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The rôle of Coulomb branches in 2D gauge theory

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Abstract. I give a simple construction of the *Coulomb branches* $\mathcal{C}_{3,4}(G;E)$ of gauge theory in three and four dimensions, defined by H. Nakajima [Adv. Theor. Math. Phys. 20 (2016)] and A. Braverman, M. Finkelberg and H. Nakajima [Adv. Theor. Math. Phys. 22 (2018)] for a compact Lie group G and a polarizable quaternionic representation E. The manifolds $\mathcal{C}(G;\mathbf{0})$ are abelian group schemes over the bases of regular adjoint $G_{\mathbb{C}}$-orbits, respectively conjugacy classes, and $\mathcal{C}(G;E)$ is glued together over the base from two copies of $\mathcal{C}(G;\mathbf{0})$ shifted by a rational Lagrangian section ε_V, representing the Euler class of the *index bundle* of a polarization V of E. Extending the interpretation of $\mathcal{C}_3(G;\mathbf{0})$ as "classifying space" for topological 2D gauge theories, I characterize functions on $\mathcal{C}_3(G;E)$ as operators on the equivariant quantum cohomologies of $M \times V$, for compact symplectic G-manifolds M. The non-commutative version has a similar description in terms of the Γ-class of V.

Keywords. Coulomb branch, Gromov–Witten theory, boundary conditions

1. Introduction

Associated to a compact connected Lie group G and a quaternionic representation E, there are expected to be *Coulomb branches* $\mathcal{C}_{3,4}(G;E)$ of $N = 4$ SUSY gauge theory in dimensions 3 and 4, with matter fields in the representation E. They ought to be components of the moduli space of vacua, representing solutions of the monopole equations with singularities. Following early physics leads [8, 17] and more recent calculations [9], a precise definition for these spaces was proposed in the series of papers [6, 14] by Nakajima and collaborators in the case when E is *polarizable* (isomorphic to $V \oplus V^\vee$ for some complex representation V). Abelian groups were handled independently by Bullimore, Dimofte and Gaitto [7] from a physics perspective, while the case of the zero representation had been developed in [4], although only later recognized as such [19, 20].

The $\mathcal{C}_{3,4}$ are expected to be hyperkähler (insofar as this makes sense for singular spaces), with $\mathcal{C}_3$ carrying an SU(2) hyperkähler rotation. They are constructed in [6] as algebraic Poisson spaces, with $\mathbb{C}^\times$-action in the case of $\mathcal{C}_3$. We shall rediscover them as such in a simpler construction, which illuminates their relevance to 2-dimensional gauge

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theory: the $\mathcal{C}_{3,4}$ for polarized E are built from their more basic versions for the zero representation $E = \mathbf{0}$. Specifically, they are affinizations of a space constructed by partial identification of two copies of $\mathcal{C}(G;\mathbf{0})$. The identification is implemented by a Lagrangian shift along the fibers of the (Toda) integrable system structure of the $\mathcal{C}_{3,4}$, and its effect is to impose growth conditions, selecting a subring of regular functions. The non-commutative versions quantize this Lagrangian shift of the $\mathcal{C}_3$ into conjugation by the Γ-class of the representation (respectively, a specialization of its Jackson-Γ_p version for $\mathcal{C}_4$).

The reconstruction results, Theorems 1, 2 and 4, are more elementary than their 2D gauge theory interpretation, but it is the latter which seems to give them meaning. In compromise, I have attempted to isolate the gauge theory comments (for which a rigorous treatment has not yet been published) into paragraphs whose omission does not harm the remaining mathematics. I have also separated the non-commutative version of the story into the final section: its meshing with quantum cohomology theory is still incomplete.

A pedestrian angle on this paper's results is the *abelianization* underlying the calculations – a reduction to the Cartan subgroup H and Weyl group W. This is seen in the description of the Euler Lagrangians (4.1) which are used to build the "material" Coulomb branches from $\mathcal{C}(G;\mathbf{0})$, and is closely related to the abelianized index formula in [22], which ends up governing the Gauged Linear Sigma model (GLSM). Oversimplifying a bit, the interesting difference between G and its abelian reduction is already contained in $\mathcal{C}(G;\mathbf{0})$, the effect of adding a polarized representation being captured by a calculus reminiscent of toric geometry. Abelianization also has an explicit manifestation, similar to the Weyl character formula, in an isomorphism

$$\mathcal{C}_{3,4}(G;E) \cong \mathcal{C}_{3,4}\big(H; E \ominus (\mathfrak{g}/\mathfrak{h})^{\oplus 2})\big)/W,$$

whenever the formal difference on the right is a genuine representation of H; a quick argument has been included in the appendix, as it appeared not to be well known.

A qualification is in order: the simple characterization above, although *not* the abelianization formula, apply to the variants of $\mathcal{C}_{3,4}$ enhanced by the (complex) *mass parameters* [7], or by the more general *flavor symmetries* [6, Section 3 (v)]. The original spaces are subsequently recovered by setting the mass parameters to zero; however, at least one parameter, effecting a compactification of V, must be initially turned on. The moral explanation is easily expressed in physics language, and in a way that can be made mathematically precise. What my construction does is characterize the 3-dimensional topological gauge theories underlying the $\mathcal{C}_{3,4}$ by means of their 2D *topological boundary theories* – a characterization accurate enough, at least, to determine their expected Coulomb branches. For pure gauge theory ($E = \mathbf{0}$), I explained in [20, Section 6] in what sense the (A-models of) flag varieties of G supply a complete family of boundary theories, the Coulomb branch being akin to a direct integral of those: more precisely, it has a Lagrangian foliation by the mirrors of flag varieties. With E-matter added, a new boundary theory, the GLSM of V by G (again in the A-version) must be introduced as a factor, carrying the action of the matter fields. Since V is not compact, this model must be regularized by the inclusion of mass parameters. There is a mathematically sound version of this statement: the GLSM is a 2D TQFT over the ring of rational functions in the complex mass parameters, and has singularities at zero mass.

The same perspective points to a difficulty in extending these constructions when E cannot be polarized. There is no *a priori* reason why a 3D TQFT should be characterized by its topological boundary theories; Chern–Simons theory (for general levels) is a notorious counter-example [13]. A G-invariant Lagrangian $V \subset E$ seems to provide (in addition to the flag varieties) a generating boundary condition for the 3D gauge theory with matter – specifically, it is a domain wall between G-gauge theory with and without matter. No substitute is apparent in general. Clearly this deserves further thought. One obstacle is that 3D gauge theory gives only a partially defined TQFT, so its mathematical structure is incompletely settled, and the list of desiderata for a presumptive reconstruction is not known with clarity.

2. Overview and key examples

This section reviews the basic ingredients of the story and indicates the construction of Coulomb branches using U_1 as an example. The full statements require more preparation, and are found in Section 4.

2.1. Background

The complex-algebraic symplectic manifold $\mathcal{C}_3(G; \mathbf{0})$ was introduced for general G in [4]; for $G = SU_n$, it had been studied in [1], in the guise of the moduli space of SU_2 monopoles of charge n. The description most relevant for us is Spec $H_*^G(\Omega G; \mathbb{C})$, the conjugation-equivariant homology of the based loop group ΩG, with its Pontryagin product. From here, its rôle as a *classifying space* for topological 2-dimensional gauge theories was developed in [19, 20], where the space was denoted $\mathrm{BFM}(G^\vee)$. As we now recall, this virtue of $\mathcal{C}_3(G; \mathbf{0})$ must be read in the sense of semiclassical symplectic calculus, and not as a spectral theorem à la Gelfand–Naimark. It gives the "mirror description" of the gauged A-models in two dimensions.

2.2. Relation with quantum cohomology

A partial summary of the classifying property of $\mathcal{C}_3(G; \mathbf{0})$ is that its regular functions (sometimes called the *ring of chiral operators*) act on the equivariant quantum cohomologies $QH_G^*(M)$ of compact G-Hamiltonian symplectic manifolds M, in a manner making the E_2 structure[1] on $QH_G^*(M)$ compatible with the E_3 structure defined by the Poisson tensor on $\mathcal{C}_3$. This lays out $QH_G^*(M)$ as a sheaf over $\mathcal{C}_3$, which turns out to have Lagrangian support (Remark 2.1). This construction generalizes Seidel's theorem [18] on the action of $\pi_1 G$ on $QH^*(M)$, as well as the shift operators on QH^* and their equivariant extensions [15]. In fact, these latter ingredients are the "leading order" description of the story of [20] in the case of torus actions. A similar narrative applies to $\mathcal{C}_4(G; \mathbf{0})$ and

[1] Understood in the derived sense.

equivariant quantum K-theory (minding, however, the orbifold nature of $\mathcal{C}_4$ for general G, see Section 3) even though the general framework for K-theoretic mirror symmetry is incompletely understood.

Remark 2.1. The shortest argument for the Lagrangian property of $QH_G^*(M)$ passes to the non-commutative Coulomb branches of Section 7, over which the versions of the equivariant quantum cohomologies $QH_G^*(M)$ equivariant under loop rotation (which are related to *cyclic homology* of the Fukaya category) are naturally modules. The Lagrangian property is now a consequence of the integrability of characteristics [10] supplemented by finiteness of $QH_G^*(M)$ over $H^*(BG)$.

2.3. *Coulomb branches with matter*

The universal property of the $\mathcal{C}(G;\mathbf{0})$ leaves the spaces $\mathcal{C}(G;E)$ in search of a rôle. Their new characterization addresses this riddle. Namely, the Seidel shift operators act on $QH^*(M)$ only when M is *compact*; for more general spaces, the most we expect is an action on the *symplectic cohomology*, when the latter is defined [16]. Equivariant symplectic cohomology $SH_G^*(X)$ is sometimes a localization of $QH_G^*(X)$, in which case the space $\mathcal{C}_3(G;\mathbf{0})$ will capture a dense open part of $QH_G^*(X)$, with portions lost at infinity. Notably, this is the case when $X = M \times V$, with compact M and a linear G-space V. The lost part of $QH_G^*(M \times V)$ can be captured in a second chart of $\mathcal{C}(G;\mathbf{0})$, shifted from the original by the effect of the functor $M \mapsto M \times V$.

This shift is implemented as follows. The tensor product defines a symmetric monoidal structure on 2-dimensional TQFTs with G-gauge symmetry. This structure is mirrored in the classifying space $\mathcal{C}_3(G;\mathbf{0})$ into a *multiplication* along an abelian group structure over Spec H_*^G(point). (The latter is isomorphic to the space $\mathfrak{g}_{\mathbb{C}}^{\mathrm{reg}}/G_{\mathbb{C}}$ of regular adjoint orbits, and the projection exhibits $\mathcal{C}_3(G;\mathbf{0})$ as a fiberwise group-completion of the classical Toda integrable system; see Section 3.2.) The operation $QH_G^*(M) \rightsquigarrow SH_G^*(M \times V)$ is implemented by multiplication by a certain rational Lagrangian section ε_V of this group scheme, whose structure sheaf is $SH_G^*(V)$. The Lagrangian ε_V should be regarded as the gauged B-model mirror of V: see Remark 4.2.

The precise statement of the main results requires preparation and is postponed to Section 4; the remainder of this section develops two key examples.

2.4. *Example I: $G = \mathrm{U}_1$, with the standard representation L*

We have

$$\mathcal{C}_3(\mathrm{U}_1;\mathbf{0}) = \mathrm{Spec}\, H_*^{\mathrm{U}_1}(\Omega\mathrm{U}_1;\mathbb{C}) = \mathbb{C}\times\mathbb{C}^\times \cong T^\vee\mathbb{C}^\times, \tag{2.2}$$

with $\mathbb{C}^\times$ *dual* to U_1: the coordinates τ and z on the two factors generate $H^2(B\mathrm{U}_1)$ and $\pi_1\mathrm{U}_1$. The canonical symplectic form $d\tau \wedge dz/z$ also admits an intrinsic topological definition, in terms of a natural circle action on $B\mathrm{U}_1 \times \Omega\mathrm{U}_1$ (cf. Sections 3.1 and 7.2).

One usually defines the *toric mirror* of the space L as the function (*super-potential*) $\psi(z) = z$ on the space $\mathbb{C}^\times$. The differential $d\psi$ defines the Lagrangian

$$\varepsilon_L := \{\tau = z\} \subset T^\vee\mathbb{C}^\times.$$

View ε_L instead as the rational section $\tau \mapsto z = \tau$ of the projection $T^\vee\mathbb{C}^\times \to \mathbb{C}$ to the τ-coordinate, and note in passing the Legendre transform $\psi^*(\tau) = \tau(\log\tau - 1)$ of ψ, in the sense that $\varepsilon_L = \exp(d\psi^*)$.

Functions on ε_L are identified with $\mathbb{C}[\tau^\pm]$; this is the U_1-equivariant symplectic cohomology of L, rather than its quantum cohomology $\mathbb{C}[\tau]$. We can recover the full quantum cohomology by gluing, onto the open set $\tau \neq 0$ in (2.2), a second copy $T^\vee\mathbb{C}^\times$, with coordinates τ and $z' = z/\tau$. This gluing is compatible with projection to the τ-coordinate and leads to the space $\mathbb{C}^2 \setminus \{0\}$, with coordinates $(x, y) = (z, \tau/z)$, living over the line $\tau = xy$. The section ε_L closes now to the line $y = 1$, identified by projection with the full τ-axis.

In [6, 7], $\mathcal{C}_3(U_1; L \oplus L^\vee)$ is taken to be the affine completion $\mathbb{C}^2 = \operatorname{Spec}\mathbb{C}[x, y]$. The following characterization is now obvious.

Proposition 2.3. *The ring $\mathbb{C}[x, y]$ is the subring of regular functions $f(\tau, z)$ on $T^\vee\mathbb{C}^\times$ with the property that $f(\tau, z\tau)$ is also regular.* ■

Our Lagrangian ε_L is related to the *Euler class of the index bundle* as follows. Denote by $\mathfrak{Pic}(\mathbb{P}^1)$ the moduli stack of holomorphic line bundles on $\mathbb{P}^1$; its equivariant homotopy type is the stack $BU_1 \times \Omega U_1$ implicit in (2.2). Over $\mathbb{P}^1 \times \mathfrak{Pic}(\mathbb{P}^1)$ lives the universal line bundle, with fiber the standard representation L. Its index along $\mathbb{P}^1$, with a simple vanishing constraint at a single marked point, is a virtual bundle Ind_L over $\mathfrak{Pic}(\mathbb{P}^1)$, with equivariant Euler class $\mathrm{e}_L \in H^*(\mathfrak{Pic}(\mathbb{P}^1))[\tau^{-1}]$ in the localized equivariant cohomology ring. Specifically, $\mathrm{Ind}_L = L^{\oplus n}$ and $\mathrm{e}_L = \tau^n$ on the component $\mathfrak{Pic}_n$, $n \in \mathbb{Z} = \pi_1 U_1$. The following is clear from these constructions.

Proposition 2.4. *The rational automorphism of multiplication by ε_L on $T^\vee\mathbb{C}^\times$, $z \mapsto \tau z$, corresponds to the cap-product action of e_L on $H_*^{U_1}(\Omega U(1); \mathbb{C})[\tau^{-1}]$.* ■

These propositions capture the rôle of $\mathcal{C}(G; L \oplus L^\vee)$ in quantum cohomology: the condition of regularity under capping with the Euler class picks out precisely those equivariant Seidel shift operators which act on $QH^*_{U_1}(L)$. More generally, we have:

Proposition 2.5. *The subring $\mathbb{C}[x, y] \subset \mathbb{C}[\tau, z^\pm]$ acts on $QH^*_{U_1}(M \times L)$ for any compact U_1-Hamiltonian symplectic manifold M, and it is the largest subring with that property.*

Proof. The subring $\mathbb{C}[\tau] \cong H_*^{U_1}(\text{point})$ acts in the natural way. Recall now (for instance, [11, 15]) that the Seidel element σ_n associated with z^n (which is a co-character of the original U_1) is the following "twisted 1-point function": namely the element in $QH^*_{U_1}(X)$ defined by the evaluation ev_∞ at ∞ of stable sections of the X-bundle over $\mathbb{P}^1$ associated to $\mathcal{O}(-n)$. All is well when X is compact: σ_n is a unit in $QH^*_{U_1}(X)$, with inverse σ_{-n}. (Without equivariance, this goes back to Seidel's original paper [18].) For $X = M \times L$ though, we have a problem when $n < 0$: equivariant integration along the fibers of ev_∞ incorporates integration along Ind_L, the kernel of $H^0(\mathbb{P}^1; \mathcal{O}(-n) \otimes_{\mathbb{C}^\times} L) \to L$, with dimension $(-n)$; the operation contributes its Euler class as a denominator, a factor of τ^n. The factor τ in $y = \tau z^{-1}$ precisely cancels the denominator. ■

2.5. *Generalization*

Propositions 2.3–2.5 extend to all G and representations V, as Theorems 1 and 3 in Section 4; Theorem 2 is the K-theory analogue. Non-commutative versions of Coulomb branches are described in Section 7. One required change throughout is the inclusion in the ground ring of an additional equivariant parameter μ, from the natural $\mathbb{C}^\times$-scaling of V. The need for this will become evident in the example that follows. One can indeed include the full G-automorphism group of V (the flavor symmetries), but any single scaling symmetry that is *compactifying* – fully expanding or fully contracting – suffices. I will spell out the case of the overall scaling.

2.6. *Example II:* U_1 *with a general representation* V

For a d-dimensional representation V of U_1 with weights $n_1, \ldots, n_d \in \mathbb{Z}$, the super-potential $\psi_V : \mathbb{C}^\times \to \mathbb{C}$ for its mirror is computed by the following adaptation of the Givental–Hori–Vafa recipe.[2] The defining homomorphism $\rho_V : \mathrm{U}_1 \to \mathrm{U}_1^d$ of V dualizes to $\rho_V^\vee : (\mathbb{C}^\times)^d \to \mathbb{C}^\times$. The standard toric super-potential for $\mathbb{C}^d$ on the source $(\mathbb{C}^\times)^d$,

$$\Psi(z_1, \ldots, z_d) = z_1 + \cdots + z_d,$$

"pushes down" to the multi-valued function $\psi_V(z)$ on the target $\mathbb{C}^\times$ whose multi-values are the critical values of Ψ along the fibers of $\rho_V^\vee$. A clean restatement is that the Legendre transform $\psi_V^*(\tau)$ is the restriction, under the infinitesimal representation $d\rho_V$, of the Legendre transform of Ψ: in obvious notation,

$$\Psi^*(\tau_1, \ldots, \tau_d) = \sum_k \tau_k(\log \tau_k - 1), \qquad \psi_V^* = \Psi^* \circ d\rho_V.$$

Our Lagrangian ε_V is the graph of $\exp(d\psi_V^*)$, namely $\tau \mapsto z = \prod_k (n_k \tau)^{n_k}$. The reader should meet no difficulty in comparing this ε_V with the Euler class e_V of the respective index bundle over $\mathfrak{Pic}$, as in Proposition 2.4. It should be equally clear how to extend this prescription to the case of a higher-rank torus and a general representation.

However, literal application of the lesson from Example 2.4 runs into trouble, already for U_1 with $V = L \oplus L^\vee$. In the GHV construction, the super-potential $\Psi = z_1 + z_2$ has no critical points along the fibers of $\rho_V(z_{1,2}) = z_1/z_2$. We have better luck with the Legendre transform,

$$\psi_V^*(\tau) = \tau(\log \tau - 1) - \tau(\log(-\tau) - 1) = \pi i \tau,$$

which identifies ε_V with the cotangent fiber over $\exp(\pi i) = -1 \in \mathbb{C}^\times$, and induces the automorphism $z \leftrightarrow (-z)$ of $T^\vee\mathbb{C}^\times$. While this does match Proposition 2.4, thanks to the Euler class cancellation $\mathrm{e}_{L\oplus L^\vee} = \mathrm{e}_L \cup \mathrm{e}_{L^\vee} = (-1)^n$ on $\mathfrak{Pic}_n$, raw application of Proposition 2.3 would falsely predict that $\mathcal{C}_3(\mathrm{U}_1, V \oplus V^\vee) = \mathcal{C}_3(\mathrm{U}_1, \mathbf{0})$, because ε_V is now regular.

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[2] The recipe is justified in the SYZ construction by the count of holomorphic disks bounding the standard coordinate tori. We are omitting the small quantum parameters, one coupled to each coordinate z_k.

The remedy incorporates scaling-equivariance into the Euler index class, converting it into the μ-homogenized total Chern class. As a Laurent series in μ^{-1}, the latter is defined for arbitrary virtual bundles. For the index bundles over $G \ltimes \Omega G$ of representations of general compact groups G, we will always find *rational functions*. With $V = L \oplus L^\vee$, we get $(\mu+\tau)^n(\mu-\tau)^{-n}$ on $\mathfrak{Pic}_n$, and the earlier cancellation in the Euler class is now seen to be "fake", arising from premature specialization to $\mu = 0$. The Coulomb branch is spelt out in Example 5.2.

Algebraically, μ is to be treated as an independent parameter. It changes the superpotential Ψ by subtracting $\mu \sum \log z_k$; this adds scale-equivariance to the mirror of $\mathbb{C}^d$. The Legendre transform Ψ^* is modified by the substitution $\tau_k \mapsto \tau_k + \mu$, and the topological origin as a scale-equivariant promotion of the Chern class is now clearly displayed. For a general V, the remedied Lagrangian is defined by $z = \prod_k (\mu + n_k\tau)^{n_k}$; in particular, it determines the representation.

Extension to a higher-rank torus, with arbitrary representations, is now a simple matter, and it should also be clear how to incorporate the entire *flavor symmetry group* (the G-automorphism group of V), if desired, by equivariant enhancements of the Lie algebra coordinates τ_k. There is a characterization of $\mathcal{C}_3$ analogous to Proposition 2.3, as the subring of regular functions on $\mathcal{C}_3(T;\mathbf{0})$ which survive translation by the newly μ-remedied ε_V, and it is easy to relate it with the abelian presentations in [7, 14]. The contribution of this paper is the non-abelian generalization.

Remark 2.6. The remedy of scale-equivariance should not surprise readers versed in toric mirror symmetry: naïve application of the GHV recipe is problematic for toric actions with non-compact quotients – which is when our fake cancellations can happen – and the recipe can be corrected by including equivariance under the full torus.

3. Background on Coulomb branches

We recall here the construction and properties of Coulomb branches; this mostly condenses material from [3, 4, 6]. I will write $\mathcal{C}_{3,4}$ for $\mathcal{C}_{3,4}(G;\mathbf{0})$ when no confusion arises. Denote by $H \subset G$ a maximal torus and by $H^\vee, G^\vee$ the Langlands dual groups, $\mathfrak{g}, \mathfrak{h}$ the Lie algebras, W the Weyl group.

3.1. The basic Coulomb branches [4]

The space $\mathcal{C}_3 := \operatorname{Spec} H_*^G(\Omega G;\mathbb{C})$ is an affine symplectic resolution of singularities of the Weyl quotient $T^\vee H_{\mathbb{C}}^\vee/W$. It arises by adjoining to $T^\vee H_{\mathbb{C}}^\vee$, prior to Weyl division, the functions $(e^{\alpha^\vee}-1)/\alpha$ for all root–coroot pairs $\alpha, \alpha^\vee$ of G. The $\mathbb{C}^\times$-action on the cotangent fibers arises from the homology grading and scales the symplectic form. The underlying Poisson structure is the leading term of a non-commutative deformation over the ring $\mathbb{C}[h] = H^*(BR)$, obtained by incorporating in to $\mathcal{C}_3$ the equivariance under the loop-rotation circle R. The loop rotation is revealed by writing $\Omega G \cong LG/G$.

For simply connected G, the spectrum of $K_*^G(\Omega G;\mathbb{C})$ is also a symplectic manifold giving an affine resolution of $(H_{\mathbb{C}} \times H_{\mathbb{C}}^\vee)/W$. This is now accomplished by adjoining the functions $(e^{\alpha^\vee} - 1)/(e^\alpha - 1)$ before Weyl division. However, the space has singularities when $\pi_1 G$ has torsion. Write $G = \tilde{G}/\pi$ for the torsion subgroup $\pi \subset \pi_1 G$, $H = \tilde{H}/\pi$. As a subgroup of $Z(\tilde{G})$, π acts by automorphisms of $K_{\tilde{G}}(X) \otimes \mathbb{C}$ for any G-space X: to see this, decompose a class in $K_{\tilde{G}}(X)$ into π-eigen-bundles, and multiply each of them by the corresponding character of π, before re-summing to a complex K-class. We adopt the smooth symplectic orbifold $\pi \ltimes \operatorname{Spec} K_*^{\tilde{G}}(\Omega G; \mathbb{C})$ as the definition of $\mathcal{C}_4$.

Remark 3.1 (Sphere topology). Some features of $\mathcal{C}_{3,4}$ are explained by *Chas–Sullivan theory* in dimension 3, one higher than usual. The underlying topological object is the mapping space from S^2 to the stack BG; it has a natural E_3 structure, which turns out to correspond to the Poisson form on $\mathcal{C}_{3,4}(G; \mathbf{0})$. Loop rotation is seen in the presentation as the two-sided groupoid $G \ltimes LG \rtimes G$, with Hecke-style product (see Remark 3.6). Tracking the loop rotation breaks E_3 down to E_1, because rotating spheres in an ambient $\mathbb{R}^3$ may be strung together linearly as beads on the rotation axis, but can no longer move around each other. This leads to the non-commutative Coulomb branches we shall review in Section 7.

3.2. Group scheme structure

The Hopf algebra structures of $H_*^G(\Omega G)$, $K_*^G(\Omega G)$ over the ground rings H_G^*, K_G of a point lead to relative abelian group structures

$$\mathcal{C}_3(G; \mathbf{0}) \xrightarrow{\chi} \mathfrak{h}_{\mathbb{C}}/W, \quad \mathcal{C}_4(G; \mathbf{0}) \xrightarrow{\kappa} \pi \ltimes (\tilde{H}_{\mathbb{C}}/W). \tag{3.2}$$

When $\pi_1 G$ has torsion, the second base is an affine orbifold whose ring of functions is $K_G(\text{point})$. (The abelian property is a piece of characteristic-zero good fortune: the correct commutativity structure is E_3, as explained in Remark 3.1, but this decouples into a strictly commutative and a graded Poisson structure.) These maps define integrable systems: χ is a partial completion of the classical Toda system[3] [3], whereas κ is its finite-difference version.

Remark 3.3 (Adjoint and Whittaker descriptions). As an algebraic symplectic manifold, $\mathcal{C}_3$ is the algebraic symplectic reduction $T^\vee_{\text{reg}} G^\vee_{\mathbb{C}} // G^\vee_{\mathbb{C}}$ of the fiberwise-regular part of the cotangent bundle under conjugation. There is a similar description of $\mathcal{C}_4$ using the Langlands dual Kac-Moody group (*not* the loop group of $G^\vee$), capturing the holomorphic (but not algebraic) symplectic structure.

The space $\mathcal{C}_3$ has another description as the two-sided symplectic reduction of $T^\vee G^\vee_{\mathbb{C}}$ by N, at the regular nilpotent character. Clearly, this is algebraic symplectic; much less obviously, it is hyperkähler, thanks to work of Bielawski on the Nahm equation [5]. The non-commutative deformation has a corresponding description in terms of $N \times N$ monodromic differential operators on $G^\vee_{\mathbb{C}}$ [3].

In both descriptions, multiplication along the group $G^\vee$ induces the group scheme structure of Section 3.2. Commutativity is more evident in the adjoint description, where the Toda fibers are the centralizers of regular co-adjoint orbits in $\mathfrak{g}^\vee_{\mathbb{C}}$.

[3] This was rediscovered in [20]; I thank H. Nakajima for pointing me to the original reference.

3.3. Coulomb branches for $E = V \oplus V^\vee$

To build the spaces $\mathcal{C}_{3,4}(G; E)$, we follow [6], to which we refer for full details, and replace ΩG in the original $\mathcal{C}$ by a *linear space* $L_V \to \Omega^a G$, a stratified space whose fibers are vector bundles over the Schubert strata of the algebraic model

$$\Omega^a G := G_{\mathbb{C}}((z))/G_{\mathbb{C}}[\![z]\!]$$

of ΩG. The fiber of L_V over a Laurent loop $\gamma \in \Omega^a G$ is the kernel of the difference

$$L_V|_\gamma \longrightarrow V[\![z]\!] \oplus V[\![z]\!] \xrightarrow{\mathrm{Id}-\gamma} V((z)). \tag{3.4}$$

Projection embeds L_V in either factor $V[\![z]\!]$ with finite co-dimension, which is bounded on any finite union of strata in $\Omega^a G$. More precisely, the complex in (3.4) descends to $G[\![z]\!]\backslash\Omega^a G$, with the left and right copies of $G[\![z]\!]$ acting on the respective factors $V[\![z]\!]$, and the left one alone acting on $V((z))$. Over any finite union of strata, L_V contains two sub-bundles of finite co-dimension, coming from a left and a right $z^n V[\![z]\!]$, for sufficiently large n. This stratified finiteness lets one define the Borel–Moore (K-)homologies $BMH_*^G(L_V)$, $BMK_*^G(L_V)$, renormalising the grading as if $\dim V[\![z]\!]$ were zero.

The normalized grading is compatible with the multiplication defined by the following correspondence diagram on the fibers of L_V, which lives over the multiplication of two loops $\gamma, \delta \in \Omega^a G$:

$$L_V|_\gamma \oplus L_V|_\delta \hookleftarrow L_V|_\gamma \oplus_{V[\![z]\!]} L_V|_\delta \rightarrowtail L_V|_{\gamma\cdot\delta}; \tag{3.5}$$

the sum in the middle is fibered over the right component of $L_V|_\gamma$ and the left one of $L_V|_\delta$, while the right embedding is the projection to the outer $V[\![z]\!]$ summands. The wrong-way map in homology along the first inclusion is well-defined, over γ, δ in a finite range of Schubert cells, after modding out by a common subspace $z^n V[\![z]\!]$, and the result is independent of n.

As before, non-commutative deformations arise by including the loop rotation R-action on ΩG and on $V[\![z]\!]$; their leading terms define Poisson structures.

Remark 3.6 (E_3 Hecke property). A Laurent loop defines a transition function for a principal $G_{\mathbb{C}}$-bundle over the non-separated disk –:– with doubled origin. The multiplications have a Hecke interpretation as correspondences on $G \ltimes \Omega G$ and L_V, induced by following left-to-right the maps relating non-separated disks with doubled and tripled centers:

$$\begin{matrix}(–:–)\\(–:–)\end{matrix} \overset{g}{\twoheadrightarrow} (–\vdots–) \overset{i}{\hookleftarrow} (–:–).$$

The map g glues the bottom sheet of the first disk to the top sheet of the second, while i hits the outer centers of the triple-centered disk. The E_3 property comes from sliding the multiple centers around, as in Chas–Sullivan sphere topology. With rotation-equivariance, this freedom is lost and we are reduced to an E_1 multiplication.

On G-bundles, the Hecke operation is represented by multiplication of transition functions, once we identify, on the left side, the top bundle on its bottom sheet with the bottom bundle on its top sheet. Next, associated to the representation V is a vector bundle over –:–, whose space of global sections is L_V. The correspondence (3.5) arises by retaining those pairs of global sections on the left which match on the glued pair of sheets, and then restricting them to the top and bottom sheets of the triple-centered disk.

3.4. *Massive versions*

We enhance the Coulomb branches by the addition of a symmetry in which $\mathbb{C}^\times \supset S^1$ scales the fibers of L_V:

$$\mathcal{C}_3^\circ(G;E) := \operatorname{Spec} BMH_*^{G\times S^1}(L_V;\mathbb{C}), \qquad \text{projecting to } \mathfrak{h}_\mathbb{C}/W \times \mathbb{C},$$
$$\mathcal{C}_4^\circ(G;E) := \pi \ltimes \operatorname{Spec} BMK_*^{\tilde{G}\times S^1}(L_V;\mathbb{C}), \quad \text{projecting to } \pi \ltimes (\tilde{H}_\mathbb{C}/W) \times \mathbb{C}^\times.$$

The projections to the massive Toda bases are defined as in (3.2), and denoted by $\chi(\mu)$ and $\kappa(m)$, with generators $\mu \in H^2(BS^1), m^\pm \in K_{S^1}(\text{point})$. The fibers over fixed values of the parameters μ, m are total spaces of (usually singular) integrable systems; this will follow from flatness of the projections (Section 5.7). The scaling is trivial when $E = \mathbf{0}$ and $L_V = \Omega^a G$, but it will couple to the Euler class of the index bundle over $G \ltimes \Omega G$, promoting it to the total Chern class.

The notation is subtly abusive: the $\mathcal{C}^\circ$ depend on the polarization V and not just on E. For instance, switching $V \leftrightarrow V^\vee$ leads an isomorphic space only if we also change the orientation of the rotating circle. This V-dependence disappears at $\mu = 0$ or $m = 1$. We will see in Section 6 that the $\mathcal{C}^\circ(G;E)$ are flat over $\mathbb{C}[\mu]$, $\mathbb{C}[m^\pm]$, and that the same spaces $\mathcal{C}_{3,4}(G;E)$, as defined earlier in this section, appear by specializing to $\mu = 0$ or $m = 1$, independently of the choice of V.

4. Main results

We are finally in a position to state Theorems 1–3; the non-commutative analogues of Theorems 1 and 2 will wait until Section 7. First, I describe the Lagrangians generalizing the massive ε_V of Example 2.4. Their Euler class interpretation, already mentioned following Proposition 2.3, will be spelt out in Section 6.

4.1. *The Euler Lagrangians*

For $w \in \mathbb{C}^\times$ and ν a weight of H, $w^\nu := \exp(\nu \log w)$ determines a point in $H_\mathbb{C}^\vee$. Consider the following rational maps from $\mathfrak{h}_\mathbb{C} \times \mathbb{C}$ and $H_\mathbb{C} \times \mathbb{C}^\times$ to $H_\mathbb{C}^\vee$, defined in terms of the weights ν of V, which are to be included with their multiplicities:

$$\varepsilon_V : (\xi, \mu) \mapsto \prod_\nu (\mu + \langle \nu | \xi \rangle)^\nu, \quad \lambda_V : (x, m) \mapsto \prod_\nu (1 - (mx^\nu)^{-1})^\nu. \tag{4.1}$$

(In parsing each formula, note the double use of ν, first as infinitesimal character of H and then as co-character of $H^\vee$.) The maps are Weyl-equivariant and their graphs are regular, away from a co-dimension 2 locus over their domains (cf. Section 5.1); their closures define Lagrangian sub-varieties $\bar{\varepsilon}_V \subset \mathcal{C}_3^\circ(G;\mathbf{0})$ and $\bar{\lambda}_V \subset \mathcal{C}_4^\circ(G;\mathbf{0})$ over their respective ground rings $\mathbb{C}[\mu], \mathbb{C}[m^\pm]$.

Remark 4.2 (Broader picture). For generic elements μ and m (but most meaningfully, near $\mu, m = \infty$), the maps (4.1) are the exponentiated differentials of the following functions, in which $\xi \in \mathfrak{g}_\mathbb{C}$ and $x \in G_\mathbb{C}$ are the arguments while μ, m are treated as parameters:

$$\xi \mapsto \mathrm{Tr}_V[(\xi \oplus \mu)\cdot(\log(\xi \oplus \mu) - 1)], \quad x \mapsto \mathrm{Tr}_V \, \mathrm{Li}_2((x \times m)^{-1}).$$

The first function appeared as the "$\Sigma \log \Sigma$ Landau–Ginzburg B-model mirror" of the abelian GLSM on V: [23], and see also Remark 7.5. The Lagrangian λ_V and its primitive appeared[4] in the index formula for Kähler differentials over the moduli of G-bundles on curves [22, (6.2) and Theorem 6.4], with the powers of m^{-1} tracking the degree of the forms. The relation with Coulomb branches was not known at the time. Today, we would express that index formula in terms of Lagrangian calculus in $\mathcal{C}_4^\circ(G,\mathbf{0})$, namely the intersection of λ_V with the graphs of certain isogenies $H_\mathbb{C} \to H_\mathbb{C}^\vee$, defined from the *levels* of central extensions of the loop group LG. Those isogenies correspond to the Theta line bundles on the moduli of $G_\mathbb{C}$-bundles on curves; they are semiclassical limits of Theta-functions – in the same sense that the Lagrangians ε_V, λ_V are semiclassical Γ-functions, see Section 7 – and are also twists of the unit section by the discrete Toda Hamiltonian of $\mathcal{C}_4$.

4.2. *Algebraic description of the Coulomb branches*

The first two results generalize to non-abelian G the explicit presentations of Coulomb branches given in [7, 14] for torus groups. Their proofs, in Section 6, are straightforward; more intriguing are the non-commutative generalizations in Section 7. To state the theorems, note that translation on the group schemes by the section ε_V, respectively λ_V, gives a rational symplectomorphism of $\mathcal{C}_3^\circ, \mathcal{C}_4^\circ$, relative to the massive Toda projection of Section 3.4.

Theorem 1. *The space $\mathcal{C}_3^\circ(G;E) \to \mathfrak{h}_\mathbb{C}/W \times \mathbb{C}$ is the affinization of two copies of the space $\mathcal{C}_3^\circ(G;\mathbf{0})$ glued together by means of ε_V-translation. In other words: regular functions on $\mathcal{C}^\circ(G;E)$ are those regular functions on $\mathcal{C}^\circ(G;\mathbf{0})$ which remain regular after translation by ε_V.*

Theorem 2. *The orbifold $\mathcal{C}_4^\circ(G;E) \to \pi \ltimes (\tilde{H}_\mathbb{C}/W) \times \mathbb{C}^\times$ is the relative affinization of two copies of $\mathcal{C}_4^\circ(G;\mathbf{0})$ glued together by means of λ_V-translation.*

Abstractly, the spaces are the quotients, in affine schemes over the massive Toda bases, of an equivalence relation on $\mathcal{C}^\circ \coprod \mathcal{C}^\circ$ defined from $\bar{\varepsilon}_V, \bar{\lambda}_V$. The relation is not very healthy, being neither proper nor open. Concretely, note that the surviving condition can equally well be imposed prior to Weyl division, giving the following moderately explicit description.

[4]For the adjoint representation, but the discussion in [23] applies to any V.

Corollary 4.3. *The regular functions on* $\mathcal{C}^\circ_{3,4}(G;E)$ *are those Weyl-invariant elements of*

$$\mathbb{C}[T^\vee H^\vee_{\mathbb{C}}][\mu]\big[(e^{\alpha^\vee}-1)/\alpha\big], \text{ respectively } \mathbb{C}[H_{\mathbb{C}}\times H^\vee_{\mathbb{C}}][m]\big[(e^{\alpha^\vee}-1)/(e^\alpha-1)\big]$$

(ranging over the roots α*) which survive translation by* ε_V*, respectively by* λ_V*.* ■

Survival can be restated in terms of growth constraints along the Toda fibers over the locus of zeroes and poles of ε_V, λ_V; we shall do that in the next section, as we review more of the algebraic geometry. Meanwhile, the next theorem, characterizing the regular functions $\mathcal{C}^\circ(G;E)$ in terms of quantum cohomology, is simple enough to prove here.

Theorem 3. *The ring* $\mathbb{C}[\mathcal{C}^\circ_3(G;E)]$ *comprises those functions on* $\mathcal{C}^\circ_3$ *which act regularly on the equivariant quantum cohomologies* $QH^*_{G\times S^1}(M\times V)$*, for compact Hamiltonian* G*-manifolds* M*.*

The ring $\mathbb{C}[\mathcal{C}^\circ_4(G;E)]$ *comprises those regular functions on* $\mathcal{C}^\circ_4$ *which act on the equivariant quantum* K*-theories* $QK^*_{G\times S^1}(M\times V)$*, for compact Hamiltonian* G*-manifolds* M*.*

Proof of Theorem 3. Away from the root hyperplanes on the massive Toda base (or the singular conjugacy locus, respectively), the statement follows by abelianization from the calculation of Proposition 2.5. On the other hand, away from $\mu=0$ (or $m=1$), the fixed-point theorem allows us to ignore E and V, and we are reduced to the action of $H^G_*(\Omega G)$ on equivariant quantum cohomology (see [20,21]). The remaining locus has co-dimension 2 on the base, over which $QH^*_{G\times S^1}(M)$ is finite and free as a module. ■

5. Some consequences

We discuss briefly some geometry of the Coulomb branches as it emerges from their description in Section 4. Flatness and normality were already established in [6], but it may be helpful to review them in the new construction.

5.1. Generic geometry of the Coulomb branches

The divisor S of singularities of the section ε_V, resp. λ_V is the unions of hyperplanes S_ν defined by the monomial factors in (4.1). The pairwise intersections of the S_ν contain the indeterminacy locus I. Away from I, each $\mathcal{C}^\circ(G;E)$ is the affinization of a smooth space, obtained by gluing two open charts $\mathcal{C}^\circ$ with a vertical relative shift over the Toda base. Away from S, the glued space is of course isomorphic to the original $\mathcal{C}^\circ$; whereas, near each $S_\nu\setminus I$, the Toda fibers undergo a nodal degeneration along the $\mathbb{C}^\times$ factor $\mathbb{C}^\nu$, modeled on $\mathbb{C}^\times \rightsquigarrow \mathbb{C}\sqcup_0\mathbb{C}$ in the fibers of the A_{n-1}-singularity $(x,y)\mapsto t=(xy)^{1/n}$. (The number n is computed from the divisibility and the multiplicities of the weight ν.) The appearance of the nodal locus, along which $\mathcal{C}^\circ(G;E)$ is singular when $n>1$, is a consequence of affinization: the smooth charts $\mathcal{C}^\circ$ cover the complement, as in Example 2.4. From here, Hartogs' theorem determines $\mathcal{C}^\circ(G;E)$ completely; but we can be more specific in concrete cases. Thus, some fibers of $\mathcal{C}^\circ(G;\mathbf{0})$ are crushed in co-dimension 2, over I.

5.2. Example: U_1 *with* $L \oplus L^\vee$

The space $\mathcal{C}_3^\circ(\mathrm{U}_1; L \oplus L^\vee)$ is the quadric cone $xy = \mu^2 - \tau^2$. In the original coordinates $\{\tau, z^\pm, \mu\}$, the rational automorphism $z \mapsto z(\mu+\tau)/(\mu-\tau)$ preserves precisely the subring generated by $\mu, \tau, x = (\mu-\tau)z, y = (\mu+\tau)z^{-1}$. The two copies of $\mathcal{C}_3^\circ$ map to the constructible subsets

$$\{\mu^2 \neq \tau^2\} \sqcup \{\mu = \tau, y \neq 0\} \sqcup \{\mu = -\tau, x \neq 0\} \sqcup \{0\},$$
$$\{\mu^2 \neq \tau^2\} \sqcup \{\mu = \tau, x \neq 0\} \sqcup \{\mu = -\tau, y \neq 0\} \sqcup \{0\},$$

whose union misses the nodal lines $x = y = 0$ in the fibers over $\mu = \tau$ and $\mu = -\tau$, with the exception of their intersection at the vertex 0, onto which the zero-fiber of each $\mathcal{C}_3^\circ$ gets crushed.

5.3. Example: SU_2 *with the standard representation*

Consider the Weyl double cover $\tilde{\mathcal{C}}_3^\circ$ of $\mathcal{C}_3^\circ$, defined from Corollary 4.3 before Weyl division. In the z, τ-notation already used for the maximal torus of SU_2, the functions over $\tilde{\mathcal{C}}_3^\circ$ are generated over $\mathbb{C}[\mu, \tau]$ by $u = (z-1)/\tau$ and $v = (1-1/z)/\tau$, with the single relation $u - v = \tau uv$. The Weyl action switches u and v and changes the sign of τ. Translation by ε_V sends z to $((\mu+\tau)/(\mu-\tau))z$. Let $x := \mu u - z$, $y := \mu v - z^{-1}$ and $w := (x-y)/\tau$; the surviving subring is described by generators and relations over the ring $\mathbb{C}[\mu, \tau]$ as

$$\{x, y, w\}, \quad \text{with relations } x - y = \tau w,\ xy = 1 + \mu w.$$

(We justify the generators in the example in Section 5.4.) Setting $\mu = 0$ yields the ring $\mathbb{C}[\tau, z^\pm, (z-1/z)/\tau]$. This is $\mathbb{C} \times \mathbb{C}^\times$, with the points $(0, \pm 1)$ blown up and the proper transform of $\tau = 0$ removed. Each of the two $\tilde{\mathcal{C}}_3^\circ$ charts covers one of the exceptional divisors and misses the other.

5.4. Example: SU_2 *with a general representation*

Factor $\varepsilon_V(\mu, \tau) = \phi(\mu, \tau)\phi^{-1}(\mu, -\tau) = \phi_+\phi_-^{-1}$, with a homogeneous polynomial ϕ of degree N, and let $x = (z\phi_- - \mu^N)/\tau$, $y = (\mu^N - z^{-1}\phi_+)/\tau$ and $w = (x-y)/\tau$ as before. Generators and relations for the surviving subring are

$$\{x, y, w\}, \quad \text{with relations } x - y = \tau w,\ xy = \frac{\mu^{2N} - \phi_+\phi_-}{\tau^2} + \mu^N w. \tag{5.1}$$

Setting $\mu = 0$ gives the subring generated by the relations $\tau^{N-1}(z - (-1)^N z^{-1})$ and $\tau^{N-2}(z + (-1)^N z^{-1})$. This reproduces the result of [6, Example 6.9].

For instance, choosing the adjoint representation gives $N = 2$ and the Weyl invariant ring is $\mathbb{C}[\tau, z + z^{-1}, \tau(z - z^{-1})]$, defining the quotient $T^\vee\mathbb{C}^\times/\{\pm 1\}$. This is the Coulomb branch for the zero representation of U_1, Weyl quotiented by ± 1. More generally, any representation with $N > 1$ leads to the Weyl quotient of the U_1 Coulomb branch for a representation with an N that is lower by 2, such as $V \ominus \mathfrak{g}/\mathfrak{h}$ if V happened to contain the adjoint representation. We generalize this in the Appendix.

5.5. *Checking the* SU_2 *example*

Let A be the surviving subring, and $A' \subset A$ the subring generated by (5.1); let us check that $A' = A$. This is clear with τ inverted, by reduction to the case of U_1, when $z\phi_-$ and $z^{-1}\phi_+$ generate $\tilde{\mathcal{C}}_3^\circ$ over $\mathbb{C}[\mu, \tau^\pm]$. Upon formal completion near $\tau = 0$, the statement is equally clear with μ inverted, when $\phi_\pm$ become units. This shows that A/A' is a quasi-coherent torsion sheaf on the (μ, τ)-plane supported at $\mu = \tau = 0$. But such a sheaf would yield a Tor_2 group against the sky-scraper at $\mu = \tau = 0$, which is forbidden, because (I claim) both A' and A are flat over $\mathbb{C}[\tau, \mu]$. Flatness A' is checked easily from the 3-step resolution built from (5.1); that of A is discussed below.

5.6. *Normality*

Our description of $\mathcal{C}^\circ(G; E)$ implies its normality: indeed, if a function f is integral over the surviving subring, then $f \circ (\varepsilon_V \cdot)$ is integral over $\mathcal{C}^\circ$, so it is regular, and so f survives. Alternatively, granting flatness of the Toda projections (to be discussed below), one sees the desired regularity in co-dimension 1 from the generic geometric behavior described in Section 5.1. Normality of the massless specialization can be extracted from the flatness discussion below, where we build $\mathcal{C}^\circ(G; E)$ from $\mathcal{C}^\circ$ by blow-ups and contractions along loci transversal to $\mu = 0$. Alternatively, granting flatness, we can again check regularity in co-dimension 1: the generic abelian description applies away from the root hyperplanes, while on the generic part of a root hyperplane the SU_2 description of Example 5.4 takes its place.

5.7. *More geometry*

Flatness of the Coulomb branches over the massive Toda bases (freedom, in fact) is wrapped into the proof of Theorem 1 in the next section. However, we can also extract it from our algebraic description; we outline the argument here, as it points a way to a more geometric description of their Weyl covers. It does suffice to treat the Weyl cover: the Toda base for the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is flat[5] over that of G, and extracting Weyl invariants does not spoil flatness (or normality).

Choose a smooth Weyl-invariant toric compactification $\bar{H}_{\mathbb{C}}^\vee$ of $H_{\mathbb{C}}^\vee$, requiring that the weights ν appearing in (4.1) should define boundary divisors B_ν.[6] The latter assemble to an ample relative boundary divisor B for the compactified projection

$$\bar{\chi} : \bar{H}_{\mathbb{C}}^\vee \times \mathfrak{h}_{\mathbb{C}} \times \mathbb{C} \to \mathfrak{h}_{\mathbb{C}} \times \mathbb{C}.$$

We will create a space leading to $\mathcal{C}_3^\circ(G; E)$ by blowing up products of pairs divisors (in the base and fiber), then remove a boundary, and finally perform an affinizing contraction over the Toda base.

[5] If we mind the orbifolding, for $\mathcal{C}_4$.

[6] Working with spaces, the construction will produce finite cyclic singularities, stemming from the multiplicities in (4.1); these can be avoided at the price of working with a suitable orbifold compactification instead.

First, we prepare to create the Weyl cover $\tilde{\mathcal{C}}^\circ$ by blowing up the loci ($\exp\alpha^\vee = 1$, $\alpha = 0$) – or rather, their successive proper transforms in a chosen sequence.[7] Then we prepare the surviving growth constraints by performing further blow-ups along (the successive proper transforms of) products $B_\nu \times S_\nu$, with an appropriate multiplicity on the hyperplane S_ν. Cyclic singularities appear here. Nonetheless, the co-dimension $(1, 1)$ of the blowing up loci and with respect to the Toda projection, and of their successive proper transforms, ensures that we get a flat modification $\tilde{\chi}$ of our projection $\bar{\chi}$.

The final step is the removal of the boundary and the collapse of proper components by taking fiberwise global functions along the projection $\tilde{\chi}$. The boundary comprises the proper transforms $\tilde{B}$ of B and $\tilde{R}$ of the root hyperplanes (the latter is to produce the original $\mathcal{C}^\circ$). The resulting ring is the colimit, as $N \to \infty$, of $R\tilde{\chi}_*(\mathcal{O}(N\tilde{B} + N\tilde{R}))$. The sheaf in the total direct image is a line bundle[8] with no higher direct images, because of its quasi-positivity and the negativity of K. Thus, each term in the limit is free over the base and so the colimit is flat.

6. Proof of Theorems 1 and 2

We use the Schubert stratification of $\Omega^a G$ into $G[[z]]$-orbits. Even-dimensionality collapses the associated spectral sequences and leads to ascending filtrations on the rings $\mathbb{C}[\mathcal{C}^\circ(G; E)]$. The associated graded components are easily described (Section 6.4), and are locally free over the Toda bases. This makes the original rings locally free as well; in particular, they are flat over $\mathbb{C}[\mu], \mathbb{C}[m^\pm]$.

I write out the proof for $\mathcal{C}_3^\circ$; the K-theory case is entirely parallel. Call A_V the ring, implied in Theorem 1, of regular functions on $\mathcal{C}_3^\circ$ which survive ε_V-translation. We will see from topology how this last operation is compatible with the Schubert filtration, so that we can also define the subring $\Sigma_V \subset \operatorname{Gr}\mathbb{C}[\mathcal{C}_3^\circ]$ of symbols which remain regular after ε_V-translation. Clearly, $\operatorname{Gr} A_V \subset \Sigma_V$. The theorems will follow from two observations:

(i) $\mathbb{C}[\mathcal{C}_3^\circ(G; E)] \subset A_V$,

(ii) $\operatorname{Gr}\mathbb{C}[\mathcal{C}_3^\circ(G; E)] = \Sigma_V$.

6.1. *The index bundle*

Over the stack $\mathfrak{Bun}_G(\mathbb{P}^1)$ of principal $G_\mathbb{C}$-bundles over $\mathbb{P}^1$, there lives the virtual index bundle Ind_V, the holomorphic Euler characteristic of the sheaf of sections of V over $\mathbb{P}^1$ with simple vanishing condition at one marked point ∞. It is a class in $K^0_{G\times S^1}(\Omega G)$, after incorporating the mass parameter μ (equivariance under scaling of V). Call e_V its equivariant Euler class, more accurately defined as the μ-homogenized G-equivariant total Chern class of Ind_V. The following two propositions are understood after suitable localization on the massive Toda base $\mathfrak{h}_\mathbb{C}/W_\mathbb{C} \times \mathbb{C}$.

[7] This step could be averted by the use of a “wonderful” normal-crossing compactification of $\mathcal{C}^\circ$ over its Toda base.

[8] For suitably divisible N, to cancel the effect of the cyclic singularities.

Proposition 6.1. *Translation by the section εA_V on $\mathcal{C}_3^\circ$ corresponds to cap-product with* e_V *on* $H_*^{G\times S^1}(\Omega G)$.

Remark 6.2. Cap-product with e_V *must a priori* correspond to translation by some rational section: the index bundle is additive for the sphere multiplication in $G \ltimes \Omega G$, so its Euler class is multiplicative. As a group-like element in the dual Hopf algebra, it represents a (rational) section of the group scheme $\mathcal{C}^\circ$ over its Toda base. We identify this section by abelianization.

Proof. Localize to the complement of the root hyperplanes on the Toda base to reduce, by the fixed-point theorem, to the case of a torus, where Proposition 2.4 applies (as enhanced in Example 2.6). ∎

Corollary 6.3. *The Schubert filtration is preserved by ε_V-translation.* ∎

6.2. Two embeddings of $\mathbb{C}[\mathcal{C}_3^\circ(G;E)]$

Refer to the notation in Section 3.3 and Remark 3.6. The Hecke construction at the point $0 \in \mathbb{P}^1$ maps the stack $\mathfrak{Bun}_G(-\!:\!-) = G[\![z]\!]\backslash\Omega^a G$ of $G_\mathbb{C}$-bundles over the double-centered disk to $\mathfrak{Bun}_G(\mathbb{P}^1)$. This gives an equivariant homotopy equivalence and in particular a (K-)homology equivalence. The key observation is that, restricted to $G[\![z]\!]\backslash\Omega^a G$, Ind_V is the "left minus right" copy of $V[\![z]\!]$.

More precisely, note the two inclusions $\iota_{l,r} : L_V \hookrightarrow V[\![z]\!]$, and recall that over any finite union of strata, the space L_V contains a finite co-dimension sub-bundle. Quotienting it out regularizes the difference of $V[\![z]\!]$-bundles into a class in $K_{G\times S^1}(\Omega G)$. A moment's thought identifies this with Ind_V, as the index of the Hecke transform of the trivial V-bundle on $\mathbb{P}^1$, minus that of the trivial V-bundle.

Each inclusion $\iota_{l,r}$ defines a graded ring homomorphism $\varphi_{l,r} : \mathbb{C}[\mathcal{C}_3^\circ(G;E)] \rightarrowtail \mathbb{C}[\mathcal{C}_3^\circ]$ by intersecting with the zero-section in the ambient bundle. Per our discussion, we have $\varphi_l = \mathrm{e}_V \cap \varphi_r$. By using φ_r to pin down $\mathcal{C}_3^\circ(G;E)$, Proposition 6.1 now settles observation (i).

6.3. Working out Σ_V

For a 1-parameter subgroup $z^\eta \in \Omega H$, with Schubert stratum C_η and Levi centralizer $Z(\eta) \subset G$, split $V = V_+ \oplus V_0 \oplus V_-$ following the sign of the η-eigenvalue. The index bundle then splits as $\mathrm{Ind}_V = I_+(\eta) \ominus I_-(\eta)$, with the ν-weight space of $V_\pm$ appearing $\pm\langle\nu|\eta\rangle$ times in $I_\pm(\eta)$. The Euler class e_V factors at z^η as

$$\mathrm{e}_V|_{z^\eta} = \mathrm{e}_+(\eta)\cdot \mathrm{e}_-(\eta)^{-1}, \quad \text{with } \mathrm{e}_\pm(\eta) = \prod_\nu (\mu+\nu)^{|\langle\nu|\eta\rangle|}.$$

There is a (degree-shifting) isomorphism

$$\mathrm{Gr}_\eta\, \mathbb{C}[\mathcal{C}^\circ(G;\mathbf{0})] = BMH_*^{G\times S^1}(C_\eta) \cong H_*^{Z(\eta)\times S^1}(\text{point}),$$

and the η-graded component of Σ_V is the subspace $\mathrm{e}_- \cap \mathrm{Gr}_\eta\, \mathbb{C}[\mathcal{C}^\circ(G;\mathbf{0})]$.

6.4. *Working out* $\mathrm{Gr}\,\mathbb{C}[\mathcal{C}_3^\circ(G;E)]$

Collapse of the Schubert spectral sequence implies that

$$\mathrm{Gr}_\eta\,\mathbb{C}[\mathcal{C}_3^\circ(G;E)] = BMH_*^{G\times S^1}(L_V|_\eta).$$

Now, the homology group is generated over $H_*^{Z(\eta)\times S^1}$(point) by the fundamental class of the total space of L_V over C_η, whose complement in the right $V[\![z]\!]$ of (3.4) is precisely $I_-(\eta)$; therefore

$$\mathrm{Gr}_\eta\,\mathbb{C}[\mathcal{C}_3^\circ(G;E)] = \mathrm{e}_-(\eta)\cap \mathrm{Gr}_\eta\,\mathbb{C}[\mathcal{C}^\circ(G;\mathbf{0})],$$

in agreement with the η-component of Σ_V above. This settles observation (ii).

7. Non-commutative Coulomb branches

Recall that incorporating the loop rotation circle R in the previous constructions leads to non-commutative deformations $\mathcal{NC}_{3,4}^\circ(G;E)$ of the Coulomb branches over the ground rings $\mathbb{C}[h] = H^*(BR)$ and $\mathbb{C}[q^\pm] = K_R$(point), respectively. The geometric objects exist in the formal neighborhoods of $h = 0$ and $q = 1$; away, only their function rings $\mathcal{A}_{3,4}$ survive. Nonetheless, we sometimes keep the convenient conversational pretence of underlying spaces $\mathcal{NC}$. The calculation in Section 6 for their description applies with only minor changes: we are only missing the good statements, which we summarize below before spelling out the argument.

This section is rather sketchy; a development spelling out the rôle of our non-commutative solutions, the Γ-functions, in connection with the GLSM, is planned for a follow-up paper.

7.1. *Summary*

The integrable abelian group structure of $\mathcal{C}^\circ$ over their Toda bases deforms to a symmetric tensor structure[9] on $\mathcal{A}$-modules, induced from the diagonal inclusion $\Omega G \rightarrowtail \Omega G\times\Omega G$. Restricting the module structure to the Toda base, this is the ordinary tensor product, with tensor unit the structure sheaf $\mathcal{O}_1$ of the identity section. For $\mathcal{C}_3^\circ$ in the Whittaker presentation (Remark 3.3), the operation comes from convolution of $\mathcal{D}$-modules on the Langlands dual group $G^\vee$: from this stance, the symmetric monoidal structure is developed in [2].

The Lagrangians ε_V, λ_V deform to modules E_V, Λ_V over $\mathcal{A}_{3,4}$, and the (rational) automorphisms of $\mathcal{C}$ defined by ε_V, λ_V-translation become, on $\mathcal{A}$-modules, the functors of convolution with E_V, Λ_V. The Hamiltonian nature of the translations renders these functors (generically) trivializable by (singular) inner automorphisms of $\mathcal{A}$. In Theorem 4, I characterize the Coulomb branches $\mathcal{NC}^\circ(G;E)$ as the subrings of elements of $\mathcal{A}$ which survive these inner automorphisms (that is, remain regular).

[9] I thank David Ben-Zvi for pointing out to me the generality of this statement.

While this loose description of the branches $\mathcal{NC}^\circ(G;E)$ appears uniform, a distinction arises between formal and genuine deformations. Formally, the modules E_V and Λ_V are generically invertible, analogous to flat line bundles with singularities, with the latter located on the singular loci of the sections ε_V, λ_V. If, following the language of $\mathcal{D}$-modules, we call *solutions* the $\mathcal{A}$-module morphisms to the identity section $\mathcal{O}_1$, then the super-potentials that were introduced in Remark 4.2 are the leading $h \to 0$ asymptotics of the logarithms of the solutions (cf. Remark 7.5).

With the deformation parameters turned on, these asymptotics become meromorphic solutions that are easily found. For E_V on $\mathcal{C}_3$, a solution is the Γ-function of the representation V (recalled in Section 7.3), while a q-analogue solves Λ_V on $\mathcal{C}_4$. Conjugation by these solutions impose the defining regularity constraints for $\mathcal{NC}^\circ(G;E)$. Outside the range the formal limit, the modules E_V, Λ_V can be defined by these solutions, which thus become the primary objects. We may prefer, for convenience, the (tensor) inverse modules and their holomorphic solutions; thus, E_V^{-1} is the quotient of $\mathcal{A}_3$ by the annihilator $\mathcal{I}_V$ of the (holomorphic) solution Γ_V^{-1}:

$$\begin{array}{ccc} E_V^{-1} := \mathcal{A}_3/\mathcal{I}_V & \xrightarrow{\cdot\,\Gamma_V^{-1}} & \mathcal{O}_1 = \mathcal{A}_3/\mathcal{I}_1. \\ \cup & & \\ \mathcal{O}_1 & & \end{array}$$

If we regard the quotient $\mathcal{A}_3/\mathcal{I}_V$ as an analytic sheaf over the Toda base, the solution map Γ_V^{-1} is an isomorphism. (Otherwise, its infinitely many zeroes prevent it from surjecting onto $\mathcal{O}_1$.) We can then characterize $\mathcal{NC}^\circ(G;E)$ in three equivalent ways, the last two of which are Γ-conjugate, namely as the subring of elements of $\mathcal{A}_3$

(i) which survive conjugation by Γ_V^{-1},

(ii) whose multiplicative action preserves the inclusion $\mathcal{O}_1 \subset E_V^{-1}$,

(iii) whose multiplicative action preserves the inclusion $\Gamma_V^{-1}\mathcal{O}_1 \subset \mathcal{O}_1$.

There is a parallel story for $\mathcal{NC}_4$. Before spelling out the details, let us revisit the case of U_1.

7.2. *Example I:* U_1 *with its standard representation*

The symplectic space $T^\vee\mathbb{C}^\times = \operatorname{Spec} H_*^{\mathrm{U}_1}(\Omega\mathrm{U}_1)$ has a natural non-commutative deformation, realized topologically by the Pontryagin ring $H_*^{\mathrm{U}_1\times R}(\Omega\mathrm{U}_1)$. Indeed, on $\pi_1\mathrm{U}_1$, z-multiplication is the shift $n \mapsto n+1$, at which point the R-rotation collects an extra U_1-weight. We compute from here the Pontryagin ring as $\mathbb{C}[h]\langle z^\pm, \tau\rangle$ with the relation $z\tau = (\tau+h)z$. We now identify the non-commutative Coulomb branch $H_*^{\mathrm{U}_1\times R}(L_L)$ for the standard representation L:

Lemma 7.1. *The non-commutative deformation* $\mathcal{NC}_3(\mathrm{U}_1; L\oplus L^\vee)$ *is the subring of* $H_*^{\mathrm{U}_1\times R}(\Omega\mathrm{U}_1)$ *generated over* $\mathbb{C}[h]$ *by* $z, z^{-1}\tau$.

Remark 7.2. By setting $X = z$ and $Y = z^{-1}\tau$, this ring is $\mathbb{C}[h]\langle X, Y\rangle/([X,Y]-h)$, as one could have guessed from the Poisson relation $\{x,y\} = h$ in $\mathbb{C}[x,y]$ (notation as in Example 2.4).

Proof. Using the right inclusion in Section 6.2 to embed the ring, we find at the winding mode $n \geq 0$ the summand $z^n \cdot \mathbb{C}[h,\tau]$; whereas at a negative winding mode $(-n)$, we find

$$z^{-n}\mathrm{e}_- = z^{-n}\tau(\tau+h)\cdots(\tau+(n-1)h) = (z^{-1}\tau)^n$$

from the Euler class $\mathrm{e}_-(-n)$ of I_-, which is the summand missing from the right copy of $V[\![z]\!]$. ■

Recall now the h-periodic Gamma-function

$$\Gamma(w;h) := h^{\frac{w}{h}-1}\Gamma\left(\frac{w}{h}\right).$$

It satisfies $\Gamma(w+h;h) = w\Gamma(w;h)$ and $\Gamma(h;h) = 1$. From $z\tau z^{-1} = \tau + h$ we get

$$\Gamma(\tau;h)\cdot z\cdot\Gamma(\tau;h)^{-1} = \tau^{-1}z, \tag{7.3}$$

which exhibits $\Gamma(\tau;h)$ as a solution to the module $\mathcal{A}_3/(z-\tau)$, the obvious quantization of ε_V:

Corollary 7.4. *Away from the poles, sending* 1 *to* $\Gamma(\tau;h)$ *maps* $\mathcal{A}_3/(z-\tau)$ *into the module* $\mathcal{O}_1 = \mathcal{A}_3/(z-1)$. ■

Holomorphy of the reciprocal function Γ^{-1} is a reason to prefer the inverse module $\mathcal{A}_3/(1-\tau z)$.

Remark 7.5. As $h \to 0$, Stirling's approximation gives (when $|\arg(\tau/h)| < \pi^-$)

$$\log\Gamma(\tau;h) = \frac{\tau}{h}(\log\tau - 1) - \frac{1}{2}\log h + \frac{1}{2}\log\left(\frac{2\pi}{\tau}\right) + O\left(\frac{h}{\tau}\right),$$

and we find in the leading h^{-1} coefficient the Legendre transform $\psi^*(\tau)$ of $\psi(z) = z$. The Legendre correspondence quantizes to the Laplace transform: viewing $\mathcal{A}_3$ as the ring of $\mathcal{D}_h$-modules on $\mathbb{C}^\times$, with $\tau = h\cdot z\frac{\partial}{\partial z}$, we find that the function $\exp(-z/h)$ on $\mathbb{C}^\times$ is the solution to the module $\mathcal{D}_h/(\tau+z)$, Laplace transformed from the one in Corollary 7.4.

Proposition 7.6. *The non-commutative deformation* $\mathcal{NC}_3(\mathrm{U}_1; L\oplus L^\vee)$ *is the subring of elements of* $H_*^{\mathrm{U}_1\times R}(\Omega\mathrm{U}_1)$ *which survive conjugation by* $\Gamma(\tau;h)^{-1}$. ■

Proof. Survival of z and $z^{-1}\tau$ is clear from (7.3). To show the converse inclusion, choose an $\mathcal{A}_3$-element of negative z-degree $(-n)$. Reordering factors expresses it uniquely in monomials of the form

$$(z^{-1}\tau)^n\tau^m,\quad m\geq 0,\qquad\text{and}\qquad (z^{-1}\tau)^a z^{a-n},\quad 0\leq a<n.$$

The former survive Γ^{-1}-conjugation. To rule out the latter, note that conjugation converts them to $z^{-a}(\tau z)^{a-n}$. These monomials are not regular in any $\mathbb{C}[h]$-linear combination, or else a right multiplication by $(\tau z)^n$ would lead to a linear dependence among the monomials

$$\begin{aligned} z^{-a}(\tau z)^a &= (\tau-h)\cdots(\tau-ah), & 0\leq a<n,\\ z^{-n}\tau^m(\tau z)^n &= (\tau-nh)^m\cdot(\tau-h)\cdots(\tau-nh), & m\geq 0,\end{aligned}$$

which is pre-empted by their τ-degree. ■

7.3. The Γ_V-class

Generalizing this involves promoting Γ to a multiplicative characteristic class of complex vector bundles. This requires some care: the Hirzebruch construction, the product $\Gamma(F;h) := \prod_\rho \Gamma(\rho;h)$ over the Chern roots ρ of F, is ill-defined, as Γ has a pole at the point 0. The reciprocal $1/\Gamma$ is entire holomorphic, but its vanishing at 0 would lead to an unstable class, undefined for virtual bundles. One remedy is to include the equivariant scaling (mass) parameter μ, resulting in a μ-meromorphic calculus for the classes $\prod_\rho \Gamma(\mu+\rho;h)$. Thus, a representation V of G leads to the entire holomorphic (in μ,ξ) reciprocal function

$$\Gamma_V(\xi,\mu;h)^{-1} : \mathfrak{h}_{\mathbb{C}}/W \times \mathbb{C} \to \mathbb{C}, \quad (\xi,\mu) \mapsto \det{}_V \Gamma(\xi \oplus \mu;h)^{-1}.$$

Remark 7.7 (Massless specialization.). The correct massless specialization is $\mu = \frac{1}{2}h$ (not $\mu = 0$). In the construction of [6], this should be interpreted as inserting a square root of the canonical bundle on the doubled disk –:–. The same insertion within the index bundle does away with the vanishing condition at ∞ in the constructions of Section 6. The specialization is illustrated by the identity

$$\Gamma\left(\frac{h}{2}+\tau;h\right)\Gamma\left(\frac{h}{2}-\tau;h\right) = \frac{\pi}{h}\sec\left(\frac{\pi\tau}{h}\right);$$

the product is therefore anti-central in $\mathcal{NC}(\mathrm{U}_1)$ (it conjugates z to $(-z)$), generalizing the identity $\mathrm{e}_L \cup \mathrm{e}_{L^\vee} = (-1)^n$ of Example II in Section 2.6.

Remark 7.8. Interpreting h as the equivariant parameter of the loop rotation group R, the Weierstraß product expansion portrays Γ_V^{-1} as a regularized Euler class of the space of Taylor loops $V[\![z]\!]$. This interpretation also makes sense over certain stacks with a circle action, such as $G[\![z]\!]\backslash\Omega^a G$: a reasonable demand is that their R-equivariant homology is free over $\mathbb{C}[h]$, so that extension of scalars to functions of h holomorphic off the negative real axis (and allowing poles in μ,ξ as needed) is a faithful operation. For a torus, we can always pretend that h is a numerical parameter, because the R-action on the stack $\mathfrak{Pic}(\mathbb{P}^1)$ is trivializable.

7.4. Example II: U_1 *with a representation* V

Split $V = V_+ \oplus V_-$ according to z-exponents.[10] Writing $\Gamma_V = \Gamma_+\Gamma_-$, we have

$$\begin{aligned}\Gamma_V^{-1}\cdot z\cdot\Gamma_V &= (\Gamma_+\Gamma_-)^{-1}\cdot z\cdot\Gamma_+\Gamma_- \\ &= \Gamma_+^{-1}z\Gamma_+z^{-1}\cdot z\cdot z^{-1}\Gamma_-^{-1}z\Gamma_- \\ &= \mathrm{e}_+(1)\cdot z\cdot \mathrm{e}_-(1)^{-1}, \end{aligned} \tag{7.9}$$

with $\mathrm{e}_\pm(1)$ the R-equivariant extensions of the index Euler classes of Section 6.3 at $\eta = 1$. Repeating the computation in the proof of Lemma 7.1,

$$[z\mathrm{e}_-(1)]^n = z^n\mathrm{e}_-(n), \quad [\mathrm{e}_+(1)z]^n = \mathrm{e}_+(n)z^n, \quad n \geq 0. \tag{7.10}$$

[10] A trivial representation summand V_0 does not affect the Coulomb branch.

Proposition 7.11. *The non-commutative deformation* $\mathcal{NC}_3^\circ(\mathrm{U}_1; V \oplus V^\vee)$ *is generated over* $\mathbb{C}[\mu, \tau, h]$ *by* $z\mathrm{e}_-(1)$, $z^{-1}\mathrm{e}_+(1)$, *and is the subring of* $H_*^{\mathrm{U}_1 \times S^1 \times R}(\Omega \mathrm{U}_1)$ *surviving conjugation by* Γ_V.

Proof. From (7.9) we see that the listed generators survive, and (7.10) shows that their nth power generates the summand of degree $\pm n$ over the Toda base. Fix now $n > 0$ say. The need for the $z^n\mathrm{e}_-(n)$ factor in a surviving element follows from unique factorization in the ground ring $\mathbb{C}[\mu, \tau, h]$. Namely, $z^n f(\mu, \tau, h)$ conjugates to $\mathrm{e}_+(n)z^n\mathrm{e}_-(n)^{-1} f$. The linear factors of $z^{-n}\mathrm{e}_+z^n$ have the form $(\mu + k\tau - ph)$ with $p > 0$, and are prime to the denominator $\mathrm{e}_-(n)$, whose factors carry non-negative multiples of h going with μ; so all canceling factors must come from f. ∎

Localizing on the Toda base, we find from the abelian calculation, formally close to $h = 0$:

Corollary 7.12. *The map* Γ_V *conjugates the unit module of* $\mathcal{NC}_3^\circ$ *into a module* E_V *with support* $\bar{\varepsilon}_V$. ∎

Away from formal $h = 0$, we can define the convolution-inverse module E_V^{-1} as the quotient of $\mathcal{A}_3$ by the annihilator of Γ_V^{-1}. Sending $1 \in \mathcal{A}_3$ to Γ_V^{-1} identifies it with the module $\mathcal{O}_1$.

7.5. *Description of the* $\mathcal{NC}$*-spaces*

Theorems 4 and 5 below are quantum versions of the Lagrangian-shift description of the Coulomb branches. The proof follows the commutative argument, with its core relying on the Euler interpretation of Γ_V (Remark 7.8): conjugation by Γ_V^{-1} becomes capping with (the R-equivariant) e_V. The capping operation is of course canonical, but the left and right module structures of $\mathcal{A}_3$ over $H^*_{G\times R}$ differ, as they come from left and right pull-backs from $B(G \times R)$ to the stack $R \ltimes (G \ltimes LG \rtimes G)$. Of course, this is why Γ_V-conjugation is not trivial.

Theorem 4. *The non-commutative deformation* $\mathcal{NC}_3^\circ(G; E)$, *defined as* $H_*^{G\times S^1 \times R}(L_V)$, *comprises those elements of* $H_*^{G\times S^1\times R}(\Omega G)$ *which survive conjugation by* Γ_V^{-1}.

Proof. Incorporate the R-action in the embeddings $\varphi_{l,r}$ of Section 6.2. I claim that conjugation switches φ_r to φ_l: this need only be checked generically on the Coulomb branch, and can be seen by restriction to the maximal torus, reducing to the abelian calculation in Example II above.

It follows that φ_r places $\mathcal{NC}_3(G; E)$ within the surviving subring, and the argument closes by quoting Proposition 7.6 on each Schubert stratum C_η, with z^η in lieu of z, to conclude that $\mathrm{Gr}_\eta\, \mathcal{NC}_3^\circ(G; E)$ exhausts the surviving part of $\mathrm{Gr}_\eta\, \mathcal{NC}_3^\circ$. ∎

7.6. *The space* $\mathcal{NC}_4^\circ$

In the Key Example of U_1, the Pontryagin ring $K_*^{\mathrm{U}_1\times R}(\Omega \mathrm{U}_1)$ is the standard non-commutative (complexified) torus, $\mathbb{C}[q^{\pm}]\langle t^{\pm}, z^{\pm}\rangle$ with relation $zt = qtz$. To proceed, we need

Jackson's p-Gamma function [12]. In terms of p-Pochhammer symbols

$$(x;p)_\infty = \prod_{n\geq 0}(1-xp^n),$$

convergent for $|p|<1$, this is

$$\Gamma_p(w;h) = (1-p)^{1-\frac{w}{h}}\frac{(p^h;p^h)_\infty}{(p^w;p^h)_\infty},$$

satisfying

$$\Gamma_p(w+h;h) = \frac{1-p^w}{1-p}\Gamma_p(w;h).$$

The requisite version of the function Γ_p arises in the limit $p,h\to 0$, as the expansion variables $q := p^{-h}$, $t := p^{-w} = p^{-\tau}$ are kept finite.[11] Set

$$\Gamma_0(t) := (q^{-1};q^{-1})_\infty/(t^{-1};q^{-1})_\infty$$

and note the conjugation

$$\Gamma_0(t)\cdot z\cdot\Gamma_0(t)^{-1} = (1-t^{-1})^{-1}z,$$

with the K-theoretic Euler class $(1-t^{-1})$ replacing τ in (7.3).

Remark 7.13. In analogy with Remark 7.5, the Laplace transform of our solution Γ_0 is expressed in terms of the q-exponential function e_q, namely $e_q(\frac{z}{1-q}) = (z;q)_\infty^{-1}$.

Define now the multiplicative class $\Gamma_{0;V}$ for vector bundles, valued in localized equivariant K-theory, as in Section 7.3; formally, near $q=1$, we then have the following proposition.

Proposition 7.14. *The multiplicative class $\Gamma_{0;V}$ conjugates the unit module of $\mathcal{NC}_4^\circ$ into a module Λ_V with support λ_V.* ■

Finally, the argument used for $\mathcal{NC}_3$ applies, after working locally on the Toda base, to give the following theorem.

Theorem 5. *The non-commutative deformation $\mathcal{NC}_4^\circ(G;V\oplus V^\vee)$, which is defined by the Pontryagin ring $K_*^{G\times S^1\times R}(L_V)$, comprises those elements of $K_*^{G\times S^1\times R}(\Omega G)$ which remain regular after conjugation by $\Gamma_{0;V}^{-1}$.* ■

Appendix A. A Weyl character formula for certain Coulomb branches

Here, I verify the abelianization result mentioned in the introduction, which describes "most" Coulomb branches for G in terms of those for the Cartan subgroup, with their Weyl group symmetry. There is also a non-commutative version, as in Section 7; I will return to it in a future paper.

[11] The signs keep the series convergent for $|q|>1$, matching $h>0$ in the additive case.

Theorem 6. *For any representation V of G whose weights contain the roots of $\mathfrak{g}$, we have*

$$\mathcal{C}_{3,4}(G;E) \cong \mathcal{C}_{3,4}\big(H;E \ominus (\mathfrak{g}/\mathfrak{h})^{\oplus 2}\big)/W,$$

compatibly with the embeddings of Section 6.2 and the morphism

$$\mathcal{C}_{3,4}(G;\mathbf{0}) \to \mathcal{C}_{3,4}(H;\mathbf{0})/W.$$

Proof. Working over the common bases $\mathfrak{h}/W$ and H/W, H-fixed point localization shows that the map induced by the named morphisms is an isomorphism away from the root hyperplanes; whereas, generically on the root hyperplanes, the SL_2 calculation of Section 5.4 confirms isomorphy. This settles the matter, because the algebras are free $\mathcal{O}$-modules over the Toda base and agree in co-dimension 2. ∎

Remark A.1. The calculation for $\mathcal{C}_3$ of SL_2 was seen to hold more generally, for all but a few choices of E. This generalizes to all groups, by the argument above: however, the formulation of the right-hand side needs more care. Exploiting the local descriptions of the $\mathcal{C}_4$ Toda bases in terms of $\mathcal{C}_3$, one can then push this to an awkward but effective calculation of most $\mathcal{C}_4$ Coulomb branches. It would be truly useful to find the formulation which dispenses with all constraints on E: this might allow an abelianized calculation of Coulomb branches with non-linear matter.

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