ , and  $W \subseteq \mathfrak{R}$  a definable open neighborhood of  $A_0$ . Fix also a definable family  $\{f_t : t \in T\}$  of  $\mathcal{D}$ -functions, realizing  $W$ , provided by Proposition 7.11. Let  $a_0$  be generic in  $T$  such that  $J_0 f_{a_0} = A_0$ . We may assume that  $T$  is a cell in some  $R^k$ , and by definable choice in o-minimal structures further assume that the map  $t \mapsto J_0 f_t$  is a homeomorphism of  $T$  and  $W$ . By Proposition 7.12,  $\dim T = \dim W = \dim \mathfrak{R} \geq 2$ .

For every  $t \in T$ , let  $U_t \subseteq G$  be the domain of  $f_t$  (containing 0). We can find a definably connected neighborhood  $U_0 \ni 0$  and a definably connected neighborhood  $T \supseteq T_0 \ni a_0$  such that  $U_0 \subseteq U_t$  for every  $t \in T_0$ . The definitions of the sets  $U_0$  and  $T_0$  may use additional parameters but we may choose them so that  $a_0$  is still generic in  $T_0$  over those parameters. Let  $W_0 = \{J_0 f_t : t \in T_0\}$  be the corresponding neighborhood of  $A_0$  in  $\mathfrak{R}$ .

Consider now the set of matrices  $\hat{W}_0 = W_0 - A_0 \subseteq \mathfrak{R}$ . It is an open neighborhood of 0 in  $\mathfrak{R}$ , which is realized by the family  $\{f_t \ominus f_{a_0} : t \in T_0\}$ .

Our goal is to show that every invertible matrix in  $\hat{W}_0$  has positive determinant. Let us first see that they all have the same determinant sign. Note that for all  $t \in T_0 \setminus \{a_0\}$ , the function  $f_t \ominus f_{a_0}$  is non-constant on  $U_0$ , thus by Theorem 4.17 it is an open map. We now show, using the above notation, that  $\sigma(f_t \ominus f_{a_0})$  is constant as  $t$  varies in a punctured neighborhood of  $a_0$ .

Fix  $x_0 \in U_0$  which is generic over  $a_0$ . Since  $(a_0, x_0)$  is generic in  $T_0 \times U_0$ , there exist an open  $T'_0 \ni a_0$  inside  $T_0$  and an open  $\hat{U}_0 \ni x_0$  inside  $U_0$  such that the map  $F(t, x) = f_t(x) \ominus f_{a_0}(x)$  is continuous on  $T'_0 \times \hat{U}_0$ . Because  $\dim T_0 = \dim W \geq 2$ , the set  $\hat{T}_0 = T'_0 \setminus \{a_0\}$  is still definably connected, and for each  $t \in \hat{T}_0$ , the function  $f_t \ominus f_{a_0}$  is open on  $\hat{U}_0$ . Given  $t_1 \neq t_2 \in \hat{T}_0$ , there exists a definable path  $p : [0, 1] \rightarrow \hat{T}_0$  connecting  $t_1$  and  $t_2$ , and by possibly shrinking  $\hat{U}_0$ , the induced map  $(s, x) \mapsto F(p(s), x)$  is a definable proper homotopy (see [27, Section 3.5.1]) of  $f_{t_1} \ominus f_{a_0}$  and  $f_{t_2} \ominus f_{a_0}$ , hence

by [27, Theorem 3.19], for every  $x$  generic in  $\hat{U}_0$ ,  $|J_x(f_{t_1} \ominus f_{a_0})|$  and  $|J_x(f_{t_2} \ominus f_{a_0})|$  have the same sign. It follows that

$$\sigma(f_{t_1} \ominus f_{a_0}) = \sigma(f_{t_2} \ominus f_{a_0}).$$

Thus, every invertible matrix in  $\hat{W}_0$  has the same determinant sign.

Next, note that for every invertible  $A \in \hat{W}_0$  sufficiently close to 0, the matrix  $A^2$  is also in  $\hat{W}_0$  and clearly has positive determinant. Thus all invertible matrices in  $\hat{W}_0$  have positive determinant.

Finally, as we saw in the proof of Proposition 7.11,  $\mathfrak{R} = \{AB^{-1} : A, B \in \hat{W}_0, B \text{ invertible}\}$ , and hence all invertible matrices in  $\mathfrak{R}$  have positive determinant. ■

*Proof of Theorem 7.13.* Assume first that every non-zero  $A \in \mathfrak{R}$  is invertible, so  $\mathfrak{R}$  is a definable division ring. It follows from [30, Theorem 4.1] that  $\mathfrak{R}$  is definably isomorphic to either  $R$  or  $R(\sqrt{-1})$  or the ring of quaternions over  $R$ . Because  $\dim \mathfrak{R} \geq 2$ , we are left with the last two possibilities. The ring of quaternions,  $H(R)$ , is not definably isomorphic to a definable subring of  $M_2(R)$ . Indeed, if  $A \subseteq M_2(R)$  is such a ring then, as we have seen in the proof of Proposition 7.11, it contains all scalar matrices. Because  $H(R)$  has o-minimal dimension 4 the same must be true of  $A$ , so  $A$  is a 4-dimensional  $R$ -vector subspace of  $M_2(R)$ . So  $A = M_2(R)$ , but the latter is not isomorphic to  $H(R)$  since  $M_2(R)$  is not a division ring.

So  $\mathfrak{R}$  is necessarily isomorphic to  $R(\sqrt{-1})$ . Since  $R(\sqrt{-1}) \cong R \oplus iR$  and  $\mathfrak{R}$  is a subring of  $M_2(R)$ , we immediately see that  $\mathfrak{R}$  is generated, as a vector space over  $R$ , by the diagonal matrices and some matrix  $M(i)$  such that  $M(i)^2 = -1$ . It follows that the eigenvalues of  $M(i)$  are  $\pm i$ , so  $M(i)$  is diagonalizable and conjugate to  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , say via some matrix  $M$ . It is now immediate that  $\mathfrak{R}$  is of the form (1) with that  $M$ .

Now assume that there exists at least one matrix  $A$  that is not invertible, of rank 1. We want to show that there exists an invertible  $M \in M_2(R)$  such that  $\mathfrak{R}$  is as in (2).

We conjugate  $\mathfrak{R}$  by some fixed matrix so that  $A$ , written in columns, has the form  $(w, 0)$  for some  $w \in R^2$ . We now show that every matrix in  $\mathfrak{R}$  is of the form  $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$  for some  $a, b \in R$ . Consider the set

$$H = \{(u, 0) \in \mathfrak{R} : u \in R^2\}.$$

It is a definable  $R$ -vector subspace of  $\mathfrak{R}$  that is also closed under ring multiplication. As an  $R$ -vector space it has positive dimension over  $R$  (since  $H$  is a non-trivial subring of  $\mathfrak{R}$ ) and  $\dim_R H \leq 2$ .

**Claim 1.**  $\dim_R H = 1$ .

*Proof of Claim 1.* Write the matrices in  $H$  in the form  $B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ , and note that for  $C = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$ ,

$$|B + C| = |C| + (af - bd). \tag{7.1}$$

Assume towards a contradiction that  $\dim H = 2$ , so  $H$  consists of *all* matrices of the form  $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ . We may now take  $C = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathfrak{R}$  invertible, sufficiently close to 0, and since

$d, f$  cannot both be 0, it is easy to see that by choosing  $a, b$  appropriately, we may obtain a matrix  $B + C \in \mathfrak{R}$  whose determinant is negative, a contradiction. ■

Thus,  $H$  is a 1-dimensional  $R$ -vector space.

**Claim 2.** *There is no  $\alpha \in \mathbb{R}$  such that all matrices in  $H$  are of the form  $B = \begin{pmatrix} a & 0 \\ \alpha a & 0 \end{pmatrix}$ .*

*Proof of Claim 2.* Assume that there is such an  $\alpha$ . Take any invertible  $C = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathfrak{R}$ . Then for every  $B = \begin{pmatrix} a & 0 \\ \alpha a & 0 \end{pmatrix} \in H$ , we have  $|B + C| = |C| + (af - \alpha ad)$ . By choosing  $a$  appropriately, we obtain  $|B + C| < 0$  (contradicting Proposition 7.15), unless  $f = \alpha d$ . Hence  $f = \alpha d$  and  $C = \begin{pmatrix} c & d \\ e & \alpha d \end{pmatrix} \in \mathfrak{R}$ . We have

$$\begin{pmatrix} c & d \\ e & \alpha d \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ \alpha a & 0 \end{pmatrix} = \begin{pmatrix} a(c + \alpha d) & 0 \\ a(e + \alpha^2 d) & 0 \end{pmatrix}.$$

But the left hand side is in  $\mathfrak{R}$ , and  $H$  is the collection of all matrices in  $\mathfrak{R}$  of the form  $(u, 0)$  for  $u \in R^2$ . Hence,  $\begin{pmatrix} a(c + \alpha d) & 0 \\ a(e + \alpha^2 d) & 0 \end{pmatrix} \in H$ . By assumption,  $\alpha(c + \alpha d) = e + \alpha^2 d$ , implying that  $e = \alpha c$ . However, this would make  $C$  non-invertible, a contradiction. ■

We are thus left with the case that all matrices in  $H$  are of the form  $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ . If we now take an arbitrary  $\begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathfrak{R}$  and multiply it on the right by a non-zero element of  $H$ , we obtain another element of  $H$ , forcing  $d$  to be 0. Thus all matrices in  $\mathfrak{R}$  are lower triangular. Because  $H$  contains all matrices of the form  $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ , every matrix in  $\mathfrak{R}$  can be written as the sum of a diagonal matrix in  $\mathfrak{R}$  and a matrix in  $H$ .

**Claim 3.** *The diagonal matrices in  $\mathfrak{R}$  are precisely the scalar matrices.*

*Proof of Claim 3.* The set of diagonal matrices in  $\mathfrak{R}$  is a definable additive subgroup of the ring of all diagonal matrices

$$\Delta := \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in R \right\}.$$

We have already seen that all scalar matrices are in  $\mathfrak{R}$ , so if  $\mathfrak{R}$  contains any matrix in  $\Delta$  that is not scalar, it contains all of  $\Delta$ , contradicting Proposition 7.15. ■

It follows that all matrices in  $\mathfrak{R}$  are of the form  $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ , as required. This ends the proof of Theorem 7.13. ■

#### 7.4. From ring to field

**Definition 7.16.** We say that  $\mathfrak{R}$  is of *analytic form* if it satisfies (1) of Theorem 7.13.

Our goal in this section is to prove that case (2) of Theorem 7.13 contradicts the strong minimality of  $\mathcal{D}$ . Thus, our negation assumption is that there exists a matrix  $M \in \text{GL}(2, R)$  such that all matrices in  $M^{-1}\mathfrak{R}M$  are of the form

$$\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \quad \text{for } a, b \in R. \tag{7.2}$$

Let us first note that we may assume that all matrices in  $\mathfrak{R}$  itself are of the form (7.2). Indeed, if  $f \in \mathfrak{F}$ , we have defined  $J_0 f$  with respect to some fixed atlas giving  $G$  its definable differentiable manifold structure. Let  $g$  be the chart in that atlas mapping a neighborhood  $U \subseteq R^2$  onto a neighborhood of 0. Consider  $h : M^{-1}U \rightarrow G$  given by  $x \mapsto g(Mx)$ . Since  $h$  is a diffeomorphism, we can replace  $g$  in our atlas with  $h$ . Denoting by  $\mathfrak{R}_h$  the ring  $\{J_0 f : f \in \mathfrak{F}\}$  with respect to this new atlas we see that  $\mathfrak{R}_h = M^{-1}\mathfrak{R}M$ , as needed.

For  $G = \langle R^2, + \rangle$ , [5, Corollary 2.18] immediately eliminates the possibility that  $\mathfrak{R}$  is the ring of upper triangular matrices. The goal of this subsection is to prove an analogue of that result in the context of an arbitrary group  $G$ .

We first need the following version of the uniqueness of definable solutions to definable ODEs. It can be easily deduced from [22, Theorem 2.3].

**Proposition 7.17.** *Let  $\text{Gr}(k, n)$  be the space of all  $k$ -dimensional linear subspaces of  $R^n$ . Let  $U \subseteq R^n$  be an open set and assume that  $L : U \rightarrow \text{Gr}(k, n)$  is a definable  $C^3$ -function assigning to each  $p \in U$  a  $k$ -dimensional space  $L_p$ . Assume that  $C_1, C_2 \subseteq U$  are definable  $k$ -dimensional smooth manifolds such that for every  $p \in C_1 \cap C_2$ , the tangent space of  $C_i$  at  $p$  equals  $L_p$ . Then for every  $p \in C_1 \cap C_2$  there exists a neighborhood  $V \ni p$  such that  $C_1 \cap V = C_2 \cap V$ .*

**Definition 7.18.** *A definable vector field on an open  $U \subseteq G$  is given by a definable partial function  $F : U \rightarrow T(U)$  from  $U$  to its tangent bundle  $T(U)$  such that  $F(g) \in T_g(G)$  for every  $g \in G$ .*

Every definable non-vanishing vector field  $F$  on  $U$  gives rise to a definable line field, still denoted by  $F$ , where to each  $g \in U$  we assign the 1-dimensional subspace of  $T_g(U)$  spanned by  $F(g)$ .

We say that a line field  $F$  is (left)  $G$ -invariant if for all  $g, h \in U$ ,

$$F(h) = d_g(\ell_{hg^{-1}}) \cdot F(g).$$

Given a line field  $F$ , we say that a definable smooth 1-dimensional set  $C \subseteq U$  is a trajectory of  $F$  if for every  $g \in C$ , the tangent space to  $C$  at  $g$  is  $F(g)$ .

**Lemma 7.19.** *Let  $F$  be a definable non-vanishing  $G$ -invariant line field. Assume that  $C \subseteq G$  is a definably connected smooth 1-dimensional trajectory of  $F$ . Then  $C$  is a coset of a definable local subgroup of  $G$ .*

*Proof.* Recall that we identify an open neighborhood  $U$  of 0 with an open subset of  $R^2$ , and  $T(U)$  is identified with  $U \times R^2$ . The line field can be viewed as a map  $F : U \rightarrow \text{Gr}(1, 2)$ .

It will suffice to show that if  $h \in C$ , then  $h \oplus C$  is a local subgroup. Hence we may assume that  $0 \in C$ . Since  $F$  is left-invariant, for any  $g \in G$ ,  $g \oplus C$  is also a trajectory of  $F$ . By Proposition 7.17,  $C$  and  $g \oplus C$  coincide on some neighborhood of  $g$ , provided that  $g \in C$ . It follows that for every  $x \in C$  and  $g \in C$  sufficiently small, we also have  $x \oplus g \in C$ . Thus  $C$  is a local subgroup of  $G$ . ■

We can now return to our main goal: proving that  $\mathfrak{R}$  is of analytic form. Recall that we assume that for a  $\mathcal{D}$ -function  $f$  and  $b \in \text{dom}(f)$  we can write

$$\tilde{J}_b f = \begin{pmatrix} \alpha_f(b) & 0 \\ \beta_f(b) & \alpha_f(b) \end{pmatrix}. \tag{7.3}$$

When  $f$  is clear from the context we omit the subscript  $f$ .

Let  $v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in T_0(G)$  and consider the non-vanishing  $G$ -invariant vector field  $F$  given by

$$\{d_0 \ell_b \cdot v_0 : b \in G\}.$$

For  $b \in G$ , let  $v_b = d_0 \ell_b \cdot v_0 \in T_b(G)$ .

**Lemma 7.20.** *For every  $\mathcal{D}$ -function  $f : U \rightarrow G$  and  $b \in \text{dom}(f)$ , we have*

$$d_b f \cdot v_b = \alpha(b)v_{f(b)} \in Rv_{f(b)},$$

so the line field induced by  $F$  is invariant under  $df$ .

If in addition  $\alpha(b) = 0$ , then

$$d_b f \cdot T_b(G) \subseteq Rv_{f(b)}.$$

*Proof.* By assumption on the form of matrices in  $\mathfrak{R}$ , we have  $\tilde{d}_b f \cdot v_0 = \alpha(b) \cdot v_0$ . Writing  $\tilde{d}_b f$  explicitly (and composing on the left with  $d_0 \ell_{f(b)}$ ), we obtain

$$(d_b f)(d_0 \ell_b) \cdot v_0 = \alpha(b)(d_0 \ell_{f(b)} \cdot v_0),$$

which implies the first clause.

For the second clause, notice that the special form of  $\tilde{d}_b f$  implies that if  $\alpha(b) = 0$ , then  $\tilde{d}_b f \cdot v \in Rv_0$  for every  $v \in T_0(G)$ . The result easily follows. ■

**Lemma 7.21.** *Assume that  $f : U \rightarrow G$  is a  $\mathcal{D}$ -function, and that  $C \subseteq U$  is a definable smooth curve which is a trajectory of  $F$ . Then so is  $f(C)$ .*

*Proof.* By the first clause of Lemma 7.20, the image of  $C$  under  $f$  is also a trajectory of  $F$ . ■

**Lemma 7.22.** *Assume that  $f$  is a  $\mathcal{D}$ -function, and  $C \subseteq \text{dom}(f)$  is a definable smooth curve such that  $\alpha(b) = 0$  at every  $b \in C$  (in the above notation). Then for every generic  $b \in C$ , the tangent space of  $f(C)$  at  $f(b)$  is the  $R$ -span of  $v_{g(b)}$ , so  $f(C)$  is a trajectory of  $F$  in a neighborhood of  $f(b)$ .*

*Proof.* Consider the restriction of  $f$  to  $C$ , and pick a generic  $b \in C$ . Since  $b$  is generic, the map  $f|_C : C \rightarrow f(C)$  is a submersion, so  $T_{f(b)}(C) = d_b f \cdot T_b(C)$ . By the second clause of Lemma 7.20, we conclude that  $T_{f(b)}(f(C))$  equals  $Rv_{f(b)}$ . ■

**Lemma 7.23.** *There exists a  $\mathcal{D}$ -function  $h$  and a definable curve  $C \subseteq G$  such that  $h(C)$  is a trajectory of  $F$ .*



*Proof.* This is similar to the proof of the claim in Proposition 7.12. Fix any  $\mathcal{D}$ -function  $f$ , and  $\alpha_f$  as in (7.3) above.

**Claim.** *There is  $a_0 \in G$  such that  $\alpha_f(b) = \alpha_f(b \oplus a_0)$  for infinitely many  $b \in G$ .*

*Proof of Claim.* For  $r \in R$ , let  $C_r = \{b \in G : \alpha_f(b) = r\}$ . Pick  $r$  generic in the image of  $\alpha_f$ . By Proposition 7.12 (and in the notation of that proposition),  $\dim J(U) = 2$ . Therefore, genericity of  $r$  implies that  $C_r$  is 1-dimensional. Consider  $D_r = C_r \oplus C_r$ . If  $D_r$  is still 1-dimensional, then as we have already seen several times,  $C_r$  is contained in a coset of a  $\checkmark$ -definable subgroup  $H_r$ , and then picking  $a_0 \in H_r$  small enough will work with any  $b \in C_r$ .

Otherwise,  $D_r$  is 2-dimensional. We may now pick  $a_0 \in D_r$  generic over  $r$ . Since  $r$  is still generic over  $a_0$ , there are infinitely many  $r'$  such that  $a_0 \in D_{r'}$ . For each such  $r'$ , there exists  $b \in C_{r'}$  with  $b \oplus a_0 \in C_{r'}$ . ■

Fix  $a_0$  as above, and consider the  $\mathcal{D}$ -function  $h(x) = f(x \oplus a_0) \ominus f(x)$ . It is easily verified that for each  $b \in G$  we have

$$\check{J}_b h = \check{J}_b(f(x \oplus a_0) \ominus f(x)) = \check{J}_b f(x \oplus a_0) - \check{J}_b f(x).$$

It follows that  $\alpha_h(b) = 0$  for every  $b \in G$  such that  $\alpha_f(b \oplus a_0) = \alpha_f(b)$ . Let  $C$  be the collection of all those elements  $b$ . By the claim,  $C$  is a curve. By Lemma 7.22, the curve  $h(C)$  is a trajectory of  $F$  near  $b$ . ■

We can now deduce the following theorem.

**Theorem 7.24.** *The ring  $\mathfrak{R}$  is of analytic form.*

*Proof.* We still work under the negation assumption that we are in Case (2) of Theorem 7.13. Using Lemmas 7.23 and 7.19 we obtain a definable local subgroup  $H$  which is a trajectory of the vector field  $F$ , and thus all of its cosets are also trajectories of  $F$ . Let  $U$  be a neighborhood of 0 which can be covered by cosets of  $H$ , all trajectories of  $F$ .

Fix any  $\mathcal{D}$ -function  $f \in \mathfrak{F}$  which is not  $G$ -affine. By Lemma 7.21, for every  $a \in U$  such that  $f(a) \in U$ , the image  $f(H \oplus a)$  is also a coset of  $H$ . Fix  $a_0 \in H$  close enough to 0 and consider the  $\mathcal{D}$ -function  $k(x) = f(x) \ominus f(x \oplus a_0)$ . Since  $f$  is not  $G$ -affine, the function  $k$  is not constant.

Notice that for every  $x$  sufficiently close to 0, the elements  $x$  and  $x \oplus a_0$  belong to the coset  $x \oplus H$ , and therefore as we just noted,  $f(x)$  and  $f(x \oplus a_0)$  belong to the same coset of  $H$ . It follows that  $k(x) \in H$  and therefore  $k$  sends an open subset of  $G$  into  $H$ , contradicting strong minimality (the preimage of some point will be infinite). ■

Note that the above argument does not really use the definability of the trajectory  $C$  but merely its existence. Thus, if we worked over the reals, then we could have used the usual existence theorem for solutions to differential equations in order to derive a contradiction.

### 8. Some intersection theory for $\mathcal{D}$ -curves

Our ultimate goal is to show, under suitable assumptions, that if two plane curves  $C, D \subseteq G^2$  are tangent at some point  $p$ , and  $C$  belongs to a  $\mathcal{D}$ -definable family  $\mathcal{F}$  of plane curves, then by varying  $C$  within  $\mathcal{F}$  one gains additional intersection points with  $D$ , near the point  $p$  (see Proposition 8.12(2)). This will allow us to detect tangency  $\mathcal{D}$ -definably.

The main tool towards this end is the following theorem, whose proof will be carried out in this section via a sequence of lemmas.

**Theorem 8.1.** *Assume that  $f$  is in  $\mathfrak{F}$ . If  $J_0(f) = 0$ , then there is no neighborhood of 0 on which  $f$  is injective.*

We now digress to report on an unsuccessful strategy, which nevertheless may be of some interest.

#### 8.1. Digression: on almost complex structures

Let  $K = R(\sqrt{-1})$ . In analogy to the notion of an almost complex structure on a real manifold, we may define a *definable almost  $K$ -structure* on a definable  $R$ -manifold  $M$  to be a definable smooth linear  $J : TM \rightarrow TM$  sending each  $T_x(M)$  to  $T_x(M)$  and such that  $J^2 = -1$ .

Note that every definable  $K$ -manifold admits a natural almost  $K$ -structure, induced by multiplication of each  $T_x(M)$  by  $i = \sqrt{-1}$ . It is known that when  $K = \mathbb{C}$  any 2-dimensional almost complex structure is isomorphic, as an almost complex structure, to a complex manifold. The proof of this result seems to be using integration and thus we do not expect it to hold for almost  $K$ -structures in arbitrary o-minimal expansions of real closed fields.

Returning now to our 2-dimensional group  $G$ , we can endow  $G$  with a definable almost  $K$ -structure in the following way. Just as we did at the beginning of Section 7.4, we may first assume that every matrix in  $\mathfrak{R}$  has the form  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ . Next, we naturally identify  $T_0(G)$  with  $R^2 \sim K$  and let  $J : T_0(G) \rightarrow T_0(G)$  be defined by  $J(x, y) = (x, -y)$ . Next, use the differential of  $\ell_a$  to obtain  $J : TG \rightarrow TG$  as required. Note that since  $TG$  is a trivial tangent bundle, this step can be carried out for any definable group of even dimension. However in the case of  $G$ , our choice of  $J$  and the fact that for each  $\mathcal{D}$ -function  $f$ ,  $\tilde{J}_a f$  has analytic form, imply that  $f$  is  *$J$ -holomorphic*: for each  $a \in \text{dom}(f)$  we have

$$J \circ d_a f = d_{f(a)} \circ J.$$

Now, if our underlying real closed field  $R$  were the field of real numbers, then  $G$  would be isomorphic as an almost complex structure to a complex manifold  $\hat{G}$ , and this isomorphism would send every  $J$ -holomorphic function from  $G$  to  $G$  to a holomorphic function from  $\hat{G}$  to  $\hat{G}$ . In particular, by our observation above, every  $\mathcal{D}$ -function would be sent to a holomorphic function. This would give an immediate proof of Theorem 8.1, due to the fact that the result is true for holomorphic maps.

Unfortunately, we do not know how to prove for arbitrary  $K$  that every 2-dimensional almost  $K$ -manifold is (definably) isomorphic to a  $K$ -manifold, and hence we cannot use

the theory of  $K$ -holomorphic maps in order to deduce Theorem 8.1. We thus use a different strategy.

8.2. *A motivating example*

If  $f$  were holomorphic, then the above theorem would follow from the argument principle and the open mapping theorem. Since our functions are not necessarily holomorphic, we describe a different, more topological proof of Theorem 8.1 for an analytic function  $f$ : we let  $h(z) = f(z)/z$  (complex division) for  $z \neq 0$  and  $h(0) = 0$ . The assumption that  $J_0 f = 0$  implies that  $h$  is continuous at 0 and hence holomorphic. Thus  $h$  is either locally constant or an open map in a neighborhood of 0. Now, if  $h$  were locally constant, then  $f \equiv 0$  near 0 and thus is clearly non-injective, so assume that  $h$  is an open map.

We now consider the complex function  $M(z, w) = z \cdot w$ , and for  $a, b \in \mathbb{C}$  near 0, let  $M_{a,b}(z) = M(z - a, h(z) - b)$ . Notice that  $M_{0,0}(z) = f(z)$ . Let  $\text{deg}_0(f)$  be the local degree of  $f$  at 0 (see details below). Since the local degree is preserved under definable homotopy (see Fact 8.2 below), it follows from the general theory that  $\text{deg}_0(M_{a,b}) = \text{deg}_0(f)$  for sufficiently small  $a, b$ . Because each  $M_{a,b}$  is holomorphic, the sign of  $|J_z M_{a,b}|$  is positive at a generic  $z$  in a small disc around 0, and therefore

$$\text{deg}_0(M_{a,b}) \geq |M_{a,b}^{-1}(w)|$$

for all  $w$  close to 0.

If we take  $w = 0$ , then we get

$$|M_{a,b}^{-1}(0)| \geq 2$$

(the points  $a$  and  $h^{-1}(b)$  being two such preimages), implying  $\text{deg}_0(f) = \text{deg}_0(M_{a,b}) \geq 2$ . This implies that  $f$  is not locally injective near 0.

Our objective is to imitate the above proof, using  $\mathcal{D}$ -functions instead of holomorphic ones. The main obstacle is that we do not have multiplication or division in  $\mathcal{D}$ , so we want to produce a  $\mathcal{D}$ -function which sufficiently resembles the multiplication function  $M$ .

8.3. *Topological preliminaries*

Throughout this section we will be implicitly using the o-minimal version of Jordan’s plane curve theorem (see [40]). We recall some definitions and results (see [26, Sections 2.2–2.3]). Given a circle  $C \subseteq \mathbb{R}^2$ , a definable continuous  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $w \notin f(C)$ , we let  $W_C(f, w)$  denote the winding number of  $f$  along  $C$  around  $w$ . If  $f^{-1}(w)$  is finite,  $p \in \mathbb{R}^2$  and  $f(p) = w$ , then  $\text{deg}_p(f)$  is defined to be  $W_C(f, f(p))$  for all sufficiently small  $C$  around  $p$ . We need the following results.

**Fact 8.2.** *Let  $C \subseteq \mathbb{R}^2$  be a circle oriented counterclockwise.*

- (1) *If  $\{f_t : t \in T\}$  is a definable continuous family of functions with  $w \notin f_t(C)$  for any  $t \in T$  and  $T$  definably connected, then  $W_C(f_{t_1}, w) = W_C(f_{t_2}, w)$  for all  $t_1, t_2 \in T$ .*

- (2) Assume that  $C$  is a circle around  $p$ ,  $f : C \rightarrow R^2$  definable and continuous, and  $w_1, w_2$  are in the same component of  $R^2 \setminus f(C)$ . Then  $W_C(f, w_1) = W_C(f, w_2)$ .
- (3) If  $f$  is definable and  $R$ -differentiable at  $p$  and  $J_p f$  is invertible, then  $\deg_p(f)$  is either 1 or  $-1$ , depending on whether  $|J_p(f)|$  is positive or negative.
- (4) Assume that  $f$  is a definable  $\mathcal{M}$ -smooth, open map, finite-to-one in a neighborhood  $U$  of  $p$  and that  $f(z) \neq f(p)$  for all  $z \neq p$  in  $U$ . Assume also that  $J_z f$  is invertible of positive determinant for all generic  $z \in U$ .

Let  $C \subseteq U$  be a circle around  $p$ . Then for all  $w \in f(\text{int}(C))$ , if  $w$  and  $f(p)$  are in the same component of  $R^2 \setminus f(C)$ , then  $W_C(f, f(p)) \geq |f^{-1}(w) \cap \text{int}(C)|$ , and if  $w$  is also generic, then  $W_C(f, f(p)) = |f^{-1}(w) \cap \text{int}(C)|$ .

*Proof.* (1) follows from [26, Lemma 2.13(4)]. (2) is just [26, Lemma 2.15]. The proof of (3) is the same as the classical one, so we omit it.

(4) It follows from (2) that  $W_C(f, f(p)) = W_C(f, w)$ . We let  $\{z_1, \dots, z_k\} = f^{-1}(w) \cap \text{int}(C)$ . By [26, Lemma 2.25],  $W_C(f, w) = \sum_{i=1}^k \deg_{z_i}(f)$ , so it is sufficient to see that  $\deg_{z_i}(f) \geq 1$  for each  $i$ . We fix a small circle  $C_i$  around  $z_i$  such that  $W_{C_i}(f, w) = \deg_{z_i}(f)$ , and then fix a generic  $w_0 \in f(\text{int}(C_i))$  sufficiently close to  $w$ , so in particular the Jacobian of  $f$  at each preimage of  $w_0$  is invertible of positive determinant. By [26, Lemma 2.25],  $\deg_{z_i}(f) = \sum_j \deg_{p_j}(f)$ , where the  $p_j$  are the preimages of  $w_0$  in  $\text{int}(C)$ . By (3), for each  $p_j$ , we have  $\deg_{p_j}(f) = 1$ , thus  $\deg_{z_i}(f) = |f^{-1}(w_0)| \geq 1$ .

The same argument shows that for generic  $w_0$  near  $p$ , we have  $W_C(f, f(p)) = |f^{-1}(w_0)|$ . ■

### 8.4. Back to $\mathcal{D}$ -functions

We still identify an open neighborhood of  $G$  with an open subset of  $R^2$  and identify  $0_G$  with  $0 = (0, 0)$ . For an open set  $U \subseteq G$  and a function  $f : U \rightarrow G$ , sending  $x_0$  to  $y_0$ , we say that  $f$  is *generically  $k$ -to-one at  $x_0$*  if for every open  $V \ni x_0$  and  $W \ni y_0$  there exists an open  $y_0 \in W_0 \subseteq W$  such that  $|f^{-1}(y) \cap V| = k$  for any generic  $y \in W_0$ .

Below we use the notion of a  $\mathcal{D}$ -function  $M$  from an open  $U \subseteq G^2$  into  $G$ . By that we mean that there exists a  $\mathcal{D}$ -definable set  $S \subseteq G^2 \times G$  of Morley rank 2 containing the graph of  $M$ .

**Lemma 8.3.** *Let  $U \subseteq G^2$  be a definable open neighborhood of  $(0, 0)$ . Assume that  $M : U \rightarrow G$  is a continuous  $\mathcal{D}$ -function such that  $M(0, y) = M(x, 0) = 0$  for all  $x, y$  close enough to 0. Assume that  $f, h \in \mathfrak{F}$  and:*

- (1) *For any  $a, b$  in some neighborhood of 0, the function  $g_{a,b}(x) = M(f(x) \ominus a, h(x) \ominus b)$  is not locally constant near 0.*
- (2)  *$f$  and  $h$  are, respectively, generically  $k$ -to-one and  $m$ -to-one near 0.*

*Then  $g(x) = M(f(x), h(x))$  is, generically, at least  $k + m$ -to-one near 0.*

*Proof.* By the assumptions on  $f, h$  and  $M$ , for  $a, b$  in some neighborhood of 0, the function  $g_{a,b}$  is a non-locally-constant  $\mathcal{D}$ -function on some neighborhood of 0, so it is

continuous and its graph is contained in a rank-1  $\mathcal{D}$ -definable set. By Theorem 4.17, it is open as well. Since it is definable in  $\mathcal{D}$  and not locally constant, it is finite-to-one near 0. Also, it follows from Corollary 7.15 and Proposition 7.24 that the Jacobian matrix of  $g_{a,b}$  has positive determinant at every point where the matrix does not vanish, which by Lemma 7.1 is a co-finite set.

We now fix a simple closed curve  $C$  around 0 such that  $0 \notin g(C) = g_{0,0}(C)$  and  $\text{deg}_0(g) = W_C(g, 0)$ . By continuity of  $M$  and  $g$  we can find an open  $U_1 \ni 0$  and an open disc  $U_2 \ni 0$  such that for all  $a, b \in U_1$ ,  $g_{a,b}(0) \in U_2$  and  $g_{a,b}(C) \cap U_2 = \emptyset$ . It follows that  $g_{a,b}(0)$  and 0 are in the same component of  $R^2 \setminus g_{a,b}(C)$ .

Take independent generics  $a, b \in U_1$ . By Fact 8.2,

$$\text{deg}_0(g) = W_C(g, 0) = W_C(g_{a,b}, 0) = W_C(g_{a,b}, g_{a,b}(0)) \geq |g_{a,b}^{-1}(0)|.$$

Because  $a, b$  are independent generics,  $f^{-1}(a) \cap h^{-1}(b) = \emptyset$ . Also, by our assumptions on  $M$  and the definition of  $g_{a,b}$ , we have

$$f^{-1}(a) \cup h^{-1}(b) \subseteq g_{a,b}^{-1}(0).$$

Hence,  $|g_{a,b}^{-1}(0)| \geq m + k$ . It follows from Fact 8.2(4) that  $\text{deg}_0(g) \geq m + k$  and  $g$  is generically at least  $k + m$ -to-one near 0. ■

### 8.5. Producing the function $M$

We now proceed to construct the desired  $\mathcal{D}$ -function  $M$  as in Lemma 8.3. We start with a  $\mathcal{D}$ -function  $k(x)$  which is not  $G$ -affine and fix a generic  $a_0 \in \text{dom } k$ . Define

$$M(x, y) = (k(a_0 \oplus x \oplus y) \ominus k(a_0 \oplus x)) \ominus (k(a_0 \oplus y) \ominus k(a_0)).$$

We write  $M_a(y) = M(a, y)$ .

By definition,

(A) For  $x, y$  near 0,  $M(0, y) = M(x, 0) = 0$ .

Our next goal is to show that  $M$  can be used, similarly to multiplication, to “divide (an appropriate) function  $f$  by  $x$ ”, so that we can implicitly solve  $M(x, y) = f(x)$  in some neighborhood of  $x = 0$ . This is the purpose of the next few results.

By Theorem 7.24 and the discussion in Section 7.4, we may assume that for a smooth  $f \in \mathfrak{F}$ , the matrix  $J_0 f$  has the form

$$\begin{pmatrix} c & -e \\ e & c \end{pmatrix} \quad \text{with } c, e \in R.$$

We consider the partial definable map  $d : G \rightarrow R^2$ , mapping  $a$  to the first column of the Jacobian matrix  $J_0 M_a$  (so if  $J_0 M_a = \begin{pmatrix} c & -e \\ e & c \end{pmatrix}$ , then  $d(a) = (c, e)$ ). Note that  $d(a)$  completely determines  $J_0 M_a$ , and in particular  $d(0) = 0$  if and only if  $J_0 M = 0$ . By Lemma 7.5 (and using the fact that  $\tilde{J}_0 f = J_0 f$ ), we find that

$$\begin{aligned} J_0 M_a &= \tilde{J}_0([k(a_0 \oplus a \oplus y) \ominus k(a_0 \oplus a)]) - \tilde{J}_0([k(a_0 \oplus y) \ominus k(a_0)]) \\ &= \tilde{J}_{a_0 \oplus a} k - \tilde{J}_{a_0} k. \end{aligned} \tag{†}$$

By Proposition 7.12 applied to  $k(x)$ , the image of every open  $U \ni 0$  under  $x \mapsto \tilde{J}_x k$  is a 2-dimensional subset of  $\mathfrak{R}$ , hence by o-minimality this map is locally injective near the generic  $a_0$ . Equivalently, the map  $x \mapsto \tilde{J}_{a_0 \oplus x} k$  is locally injective near 0. Since  $\tilde{J}_{a_0} k$  is constant, it follows that  $d(x)$  is locally injective at 0. In particular,

(B)  $d(0) = 0$ , and there is a neighborhood of 0 where  $d(a) \neq 0$  for all  $a \neq 0$ .

We are going to use several different norms in the next argument, so we set

$$\|(x, y)\| = \sqrt{x^2 + y^2},$$

and for a linear map  $T$  we denote the operator norm by

$$\|T\|_{\text{op}} = \max \{\|T(x)\|/\|x\| : x \neq 0\}.$$

Observe that  $\|d(a)\| = \|J_0 M_a\|_{\text{op}}$ . It is well-known (and easy to see) that if we identify every linear map with a  $2 \times 2$  matrix, then  $\|T\|_{\text{op}}$  and  $\|T\|$  are equivalent norms.

We need an additional property of  $M$ . Given two functions  $\alpha, \beta : U^* \rightarrow R^{\geq 0}$  on a punctured neighborhood  $U^* \subseteq R^2$  of 0, we write  $\alpha \sim \beta$  if  $\lim_{t \rightarrow 0} \alpha(t)/\beta(t)$  is a positive element of  $R$ . We will show:

(C) There are definable  $R^{>0}$ -valued functions  $e(a)$  and  $\delta(a)$ , in some punctured neighborhood  $U^*$  of 0, with  $e(a) \sim \|d(a)\|$  and  $\delta(a) \sim \|d(a)\|^2$ , such that for every  $a \in U^*$ , the function  $M_a = M(a, -)$  is invertible on the disc  $B_{e(a)}$  and its image contains the disc  $B_{\delta(a)}$  (recall that for  $a = 0$  we have  $M_a(x) \equiv 0$  near 0).

In order to prove (C), we use an effective version of the inverse function theorem, as appearing in [38, §7.2]. We give the details, with references to [38].

**Proposition 8.4.** *There exists a constant  $C > 0$  such that if  $e(a) = \|d(a)\|/(4C)$  and  $\delta(a) = e(a)^2/2$ , then for all  $a$  in a small punctured neighborhood of 0 the function  $M_a(y)$  is injective on  $B_{e(a)}$  and its image contains a ball of radius  $\delta(a)$  around 0.*

*Proof.* We start with some observations. If

$$A = J_0 M_a = \begin{pmatrix} c & -e \\ -e & c \end{pmatrix}$$

then  $\|A\|_{\text{op}} = \sqrt{c^2 + e^2} = \|d(a)\|$ . And if  $A$  is invertible, then  $\|A^{-1}\|_{\text{op}} = 1/\|d(a)\|$ .

Consider the partial map  $D : G \times G \rightarrow R^4$  defined by  $D(a, y) = J_y M_a \in M_2(R)$ . For any  $a, y$ , we view  $D(a, y)$  both as a linear operator and a vector in  $R^4$ . Since  $M$  is a  $C^2$ -function,  $\|J_{(a,y)} D\|_{\text{op}}$  is bounded by some constant  $C$  as  $(a, y)$  varies in a neighborhood  $B_1 \times B_2$  of  $(0, 0)$ , and we may assume that  $C > 1$ . By [38, Lemma 7.2.8] applied to  $D$ , for all  $(a_1, y_1), (a_2, y_2) \in B_1 \times B_2$  we have

$$\|J_{y_1} M_{a_1} - J_{y_2} M_{a_2}\| < C \|(a_1, y_1) - (a_2, y_2)\|. \tag{*}$$

Note also that  $D(0, 0) = J_0 M_0 = 0$ , so restricting further  $B_1, B_2$  we may also assume that  $\|D(a, y)\| < 1$  for all  $(a, y) \in B_1 \times B_2$ .

We now need a version of [38, Lemma 7.2.10].

**Lemma 8.5.** *For every  $a \in B_1$  such that  $J_0M_a$  is invertible, and for all  $y_1, y_2 \in B_2$ , if  $\|y_1\|, \|y_2\| \leq e(a)$ , then:*

(1) *The matrices  $J_{y_1}M_a$  and  $J_{y_2}M_a$  are invertible.*

(2)  $\|M_a(y_1) - M_a(y_2)\| \geq e(a)\|y_1 - y_2\|$ . *In particular,  $M_a$  is injective on  $B_{e(a)}$ .*

*Proof.* We fix  $a$  with  $J_0M_a$  invertible and we write  $J_0M_a = \begin{pmatrix} c & e \\ -e & c \end{pmatrix}$ . By (\*), for every  $y \in B_2$  and every  $E > 0$ , if  $\|y\| < E/(2C)$ , then

$$\|J_yM_a - J_0M_a\| \leq C\|y\| \leq C\|(y, a) - (0, a)\| < E/2.$$

In particular, since  $J_0M_a \neq 0$ , we may take  $E < \|d(a)\| = \|J_0M_a\|_{\text{op}}$  and then  $J_yM_a$  must be non-zero. Because  $M_a$  is a  $\mathcal{D}$ -function it follows that  $J_yM_a$  is invertible.

Let  $c' = 1/\|(J_0M_a)^{-1}\|_{\text{op}}$ . As pointed out earlier, in our case

$$\|J_0(M_a^{-1})\|_{\text{op}} = \|(J_0M_a)^{-1}\|_{\text{op}} = 1/\|d(a)\|,$$

hence  $c' = \|d(a)\|$ . Now, for all non-zero vectors  $w$ , we have  $\|(J_0M_a)^{-1}(w)\| \leq \frac{1}{c'}\|w\|$ , so if we replace  $w$  with  $(J_0M_a)^{-1}(z)$ , we get  $c'\|z\| = \|d(a)\| \cdot \|z\| \leq \|J_0M_a(z)\|$ .

Hence, for any  $y_1, y_2 \in R^2$ ,

$$\|J_0M_a(y_1 - y_2)\| \geq \|d(a)\| \cdot \|y_1 - y_2\|. \tag{**}$$

By [38, Lemma 7.2.9], applied to the function  $M_a$ , we also have for all  $y_1, y_2 \in B_2$ ,

$$\|M_a(y_1) - M_a(y_2) - J_0M_a(y_1 - y_2)\| \leq \|y_1 - y_2\| \max_{t \in [y_1, y_2]} \|J_tM_a - J_0M_a\|_{\text{op}},$$

where  $[y_1, y_2]$  is the line segment in  $R^2$  connecting  $y_1$  and  $y_2$ . Hence, by the triangle inequality,

$$\|M_a(y_1) - M_a(y_2)\| \geq \|J_0M_a(y_1 - y_2)\| - \|y_1 - y_2\| \max_{t \in [y_1, y_2]} \|J_tM_a - J_0M_a\|_{\text{op}}.$$

Putting this together with (\*) and (\*\*), we find that if  $y_1, y_2 \in B_2$  and  $\|y_i\| < E/(2C)$  for  $i = 1, 2$ , then

$$\|M_a(y_1) - M_a(y_2)\| \geq (\|d(a)\| - C\|y_1 - y_2\|)\|y_1 - y_2\|.$$

If in addition  $\|y_1 - y_2\| < \|d(a)\|/(2C)$ , then

$$\|M_a(y_1) - M_a(y_2)\| \geq \left(\|d(a)\| - \frac{\|d(a)\|}{2}\right)\|y_1 - y_2\| = \frac{\|d(a)\|}{2}\|y_1 - y_2\|. \tag{***}$$

We summarize what we have shown so far: for any  $E < \|d(a)\|$ , if  $\|y_1\|, \|y_2\| < E/(2C)$  and  $\|y_1 - y_2\| < \|d(a)\|/(2C)$ , then  $J_yM_a$  is invertible and (\*\*\*) holds.

We now fix the parameters as follows: set  $E = \|d(a)\|/2$ ,  $e(a) = \|d(a)\|/(4C) = E/(2C)$ . So, if  $\|y_1\|, \|y_2\| < E/(2C)$ , then  $\|y_1 - y_2\| < E/C = \|d(a)\|/(2C)$ , so we may apply (\*\*\*) and conclude that  $J_{y_i}$  is invertible for  $i = 1, 2$  and

$$\|M_a(y_1) - M_a(y_2)\| \geq \frac{\|d(a)\|}{2}\|y_1 - y_2\| \geq e(a)\|y_1 - y_2\|. \quad \blacksquare$$

By the proof of [38, Theorem 2.11],

$$\{y : \|y - M_a(0)\| < e(a)^2/2\} \subseteq \{M_a(z) : \|z\| < e(a)\}$$

(apply the claim of [38, p. 113, second line] with  $\epsilon, c$  there both replaced with  $e(a)$  here, and our  $M_a$  substituted for  $f$  there). Thus, the image of the disc  $B_{e(a)}$  under  $M_a$  contains a disc of radius  $e(a)^2/2$  around  $M_a(0) = 0$ . We do not repeat the proof here. ■

### 8.6. Proving Theorem 8.1

We now fix a  $\mathcal{D}$ -function  $M : G^2 \rightarrow G$  satisfying conditions (A)–(C) above, with  $d(x)$  the first column of  $J_0 M_x$ . We first need a simple observation.

**Fact 8.6.** *Assume that  $f : R^2 \supseteq U \rightarrow R^2$  is a definable  $C^2$ -function sending 0 to 0. If  $J_0 f = 0$ , then  $\lim_{x \rightarrow 0} \|f \circ f(x)\|/\|x\|^2 = 0$ .*

*Proof.* As already mentioned, the operator norm and the Euclidean norm on  $R^2$  are equivalent – so we may work with either.

We first claim that there exists some neighborhood  $U$  of 0 and a constant  $C$  such that  $\|f(x)\| \leq C\|x\|^2$  for all  $x \in U$ .

We use the following corollary of [38, Lemma 7.2.8]: Let  $g$  be a definable  $C^1$ -map from an open ball  $B \subseteq R^m$  centered at 0 into  $R^n$  with  $g(0) = 0$ . Then for all  $x \in B$ ,

$$\|g(x)\| \leq \sup_{a \in B(0; \|x\|)} \|J_a g\| \|x\|. \tag{8.1}$$

We now consider the map  $\alpha(a) = J_a f$ , as a map from an open ball  $B$  around  $0 \in G$  (identified with a ball centered at  $(0, 0) \in R^2$ ) into  $R^4$ . Since  $f$  is a  $C^2$ -map the map  $\alpha$  is a  $C^1$ -map and hence, by (8.1), there is some constant  $C$  (a bound on the norm of  $d_a(\alpha)$  as  $a$  varies in  $B$ ) such that for all  $a \in B$ ,

$$\|J_a f\| = \|\alpha(a)\| \leq C\|a\|.$$

It follows that  $\sup_{a \in B(0; \|x\|)} \|J_a f\| \leq C\|x\|$  for all  $x \in B$ .

Next, we apply (8.1) to the map  $f$  itself and conclude, using what we have just shown, that for all  $x \in B$ ,

$$\|f(x)\| \leq \sup_{a \in B(0; \|x\|)} \|J_a f\| \|x\| \leq C\|x\|^2.$$

This ends the proof of our first claim.

It now follows that  $\|f(f(x))\| \leq C\|f(x)\|^2 \leq C^2\|x\|^4$ . This yields the desired conclusion. ■

We also need the following lemma.

**Lemma 8.7.** *If  $x(t) : (a, \epsilon) \rightarrow R^2$  is a definable curve tending to 0 as  $t \rightarrow 0$ , then  $\lim_{t \rightarrow 0} \|d(x(t))\|/\|x(t)\| \neq 0$ .*



*Proof.* Recall that  $d$  is a map from  $U$  into  $R^2$  mapping  $a$  to the first column of  $J_0M_a$ , and recall that  $d(a)$  completely determines  $J_0M_a$ .

We claim that  $J_0(d)$  is invertible. Indeed, we have seen in (†) above (Section 8.5) that  $J_0M_a = \tilde{J}_{a_0 \oplus a}k - \tilde{J}_{a_0}k$ . By Proposition 7.12, the function  $a \mapsto \tilde{J}_a f$  is a diffeomorphism in a small neighborhood of the generic point  $a_0$  onto an open subset of  $\mathfrak{R}$ . Since  $x \mapsto a_0 \oplus x$  is a diffeomorphism (between open subsets of  $G$ ) in a neighborhood of 0, we get that  $a \mapsto \tilde{J}_{a_0 \oplus a}k$  is a diffeomorphism near 0 between an open subset of  $G$  and  $\mathfrak{R}$ . Since  $\tilde{J}_{a_0}k$  is a constant matrix, it follows that  $a \mapsto J_0M_a$  is a diffeomorphism near 0. Thus, from the special form of  $J_0M_a$  and the definition of  $d$  we deduce that  $J_0d$  is invertible.

It follows from the definition of the differential that

$$\lim_{t \rightarrow 0} \left( \frac{d(x(t))}{\|x(t)\|} - \frac{J_0d \cdot x(t)}{\|x(t)\|} \right) = 0.$$

As  $J_0d$  is invertible, the limit of  $\frac{J_0d \cdot x(t)}{\|x(t)\|}$  is a non-zero vector, and so

$$\lim_{t \rightarrow 0} \frac{\|d(x(t))\|}{\|x(t)\|} \neq 0. \quad \blacksquare$$

**Corollary 8.8.** *Let  $e(a)$  and  $\delta(a)$  be as in Proposition 8.4. Assume that  $f : G \rightarrow G$  is a smooth non- $G$ -linear  $\mathcal{D}$ -function such that  $f(0) = 0$  and  $J_0f = 0$ . Let  $g = f \circ f$ . Then there is an open neighborhood  $U \ni 0$  such that for all non-zero  $a \in U$ , we have:*

- (i)  $\|g(a)\| < \delta(a)$ .
- (ii) *There exists a unique  $y \in B_{e(a)}$  such that  $M(a, y) = g(a)$ .*

*Proof.* Assume that (i) fails. Then there exists a definable function  $x(t)$  tending to 0 in  $G$  such that for all  $t$ ,

$$\|g(x(t))\| \geq \delta(x(t)) = \|d(x(t))\|^2 / (32C^2).$$

Because  $J_0f = 0$ , Fact 8.6 implies that  $\lim_{t \rightarrow 0} \|g(x(t))\| / \|x(t)\|^2 = 0$ . Combined with the above inequality we get  $\lim_{x(t) \rightarrow 0} \|d(x(t))\|^2 / \|x(t)\|^2 = 0$ , hence  $\lim_{t \rightarrow 0} \|d(x(t))\| / \|x(t)\| = 0$ , contradicting Lemma 8.7. Thus there exists  $U \ni 0$  such that for all  $a \in U \setminus \{0\}$  we have  $\|g(a)\| < \delta(a)$ . It now follows from our choice of  $\delta(a)$  that there exists a unique  $y \in B_{e(a)}$  such that  $M(a, y) = g(a)$ . ■

**Corollary 8.9.** *Let  $f$  and  $g$  be as above, and  $e(a)$ ,  $\delta(a)$  as in (C) above. Let  $U = \{x : \|g(x)\| < \delta(a)\}$  and  $U^* = U \setminus \{0\}$ . For every  $x \in U^*$ , let  $h(x)$  be the unique  $y$  in  $B_{e(a)}$  such that  $M(x, y) = g(x)$ . Then:*

- (i)  $U$  contains an open disc around 0.
- (ii)  $h$  is differentiable on  $U^*$  and  $\lim_{x \rightarrow 0} h(x) = 0$  (so it extends continuously to 0). Moreover, if  $f$  is not constant, then neither is  $h$ .
- (iii) *The continuous extension of  $h$  to  $U$  is a  $\mathcal{D}$ -function.*

*Proof.* Clause (i) is just Corollary 8.8. To see that  $h$  is differentiable everywhere we apply the Implicit Function Theorem to  $M(x, y) - g(x)$ . By Lemma 8.5,  $J_y M_x$  is invertible for every  $x \in U^*$  and  $|y| < e(a)$ , so indeed  $h(x)$ , the solution to  $M(x, y) - g(x) = 0$ , is differentiable at  $x$ .

To see that the limit of  $h$  at 0 is 0, we compute the limit along an arbitrary curve  $x(t)$  tending to 0. By definition,  $|h(x(t))| < \|e(t)\| \sim \|d(t)\|$ , so since  $d(0) = 0$ , also  $h(x(t))$  must tend to 0. The second clause of (ii) follows since if  $h$  were constant with  $h(0) = 0$  then necessarily  $h$  would vanish on its domain, implying that  $g$  was identically 0 (because  $M(x, 0) = 0$  for all  $x$ ). Because  $f$  is not constant its image is infinite, and because it is a  $\mathcal{D}$ -function it follows that also  $g = f \circ f$  is non-constant.

For (iii), note that the graph of  $h$  is contained in the plane curve  $B = \{(x, y) : (x, y, g(x)) \in \hat{M}(x, y)\}$  where  $\hat{M}$  is a  $\mathcal{D}$ -definable set of Morley rank 2 containing the graph of  $M$ . ■

We note that locally near the point  $(0, 0)$  itself, the  $\mathcal{D}$ -definable set  $B$  need not be the graph of a function, but this does not come up in the argument.

*Proof of Theorem 8.1.* Assume that  $f \in \tilde{\mathfrak{F}}$  and  $J_0(f) = 0$ . We will show that  $f$  is not injective near 0.

Consider  $g(x) = f(f(x))$ , and assume towards a contradiction that  $f$  and thus also  $g$  is injective near 0. By Corollary 8.9, there exists a  $\mathcal{D}$ -function  $h$  in a neighborhood  $U$  of 0, with  $h(0) = 0$ , such that for all  $x \in U$ ,

$$M(x, h(x)) = g(x).$$

We now wish to apply Lemma 8.3 to the functions  $x \mapsto x$  and  $x \mapsto h(x)$ . For that we just need to note that for  $a$  and  $b$  near 0 the function  $g_{a,b}(x) = M(x \ominus a, h(x) \ominus b)$  is non-constant near 0. Indeed, we can find a fixed definably connected open  $W \ni 0$  such that  $W \subseteq \text{dom}(g_{a,b})$  for all  $a, b$  close to 0. Since each  $g_{a,b}$  is a  $\mathcal{D}$ -function, its graph is contained in a strongly minimal set, and hence if it were constant near 0, it would have to be constant on the whole of  $W$ . But then, by the continuity of  $M$ , the function  $g = g_{0,0}$  must also be constant on  $W$ , a contradiction.

By applying Lemma 8.3, we conclude that  $g(x)$  is at least  $1 + k$ -to-one near 0, where  $k \geq 1$ . This contradicts the assumption that  $f$  and thus  $g$  were locally injective. ■

The following example shows that the proof of Theorem 8.1 uses more than just the basic geometric properties of the function  $f$ .

**Example 8.10.** A crucial point in our above argument was that  $f(z)/z$ , or in the language of our proof, the implicitly defined function  $h$ , is an open map. This followed from the fact that it was a  $\mathcal{D}$ -function.

Consider the function  $f(z) = |z|^2 z$  from  $\mathbb{C}$  to  $\mathbb{C}$ . It is smooth everywhere,  $J_0 f = 0$ , and yet it is injective everywhere. However, the function  $f(z)/|z|$  is clearly not an open map.

8.7. Intersection theory in families

Based on the topological properties we have established thus far we can develop some intersection theory resembling that of complex analytic curves.

**Definition 8.11.** Let  $X, Y$  be two plane curves, and  $p = (p_1, p_2) \in X \cap Y$ . We say that  $X$  and  $Y$  are *tangent at  $p$*  if there are  $\mathcal{D}$ -functions  $f, g$  which are  $C^1$  in a neighborhood of  $p_1$ , with  $\Gamma_f \subseteq X$  and  $\Gamma_g \subseteq Y$ , such that

$$f(p_1) = p_2 = g(p_1) \quad \text{and} \quad J_{p_1} f = J_{p_1} g.$$

The following proposition is the key technical tool for identifying tangency in the reduct  $\mathcal{D}$ . The first part of the proposition uses mainly the topological properties of  $\mathcal{D}$ -functions to show that if  $X, E$  are  $\mathcal{D}$ -plane curves intersecting generically enough, then the number of intersection points cannot drop under slight perturbations of the curves. The second part of the proposition uses the differential properties of  $\mathcal{D}$ -functions (and in particular Theorem 8.1) to show that if  $X$  and  $E$  are tangent at a point, then the number of intersection points is expected to increase under slight perturbations.

**Proposition 8.12.** Let  $\mathcal{F} = \{E_a : a \in T\}$  be a  $\mathcal{D}$ -definable almost faithful family of plane curves,  $\mathcal{D}$ -definable over  $\emptyset$ , and let  $X$  be a strongly minimal plane curve not almost a straight line.

Assume that  $a$  is generic in  $T$  over  $\emptyset$ ,  $E_a$  strongly minimal,  $X \cap E_a$  is finite and  $p = (x_0, y_0) \in E_a \cap X$ .

- (1) If  $p$  is  $\mathcal{D}$ -generic in  $E_a$  over  $a$ , non-isolated on  $E_a$ , non-isolated in  $X$  and also  $\mathcal{D}$ -generic in  $X$  over  $[X]$ , then for every neighborhood  $U \ni p$ , there is a neighborhood  $V \ni a$  in  $T$  such that for every  $a' \in V$ ,  $E_{a'}$  intersects  $X$  in  $U$ .
- (2) (Here we do not make any genericity assumptions on  $p$ .) Assume that for some open  $W \ni a$ , whenever  $a' \in W$  the set  $E_{a'}$  represents a  $\mathcal{D}$ -function  $f_{a'}$  in a neighborhood of  $(x_0, y_0)$  and the map  $(a', x) \mapsto f_{a'}(x)$  is continuous at  $(a, x_0)$ . Assume also that  $X$  represents a function  $g$  at  $p$  and  $J_{x_0} f_a = J_{x_0} g$ . Then for every neighborhood  $U \ni p$  there is a neighborhood  $V \ni a$  in  $T$  such that for every  $a' \in V$ , either  $E_{a'}$  and  $X$  are tangent at some point in  $U$  or  $|E_{a'} \cap X \cap U| > 1$ .

*Proof.* (1) Fix an open  $U = U_1 \times U_2 \ni p$  definably connected. Since  $p$  is non-isolated and  $\mathcal{D}$ -generic in  $E_a$  over  $a$ , it follows from Corollary 6.3, applied to  $E_a$ , that there are three possibilities: (i)  $E_a$  is locally at  $p$  the graph of a constant function in the first variable, (ii)  $E_a$  is locally at  $p$  the graph of a constant function in the second variable, or (iii)  $E_a$  is locally at  $p$  the graph of a homeomorphism.

In all cases,  $E_a$  is locally at  $p$  either the graph of a continuous function mapping  $x_0$  to  $y_0$  or vice versa. Since our assumptions on  $p$  are symmetric with respect to the coordinates, we may assume that there is an open  $U = U_1 \times U_2 \ni p$  such that  $E_a$  is locally the graph of a continuous function  $f_a : U_1 \rightarrow U_2$ .

Since, in addition,  $a$  is generic in  $T$  over  $\emptyset$ , we may shrink  $U$  and find an open definably connected  $V_0 \ni a$  in  $T$  such that for every  $a' \in V_0$ , the set  $E_{a'} \cap U_1 \times U_2$  is the graph

of a  $\mathcal{D}$ -function  $f_{a'} : U_1 \rightarrow U_2$  and furthermore, the map  $(a', x) \mapsto f_{a'}(x)$  is continuous on  $V_0 \times U_1$ .

Since  $p$  is not isolated in  $X$ ,  $\mathcal{D}$ -generic in  $X$  over  $[X]$ , and the projections of  $X$  on both coordinates are finite-to-one, it follows from Corollary 6.3, applied to  $X$ , that, after possibly shrinking  $U$  further, the set  $X \cap U$  is the graph of an open continuous map  $g : U_1 \rightarrow U_2$ .

Notice that for every  $a' \in V_0$  and  $(x, y) \in U$ ,

$$(x, y) \in E_{a'} \cap X \iff f_{a'}(x) \ominus g(x) = 0.$$

Because  $X \cap E_a$  is finite, the function  $f_a \ominus g$  is not constant on its domain, so by Theorem 4.17,  $f_a \ominus g$  is open on  $U_1$ .

**Claim.** *There exists  $V \ni a$  such that for every  $a' \in V \setminus \{a\}$ , the function  $f_{a'} \ominus g$  is an open map on  $U_1$ .*

*Proof of Claim.* Indeed, assume towards a contradiction that for  $a' \in V_0$  arbitrarily close to  $a$  the map  $f_{a'} \ominus g$  is not open. Thus, by Theorem 4.17, it is constant on  $U_1$ . It follows from continuity that  $f_a \ominus g$  is constant on  $U_1$ , contradicting our assumption. ■

Thus, we have shown that there exists  $V \ni a$  such that for all  $a' \in V$ , the function  $f_{a'} \ominus g$  is open and finite-to-one on  $U_1$ . In addition, the map  $(a', x) \mapsto (f_{a'} \ominus g)(x)$  is continuous in a neighborhood  $(a, x_0)$ . Because  $0 \in (f_a \ominus g)(U)$  it follows from Fact 8.2(1,4) that for some open  $V_0 \ni a$  small enough and for all  $a' \in V_0$ , the set  $(f_{a'} \ominus g)(U)$  contains 0, so  $X \cap E_{a'} \cap U \neq \emptyset$ . This ends the proof of (1).

(2) Let  $g$  be a  $\mathcal{D}$ -function with  $g(x_0) = y_0$  such that  $\Gamma_g \subseteq X$  and  $J_{x_0} f_a = J_{x_0} g$ . Note that  $(f_a \ominus g)(x_0) = 0$  and  $f_a \neq g$ . So, for  $C \subseteq G$  a sufficiently small circle around  $x_0$  the only zero of  $f_a \ominus g$  in the closed ball  $B$  determined by  $C$  is  $x_0$ . By continuity of  $(x, a') \mapsto f_{a'}(x)$ , we may find some neighborhood  $V \subseteq W$  of  $a$  such that for every  $a' \in V$ ,  $0 \notin (f_{a'} \ominus g)(C)$ . It follows from Fact 8.2(1) that

$$W_C(f_{a'} \ominus g, 0) = W_C(f_a \ominus g, 0)$$

for every  $a' \in V$ .

By our assumptions,  $J_{x_0}(f_a \ominus g) = 0$  and therefore by Theorem 8.1,  $f_a \ominus g$  is not injective in any neighborhood of  $x_0$ , that is, for every generic  $y$  near 0,

$$|(f_a \ominus g)^{-1}(y)| > 1.$$

It follows from Fact 8.2(4) that  $W_C(f_a \ominus g, 0) > 1$ . Thus,  $W_C(f_{a'} \ominus g, 0) > 1$  for all  $a' \in V$ .

We can now conclude that for every  $a'$ , either 0 is a regular value of the function  $f_{a'} \ominus g$  on  $\text{int}(C)$ , in which case it has more than one preimage and then  $E_{a'}$  and  $X$  intersect more than once in  $\text{int}(C)$ , or 0 is a singular value, in which case the curves  $E_{a'}$  and  $X$  are tangent at some point in  $\text{int}(C)$ . ■

**9. The main theorem**

We are now ready to prove our main result. Our proof follows that of [5, Theorem 7.3]. We begin with a series of useful technical facts. Throughout this section we let  $K := \mathfrak{R}$ .

**Lemma 9.1.** *There exist  $\mathcal{D}$ -definable families  $\mathcal{C}_0 = \{E_a^0 : a \in T_0\}$ ,  $\mathcal{C}_1 = \{E_b^1 : b \in T_1\}$  of plane curves all passing through  $(0, 0)$  such that:*

- (1) *For  $i = 0, 1$ ,  $T_i$  is strongly minimal and  $\mathcal{C}_i$  is almost faithful.*
- (2) *Every generic curve in  $\mathcal{C}_i$ ,  $i = 0, 1$ , is closed, strongly minimal and has no isolated points.*
- (3) *There are definable open neighborhoods  $U \subseteq G$  of  $0$  and definable open sets  $T'_0 \subseteq T_0$  and  $T'_1 \subseteq T_1$  such that for every  $i = 0, 1$  and  $a \in T'_i$ , the curve  $E_a^i$  represents a function  $f_a^i : U \rightarrow G$  in  $\mathfrak{F}$ .*
- (4) *For  $i = 0, 1$ , the sets*

$$W_i := \{J_0 f_a^i : a \in T'_i\}$$

*are open subsets of  $K$  with  $0 \in \text{cl}(W_0)$  and  $1 \in \text{cl}(W_1)$ .*

- (5) *For each  $i = 0, 1$ , the map  $(a, x) \mapsto f_a^i(x)$  is continuous on  $T'_i \times U$ .*

*Proof.* By Claim 7.9, there exists a  $\mathcal{D}$ -function  $f : U \rightarrow G$  which is not  $G$ -affine and such that  $J_0 f = 0$ . Let  $S \subseteq G^2$  be a strongly minimal set representing  $f$ . By Theorem 4.9, we may assume that  $S$  is closed, and by allowing parameters we may assume that  $S$  has no isolated points. Let

$$\mathcal{C}_0 = \{S \oplus p : p \in S\}.$$

Let  $T_0 := S$  and for  $a \in S$  let  $E_a^0 := S \oplus a$ .

For every  $a = (x_0, f(x_0)) \in S$ , the curve  $S_a$  represents the  $\mathcal{D}$ -function  $f(x \oplus x_0) \oplus f(x_0)$ . By Proposition 7.12, the set

$$W = \{J_0(f(x \oplus x_0) \oplus f(x_0)) : x_0 \in U\}$$

of elements of  $K$  has dimension 2, and by applying the same proposition to a smaller  $U$ , we see that  $J_0 f = 0$  is in the closure of a 2-dimensional component of  $W$ . By o-minimality, we can find an open  $U' \subseteq U$  such that the set  $W_0 = \{J_0(f(x \oplus x_0) \oplus f(x_0)) : x_0 \in U'\}$  is an open subset of  $K$  with the zero matrix in its closure. We let  $T'_0 := \{(x_0, f(x_0)) : x_0 \in U'\}$ . By its definition, the sets  $U'$ ,  $T'_0$  and  $W_0$  satisfy all clauses of the lemma.

In order to obtain  $\mathcal{C}_1$ , we replace  $f$  with the function  $h(x) = f(x) \oplus x$ . It is a  $\mathcal{D}$ -function which is not  $G$ -affine, with  $J_0 h = 1 \in K$ . We repeat the above process and obtain the rest of the lemma. ■

Our aim is to construct a field configuration in  $\mathcal{D}$  (see Definition 2.1). We will pull a field configuration from  $K$  into  $\mathcal{D}$  by using the properties of Jacobians of  $\mathcal{D}$ -functions as studied in the previous sections. Lemma 9.1 provides us with the families of curves we will be using to construct the field configuration. For simplicity of notation we will

absorb into the language all the parameters needed to define all the objects appearing in Lemma 9.1.

Observe that although  $W_0$  and  $W_1$  from Lemma 9.1, are not neighborhoods of 0 and 1, respectively, it is still the case that for every  $B \in W_0$ , if  $A \in W_1$  and  $C \in W_0$  are sufficiently close to 1 and 0, respectively, then  $AB + C$  is still in  $W_0$  (since  $W_0$  is open). Similarly, for every  $A \in W_1$ , if  $C \in W_1$  is sufficiently close to 1, then  $AC \in W_1$ .

Let  $e = (1, 0)$  be the identity of  $\mathbb{G}_m \times \mathbb{G}_a$ , and choose  $b \in W_0$  and  $h, g$  in  $W_1 \times W_0 \subseteq \mathbb{G}_m \times \mathbb{G}_a$  sufficiently close to  $e$  so that  $gh \in W_1 \times W_0$ , and  $h \cdot b$  and  $hg \cdot b$  are in  $W_0$ . Note that we may choose  $g, h, b$  to be independent generics in the sense of  $\mathcal{M}$  (and thus also independent in the sense of  $K$ ).

To simplify notation, we denote the functions in  $\mathcal{C}_0$  by  $f_i$  and the functions in  $\mathcal{C}_1$  by  $g_s$ , and abusing notation, we will sometimes write  $f \in \mathcal{C}_i$  for a  $\mathcal{D}$ -function  $f$  which is represented by a curve in  $\mathcal{C}_i$ . In particular, let us denote, for  $i = 1, 2$ ,

$$\mathcal{C}'_i = \{f'_t : t \in T'_i\}.$$

We will construct a field configuration of Jacobian matrices of  $\mathcal{D}$ -functions in  $\mathcal{C}'_0$  and  $\mathcal{C}'_1$ , and show that the corresponding configuration of parameters of  $\mathcal{D}$ -definable curves representing those  $\mathcal{D}$ -functions is a field configuration in  $\mathcal{D}$ .

We get the following corollary to Lemma 9.1.

**Corollary 9.2.** *There are  $a_1, a_2 \in W_1 \subseteq K$  and  $b, b_1, b_2 \in W_0 \subseteq K$  such that  $g = (a_1, b_1), h = (a_2, b_2) \in W_1 \times W_0$  and the following hold:*

- (1) *There exist  $g_1, g_2 \in \mathcal{C}'_1$  and  $f_1, f_2, k_1 \in \mathcal{C}'_0$  with  $J_0g_i = a_i$  (for  $i = 1, 2$ ) and  $J_0f_i = b_i$  (for  $i = 1, 2$ ) and  $J_0k_1 = b$ .*
- (2)  *$hg \in W_1 \times W_0$ , and there are  $f_3 \in \mathcal{C}'_0$  and  $g_3 \in \mathcal{C}'_1$  with  $(J_0g_3, J_0f_3) = hg$ .*
- (3) *There are  $k_2, k_3 \in \mathcal{C}'_0$  such that  $J_0k_2 = h \cdot b$  and  $J_0k_3 = hg \cdot b$ .*

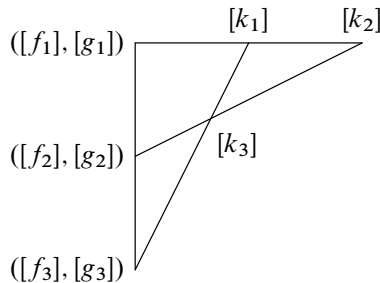
For a  $\mathcal{D}$ -function  $\Psi$ , we denote by  $[\Psi]$  the  $\mathcal{D}$ -canonical parameter of some fixed strongly minimal set representing it. Our goal is to prove the following proposition.

**Proposition 9.3.** *In the above notation,*

$$\mathcal{Y} := \{([f_1], [g_1]), ([f_2], [g_2]), ([f_3], [g_3]), [k_1], [k_2], [k_3]\} \tag{*}$$

*is a field configuration in  $\mathcal{D}$ .*

*Proof.* We have to verify that the following diagram satisfies (1)–(4) of Definition 2.1:



The families  $\mathcal{C}_0$  and  $\mathcal{C}_1$  are almost faithful, so for a function  $f_a \in \mathcal{C}'_0$  we have  $\text{acl}_{\mathcal{D}}(a) = \text{acl}_{\mathcal{D}}([f_a])$ . Since field configurations are stable with respect to  $\mathcal{D}$ -interalgebraicity (over  $\emptyset$ ), we may assume that  $a = [f_a]$ . The same is true for  $\mathcal{C}'_1$ .

By construction,  $\mathcal{Y}$  satisfies the assumptions of Lemma 2.4. So we are reduced to proving (3) of Definition 2.1. That is, we have to show that all lines in the above diagram represent  $\mathcal{D}$ -dependencies. For example, we have to show that  $\{[k_2], [k_3], ([f_2], [g_2])\}$  is  $\mathcal{D}$ -dependent, and similarly  $\{[k_1], [k_3], ([f_3], [g_3])\}$ , etc. Since all the arguments are similar, we only prove in detail the latter case.

It will suffice to show the following statement.

**Lemma 9.4.**  $k_3 \in \text{acl}_{\mathcal{D}}([f_3], [k_1], [g_3])$ .

*Proof.* The geometric idea behind it goes back to Eugenia Rabinovich’s work [35]. Write  $f_a = k_3$  with  $a \in T_0$ . By our assumptions,  $J_0 f_a$  is generic in  $K$  over  $\emptyset$ . To simplify the notation, we denote the curves in  $\mathcal{C}_0$  by  $E_{a'}$ ,  $a' \in T_0$ , and the curves in  $\mathcal{C}_1$  by  $C_g$ ,  $g \in T_1$ .

Let  $X$  be a strongly minimal subset of  $S := (E_{f_3} \circ E_{k_1}) \boxplus C_{g_3}$ , representing the function  $(f_3 \circ k_1) \oplus g_3$  (see Lemma 7.3 for the notation). We want to show that  $a \in \text{acl}_{\mathcal{D}}([S])$ . Assume towards a contradiction that this is not the case.

**Claim 1.** *The projections of  $X$  on both coordinates are infinite and all isolated points of  $X$  are in  $\text{acl}_{\mathcal{D}}([S])$ .*

*Proof of Claim 1.* By our choice of  $\mathcal{C}_0$ , the curves  $E_{f_3}$  and  $E_{k_1}$  are strongly minimal without isolated points. It follows that each of these curves has a finite intersection with every straight line, and thus  $E_{f_3} \circ E_{k_1}$  has no isolated points. Indeed, if  $(a, b) \in E_{f_3} \circ E_{k_1}$  there is some  $c$  such that  $(a, c) \in E_{k_1}$  and  $(c, b) \in E_{f_3}$ . Since these curves are not straight lines and have no isolated points, they are open over  $c$  at  $(a, c)$  and  $(c, b)$ , respectively (Corollary 6.3). So for every open  $U \ni (a, c)$  there is  $c' \in \pi_2(U \cap E_{k_1})$  distinct from  $c$ . Hence there is some  $a'$  such that  $(a', c') \in U \cap E_{k_1}$ . A similar argument will provide us with some  $(c', b') \in E_{f_3}$  so  $(a', b') \in E_{f_3} \circ E_{k_1}$  with  $(a', b')$  arbitrarily close to  $(a, b)$ .

Note also that the curve  $C_{g_3}$  has no isolated points. An argument similar to the one in the previous paragraph shows that the  $\boxplus$ -sum  $S$  of  $E_{f_3} \circ E_{k_1}$  and  $C_{g_3}$  has no isolated points either. Let  $I(X)$  be the set of isolated points of  $X$ . Let  $X' := X \setminus I(X)$ . Then, as  $S$  has no isolated points,  $I(X) \subseteq \text{cl}(S \setminus I(X))$ . But  $\text{cl}(S \setminus I(X)) = \text{cl}(S \setminus X) \cup \text{cl}(X')$ , and since  $I(X) \cap \text{cl}(X') = \emptyset$  we find that  $I(X) \subseteq \text{fr}(S \setminus X)$ . Since  $[X] \in \text{acl}_{\mathcal{D}}(S)$ , we deduce from Theorem 4.9 that  $\text{fr}(S \setminus X) \subseteq \text{acl}_{\mathcal{D}}([S])$ .

Since  $X$  is a strongly minimal set representing a function  $(f_3 \circ k_1) \oplus g_3$ , its projection on the first coordinate is finite-to-one. Since the function has a non-zero Jacobian matrix at 0, it is non-constant and hence has an infinite projection in the second coordinate as well. ■

It follows from Claim 1 that the (finite) set of isolated points of  $X$  is contained in a  $\mathcal{D}$ -algebraic set  $\mathcal{D}$ -definable over  $[S]$ . Thus, by removing this  $\mathcal{D}$ -definable set there is no loss of generality in assuming that  $X$  contains no isolated points.

Note that the assumption that  $a \notin \text{acl}_{\mathcal{D}}([S])$  implies that  $a \notin \text{acl}_{\mathcal{D}}([X])$ . We will ultimately show that this leads to a contradiction. Because  $E_a$  is strongly minimal, it follows that  $E_a \cap X$  is finite.

Since  $T_0$  is strongly minimal, there exists some natural number  $n$  such that  $|X \cap E_b| = n$  for all  $b \in T_0$  which are  $\mathcal{D}$ -generic over  $[X]$ . Thus, the set

$$F = \{b \in T_0 : |X \cap E_b| < n\}$$

is finite, defined in  $\mathcal{D}$  over  $[X]$ . We will show that  $a \in F$ , thus reaching a contradiction.

By our choice of  $a$ ,  $\dim(J_0 f_a / \emptyset) = 2 = \dim G$  and since  $J_0 f_a \in \text{dcl}(a)$ , we also have  $\dim(a / \emptyset) = 2$ . Thus we also have  $a \in \text{dcl}(J_0 f_a)$ .

**Claim 2.** *Let  $\{x_1, \dots, x_k\} := X \cap E_a$ . Then for every  $i = 1, \dots, k$ , either  $\text{RM}(x_i/a) = 1$ , or  $x_i \in \text{acl}_{\mathcal{D}}(\emptyset)$ .*

*Proof of Claim 2.* We consider the family

$$\mathcal{F}' = \{(E_{a_1} \circ E_{a_2}) \boxplus C_b : a_1, a_2 \in T_0, b \in T_1\},$$

and for simplicity write the members of  $\mathcal{F}'$  as  $\{X_t : t \in T\}$ . By our choice of  $X$ , there is  $t_0 \in T$  generic such that  $X$  is a strongly minimal subset of  $X_{t_0}$ , so definable over  $\text{acl}_{\mathcal{D}}(t_0)$ . We may now replace  $\mathcal{F}'$  by another family of the same dimension, defined over  $\emptyset$ , such that the generic member of  $\mathcal{F}'$  is strongly minimal and  $X$  belongs to the family. We call this new family  $\mathcal{F}$ .

Thus  $X = X_{t_0}$ , with  $\mathcal{F} = \{X_t : t \in T\}$  a  $\mathcal{D}$ -definable almost faithful family of plane curves, and  $t_0$  generic in  $T$  over  $\emptyset$ . Our underlying negation assumption implies that  $\text{RM}(a/t_0) = 1$ .

Assume now that  $\text{RM}(x_i/a) \neq 1$ . Since  $x_i \in E_a$  it follows that  $x_i \in \text{acl}_{\mathcal{D}}(a)$ . Because  $\text{RM}(a/t_0) = 1$  it follows that  $t_0$  is  $\mathcal{D}$ -generic in  $T$  over  $a$  and hence also over  $x_i$ . But then  $x_i$  is in  $X_t$  for every  $t$  which is  $\mathcal{D}$ -generic in  $T$ . This necessarily implies that  $x_i \in \text{acl}_{\mathcal{D}}(\emptyset)$  because there can only be finitely many points in  $G \times G$  belonging to every  $\mathcal{D}$ -generic curve  $X_t$ . This ends the proof of Claim 2. ■

We now return to the proof of Lemma 9.4. By Claim 2 we may assume that  $\text{RM}(x_i/a) = 1$  for  $i = 1, \dots, r$ , and  $x_i \in \text{acl}_{\mathcal{D}}(\emptyset)$  for  $i = r + 1, \dots, k$ . Without loss of generality,  $x_k = 0$ .

In order to show that  $a \in F$ , we have to show that  $k < n$ . Towards that end, we will show that there are infinitely many  $a' \in T_0$  such that  $n = |X \cap E_{a'}| \geq k + 1$ .

Let  $U_1, \dots, U_r, U_k$  be pairwise disjoint open neighborhoods of  $x_1, \dots, x_r, x_k$ , respectively. Since  $x_{r+1}, \dots, x_k$  are in  $\text{acl}_{\mathcal{D}}(\emptyset)$ , each of these points belongs to all but finitely many  $E_{a'}$ .

Because  $X$  and  $E_a$  have no isolated points, we can apply Proposition 8.12. We first apply Proposition 8.12(2) to  $0 = x_k$ , and obtain  $V \ni a$  such that for every  $a' \in V$ ,  $|E_{a'} \cap X \cap U_k| \geq 2$ , counted with multiplicity. Because  $J_0 E_a$  is generic in  $K$ , it is attained at most finitely many times and hence by choosing  $V$  sufficiently small and



$a' \in V, a' \neq a$ , the curves  $E_{a'}$  and  $X$  are not tangent at 0, so there exists  $p \in E_{a'} \cap X \cap U_k$  which is non-zero. It follows that for all but finitely many  $a' \in V, |E_{a'} \cap X \cap U_k| \geq 2$ .

We now apply Proposition 8.12(1) to  $x_1, \dots, x_r$ , and obtain a subneighborhood  $V'$  of  $a$  such that for every  $a' \in V'$  and  $i = 1, \dots, r, E_{a'} \cap X \cap U_i \neq \emptyset$ .

Summarizing, we see that for every  $a' \neq a$  close to  $a$ , we have  $|E_{a'} \cap X| \geq k + 1$ , and therefore  $a$  is in the finite set  $F$  defined above. This ends the proof of Lemma 9.4, and also the proof of Proposition 9.3. ■

We can now prove our main result.

**Theorem 9.5.** *Let  $\mathcal{D} = \langle G; \oplus, \dots \rangle$  be a strongly minimal expansion of a group  $G$ , interpretable in an o-minimal expansion  $\mathcal{M}$  of a field  $R$ , with  $\dim_{\mathcal{M}}(G) = 2$ . If  $\mathcal{D}$  is not locally modular, then there exists in  $\mathcal{D}$  an interpretable algebraically closed field  $K \simeq R(\sqrt{-1})$ , and there exists a  $K$ -algebraic group  $H$  such that  $G$  and  $H$  are definably isomorphic in  $\mathcal{D}$  and every  $\mathcal{D}$ -definable subset of  $H^n$  is  $K$ -constructible.*

*Moreover, the structure  $\mathcal{D}$  and the field  $K$  are bi-interpretable.*

*Proof.* By Proposition 9.3, the configuration  $\mathcal{Y}$  of  $(*)$  is a field configuration in  $\mathcal{D}$ .

By Fact 2.3, an algebraically closed field  $K$  is interpretable in  $\mathcal{D}$ . By strong minimality, there exists a  $\mathcal{D}$ -definable function  $f : G \rightarrow K$  with finite fibers (this is standard using the symmetric functions on  $K$ ). By [24, Lemma 4.6] (and using strong minimality of  $G$ ), there exists a finite subgroup  $F \leq G$  such that  $G/F$  is internal to  $K$  in the structure  $\mathcal{D}$  (it is in fact the proof of the lemma that provides us with the finite subgroup  $F$ ). By [26, Theorem 3.1], every  $\mathcal{D}$ -definable subset of  $K^n$  is  $K$ -constructible, and therefore  $G/F$  is  $\mathcal{D}$ -definably isomorphic to a  $K$ -constructible group. By Weil–Hrushovski [3, Theorem 1], it is therefore definably isomorphic to a  $K$ -algebraic group  $H$  (of algebraic dimension 1). It is known that  $H$ , as an algebraic curve with all its induced  $K$ -algebraic structure, is bi-interpretable with  $K$  (this follows, for example, from the main result of [13]). For the sake of completeness let us sketch this argument.

If  $C$  is an algebraic curve in  $K$ , then clearly  $C$  is interpretable in  $K$ . Since, up to finitely many points,  $C$  is affine, it is interalgebraic in  $K$  with  $K$ . Using this interalgebraicity, we can pull back any field configuration from  $K$  to  $C$  allowing us to interpret a field  $K'$  in  $C$  (with its  $K$ -induced structure). By [34, Theorem 4.15],  $K'$  is  $K$ -definably isomorphic to  $K$ , so in particular  $K$  is interpretable in  $C$ . Finally, the isomorphism from  $K$  to  $K'$  takes  $C$  to a  $C$ -interpretable curve  $C'$ . The induced map from  $C$  to  $C'$  is  $K$ -definable, hence it is definable in  $C$ . This shows that  $C$  and  $K$  are bi-interpretable.

So  $H$  and hence also  $G/F$ , with all its induced  $\mathcal{D}$ -structure, is bi-interpretable with  $K$ . By Lemma 3.12, the structure  $\mathcal{D}$  is also bi-interpretable with  $K$ . ■

### 9.1. Concluding remarks

Note that as a result of the main theorem, the almost  $K$ -structure on  $G$  which we introduced in Section 8.1 turns out to be definably isomorphic to the  $K$ -structure of the algebraic group  $H$ . Thus, in this very special setting, we are able to mimic the classical result about the integrability of 2-dimensional almost complex curves.

Also, note that the general o-minimal version of Zilber's conjecture remains open for general strongly minimal structures whose universe has dimension 2. As noted earlier, the more general conjecture, allowing underlying sets of arbitrary dimension, is open even for reducts of the complex field.

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