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Stability and collapse of the Lyapunov spectrum for Perron–Frobenius operator cocycles

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Abstract. In this paper, we study random Blaschke products, acting on the unit circle, and consider the cocycle of Perron–Frobenius operators acting on Banach spaces of analytic functions on an annulus. We completely describe the Lyapunov spectrum of these cocycles. As a corollary, we obtain a simple random Blaschke product system where the Perron–Frobenius cocycle has infinitely many distinct Lyapunov exponents, but where arbitrarily small natural perturbations cause a complete collapse of the Lyapunov spectrum, except for the exponent 0 associated with the absolutely continuous invariant measure. That is, under perturbations, the Lyapunov exponents become 0 with multiplicity 1, and $-\infty$ with infinite multiplicity. This is superficially similar to the finite-dimensional phenomenon, discovered by Bochi [4], that away from the uniformly hyperbolic setting, small perturbations can lead to a collapse of the Lyapunov spectrum to zero. In this paper, however, the cocycle and its perturbation are explicitly described; and further, the mechanism for collapse is quite different.

We study stability of the Perron–Frobenius cocycles arising from general random Blaschke products. We give a necessary and sufficient criterion for stability of the Lyapunov spectrum in terms of the derivative of the random Blaschke product at its random fixed point, and use this to show that an open dense set of Blaschke product cocycles have hyperbolic Perron–Frobenius cocycles.

In the final part, we prove a relationship between the Lyapunov spectrum of a single cocycle acting on two different Banach spaces, allowing us to draw conclusions for the same cocycles acting on C^r function spaces.

Keywords. Lyapunov exponents, transfer operators, random dynamical systems

1. Introduction

A well known technique for computing rates of decay of correlation for hyperbolic dynamical systems is based on identifying the spectral gap for its Perron–Frobenius operator, acting on a suitable Banach space of functions (different function spaces give rise to

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different spectral gaps, and so to different rates of decay). In particular, exceptional eigenvalues (those outside the essential spectral radius) play a key role in determining rates of decay of correlation.

In the autonomous case, Keller and Liverani [14] established, in a landmark paper based on spectral techniques, robust checkable conditions under which the exceptional eigenvalues, and their corresponding eigenspaces, vary continuously in response to small perturbations of the dynamics, or more precisely, of its Perron–Frobenius operator.

With Froyland, we have been working towards random analogues of the result of Keller and Liverani, where instead of the Perron–Frobenius operator of a single map, one has a random dynamical system (that is, a skew product of the *base dynamics* $\sigma : \Omega \to \Omega$ with ω-dependent maps, T_ω, in the fibres). One then forms the random linear dynamical system, or cocycle, where the fibre maps are now the Perron–Frobenius operators, $\mathcal{L}_{T_\omega}$, acting on a Banach X. In the case of cocycles, spectral techniques no longer apply. A significant motivation for this line of research comes from the fact that Froyland and collaborators have developed effective tools making use of finite-dimensional approximations of Perron–Frobenius cocycles to identify regions of interest in environmental dynamical systems (see e.g. [24]). Our program is aimed towards assessing the robustness of these methods.

The non-autonomous analogue of eigenvalues, Lyapunov exponents, are known to be far more sensitive to perturbations than eigenvalues. Based on an outline proposed by Mañé, Bochi showed in his thesis [4] that on any compact surface there is a residual set of C^1 area-preserving diffeomorphisms that either are Anosov or have all Lyapunov exponents equal to zero. Similar results are established for two-dimensional matrix cocycles. Bochi and Viana [5] extended this to higher-dimensional systems and cocycles, so that these results show that Lyapunov exponents are highly unstable.

On the other hand, Ledrappier and Young [16] showed that if one makes *absolutely continuous* perturbations to invertible matrix cocycles, small perturbations lead to small changes in the Lyapunov exponents. Ochs [21] then showed that in the finite-dimensional invertible case, small changes in Lyapunov exponents lead to small changes (in probability) in the Oseledets spaces, the non-autonomous analogue of (generalized) eigenspaces.

In [7], with Froyland, we gave fairly general conditions under which the top Oseledets space responds continuously to perturbations of the Perron–Frobenius operator cocycle. In [8], we extended the results of Ledrappier and Young, and of Ochs to the semi-invertible setting (where the base dynamics are assumed to be invertible, but no injectivity or invertibility assumptions are made on the operators). In [9], we (again with Froyland) also gave the first result on stability of Lyapunov exponents in an infinite-dimensional setting: we consider Hilbert–Schmidt cocycles on a separable Hilbert space with exponential decay of the entries. In this case, there is no single natural notion of noise, but we consider perturbations by additive noise with faster exponential decay. Again, we recover in [9] the stability of the Lyapunov exponents and Oseledets spaces. Another related result is due to Nakano and Wittsten [20], who study random perturbations of partially expanding maps of the torus (skew products of circle rotations over uniformly expanding maps of the circle) whose transfer operators have a spectral gap. Using semi-classical analysis, they show that the spectral gap is preserved under small random perturbations.

One could interpret the available results collectively as saying that carefully chosen perturbations may lead to radical change to the Lyapunov spectrum, while noise-like perturbations of the cocycle tend to lead to small changes to the Lyapunov spectrum.

The examples that we focus on in this work are expanding finite Blaschke products, a class of analytic maps from the unit circle to itself. The Perron–Frobenius operators for single maps of this type, acting on the Hardy space of a suitable annulus, were studied by Bandtlow, Just and Slipantschuk [3], where they used results on composition operators to obtain a precise description of the set of eigenvalues. Indeed, the eigenvalues they obtain are precisely the non-negative powers of the derivative of the underlying Blaschke product at its unique (attracting) fixed point in the unit disc, and their complex conjugates. We study a random version, where instead of a single Blaschke product, T, one applies a Blaschke product T_ω that is selected by the base dynamics.

In Section 4, we generalize the results of [3] to the non-autonomous setting, and show that the Lyapunov spectrum of the Perron–Frobenius cocycle is given by the non-negative multiples of the Lyapunov exponent of the underlying Blaschke product cocycle at the random attracting fixed point in the unit disc (with multiplicity 2 for all positive multiples of this exponent). To our knowledge, this is the first time that a complete description of an infinite Lyapunov spectrum of a Perron–Frobenius cocycle has been given. We find it quite remarkable that the Perron–Frobenius spectrum (describing what happens to densities on the unit circle) is governed in this way by the derivative at the random fixed point in the interior of the unit disc.

In Section 5, we show, to our surprise, that there are natural examples of random dynamical systems, where natural perturbations of the Perron–Frobenius cocycle lead to a collapse of the Lyapunov spectrum. We focus on a particular Blaschke product cocycle (with maps T_0 and T_1 applied in an i.i.d. way, where $T_0(z) = z^2$ and $T_1(z) = [(z + 1/4)/(1 + z/4)]^2$). The Perron–Frobenius operator of T_0 is known to be highly degenerate [1, Exercise 2.14]. If the frequency of applying T_0 is p, then we find a phase transition: for $p \geq 1/2$, the Lyapunov spectrum collapses (so that there is an exponent 0 with multiplicity 1, and all other Lyapunov exponents are $-\infty$), while for $p < 1/2$, there is a complete (infinite) Lyapunov spectrum. We then consider normal perturbations (corresponding to adding random normal noise to the dynamical system), $\mathcal{L}_0^\epsilon$ and $\mathcal{L}_1^\epsilon$ and show that for $p \geq 1/4$, there is collapse of the Lyapunov spectrum for all $\epsilon > 0$.

In particular, for $1/4 \leq p < 1/2$, the unperturbed system has a complete Lyapunov spectrum, while arbitrarily small normal perturbations have a collapsed Lyapunov spectrum. We also show that collapse can occur for every $p > 0$ in the setting of uniform perturbations. Unlike in Bochi's setting, our perturbations are explicitly described and arise naturally in the area.

In Section 6, we show that, under natural conditions on the underlying Blaschke product cocycle, the corresponding Perron–Frobenius cocycle has some hyperbolicity properties, guaranteeing a uniform separation between fast and slow Oseledets subspaces. This allows us to give a simple necessary and sufficient condition for stability of the Lyapunov spectrum. In particular, we show that the set of stable cocycles is open and dense.

In the Appendix we compare Lyapunov exponents and Oseledets splittings of random linear dynamical systems that arise from restricting the cocycle to a finer subspace. This allows us to study the action of the Perron–Frobenius cocycle on coarser Banach spaces such as C^r. (In the case of autonomous systems, this question has been addressed by Baladi and Tsujii [2].) We construct an explicit example of a cocycle of analytic maps of the circle (indeed, of Blaschke products) whose Perron–Frobenius operator, when acting on a C^r function space, has negative exceptional Lyapunov exponents. The corresponding situation in the autonomous case was first established by Keller and Rugh [15], and it was suggested in [3] that Blaschke products could also exhibit this phenomenon.

2. Statement of theorems

In this section, we state our main theorems. We will assume standard definitions, but for completeness, some terms used here will be defined in the next section.

For a finite Blaschke product, set $r_T(R) := \max_{|z|=R} |T(z)|$. By [27] if the restriction of a Blaschke product to the unit circle is expanding, then $r_T(R) < R$ for some $R < 1$. By the Hadamard three circle theorem, $R \mapsto r_T(R)$ satisfies the log-convexity property: $r_T(R_1^{1-\lambda} R_2^{\lambda}) \le r_T(R_1)^{1-\lambda} r_T(R_2)^{\lambda}$ for $0 < \lambda < 1$. In particular for Blaschke products $r_T(1) = 1$ so if $r_T(R) < R$ then $r_T(R') < R'$ for all $R' \in (R, 1)$. We let $\beta(T) = \inf\{R > 0 \colon r_T(R) < R\}$. Then $r_T(R) < R$ if and only if $R \in (\beta(T), 1)$.

For a Blaschke product cocycle $\mathcal{T} = (T_\omega)_{\omega \in \Omega}$, we define $r_{\mathcal{T}}(R) = \operatorname{ess\,sup}_{\omega \in \Omega} r_{T_\omega}(R)$. If $\mathcal{T}$ is finite, for $R \in (\operatorname{ess\,sup}_\omega \beta(T_\omega), 1)$ we have $r_{\mathcal{T}}(R) < R$. The condition $r_{\mathcal{T}}(R) < R$ will be imposed throughout the paper.

Theorem 2.1 (Lyapunov spectrum of a Blaschke product cocycle). *Let σ be an invertible ergodic measure-preserving transformation of a probability space $(\Omega, \mathbb{P})$. Let $R < 1$ and let $\mathcal{T} = (T_\omega)_{\omega \in \Omega}$ be a family of finite Blaschke products, depending measurably on ω, satisfying $r := r_{\mathcal{T}}(R) < R$. Let $\mathcal{L}_\omega$ denote the Perron–Frobenius operator of T_ω, acting on the Hilbert space $H^2(A_R)$ on the annulus $A_R \colon = \{z \colon R < |z| < 1/R\}$.*

Then the cocycle is compact and the following hold:

(1) (Random fixed point) *There exists a measurable map $x \colon \Omega \to \bar{D}_r$ (with $x(\omega)$ written as x_ω) such that $T_\omega(x_\omega) = x_{\sigma(\omega)}$. For all $z \in D_R$, $T^{(N)}_{\sigma^{-N}\omega}(z) := T_{\sigma^{-1}\omega} \circ \cdots \circ T_{\sigma^{-N}\omega}(z) \to x_\omega$.*

(2) (Critical random fixed point) *If $\mathbb{P}(\{\omega \colon T'_\omega(x_\omega) = 0\}) > 0$, then the Lyapunov spectrum of the cocycle is 0 with multiplicity 1, and $-\infty$ with infinite multiplicity.*

(3) (Generic case) *If $\mathbb{P}(\{\omega \colon T'_\omega(x_\omega) = 0\}) = 0$, then define $\Lambda = \int \log |T'_\omega(x_\omega)|\, d\mathbb{P}(\omega)$. This satisfies $\Lambda \le \log(r/R) < 0$.*

If $\Lambda = -\infty$, then the Lyapunov spectrum of the cocycle is 0 with multiplicity 1, and $-\infty$ with infinite multiplicity.

If $\Lambda > -\infty$, then the Lyapunov spectrum of the cocycle is 0 with multiplicity 1, and $n\Lambda$ with multiplicity 2 for each $n \in \mathbb{N}$. The Oseledets space with exponent 0 is spanned by $1/(z - x_\omega) - 1/(z - 1/\bar{x}_\omega)$. The Oseledets space with exponent $j\Lambda$ is spanned by two functions: one a linear combination of $1/(z - x_\omega)^2, \ldots, 1/(z - x_\omega)^{j+1}$ with a pole of order $j + 1$ at x_ω; the other a linear combination of $z^{k-1}/(1 - \bar{x}_\omega z)^{k+1}$ for $k = 1, \ldots, j$ with a pole of order $j + 1$ at $1/\bar{x}_\omega$.

Corollary 2.2. *Let $\Omega = \{0,1\}^{\mathbb{Z}}$, σ be the shift map, and $\mathbb{P}_p$ be the Bernoulli measure where $\mathbb{P}_p([0]) = p$. Let $\mathcal{L}_0$ be the Perron–Frobenius operator of $T_0\colon z \mapsto z^2$, $\mathcal{L}_1$ be the Perron–Frobenius operator of $T_1\colon z \mapsto [(z+1/4)/(1+z/4)]^2$, and consider the operator cocycle generated by $\mathcal{L}_\omega := \mathcal{L}_{\omega_0}$, acting on $H^2(A_R)$, where R satisfies $r_{\mathcal{T}}(R) < R$.*

(a) *If $p < 1/2$, then the cocycle has countably infinitely many finite Lyapunov exponents.*

(b) *If $p \geq 1/2$, then $\lambda_1 = 0$ and all remaining Lyapunov exponents are $-\infty$.*

We define an operator on $H^2(A_R)$ by

$$(\mathcal{N}_\epsilon f)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ze^{-2\pi i\epsilon t}) e^{-2\pi i\epsilon t - t^2/2}\, dt.$$

We show below that this operator corresponds to the operator on functions on $\mathbb{R}/\mathbb{Z}$ given by $\mathcal{N}_\epsilon^{\mathbb{R}/\mathbb{Z}} f(x) = \mathbb{E} f(x+\epsilon N)$, where N is a standard normal random variable. That is, $\mathcal{N}_\epsilon^{\mathbb{R}/\mathbb{Z}}$ acts on densities by convolution with a Gaussian with variance ϵ^2. Let $\mathcal{L}_\omega$ be the cocycle in the theorem above and consider the perturbation $\mathcal{L}_\omega^\epsilon$ of $\mathcal{L}_\omega$ given by $\mathcal{L}_\omega^\epsilon = \mathcal{N}_\epsilon \circ \mathcal{L}_\omega$.

Corollary 2.3 (Collapse of Lyapunov spectrum). *Let $\Omega = \{0,1\}^{\mathbb{Z}}$, equipped with the map σ and measure $\mathbb{P}_p$ as above. If $p \geq 1/4$, the perturbed cocycle $(\mathcal{L}_\omega^\epsilon)_{\omega\in\Omega}$ has $\lambda_1 = 0$ and $\lambda_j = -\infty$ for all $j > 1$. In particular, if $1/4 \leq p < 1/2$, then the unperturbed cocycle has a complete Lyapunov spectrum, but for each $\epsilon > 0$, the Lyapunov spectrum collapses.*

Remark. We remark that in the context of this corollary, we write down an explicit expression for the one-dimensional space with Lyapunov exponent 0.

Similarly, we define $\mathcal{L}_i^{U,\epsilon} := \mathcal{U}_\epsilon \circ \mathcal{L}_i$, where $\mathcal{U}_\epsilon^{\mathbb{R}/\mathbb{Z}}$ convolves densities with a bump function of integral 1 supported on $[-\epsilon, \epsilon]$,

$$(\mathcal{U}_\epsilon f)(z) = \frac{1}{2} \int_{-1}^{1} f(ze^{-2\pi i\epsilon t}) e^{-2\pi i\epsilon t}\, dt.$$

Corollary 2.4. *Let $\Omega = \{0,1\}^{\mathbb{Z}}$, equipped with the map σ and measure $\mathbb{P}_p$ as in Corollary 2.2. For every $p > 0$, there are arbitrarily small values of ϵ for which the perturbed cocycle $(\mathcal{L}_\omega^{U,\epsilon})_{\omega\in\Omega}$ has $\lambda_1 = 0$ and $\lambda_j = -\infty$ for all $j > 1$.*

The above corollaries give simple explicit examples and perturbations of Perron–Frobenius cocycles for which the Lyapunov spectrum collapses. We now give general conditions for stability and instability.

Theorem 2.5 (Stability of Lyapunov spectrum). *Let σ be an ergodic invertible measure-preserving transformation of $(\Omega, \mathbb{P})$. Let $R < 1$ and let $\mathcal{T} = (T_\omega)_{\omega\in\Omega}$ be a Blaschke product cocycle satisfying $r_{\mathcal{T}}(R) < R$.*

(a) *Suppose $\operatorname{ess\,inf}_\omega |T_\omega'(x_\omega)| > 0$. Then if $(\mathcal{L}_\omega)$ is the Perron–Frobenius cocycle of (T_ω) and $(\mathcal{L}_\omega^\epsilon)$ is a family of Perron–Frobenius cocycles such that $\operatorname{ess\,sup}_\omega \|\mathcal{L}_\omega^\epsilon - \mathcal{L}_\omega\|$*

$\to 0$ *as* $\epsilon \to 0$*, then the perturbed cocycle is quasi-compact and* $\mu_k^\epsilon \to \mu_k$ *as* $\epsilon \to 0$*, where* (μ_k) *is the sequence of Lyapunov exponents of* $(\mathcal{L}_\omega)$*, listed with multiplicity, and* (μ_k^ϵ) *is the sequence of Lyapunov exponents of* $(\mathcal{L}_\omega^\epsilon)$*.*

In addition, the Oseledets subspaces converge as $\epsilon \to 0$ *to those of the unperturbed cocycle.*

(b) *Suppose* $\operatorname{ess\,inf}_\omega |T_\omega'(x_\omega)| = 0$*. Then there exists a family of Blaschke product cocycles* $\mathcal{T}^\epsilon = (T_\omega^\epsilon)_{\omega\in\Omega}$ *such that* $\operatorname{ess\,sup}_\omega \|\mathcal{L}_\omega^\epsilon - \mathcal{L}_\omega\| \to 0$ *as* $\epsilon \to 0$ *with the property that the Lyapunov exponents of* $(\mathcal{L}_\omega^\epsilon)$ *are* 0 *with multiplicity* 1*, and* $-\infty$ *with infinite multiplicity, for all* $\epsilon > 0$*.*

Remark 1. We remark that it follows from [9] that the Oseledets spaces also converge in probability to those of the unperturbed cocycle.

Remark 2. Note that part (a) allows quite general perturbations to the Perron–Frobenius cocycle, while part (b) shows that if the random fixed point has unbounded contraction, there are counterexamples even within the class of Perron–Frobenius operators of Blaschke products. For these counterexamples, the corresponding Blaschke products satisfy $\operatorname{ess\,sup}_\omega \max_{z\in\bar{D}_1} |T_\omega^\epsilon(z) - T_\omega(z)| < \epsilon$.

Let $\mathsf{Blaschke}_\mathsf{R}(\text{-})$ denote the set of measurable maps $\mathcal{T} : \omega \mapsto T_\omega$ from Ω to the collection of Blaschke products such that $r_\mathcal{T}(R) < R$. We equip $\mathsf{Blaschke}_\mathsf{R}(\text{-})$ with the distance $d(\mathcal{T}, \mathcal{S}) = \operatorname{ess\,sup}_\omega \max_{z\in\bar{D}_1} |T_\omega(z) - S_\omega(z)|$ (which, by the maximum principle, is the same as $\operatorname{ess\,sup}_\omega \max_{z\in C_1} |T_\omega(z) - S_\omega(z)|$). We say that $\mathcal{T} \in \mathsf{Blaschke}_\mathsf{R}(\text{-})$ is *stable* if the conditions of Theorem 2.5(a) apply.

Corollary 2.6. *Let* σ *be an invertible ergodic measure-preserving transformation of* $(\Omega, \mathbb{P})$*. The stable Blaschke product cocycles form an open dense subset of* $\mathsf{Blaschke}_\mathsf{R}(\text{-})$*.*

The following theorem allows us to analyse the Lyapunov spectrum of Perron–Frobenius cocycles of Blaschke products acting on coarser Banach spaces than $H^2(A_R)$.

Theorem 2.7. *Let* $\mathcal{R} = (\Omega, \mathbb{P}, \sigma, X, \mathcal{L})$ *be a random linear dynamical system let* X' *be a dense subspace of* X *equipped with a finer norm such that* $\mathcal{L}_\omega(X') \subset X'$ *for a.e.* $\omega \in \Omega$*. Let* $\mathcal{R}' = (\Omega, \mathbb{P}, \sigma, X', \mathcal{L}|_{X'})$ *be the restriction of* $\mathcal{R}$ *to* X'*. Suppose that* $\mathcal{R}$ *and* $\mathcal{R}'$ *both satisfy the conditions of Theorem* 3.4 (*Multiplicative Ergodic Theorem*)*. Then the exceptional Lyapunov exponents of* $\mathcal{R}$ *and* $\mathcal{R}'$ *that exceed* $\max(\kappa(\mathcal{R}), \kappa(\mathcal{R}'))$ *coincide, as do the corresponding Oseledets spaces.*

3. Background

Recall that a (finite) *Blaschke product* is a map from $\hat{\mathbb{C}}$ to itself of the form

$$T(z) = \zeta \prod_{j=1}^{n} \frac{z - \zeta_j}{1 - \bar{\zeta}_j z},$$

where the ζ_j's lie in D, the open unit disc, and $|\zeta| = 1$. We say a function from Ω into the collection of Blaschke products is measurable if each of the parameters, $n, \zeta, \zeta_1, \dots, \zeta_n$, is a Borel measurable function of ω. We note also that if $S(z) = \xi \prod_{j=1}^{n}(z - \xi_j)(1 - \bar{\xi}_j z)^{-1}$ and $(\xi, \xi_1, \dots, \xi_n)$ is sufficiently close to $(\zeta, \zeta_1, \dots, \zeta_n)$, then $\max_{z \in C_1} |S(z) - T(z)| < \epsilon$.

We record the following:

Lemma 3.1 (Properties of finite Blaschke products). *Let T be a Blaschke product. Then:*

(a) $T(C_1) = C_1$.

(b) $T \circ I = I \circ T$, *where I is the inversion map $I(z) = 1/\bar{z}$.*

(c) *T maps D_1 to itself* (*and hence T maps $\hat{\mathbb{C}} \setminus \bar{D}_1$ to itself*).

(d) *if T is a non-constant map from the closed unit disc to itself that is analytic in the interior and maps the boundary to itself, then T is a finite Blaschke product.*

(e) *The composition of two finite Blaschke products is again a finite Blaschke product.*

We are interested in finite Blaschke products whose restriction to the unit circle is expanding. A simple sufficient condition for this, namely that $\sum_{j=1}^{n} \frac{1-|\zeta_j|}{1+|\zeta_j|} > 1$, may be found in the work of Martin [19].

Let $\pi(x) = e^{2\pi i x}$ be the natural bijection between $\mathbb{R}/\mathbb{Z}$ and C_1, the unit circle in the complex plane. Let S be an orientation-preserving expanding real analytic map from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}$ and let $T = \pi S \pi^{-1}$ be its conjugate mapping from C_1 to C_1. Then there exist $r < R < 1$ such that T maps the annulus $A_R = \{z : R < |z| < 1/R\}$ over A_r in a k-to-1 way (where k is the absolute value of the degree of S). We will work with a family of expanding analytic maps of C_1, all mapping C_R into $\bar{D}_r$ (and hence $C_{1/R}$ into $D^c_{1/r}$). We consider the separable Hardy–Hilbert space $H^2(A_R)$ of analytic functions on A_R with an L^2 extension to ∂A_R. $H^2(A_R)$ is a Hilbert space with respect to the inner product $\langle f, g\rangle = \frac{1}{2\pi}\big(\int_{\partial A_R} f(z)\overline{g(z)}\,\frac{|dz|}{|z|}\big)$. An orthonormal basis for the Hilbert space is $\{d_n z^n : n \in \mathbb{Z}\}$, where $d_n = R^{|n|}(1 + R^{4|n|})^{-1/2} = R^{|n|}(1 + o(1))$. More details can be found in [3].

Let $(\sigma_i)_{i=1}^{k}$ be a family of inverse branches of S and $(\tau_i)_{i=1}^{k}$ be a family of inverse branches of $T|_{C_1}$.

If $f \in C(C_1)$ and $g \in C(\mathbb{R}/\mathbb{Z})$, define

$$\begin{aligned} \mathcal{L}_T f(z) &= \sum_{i=1}^{k} f(\tau_i(z))\tau_i'(z), \\ \mathcal{L}_S g(x) &= \sum_{i=1}^{k} g(\sigma_i(x))\sigma_i'(x). \end{aligned} \tag{3.1}$$

In fact, rather than acting on $C(C_1)$, we think of $\mathcal{L}_T$ as acting on $H^2(A_R)$. This corresponds to an action of $\mathcal{L}_S$ on the strip $\mathbb{R}/\mathbb{Z} \times (\log R/(2\pi), -\log R/(2\pi))$. Note that there is a one-one correspondence between elements of $H^2(A_R)$ and their restrictions to C_1 (which are necessarily continuous).

We record the following lemma which is a version of [25, Remark 4.1]

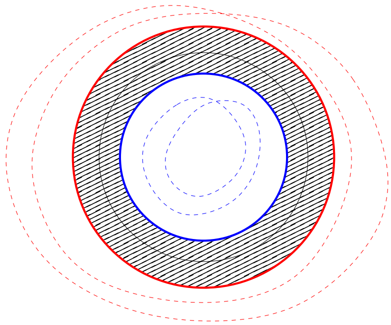

Fig. 1. Schematic diagram showing the annulus A_R (shaded); the inner boundary, C_R (blue); its image under T (dashed, blue); the outer boundary $C_{1/R}$ (red); and its image (dashed, red).

Lemma 3.2 (Correspondence between Perron–Frobenius operators). *The operators $\mathcal{L}_S$ and $\mathcal{L}_T$ defined above are conjugate by the map $\mathcal{Q}: C(C_1) \to C(\mathbb{R}/\mathbb{Z})$ given by*

$$(\mathcal{Q} f)(x) = f(e^{2\pi i x})e^{2\pi i x}.$$

In particular, the spectral properties of $\mathcal{L}_S$ and $\mathcal{L}_T$ are the same, so that even though $\mathcal{L}_S$ is the more standard object in dynamical systems, we will study $\mathcal{L}_T$ since this will allow us to directly apply tools of complex analysis. Further, if $S_{i_1}, \ldots, S_{i_n}$ are expanding maps of $\mathbb{R}/\mathbb{Z}$ and $T_{i_1}, \ldots, T_{i_n}$ are their conjugates, acting on C_1, then $\mathcal{L}_{S_{i_n}} \circ \cdots \circ \mathcal{L}_{S_{i_1}} = \mathcal{Q}^{-1} \mathcal{L}_{T_{i_n}} \circ \cdots \circ \mathcal{L}_{T_{i_1}} \mathcal{Q}$, so the Lyapunov exponents of a cocycle of $\mathcal{L}_S$ operators are the same as the Lyapunov exponents of the corresponding cocycle of $\mathcal{L}_T$ operators (provided that $\mathcal{Q}$ is an isomorphism of the two Banach spaces on which the operators are acting).

We record the following well known lemma.

Lemma 3.3 (Duality relations). *Let T be an expanding analytic map from C_1 to C_1 and let $\mathcal{L}_T$ be as above. If $f \in C(C_1)$, $g \in L^\infty(C_1)$, then*

$$\frac{1}{2\pi i}\int_{C_1} f(z)g(Tz)\,dz = \frac{1}{2\pi i}\int_{C_1} \mathcal{L}_T f(z)g(z)\,dz.$$

That is, if a linear functional θ is defined by integrating against g, then $\mathcal{L}_T^*\theta$ is given by integrating against $g \circ T$.

Let σ be an ergodic measure-preserving transformation of $(\Omega, \mathbb{P})$. Let X be a Banach space and suppose that $(\mathcal{L}_\omega)_{\omega\in\Omega}$ is a family of linear operators on X that is *strongly measurable*, that is, for any fixed $x \in X$, $\omega \mapsto \mathcal{L}_\omega(x)$ is $(\mathcal{F}_\Omega, \mathcal{F}_X)$-measurable, where $\mathcal{F}_\Omega$ is the σ-algebra on Ω and $\mathcal{F}_X$ is the Borel σ-algebra on X. In this case we say that the tuple $\mathcal{R} = (\Omega, \mathbb{P}, \sigma, X, \mathcal{L})$ is a *random linear dynamical system* and we define $\mathcal{L}_\omega^{(n)} = \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_\omega$.

A cocycle analogue to the (logarithm of the) spectral radius of a single operator is the quantity

$$\lambda_1(\omega) = \lim_{n\to\infty} \tfrac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|.$$

If one assumes that $\int \log \|\mathcal{L}_\omega\| \, d\mathbb{P} < \infty$, then the Kingman subadditive ergodic theorem guarantees the $\mathbb{P}$-a.e. convergence of this limit to a value in $[-\infty, \infty)$. Ergodicity also ensures that $\lambda_1(\cdot)$ is almost everywhere constant, so that we just write λ_1.

A second quantity of interest is the analogue of the (logarithmic) essential spectral radius, the *asymptotic index of compactness* [26]

$$\kappa(\omega) = \lim_{n\to\infty} \tfrac{1}{n} \log \alpha(\mathcal{L}_\omega^{(n)}),$$

where $\alpha(L)$ is the *index of compactness* of an operator L, the infimum of those real numbers t such that the image of the unit ball in X under L may be covered by finitely many balls of radius t, so that L is a compact operator if and only if $\alpha(L) = 0$. The quantity $\alpha(L)$ is also submultiplicative, so that Kingman's theorem again implies $\kappa(\omega)$ exists for $\mathbb{P}$-a.e. $\omega \in \Omega$ and is independent of ω, so that we just write κ. The cocycle will be called *quasi-compact* if $\kappa < \lambda_1$. The first Multiplicative Ergodic Theorem in the context of quasi-compact cocycles of operators on Banach spaces was proved by Thieullen [26]. We require a semi-invertible version (that is: although the base dynamical system is required to be invertible, the operators are not required to be injective) of a result of Lian and Lu [18].

Theorem 3.4 ([12]). *Let σ be an invertible ergodic measure-preserving transformation of a probability space $(\Omega, \mathbb{P})$ and let $\omega \mapsto \mathcal{L}_\omega$ be a quasi-compact strongly measurable cocycle of operators acting on a Banach space X with a separable dual satisfying $\int \log \|\mathcal{L}_\omega\| \, d\mathbb{P}(\omega) < \infty$.*

Then there exist $1 \le \ell \le \infty$, exponents $\lambda_1 > \cdots > \lambda_\ell > \kappa \ge -\infty$, finite multiplicities $m_1, \ldots, m_\ell$ and subspaces $V_1(\omega), \ldots, V_\ell(\omega), W(\omega)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$:

(a) $\dim(V_i(\omega)) = m_i$.

(b) $\mathcal{L}_\omega V_i(\omega) = V_i(\sigma(\omega))$ *and* $\mathcal{L}_\omega W(\omega) \subseteq W(\sigma(\omega))$.

(c) $V_1(\omega) \oplus \cdots \oplus V_\ell(\omega) \oplus W(\omega) = X$.

(d) *For* $x \in V_i(\omega) \setminus \{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} x\| = \lambda_i$.

(e) *For* $x \in W(\omega)$, $\limsup \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} x\| \le \kappa$.

For a bounded linear operator A from X to itself, we defined the following crude notion of volume growth in [12]:

$$\mathcal{D}_k(A) = \sup_{x_1,\ldots,x_k} \prod_{j=1}^{k} d(Ax_j, \operatorname{lin}\{Ax_i : i < j\}),$$

where the supremum is taken over x's of norm 1; $\operatorname{lin}\{y_1, \ldots, y_n\}$ denotes the linear span of the vectors $y_1, \ldots, y_n$; the linear span of the empty set is taken to be $\{0\}$; and $d(x, S) := \inf_{y\in S} \|x - y\|$.

Lemma 3.5. *Let σ, $(\Omega, \mathbb{P})$ and $\omega \mapsto \mathcal{L}_\omega$ be as in the statement of Theorem 3.4. Let $\mu_1 \ge \mu_2 \ge \cdots$ be the sequence of λ's in decreasing order* with repetition, *so that λ_i occurs m_i times in the sequence.*

(a) $\mathcal{D}_k$ *is submultiplicative:* $\mathcal{D}_k(AB) \le \mathcal{D}_k(A)\mathcal{D}_k(B)$ *if* A *and* B *are bounded linear operators on* X.

(b) *There exists a constant* c_k *such that if* Y *is a closed subspace of* X *of codimension* 1 *and* A *is a linear operator on* X*, then* $\mathcal{D}_k(A) \le c_k \|A\| \, \|A|_Y\|^{k-1}$.

(c) $\frac{1}{n} \log \mathcal{D}_k(\mathcal{L}_\omega^{(n)}) \to \mu_1 + \cdots + \mu_k$ *for* $\mathbb{P}$*-almost every* $\omega \in \Omega$.

The proof of this is in [12, Lemmas 1, 8 and 12].

4. Lyapunov spectrum for expanding Blaschke products

Lemma 4.1. *Let* $R < 1$ *and let* T *be a Blaschke product satisfying* $r := r_T(R) < R$. *Let* d_R *be the hyperbolic metric on* D_R: $d_R(z, w) = d_H(z/R, w/R)$*, where* d_H *is the standard hyperbolic metric on the unit disc. Then*

$$d_R(T(z), T(w)) \le \tfrac{r}{R}\, d_R(z, w) \quad \text{for all } z, w \in D_R.$$

Proof. We may write T as $Q \circ S$ where $Q(z) = rz/R$ and $S(z) = RT(z)/r$, so that S maps D_R to itself. By the Schwarz–Pick theorem, $d_R(S(z), S(w)) \le d_R(z, w)$ for all $z, w \in D_R$, so it suffices to show that $d_R(Q(z), Q(w)) \le \frac{r}{R}\, d_R(z, w)$ for all $z, w \in D_R$. The metric d_R is given, up to a constant multiple, by

$$d_R(z, w) = \inf_\gamma \int \frac{|d\xi|}{1 - |\xi|^2/R^2}$$

where the infimum is taken over paths γ from z to w. Given z and w, let γ be the geodesic joining them. Now $(r/R)\gamma(t)$ is a path (generally not a geodesic) joining $Q(z)$ and $Q(w)$. The length element is scaled by a factor of r/R and the integrand is decreased, so that $d_R(Q(z), Q(w)) \le \frac{r}{R}\, d_R(z, w)$ as claimed. ■

Corollary 4.2. *Let* $R < 1$ *and* $\mathcal{T} = (T_\omega)_{\omega\in\Omega}$ *be a measurable cocycle of expanding finite Blaschke products satisfying* $r := r_{\mathcal{T}}(R) < R$. *Then there exists a measurable random fixed point* x_ω *(that is, a point such that* $T_\omega(x_\omega) = x_{\sigma(\omega)}$*) in* $\bar{D}_r$ *such that for all* $\epsilon > 0$*, there exists* n *such that for all* $z \in \bar{D}_R$ *and a.e.* $\omega \in \Omega$*,* $|T^{(n)}_{\sigma^{-n}\omega}(z) - x_\omega| < \epsilon$.

Proof. The set $\bar{D}_r$ has bounded diameter, L say, in the d_R metric, and by assumption, for a.e. $\omega \in \Omega$, the sets $T^{(n)}_{\sigma^{-n}\omega}(\bar{D}_R)$ are nested. By Lemma 4.1, $T^{(n)}_{\sigma^{-n}\omega}(\bar{D}_R)$ has d_R-diameter at most $L(r/R)^{n-1}$. By completeness, $\bigcap T^{(n)}_{\sigma^{-n}\omega}(\bar{D}_R)$ is a singleton, $\{x_\omega\}$. Since $x_\omega = \lim_{n\to\infty} T^{(n)}_{\sigma^{-n}\omega}(0)$, and so is the limit of measurable functions, we see that x_ω depends measurably on ω. This equality also implies that $x_{\sigma(\omega)} = T_\omega(x_\omega)$. Since on D_r, d_R is within a bounded factor of Euclidean distance, we obtain the required uniform convergence in the Euclidean distance. ■

We introduce a non-standard definition of order of singularity for rational functions on the Riemann sphere: for $x \in \mathbb{C}$, $\operatorname{ord}_x f$ is n if $f(z) \sim a/(z-x)^n$ as $z \to x$ for some $n \geq 1$, and 0 otherwise. If $f(z) \sim bz^{n-2}$ for some $n \geq 1$ as $z \to \infty$ then $\operatorname{ord}_\infty f = n$ or $\operatorname{ord}_\infty f = 0$ otherwise. In particular, with this definition $\operatorname{ord}_\infty 1 = 2$. If f is a non-zero rational function on the Riemann sphere, then $\sum_{z\in\hat{\mathbb{C}}} \operatorname{ord}_z f \geq 2$: if f is a non-zero constant function, it has a singularity of order 2 at ∞; otherwise f must have at least one pole by Liouville's theorem. If f has exactly one pole, of order 1 at x say, then $f - a/(z-x)$ (where a is the residue) is bounded and hence constant, but $f(z) = c + a/(z-x)$ has order 1 at ∞ if $c = 0$, and order 2 at ∞ otherwise.

We previously gave a pointwise definition of Perron–Frobenius operators for expanding maps of the circle in (3.1). In fact, the same definition can be extended to rational maps.

Lemma 4.3. *If Q is a Möbius transformation (that is, $Q(z) = \frac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$), then $\mathcal{L}_Q$ maps the collection of rational functions on $\hat{\mathbb{C}}$ into itself. Further, $\operatorname{ord}_{Q(x)} \mathcal{L}_Q f = \operatorname{ord}_x f$ for all $x \in \hat{\mathbb{C}}$.*

Proof. It suffices to check that $\mathcal{L}_Q f$ is meromorphic on a neighbourhood of all points in $\hat{\mathbb{C}}$. If $Q(w) = z$ with $w, z \neq \infty$, then it is clear that $\mathcal{L}_Q f$ is meromorphic near z and $\operatorname{ord}_{\mathcal{L}_Q f} z = \operatorname{ord}_f w$. If $Q(z) = 1/z$, then a calculation shows $\mathcal{L}_Q f(z) = -f(1/z)/z^2$ and we can check that $\operatorname{ord}_0 \mathcal{L}_Q f = \operatorname{ord}_\infty f$ and $\operatorname{ord}_\infty \mathcal{L}_Q f = \operatorname{ord}_0 f$. Since $\mathcal{L}_{S\circ T} = \mathcal{L}_S \circ \mathcal{L}_T$ and every Möbius transformation can be expressed as a composition of maps of the form $z \mapsto az + b$ with $a \neq 0$ and $z \mapsto 1/z$, the proof is complete. ∎

Theorem 4.4. *Let T be a rational map. Then $\mathcal{L}_T$ maps the collection of rational functions on $\hat{\mathbb{C}}$ into itself. $\mathcal{L}_T$ does not increase orders of singularities and may decrease them: $\operatorname{ord}_x \mathcal{L}_T f \leq \max_{y\in T^{-1}x} \operatorname{ord}_y f$ for each $x \in \hat{\mathbb{C}}$.*

Further, if T has a critical point at x, and f has a singularity of order greater than 1 at x and no singularity at any other point of $T^{-1}(Tx)$, then $\operatorname{ord}_{T(x)} \mathcal{L}_T f < \operatorname{ord}_x f$.

The inequality in the first paragraph is an equality except at points x such that $T^{-1}(Tx)$ contains a critical point at which f has a singularity, or contains multiple singularities of f. We remark that the theorem seems quite surprising since the inverse branches of rational functions are typically not rational. We have since learned from the referee that this theorem is a corollary of a result in Levin, Sodin and Yuditskiĭ [17].

Proof of Theorem 4.4. By Lemma 4.3, it suffices, by pre- and post-composing T with Möbius transformations if necessary, to consider the case $z \neq \infty$ and $T(\infty) \neq z$. First, notice that if T has no critical points or poles in $T^{-1}z$, then it is clear from the definition of $\mathcal{L}_T$ that $\mathcal{L}_T f$ is analytic in a neighbourhood of z. In particular, $\mathcal{L}_T f$ is analytic off the finite set of images of critical points of T and poles of f.

Now let $T^{-1}z = \{y_1, \ldots, y_k\}$ and let $T(w) - z$ have a zero of order $m_i \geq 1$ at y_i for $i = 1, \ldots, k$. Suppose further that $\operatorname{ord}_{y_i} f = o_i$. Let δ be sufficiently small that $T^{-1} B_\delta(z)$ consists of k disjoint neighbourhoods $N_1, \ldots, N_k$ of $y_1, \ldots, y_k$. Now for $0 < |h| < \delta$, we have

$$\mathcal{L}_T f(z+h) = \sum_{j=1}^{k} \sum_{y \in T^{-1}(z+h) \cap N_j} \frac{f(y)}{T'(y)}.$$

If $y \in T^{-1}(z+h) \cap N_j$, we have $|y - y_j| = O(h^{1/m_j})$, $|1/T'(y)| = O(h^{-1+1/m_j})$ and $|f(y)| = O(h^{-o_j/m_j})$. Hence $\mathcal{L}_T f(z+h) = O(h^{-1-\max_j (o_j-1)/m_j})$. This guarantees that $\mathcal{L}_T f$ does not have an essential singularity at z, so that $\mathcal{L}_T f$ is meromorphic on a neighbourhood of z as claimed. Further, since meromorphic functions cannot exhibit fractional power growth rates, we have $\operatorname{ord}_z \mathcal{L}_T f \le 1 + \max_j \lfloor \frac{o_j - 1}{m_j} \rfloor \le \max o_j$ as required. ∎

Corollary 4.5. *Let T be a rational function and let $x \in \mathbb{C}$ satisfy $T(x) \in \mathbb{C}$. If $f(z) = 1/(z-x)^{n+1}$ for some $n \ge 1$, then $\mathcal{L}_T f$ is a linear combination of $\{1/(z - T(x))^{j+1} \colon 1 \le j \le n\}$.*

Proof. Since f has only one singular point, at x, of order $n+1$, it follows that $\mathcal{L}_T f$ is a rational function on the sphere with only one singular point, at $T(x)$, of order at most $n+1$. In particular, $\mathcal{L}_T f = O(z^{-2})$ in a neighbourhood of ∞, which ensures that there is no $1/(z - T(x))$ term in $\mathcal{L}_T f$. ∎

We now introduce an operator that commutes with the Perron–Frobenius operators of Blaschke products, performing inversion at the level of meromorphic functions. This will allow us to focus on poles inside the unit disc, and avoid dealing separately with poles at ∞. A precursor appears in [25, Lemma 3.1c]. Define

$$\mathcal{L}_I f(z) = \bar{f}(I(z))/z^2.$$

Lemma 4.6. *The operator $\mathcal{L}_I$ has the following properties:*

(a) *If T is a finite Blaschke product, then $\mathcal{L}_T \mathcal{L}_I = \mathcal{L}_I \mathcal{L}_T$.*

(b) *$\mathcal{L}_I$ maps meromorphic functions to meromorphic functions.*

(c) *$\mathcal{L}_I$ is a bounded anti-linear involution.*

(d) $\operatorname{ord}_{I(z)} \mathcal{L}_I f = \operatorname{ord}_z f$ *for f meromorphic and $z \in \hat{\mathbb{C}}$.*

Proof. We first show (a). Using the identity $I \circ T = T \circ I$, we have

$$\begin{aligned} T'(z) &= \lim_{h \to 0} \frac{T(z+h) - T(z)}{h} = \lim_{h \to 0} \frac{\frac{1}{\overline{\bar{T}(1/(\bar{z}+\bar{h}))}} - \frac{1}{\overline{\bar{T}(1/\bar{z})}}}{h} \\ &= \lim_{h \to 0} \frac{\bar{T}(1/\bar{z}) - \bar{T}(1/(\bar{z}+\bar{h}))}{h \bar{T}(1/\bar{z})^2} = \frac{\overline{T'(1/\bar{z})}}{z^2 \bar{T}(1/\bar{z})^2}. \end{aligned}$$

We deduce $\overline{T'(I(y))} = y^2 T'(y)/T(y)^2$. Now we have

$$\mathcal{L}_T \mathcal{L}_I f(z) = \sum_{y \in T^{-1} z} \frac{\mathcal{L}_I f(y)}{T'(y)} = \sum_{y \in T^{-1} z} \frac{\bar{f}(I(y))}{y^2 T'(y)};$$

and

$$\begin{aligned}\mathcal{L}_I\mathcal{L}_T f(z) &= \overline{\mathcal{L}_T f(I(z))}/z^2 \\ &= \frac{1}{z^2}\sum_{y\in T^{-1}(I(z))}\frac{\bar{f}(y)}{\overline{T'(y)}} = \frac{1}{z^2}\sum_{y\in T^{-1}(z)}\frac{\bar{f}(I(y))}{\overline{T'(I(y))}} \\ &= \frac{1}{z^2}\sum_{y\in T^{-1}(z)}\frac{\bar{f}(I(y))T(y)^2}{y^2T'(y)} = \mathcal{L}_T\mathcal{L}_I f(z).\end{aligned}$$

Part (b) is standard; the boundedness follows from the fact that $|1/z^2| \le 1/R^2$ on ∂A_R; that $\mathcal{L}_I$ is an involution and that it preserves orders are simple computations. ■

Lemma 4.7. *Let T be a finite Blaschke product, let $x \in \mathbb{C}\setminus C_1$ and let $f(z) = 1/(z-x)$. Then*

$$\mathcal{L}_T f(z) = \frac{1}{z-T(x)} - \frac{1}{z-T(\infty)},$$

where $1/(z-\infty)$ is interpreted as the constant 0 function.

Proof. It follows from Theorem 4.4 that $\mathcal{L}_T f$ is a rational function with singularities of order 1 at $T(x)$ and $T(\infty)$:

$$\mathcal{L}_T f = \frac{a}{z-T(x)} + \frac{b}{z-T(\infty)}.$$

In order that $\mathcal{L}_T f$ have no singularity at ∞, we must have $b = -a$.

If $|x| < 1$, we see from Lemma 3.3 (taking the function g to be 1) that $a = 1$, as required.

If $|x| > 1$, then $\mathcal{L}_T f = \mathcal{L}_I\mathcal{L}_T\mathcal{L}_I f$. We calculate $\mathcal{L}_I f(z) = 1/z - 1/(z - I(x))$, and using the first part $\mathcal{L}_T\mathcal{L}_I f(z) = 1/(z-T(0)) - 1/(z-T(I(x)))$. Applying $\mathcal{L}_I$ again, we arrive at

$$\mathcal{L}_T f(z) = \frac{1}{z-I(T(I(x)))} - \frac{1}{z-I(T(0))} = \frac{1}{z-T(x)} - \frac{1}{z-T(\infty)},$$

as required. ■

Corollary 4.8. *Let $\mathcal{T} = (T_\omega)$ be a cocycle of expanding finite Blaschke products. Let $x_\omega \in D_r$ be the random fixed point. The space $W_0(\omega)$ spanned by $\hat{e}_{0,\omega}$ where*

$$\hat{e}_{0,\omega}(z) = \frac{1}{z-x_\omega} - \frac{1}{z-I(x_\omega)}$$

is a one-dimensional equivariant subspace with Lyapunov exponent 0.

If $x \in D_r$ and $f(z) = 1/(z-x)$, then $\|\mathcal{L}_\omega^{(n)} f - \hat{e}_{0,\sigma^n\omega}\| \to 0$.

This corollary may be seen as a random version of a result of Martin [19], expressing the invariant measure of an expanding Blaschke product as a Poisson kernel.

Proof of Corollary 4.8. By Lemma 4.7 and the fact that $\mathcal{L}_{T_1\circ T_2} = \mathcal{L}_{T_1} \circ \mathcal{L}_{T_2}$, we see

$$\mathcal{L}_\omega^{(n)} f(z) = \frac{1}{z - T_\omega^{(n)}(x)} - \frac{1}{z - T_\omega^{(n)}(\infty)}.$$

The claimed equivariance follows. Since $|T_\omega^{(n)}(x) - x_{\sigma^n\omega}| \to 0$, we see that

$$\left\| \frac{1}{z - T_\omega^{(n)}(x)} - \frac{1}{z - x_{\sigma^n\omega}} \right\| \to 0.$$

Similarly, since $T_\omega^{(n)}(\infty) = I(T_\omega^{(n)}(0))$, Corollary 4.2 yields $d_{\hat{\mathbb{C}}}(T_\omega^{(n)}(\infty), I(x_{\sigma^n\omega})) \to 0$, where $d_{\hat{\mathbb{C}}}$ is the standard metric on the Riemann sphere. It follows that

$$\left\| \frac{1}{z - T_\omega^{(n)}(\infty)} - \frac{1}{z - I(x_{\sigma^n\omega})} \right\| \to 0,$$

as required. ■

We now show that the hypotheses of Theorem 3.4 are satisfied by a cocycle of Perron–Frobenius operators of Blaschke products satisfying $r_{\mathcal{T}}(R) < R$.

Lemma 4.9. *Let $0 < R < 1$. If $r_T(R), r_{\tilde{T}}(R) \le r < R$, then*

$$\|\mathcal{L}_T - \mathcal{L}_{\tilde{T}}\| \le C \max_{x\in C_1} |T(x) - \tilde{T}(x)|,$$

where C is a constant depending only on r and R. In particular, restricted to Blaschke products satisfying $r_T(R) < R$, the map $T \mapsto \mathcal{L}_T$ is continuous.

Proof. Recall that $H^2(A_R)$ is a Hilbert space with respect to the inner product $\langle f, g\rangle = \frac{1}{2\pi}\left(\int_{\partial A_R} f(z)\overline{g(z)}\,\frac{|dz|}{|z|}\right)$. With respect to this inner product, the functions $e_n(z) = d_n z^n$ form an orthonormal basis of $H^2(A_R)$, where $d_n = (R^{2n} + R^{-2n})^{-1/2}$, so that $d_n \sim R^{|n|}$.

We now compute

$$\begin{aligned}
\langle \mathcal{L}_T(f), e_n\rangle &= \frac{d_n}{2\pi i}\int_{\partial A_R} \mathcal{L}_T f(z)\bar{z}^n\,\frac{dz}{z} \\
&= \frac{d_n}{2\pi i}\left(\int_{C_R} \frac{\mathcal{L}_T f(z) R^{2n}}{z^n}\,\frac{dz}{z} + \int_{C_{1/R}} \frac{\mathcal{L}_T f(z) R^{-2n}}{z^n}\,\frac{dz}{z}\right) \\
&= \frac{d_n(R^{2n} + R^{-2n})}{2\pi i}\int_{C_1} \frac{\mathcal{L}_T f(z)}{z^{n+1}}\,dz \\
&= \frac{1}{2\pi i d_n}\int_{C_1} \frac{f(z)}{T(z)^{n+1}}\,dz.
\end{aligned}$$

Let $f \in H^2(A_R)$ be arbitrary and let $n \ge 0$. Let T and $\tilde{T}$ be any two Blaschke products satisfying $r_T(R) \le r$ and $r_{\tilde{T}}(R) \le r$ for some $r < R$, and let $\delta = \max_{z\in C_1} |T(z) - \tilde{T}(z)|$.

Note by the maximum modulus principle, $|T(z) - \tilde{T}(z)| \leq \delta$ for all $z \in D_1$ also. Then deforming the contour to $C_{1/R}$, we see that

$$\begin{aligned}
|\langle(\mathcal{L}_T - \mathcal{L}_{\tilde{T}})f, e_n\rangle| &\leq \frac{1}{2\pi d_n}\int_{C_{1/R}} |f(z)|\left|\frac{1}{T(z)^{n+1}} - \frac{1}{\tilde{T}(z)^{n+1}}\right| |dz| \\
&\leq \frac{1}{Rd_n}\|f\| \max_{z\in C_{1/R}}\left|\frac{1}{T(z)^{n+1}} - \frac{1}{\tilde{T}(z)^{n+1}}\right| \\
&= \frac{1}{Rd_n}\|f\| \max_{z\in C_R} |T(z)^{n+1} - \tilde{T}(z)^{n+1}| \\
&\leq \frac{(n+1)r^n}{Rd_n}\|f\| \max_{z\in C_R} |T(z) - \tilde{T}(z)| \\
&\leq \frac{2(n+1)}{R}\left(\frac{r}{R}\right)^n \delta\|f\|,
\end{aligned}$$

where for the third line, we use the fact that Blaschke products commute with inversion, and for the fourth line, we use $r_T(R), r_{\tilde{T}}(R) \leq r$. If $n = -k$ with $k \geq 1$, then an analogous computation, deforming the contour to C_R, shows

$$|\langle(\mathcal{L}_T - \mathcal{L}_{\tilde{T}})f, e_n\rangle| \leq \frac{2(k-1)}{R}\left(\frac{r}{R}\right)^{k-2} \delta\|f\|.$$

In particular, since the (e_n) form an orthonormal basis, we deduce that

$$\|(\mathcal{L}_T - \mathcal{L}_{\tilde{T}})f\| \leq C\delta\|f\|,$$

where C depends only on r and R, as required.

We note that $r_T(R)$ depends continuously on T. Hence, if $r_T(R) < R$ and $\tilde{T}$ is sufficiently close to T, then $r_{\tilde{T}}(R) \leq (R + r_T(R))/2 < R$ and the last statement of the lemma follows from the argument above. ■

Corollary 4.10. *Let $r < \rho < R < 1$. There exists $C > 0$ such that if the Blaschke product T satisfies $r_T(R) \leq r$, then $\|\mathcal{L}_T\|_{H^2(A_R)\to H^2(A_\rho)} \leq C$. In particular, $\mathcal{L}_T$ is compact as an operator from $H^2(A_R)$ to itself.*

Proof. First, notice by the proof of Theorem 4.4 that $\mathcal{L}_T f$ is analytic on A_r. As in the above proof, $\tilde{e}_n(z) = \tilde{d}_n z^n$ is an orthonormal basis for $H^2(A_\rho)$, where $\tilde{d}_n = (\rho^{2n} + \rho^{-2n})^{-1/2}$. As above $\langle\mathcal{L}_T f, \tilde{e}_n\rangle_{H^2(A_\rho)} = \frac{1}{2\pi i\tilde{d}_n}\int_{C_1} f(z)/T(z)^{n+1}\,dz$. Deforming the contour to $C_{1/R}$ in the case where $n \geq 0$ and to C_R when $n < 0$, we obtain $|\langle\mathcal{L}_T f, \tilde{e}_n\rangle_{H^2(A_\rho)}| \leq C(r/\rho)^{|n|}\|f\|$. Since this is square summable, the result follows. ■

In the context of Theorem 2.1, the map $\omega \mapsto T_\omega$ is measurable, and the map $T \mapsto \mathcal{L}_T$ is continuous, so that the composition, $\omega \mapsto \mathcal{L}_{T_\omega}$, satisfies the hypotheses of Theorem 3.4. We recall that if U and V are closed complementary subspaces of a Banach space X, then $\Pi_{U\|V}$ denotes the projection onto U with kernel V. Also, we use the notation $S(X)$ to denote the unit sphere of X.

Lemma 4.11. *Let $\mathcal{R}$ be a random linear dynamical system satisfying the conditions of Theorem 3.4. Let $E_j(\omega)$ be the jth "fast space" $V_1(\omega) \oplus \cdots \oplus V_j(\omega)$ and let $F_j(\omega)$ be the complementary "slow space". If V is a subspace of X satisfying $\Pi_{E_j(\omega)\|F_j(\omega)}(V) = E_j(\omega)$, then*

$$\sup_{x \in E_j(\sigma^n\omega)\cap S(X)} d(x, \mathcal{L}_\omega^{(n)} V) \to 0 \quad \text{as } n \to \infty.$$

Proof. We write E and F for $E_j(\omega)$ and $F_j(\omega)$. Let W be a subspace of V of the same dimension as E such that $\Pi_{E\|F}(W) = E$. Let $Q = (\Pi_{E\|F}|_W)^{-1}$. Let $0 < 2\epsilon < \lambda_j - \lambda_{j+1}$ and $C_\omega > 0$ satisfy, for every $x \in E_j(\sigma^n\omega)$, and $u \in E$ such that $\mathcal{L}_\omega^{(n)} u = x$,

$$\|u\| \le C_\omega e^{-(\lambda_j - \epsilon)n} \|x\|;$$

and for every $f \in F$,

$$\|\mathcal{L}_\omega^{(n)} f\| \le C_\omega e^{(\lambda_{j+1}+\epsilon)n} \|f\|.$$

Now $Qu - u = Qu - \Pi_{E\|F} Qu \in F$, so that $\|\mathcal{L}_\omega^{(n)}(Qu - u)\| \le C_\omega e^{(\lambda_{j+1}+\epsilon)n} \|Qu - u\|$. Hence,

$$\|\mathcal{L}_\omega^{(n)}(Qu) - x\| \le C_\omega^2 e^{-(\lambda_j - \lambda_{j+1} - 2\epsilon)n} (\|Q\| + 1)\|x\|.$$

Since $\mathcal{L}_\omega^{(n)}(Qu) \in \mathcal{L}_\omega^{(n)} W \subset \mathcal{L}_\omega^{(n)} V$, the proof is complete. ∎

Lemma 4.12. *Let the measure-preserving transformation and cocycles satisfy $r_{\mathcal{T}}(R) < R$ as in the statement of Theorem 2.1. Let V be the subspace of $H^2(A_R)$ spanned by the Laurent polynomials $z^{-(j+1)}$ for $j = 1, \ldots, N$. Then $\mathcal{L}_\omega^{(n)} V$ is spanned by $1/(z - T_\omega^{(n)}(0))^{j+1}$ for $j = 1, \ldots, N$.*

In particular, the sequence $\mathcal{L}_\omega^{(n)} V$ approaches the equivariant sequence of subspaces

$$P_N^-(\sigma^n\omega) := \operatorname{lin}\left\{\frac{1}{(z - x_{\sigma^n\omega})^{j+1}} : 1 \le j \le N\right\}.$$

Proof. It suffices to show that if $f(z) = 1/(z-x)^{j+1}$ with $x \in D$, then for any finite expanding Blaschke product, $\mathcal{L}_\omega f$ is a linear combination of $1/(z - T(x))^{k+1}$ for k in the range 1 to j, but that was established in Corollary 4.5. ∎

Corollary 4.13. *Let the measure-preserving transformation and cocycles be as above. Let W be the subspace of $H^2(A_R)$ spanned by the Laurent polynomials z^{j-1} for $j = 1, \ldots, N$. Then $\mathcal{L}_\omega^{(n)} W$ is spanned by $z^{j-1}/(1 - \overline{T_\omega^{(n)}(0)} z))^{j+1}$ for $j = 1, \ldots, N$.*

In particular, the sequence $\mathcal{L}_\omega^{(n)} W$ approaches the equivariant sequence of subspaces,

$$P_N^+(\sigma^n\omega) = \operatorname{lin}\left\{\frac{z^{j-1}}{(1 - \bar{x}_{\sigma^n\omega} z)^{j+1}} : 1 \le j \le N\right\}.$$

Proof. Notice that $\mathcal{L}_I$ maps $z^{-(j+1)}$ to z^{j-1} (and vice versa), and is a continuous operator on $H^2(A_R)$. So $\mathcal{L}_I(V) = W$, where V is as in the statement of Lemma 4.12. Since $\mathcal{L}_\omega$ and $\mathcal{L}_I$ commute, we see $\mathcal{L}_\omega^{(n)} W = \mathcal{L}_\omega^{(n)} \mathcal{L}_I(V) = \mathcal{L}_I(\mathcal{L}_\omega^{(n)}(V))$. A computation shows that if $f(z) = 1/(z-x)^{j+1}$, then $\mathcal{L}_I f(z) = z^{j-1}/(1 - \bar{x}z)^{j+1}$. ∎

Corollary 4.14. *Let the dynamical system, Blaschke product cocycle and family of Perron–Frobenius operators satisfy* $r_{\mathcal{T}}(R) < R$ *as above. Let* $E(\omega)$ *be an equivariant family of finite-dimensional fast spaces for the cocycle. Then there exists an* N *such that for* $\mathbb{P}$*-a.e.* ω,

$$E(\omega) \subset P_N^-(\omega) \oplus W_0(\omega) \oplus P_N^+(\omega),$$

where $W_0(\omega)$ *is as defined in Corollary 4.8.*

Proof. Let $F(\omega)$ be the corresponding slow subspace to $E(\omega)$. Since the (finite term) Laurent polynomials, L, form a dense subspace of $H^2(A_R)$, and $\Pi_{E(\omega)\|F(\omega)}$ is bounded, we see that $\Pi_{E(\omega)\|F(\omega)}(L)$ is a dense subspace of $E(\omega)$ and hence is equal to $E(\omega)$. Pick out a finite-dimensional subspace L_1 of L such that $\Pi_{E(\omega)\|F(\omega)}(L_1) = E(\omega)$. Hence there exists an N such that $V = \operatorname{lin}\{z^{-1+j} : |j| \le N\}$ satisfies the hypothesis of Lemma 4.11. By Lemma 4.12 and Corollaries 4.8 and 4.13, we see

$$\sup_{x \in E(\sigma^n\omega) \cap S(X)} d\big(x, P_N^-(\sigma^n\omega) \oplus W_0(\sigma^n\omega) \oplus P_N^+(\sigma^n\omega)\big) \to 0. \tag{4.1}$$

For a fixed N, let A_N be the set of ω for which (4.1) is satisfied and notice that A_N is a σ-invariant measurable subset of Ω. Hence there exists an $N > 0$ such that for $\mathbb{P}$-a.e. ω,

$$\sup_{x \in E(\sigma^n\omega) \cap S(X)} d\big(x, P_N^-(\sigma^n\omega) \oplus W_0(\sigma^n\omega) \oplus P_N^+(\sigma^n\omega)\big) \to 0.$$

It follows from the Poincaré recurrence theorem that if $\kappa\colon \Omega \to [0,\infty)$ is a measurable function such that $\kappa(\sigma^n\omega) \to 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, then $\kappa(\omega) = 0$ for almost every ω. We apply this to

$$\kappa(\omega) = \sup_{x \in E(\omega) \cap S(X)} d\big(x, P_N^-(\omega) \oplus W_0(\omega) \oplus P_N^+(\omega)\big),$$

to deduce that $E(\omega) \subset P_N^-(\omega) \oplus W_0(\omega) \oplus P_N^+(\omega)$ for $\mathbb{P}$-a.e. ω, as required. ∎

Proof of Theorem 2.1. The compactness of the cocycle follows from Corollary 4.10 and statement (1) follows from Corollary 4.2.

In the light of Corollary 4.14, it suffices to evaluate the Lyapunov exponents when the system is restricted to the finite-dimensional equivariant subspaces $P_N(\omega) = P_N^-(\omega) \oplus W_0(\omega) \oplus P_N^+(\omega)$. Notice that since each of $E_0(\omega)$ and $P_N^\pm(\omega)$ is equivariant, the Lyapunov exponents of the cocycle restricted to P_N are simply the combination of the Lyapunov exponents of $E_0(\omega)$, $P_N^+(\omega)$ and $P_N^-(\omega)$. Since $\mathcal{L}_I(P_N^\pm(\omega)) = P_N^\mp(\omega)$, $\mathcal{L}_I$ is a bounded involution, and $\mathcal{L}_{T_\omega^{(n)}} \circ \mathcal{L}_I = \mathcal{L}_I \circ \mathcal{L}_{T_\omega^{(n)}}$, we deduce the Lyapunov exponents of the restriction of the cocycle to $P_N^+(\omega)$ are the same as those of the restriction to $P_N^-(\omega)$. As noted in Corollary 4.8, the exponent of the cocycle restricted to the equivariant space $E_0(\omega)$ is 0 (this will turn out to be the leading exponent). It suffices to compute the Lyapunov exponents of the restriction of the cocycle to $P_N^-(\omega)$. Each of these Lyapunov exponents will then have multiplicity 2 for the full cocycle, being repeated as a Lyapunov exponent in the restriction to $P_N^+(\omega)$.

It follows from Corollary 4.5 that the matrix representing the restriction of the cocycle to $P_N^-(\omega)$ is upper triangular with respect to the natural family of bases, $(z-x_\omega)^{-(j+1)}$ for $j=1,\dots,N$.

If $T'_\omega(x_\omega)=0$, then all the diagonal terms of the matrix are 0 by Theorem 4.4. Hence if $T'_\omega(x_\omega)=0$ for a set of ω's of positive measure, we see that the Lyapunov spectrum is 0 with multiplicity 1 and $-\infty$ with infinite multiplicity.

Otherwise, we compute the leading term of $\mathcal{L}_\omega f(z)$ near $x_{\sigma\omega}$ for $f(z)=1/(z-x_\omega)^{j+1}$. Let $\alpha=T'_\omega(x_\omega)$. From Theorem 4.4, $\mathcal{L}_\omega f(z)$ has a leading term of the form $c/(z-x_{\sigma\omega})^{j+1}$. The corresponding diagonal entry of the matrix is then c. We compute c as follows:

$$c=\int_{C_1}\mathcal{L}_\omega f(z)(z-x_{\sigma\omega})^j\,dz=\int_{C_1}\frac{(T(z)-T(x_\omega))^j}{(z-x_\omega)^{j+1}}\,dz.$$

Since the only pole inside C_1 is at x_ω, we may deform the contour to a small contour around x_ω. Since the numerator is $\alpha^j(z-x_\omega)^j+O((z-x_\omega)^{j+1})$, we see that $c=\alpha^j=T'_\omega(x_\omega)^j$.

We also verify that the off-diagonal elements of the matrix are bounded: If $i<j$, then the (i,j) entry of the matrix is given by $\frac{1}{2\pi i}\int\mathcal{L}_\omega[f](z)(z-x_{\sigma\omega})^i\,dz$, where $f(z)=(z-x_\omega)^{-(j+1)}$ and the integral is over the unit circle. Since by Corollary 4.10, the operators $\mathcal{L}_\omega$ are a uniformly bounded family on $H^2(A_R)$, we see that the entries of the $N\times N$ matrix, representing the restriction to $P_N^-(\omega)$, are uniformly bounded. (In fact, we give more refined estimates in Section 6.)

The Lyapunov exponents of the cocycle restricted to $P_N^+(\omega)$ are therefore given by the values

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log\prod_{k=0}^{n-1}|T'_{\sigma^k\omega}(x_{\sigma^k\omega})|^j&=j\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log|T'_{\sigma^k\omega}(x_{\sigma^k\omega})|\\&=j\int\log|T'_\omega(x_\omega)|\,d\mathbb{P}(\omega)=:j\Lambda,\end{aligned}$$

where j ranges from 1 to N, and we use the Birkhoff ergodic theorem in the last line.

Finally, to show that $\Lambda\le\log(r/R)$, notice that the restriction of d_R to $\bar{D}_r$ agrees with Euclidean distance up to a bounded factor. The above shows that $\Lambda=\lim_{n\to\infty}\log|(T_\omega^{(n)})'(x_\omega)|$. Lemma 4.1 shows that $d_R(T_\omega^{(n)}(x_\omega+h),T_\omega^{(n)}(x_\omega))\le a|h|(r/R)^n$, where $a=d_R(x_\omega+h,x_\omega)/|h|$ is a uniformly bounded quantity. Hence $|T_\omega^{(n)}(x_\omega+h)-T_\omega^{(n)}(x_\omega)|/h\le c(r/R)^n$. The fact that $\Lambda\le\log(r/R)$ follows. ■

5. Spectrum collapse

In this section, we focus on an example. Let

$$T_0(z)=z^2\quad\text{and}\quad T_1(z)=\left(\frac{z+1/4}{1+z/4}\right)^2,$$

so that both T_0 and T_1 are expanding degree 2 maps of the unit circle, mapping the unit disc to itself in a 2-to-1 way. We take the base dynamical system to be the full shift σ on $\Omega = \{0,1\}^{\mathbb{Z}}$ with invariant measure $\mathbb{P}_p$, the Bernoulli measure where each coordinate takes the value 0 with probability p and 1 with probability $1-p$.

We let $\mathcal{L}_0$ and $\mathcal{L}_1$ be the Perron–Frobenius operators corresponding to T_0 and T_1 acting on the unit circle with respect to the signed measure, dz, we consider the cocycle $\mathcal{L}_\omega := \mathcal{L}_{\omega_0}$ and we study the properties of $\mathcal{L}_\omega^{(n)} := \mathcal{L}_{\omega_{n-1}} \circ \cdots \circ \mathcal{L}_{\omega_0}$.

Lemma 5.1. *Let T_0 and T_1 be defined as above. Then:*

(a) *T_0 fixes 0 and T_1 fixes $a = \frac{1}{2}(7 - 3\sqrt{5}) \approx 0.146$.*

(b) *T_0 and T_1 both map the subset $[0,a]$ of the unit disc in an increasing way into itself (with disjoint ranges).*

(c) *Both maps act as contractions on $[0,a]$: $\frac{15}{32} \le T_1' \le \frac{2}{3}$ on $[0,a]$ and $0 \le T_0' \le 2a$ on $[0,a]$.*

For $\omega \in \Omega$, let x_ω denote the random fixed point as described in Theorem 2.1.

(d) *If $\omega = \dots 10^n \cdot 0 \dots$, then $2b^{2^n} \le T_{\omega_0}'(x_\omega) \le 2a^{2^n}$, where $b = T_1(0)$.*

Proof. We just prove statement (d). Let $\omega_{-(n+1)} = 1$ and $\omega_{-n} = \cdots = \omega_{-1} = \omega_0 = 0$. Then since $x_{\sigma^{-n}\omega} = T_1(x_{\sigma^{-(n+1)}\omega})$, we have $b \le x_{\sigma^{-n}\omega} \le a$. Since $x_\omega = T_0^n(x_{\sigma^{-n}\omega})$, we have $b^{2^n} \le x_\omega \le a^{2^n}$ and $2b^{2^n} \le T_\omega'(x_\omega) \le 2a^{2^n}$. ∎

Proof of Corollary 2.2. For (a), using Theorem 2.1, it suffices to prove $\Lambda > -\infty$, where $\Lambda = \int \log|T_\omega'(x_\omega)|\, d\mathbb{P}(\omega)$. The set Ω may be countably partitioned (apart from the fixed point of all 0's) into $[\cdot 1] := \{\omega \in \Omega\colon \omega_0 = 1\}$ and the sets $[10^n \cdot 0] := \{\omega \in \Omega\colon x_{-(n+1)} = 1, x_{-n} = \cdots = x_0 = 0\}$ for $0 \le n < \infty$. On $[\cdot 1]$, by Lemma 5.1(c), $\log T_\omega'(x_\omega) \ge \log \frac{15}{32}$. On $[10^n \cdot 0]$, $\log T_\omega'(x_\omega) \ge 2^n \log b$ by Lemma 5.1(d). Since $\mathbb{P}([10^n \cdot 0]) = (1-p)p^{n+1}$, we see

$$\begin{aligned}\int \log|T_\omega'(x_\omega)|\, d\mathbb{P}(\omega) &= \int_{[\cdot 1]} \log|T_\omega'(x_\omega)|\, d\mathbb{P} + \sum_{n=0}^{\infty} \int_{[10^n\cdot 0]} \log|T_\omega'(x_\omega)|\, d\mathbb{P} \\ &\ge (1-p)\log\tfrac{15}{32} + p(1-p)\log b \sum_{n=0}^{\infty} (2p)^n > -\infty.\end{aligned}$$

For (b), arguing as above, we see that on $[10^n \cdot 0]$, $\log|T_\omega'(x_\omega)| \le 2^n \log a + \log 2$ (where $\log a \approx -1.925$). Hence

$$\begin{aligned}\Lambda &\le \mathbb{P}_p([1])\big(\log\tfrac{2}{3}\big) + \sum_{n=0}^{\infty} (2^n \log a + \log 2)\mathbb{P}_p([10^n \cdot 0]) \\ &= (1-p)\log\tfrac{2}{3} + p\log 2 + p(1-p)\log a \sum_{n=0}^{\infty} (2p)^n = -\infty.\end{aligned}$$ ∎

5.1. Gaussian perturbations

We now consider a perturbed version of the cocycle, where $\mathcal{L}_i$ is replaced by $\mathcal{L}_i^\epsilon := \mathcal{N}_\epsilon \circ \mathcal{L}_i$, where $\mathcal{N}_\epsilon$ has the effect of convolving a density with a Gaussian with mean 0 and variance ϵ^2. On $\mathbb{R}/\mathbb{Z}$, we have

$$(\mathcal{N}_\epsilon^{\mathbb{R}/\mathbb{Z}} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\epsilon t)e^{-t^2/2}\,dt = \mathbb{E} f(x+\epsilon N),$$

where N is a standard normal random variable. The corresponding conjugate operator on $C(C_1)$ is $\mathcal{N}_\epsilon := \mathcal{N}_\epsilon^{C_1} = Q^{-1}\mathcal{N}_\epsilon^{\mathbb{R}/\mathbb{Z}} Q$, where Q is as in Lemma 3.2. A calculation using that lemma shows

$$(\mathcal{N}_\epsilon f)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ze^{-2\pi i\epsilon t})e^{-2\pi i\epsilon t - t^2/2}\,dt.$$

Proof of Corollary 2.3. From the definition of $\mathcal{L}_0$, we check

$$\mathcal{L}_0(f)(z) = \frac{1}{2}\left(\frac{f(\sqrt{z})}{\sqrt{z}} + \frac{f(-\sqrt{z})}{-\sqrt{z}}\right),$$

where $\pm\sqrt{z}$ are the two square roots of z. We define $\hat{e}_n(z) = z^{n-1}$ and verify that

$$\mathcal{L}_0(\hat{e}_n) = \begin{cases} \hat{e}_{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

We compute

$$\begin{aligned} \mathcal{N}_\epsilon(\hat{e}_n)(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{e}_n(ze^{-2\pi i\epsilon t})e^{-2\pi i\epsilon t - t^2/2}\,dt \\ &= z^{n-1}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\pi i n\epsilon t - t^2/2}\,dt \\ &= e^{-2\pi^2 n^2\epsilon^2}\hat{e}_n(z). \end{aligned}$$

Combining the two, we have

$$(\mathcal{L}_0^\epsilon)^n \hat{e}_\ell = \begin{cases} \exp(-2\pi^2\epsilon^2 m^2(4^{n-1}+\cdots+4+1))\hat{e}_m & \text{if } \ell = 2^n m, \\ 0 & \text{otherwise.} \end{cases}$$

We let $H_0^2(A_R)$ be the subspace of $H^2(A_R)$ consisting of those functions whose Laurent expansions have a vanishing z^{-1} term. Let $f \in H_0^2(A_R)$ be of norm 1 and let $f = \sum_{n\in\mathbb{Z}} a_n z^n$ be its Laurent expansion. We recall $|a_n| \le R^{|n|} \le 1$ for all $n \in \mathbb{Z}$ and $a_{-1} = 0$.

Now

$$(\mathcal{L}_0^\epsilon)^n f(z) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \exp(-2\pi^2\epsilon^2 m^2(4^{n-1}+\cdots+4+1))a_{2^n m-1}z^{m-1}.$$

so that for $z \in A_R$ and $n > 0$,

$$|(\mathcal{L}_0^\epsilon)^n f(z)| \le \sum_{m\in\mathbb{Z}\setminus\{0\}} \exp(-2\pi^2\epsilon^2 m^2(4^{n-1}+\cdots+4+1))R^{|2^n m-1|}R^{-|m-1|}$$
$$\le \frac{2}{1-R}\exp(-2\pi^2\epsilon^2 4^{n-1}).$$

Since if g is a bounded analytic function on A_R, $\|g\|_{H^2(A_R)} \le 2\|g\|_\infty$, we see that $\|(\mathcal{L}_0^\epsilon)^n|_{H_0^2(A_R)}\| \le \frac{4}{1-R}\exp(-2\pi^2\epsilon^2 4^{n-1})$. By Lemma 3.5(b), $\mathcal{D}_2((\mathcal{L}_0^\epsilon)^n\mathcal{L}_1^\epsilon) \le A\exp(-2\pi^2\epsilon^2 4^{n-1})$, where $A = 4\|\mathcal{L}_1\|^2 c_2/(1-R)$.

Now let $N(\omega) = \min\{n > 0 : \omega_n = 1\}$. We consider the induced map on [1]: $\tilde\sigma(\omega) = \sigma^{N(\omega)}(\omega)$. The induced cocycle is defined for $\omega \in [1]$ by $\tilde{\mathcal{L}}_\omega^\epsilon = \mathcal{L}_\omega^{\epsilon\,(N(\omega))}$, so that $\tilde{\mathcal{L}}_\omega^\epsilon = (\mathcal{L}_0^\epsilon)^{N(\omega)-1}\mathcal{L}_1^\epsilon$. By $\tilde{\mathcal{L}}_\omega^{\epsilon\,(n)}$ we mean $\tilde{\mathcal{L}}_{\tilde\sigma^{n-1}(\omega)}^\epsilon \circ\cdots\circ \tilde{\mathcal{L}}_\omega^\epsilon$, and by $\tilde{\mathbb{P}}$ the normalized restriction of $\mathbb{P}$ to [1] (so the convention is that quantities marked with tildes refer to the induced system).

We define the return times for $\omega \in [1]$ by $N_1(\omega) = N(\omega)$ and $N_{n+1}(\omega) = N_n(\omega) + N(\sigma^{N_n(\omega)}(\omega))$ for $n \ge 1$. Now we have, using Lemma 3.5(a),

$$\begin{aligned}\frac{1}{N_n(\omega)}\log\mathcal{D}_2(\mathcal{L}_\omega^{\epsilon\,(N_n(\omega))}) &= \frac{1}{N_n(\omega)}\log\mathcal{D}_2(\tilde{\mathcal{L}}_\omega^{\epsilon\,(n)})\\ &= \frac{n}{N_n(\omega)}\frac{1}{n}\log\mathcal{D}_2(\tilde{\mathcal{L}}^\epsilon_{\tilde\sigma^{n-1}\omega}\circ\cdots\circ\tilde{\mathcal{L}}^\epsilon_\omega)\\ &\le \frac{n}{N_n(\omega)}\frac{1}{n}\log\big(\mathcal{D}_2(\tilde{\mathcal{L}}^\epsilon_{\tilde\sigma^{n-1}\omega})\cdot\ldots\cdot\mathcal{D}_2(\tilde{\mathcal{L}}^\epsilon_\omega)\big)\\ &= \frac{n}{N_n(\omega)}\frac{1}{n}\sum_{i=0}^{n-1}\log\mathcal{D}_2(\tilde{\mathcal{L}}^\epsilon_{\tilde\sigma^i\omega})\\ &\le \frac{n}{N_n(\omega)}\frac{1}{n}\sum_{i=0}^{n-1}(-2\pi^2\epsilon^2 4^{N(\tilde\sigma^i\omega)-1}+\log A).\end{aligned}$$

Since $\int_{[1]} 4^{N(\omega)}\,d\tilde{\mathbb{P}}_p(\omega) = \sum_{n=1}^\infty 4^n p^{n-1}(1-p) = \infty$, we see that the average $\frac{1}{n}\sum_{i=0}^{n-1}(\ldots)$ in the last displayed line converges to $-\infty$ almost surely by Birkhoff's theorem applied to the ergodic transformation $\tilde\sigma$ of $([1],\tilde{\mathbb{P}}_p)$. As $n/N_n(\omega) \to 1/\mathbb{P}_p([1])$ for $\tilde{\mathbb{P}}_p$-almost every $\omega \in [1]$, we see that $\frac{1}{N_n(\omega)}\log\mathcal{D}_2(\mathcal{L}_\omega^{\epsilon\,(N_n(\omega))}) \to -\infty$ for $\tilde{\mathbb{P}}_p$-a.e. $\omega \in [1]$. Since this is a subsequence of the convergent sequence $\frac{1}{n}\log\mathcal{D}_2(\mathcal{L}_\omega^{\epsilon\,(n)})$, it follows that $\frac{1}{n}\log\mathcal{D}_2(\mathcal{L}_\omega^{\epsilon\,(n)}) \to -\infty$ for $\mathbb{P}_p$-a.e. ω.

This establishes, by Lemma 3.5(c), that $\lambda_2 = -\infty$. We recall from [12, Theorem 13] that the exceptional Lyapunov exponents of a cocycle and its adjoint cocycle coincide. By Lemma 3.3 (taking $g = 1$) and a calculation showing $\int_{C_1}\mathcal{N}^\epsilon(f)(z)\,dz = \int_{C_1} f(z)\,dz$ for all $f \in H^2(A_R)$, we see that $\phi(f) = \frac{1}{2\pi i}\int_{C_1} f(z)\,dz$ satisfies $\phi(f) = \phi(\mathcal{L}_\omega^\epsilon f)$, so that $(\mathcal{L}_\omega^\epsilon)^*\phi = \phi$. Hence $\lambda_1 = \lambda_1^* = 0$. ■

Remark. We briefly explain how to identify the family of equivariant functions, or top Oseledets spaces $V_1(\omega)$ for $\mathcal{L}^\epsilon$. For $\omega \in \Omega$ and $\mathbf{t} = (t_n)_{n\in\mathbb{Z}^-} \in \mathbb{R}^{\mathbb{Z}^-}$, we define

$$\Phi(\omega, \mathbf{t}) = \lim_{k\to\infty} R_{2\pi\epsilon t_{-1}} T_{\sigma^{-1}\omega} \circ \cdots \circ R_{2\pi\epsilon t_{-k}} T_{\sigma^{-k}\omega}(0),$$

where $R_\theta(z) = e^{i\theta} z$. Existence of the limit follows as in Corollary 4.2. The equivariant function is then given by

$$f_\omega(z) = \int_{\mathbb{R}^{\mathbb{Z}^-}} \left(\frac{1}{z - \Phi(\omega, \mathbf{t})} - \frac{1}{z - I(\Phi(\omega, \mathbf{t}))} \right) d\Gamma(\mathbf{t}),$$

where Γ is the measure on $\mathbb{R}^{\mathbb{Z}^-}$ where each coordinate is an independent standard normal random variable. The fact that $\mathcal{L}^\epsilon_\omega f_\omega = f_{\sigma\omega}$ essentially follows from Lemma 4.7 and Corollary 4.8. Since the range of Φ is contained in $\bar{D}_r$, it is easy to see that f_ω lies in $H^2(A_R)$ for all $\omega \in \Omega$.

5.2. *Uniform perturbations*

We now consider another perturbed version of the cocycle, where $\mathcal{L}_i$ is replaced by $\mathcal{L}_i^{U,\epsilon} := \mathcal{U}_\epsilon \circ \mathcal{L}_i$, where $\mathcal{U}_\epsilon^{\mathbb{R}/\mathbb{Z}}$ has the effect of convolving a density with a bump function of support $[-\epsilon, \epsilon]$. On $\mathbb{R}/\mathbb{Z}$, we have

$$(\mathcal{U}_\epsilon^{\mathbb{R}/\mathbb{Z}} f)(x) = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x - t)\, dt = \frac{1}{2} \int_{-1}^{1} f(x - \epsilon t)\, dt = \mathbb{E} f(x + \epsilon U),$$

where U is a uniformly distributed random variable on $[-1, 1]$. The corresponding conjugate operator on $C(C_1)$ is $\mathcal{U}_\epsilon := \mathcal{U}_\epsilon^{C_1} = Q^{-1}\mathcal{U}_\epsilon^{\mathbb{R}/\mathbb{Z}} Q$, where Q is as in Lemma 3.2. A calculation shows

$$(\mathcal{U}_\epsilon f)(z) = \frac{1}{2} \int_{-1}^{1} f(ze^{-2\pi i\epsilon t}) e^{-2\pi i\epsilon t}\, dt.$$

As before, we let $\hat{e}_n(z) = z^{n-1}$ and compute, for $n \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned} \mathcal{U}_\epsilon(\hat{e}_n)(z) &= \frac{1}{2} \int_{-1}^{1} \hat{e}_n(ze^{-2\pi i\epsilon t}) e^{-2\pi i\epsilon t}\, dt \\ &= \frac{z^{n-1}}{2} \int_{-1}^{1} e^{-2\pi i n\epsilon t}\, dt = \frac{\sin(2\pi n\epsilon)}{2\pi n\epsilon} \hat{e}_n(z). \end{aligned}$$

Also, $\mathcal{U}_\epsilon(\hat{e}_0)(z) = \hat{e}_0(z)$. Hence, for $\ell \in \mathbb{Z} \setminus \{0\}$,

$$(\mathcal{L}_0^{U,\epsilon})^n \hat{e}_\ell = \begin{cases} (2\pi m\epsilon)^{-n} 2^{-n(n-1)/2} \prod_{j=1}^{n} \sin(2^j m\pi\epsilon) \hat{e}_m & \text{if } \ell = 2^n m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Corollary 2.4. Let $\epsilon = b/2^k$ for some odd integer b and $k \in \mathbb{N}$. Then for every $\ell \in \mathbb{Z} \setminus \{0\}$ and $n \geq k$ we get $(\mathcal{L}_0^{U,\epsilon})^n \hat{e}_\ell = 0$. This immediately implies that $(\mathcal{L}_0^{U,\epsilon})^n|_{H_0^2(A_R)} = 0$, and so $\lambda_2(\mathcal{L}_\omega^{U,\epsilon}) = -\infty$. The fact that $\lambda_1(\mathcal{L}_\omega^{U,\epsilon}) = 0$ follows exactly as in Corollary 2.3. ∎

6. Lyapunov spectrum stability

It is natural to ask for an explanation of the instability of the Lyapunov spectrum, exhibited in Corollaries 2.3 and 2.4. From the finite-dimensional theory of hyperbolic dynamical systems, we know that a key issue in the stability is control of the angle between the fast and slow subspaces. In general, the various versions of the Multiplicative Ergodic Theorem show that for Oseledets spaces with different exponents, the angle between the subspaces is bounded away from 0 at least by a quantity that is at worst sub-exponentially small in n, the number of iterations. In the uniformly hyperbolic situation, this angle is uniformly bounded away from 0.

One way to quantify the angle between complementary closed subspaces that is particularly well suited to the infinite-dimensional case is to compute $\|\Pi_{E\|F}\|$, where $\Pi_{E\|F}$ is the projection that fixes E and annihilates F: for Hilbert spaces the norm of the projection is the reciprocal of the sine of the angle between the spaces. (See [12] for an alternative way of measuring angles.)

With this in mind, we study $\Pi_{E_k(\omega)\|F_k(\omega)}$ where $E_k(\omega)$ is the span of the Oseledets vectors with exponents $\lambda_1, \dots, \lambda_k$ and $F_k(\omega)$ is the complementary space of vectors that expand at rate λ_{k+1} or slower (the $(2k-1)$-dimensional *fast space* and $(2k-1)$-codimensional *slow space* respectively).

For the unperturbed cocycle appearing in Corollaries 2.3 and 2.4, we claim that $\|\Pi_{E_k(\omega)\|F_k(\omega)}\|$ is essentially unbounded in ω. To see this, let σ be the shift map on $\{0,1\}^{\mathbb{Z}}$ equipped with the Bernoulli probability measure $\mathbb{P}_p$ and $(\mathcal{L}_\omega)$ as before. Set $h_\omega(z) = (z-x_\omega)^{-2}$ and $g(z) = z^{-2}$. Note that if $\omega_0 = 0$, then $\mathcal{L}_\omega g = 0$ (see (5.1)), so that $g \in F_2(\omega)$. Also $h_\omega \in E_2(\omega)$ by Lemma 4.12, and if $\omega_{-n} = \cdots = \omega_{-1} = \omega_0 = 0$, then $x_\omega \le a^{2^n}$, where $a < 1$ is as in Lemma 5.1. In particular $\operatorname{ess\,inf}_\omega \|h_\omega - g\| = 0$. However, since $\Pi_{E_2(\omega)\|F_2(\omega)}(h_\omega - g) = h_\omega$ and $\|h_\omega\|$ is bounded away from 0, we see that $\|\Pi_{E_2(\omega)\|F_2(\omega)}\|$ is essentially unbounded for the cocycle, so that the fast and slow spaces become arbitrarily close.

The core of the issue is that the kernel of $\mathcal{L}_0$ includes all even integer powers of $z-0$ (0 being the critical point of T_0), while combinations of negative powers of $z - x_\omega$ appear in the fast spaces. To avoid the situation above, one is led to consider situations in which the critical point(s) of T_ω are bounded away from the random fixed point x_ω. We impose this by assuming a lower bound on $|T'_\omega(x_\omega)|$.

In this section, we show that if we impose the condition that $|T'_\omega(x_\omega)|$ is bounded below, then the projections $\Pi_{E_k(\omega)\|F_k(\omega)}$ are uniformly bounded, and hence the fast and slow spaces are uniformly transverse. We deduce the existence of a field of cones around $E_k(\omega)$, and use this to prove Theorem 2.5, showing that the Lyapunov spectrum of the Perron–Frobenius cocycle is stable under small perturbations. Conversely, if $T'_\omega(x_\omega)$ is not bounded below, we show that small perturbations to the Perron–Frobenius cocycle, even within the class of Perron–Frobenius operators of Blaschke products, can lead to collapse of the Lyapunov spectrum.

In the first part of the section, we replace the Blaschke product cocycle (T_ω) with a conjugate cocycle $(\tilde{T}_\omega)$, almost all of whose elements fix the origin. We first prove the theorem in that context, and then show how the full theorem follows.

6.1. Bounded projections

Let $\sigma: (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$ be an invertible ergodic measure-preserving transformation and let $\mathcal{T} = (T_\omega)_{\omega\in\Omega}$ be a Blaschke product cocycle satisfying the following conditions:

(a) $T_\omega(0) = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

(b) $\operatorname{ess\,inf}_\omega |T'_\omega(0)| > 0$.

(c) $r_{\mathcal{T}}(R) < R$.

Notice that for Blaschke product cocycles satisfying these conditions, Theorem 2.1 applies and the quantity $\Lambda = \int \log |T'_\omega(0)|\, d\mathbb{P}$ is finite (and negative), so that the Lyapunov exponents are $\lambda_j = (j-1)\Lambda$, where $\lambda_1 = 0$ has multiplicity 1 and the remaining exponents have multiplicity 2. Notice that (b) gives uniform control on $|T'_\omega(x_\omega)|$, while the condition $\Lambda > -\infty$ in Theorem 2.1 for a non-trivial Lyapunov spectrum gives only average control on $|T'_\omega(x_\omega)|$.

The proof of Theorem 2.1 (in the special case $x_\omega = 0$) shows that for $j > 1$, one can identify a natural basis for $V_j(\omega)$ consisting of a Laurent polynomial $f_{\omega,j}$ with $z^{-2}, \dots, z^{-j}$ terms and its inversion $\mathcal{L}_I f_{\omega,j}$ with $z^0, \dots, z^{j-2}$ terms.

We set $r = r_{\mathcal{T}}(R)$. We shall also require that $|T''_\omega|$ is uniformly bounded above on D_r, but in fact, this condition is true automatically since

$$T''(z) = \frac{2!}{2\pi i} \int_{C_1} \frac{T(w)}{(w-z)^3}\, dw$$

and $|T(w)| = 1$ whenever $|w| = 1$, so that $|T''(w)| \le 2/(1-r)^3$ on D_r.

Let $\lambda^{(n)}_\omega$ denote $|(T^{(n)}_\omega)'(0)|$ in all what follows.

Lemma 6.1 (Random fixed point distortion estimate). *Under the conditions above, there exists a c_1 such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, for all $z \in C_R$,*

$$|T^{(n)}_\omega(z)| \le c_1 \lambda^{(n)}_\omega.$$

Proof. Let $\gamma_{\omega,n}(z) = |T^{(n)}_\omega z|/\lambda^{(n)}_\omega$ and $r = r_{\mathcal{T}}(R) < R$. Then

$$\begin{aligned}\gamma_{\omega,n+1}(z) &= \frac{|T^{(n+1)}_\omega z - 0|}{\lambda^{(n+1)}_\omega} \\ &\le \frac{\max_{y\in[0,T^{(n)}_\omega z]} |T'_{\sigma^n\omega}(y)| \cdot |T^{(n)}_\omega z - 0|}{\lambda^{(n)}_\omega T'_{\sigma^n\omega}(0)} \\ &= \gamma_{\omega,n}(z) \max_{y\in[0,T^{(n)}_\omega z]} |T'_{\sigma^n\omega}(y)|/|T'_{\sigma^n\omega}(0)|,\end{aligned}$$

where $[a, b]$ denotes the line segment joining a and b. Since we showed in Lemma 4.1 the existence of a c such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $|T^{(n)}_\omega(z) - 0| \le c(r/R)^{n-1}$, we see that for $y \in [0, T^{(n)}_\omega(z)]$, using the uniform boundedness of T''_ω, we obtain the inequality $|T'_{\sigma^n\omega}(y)/T'_{\sigma^n\omega}(0) - 1| \le c'(r/R)^n$, so that $\gamma_{\omega,n}(z)$ is bounded essentially uniformly in ω and uniformly in n and z as z runs over C_R. ∎

Let $H^2(A_R)^-$ be $\mathrm{lin}\{z^{-j-1}\colon j > 0\}$ and $H^2(A_R)^+$ be $\mathrm{lin}\{z^{j-1}\colon j > 0\}$. (We choose to offset the indices by 1 as we have seen that this is well suited to the calculations in the rest of the paper). We set $\hat{e}_j(z) = R^{|j-1|}z^{j-1}$, as a convenient rescaling of the standard orthogonal basis of $H^2(A_R)$ described earlier (in particular their norms are between 1 and 2).

Lemma 6.2. *Let $\rho < R$ and let f lie in the unit sphere of $H^2(A_\rho)^-$. Then f may be expanded as $f(z) = \sum_{m>0} a_{-m}\hat{e}_{-m}$ where $|a_{-m}| \leq (\rho/R)^m$. In particular, the partial sums of the series for f given above are convergent in $H^2(A_R)$.*

Let U_k^- be the subspace of $H^2(A_R)^-$ spanned by $\hat{e}_{-1}, \ldots, \hat{e}_{-(k-1)}$ and define

$$V_k^-(\omega) = \left\{f \in H^2(A_R)^-\colon\ \limsup \tfrac{1}{n}\log\|\mathcal{L}_\omega^{(n)} f\| \leq k\Lambda\right\},$$

so that $H^2(A_R)^- = U_k^- \oplus V_k^-(\omega)$. In particular, U_k^- is the span of those Oseledets vectors, mentioned above, with Lyapunov exponents $\lambda_2, \ldots, \lambda_k$ that are polynomials in z^{-1}. Let Q^- denote the orthogonal projection of $H^2(A_R)$ onto $H^2(A_R)^-$ and let Q^+ be the orthogonal projection of $H^2(A_R)$ onto $H^2(A_R)^+ = \mathrm{lin}\{z^{n-1}\colon n \geq 1\}$. We write $H^2(A_R)^\pm$ for $H^2(A_R)^+ \oplus H^2(A_R)^-$ and note that $H^2(A_R)^\pm = (\mathrm{lin}\{z^{-1}\})^\perp$. We now show that under the conditions above, U_k^- and $V_k^-(\omega)$ are uniformly transverse.

Lemma 6.3. *Let the Blaschke product cocycle satisfy conditions (a)–(c). Then for each $k \in \mathbb{N}$, there exists a constant $M > 0$ such that $\|\Pi_{U_k^-\|V_k^-(\omega)}\| \leq M$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. Let c_1 be as in Lemma 6.1, and let $\hat{e}_{-j}$ be as defined earlier. We compute the matrix of $\mathcal{L}_\omega$ with respect to the $(\hat{e}_{-j})$ basis.

For $j > 0$, write $\mathcal{L}_\omega(\hat{e}_{-j})$ as $\sum_{i=1}^{j} a_{ij}\hat{e}_{-i}$ (such an expansion exists with no higher order terms by Corollary 4.5). We now have, for $0 < i < j$,

$$\begin{aligned} a_{ij} &= \frac{1}{2\pi i}\int_{C_1} \mathcal{L}_\omega(\hat{e}_{-j})(z)R^{-i-1}z^i\,dz = \frac{R^{-i-1}}{2\pi i}\int_{C_1}\hat{e}_{-j}(z)(T_\omega(z))^i\,dz \\ &= \frac{R^{j-i}}{2\pi i}\int_{C_R}\frac{(T_\omega(z))^i}{z^{j+1}}\,dz. \end{aligned}$$

For $i = j$, we have $a_{jj} = (\lambda_\omega)^j$ as shown in Theorem 2.1.

Similarly, using Lemma 6.1, we see that for a.e. $\omega \in \Omega$, $a_{\omega,ij}^{(n)}$, the coefficient of $\mathcal{L}_\omega^{(n)}$ with respect to $(\hat{e}_{-j})$ and $(\hat{e}_{-i})$, satisfies

$$\begin{aligned} |a_{\omega,ij}^{(n)}| &= \left|\frac{R^{j-i}}{2\pi i}\int_{C_R}\frac{(T_\omega^{(n)}(z))^i}{z^{j+1}}\,dz\right| \\ &\leq (c_1\lambda_\omega^{(n)}/R)^i \quad \text{for all } 1 \leq i \leq j. \end{aligned} \tag{6.1}$$

For $i = j$ we have $a_{jj}^{(n)} = (\lambda_\omega^{(n)})^j$, and for $i > j$, $a_{ij}^{(n)} = 0$. Letting $c_2 = (c_1/R)^{k-1}$, we see that for a.e. $\omega \in \Omega$,

$$|a_{\omega,ij}^{(n)}| \leq c_2(\lambda_\omega^{(n)})^i \quad \text{for all } 1 \leq i \leq k-1 \text{ and all } j \in \mathbb{N}. \tag{6.2}$$

Define Π_{-k} to be the orthogonal projection from $H^2(A_R)^-$ onto U_k^-. Now consider the operators $\Lambda^{(n)}_{\omega,k} = (\mathcal{L}^{(n)}_\omega|_{U_k^-})^{-1} \circ \Pi_{-k} \circ \mathcal{L}^{(n)}_\omega$. We write

$$\Lambda^{(n)}_{\omega,k} = (\mathcal{L}_\omega|_{U_k^-})^{-1} \circ (\mathcal{L}^{(n-1)}_{\sigma\omega}|_{U_k^-})^{-1} \circ \Pi_{-k} \circ \mathcal{L}^{(n-1)}_{\sigma\omega} \circ \mathcal{L}_\omega.$$

By (6.2), the matrix representing the restricted operator $(\mathcal{L}^{(n-1)}_{\sigma\omega}|_{U_k^-})$ with respect to the bases $\hat{e}_{-1}, \dots, \hat{e}_{-(k-1)}$ and $\hat{e}_{-1}, \dots, \hat{e}_{-(k-1)}$ may be factorized as DA where D is the diagonal matrix with entries $(\lambda^{(n-1)}_{\sigma\omega})^j$ for $j = 1, \dots, k-1$ and A is an upper triangular matrix with 1's on the diagonal and with entries with absolute value bounded above by c_2 above the diagonal for a.e. $\omega \in \Omega$. Of course we have $(DA)^{-1} = A^{-1}D^{-1}$, and by the above, A^{-1} has uniformly bounded entries (i.e. with entries that are uniformly bounded for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $n > 0$ by a constant depending only on k). The matrix $B = (b_{ij})_{1\le i\le k-1; j\in\mathbb{N}}$ of $(\mathcal{L}^{(n-1)}_{\sigma\omega}|_{U_k^-})^{-1}\Pi_{-k}\mathcal{L}^{(n-1)}_{\sigma\omega}$ is $B = A^{-1}D^{-1}PL^{(n-1)}$, where P is the $(k-1)\times\infty$ matrix with 1's on the diagonal and the remaining entries 0, and $L^{(n-1)}$ is the matrix of $\mathcal{L}^{(n-1)}_{\sigma\omega}$. Combining (6.2) with the expression for D, we obtain a constant c_3 such that for a.e. $\omega \in \Omega$ and all n, $|b_{ij}| \le c_3$ for $1 \le i \le k-1$ and $j \in \mathbb{N}$. (Note that c_3 does depend on k.)

Let $r < \rho < R$. By Corollary 4.10, there a constant c_4 such that $\|\mathcal{L}_\omega\|_{H^2(A_R)\to H^2(A_\rho)} \le c_4$ for almost all $\omega \in \Omega$. Now if $f \in H^2(A_R)^-$ satisfies $\|f\|_{H^2(A_R)} = 1$, we have $\|\mathcal{L}_\omega f\|_{H^2(A_\rho)} \le c_4$, so that by Lemma 6.2, for $j > 0$, the coefficient of $\hat{e}_{-j}$ in the expansion of $\mathcal{L}_\omega f$ is at most $c_4(\rho/R)^j$.

Combining this with the estimate for b_{ij} above, we see that the $\hat{e}_{-i}$ coefficient of $(\mathcal{L}^{(n-1)}_{\sigma\omega}|_{U_k^-})^{-1}\Pi_{\sigma^n\omega,k}\mathcal{L}^{(n)}_\omega f$ is bounded above by $c_3c_4\sum_{j=i}^\infty(\rho/R)^j = c_3c_4(\rho/R)^i/(1-\rho/R)$. Since $(\mathcal{L}_\omega|_{U_k^-})^{-1}$ is essentially uniformly bounded in ω (because it is upper triangular with uniformly bounded entries, and has diagonal elements uniformly bounded away from 0), we deduce that the operators $\Lambda^{(n)}_{\omega,k}$ are uniformly bounded in n and ω (but not in k), say $\|\Lambda^{(n)}_{\omega,k}\| \le M$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$. It is immediate that $\Lambda^{(n)}_{\omega,k} f = f$ for all $f \in U_k^-$ and it is not hard to verify from the definition of $\Lambda^{(n)}_{\omega,k}$ that $\Lambda^{(n)}_{\omega,k} f \to 0$ for $f \in V_k^-(\omega)$. Hence $\Lambda^{(n)}_{\omega,k}$ converges strongly to $\Pi_{U_k^-\|V_k^-(\omega)}$, so that $\|\Pi_{U_k^-\|V_k^-(\omega)}\| \le M$. ∎

We define further spaces: $U_k^+ = \mathcal{L}_I U_k^-$, $V_k^+(\omega) = \mathcal{L}_I V_k^-(\omega)$ and $W_0 = \text{lin}\,\{\hat{e}_0\}$. Since $\mathcal{L}_I$ commutes with $\mathcal{L}_\omega$, the $\mathcal{L}_\omega$ equivariance of U_k^- implies that of U_k^+ and similarly equivariance of $V_k^+(\omega)$ follows from that of $V_k^-(\omega)$. The equivariance of W_0 was noted in Corollary 4.8 (note that $x_\omega = 0$ for the cocycles we are considering). Since $H^2(A_R)^+ = \mathcal{L}_I(H^2(A_R)^-)$, we see that $H^2(A_R)^+ = U_k^+ \oplus V_k^+(\omega)$.

Corollary 6.4. *There exists $M' > 0$ such that $\|\Pi_{U_k^+\|V_k^+(\omega)}\| \le M'$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. We can verify that $\Pi_{U_k^+\|V_k^+(\omega)} = \mathcal{L}_I \circ \Pi_{U_k^-\|V_k^-(\omega)} \circ \mathcal{L}_I$. Since $\mathcal{L}_I$ is bounded, the result follows. ∎

Next, let $E_k = U_k^- \oplus W_0 \oplus U_k^+$ and $F_k(\omega) = V_k^-(\omega) \oplus V_k^+(\omega)$. These are the $(2k-1)$-dimensional and $(2k-1)$-codimensional fast and slow spaces for the cocycle respectively.

We observe that $\Pi_0 := \Pi_{W_0 \| H^2(A_R)^\pm}$ is just the orthogonal projection onto W_0, so that $\|\Pi_{W_0\|H^2(A_R)^\pm}\| = 1$.

Corollary 6.5. *There exists $M > 0$ such that $\|\Pi_{E_k\|F_k(\omega)}\| \leq M$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. We have

$$\Pi_{E_k\|F_k(\omega)} = \Pi_0 + \Pi_{U_k^-\|V_k(\omega)^-} \circ Q^- + \Pi_{U_k^+\|V_k(\omega)^+} \circ Q^+.$$

Since all the operators appearing on the right hand side are uniformly bounded in ω, so is $\Pi_{E_k\|F_k(\omega)}$. ∎

6.2. *Invariant cone field*

In this subsection, we shall show that there is a cone around E_k attracting a neighbourhood in $H^2(A_R)$. Consider the cone

$$\mathcal{C}_{\omega,\eta} = \{f \in H^2(A_R) \colon \|\Pi_{F_k(\omega)\|E_k} f\| \leq \eta \|\Pi_{E_k\|F_k(\omega)} f\|\}.$$

Lemma 6.6. *Let σ be an ergodic invertible measure-preserving transformation of $(\Omega, \mathbb{P})$ and let the Blaschke product cocycle satisfy conditions (a)–(c) from §6.1. Then for each $k \in \mathbb{N}$, there exists an N such that for each $n \geq N$ and a.e. $\omega \in \Omega$, $\mathcal{L}_\omega^{(n)}(\mathcal{C}_{\omega,\eta}) \subset \mathcal{C}_{\sigma^n\omega,\eta/2}$ for all $\eta > 0$.*

Proof. Let Π_{-k} be as in the proof of Lemma 6.3, the orthogonal projection onto $\operatorname{lin}\{\hat{e}_{-1}, \ldots, \hat{e}_{-(k-1)}\}$, and let Π_k be the orthogonal projection onto $\operatorname{lin}\{\hat{e}_1, \ldots, \hat{e}_{k-1}\}$. Let $S_k = \Pi_{-k} + \Pi_k$ and let Π_0 be the orthogonal projection onto $\operatorname{lin}\{\hat{e}_0\}$.

We make the following claims:

(i) There exists a constant $K > 0$ such that for a.e. $\omega \in \Omega$, $n \in \mathbb{N}$ and $f \in U_k^-$,

$$\|\mathcal{L}_\omega^{(n)} f\| \geq K(\lambda_\omega^{(n)})^{k-1} \|f\|.$$

(ii) There exists $c_5 > 0$ such that for a.e. $\omega \in \Omega$, $n \in \mathbb{N}$ and $f \in E_k$,

$$\|\mathcal{L}_\omega^{(n)} f\| \geq c_5(\lambda_\omega^{(n)})^{k-1} \|f\|.$$

(iii) There exist $c > 0$ and $n_0 \in \mathbb{N}$ such that for a.e. $\omega \in \Omega$, $n \geq n_0$ and $f \in H^2(A_R)^-$,

$$\|(I - \Pi_{-k})\mathcal{L}_\omega^{(n)} f\| \leq c(\lambda_\omega^{(n)})^k \|f\|.$$

(iv) There exist $c_6 > 0$ and $n_0 \in \mathbb{N}$ such that for a.e. $\omega \in \Omega$, $n \geq n_0$ and $f \in H^2(A_R)^\pm$,

$$\|(I - S_k)\mathcal{L}_\omega^{(n)} f\| \leq c_6(\lambda_\omega^{(n)})^k \|f\|.$$

To establish (i), first notice that there is a constant $c > 1$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all vectors $(a_1, \dots, a_{k-1})$,

$$\|(a_1, \dots, a_{k-1})\|_2 / c \le \left\| \sum_{i=1}^{k-1} a_i \hat{e}_{-i} \right\| \le c \|(a_1, \dots, a_{k-1})\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^k$. Hence it suffices to demonstrate that there exists $c > 0$ such that $\|L_\omega^{(n)} v\| \ge c(\lambda_\omega^{(n)})^{k-1} \|v\|$ for all $v \in \mathbb{R}^k$, where $L_\omega^{(n)}$ is the matrix of $\mathcal{L}_\omega^{(n)}$ with respect to $(\hat{e}_{-j})_{j=1}^{k-1}$. This is equivalent to showing the existence of a $c > 0$ such that $\|(L_\omega^{(n)})^{-1}\| \le c(\lambda_\omega^{(n)})^{-(k-1)}$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$. We have shown that $L_\omega^{(n)}$ may be expressed as DA where D is the diagonal matrix with entries $(\lambda_\omega^{(n)})^i$, with i going from 1 to $k-1$; and A is an upper triangular matrix with bounded entries and 1's on the diagonal. It follows that $\|(L_\omega^{(n)})^{-1}\| \le c(\lambda_\omega^{(n)})^{-(k-1)}$ as required, so that we have demonstrated (i).

To show (ii), let $f \in E_k$, and write $f = f^+ + f^0 + f^-$ where the components lie in U_k^+, W_0 and U_k^- respectively. Clearly at least one of the components has norm at least $\|f\|/3$. Since $\mathcal{L}_\omega^{(n)} f^\star = \Pi_\omega^\star \mathcal{L}_\omega^{(n)} f$, where $\star$ is each of $+$, 0 or $-$, we have

$$\|\mathcal{L}_\omega^{(n)} f\| \ge \max(\|\mathcal{L}_\omega^{(n)} f^+\|, \|\mathcal{L}_\omega^{(n)} f^0\|, \|\mathcal{L}_\omega^{(n)} f^-\|).$$

Hence it suffices to show that for each of $-$, 0 and $+$ there exists a $c^\star > 0$ such that for all $\omega \in \Omega$ and $n \in \mathbb{N}$,

$$\|\mathcal{L}_\omega^{(n)} f\| \ge c^\star (\lambda_\omega^{(n)})^{k-1} \|f\| \tag{6.3}$$

for f lying in $U_k^\star$.

The first of these was demonstrated in (i). If $f \in W_0$, then $\mathcal{L}_\omega^{(n)} f = f$, so that $\|\mathcal{L}_\omega^{(n)} f\| = \|f\|$. Hence (6.3) is satisfied when $\star$ is 0. If $f \in U_k^+$, then $\mathcal{L}_\omega^{(n)} f = \mathcal{L}_I \mathcal{L}_\omega^{(n)} \mathcal{L}_I f$. Since $\mathcal{L}_I$ is a bounded involution, it follows that there exists $c > 0$ such that $\|\mathcal{L}_I f\| \ge c\|f\|$ for all $f \in H^2(A_R)$. Combining this with (i) establishes the required result for f lying in U_k^+.

We now demonstrate (iii). Let ρ be as chosen in the proof of Lemma 6.3 and let $f \in H^2(A_R)^-$ be of norm 1. We showed in the proof of Lemma 6.3 that the $\hat{e}_{-j}$ coefficient of $\mathcal{L}_\omega f$ is at most $c_4(\rho/R)^j$. Applying $\mathcal{L}_{\sigma\omega}^{(n-1)}$ and recalling the estimate (6.1), we find that the coefficient of $\hat{e}_{-i}$ is at most $\sum_{j \ge i} c_4(\rho/R)^j (c_1 \lambda_{\sigma\omega}^{(n-1)}/R)^i$. This gives the estimate

$$\begin{aligned}
\|(I - \Pi_{-k})\mathcal{L}_\omega^{(n)} f\| &\le \sum_{i \ge k} \sum_{j \ge i} c_4(\rho/R)^j (c_1 \lambda_{\sigma\omega}^{(n-1)}/R)^i \|\hat{e}_{-i}\| \\
&\le \frac{2c_4}{1-\rho/R} \sum_{i \ge k} (\rho c_1 \lambda_{\sigma\omega}^{(n-1)}/R^2)^i \\
&= \frac{2c_4}{1-\rho/R} (\rho c_1/R^2)^k (\lambda_{\sigma\omega}^{(n-1)})^k \sum_{i \ge 0} (\rho c_1 \lambda_{\sigma\omega}^{(n-1)}/R^2)^i.
\end{aligned}$$

From the final part of the proof of Theorem 2.1, there exists $c' > 0$ such that for a.e. $\omega \in \Omega$, $\lambda_\omega^{(n)} \le c'(r/R)^n$ for all n. Thus, there exists an n_0 such that for all $n \ge n_0$ and a.e. $\omega \in \Omega$, $\rho c_1 \lambda_\omega^{(n-1)}/R^2 < 1/2$. Since $|T'_\omega(0)|$ is essentially uniformly bounded below, we have $\lambda_{\sigma\omega}^{(n-1)} \le \lambda_\omega^{(n)}/\operatorname{ess\,inf}_\omega |T'_\omega(0)|$ a.e. Now for $n \ge n_0$, and all f in the unit sphere of $H^2(A_R)^-$,

$$\|(I - \Pi_{-k})\mathcal{L}_\omega^{(n)} f\| \le c(\lambda_\omega^{(n)})^k,$$

where $c = 4c_4(\rho c_1/R^2)^k/((1-\rho/R)\inf_\omega |T'_\omega(0)|^k)$, proving (iii).

Finally, to show (iv), let $f \in H^2(A_R)^\pm$, and write $f = f^+ + f^-$. Since Q^+ and Q^- are orthogonal projections, both $\|f^+\|$ and $\|f^-\|$ are bounded above by $\|f\|$. The above shows $\|(I - S_k)\mathcal{L}_\omega^{(n)} f^-\| \le c(\lambda_\omega^{(n)})^k \|f\|$. Also $(I - S_k)\mathcal{L}_\omega^{(n)} f^+ = \mathcal{L}_I(I - S_k)\mathcal{L}_\omega^{(n)} \mathcal{L}_I f^+$. Since $\mathcal{L}_I f^+ \in H^2(A_R)^-$, we see

$$\|\mathcal{L}_I(I - S_k)\mathcal{L}_\omega^{(n)} \mathcal{L}_I f^+\| \le \|\mathcal{L}_I\|^2 c(\lambda_\omega^{(n)})^k \|f\|,$$

so that the desired conclusion follows by summing the two estimates.

Now let $f \in \mathcal{C}_{\omega,\eta}$ and let $f = u + v$ where $u \in E_k$ and $v \in F_k(\omega)$. Applying the above inequalities, we have

$$\|\mathcal{L}_\omega^{(n)} u\| \ge c_5(\lambda_\omega^{(n)})^{k-1}\|u\|;$$

and since $F_k(\omega) \subset H^2(A_R)^\pm$, for $n \ge n_0$,

$$\|(1 - S_k)\mathcal{L}_\omega^{(n)} v\| \le c_6(\lambda_\omega^{(n)})^k \|v\|.$$

Notice that $\Pi_{E_k \| F_k(\sigma^n\omega)}(1 - S_k)\mathcal{L}_\omega^{(n)} v = -S_k \mathcal{L}_\omega^{(n)} v$, so that by Lemma 6.3, $\|S_k \mathcal{L}_\omega^{(n)} v\| \le M\|(1 - S_k)\mathcal{L}_\omega^{(n)} v\|$. Hence

$$\|\mathcal{L}_\omega^{(n)} v\| \le (M + 1)\|(1 - S_k)\mathcal{L}_\omega^{(n)} v\| \le c_6(M + 1)(\lambda_\omega^{(n)})^k \|v\|.$$

Now there exists an n_1 such that for all $n \ge n_1$ and a.e. $\omega \in \Omega$, $c_6(M + 1)\lambda_\omega^{(n)} \le \frac{1}{2}c_5$. Hence if $n \ge \max(n_0, n_1)$, we see that $\mathcal{L}_\omega^{(n)} f \in \mathcal{C}_{\sigma^n\omega,\eta/2}$. ■

6.3. Stability of exponents

Our theorem in this section is similar to a theorem of Bogenschütz [6]. In his setting, there were two key assumptions: uniformity of the splitting and uniformity of convergence to the Lyapunov exponents. The first of these is satisfied in our setting by the above, while we relax the second condition by not imposing any convergence conditions on the Lyapunov exponents.

Lemma 6.7. *Let σ be an ergodic invertible measure-preserving transformation of $(\Omega, \mathbb{P})$, let the Blaschke product cocycle satisfy conditions (a)–(c) from §6.1, and let $\mathcal{L}_\omega$ be the corresponding family of Perron–Frobenius operators. For each $\epsilon > 0$, let $\mathcal{L}_\omega^\epsilon$ be a family of operators such that* $\operatorname{ess\,sup}_{\omega\in\Omega} \|\mathcal{L}_\omega^\epsilon - \mathcal{L}_\omega\| \to 0$ *as $\epsilon \to 0$. Then the perturbed cocycle is quasi-compact. If (μ_n) are the Lyapunov exponents of the unperturbed cocycle listed with multiplicity, then for each n, $\mu_n^\epsilon \to \mu_n$ as $\epsilon \to 0$ where (μ_n^ϵ) are the exponents of the perturbed cocycle.*

Proof. Notice that $\kappa(\mathcal{L}_\omega^\epsilon) \le \kappa(\mathcal{L}_\omega) + \|\mathcal{L}_\omega^\epsilon - \mathcal{L}_\omega\|$, which, together with the hypothesis $\operatorname{ess\,sup}_{\omega\in\Omega} \|\mathcal{L}_\omega^\epsilon - \mathcal{L}_\omega\| \to 0$ as $\epsilon \to 0$, implies the quasi-compactness of the perturbed cocycle.

Let $\Lambda = \int \log |T_\omega'(0)|\, d\mathbb{P}(\omega) < 0$, so that by Theorem 2.1, the Lyapunov exponents are $\lambda_j = (j-1)\Lambda$ for $j = 1, 2, \ldots$, where $\lambda_1 = 0$ has multiplicity 1 and the remaining exponents have multiplicity 2. List the exponents with multiplicity as $\mu_1 = 0$, and $\mu_{2k} = \mu_{2k+1} = k\Lambda$ for each $k \in \mathbb{N}$.

Let N be as in the statement of Lemma 6.6 and M be as in the statement of Lemma 6.3 and c_5 be as in the proof of Lemma 6.6. Let $\eta > 0$. Pick $n > N$ so that $(\frac{3}{4}c_5)^{1/n} > e^{-\eta}$. Now by Corollary 4.10, $\operatorname{ess\,sup}_{\omega\in\Omega} \|\mathcal{L}_\omega\|$ is finite. By an application of the triangle inequality, $\operatorname{ess\,sup}_{\omega\in\Omega} \|\mathcal{L}_\omega^{\epsilon\,(n)} - \mathcal{L}_\omega^{(n)}\| \to 0$ as $\epsilon \to 0$. Pick $\epsilon_0 > 0$ so that $\epsilon < \epsilon_0$ implies

$$\operatorname*{ess\,sup}_\omega \|\mathcal{L}_\omega^{\epsilon\,(n)} - \mathcal{L}_\omega^{(n)}\| < \frac{c_5 \operatorname{ess\,inf}_\omega |T_\omega'(0)|^{n(k-1)}}{8(M+1)}. \tag{6.4}$$

Suppose that $\epsilon < \epsilon_0$ and let $u + v \in \mathcal{C}_{\omega,1}$, where $u \in E_k$ and $v \in F_k(\omega)$. We claim that for a.e. $\omega \in \Omega$,

$$\mathcal{L}_\omega^{\epsilon\,(n)}(u+v) \in \mathcal{C}_{\sigma^n\omega,1}, \tag{6.5}$$

$$\|\Pi_{E_k\|F_k(\sigma^n\omega)} \mathcal{L}_\omega^{\epsilon\,(n)}(u+v)\| \ge \frac{3c_5}{4} (\lambda_\omega^{(n)})^{k-1} \|u\|. \tag{6.6}$$

Writing $\Pi_{E\|F}^{(n)}$ for $\Pi_{E_k\|F_k(\sigma^n\omega)}$ and $\Pi_{F\|E}^{(n)}$ for $I - \Pi_{E\|F}^{(n)}$, we have

$$\Pi_{E\|F}^{(n)} \mathcal{L}_\omega^{\epsilon\,(n)}(u+v) = \mathcal{L}_\omega^{(n)} u + \Pi_{E\|F}^{(n)} (\mathcal{L}_\omega^{\epsilon\,(n)} - \mathcal{L}_\omega^{(n)})(u+v),$$

so that

$$\begin{aligned}\|\Pi_{E\|F}^{(n)} \mathcal{L}_\omega^{(n)}(u+v)\| &\ge c_5 (\lambda_\omega^{(n)})^{k-1}\|u\| - M \frac{c_5 \operatorname{ess\,inf}_\omega |T_\omega'(0)|^{n(k-1)}}{8(M+1)} 2\|u\| \\ &\ge \tfrac{3}{4} c_5 (\lambda_\omega^{(n)})^{k-1} \|u\|,\end{aligned}$$

establishing (6.6). On the other hand,

$$\Pi_{F\|E}^{(n)} \mathcal{L}_\omega^{\epsilon\,(n)}(u+v) = \mathcal{L}_\omega^{(n)} v + \Pi_{F\|E}^{(n)} (\mathcal{L}_\omega^{\epsilon\,(n)} - \mathcal{L}_\omega^{(n)})(u+v),$$

so that

$$\begin{aligned}\|\Pi_{F\|E}^{(n)} \mathcal{L}_\omega^{\epsilon\,(n)}(u+v)\| &\le \tfrac{1}{2} c_5 (\lambda_\omega^{(n)})^{k-1}\|v\| + (M+1) \frac{c_5 \operatorname{ess\,inf}_\omega |T_\omega'(0)|^{n(k-1)}}{8(M+1)} 2\|u\| \\ &\le \tfrac{3}{4} c_5 (\lambda_\omega^{(n)})^{k-1} \|u\|.\end{aligned}$$

This implies $\mathcal{L}_\omega^{\epsilon\,(n)}(u+v) \in \mathcal{C}_{\sigma^n\omega,1}$.

Using (6.5) inductively, we see that $\mathcal{L}_\omega^{\epsilon\,(mn)}(u+v) \in \mathcal{C}_{\sigma^{mn}\omega,1}$ for all $m \in \mathbb{N}$. Then an inductive application of (6.6) shows that for any $m \in \mathbb{N}$,

$$\begin{aligned}\|\Pi_{E_k\|F_k(\sigma^{mn}\omega)}\mathcal{L}_\omega^{\epsilon\,(mn)}(u+v)\| &\geq (3c_5/4)^m(\lambda_\omega^{(mn)})^{k-1}\|u\| \\ &\geq e^{-\eta mn}(\lambda_\omega^{(mn)})^{k-1}\|u\|.\end{aligned}$$

Since $(mn)^{-1}\log(e^{-\eta mn}(\lambda_\omega^{(mn)})^{k-1})$ converges to $\lambda_k - \eta$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have identified a $(2k-1)$-dimensional subspace, namely E_k, on which every vector has Lyapunov exponent at least $\lambda_k - \eta$, so that the Lyapunov exponents of the perturbed cocycle, μ_j^ϵ (again listed with multiplicity), satisfy $\mu_{2k-1}^\epsilon > \mu_{2k-1} - \eta$ and $\mu_{2k-2}^\epsilon \geq \mu_{2k-1}^\epsilon > \mu_{2k-1} - \eta = \mu_{2k-2} - \eta$. Since η and k are arbitrary, this establishes lower semicontinuity of each Lyapunov exponent for arbitrary small perturbations of the original cocycle.

To prove the continuity of the exponents for perturbations of the cocycle, it suffices to show the standard property of upper semicontinuity of the partial sums of the Lyapunov exponents: for each l and each $\eta > 0$, for all sufficiently small $\epsilon > 0$, one has

$$\mu_1^\epsilon + \cdots + \mu_l^\epsilon < \mu_1 + \cdots + \mu_l + \eta.$$

To show this, define

$$\mathcal{E}_l(\mathcal{L}) = \sup_{f_1,\ldots,f_l;\,\phi_1,\ldots,\phi_l} \det(\phi_i(\mathcal{L}f_j))_{1\leq i,j\leq l},$$

where $f_1,\ldots,f_l$ and $\phi_1,\ldots,\phi_l$ run over the unit sphere of $H^2(A_R)$ and the unit sphere of the dual space respectively. The quantity $\mathcal{E}_l$ is submultiplicative (this is standard for Hilbert spaces, and was demonstrated for arbitrary Banach spaces in [22]). Results of [12] combined with the Kingman subadditive ergodic theorem show that

$$\inf_n \frac{1}{n}\int \log \mathcal{E}_l(\mathcal{L}_\omega^{(n)})\,d\mathbb{P}(\omega) = \mu_1 + \cdots + \mu_l.$$

In particular, for any η and any l there exists an $n > 0$ such that $\frac{1}{n}\int \log \mathcal{E}_l(\mathcal{L}_\omega^{(n)})\,d\mathbb{P}(\omega) < \mu_1 + \cdots + \mu_l + \eta/2$.

For any collections $F = (f_1,\ldots,f_l)$ of functions in the unit sphere of $H^2(A_R)$ and $\Phi = (\phi_1,\ldots,\phi_l)$ of elements of the unit sphere of $H^2(A_R)^*$, let $E_{F,\Phi}(\mathcal{L}) = \det(\phi_i(\mathcal{L}f_j))$. These maps are equicontinuous, and indeed uniformly equicontinuous when restricted to $\{\mathcal{L}\colon H^2(A_R) \to H^2(A_R)\colon \|\mathcal{L}\| \leq K\}$ for any K, so that $\mathcal{L} \mapsto \mathcal{E}_l(\mathcal{L})$ is continuous when restricted to operators of norm at most K. It follows using Lemma 4.9 that there exists $\epsilon > 0$ such that for a.e. $\omega \in \Omega$, $|\mathcal{E}_l(\mathcal{L}_\omega^{\epsilon\,(n)}) - \mathcal{E}_l(\mathcal{L}_\omega^{(n)})| < \eta/2$. Hence for sufficiently small $\epsilon > 0$,

$$\frac{1}{n}\int \log \mathcal{E}_l(\mathcal{L}_\omega^{\epsilon\,(n)})\,d\mathbb{P}(\omega) < \mu_1 + \cdots + \mu_l + \eta,$$

so that the sum of the first l Lyapunov exponents of the $\mathcal{L}_\omega^\epsilon$ cocycle is at most $\mu_1 + \cdots + \mu_l + \eta$. This establishes, for each l, the upper semicontinuity of the sum of the first l Lyapunov exponents under perturbations of the original cocycle as required. ∎

We now show that we can deduce Theorem 2.5 as a corollary of the above.

Lemma 6.8. *Let σ be an ergodic transformation of $(\Omega, \mathbb{P})$ and let $n \in \mathbb{N}$. Then there exists k, a factor of n, and a σ^n-invariant subset B of Ω of measure $1/k$ such that $\Omega = \bigcup_{i=0}^{k-1} \sigma^{-i} B$ and $\sigma^n|_B$ is ergodic. The ergodic components of $\mathbb{P}$ under σ^n are the restrictions of $\mathbb{P}$ to the sets $\sigma^{-i} B$. If (A_ω) is a matrix or operator cocycle over σ, then the cocycle $(A_\omega^{(n)})$ over σ^n restricted to $\sigma^{-i} B$ has Lyapunov exponents $(n\lambda_j)$ where (λ_j) are the exponents of the original cocycle.*

For a proof, one finds the largest factor k of n such that $e^{2\pi i/k}$ is an eigenvalue of the operator $f \mapsto f \circ \sigma^n$ on $L^2(\Omega)$. The set B is a level set of the eigenvector.

For $a \in D_1$, let M_a be the Möbius transformation $M_a(z) = (z + a)/(1 + \bar{a}z)$, sending 0 to a and preserving the unit circle, so that in particular these transformations are Blaschke products. We record without proof the following straightforward facts about Möbius transformations.

Lemma 6.9. *Let $|a| < 1$ and let the M_a be as above. Then:*

(a) $M_a^{-1} = M_{-a}$.

(b) *For z in the closed unit disc, $|M_a(z) - z| \le 2|a|/(1 - |a|)$. In particular if $|a| < 1/3$, then $|M_a(z) - z| < 3|a|$ whenever $|z| \le 1$.*

(c) $|M_a'(z)| \le \frac{1+|a|}{1-|a|}$ *for all z in the closed unit disc.*

Proof of Theorem 2.5. We first prove part (a). Let σ, $(\Omega, \mathbb{P})$ and (T_ω) be as in the statement and let $R < 1$ satisfy $r := r_T(R) < R$. Let x_ω be the random fixed point of (T_ω), as guaranteed by Theorem 2.1. We now define a new conjugate family of cocycles:

$$\tilde{T}_\omega = M_{x_{\sigma\omega}}^{-1} \circ T_\omega \circ M_{x_\omega},$$

so that $\tilde{T}_\omega(0) = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\tilde{T}_\omega^{(n)} = M_{x_{\sigma^n\omega}}^{-1} \circ T_\omega^{(n)} \circ M_{x_\omega}$. If $|z| \le A := (R - r)/(1 - rR)$ then for any $x \in \bar{D}_r$, $|M_x(z)| \le R$, so that $T_\omega^{(n)} \circ M_{x_\omega}(\bar{D}_A)$ is contained in the intersection of $\bar{D}_r$ with a disc of radius $c(r/R)^n$ about $x_{\sigma^n\omega}$. Since the Lipschitz constant of M_x is $\frac{1+|x|}{1-|x|}$ and $M_{x_{\sigma^n\omega}}^{-1}(x_{\sigma^n\omega}) = 0$, we see that $\tilde{T}_\omega^{(n)}(\bar{D}_A) \subset \bar{D}_a$ where $a = \frac{1+r}{1-r} c(r/R)^n$. Let n be chosen so that $a < A$. Since neither a nor A depend on ω, nor does n. Let B be an ergodic component of σ^n as guaranteed by Lemma 6.8 and consider the cocycle $\tilde{\mathcal{L}}_\omega^{(n)}$ restricted to B. For $\mathbb{P}$-a.e. $\omega \in \Omega$, condition (c) of Lemma 6.7 is satisfied. Condition (a) is clearly satisfied and the above Lipschitz estimates combined with the assumption that $\operatorname{ess\,inf}_\omega |T_\omega'(x_\omega)| > 0$ show that $\operatorname{ess\,inf}_\omega |(\tilde{T}_\omega^{(n)})'(0)| > 0$. Hence the Lyapunov spectrum for the cocycle $(\tilde{\mathcal{L}}_\omega^{(n)})_{\omega \in B}$, with base dynamics $\sigma^n : B \to B$, is stable as shown in Lemma 6.7. Together with the final part of Lemma 6.8, this implies stability of the Lyapunov spectrum for the cocycle $(\tilde{\mathcal{L}}_\omega)$ over Ω. Since this cocycle is conjugate to the original cocycle: $\mathcal{L}_\omega^{(n)} = \mathcal{L}_{M_{x_{\sigma^n\omega}}} \circ \tilde{\mathcal{L}}_\omega^{(n)} \circ \mathcal{L}_{M_{x_\omega}^{-1}}$, and $\mathcal{L}_{M_x^{\pm 1}}$ is uniformly bounded as x runs over D_r, we deduce the stability of the Lyapunov spectrum of the original cocycle.

To show part (b) by Theorem 2.5, we first conjugate the Blaschke product cocycle to a new cocycle with 0 as the random fixed point. Write $(\tilde{T}_\omega)$ for the conjugate Blaschke product cocycle $\tilde{T}_\omega(z) = M^{-1}_{x_{\sigma\omega}} \circ T_\omega \circ M_{x_\omega}(z)$, that is, the cocycle where the random fixed point is conjugated to 0, so that $\tilde{T}_\omega(0) = 0$ and $\tilde{T}^{(n)}_\omega(z) = M^{-1}_{x_{\sigma^n\omega}} \circ T^{(n)}_\omega \circ M_{x_\omega}(z)$.

The proof will work by modifying the Blaschke product cocycle $(\tilde{T}_\omega)$ to give a new nearby Blaschke product cocycle $(\tilde{S}_\omega)$ where the random fixed point is still 0. In the last step, we invert the conjugacy operation to give a new Blaschke product cocycle $(S_\omega) = (M_{x_{\sigma\omega}} \circ \tilde{S}_\omega \circ M^{-1}_{x_\omega})$ that has the same random fixed point, (x_ω), as (T_ω).

Notice that since $\tilde{T}_\omega(0) = 0$, $\tilde{T}_\omega(z)$ may be written as $\tilde{T}_\omega(z) = zP_\omega(z)$ for a rational function P_ω that is analytic on the unit disc. We see that P_ω maps the unit circle to itself so that by Lemma 3.1(d), $P_\omega(z)$ is another Blaschke product.

We now let $0 < \epsilon < 1$ and set $\delta = \epsilon(1-r)/(3(1+r))$. Define a new family of Blaschke products by

$$Q_\omega = \begin{cases} M^{-1}_{P_\omega(0)} \circ P_\omega & \text{if } |P_\omega(0)| < \delta, \\ P_\omega & \text{otherwise,} \end{cases}$$

where the fact that Q_ω is a Blaschke product follows from Lemma 3.1(e). We can check that $Q_\omega(0) = 0$ whenever $|P_\omega(0)| < \delta$. By Lemma 6.9, we see that $|Q_\omega(z) - P_\omega(z)| \le 3\delta$ for each z in C_1. Now set $\tilde{S}_\omega(z) = zQ_\omega(z)$ so $|\tilde{S}_\omega(z) - \tilde{T}_\omega(z)| \le 3\delta$ for each $z \in C_1$. Next, observe (from the product rule) that $\tilde{T}'_\omega(0) = P_\omega(0)$ and $\tilde{S}'_\omega(0) = Q_\omega(0)$ so that $\tilde{S}'_\omega(0) = 0$ whenever $|\tilde{T}'_\omega(0)| < \delta$. Now

$$\begin{aligned} \max_{z\in C_1} |S_\omega(z) - T_\omega(z)| &= \max_{z\in C_1} |M_{x_{\sigma\omega}} \circ \tilde{S}_\omega \circ M^{-1}_{x_\omega}(z) - M_{x_{\sigma\omega}} \circ \tilde{T}_\omega \circ M^{-1}_{x_\omega}(z)| \\ &= \max_{z\in C_1} |M_{x_{\sigma\omega}} \circ \tilde{S}_\omega(z) - M_{x_{\sigma\omega}} \circ \tilde{T}_\omega(z)| \\ &\le \operatorname{Lip}(M_{x_{\sigma\omega}}) \max_{z\in C_1} |\tilde{S}_\omega(z) - \tilde{T}_\omega(z)|. \end{aligned}$$

By Lemma 6.9(a, c) and since $|x_{\sigma\omega}| \le r$, we have $\operatorname{Lip}(M^{-1}_{x_{\sigma\omega}}) \le \frac{1+r}{1-r}$ so that $\max_{z\in C_1} |S_\omega(z) - T_\omega(z)| \le \epsilon$. By Lemma 4.9, $\operatorname{ess\,sup} \|\mathcal{L}^\epsilon_\omega - \mathcal{L}_\omega\| \le C\epsilon$. The new cocycle has the same random fixed point as the old one, but for a subset of Ω of positive measure we have $S'_\omega(x_\omega) = 0$, so that the perturbed cocycle is in case (2) of Theorem 2.1 as required. ∎

Proof of Corollary 2.6. We first show that the stable elements of $\mathsf{Blaschke}_\mathsf{R}(\text{-})$ form an open subset. Let $\mathcal{T} \in \mathsf{Blaschke}_\mathsf{R}(\text{-})$ be stable. Let $r = r_\mathcal{T}(R) < R$, let $r < \rho < R$, and let C be such that $|w - z| \le d_R(w, z) \le C|w - z|$ for all w, z lying in $\bar{D}_\rho$.

Now let $\epsilon = \operatorname{ess\,inf}_\omega |T'_\omega(x_\omega)|/3$. Pick

$$\delta < \min\big(\rho - r, (1-r)^2\epsilon, \epsilon(1-\rho)^3(1 - r/R)/(2C)\big)$$

and let $d(\mathcal{S}, \mathcal{T}) < \delta$. We first show that the random fixed point, y_ω, for the $\mathcal{S}$ cocycle satisfies $d_R(y_\omega, x_\omega) \le C\delta/(1 - r/R)$. To see this, suppose y and x satisfy $d_R(y, x) \le$

$C\delta/(1-r/R)$, $T_\omega(\bar D_R) \subset \bar D_r$ and $\max_{z\in C_1} |S_\omega(z) - T_\omega(z)| < \delta$. Then

$$\begin{aligned} d_R(S_\omega(y), T_\omega(x)) &\le d_R(S_\omega(y), T_\omega(y)) + d_R(T_\omega(y), T_\omega(x)) \\ &\le C|S_\omega(y) - T_\omega(y)| + \frac{r}{R} d_R(y, x) \\ &\le C\delta + \frac{r}{R}\frac{C\delta}{1-r/R} = C\frac{\delta}{1-r/R}. \end{aligned}$$

Applying this inductively, we see that for a.e. $\omega \in \Omega$, $d_R(S^{(n)}_{\sigma^{-n}\omega}(0), T^{(n)}_{\sigma^{-n}\omega}(0)) \le C\delta/(1-r/R)$ for all n. Since for a.e. $\omega \in \Omega$, $S^{(n)}_{\sigma^{-n}\omega}(0) \to y_\omega$ and $T^{(n)}_{\sigma^{-n}\omega}(0) \to x_\omega$, we see that $d_R(y_\omega, x_\omega) \le C\delta/(1-r/R)$ a.s. Hence $|y_\omega - x_\omega| \le C\delta/(1-r/R)$.

Now we have, for a.e. $\omega \in \Omega$,

$$\begin{aligned} |S'_\omega(y_\omega)| &\ge |T'_\omega(x_\omega)| - |S'_\omega(x_\omega) - T'_\omega(x_\omega)| - |S'_\omega(y_\omega) - S'_\omega(x_\omega)| \\ &\ge 3\epsilon - \frac{\delta}{(1-r)^2} - \frac{2}{(1-\rho)^3}\frac{C\delta}{1-r/R} \ge \epsilon, \end{aligned}$$

where we use Cauchy's formula and the fact that $|x_\omega| \le r$ to estimate $|(S'_\omega - T'_\omega)(x_\omega)|$ and the fact that $|S''_\omega(z)| \le 2/(1-\rho)^3$ on $\bar D_\rho$ for the last term. Hence $\mathcal{S}$ is stable as required.

Now for the density, let $\mathcal{T}$ be a Blaschke product cocycle and suppose $r := r_R(\mathcal{T}) < R$. Write x_ω for the random fixed point. We shall obtain a new cocycle $\mathcal{S}$ with the same random fixed point as $\mathcal{T}$, in a similar way to the proof of Theorem 2.5(b), but where we ensure that the derivative of S_ω at x_ω is bounded away from 0. Let $\epsilon < R - r$ and set $\delta = \frac{1-r}{6(1+r)}\epsilon$.

As before, define $\tilde T_\omega = M_{-x_{\sigma\omega}} \circ T_\omega \circ M_{x_\omega}$, so that $\tilde{\mathcal{T}} = (\tilde T_\omega)$ is a conjugate cocycle fixing 0. Since $\tilde T_\omega(0) = 0$, we have $\tilde T_\omega(z) = zP_\omega(z)$, where P_ω is another Blaschke product and $\tilde T'_\omega(0) = P_\omega(0)$. To form the perturbation, let

$$Q_\omega = \begin{cases} M_{2\delta} \circ P_\omega & \text{if } |P_\omega(0)| \le \delta, \\ P_\omega & \text{otherwise.} \end{cases}$$

We can check that $|Q_\omega(0)| > \delta$ for each ω. We then let $\tilde S_\omega(z) = zQ_\omega(z)$ and $S_\omega(z) = M_{x_{\sigma\omega}} \circ \tilde S_\omega \circ M_{-x_\omega}$, so that $S_\omega(x_\omega) = x_{\sigma\omega}$. As in the proof of Theorem 2.5(b), we see $|S_\omega(z) - T_\omega(z)| \le \epsilon$ for all $z \in C_1$. From the definition of $\tilde S_\omega$, $|\tilde S'_\omega(0)| > \delta$ and so, using Lemma 6.9(c), we see $|S'_\omega(x_\omega)| > \left(\frac{1-r}{1+r}\right)^2\delta$ for a.e. $\omega \in \Omega$. Hence $\mathcal{S} = (S_\omega)_{\omega\in\Omega}$ is an ϵ-perturbation of $\mathcal{T}$, which is stable. ■

6.4. *Examples*

For a simple example of a cocycle satisfying the conditions of Theorem 2.5, let a and b be any numbers in $(0, 1/3)$, and $T_1(z) = [(a+z)/(1+az)]^2$, $T_2(z) = [(b+z)/(1+bz)]^2$. Let Ω be the full shift on $\{1,2\}^{\mathbb{Z}}$ and $\mathbb{P}$ any shift-invariant ergodic probability measure. Then one can check that $x_\omega \ge 0$ for all ω while the critical points are at $-a$ and $-b$.

Now given any cocycle satisfying the above conditions, one may apply Theorem 2.5 in each of the following situations:

(1) (Static perturbation) Suppose for each $\epsilon > 0$, (T_ω^ϵ) is a cocycle of Blaschke products such that $|T_\omega(z) - T_\omega^\epsilon(z)| \leq \epsilon$ for each $z \in C_1$. Then by Lemma 4.9, we have $\sup_\omega \|\mathcal{L}_\omega^\epsilon - \mathcal{L}_\omega\| \to 0$, so that by Theorem 2.5, $\mu_n^\epsilon \to \mu_n$ as $\epsilon \to 0$.

(2) (Quenched random perturbation) -1 Let $\tau\colon \Xi \to \Xi$ be an invertible measure-preserving transformation such that $\sigma \times \tau$ is ergodic. Then consider a family of cocycles of Blaschke products $(T_{\omega,\xi}^\epsilon)$ over the transformation $\sigma \times \tau$ such that $|T_{\omega,\xi}^\epsilon(z) - T_\omega(z)| \leq \epsilon$ for each $z \in C_1$. Then the Lyapunov exponents of the perturbed cocycle converge to those of the unperturbed cocycle as ϵ is shrunk to 0. This follows by considering both cocycles as cocycles over $\sigma \times \tau$ (even though the initial cocycle depends only on the first component). The result then follows by the previous example.

(3) (Annealed random perturbation) Let $\mathcal{L}_\omega^\epsilon = \mathcal{N}^\epsilon \circ \mathcal{L}_\omega$ define a cocycle over σ, where $\mathcal{N}^\epsilon$ is convolution with a Gaussian kernel as in Section 5.1. We have $\mathcal{L}_\omega^\epsilon f - \mathcal{L}_\omega f = (\mathcal{N}^\epsilon - I)\mathcal{L}_\omega f$. By the calculations in Lemma 4.9, for any $r < \rho < R$, there exists a $c > 0$ such that the e_n coefficient of $\mathcal{L}_\omega f$ is of absolute value at most $c(\rho/R)^{|n|}\|f\|$. By the calculations in Section 5.1, $\mathcal{N}^\epsilon - I$ is a diagonal operator with respect to the e_n's, scaling e_n by $e^{-2\pi^2 n^2 \epsilon^2} - 1 \leq 2\pi^2 n^2 \epsilon^2$. Hence for any $\omega \in \Omega$ and any $f \in H^2(A_R)$, the coefficients in the Laurent expansion of $(\mathcal{L}_\omega^\epsilon - \mathcal{L}_\omega)f$ are of absolute value at most $2\pi^2\epsilon^2 c n^2(\rho/R)^{|n|}$. In particular, $\sup_{\omega\in\Omega}\|\mathcal{L}_\omega^\epsilon - \mathcal{L}_\omega\| \to 0$ as $\epsilon \to 0$, so that we can apply Theorem 2.5. Hence the Lyapunov exponents vary continuously, in strong contrast to the situation in Section 5.1 involving an application of the same perturbations to another cocycle.

Appendix A. Comparison of Lyapunov spectrum and Oseledets splittings on different function spaces

Let $\mathcal{R} = (\Omega, \mathbb{P}, \sigma, X, \mathcal{L})$ be a random linear dynamical system and suppose the Banach space X' embeds densely as a subspace of X, equipped with a norm $\|\cdot\|_{X'}$ such that $\|x\|_{X'} \geq \|x\|_X$ for all $x \in X'$ and $\mathcal{L}_\omega(X') \subset X'$. Then we say $\mathcal{R}' = (\Omega, \mathbb{P}, \sigma, X', \mathcal{L}|_{X'})$ is a *dense restriction* of $\mathcal{R}$. We restate Theorem 2.7 more precisely in this language.

Theorem A.1 (Comparison of Lyapunov exponents for Oseledets splittings). *Let $\mathcal{R} = (\Omega, \mathbb{P}, \sigma, X, \mathcal{L})$ be a random linear dynamical system with ergodic invertible base and let $\mathcal{R}'$ be its dense restriction to the Banach space X'. Suppose the two systems satisfy the assumptions of Theorem* 3.4.

Let $X = W(\omega) \oplus \bigoplus_{j=1}^{l} V_j(\omega)$ and $X' = W'(\omega) \oplus \bigoplus_{j=1}^{l'} V_j'(\omega)$ be the splittings associated to $\mathcal{R}$ and $\mathcal{R}'$, respectively, and let $\{\lambda_j\}_{j=1}^{l}$ and $\{\lambda_j'\}_{j=1}^{l'}$ be the corresponding exceptional Lyapunov exponents. Then, whenever $\max(\lambda_j, \lambda_j') > \alpha := \max(\kappa(\mathcal{R}), \kappa(\mathcal{R}'))$, *we have:*

(1) $\lambda_j = \lambda_j'$.

(2) *For $\mathbb{P}$-a.e. ω, $V_j(\omega) = V_j'(\omega)$.*

For each $\omega \in \Omega$, $f \in X$, we let $\lambda_X(\omega, f) = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} f\|_X$. If $f \in X'$, we define $\lambda_{X'}(\omega, f) = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} f\|_{X'}$. The following result will be needed in the proof.

Lemma A.2 (Coincidence of external exponents). *Let $\mathcal{R}$, $\mathcal{R}'$ and α be as in Theorem A.1. Then for every $f \in X'$ for which $\lambda_{X'}(\omega, f) > \alpha$ one has $\lambda_X(\omega, f) = \lambda_{X'}(\omega, f)$.*

Remark. A slightly different result was established in [10, Theorem 3.3]. We include a proof of Lemma A.2 for completeness.

Proof of Lemma A.2. Let $f \in X'$ satisfy $\lambda_{X'}(\omega, f) > \alpha$. Clearly $\lambda_X(\omega, f) \le \lambda_{X'}(\omega, f)$. Since $\lambda_{X'}(\omega, f) > \alpha$, there exists a j such that $\lambda_{X'}(\omega, f) = \lambda'_j$. Write $V'_j(\omega)$ for the corresponding Oseledets subspace of X' and note that $V'_j(\omega)$ is also a subspace of X. Since $V'_j(\omega)$ is finite-dimensional, there exists a positive measurable function $c(\omega)$ such that $\|g\|_X \ge c(\omega)\|g\|_{X'}$ for all $g \in V'_j(\omega)$. Now write $f = g + h$ with $g \in V'_j(\omega)$ and $h \in F'_j(\omega)$. We have

$$\|\mathcal{L}_\omega^{(n)} f\|_X \ge \|\mathcal{L}_\omega^{(n)} g\|_X - \|\mathcal{L}_\omega^{(n)} h\|_X \ge \|\mathcal{L}_\omega^{(n)} g\|_{X'}\, c(\sigma^n\omega) - \|\mathcal{L}_\omega^{(n)} h\|_{X'}.$$

Taking a limit along a positive density sequence of n's where $c(\sigma^n\omega)$ is bounded away from 0, we see that $\lambda_X(\omega, f) \ge \lambda_{X'}(\omega, f)$, so that the two quantities coincide. ∎

Proof of Theorem A.1. We prove the result by induction. Suppose $\lambda_i = \lambda'_i$ and $V_i(\omega) = V'_i(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ for $i = 1, \ldots, j-1$, with $j \ge 1$. If $\lambda'_j > \alpha$, then letting $f \in V'_j(\omega)$ and applying the lemma above, we see that $\lambda_j \ge \lambda'_j$.

Using continuity of $\Pi_{F_{j-1}(\omega)\|E_{j-1}(\omega)}$ and the induction hypothesis, we find that $F_{j-1}(\omega) \cap X' = \Pi_{F_{j-1}(\omega)\|E_{j-1}(\omega)}(X')$ is dense in $F_{j-1}(\omega)$ and therefore $\Pi_{E_j(\omega)\|F_j(\omega)}(X' \cap F_{j-1}(\omega)) = V_j(\omega)$. Let U be a subspace of $X' \cap F_{j-1}(\omega)$ of dimension $m_j = \dim V_j(\omega)$ such that $\Pi_{E_j(\omega)\|F_j(\omega)}(U) = V_j(\omega)$. Now if $h \in U \setminus \{0\}$, then $\lambda_{X'}(\omega, h) = \lambda_X(\omega, h) = \lambda_j$.

It follows that λ_j is an exceptional exponent of $\mathcal{R}'$ and $\lambda'_j \ge \lambda_j$, so that $\lambda_j = \lambda_{j'}$. Since U is m_j-dimensional, we claim that $V'_j(\omega)$ is of dimension at least m_j. To see this, notice that if not, there would be a non-zero element of U whose projection under $\Pi_{V'_j(\omega)}$ would be trivial, so that the growth rate of this element would be strictly smaller than λ_j, giving a contradiction. By Lemma A.2, we see that $V'_j(\omega) \subset V_j(\omega)$ and the above argument shows that $\dim V'_j(\omega) \ge \dim V_j(\omega)$, so that $V'_j(\omega) = V_j(\omega)$ as required. ∎

A.1. Example

We consider finite Blaschke products of the form

$$B(z) = z \prod_{j=1}^{n} \frac{z + \zeta_j}{1 + \bar{\zeta}_j z}.$$

Note that $B(0) = 0$ and $B'(0) = \prod_{j=1}^{n} \zeta_j$. Furthermore, [19, Proposition 1] ensures that

$$\inf_{|z|=1} |B'(z)| \geq 1 + \sum_{j=1}^{n} \frac{1 - |\zeta_j|}{1 + |\zeta_j|} > 1.$$

For $0 < a < 1$, let

$$B_a(z) = z\left(\frac{z-a}{1-az}\right)^2.$$

Notice that $B_a'(0) = a^2$ and $\inf_{|z|=1} |B_a'(z)| \geq 1 + \frac{2(1-a)}{1+a}$. Let $\Omega = \{0,1\}^{\mathbb{Z}}$, σ be the shift map, and $\mathbb{P}$ be the Bernoulli measure with $\mathbb{P}([0]) = 0.5$. Let $\mathcal{L}_0$ be the Perron–Frobenius operator of $B_{0.5}$, $\mathcal{L}_1$ be the Perron–Frobenius operator of $B_{0.6}$ and consider the operator cocycle $\mathcal{R} = (\Omega, \mathbb{P}, \sigma, X, \mathcal{L})$ generated by $\mathcal{L}_\omega := \mathcal{L}_{\omega_0}$, acting on $X = C^3(S^1)$, as well as the dense restriction $\mathcal{R}' = (\Omega, \mathbb{P}, \sigma, X', \mathcal{L}|_{X'})$, where $X' = H^2(A_R)$. To see that X' embeds densely as a subspace of $C^3(S^1)$, notice that for any $f \in C^3(S^1)$, f''' may be approximated uniformly by a trigonometric polynomial. Since both $B_{0.5}$ and $B_{0.6}$ are expanding, R may be chosen so that $r_{B_a}(R) < R$ for $a \in \{0.5, 0.6\}$. By Theorem 2.1, the Lyapunov spectrum of $\mathcal{R}'$ is $\Sigma(\mathcal{R}') = \{0\} \cup \{n\Lambda : n \in \mathbb{N}\}$, where $\Lambda = \log(0.5) + \log(0.6) > -1.204$.

Note that $\inf_{|z|=1, a\in\{0.5,0.6\}} |B_a'(z)| \geq 1 + 2\frac{0.4}{1.6} = 1.5 =: \gamma$. The work of Ruelle [23] ensures that the essential spectral radius of each of $\mathcal{L}_0, \mathcal{L}_1$ acting on the space of C^r functions satisfies $\rho_e(\mathcal{L}_B) \leq 1/\gamma^r$. This result may also be established relying on Lasota–Yorke type inequalities, for example, following the strategy presented in [13, Lemma 3.3 & Corollary 3.4] for the C^1 case. A random version of Ruelle's result hence follows from [11, Lemma C.5], which provides a random version of Hennion's theorem. Thus, we have $\kappa(\mathcal{R}) \leq -r \log \gamma = -3 \log 1.5 < -1.21 < \Lambda < 0$. Theorem A.1 implies that Λ is an exceptional Lyapunov exponent of $\mathcal{R}$.

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