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The structure of minimal surfaces in CAT(0) spaces

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Abstract. We prove that a minimal disc in a CAT(0) space is a local embedding away from a finite set of "branch points". On the way we establish several basic properties of minimal surfaces: monotonicity of area densities, density bounds, limit theorems and the existence of tangent maps. As an application, we prove Fáry–Milnor's theorem in the CAT(0) setting.

Keywords. Minimal surfaces, CAT(0), total curvature, knots

Contents

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1. Introduction

1.1. Motivation and main results

Minimal surfaces are an indispensable tool in Riemannian geometry. Part of their success relies on the well-understood structure of minimal discs. For example, by the classical Douglas–Rado Theorem, any smooth Jordan curve Γ in $\mathbb{R}^n$ bounds a least-area disc. Moreover, this disc is a smooth immersion away from a finite set of branch points. Recently, Alexander Lytchak and Stefan Wenger proved that a rectifiable Jordan curve in a proper metric space bounds a least-area disc as long as it bounds at least one disc of finite energy [33]. As in the case of Douglas and Rado, the minimal disc is obtained by minimizing energy among all admissible boundary parametrizations. The existence and regularity of energy minimizers or *harmonic maps* in metric spaces was studied earlier, usually under some kind of nonpositive curvature assumption [20, 24, 27]. For instance, Nicholas Korevaar and Richard Schoen solved the Dirichlet problem in CAT(0) spaces and showed that the resulting harmonic maps are locally Lipschitz in the interior [27].

The intrinsic geometry of minimal discs was studied in [35, 38, 43]. There, it is shown (with varying generality) that minimal discs in CAT(0) spaces are intrinsically nonpositively curved. However, apart from regularity nothing else is known about the mapping behavior of minimal discs. The first aim of this paper is to establish topological properties. We obtain the following structural result for minimal discs, similar to the classical statement that minimal surfaces are branched immersions.

Theorem 1. *Let X be a* CAT(0) *space and $\Gamma \subset X$ a rectifiable Jordan curve of finite total curvature. Let $u : D \to X$ be a minimal disc filling Γ. Then there exists a finite set $B \subset D$ such that u is a local embedding on $D \setminus B$.*

This is proven in Section 4.5 as Theorem 70. Unlike in the smooth case, the corresponding result for harmonic discs fails, even if the target is of dimension two and has only isolated singularities, see work of Ernst Kuwert [28]. We then aim at topological applications and prove the Fáry–Milnor Theorem for CAT(0) spaces, generalizing the original theorem, proven independently by Istvan Fáry [18] and John Milnor [39].

Theorem 2 (Fáry–Milnor). *Let Γ be a Jordan curve in a* CAT(0) *space X. If the total curvature of Γ is less than 4π, then Γ bounds an embedded disc.*

See Theorem 5 for a more general result and Section 2.4 for the definition of total curvature.

Our proofs of both theorems rely heavily on the monotonicity of area densities (Corollary 63 in Section 4.4):

Theorem 3 (Monotonicity). *Let X be a* CAT(0) *space. Suppose that $u : \bar{D} \to X$ is a minimal disc and p is a point in $u(\bar{D}) \setminus u(\partial \bar{D})$. Then the area density*

$$\Theta(u, p, r) := \frac{\operatorname{area}(u(D) \cap B_r(p))}{\pi r^2}$$

is a nondecreasing function of r as long as $r < |p, u(\partial D)|$.

1.2. Overview and further results

On the structure of minimal surfaces. In order to obtain control on the mapping behavior of minimal discs, we make intensive use of the intrinsic point of view, developed by Alexander Lytchak and Stefan Wenger in [34] and [35]. (See also [43].) They showed that if X is CAT(0), then any minimal disc $u : D \to X$ factors through an intrinsic space Z_u, which is itself a CAT(0) disc. Furthermore, it turns out that the factors $\pi : D \to Z_u$ and $\bar{u} : Z_u \to X$ are particularly nice. Namely, π is a homeomorphism and $\bar{u}$ preserves the length of every rectifiable curve, cf. Theorem 41. Therefore, in order to prove Theorem 1, we only need to investigate the induced map $\bar{u}$. For this purpose we introduce the notion of *intrinsic minimal surfaces* which by definition is a synthetic version of the induced map $\bar{u}$. We then prove several basic properties for intrinsic minimal surfaces, all well known in the smooth case. The most important of these is the monotonicity of area densities and a corresponding lower density bound (see Proposition 62 and Lemma 65). As in the classical case, monotonicity is accompanied by a rigidity statement (Theorem 73).

However, the proof of rigidity is more involved and we were unable to directly derive it from the monotonicity of area densities. Instead, we first investigate intrinsic minimal surfaces on an infinitesimal scale. We show that intrinsic minimal surfaces have tangent maps at all points. Tangent maps for harmonic maps into CAT(0) spaces were also investigated by Misha Gromov and Richard Schoen [20] and later by Georgios Daskalopoulos and Chikako Mese [15]. However, our results are independent, as we aim for "intrinsic tangent maps". In our setting, we show that each such tangent map is itself an intrinsic minimal map which in addition is conical and even locally isometric away from a single point. (See Lemma 68 and the prior definition.) Building on this, we prove the rigidity supplement to monotonicity and our main structural result:

Theorem 4. *Let X be a* CAT(0) *space and Z a* CAT(0) *disc. Suppose that $f : Z \to X$ is an intrinsic minimal surface. If z_0 is a point in the interior of Z, whose link $\Sigma_{z_0} Z$ satisfies $\mathcal{H}^1(\Sigma_{z_0} Z) < 4\pi$, then f restricts to a bilipschitz embedding on a neighborhood of z_0. In particular, if Z is a* CAT(0) *disc, then f is locally a bilipschitz embedding in the interior of Z away form finitely many points.*

Together with Theorem 41 this then yields Theorem 1.

On Fáry–Milnor's theorem. The original theorem of Fáry–Milnor from 1949 says that a knot in $\mathbb{R}^3$ has to be the unknot if it is of finite total curvature less or equal than 4π. The first generalization of this theorem to variable curvature came about 50 years later and is due to Stefanie Alexander and Richard Bishop ([2]). We also refer the reader to their work for the history of the problem. Their result extended the Fáry–Milnor Theorem to simply connected 3-dimensional manifolds of nonpositive sectional curvature. More precisely, it is shown in [2] that a Jordan curve of total curvature less or equal to 4π in a 3-dimensional Hadamard manifold bounds an embedded disc. In the same paper it was noticed that the analog statement cannot be true for CAT(0) spaces. There is an example of a Jordan curve of total curvature 4π in a 2-dimensional CAT(0) space which does not bound an embedded disc, see Example 1. However, Theorem 2 shows that Fáry–Milnor's theorem does hold for CAT(0) spaces, at least if the strict inequality for the total curvature is fulfilled. We actually prove the following more general result.

Theorem 5 (Rigidity case of Fáry–Milnor). *Suppose that Γ is a Jordan curve in a* CAT(0) *space X. If the total curvature of Γ is less than or equal to 4π, then either Γ bounds an embedded disc, or else the total curvature is equal to 4π and Γ bounds an intrinsically flat geodesic cone. More precisely, there is a map from a convex subset of a Euclidean cone of cone angle 4π which is a local isometric embedding away from the cone point and which fills Γ.*

Our proof relies on minimal surface theory and follows the strategy of Tobias Ekholm, Brian White, and Daniel Wienholtz in [16], where the authors show that a minimal surface Σ in $\mathbb{R}^n$ of any topological type is embedded if the total curvature κ of the boundary is less than or equal to 4π. Their approach was also used in [14] to prove the Fáry–Milnor Theorem in n-dimensional Hadamard manifolds.

We quickly recall their argument. If Σ is such a surface in $\mathbb{R}^n$, then for any point p not in the boundary of Σ one augments Σ by an exterior cone E_p over $\partial\Sigma$. More precisely, we have

$$E_p = \bigcup_{q\in\partial\Sigma} \{p + t(q-p) : t \geq 1\}.$$

The monotonicity of area densities continues to hold for $\Sigma \cup E_p$ and now it even holds for all times. Since the area growth of E_p is equal to κ, this relates the number of inverse images of p to the total curvature of $\partial\Sigma$. The completion of the proof is then based on a lower density bound.

In our case there is no exterior cone. Additionally, for an ordinary minimal disc in a CAT(0) space the required lower density bound is unclear. However, for intrinsic minimal discs the lower density bound is obvious and the above argument still shows the following (Corollary 77).

Theorem 6. *Let $\hat{X}$ be a* CAT(0) *space and $\hat{f} : \hat{Z} \to \hat{X}$ a proper intrinsic minimal plane. Suppose that the area growth of $\hat{f}$ is less than twice the area growth of the Euclidean plane. Then $\hat{f}$ is an embedding.*

In order to prove the Fáry–Milnor Theorem we then show an extension result for intrinsic minimal discs. Roughly, it says that for each intrinsic minimal disc $f : Z \to X$ with finite total curvature κ of the boundary we can embed X isometrically into a CAT(0) space $\hat{X}$ such that f extends to a proper intrinsic minimal plane $\hat{f}$ in $\hat{X}$ (Proposition 80). The space $\hat{X}$ is obtained from X by gluing a flat funnel along the boundary of f.

2. Preliminaries from metric geometry

We refer the reader to [11], respectively [9], for definitions and basics on metric geometry and to [8,10,26] for metric spaces with upper curvature bounds. However, we include this short section to agree on some terminology and notations.

2.1. Generalities

Let D be the open unit disc in the plane and denote by S^1 the unit circle.

For a metric space X we will denote the distance between two points $x, y \in X$ by $|x, y|$, i.e. $|\cdot, \cdot|$ is the metric on X. If $\lambda > 0$, we define the rescaled metric space $\lambda \cdot X$ be declaring the distance between points to be λ times their old distance.

For a subset $A \subset X$ we denote by $\bar{A}$ its closure. If $x \in X$ is a point and $r > 0$ is a radius, we denote by $B_r(x)$ the open ball of radius r around x in X. More generally, for a subset $P \subset X$ we denote by $N_r(P)$ the tubular neighborhood of radius r.

A *Jordan curve* in X is a subset $\Gamma \subset X$ which is homeomorphic to S^1. If Γ is a Jordan curve in $\mathbb{R}^2$, then its *Jordan domain* is the bounded connected component of $\mathbb{R}^2 \setminus \Gamma$.

A geodesic in X is a curve of constant speed whose length is equal to the distance of its endpoints. A space is called *geodesic* if there is a geodesic between any two of its points and it is called *uniquely geodesic* if there is only one such geodesic. If X is uniquely geodesic, then xy will denote the (image) of the unique geodesic between x and y. For a point $p \in X$ we call a geodesic *p-radial* if it starts in p. A map $f : X \to Y$ to another metric space Y is called *radial isometry (with respect to p)* if it preserves distances to p. As usual, a map $f : X \to Y$ between metric spaces X and Y is called *Lipschitz continuous* or simply *Lipschitz* if there exists a (Lipschitz-)constant $L > 0$ such that $|f(x), f(x')| \leq L \cdot |x, x'|$ holds for all $x, x' \in X$. Further, f will be called *bilipschitz* if it is bijective and has a Lipschitz continuous inverse. Distance nonincreasing maps or 1-Lipschitz maps between metric spaces will simply be called *short*. For $n \in \mathbb{N}$ we will denote by $\mathcal{H}^n$ the n-dimensional Hausdorff measure on X.

A *surface* is a 2-dimensional topological manifold, possibly with boundary. If Σ is a surface and $f : \Sigma \to X$ is a map into a space X, then we denote by $\partial f : \partial\Sigma \to X$ the restriction $f|_{\partial\Sigma}$.

If (X_k, x_k) denotes a pointed sequence of metric spaces, we can always take an *ultralimit* (X_ω, x_ω) with respect to some nonprincipal ultrafilter ω on the natural numbers. If (Y_k, y_k) denotes another such sequence and $f_k : X_k \to Y_k$ are Lipschitz maps with a uniform Lipschitz constant $L > 0$, then we obtain an L-Lipschitz ultralimit

$$f_\omega : (X_\omega, x_\omega) \to (Y_\omega, y_\omega).$$

For a precise definition and basics on ultralimits of metric spaces we refer the reader to [3].

2.2. Intrinsic metric spaces

A metric space X is called *intrinsic metric space* or *length space* if the distance between any two of its points is equal to the greatest lower bound of the lengths of continuous curves joining those points. Let M be a topological space and X a metric space. If $f : M \to X$ is a continuous map, then we can define on M an *intrinsic (pseudo-)metric* associated to f. Namely, the intrinsic distance between two points in M is equal to the greatest lower bound for lengths of f-images of curves joining these points. If any pair of points in M is joined by a curve whose image under f is rectifiable, then one can identify points of zero f-distance in M to obtain an associated *intrinsic metric space* M_f. For instance, this is ensured if M is a length space and f is Lipschitz continuous. We will say that a map f has some property *intrinsically* if the associated space M_f has this property. If M is equal to X and f is the identity, then we obtain the *intrinsic space* associated to X and we denote it by X^i.

2.3. CAT(κ)

For the definition and basic properties of CAT(κ) spaces we refer the reader to [10] or [9]. We just point out that all our CAT(κ) spaces are assumed to be complete. If X is a CAT(κ) space and $p, x, y \in X$ are points with $x, y \in B_{\frac{\pi}{\sqrt{\kappa}}}(p) \setminus \{p\}$, then there is a well-defined angle $\angle_p(x, y)$ between x and y at p. Each point p in a CAT(κ) space X has an associated *space of directions* or *link* $\Sigma_p X$ which is the metric completion, with respect to angles, of germs of geodesics starting at p. Recall that $\Sigma_p X$ is a CAT(1) space with respect to the intrinsic metric induced by $\angle$. The *tangent space at* p is defined as the Euclidean cone over $\Sigma_p X$ and denoted by $T_p X$. In particular, $T_p X$ is again a CAT(0) space. Note that if X is CAT(0), then there is a natural short *logarithm map* $\log_p : X \to T_p X$ which is a radial isometry and preserves initial directions of p-radial geodesics.

The following two theorems by Reshetnyak will be used repeatedly. The gluing theorem is useful in order to check if a certain space is CAT(0). For a detailed discussion and a proof we refer to [9] and [3]. The majorization theorem provides controlled Lipschitz fillings of circles.

Theorem 7 (Reshetnyak's gluing theorem). *Let X and X' be two* CAT(0) *spaces with closed convex subsets $C \subset X$ and $C' \subset X'$. If $\iota : C \to C'$ is an isometry, then the space $X \cup_\iota X'$, which results from gluing X and X' via ι, is* CAT(0) *with respect to the induced length structure.*

Theorem 8 (Reshetnyak's majorization theorem). *Let X be a* CAT(0) *space and let $c : [0, L] \to X$ be a closed curve of unit speed. Then there is a convex region $C \subset \mathbb{R}^2$, possibly degenerated, bounded by a unit speed curve $\tilde{c} : [0, L] \to \mathbb{R}^2$, and a short map $\mu : C \to X$ with $\mu \circ \tilde{c} = c$.*

As a consequence, the Euclidean isoperimetric inequality for curves holds in CAT(0) spaces.

Theorem 9 (Isoperimetric inequality). *Let X be a* CAT(0) *space and let $c : S^1 \to X$ be a Lipschitz circle. Then there exists a Lipschitz extension $\hat{c} : \bar{D} \to X$ of c with the property that* $\operatorname{area}(\hat{c}) \leq \frac{1}{4\pi}\operatorname{length}(c)^2$, *where the area of $\hat{c}$ is the Hausdorff* 2*-measure counted with multiplicities, cf. Definition* 37.

2.4. *Total curvature*

Let X be a CAT(0) space. A curve $\sigma : [a, b] \to X$ is called a *k-gon* if there is a subdivision $a = t_0 < t_1 < \cdots < t_k = b$ such that the restrictions $\sigma|_{[t_i, t_{i+1}]}$ are geodesics. Note that every ordered k-tuple $(x_1, \ldots, x_k)$ of points $x_i \in X$ determines an k-gon. The points $x_i := \sigma(t_i)$ are called *vertices* of σ. If the number of vertices is not important, we will simply call σ a *polygon*. The *total curvature* $\kappa(\sigma)$ of a k-gon σ with vertices (x_i) is defined by

$$\kappa(\sigma) := \sum_{i=2}^{k-1} (\pi - \angle_{x_i}(x_{i-i}, x_{i+1})).$$

A k-gon σ with vertices (x_i) is *inscribed* in a curve γ if there is a parametrization $\tilde{\gamma}$ of γ such that $\tilde{\gamma}^{-1}(x_i) \leq \tilde{\gamma}^{-1}(x_{i+1})$ for $0 \leq i \leq k-1$. If a polygon σ is inscribed in a polygon ρ, then $\kappa(\sigma) \leq \kappa(\rho)$ because the sum of the interior angles of triangles in CAT(0) spaces is bounded above by π. (See [2, Lemma 2.1].)

Definition 10. Let γ be a curve in a CAT(0) space X. Then its *total curvature* $\kappa(\gamma)$ is defined as

$$\kappa(\gamma) := \sup\{\kappa(\sigma) : \sigma \text{ is an inscribed polygon in } \gamma\}.$$

Note that this definition generalizes the Riemannian definition of total curvature. A curve of finite total curvature is rectifiable and has left and right directions at every interior point. Moreover, Fenchel's theorem holds, i.e. the total curvature of a closed curve γ is at least 2π and equality is attained if and only if γ bounds a flat convex subset or degenerates to an interval (cf. [2, Section 2]).

2.5. *Lipschitz maps*

We have the following version of Rademacher's theorem is contained in [29]. Since it is not explicitly stated in [29], we will provide a sketch.

Proposition 11. *Let X be a* CAT(κ) *space and let $K \subset \mathbb{R}^n$ be a measurable subset. Let $f : K \to X$ be a Lipschitz map. Then f is almost everywhere differentiable in the following sense. For almost all $p \in K$ there exists a unique map $df_p : T_p\mathbb{R}^n \to T_{f(p)}X$ whose image is a Euclidean space, which is linear and such that*

$$df_p(v) = \lim_{h\to 0} \frac{\log_{f(p)}(f(p+hv))}{h}.$$

Proof. By [25] we know that for almost every point p in K there exists a unique seminorm $|df_p(\cdot)|$ on $\mathbb{R}^n$ such that

$$\lim_{v\to 0} \frac{|f(p+v), f(p)| - |df_p(v)|}{|v|} = 0.$$

Since X is CAT(κ), this seminorm is almost everywhere induced by a possibly degenerated inner product (cf. [27, 33]). In [29, Section 12] it is first shown that any Lipschitz curve $\gamma : (0,1) \to X$ has for almost all $t \in (0,1)$ left and right derivatives $\gamma^{\pm}(t)$ which are antipodal. More precisely, we have

$$\gamma^{\pm}(t) = \lim_{h\searrow 0} \frac{\log_{\gamma(t)}(\gamma(t \pm h)}{h}$$

and $\angle(\gamma^+(t), \gamma^-(t)) = \pi$. By Fubini's theorem it is concluded that for almost all points and a countable dense set of directions there exist directional derivatives. Since f is

Lipschitz, it follows that almost everywhere f is directional differentiable. Hence at almost every point p we have an induced map $F_p : \mathbb{R}^n \to T_{f(p)}X$. If p is such that the metric differential $|df_p(\cdot)|$ comes from an inner product, then $\mathbb{R}^n$ splits orthogonally into $V^+ \oplus V_0$, where V_0 is the kernel of $|df_p(\cdot)|$. The map F_p restricts to an isometric embedding $(V^+, |df_p(\cdot)|) \to T_{f(p)}X$. In particular, $df_p := F_p$ is linear. ■

Lemma 12 ([25, Theorem 7]). *Let X be a* CAT(0) *space. Let $K \subset \mathbb{R}^2$ be measurable, let $f : K \to X$ be a Lipschitz map with $f(K) = Y$. Let $N_f : Y \to [1, \infty]$ be the multiplicity function $N_f(y) = \#\{z \in K : f(z) = y\}$. Then the following area formula holds true:*

$$\int_Y N_f(y)\, d\mathcal{H}^2(y) = \int_K \mathcal{J}(df_z)\, dz,$$

where $\mathcal{J}(df_z)$ denotes the usual Jacobian of the linear map df_z.

Corollary 13. *Let X be a metric space. Let $K \subset \mathbb{R}^2$ be measurable and let $f : K \to \mathbb{R}^2$ be a monotone Lipschitz map with $f(K) = Y$. Further, let $s : Y \to X$ be a Lipschitz map. Then* $\operatorname{area}(s \circ f) = \operatorname{area}(s)$.

Proof. We have $N_{s\circ f} \geq N_s$. Since f is monotone, it only takes the values 1 and ∞ on Y. We conclude $N_{s\circ f} = N_s$ for $\mathcal{H}^2$-almost all points in X and the claim follows from Lemma 12. ■

We will need the following special case of the general coarea formula.

Lemma 14 ([7, Theorem 9.4]). *Let Z be a countably $\mathcal{H}^2$-rectifiable* CAT(0) *space and let $g : Z \to \mathbb{R}$ be a Lipschitz function. Then, for any Borel function $\theta : Z \to [0, \infty]$,*

$$\int_Z \theta(z)|\nabla g(z)|\, d\mathcal{H}^2(z) = \int_{\mathbb{R}} \left(\int_{g^{-1}(t)} \theta(x)\, d\mathcal{H}^1(x) \right) dt.$$

Proof. Note that Z has Euclidean tangent planes at almost all points. Hence the coarea factor $C_1(d^Z g(z))$ appearing is the general coarea formula is given by $|\nabla g(z)|$ for almost all $z \in Z$, see [7, (9.2)]. ■

We will repeatedly make use of the following proposition which is a consequence of [1, Theorem 2.5].

Proposition 15. *Let $f : \bar{D} \to \mathbb{R}$ be a Lipschitz map and denote by $\Pi_y := f^{-1}(y)$ its fibers. Then for almost every value $y \in \mathbb{R}$ the following holds.*

(i) *$\mathcal{H}^1(\Pi_y)$ is finite.*

(ii) *Each connected component of Π_y which has positive length is contained in a rectifiable Jordan curve $\Gamma \subset \mathbb{R}^2$. Moreover, the union of those components has full $\mathcal{H}^1$-measure in Π_y.*

Proof. Extend f to a Lipschitz map $F : \mathbb{R}^2 \to \mathbb{R}$ with compact support. Denote by $\Pi_y^F := F^{-1}(y)$ its fibers. By [1, Theorem 2.5], almost every fiber Π_y^F of F decomposes as $\Pi_y^F = N \cup \bigcup_{i=1}^{\infty} \Gamma_i$, where $\mathcal{H}^1(N) = 0$ and each Γ_i is a rectifiable Jordan curve. Moreover, $\mathcal{H}^1(\Pi^F)$ is finite and equal to $\sum_{i=1}^{\infty} \mathcal{H}^1(\Gamma_i)$. The claim follows. ■

Definition 16. Let $f : \bar{D} \to \mathbb{R}$ be a Lipschitz map. We call $y \in \mathbb{R}$ an *quasi regular value* if the conclusion of Proposition 15 holds for the fiber Π_y of f.

2.6. Preparation for cut and paste

Lemma 17. *Let X be a* CAT(0) *space, $p \in X$ a point and $r > 0$ some radius. Suppose that $\gamma : S^1 \to X$ is a Lipschitz circle with image contained in $\bar{B}_r(p)$. Let $v : \bar{D} \to X$ be the p-radial extension of γ. Then v is Lipschitz continuous with*

$$\operatorname{area}(v) \leq \frac{r}{2} \cdot \operatorname{length}(\gamma).$$

Moreover, equality holds if and only if γ lies at constant distance r from p and the geodesic cone over γ with tip p is intrinsically a flat cone.

Proof. Recall that v maps radial geodesics in $\bar{D}$ with constant speed to p-radial geodesics in X. We first show that v is Lipschitz continuous by estimating distances using piecewise radial and spherical paths. For $x \in \bar{D}$ set $\lambda_x = |\gamma(\frac{x}{\|x\|}), p|$. If $L \geq r$ is a Lipschitz constant for γ, then we have

$$|v(x), v(y)| \leq \lambda_x \cdot \big|\|x\| - \|y\|\big| + L \cdot \angle_0(x, y) \cdot \|y\| \leq 3L \cdot \|x - y\|.$$

At almost every point $x \in \bar{D}$ we can estimate the Jacobian by $\mathcal{J}(dv_x) \leq \lambda_x \cdot |\dot{\gamma}(\frac{x}{\|x\|})|$. Hence

$$\begin{aligned}\operatorname{area}(v) &\leq \int_0^1 \int_0^{2\pi} \lambda_\theta |\dot{\gamma}(\theta)| s \, ds \, d\theta \\ &\leq r \cdot \int_0^1 s \cdot \left(\int_0^{2\pi} |\dot{\gamma}(\theta)| d\theta \right) ds = \frac{r}{2} \cdot \operatorname{length}(\gamma).\end{aligned}$$

Equality implies $\lambda_\theta \equiv r$ and that the length of distance spheres in the intrinsic space grows exactly linearly. Hence the claim follows from the rigidity statement in the Bishop–Gromov Theorem (Theorem 24). ■

Lemma 18. *Let X be a* CAT(0) *space and $\gamma : S^1 \to X$ a Lipschitz curve of length equal to 2π. Then the following holds.*

(i) *For $\theta \in [0, 2\pi)$ let $R_\theta : S^1 \to S^1$ be the counter clockwise rotation by the angle θ. For each such θ there exists a Lipschitz homotopy $h_\theta : S^1 \times [0, 1] \to X$ form γ to $\gamma \circ R_\theta$ with* $\operatorname{im}(h_\theta) = \operatorname{im}(\gamma)$.

(ii) *Let $\bar{\gamma}$ be an arc length parametrization of γ, then there is a Lipschitz homotopy $h : S^1 \times [0, 1] \to X$ form $\bar{\gamma}$ to γ with* $\operatorname{im}(h) = \operatorname{im}(\gamma)$.

In particular, all of the above homotopies have zero mapping area.

Proof. For the first part we simply set $h_\theta(e^{it}, s) = \gamma(e^{i(t+s\theta)})$. For the second part, denote by $l(\theta)$ the length of the curve $\gamma(e^{it})|_{[0,\theta]}$ and set $\tau(\theta) = e^{il(\theta)}$. Note that the map $l : [0, 2\pi] \to [0, 2\pi]$ is Lipschitz continuous and $h(\theta, s) = \bar{\gamma}(e^{i((1-s)\theta + sl(\theta))})$ provides a homotopy as required. ■

Lemma 19. *Let X be a* CAT(0) *space and let $\epsilon > 0$ be a number.*

(i) *For $i = 1, 2$, let $\gamma_i : [0, 1] \to X$ be Lipschitz curves with Lipschitz constant $L > 0$. If $\sup_{t\in[0,1]} |\gamma_1(t), \gamma_2(t)| < \epsilon$, then there exists a Lipschitz homotopy h from γ_1 to γ_2 with* $\operatorname{area}(h) \leq \frac{8}{\pi} \cdot L \cdot \epsilon$.

(ii) *Let $\gamma^+ : [0, 1] \to X$ be an L-Lipschitz curve with $L > 0$ and denote by γ^- the geodesic from $\gamma^+(1)$ to $\gamma^+(0)$. Suppose that* $\operatorname{im}(\gamma^+) \subset N_\epsilon(\operatorname{im}(\gamma^-))$. *Let γ denote the concatenation of γ^+ and γ^-. Then there exists a Lipschitz disc w filling γ and such that* $\operatorname{area}(w) \leq \frac{8}{\pi} \cdot L \cdot \epsilon$.

Proof. Choose $n \in \mathbb{N}$ such that $\frac{\epsilon}{2L} \leq \frac{1}{n} \leq \frac{\epsilon}{L}$. On each interval $\{\frac{k}{n}\} \times [0, 1]$, $1 \leq k \leq n$, we define the homotopy h to be the constant speed parametrization of the geodesic from $\gamma_1(\frac{k}{n})$ to $\gamma_2(\frac{k}{n})$. To extend the definition to the remaining domains in $[0, 1] \times [0, 1]$, we use the isoperimetric inequality 9. The boundaries of these domains map to curves of length $\leq 4\epsilon$. Therefore h is a Lipschitz map of area bounded above by $n \cdot \frac{(4\epsilon)^2}{4\pi} \leq \frac{8}{\pi} L\epsilon$. This proves part (i).

For part (ii) we choose n as above. Denote by x_k the nearest point projection of $\gamma^+(\frac{k}{n})$. Using the geodesics between $\gamma^+(\frac{k}{n})$ and x_k, we can cut γ into n Lipschitz circles of length bounded above by 4ϵ. We can now finish the proof as above, using the isoperimetric inequality. ∎

3. CAT(0) geometry

3.1. Funnel extensions of CAT(0) spaces

Definition 20. For $\alpha > 0$ denote C_α the Euclidean cone over a circle of length α. A metric space E_α is called a *flat funnel* (of angle α) if it is isometric to the complement of a relatively compact convex neighborhood of the vertex of C_α.

Definition 21. Let C_α be a Euclidean cone over a closed interval of length α. A metric space $s_{\alpha,r}$ is called a *flat sector* (of angle $\alpha \geq 0$ and radius $r > 0$) if it is isometric to the closed r-ball around the vertex of C_α. An infinite flat sector will be called an *ideal flat sector*. The *legs* l_1 and l_2 of $s_{\alpha,r}$ are the two radial geodesic segments of length r lying in the boundary of $s_{\alpha,r}$. They intersect in a single point p, the *tip* of $s_{\alpha,r}$.

Lemma 22. *Let X be a* CAT(0) *space and let q, x, y be three points in X such that $r := |qx| = |qy| > 0$ and $\alpha := \angle_q(x, y)$. Denote l_1 and l_2 the legs of a flat sector $s_{2\pi-\alpha,r}$. If $f : l_1 \cup l_2 \to X$ is an intrinsic isometry onto $qx \cup qy$, then $X \cup_f s_{2\pi-\alpha,r}$ is a* CAT(0) *space.*

Proof. By the theorem of Cartan–Hadamard, it is enough to show that $X \cup_f s_{2\pi-\alpha,r}$ is locally CAT(0). Since geodesic segments in CAT(0) spaces are convex, Reshetnyak's theorem implies the claim away from the point q. We show that the r-ball around q is CAT(0).

One can obtain $B_r(q) \subset X \cup_f s_{2\pi-\alpha,r}$ from $B_r(q) \subset X$ in two steps. First we glue a flat sector $s_{\pi-\alpha,r}$ along one of its legs to the geodesic segment qx. In the resulting

CAT(0) space the second leg of $s_{\pi-\alpha,r}$ extends the geodesic segment qy to a geodesic σ of length $2r$. In a second step we can now glue a flat half-disc of radius r to σ, thereby producing $B_r(q) \subset X \cup_f s_{2\pi-\alpha,r}$. Hence $X \cup_f s_{2\pi-\alpha,r}$ is CAT(0). ∎

Lemma 23. *Let γ be an embedded closed curve in X with finite total curvature $\kappa(\gamma)$. Then there is a flat funnel $E_{\kappa(\gamma)}$ and an intrinsic isometry $f : \gamma \to \partial E_{\kappa(\gamma)}$ such that $X \cup_f E_{\kappa(\gamma)}$ is a* CAT(0) *space.*

Proof. Let us assume that γ is a k-gon. We will glue the flat funnel E_α to X in two steps. First, let us choose for every pair of adjacent vertices x_i and x_{i+1} of γ a flat half-strip $h_i \cong [0, |x_i, x_{i+1}|] \times [0, \infty)$. We then glue these flat half-strips to X via isometries $[0, |x_i, x_{i+1}|] \times \{0\} \to x_i x_{i+1}$. In the resulting space we see two geodesic rays $r_i^- \subset h_{i-1}$ and $r_i^+ \subset h_{i+1}$ emanating from every vertex x_i. The distance between the directions of r_i^+ and r_i^- at x_i is given by $\pi + \angle_{x_i}(x_{i-1}, x_{i+1})$. As a second step we insert ideal flat sectors s_i of angle $\alpha_i \geq \pi - \angle_{x_i}(x_{i-1}, x_{i+1})$, thereby completing the angle at every vertex of γ to a total angle $\geq 2\pi$. Altogether we obtain $X \cup E_\alpha$, and from the construction it is clear that we can achieve $\alpha = \kappa(\gamma)$. The CAT(0) property follows from Lemma 22 together with the theorem of Cartan–Hadamard.

The case for general γ follows by approximating γ by inscribed polygons γ_n. For each n we choose the completing angles α_i slightly larger than necessary in order to ensure that each funnel E_n is contained in $E_{\kappa(\gamma)}$. After passing to a subsequence, we may assume that ∂E_n Hausdorff converges to $\partial E_{\kappa(\gamma)}$ and the gluing maps $\iota_n : \partial E_n \to \gamma_n$ converge pointwise to a limit map $\iota : \partial E_{\kappa(\gamma)} \to \gamma$. Since ι is short and

$$\lim_{n\to\infty} \text{length}(\gamma_n) = \text{length}(\gamma),$$

it follows that ι is an intrinsic isometry. Now any quadruple of points in $X \cup E_{\kappa(\gamma)}$ can be approximated by quadruples in $X \cup E_n$ such that pairwise distances converge. Then [10, Theorem 3.9] implies that $X \cup_f E_{\kappa(\gamma)}$ is CAT(0). ∎

3.2. CAT(0) surfaces

A CAT(0) space Z which is homeomorphic to a topological surface is called a CAT(0) surface. Since CAT(0) spaces are contractible, the topology of CAT(0) surfaces is rather simple. In particular, any compact CAT(0) surface Z is homeomorphic to the closed unit disc $\bar{D}$ in $\mathbb{R}^2$. In this case, Z is called a CAT(0) *disc*. If Z is a CAT(0) surface and z is a point in the interior of Z, then small metric balls around z are homeomorphic to $\bar{D}$ and the link $\Sigma_z Z$ is homeomorphic to a circle. Moreover, a metric ball around an interior point is even bilipschitz to the corresponding ball in the tangent space and the Lipschitz constant can be chosen arbitrary close to one as the radius of the ball tends to zero [12]. Using bilipschitz parametrizations, we can define the area of Lipschitz maps with domain a CAT(0) surface and target an arbitrary metric space. Moreover, the Hausdorff 2-measure on a CAT(0) surface behaves similarly to the Lebesgue measure on a smooth Hadamard surface. For instance, the following volume comparison holds, cf. [40, Proposition 7.4]. We will refer to it as Bishop–Gromov's theorem in analogy to the case of lower curvature bounds.

Theorem 24 (Bishop–Gromov). *Let Z be a* CAT(0) *surface. Let p be a point in Z and suppose that $|p, \partial Z| > r$. Then we have*

$$\mathcal{H}^2(B_r(p)) \geq \frac{r^2}{2} \cdot \mathcal{H}^1(\Sigma_p Z).$$

Moreover, equality holds if and only if $B_r(p)$ is isometric to the radius r ball around the tip in $T_p Z$.

Note that each CAT(0) surface is geodesically complete in the sense that any geodesic segment is contained in geodesic segment whose boundary lies in the boundary of the surface. This is immediate from the fact that a pointed neighborhood of an interior point cannot be contractible. Hence the interior of a CAT(0) surface is GCBA in the sense of Lytchak and Nagano [31].

Lemma 25. *Let (Z_k) be a sequence of* CAT(0) *discs with rectifiable boundaries. Assume that (Z_k) Gromov–Hausdorff converges to a* CAT(0) *disc Z. If boundary lengths are uniformly bounded, $\mathcal{H}^1(\partial Z_k) < C$, then the total measures converge, $\mathcal{H}^2(Z_k) \to \mathcal{H}^2(Z)$.*

Proof. By [31, Theorem 12.1] it is enough to show that no measure is concentrated near the boundary.

Let $\epsilon > 0$. Since $Z \setminus \partial Z$ is an intrinsic space, we can find a Jordan polygon σ in the set $Z \setminus \partial Z$ such that the arc length parametrization of σ is uniformly ϵ-close to an arc length parametrization of ∂Z. Then we lift σ to Jordan polygons σ_k in Z_k. Denote by $W \subset Z$ the closure of the Jordan domain of σ and by W_k the closure of the Jordan domain of $\sigma_k \subset Z_k$. Then $W_k \to W$ and it is enough to bound $\mathcal{H}^2(Z_k \setminus W_k)$ uniformly. But since $\mathcal{H}^1(\partial Z_k) < C$ and length$(\sigma_k) \to$ length(σ), it follows from Lemma 19 below that $\mathcal{H}^2(Z_k \setminus W_k) \leq C' \cdot \epsilon$ with a uniform constant $C' > 0$. Hence the claim follows. ■

If Z is a CAT(0) disc, then its interior $Y := Z \setminus \partial Z$ is a length space which is still locally CAT(0). As such it is a surface of bounded integral curvature in the sense of Alexandrov [4]. For a detailed account to surfaces of bounded integral curvature we refer the reader to [5]. Here we only collect a few facts needed later. On Y there exists a (possibly infinite) nonpositive Radon measure μ, called *curvature measure*, such that if a Jordan triangle $\triangle$ is contained in a Jordan domain O, then the excess of $\triangle$ is bounded below by $\mu(O)$. In particular, $\mu(O) = 0$ for a Jordan domain $O \subset Y$ is equivalent to O being flat. The atoms of μ correspond to points in Y where the tangent cone is not isometric to the flat plane. More precisely, it holds

$$\mu(\{y\}) = 2\pi - \mathcal{H}^1(\Sigma_y Y)$$

for $y \in Y$.

Lemma 26. *Let Z be a* CAT(0) *surface and let (z_i) be a sequence in Z. Further, let (ϵ_i) be a nullsequence of positive real numbers. If (z_i) converges to a point z in the interior of Z, then the following holds.*

(i) $\omega\text{-lim}(\frac{1}{\epsilon_i} Z, z_i) \cong \mathbb{R}^2$ *if* $\omega\text{-lim} \frac{|z,z_i|}{\epsilon_i} = +\infty$.

(ii) $\omega\text{-lim}(\frac{1}{\epsilon_i} Z, z_i) \cong T_z Z$ *if* $\omega\text{-lim} \frac{|z,z_i|}{\epsilon_i} < +\infty$.

Proof. If $\omega\text{-lim}\,\frac{|z,z_i|}{\epsilon_i} < +\infty$, then $\omega\text{-lim}(\frac{1}{\epsilon_i}Z, z_i)$ is isometric to $\omega\text{-lim}(\frac{1}{\epsilon_i}Z, z)$, although possibly not pointed-isometric. Then the second claim follows from

$$\omega\text{-lim}\left(\frac{1}{\epsilon_i}Z, z\right) \cong T_z Z.$$

Now assume that $\omega\text{-lim}\,\frac{|z,z_i|}{\epsilon_i} = +\infty$ and set

$$W := \omega\text{-lim}\left(\frac{1}{\epsilon_i}Z, z_i\right).$$

Then W is a complete CAT(0) surface without boundary and we need show that it is flat. Notice that $\lim_{r\to 0}\mu(B_r(z)\setminus\{z\}) = 0$. Hence for each fixed $R > 0$ we have

$$\lim_{i\to\infty}\mu(B_{\epsilon_i R}(z_i)) = 0.$$

Now if $\triangle$ is a geodesic triangle in W, then we can find geodesic triangles $\triangle_i$ in (Z, z_i) such that $\omega\text{-lim}\,\frac{1}{\epsilon_i}\triangle_i = \triangle$ and all three angles converge ([23, Proposition 2.4.1]). Since there exists $R_0 > 0$ such that $\triangle_i \subset B_{\epsilon_i R_0}(z_i)$ for ω-all i, we see that the excess of $\triangle_i$ goes to zero. It follows that W is flat and therefore $W \cong \mathbb{R}^2$. ■

Lemma 27. *Let Z be a* CAT(0) *disc and assume that its boundary ∂Z is a polygon with positive angles. Then Z is bilipschitz to $\bar{D}$.*

Proof. The double Y of Z is a closed Alexandrov surface of bounded integral curvature. By [13, Theorem 1] it is therefore bilipschitz to the round sphere S^2. By the Lipschitz version of the Schönflies Theorem in [47] we conclude that Z is bilipschitz to $\bar{D}$. ■

Lemma 28. *Let Z be a* CAT(0) *surface. Let Γ be a rectifiable Jordan curve in Z and denote by Ω_Γ its Jordan domain. Then $\bar{\Omega}_\Gamma$ equipped with the induced intrinsic metric is a* CAT(0) *disc. If Γ is a Jordan polygon with positive angles, then this space is even bilipschitz to $\bar{\Omega}_\Gamma$.*

Proof. To proof the first claim, we will use Theorem 39 and Theorem 41 below. By Theorem 39, there is a solution $u : \bar{D} \to Z$ to the Plateau problem for (Γ, Z). By Theorem 41, u factors over the associated intrinsic space Z_u and induces a short map $\bar{u} : Z_u \to Z$ which restricts to an arc length preserving homeomorphism $\partial Z_u \to \Gamma$. Moreover, Z_u is a CAT(0) disc. Now since Z is a surface, we infer from [36, Theorem 6.1] that u is a homeomorphism from $\bar{D}$ to $\bar{\Omega}_\Gamma$. This implies that $\bar{u}$ provides an isometry $Z_u \to \bar{\Omega}_\Gamma$.

We turn to the second claim. Any metric is bounded above by its associated intrinsic metric. By compactness, it is enough to locally control the intrinsic metric on $\bar{\Omega}_\Gamma$ by the induced metric. However, near an interior point both metrics coincide. But in a neighborhood of a boundary point the two metrics are still bilipschitz equivalent because the angles of Γ are positive. Hence the claim holds. ■

Lemma 29. *Let Z be a* CAT(0) *disc with rectifiable boundary ∂Z and suppose that $c : S^1 \to \partial Z$ is a simple L-Lipschitz parametrization. Then for every $\epsilon > 0$ there exists an $(L+\epsilon)$-Lipschitz curve $\sigma : S^1 \to Z \setminus \partial Z$ which parametrizes a Jordan polygon of positive angles and is uniformly ϵ-close to c.*

Proof. Choose $n \in \mathbb{N}$ such that $\frac{L\cdot 4\pi}{n} < \frac{\epsilon}{2}$ and then choose $\delta > 0$ such that $4 \cdot n \cdot \delta < \frac{\epsilon}{2}$. Next, choose points $\sigma(\frac{k\cdot 2\pi}{n})$ in $Z \setminus \partial Z$ such that $|\sigma(\frac{k\cdot 2\pi}{n}), c(\frac{k\cdot 2\pi}{n})| < \delta$. By the triangle inequality we obtain

$$\left|\sigma\left(\frac{(k-1)\cdot 2\pi}{n}\right), \sigma\left(\frac{k\cdot 2\pi}{n}\right)\right| < \frac{L\cdot 2\pi}{n} + 2\cdot\delta.$$

Since $Z \setminus \partial Z$ is a length space ([35, Theorem 1.3]), we can define $\sigma|_{[\frac{(k-1)\cdot 2\pi}{n}, \frac{k\cdot 2\pi}{n}]}$ to be a constant speed parametrization of a polygon in $Z \setminus \partial Z$ whose length is bounded above by $\frac{L\cdot 2\pi}{n} + 3\cdot\delta$. Hence the Lipschitz constant of σ is less than $L + \frac{3\cdot n\cdot\delta}{2\pi}$ which is less than $L + \epsilon$ by our choice of δ.

Again by the triangle inequality we obtain

$$|\sigma(t), c(t)| < \frac{L\cdot 4\pi}{n} + 4\cdot\delta < \epsilon$$

and therefore σ is uniformly ϵ-close to c.

Now let us take a sequence $\epsilon_k \searrow 0$ and polygonal circles $\sigma_k : S^1 \to Z$ which are $(L+\epsilon_k)$-Lipschitz and uniformly ϵ_k-close to c. Note that the image of σ_k is a finite planar graph. We denote by Γ_k a longest Jordan curve in $\mathrm{im}(\sigma_k)$. Set $M_k = \sigma_k^{-1}(\Gamma_k)$. Then Γ_k converges to ∂Z with respect to Hausdorff distance. Set $M_k = \sigma_k^{-1}(\Gamma_k)$ and let $J_k \subset S^1$ be a component of $S^1 \setminus M_k$. Since σ_k converges uniformly to the simple curve c, the diameters of $\sigma_k(J_k)$ go to zero. Assume that k is large enough such that the diameter of the image of any component of $S^1 \setminus M_k$ is less than $\frac{\epsilon}{2}$. Then we define a new map $\tilde{\sigma}_k$ which equals σ_k on M_k and is constant on each component of $S^1 \setminus M_k$. Note that $\tilde{\sigma}_k$ parametrizes Γ_k. By construction, $\tilde{\sigma}_k$ is uniformly $(\epsilon_k + \frac{\epsilon}{2})$-close to c and still $(L+\epsilon_k)$-Lipschitz. At last we modify $\tilde{\sigma}_k$ in order to guarantee positive angles at all vertices. Let x, v and y be consecutive vertices of $\tilde{\sigma}_k$ and $\angle_v(x,y) = 0$. Since Γ_k is a Jordan curve, the edges xv and vy intersect only in v. We take any point $v' \neq x, v$ on the geodesic xv, then $\angle_{v'}(x,y) \neq 0$ by the uniqueness of geodesics. ■

Recall that a map between topological spaces is called *monotone*, if the inverse image of every point is connected.

Lemma 30. *Let Z be a* CAT(0) *disc with rectifiable boundary ∂Z. Let $c : S^1 \to \partial Z$ be a Lipschitz parametrization. Then c extends to a Lipschitz continuous monotone map $\mu : \bar{D} \to Z$.*

Proof. Pick a point p in the interior of Z. Then extend c by mapping radial geodesics in $\bar{D}$ to constant speed p-radial geodesics in Z. Lipschitz continuity follows in the same way as in Lemma 17 above. Observe that if y and z are two points in ∂Z such that the geodesics py and pz intersect in a nontrivial geodesic segment px, then the union $xy \cup xz$ separates Z. Hence one of the two components of $\partial Z \setminus \{y, z\}$ has the property that the geodesic from any of its points to p contains px. This implies the claimed monotonicity. ■

We will need to recognize when a CAT(0) surface is flat away from a finite number of cone points. We begin with a definition.

Definition 31. Let X be a CAT(0) space. Fix a point $p \in X$ and a radius $r > 0$. Then we call an element $v \in \Sigma_p X$ *r-regular*, if there exists a unique geodesic of length r starting in p with direction v. This induces a decomposition of the link into *regular* and *irregular* directions,

$$\Sigma_p X = \Sigma_p^{\mathrm{reg}(r)} X \cup \mathcal{S}_r(p).$$

We denote by $\mathcal{R}_r(p)$ the *regular star of radius r around p*. By definition, it is the union of all p-radial geodesics of length r starting in a r-regular direction. Let $\bar{\mathcal{R}}_r(p)$ denote its closure.

Now let Z be a CAT(0) surface and let $p \in Z$ be an interior point. For $r < |p, \partial Z|$ we consider the restriction of the logarithm to the distance sphere, $\log_p : S_r(p) \to \Sigma_p Z$. Note that $S_r(p)$ is homeomorphic to a circle. The inverse image of an irregular direction $v \in \mathcal{S}_r(v)$ is a nondegenerate interval $\log_p^{-1}(v) = [x_v^-, x_v^+]$. In particular, we see that there are at most countably may irregular directions. Now consider the logarithm on the closed ball, $\log_p : \overline{B_r(p)} \to \Sigma_p Z$, and let $v \in \Sigma_p X$ be an irregular direction. On $\log_p^{-1}(v)$ define the equivalence relation

$$x \sim x' \quad \text{if } |p, x| = |p, x'|.$$

Lemma 32. *The set $\overline{B_r(p)}/\sim$ is a* CAT(0) *disc with respect to the quotient metric.*

Proof. Choose an antipode $\hat{v}$ of v and a geodesic $\hat{\gamma}$ of length r in the direction $\hat{v}$. Denote by $\gamma^\pm$ the geodesics from p to $x_v^\pm$. Let $Z^\pm \subset \overline{B_r(p)}$ be closure of the component left respectively right from the concatenation $\hat{\gamma} * \gamma^\pm$. Then the quotient space is obtained by gluing Z^+ to Z^- along the boundary geodesic and the claim follows from Reshetnyak's gluing theorem (Theorem 7) . ■

We now obtain a regularized version of the CAT(0) disc $\overline{B_r(p)}$ by dividing out the equivalence relation

$$x \approx x' \quad \text{if } \log_p(x) = \log_p(x') \text{ and } |p, x| = |p, x'|.$$

Corollary 33. *The space $Q_r(p) = \overline{B_r(p)}/\approx$ is a* CAT(0) *disc with respect to the quotient metric.*

Proof. Let us count the irregular directions, $\mathcal{S}_r(p) = \{v_1, v_2, \ldots\}$. By Lemma 32 we know that the space Z_n which results from collapsing $\log_p^{-1}(v_i)$ for $i = 1, \ldots, n$ is a CAT(0) disc. The quotient maps $\pi_n : Z_n \to Q_r(p)$ are short and the diameter of inverse images of points $\pi_n^{-1}(q)$ go to zero uniformly. It follows that Z_n converges to $Q_r(p)$ with respect to Gromov–Hausdorff distance. Hence $Q_r(p)$ is CAT(0) disc retract, see [43, 4.1 Compactness lemma]. If $\bar{p}$ denotes the image of p under the quotient map $\pi : \overline{B_r(p)} \to Q_r(p)$, then $Q_r(p)$ is equal to the closed r-ball around $\bar{p}$. Hence $\partial Q_r(p)$ is a Jordan curve and $Q_r(p)$ is a CAT(0) disc. ■

Note that the restriction of the quotient map $\pi : \overline{B_r(p)} \to Q_r(p)$ to the regular star is area preserving. By the Bishop–Gromov Theorem (Theorem 24),

$$\mathcal{H}^2(\mathcal{R}_r(p)) = \mathcal{H}^2(Q_r(p)) \geq \frac{r^2}{2}\mathcal{H}^1(\Sigma_{\pi(p)}(Q_r(p)) = \frac{r^2}{2}\mathcal{H}^1(\Sigma_p Z).$$

Moreover, equality holds if and only if $Q_r(p)$ is isometric to a Euclidean cone of cone angle $\mathcal{H}^1(\Sigma_p Z)$; or informally, if $\mathcal{R}_r(p)$ is isometric to a Euclidean cone which is cut open along some radial geodesics.

Before we come to the criterion when a CAT(0) surface is flat away from a finite set of cone points, we provide an illustrating example.

Let $C = C_{4\pi}(o)$ be a flat cone of cone angle equal to 4π. Consider two points p^- and p^+ at distance 2 from each other and suppose that o is their midpoint. Set $P = \{p^-, p^+\}$. We are interested in the tubular r-neighborhoods of P for varying r. Clearly, if $r < 1$, then $N_r(P)$ is a disjoint union of the two flat discs $B_r(p^\pm)$. In particular, we have $\mathcal{H}^2(N_r(P)) = 2\pi r^2$. On the other hand, if $r > 1$, the $p^\pm$-radial geodesics branch at o. However, the regular stars $\mathcal{R}_r(p^\pm)$ are still disjoint and the equation $\mathcal{H}^2(N_r(P)) = 2\pi r^2$ continues to hold.

Lemma 34. *Let Z be a* CAT(0) *surface and let $P = \{p_1, \dots, p_k\} \subset Z$ be a finite subset. Let $R > 0$ be such that $|p_i, \partial Z| > R$ for all $p_i \in P$. Suppose that the regular stars $\mathcal{R}_r(p_i)$ are disjoint for all $r \leq R$ and such that*

$$\mathcal{H}^2(N_r(P)) = \frac{r^2}{2} \sum_{i=1}^{k} \mathcal{H}^1(\Sigma_{p_i} Z).$$

Then $N_R(P)$ is flat away from a finite set of cone points.

Proof. By Bishop–Gromov (Theorem 24), our assumptions guarantee that each quotient $Q_r(p_i)$ of $\bar{\mathcal{R}}_r(p_i)$ is flat away from p_i. In particular, if r is smaller than the distances between the different p_i, then $N_r(P)$ is the disjoint union of the flat cones $B_r(p_i)$. Also, there exist a definite time before p_i-radial geodesic start to branch. Let v be an irregular direction at p_i and let z be the first branch point in direction v. Since $\bigcup_{i=1}^k \mathcal{R}_r(p_i)$ has full measure in $N_r(P)$, there exists $j \neq i$ such that $z \in \bar{\mathcal{R}}_r(p_j)$. Since the total angle at z as seen from within $\mathcal{R}_r(p_i)$ is 2π and $\mathcal{R}_r(p_i) \cap \mathcal{R}_r(p_j) = \emptyset$, we conclude $\mathcal{H}^1(\Sigma_x Z) \geq 3\pi$. It follows that there are only finitely many irregular directions at p_i and $\bar{\mathcal{R}}_r(p_i)$ is isometric to a flat cone cut open along finitely many p_i-radial geodesics. By a similar argument we can conclude the proof. Note that a small ball around a boundary point of $\bar{\mathcal{R}}_r(p_i)$ is either isometric to a flat half disc or a flat disc cut open along a single radial geodesic. Hence if x is any point in $N_R(P) \setminus \{P\}$, then a small ball around x is either flat or isometric to a flat cone of cone angle at least 3π. This completes the proof. ∎

4. Minimal discs

4.1. Sobolev spaces

We will collect some basic definitions and properties from Sobolev space theory in metric spaces as developed in [22, 27, 33, 45]. For more details we refer the reader to these articles. We denote by Ω an open bounded Lipschitz domain in the Euclidean plane and fix a complete metric space X. Following Reshetnyak [46], we say that a map $u : \Omega \to X$

has *finite energy*, or lies in the Sobolev space $W^{1,2}(\Omega, X)$ if

- u is measurable and has essentially separable image.
- There exists a map $g \in L^2(\Omega)$ such that the composition $f \circ u$ with any short map $f : X \to \mathbb{R}$ lies in the classical Sobolev space $W^{1,2}(\Omega)$ and the norm of the weak gradient $|\nabla(f \circ u)|$ is almost everywhere bounded above by g.

Any Sobolev map u has a well-defined trace $\mathrm{tr}(u) \in L^2(\partial\bar{\Omega})$. (Cf. [27] and [33].) If u has a representative which extends to a continuous map $\bar{u}$ on $\bar{\Omega}$, then $\mathrm{tr}(u)$ is represented by $\bar{u}|_{\partial\bar{\Omega}}$. If the domain Ω is homeomorphic to the open unit disc D, then we call a map $u \in W^{1,2}(\Omega, X)$ a *Sobolev disc*.

We say that a circle $\gamma : \partial\bar{D} \to X$ *bounds* a Sobolev disc u if $\mathrm{tr}(u) = \gamma$ in $L^2(\partial\bar{D}, X)$.

4.1.1. Energy and the Dirichlet problem. Every Sobolev map $u \in W^{1,2}(\Omega, X)$ has an approximate metric differential at almost every point. More precisely, for almost every point $z \in \Omega$ there exists a unique seminorm on $\mathbb{R}^2$, denoted $|du_z(\cdot)|$ such that

$$\mathrm{aplim}_{w \to z} \frac{|u(w), u(z)| - |du_z(w - z)|}{|w - z|} = 0,$$

where aplim denotes the approximate limit, see [17].

Definition 35. The *Reshetnyak energy* of a Sobolev map $u \in W^{1,2}(\Omega, X)$ is given by

$$E(u) := \int_{\Omega} \max_{v \in S^1} |du_z(v)|^2 \, dz.$$

Theorem 36 (Dirichlet problem, [27]). *Let γ be a circle in a* CAT(0) *space X which can be spanned by a Sobolev disc. Then there exists a unique Sobolev disc u which minimizes the energy among all Sobolev discs spanning γ. The energy minimizer u is locally Lipschitz continuous in D and extends continuously to $\bar{D}$.*

Moreover, the local Lipschitz constant of u at a point z depends only on the total energy of u and the distance of z to the boundary $\partial\bar{D}$.

4.1.2. Area and the Plateau problem.

Definition 37. The *parametrized (Hausdorff) area* of a Sobolev map $u \in W^{1,2}(\Omega, X)$ is given by

$$\mathrm{area}(u) := \int_{\Omega} \mathcal{J}(du_z) \, dz,$$

where the Jacobian $\mathcal{J}(s)$ of a seminorm s on $\mathbb{R}^2$ is the Hausdorff 2-measure of the unit square with respect to s if s is a norm and $\mathcal{J}(s) = 0$ otherwise.

For a given Jordan curve Γ we denote by $\Lambda(\Gamma, X)$ the family of all Sobolev discs $u \in W^{1,2}(D, X)$ whose traces have representatives which are monotone parametrizations of Γ, cf. [33].

Definition 38 (Area-minimizer). Let Γ be a Jordan curve and $u \in \Lambda(\Gamma, X)$ a Sobolev map. The map u will be called *area minimizing* if it has the least area among all Sobolev competitors, i.e. if $\mathrm{area}(u) = \inf\{\mathrm{area}(u') : u' \in \Lambda(\Gamma, X)\}$.

The following theorem is a special case of [33, Theorem 1.4].

Theorem 39 (Plateau's problem). *Let X be a* CAT(0) *space and $\Gamma \subset X$ a rectifiable Jordan curve. Then there exists a Sobolev disc $u \in \Lambda(\Gamma, X)$ with*

$$E(u) = \inf\{E(u') : u' \in \Lambda(\Gamma, X)\}.$$

Moreover, every such u has the following properties.

(i) *u is an area minimizer.*

(ii) *u is a conformal map in the sense that there exists a* conformal factor $\lambda \in L^2(D)$ *with $|du_z| = \lambda(z) \cdot s_0$ almost everywhere in D, where s_0 denotes the Euclidean norm on $\mathbb{R}^2$.*

(iii) *u has a locally Lipschitz continuous representative which extends continuously to $\bar{D}$.*

Definition 40. Let X be a CAT(0) space and $\Gamma \subset X$ a rectifiable Jordan curve. A map $u \in \Lambda(\Gamma, X)$ as in Theorem 39 above is called a *minimal disc* or a *solution of the Plateau problem for* (Γ, X).

4.2. *Intrinsic minimal surfaces*

The following result is a consequence of [35, Theorem 1.2] and [45, Theorem 7.1.1]. The factorization and the fact that the intrinsic space is a CAT(0) space can also be deduced from [43, Theorem 1.1].

Theorem 41 (Intrinsic structure of minimal discs). *Let X be a* CAT(0) *space and $\Gamma \subset X$ a rectifiable Jordan curve. If $u : \bar{D} \to X$ is a minimal disc filling Γ, then the following holds. There exists a* CAT(0) *disc Z_u such that u factorizes as $u = \bar{u} \circ \pi$ with continuous maps $\pi : \bar{D} \to Z_u$ and $\bar{u} : Z_u \to X$. Moreover:*

(i) *π is locally Lipschitz.*

(ii) *π is monotone and restricts to an embedding on D.*

(iii) *$\pi \in \Lambda(\partial Z_u, Z_u) \subset W^{1,2}(D, Z_u)$.*

(iv) *$\bar{u}$ is short.*

(v) *$\bar{u}$ restricts to an arc length preserving homeomorphism $\partial Z_u \to \Gamma$.*

(vi) *$\bar{u}$ preserves the lengths of all rectifiable curves.*

(vii) *For any open subset $U \subset D$ it holds*

$$\mathcal{H}^2(\pi(U)) = \operatorname{area}(u|_U) = \operatorname{area}(\pi|_U).$$

Note that Corollary 13 implies

$$\operatorname{area}(\bar{v} \circ \pi) = \operatorname{area}(\bar{v})$$

for every Lipschitz map $\bar{v} : Z_u \to X$. As a consequence, the induced map $\bar{u}$ is area minimizing in the sense of the following definition.

Definition 42 (Intrinsic minimal surface). Let Z be a CAT(0) surface and X be a CAT(0) space. A map $f : Z \to X$ will be called *intrinsic minimal surface* or *intrinsic (area) minimizer* if

(i) f is short and proper,

(ii) f is area preserving in the sense that $\operatorname{area}(f|_U) = \mathcal{H}^2(U)$ for every open set $U \subset Z$,

(iii) for each closed disc Y embedded in Z the map $f|_Y$ has the least area among all Lipschitz competitors, i.e.

$$\operatorname{area}(f|_Y) = \inf\{\operatorname{area}(f') \mid f' : Y \to X \text{ Lipschitz with } f'|_{\partial Y} = f|_{\partial Y}\}.$$

If Z is homeomorphic to a plane or a closed disc, we call f *intrinsic minimal plane*, respectively *intrinsic minimal disc*.

4.2.1. Basic properties.

Lemma 43. *Let X be a* CAT(0) *space and $f : Z \to X$ an intrinsic minimal surface. Suppose that $\Gamma \subset Z$ is a rectifiable Jordan curve with Jordan domain Ω_Γ. Then the restriction $f_{\bar{\Omega}_\Gamma}$ is an intrinsic minimal disc, where $\bar{\Omega}_\Gamma$ is equipped with the induced intrinsic metric.*

Proof. By Lemma 28, $\bar{\Omega}_\Gamma$ is a CAT(0) disc. The other properties are immediate. ∎

Lemma 44 (Convex hull property). *Let $u : \bar{D} \to X$ be a minimal disc in a* CAT(0) *space X and $p \in X$ a point. If $u(\partial \bar{D}) \subset \bar{B}_p(r)$, then $u(\bar{D}) \subset \bar{B}_p(r)$.*

Proof. Since the nearest point projection $\pi : X \to \bar{B}_p(r)$ is short, the energy of $\pi \circ u$ is bounded above by the energy of u. Note that $\pi \circ u$ and u have the same boundary values. Because u is the unique energy minimizing filling with respect to its boundary, we conclude $\pi \circ u = u$. ∎

Lemma 45 (Maximum principle). *Let $u : \bar{D} \to X$ be a harmonic disc in a* CAT(0) *space X. Let $\varphi : X \to \mathbb{R}$ be a continuous convex function. Then the function $\varphi \circ u$ attains its maximum at the boundary. If φ is even Lipschitz continuous and* 1*-convex, then the maximum can only be attained if u is constant.*

Proof. By [19, Theorem 2 b)], $\varphi \circ u$ is subharmonic. Hence the maximum principle yields the first claim. For the second claim, we use the strong maximum principle to conclude that $\varphi \circ u$ is constant. The claim follows from [32, Corollary 1.6]. ∎

We will make use of the following elementary observation.

Lemma 46. *Let $\gamma : S^1 \to X$ be a Lipschitz circle and assume that $\gamma(p) = \gamma(q)$ for $p \neq q \in S^1$. Denote $S^\pm$ the two components of $S^1 \setminus \{p, q\}$ and let $\gamma^\pm : S^\pm/\partial S^\pm \to X$ be the induced loops. Suppose that $u^\pm : \bar{D} \to X$ are Lipschitz discs filling $\gamma^\pm$. Then there exists a Lipschitz disc $u : \bar{D} \to X$ which fills γ and such that*

$$\operatorname{area}(u) \leq \operatorname{area}(u^+) + \operatorname{area}(u^-).$$

Proof. Let $\pi : S^1 \to S^1/\{p = q\}$ be the quotient map. Glue two discs $D^{\pm}$ to $S^1/\{p = q\}$ such that the resulting space Y is a union of two discs which intersect in a single point. Denote by $\iota : S^1/\{p = q\} \to Y$ the canonical embedding. Note that the mapping cylinder Π of $\iota \circ \pi$ is homeomorphic to a disc. Write

$$\Pi = S^1 \times [0, 1]/(x, 1) \sim (\iota \circ \pi(x), 1) \cup D^+ \cup D^-.$$

Then we obtain a Lipschitz map v defined on Π by setting $v|_{D^{\pm}} = u^{\pm}$ and $v|_{S^1 \times \{t\}} = \gamma$. The area of v is equal to $\text{area}(u^+) + \text{area}(u^-)$. The desired map u is then given by precomposing v with the quotient map $\bar{D} \to \Pi$. ■

We record a special case using the same notation as above.

Corollary 47. *If the image of γ^- is a tree, then*

$$\text{area}(u) \leq \frac{\text{length}(\gamma^+)^2}{4\pi}.$$

Proof. Since the filling area of a tree is equal to zero, the claim follows from Lemma 46 and the isoperimetric inequality 9. ■

The following will be used repeatedly. It is a consequence of Lemma 17.

Corollary 48. *Let $f : Z \to X$ be an intrinsic minimal surface. If Z contains a closed convex subset W which is isometric to a Euclidean disc, then f restricts to an isometric embedding $W \to X$.*

Proof. We may assume that W is isometric to the closed unit disc. Let w be the center of W, i.e. $W = \bar{B}_1(w) \subset Z$. Let $c : S^1 \to Z$ be an arc length parametrization of ∂W. If the distance between $f \circ c$ and $f(w)$ would be less than one, then by Lemma 17,

$$\mathcal{H}^2(W) = \text{area}(f|_W) < \frac{1}{2}\,\text{length}(f \circ c) \leq \pi.$$

A cut and paste argument based on Lemma 17 would then show that f is not area minimizing. Hence $f \circ c$ is at constant distance one from $f(w)$ and f restricts to a radial isometry on W. Repeating the same argument for subdiscs of W with different centers shows that f is an isometric embedding. ■

Corollary 49. *Let X be a* CAT(0) *space and $f : C_\alpha \to X$ an intrinsic minimal plane where C_α is a Euclidean cone of cone angle $\alpha \geq 2\pi$. Then the following holds.*

(i) *f is a locally isometric embedding away from the tip o of C_α. In particular, f is a radial isometry with respect to o.*

(ii) *If $f(x) = f(y)$ for $x \neq y$ and $v_x, v_y \in \Sigma_o C_\alpha$ denote the directions at o pointing to x respectively y, then the intrinsic distance between v_x and v_y is at least 2π.*

(iii) *If $\alpha < 4\pi$, then f is injective. If even $\alpha = 2\pi$, then f is an isometric embedding.*

Proof. Claim (i) is immediate from Corollary 48. If f would not be injective, then $\Sigma_{f(o)}X$ would contain a geodesic loop of length $\leq \frac{\alpha}{2}$. Since $\Sigma_{f(o)}X$ is CAT(1), we conclude claim (ii) and the first part of (iii). The supplement in the third claim follows directly from Corollary 48. ■

Remark 50. In the case $\alpha < 4\pi$ above, f does not have to be an isometric embedding, as can be seen in a product of two ideal tripods.

Lemma 51. *Let Γ be a Jordan curve in X. Let $f : Z \to X$ be an intrinsic minimal disc filling Γ. Then for $p \in Z$,*

$$|p, \partial Z|_Z \le \frac{\text{length}(\Gamma)}{2\pi}.$$

Furthermore, equality holds if and only if Z is a flat disc and f is an isometric embedding.

Proof. Set $R = |p, \partial Z|_Z$. Then $\text{length}(\Gamma) \ge \mathcal{H}^1(\partial \bar{B}_R(p))$ since the nearest point projection onto $\bar{B}_R(p)$ is short. By Bishop–Gromov (Theorem 24), we have

$$\mathcal{H}^1(\partial \bar{B}_R(p)) \ge 2\pi R.$$

This proves the inequality. The case of equality follows from the rigidity statement in Bishop–Gromov together with Corollary 48. ∎

4.2.2. Minimal vs. intrinsic minimal.

Lemma 52. *Let X be a* CAT(0) *space and Z_i, $i = 1, 2$,* CAT(0) *discs with rectifiable boundaries. Let $c_i : S^1 \to \partial Z_i$ be L_i-Lipschitz parametrizations and suppose that $f_i : Z_i \to X$ are L-Lipschitz maps. Then there exists a constant $C = C(L, L_1, L_2)$ such that the following holds. If the compositions $f_i \circ c_i$ are uniformly ϵ-close to each other, then there exists a Lipschitz map $\tilde{f_1} : Z_1 \to X$ with $\partial \tilde{f_1} = \partial f_1$ and*

$$\text{area}(\tilde{f_1}) < \text{area}(f_2) + C \cdot \epsilon.$$

Proof. By Lemma 29, we can choose parametrized Jordan polygons $\sigma_i : S^1 \to Z_i \setminus \partial Z_i$ which have positive angles and are uniformly ϵ-close to c_i. Moreover, the Lipschitz constant of σ_i is bounded above by $(L_i + \epsilon)$. We denote the associated Jordan domains by Ω_i. By Lemma 28, $\bar{\Omega}_i$ is intrinsically a CAT(0) disc and there is a bilipschitz map $\varphi_i : \bar{D} \to Z_i$. In order to obtain $\tilde{f_1}$ we will cut and paste $f_1|_{\bar{\Omega}_1}$.

By Lemma 18, there are Lipschitz homotopies h_i of zero area between $\partial(f_i \circ \varphi_i)$ and $f_i \circ \sigma_i$. By our assumptions, the properties of σ_i and the triangle inequality we conclude $\sup_{t \in S^1} |f_1 \circ \sigma_1, f_2 \circ \sigma_2| < (2L + 1) \cdot \epsilon$. Hence Lemma 19 gives a Lipschitz homotopy h between $f_1 \circ \sigma_1$ and $f_2 \circ \sigma_2$ with

$$\text{area}(h) \le C \cdot L \cdot \max_{i=1,2} L_i \cdot \epsilon.$$

Now we define a Lipschitz map $\Phi : \bar{D} \to X$ with $\partial \Phi = \partial(f_1 \circ \varphi_1)$ as follows. We partition $\bar{D}$ into a central disc and three concentric annuli. Then we use h_1 on the outmost annulus, then h, then h_2 and on the central disc we use $f_2 \circ \varphi_2$. In particular,

$$\text{area}(\Phi) \le \text{area}(f_2) + C \cdot L \cdot \max_{i=1,2} L_i \cdot \epsilon.$$

Again by Lemma 19

$$\text{area}(f_1|_{Z_1 \setminus \Omega_1}) \le L^2 \cdot \mathcal{H}^2(Z_1 \setminus \Omega_1) \le L^2 \cdot C \cdot L_1 \cdot \epsilon.$$

Now we cut $f_1|_{\bar{\Omega}_1}$ and use φ_1 to paste Φ. This defines $\tilde{f}_1$. Combining the estimates above gives the necessary area bound for $\tilde{f}_1$:

$$\operatorname{area}(\tilde{f}_1) \leq L^2 \cdot C \cdot L_1 \cdot \epsilon + \operatorname{area}(f_2) + C \cdot L \cdot \max_{i=1,2} L_i \cdot \epsilon. \qquad \blacksquare$$

The following is a special case of [36, Theorem 1.2]. It can also be deduced from [44, Theorem 1.2] and Corollary 13.

Lemma 53. *Let Z be a* CAT(0) *disc and $\Gamma \subset Z$ a rectifiable Jordan curve with Jordan domain Ω_Γ. Suppose that $u : \bar{D} \to Z$ is a minimal disc filling Γ. Then* $\operatorname{im}(u) = \bar{\Omega}_\Gamma$, $\operatorname{area}(u) = \mathcal{H}^2(\Omega_\Gamma)$ *and for any Lipschitz map $f : Z \to Y$ to a metric space Y it holds* $\operatorname{area}(f \circ u) = \operatorname{area}(f|_{\Omega_\Gamma})$.

By Theorem 41, every minimal disc yields an intrinsic minimal disc. The following proposition provides a converse.

Proposition 54. *Let X be a* CAT(0) *space and $f : Z \to X$ an intrinsic minimal surface. Suppose that $\Gamma \subset Z$ is a rectifiable Jordan curve. Let $u : D \to Z$ be a minimal disc filling Γ. Then $f \circ u$ is conformal and harmonic. If in addition f restricts to an embedding on Γ, then $f \circ u$ is a solution to the Plateau problem for $(f(\Gamma), X)$.*

Proof. Let us settle the claim on conformality first. For almost every $x \in D$ we have $|du_x| = \lambda \cdot s_0$ where s_0 denotes the Euclidean norm; $df_{u(x)}$ is a linear isometric embedding; u is differentiable at x with a linear differential and the chain rule holds [29]. Hence $f \circ u$ is conformal.

We will show that $f \circ u$ is harmonic. Choose a small $\rho > 0$ and set $u_\rho := u|_{(1-\rho)\cdot\bar{D}}$. Let $v_\rho \in W^{1,2}((1-\rho) \cdot D, X)$ be a solution to the Dirichlet problem with the property that $\operatorname{tr}(v_\rho) = \operatorname{tr}(f \circ u_\rho)$. By Theorem 36, v_ρ extends continuously to $(1-\rho) \cdot \bar{D}$. The extension will still be called v_ρ. In particular, $\partial v_\rho = \partial(f \circ u_\rho)$. Then we have

$$\operatorname{area}(v_\rho) \leq E(v_\rho) \leq E(f \circ u_\rho) = \operatorname{area}(f \circ u_\rho) = \operatorname{area}(f|_{\operatorname{im}(u_\rho)}).$$

Now u_ρ is a Lipschitz embedding. Therefore $Z_\rho := \overline{\operatorname{im}(u_\rho)}$ is intrinsically a CAT(0) disc (Lemma 28) and $f_\rho := f|_{Z_\rho}$ is an intrinsic minimal disc (Lemma 43).

By [37, Proposition 3.1], for every $\epsilon > 0$ there exists a Lipschitz disc

$$\tilde{v}_\rho : (1-\rho) \cdot \bar{D} \to X$$

with $\partial\tilde{v}_\rho = \partial v_\rho = f \circ \partial u_\rho$ and $\operatorname{area}(\tilde{v}_\rho) \leq \operatorname{area}(v_\rho) + \epsilon$. Since ∂u_ρ is a Lipschitz parametrization of ∂Z_ρ, Lemma 52 implies $\operatorname{area}(f_\rho) \leq \operatorname{area}(v_\rho)$. Hence $E(v_\rho) = E(f \circ u_\rho)$ and by uniqueness $v_\rho = f \circ u_\rho$. As $\rho > 0$ was arbitrary, we conclude that $f \circ u$ is harmonic.

If $f|_\Gamma$ is an embedding, then we use Lemma 52 and argue as above to show that $f \circ u$ is area minimizing which completes the proof. $\blacksquare$

Corollary 55. *Let X be a* CAT(0) *space and let $f : Z \to X$ be intrinsic minimal surface. Let $\Gamma \subset Z$ be a rectifiable Jordan curve and denote by Ω_Γ its Jordan domain. Let $\varphi : X \to \mathbb{R}$ be a Lipschitz continuous* 1*-convex function. Then $\varphi \circ f|_{\bar{\Omega}_\Gamma}$ cannot attain its maximum in Ω_Γ.*

Proof. By Lemma 43, $f|_{\bar{\Omega}_\Gamma}$ is an intrinsic minimal disc. Therefore we may assume that Z is a disc and $\Gamma = \partial Z$. Let u solve the Plateau problem for (Γ, Z). By [36, Theorem 1.1], u is a homeomorphism. Hence if $\varphi \circ f$ attains a maximum in Ω_Γ, then $\varphi \circ f \circ u$ attains a maximum in D. By Proposition 54, $f \circ u$ is harmonic and hence our claim follows from the maximum principle (Lemma 45). ∎

Corollary 56. *Let X be a* CAT(0) *space and let $f : Z \to X$ be a intrinsic minimal surface. Let $p \in X$ be a point and set $f_p := |f(\cdot), p|$. Let $r > 0$ be a quasi regular value of f_p with $r < |p, f(\partial Z)|$. Suppose that $\Pi_r = N \cup \bigcup_{i=1}^{\infty} \Gamma_i$ is the corresponding decomposition of the fiber, cf. Proposition 15. Then the associated Jordan domains Ω_i are all disjoint.*

Proof. Assume that $\Omega_1 \subset \Omega_2$. Then $f_p|_{\bar{\Omega}_2}$ attains its maximal value r in Ω_2. The squared distance function $|\cdot, p|^2$ is locally Lipschitz continuous and 1-convex. Hence Corollary 55 implies that f is constant equal to p on Ω_2. Contradiction. ∎

4.3. *Limits of minimal discs*

Lemma 57. *Let (X_k, x_k) be a sequence of pointed* CAT(0) *spaces. Denote by (X_ω, x_ω) their ultralimit. Let (Z_k) be a sequence of* CAT(0) *discs which Gromov–Hausdorff converge to a* CAT(0) *disc Z. Assume that each Z_k is bilipschitz to $\bar{D}$ and that the boundary lengths $\mathcal{H}^1(\partial Z_k)$ are uniformly bounded. Suppose that $f : Z \to X_\omega$ is a Lipschitz map and set $\gamma := \partial f$. Furthermore, assume that for some $L > 0$ there are L-Lipschitz circles $\gamma_k : \partial Z_k \to X_k$ with ω-$\lim \gamma_k = \gamma$. Then, for every $\epsilon > 0$ there exist Lipschitz maps $f_k : Z_k \to X_k$ with $\partial f_k = \gamma_k$ and such that for ω-all k it holds*

$$\operatorname{area}(f_k) \leq \operatorname{area}(f) + \epsilon.$$

Proof. Let $\rho : S^1 \to \partial Z$ be a constant speed parametrization. Set $c := \gamma \circ \rho$. By assumption, we can find constant speed parametrizations $\rho_k : S^1 \to \partial Z_k$ such that $c_k := \gamma_k \circ \rho_k$ ω-converges to c. By Lemma 30, ρ extends to a monotone Lipschitz map $\mu : \bar{D} \to Z$. Put $v := f \circ \mu$. By Corollary 13, we have $\operatorname{area}(v) = \operatorname{area}(f)$. For given $\epsilon > 0$, [48, Theorem 5.1] provides a sequence (v_k) of Lipschitz maps $v_k : \bar{D} \to X_k$ filling c_k and such that $\operatorname{area}(v_k) \leq \operatorname{area}(v) + \epsilon$ holds for ω-all k. The statement follows since each Z_k is bilipschitz to $\bar{D}$. ∎

Remark 58. The condition on the Z_k is necessary because if ∂Z_k has a peak, then there might not be a single Lipschitz map $Z_k \to X_k$ filling c_k.

Proposition 59. *Let (X_k, x_k) be a sequence of pointed* CAT(0) *spaces and denote by (X_ω, x_ω) their ultralimit. Further, let (Z_k, z_k) be a sequence of* CAT(0) *discs, each bilipschitz to $\bar{D}$. Suppose that (Z_k) Gromov–Hausdorff converges to a* CAT(0) *disc Z and such that the boundary lengths $\mathcal{H}^1(\partial Z_k)$ are uniformly bounded. For each $k \in \mathbb{N}$ let $f_k : Z_k \to X_k$ be an intrinsic minimal disc with $f(z_k) = x_k$. Then*

$$f_\omega := \omega\text{-}\lim f_k : Z_\omega \to X_\omega$$

is an intrinsic minimal disc with $\operatorname{area}(f_\omega) = \omega\text{-}\lim \operatorname{area}(f_k)$.

Remark 60. If we remove the condition on Z_ω being a disc, then f_ω is still area minimizing and its domain is a CAT(0) disc retract, cf. [43].

Proof. Since all the f_k are short, we obtain a well-defined short limit map $f_\omega : Z_\omega \to X_\omega$. Since each f_k is area minimizing, we conclude from Lemma 57 $\operatorname{area}(\varphi) \geq \omega\text{-}\lim \operatorname{area}(f_k)$ for any Lipschitz map φ with $\partial\varphi = \partial f_\omega$. By our assumption, Z_ω is isometric to Z. From Lemma 25 we know $\mathcal{H}^2(Z) = \lim_{k\to\infty} \mathcal{H}^2(Z_k)$. Hence

$$\operatorname{area}(f_\omega) \leq \mathcal{H}^2(Z_\omega) = \lim_{k\to\infty} \mathcal{H}^2(Z_k) = \omega\text{-}\lim \operatorname{area}(f_k).$$

Therefore, equality holds and f_ω is an intrinsic minimal disc. ∎

4.4. Monotonicity

A key property of minimal surfaces in smooth spaces is the monotonicity of area ratios. The aim of this section is to prove monotonicity in a more general setting.

Lemma 61. *Let X be a* CAT(0) *space and let $f : Z \to X$ be an intrinsic minimal disc. Suppose that there is a point p in X and a radius $r > 0$ such that $f(\partial Z) \subset \partial \bar{B}_r(p)$. Then* $\operatorname{area}(f) \leq \frac{r}{2} \cdot \mathcal{H}^1(\partial Z)$.

Proof. Let $\epsilon > 0$. By Lemma 29, we find a parametrized Jordan polygon $\sigma : S^1 \to Z$ with positive angles which is uniformly close to an arc length parametrization of ∂Z and such that $\operatorname{length}(\sigma) \leq (1+\epsilon)\cdot\mathcal{H}^1(\partial Z)$ holds. Denote by Ω_σ the associated Jordan domain. By Lemma 19, we have

$$\operatorname{area}(f|_{Z\setminus\Omega_\sigma}) = \mathcal{H}^2(Z \setminus \Omega_\sigma) \leq C \cdot \mathcal{H}^1(\partial Z)\cdot\epsilon$$

with a uniform constant $C > 0$. By Lemma 28, $\bar{\Omega}_\sigma$ is intrinsically a CAT(0) disc which is bilipschitz to $\bar{D}$. Hence Lemma 17 implies

$$\operatorname{area}(f|_{\Omega_\sigma}) \leq \frac{r}{2}\cdot\operatorname{length}(\sigma) \leq \frac{(1+\epsilon)\cdot r}{2}\cdot\mathcal{H}^1(\partial Z).$$

The claim follows since $\epsilon > 0$ was arbitrary. ∎

Proposition 62 (Intrinsic monotonicity). *Let X be a* CAT(0) *space and Z a* CAT(0) *surface. Suppose that $f : Z \to X$ is an intrinsic minimal surface. Then for any point $p \in X$ the area density*

$$\Theta(f,p,r) := \frac{\mathcal{H}^2(f^{-1}(B_r(p)))}{\pi r^2}$$

is a nondecreasing function of r as long as $r < |p, f(\partial Z)|$.

Proof. We put $f_p(x) := |f(x), p|$ and define $\Omega_r := f_p^{-1}([0,r))$ and $\Pi_r := f_p^{-1}(r)$. Moreover, we set $A(r) := \mathcal{H}^2(\Omega_r)$ and $L(r) := \mathcal{H}^1(\Pi_r)$. Then, since f_p is short, the coarea formula (Lemma 14) yields $A(r) \geq \int_{\Omega_r} |\nabla f_p| = \int_0^r L(t)\,dt$. Therefore

$$A'(r) \geq L(r) \tag{1}$$

for almost all $r < |p, f(\partial Z)|$.

The desired monotonicity follows if we can show that

$$A(r) \leq \frac{r}{2} L(r) \tag{2}$$

holds almost everywhere. By Proposition 15, almost all r are quasi regular. By Corollary 56, all Jordan domains resulting from a decomposition of a quasi regular fiber are disjoint. Hence we may assume that Π_r is equal to a single rectifiable Jordan curve. By Lemma 43, $f|_{\bar{\Omega}_r}$ is an intrinsic minimal disc and therefore the required area estimate follows from Lemma 61. ∎

Corollary 63 (Monotonicity). *Let X be a* CAT(0) *space. Suppose that $u : \bar{D} \to X$ is a minimal disc and p is a point in $u(\bar{D}) \setminus u(\partial \bar{D})$. Then the area density*

$$\Theta(u, p, r) := \frac{\operatorname{area}(u(D) \cap B_r(p))}{\pi r^2}$$

is a nondecreasing function of r as long as $r < |p, u(\partial \bar{D})|$.

Proof. Factorize u as $\bar{u} \circ \pi$ as in Theorem 41. Then

$$\operatorname{area}(u(D) \cap B_r(p)) = \mathcal{H}^2(\bar{u}^{-1}(B_r(p))),$$

by Theorem 41 (iv). Since $\bar{u}$ is an intrinsic minimizer, Proposition 62 applies. ∎

4.5. *Densities and blow-ups*

The monotonicity of area densities justifies the following definition.

Definition 64 (Density). For an intrinsic area minimizer f and a point $p \in f(Z) \setminus f(\partial Z)$ we define the *density at p* by

$$\Theta(f, p) := \lim_{r \to 0} \Theta(f, p, r).$$

If Z is compact, then the density is finite. The function $p \mapsto \Theta(f, p)$ is upper semicontinuous by monotonicity (Proposition 62).

Lemma 65. *Let X be a* CAT(0) *space. If $f : Z \to X$ is an intrinsic area minimizer, and $p \in f(Z) \setminus f(\partial Z)$, then $\Theta(f, p) \geq \# f^{-1}(p)$.*

Proof. Let $\{x_1, \ldots, x_k\}$ be a finite subset of the inverse image of the point p under f. For $r < \frac{1}{2} \min\{|x_i, x_j| : 1 \leq i < j \leq k\}$, the balls $B_r(x_i)$ are disjoint and since f is short, we have

$$\operatorname{area}(f^{-1}(B_r(p))) \geq \sum_{i=1}^{k} \operatorname{area}(B_r(x_i)).$$

The claim follows from Bishop–Gromov (Theorem 24). ∎

Corollary 66. *Let X be a* CAT(0) *space and $f : Z \to X$ an intrinsic minimal surface of finite area,* $\operatorname{area}(f) < \infty$*. Then the fiber of each point $p \in f(Z) \setminus f(\partial Z)$ is finite.*

Proof. For $r < |p, f(\partial Z)|$ it holds $\Theta(f, p, r) \leq \frac{\operatorname{area}(f)}{\pi r^2}$. Hence the claim follows from monotonicity and Lemma 65. ∎

Definition 67. Let ω be a nonprincipal ultrafilter on $\mathbb{N}$. Let X be a CAT(0) space and let $f : Z \to X$ be an intrinsic minimal surface. Fix a point $z_0 \in Z$. For any $r > 0$ we define the rescaled maps $f_r : (r \cdot Z, z_0) \to (r \cdot X, x_0)$, where $x_0 = f(z_0)$. A *tangent map at* z_0 is an ultralimit of rescalings $f_{\frac{1}{\epsilon_i}}$ for some nullsequence (ϵ_i):

$$df_{z_0} : T_{z_0} Z \to (X_\omega, x_\omega).$$

Here X_ω denotes the pointed ultralimit $\omega\text{-lim}(\frac{1}{\epsilon_i} \cdot X, x_0)$.

Lemma 68. *Let X be a* CAT(0) *space and $f : Z \to X$ an intrinsic minimal surface. Let $df_{z_0} : T_{z_0} Z \to X_\omega$ be a tangent map at a point z_0 in the interior of Z. Then df_{z_0} is an intrinsic minimal plane which is a locally isometric embedding away from the tip o_{z_0} of $T_{z_0} Z$. In particular, it is of constant area density*

$$\Theta(df_{z_0}, x_\omega, r) \equiv \frac{\mathcal{H}^1(\Sigma_{z_0} Z)}{2\pi}.$$

Moreover, if $x_0 \in f(Z) \setminus f(\partial Z)$ and $f^{-1}(x_0) = \{z_0, \dots, z_k\}$ is a finite fiber, then

$$\sum_{i=0}^{k} \Theta(df_{z_i}, x_\omega) = \Theta(f, x_0).$$

Proof. Since z_0 lies in the interior of Z, the tangent cone $T_{z_0} Z$ is isometric to a Euclidean cone C_α with $\alpha = \mathcal{H}^1(\Sigma_{z_0} Z) \geq 2\pi$. For each $r > 0$ and i large enough, $f|_{B_{\epsilon_i r}(z_0)}$ is an intrinsic minimal disc in X and $\bar{B}_{\epsilon_i r}$ is bilipschitz to $\bar{D}$. By Proposition 59, we conclude that $df_{z_0}|_{B_r(o_{z_0})}$ is an intrinsic minimal disc. Hence df_{z_0} is an intrinsic minimal plane. By Corollary 48, df_{z_0} is a locally isometric embedding away from o_{z_0}. In particular, it is radially isometric and therefore has constant area density

$$\Theta(df_{z_0}, x_\omega, r) \equiv \frac{\mathcal{H}^1(\Sigma_{z_0} Z)}{2\pi}.$$

Now assume that the fiber of x_0 is finite, $f^{-1}(x_0) = \{z_0, \dots, z_k\}$. (By Corollary 66, this is automatic if Z is compact.) For simplicity we assume that $k = 0$ so that z_0 is the only inverse image of x_0. The proof for $k > 0$ is identical. We know that

$$\Theta(df_{z_0}, x_\omega) \leq \Theta(f, x_0)$$

since f is short.

To see the converse inequality, let $r > 0$ and choose a sequence $\epsilon_i \to 0$. Now define $r_i > 0$ to be the smallest radius such that $f^{-1}(\bar{B}_{\epsilon_i r}(x_0)) \subset \bar{B}_{r_i}(z_0)$. In particular,

$$\mathcal{H}^2(f^{-1}(\bar{B}_{\epsilon_i r}(x_0))) \leq \mathcal{H}^2(\bar{B}_{r_i}(z_0)).$$

Since df_{z_0} is a radial isometry, we obtain that $\omega\text{-lim} \frac{r_i}{\epsilon_i} = r$. Since a subsequence of the $\frac{1}{\epsilon_i} \cdot \bar{B}_{r_i}(z_0)$ converges Gromov–Hausdorff to $\bar{B}_r(o_{z_0})$, we get from Lemma 25 that

$$\lim_{i \to \infty} \mathcal{H}^2\left(\frac{1}{\epsilon_i} \cdot \bar{B}_{r_i}(z_0)\right) = \mathcal{H}^2(\bar{B}_r(o_{z_0})) = \mathcal{H}^1(\Sigma_{z_0} Z) \cdot \frac{r^2}{2}.$$

This shows $\Theta(f, x_0) \leq \Theta(df_{z_0}, x_\omega)$ and completes the proof. ∎

From Lemma 68 above and [30, Proposition 1.1], we can conclude that an intrinsic minimal surface $f : Z \to X$ is a locally bilipschitz embedding on an open dense set of Z. However, our situation is more special and we actually get:

Theorem 69. *Let X be a* CAT(0) *space and $f : Z \to X$ an intrinsic minimal surface. If z_0 is a point in the interior of Z with $\mathcal{H}^1(\Sigma_{z_0} Z) < 4\pi$, then f restricts to a bilipschitz embedding on a neighborhood of z_0. In particular, if Z is a* CAT(0) *disc, then f is locally a bilipschitz embedding in the interior of Z away from finitely many points.*

Proof. Let z_0 be a point in the interior of Z with $\mathcal{H}^1(\Sigma_{z_0} Z) < 4\pi$. Assume that the claim is false. Then we can find sequences (x_k) and (y_k) in Z with $x_k \neq y_k$, and such that $x_k, y_k \in B_{\frac{1}{k}}(z_0)$ and

$$|f(x_k), f(y_k)| \leq \frac{1}{k}|x_k, y_k|.$$

We set $\epsilon_k := |x_k, y_k|$. We consider the rescaled maps

$$f_{\frac{1}{\epsilon_k}} : \left(\frac{1}{\epsilon_k} \cdot Z, x_k\right) \to \left(\frac{1}{\epsilon_k} \cdot X, f(x_k)\right)$$

and build the blow up

$$f_\omega : \omega\text{-lim}\left(\frac{1}{\epsilon_k} \cdot Z, x_k\right) \to \omega\text{-lim}\left(\frac{1}{\epsilon_k} \cdot X, f(x_k)\right).$$

By Lemma 26, we know that $\omega\text{-lim}(\frac{1}{\epsilon_k} Z, x_k)$ is isometric to a Euclidean cone C_α with cone angle $\alpha \leq \mathcal{H}^1(\Sigma_{z_0} Z)$. (Note that the tip of C_α might be different from x_ω.) As in the proof of Lemma 68, we conclude from Proposition 59 that df_{x_ω} is an intrinsic minimal plane. Corollary 49 implies that df_{x_ω} is injective. But by construction we have $f_\omega(x_\omega) = f_\omega(y_\omega)$ and $|x_\omega, y_\omega| = 1$. This contradiction completes the proof. ∎

Combining Theorem 69 with Theorem 41, we obtain our main structure result:

Theorem 70. *Let X be a* CAT(0) *space and $\Gamma \subset X$ a rectifiable Jordan curve. Let $u : D \to X$ be a minimal disc filling Γ. Then there exists a finite set $B \subset D$ such that u is a local embedding on $D \setminus B$.*

Corollary 71. *Let X be a* CAT(0) *space and $\Gamma \subset X$ a rectifiable Jordan curve. Suppose that $u : D \to X$ is a minimal disc filling Γ. Denote by Y the image of u. Then there is a finite set P in Y such that on $Y \setminus P$ the intrinsic and extrinsic metrics are locally bilipschitz equivalent.*

4.6. *Rigidity*

We will make use of the following auxiliary lemma which gives a lower bound on the size of links in domains of intrinsic minimal surfaces.

Lemma 72. *Let $f : Z \to X$ be an intrinsic area minimizer. Let $x_1 \neq x_2$ and y be points in the interior of Z with $f(x_i) = p$, $i = 1, 2$, and $f(y) = q$. Assume that f maps the geodesics $x_i y$ isometrically onto the geodesic pq. If v_i denotes the direction at y pointing to x_i, then $|v_1, v_2| \geq 2\pi$.*

Proof. Recall that $T_y Z$ is a Euclidean cone of cone angle $\alpha \geq 2\pi$. By Lemma 68, each tangent map df_y is an intrinsic minimal plane. The claim follows from Corollary 49. ■

Theorem 73 (Rigidity). *Let X be a* CAT(0) *space. Let $f : Z \to X$ be an intrinsic minimal disc and let p be a point in $f(Z) \setminus f(\partial Z)$. Assume that there exist $\Theta > 0$ and a radius $R > 0$ with $R < |p, f(\partial Z)|$ such that the area ratio with respect to p is constant,*

$$A(r) \equiv \frac{\Theta}{2} r^2 \quad \text{for all } r \leq R.$$

Then $\Omega_R := f^{-1}(B_R(p))$ is flat away from a finite set of cone points $z_1, \ldots, z_k$. Moreover, f is a locally isometric embedding on $\Omega_R \setminus \{z_1, \ldots, z_k\}$ and $f(\Omega_R)$ is a union of p-radial geodesics.

Proof. The supplement follows from Corollary 48, so it is enough to show that Ω_R is flat away form finitely many points. By Corollary 66, we know that f has finite fibers. Let $P := \{x_1, \ldots, x_n\}$ be the finitely many inverse images of p. We claim that it is enough to show that the regular stars $\mathcal{R}_R(x_i)$, $i = 1, \ldots, n$, are disjoint. Indeed, if this holds, then we get

$$\Theta = \frac{\mathcal{H}^2(\Omega_R)}{\pi R^2} \geq \sum_{i=1}^{n} \frac{\mathcal{H}^2(\mathcal{R}_R(x_i))}{\pi R^2} \geq \sum_{i=1}^{n} \frac{\mathcal{H}^1(\Sigma_{x_i} Z)}{2\pi} = \Theta.$$

The last equality follows from Lemma 68. As $\bigcup_{i=1}^{n} \mathcal{R}_R(x_i) \subset N_R(P) \subset \Omega_R$, we obtain

$$\mathcal{H}^2(N_R(P)) = \frac{R^2}{2} \sum_{i=1}^{n} \mathcal{H}^1(\Sigma_{x_i} Z)$$

and Lemma 34 applies.

To see that the regular stars are disjoint, we let m_{ij} denote the midpoint of x_i and x_j. Further, we will denote by v_i the direction at m_{ij} pointing at x_i. If we can show that $|v_i, v_j| \geq 2\pi$, then clearly the regular stars have to be disjoint. Suppose this is not the case for m_{12}. Moreover, we may assume that it holds for all (i, j) with $|x_i, x_j| < |x_1, x_2|$. Set $r := \frac{|x_1, x_2|}{2}$. Then the $\mathcal{R}_r(x_i)$ are disjoint and f restricts to a radial isometry on each of them. It follows that f maps the geodesics $x_1 m_{12}$ and $x_2 m_{12}$ isometrically to the geodesic $p f(m_{12})$. Hence Lemma 72 shows $|v_1, v_2| \geq 2\pi$. Contradiction. ■

Remark 74. Notably, the proof shows that the link at any midpoint m_{ij} as above has length at least 4π.

4.7. Extending minimal discs to planes

Recall that a map between metric spaces is called *(metrically) proper* if inverse images of bounded sets are bounded.

Let X be a CAT(0) space and $\hat{f} : \hat{Z} \to \hat{X}$ a proper intrinsic minimal plane. Then we know that the monotonicity of area densities holds for all times. More precisely, for all points $p \in \hat{f}(Z)$ the function $r \mapsto \Theta(\hat{f}, p, r)$ is nondecreasing for all $r > 0$.

Definition 75 (Area-growth). Let $\hat{f} : \hat{Z} \to \hat{X}$ be a proper intrinsic minimal plane. Then we define the *density at infinity* or *area growth* of $\hat{f}$ by

$$\Theta^\infty(\hat{f}) := \lim_{r\to\infty} \Theta(\hat{f}, p, r).$$

We say that $\hat{f}$ is of *quadratic area growth*, if $\Theta^\infty(\hat{f}) \in (0, \infty)$.

Combining the monotonicity of area densities with Lemma 65, we obtain:

Lemma 76 (Key estimate). *Let $\hat{f} : \hat{Z} \to \hat{X}$ be a proper intrinsic minimal plane in an arbitrary* CAT(0) *space $\hat{X}$. Then for every point p in the image of $\hat{f}$ we have*

$$\#\hat{f}^{-1}(p) \leq \Theta^\infty(\hat{f}).$$

In particular, if $\hat{f}$ is of quadratic area growth, then it has finite fibers.

Corollary 77. *Let $\hat{X}$ be a* CAT(0) *space and $\hat{f} : \hat{Z} \to \hat{X}$ a proper intrinsic minimal plane. Suppose that the area growth of $\hat{f}$ satisfies $\Theta^\infty(\hat{f}) < 2$. Then $\hat{f}$ is an embedding. If the area growth is even Euclidean, $\Theta^\infty(\hat{f}) = 1$, then Z is isometric to the flat Euclidean plane and $\hat{f}$ is an isometric embedding.*

Proof. The first claim is immediate from Lemma 76. We turn to the second claim. Since $\hat{f}$ is short, the area growth of Z is bounded above by $\Theta^\infty(\hat{f}) = 1$. It follows from the Bishop–Gromov Theorem (Theorem 24) that Z is isometric to the flat Euclidean plane. Then $\hat{f}$ is an isometric embedding by Corollary 48. ∎

Our goal now is to extend a minimal disc to a minimal plane such that the area growth of the minimal plane is controlled by the total curvature of the boundary of the minimal disc. If we can do this, then we can argue as above to control the mapping behavior of the minimal disc.

So let Γ be a Jordan curve of finite total curvature κ in a CAT(0) space X. Denote by $\hat{X} := X \cup_\Gamma E_\kappa$ the CAT(0) space obtained from the funnel construction, see Section 3.1. Let $f : Z \to X$ be an intrinsic minimal disc spanning Γ and such that f restricts to a homeomorphism $\partial Z \to \Gamma$. Then we can glue the space $\hat{Z} := Z \cup_{\partial Z} E_\kappa$ via f. Using the identity map on E_κ, we obtain a natural extension

$$\hat{f} : \hat{Z} \to \hat{X}.$$

Clearly, this is a proper, area preserving short map.

Lemma 78. *The map $\hat{f}$ is area minimizing. More precisely, if Y is an embedded disc in $\hat{Z}$, then $\hat{f}|_Y$ minimizes the area among all Lipschitz maps $h : Y \to \hat{X}$ with $\partial h = \partial \hat{f}|_Y$.*

Proof. Denote by d_X the distance function to $X \subset \hat{X}$. Let $r_0 > 0$. Set $A_0 := d_X^{-1}((0, r_0])$ and $Y_0 := \hat{f}^{-1}(A_0) \cup Z$. It is enough to show that $\hat{f}|_{Y_0}$ is area minimizing for all large r_0. Note that

$$\operatorname{area}(\hat{f}|_{Y_0}) = \mathcal{H}^2(A_0) + \operatorname{area}(f).$$

We claim that there is a Lipschitz continuous monotone map $\bar{D} \to Y_0$.

Recall that E_κ is the complement of an open convex neighborhood V of the tip in a Euclidean cone C_κ of cone angle κ. Hence there is a short retraction $\pi : \bar{N}_{r_0}(V) \to \bar{V}$. Let σ be an arclength parametrization of $\partial N_{r_0}(V)$. Denote by $A_{2,1}$ the closed annulus in $\mathbb{R}^2$ centered in 0 and of radii 2 and 1. We obtain a Lipschitz continuous monotone map $v_1 : A_{2,1} \to \bar{N}_{r_0}(V) \setminus V$ by sending radial geodesic with constant speed to the unique geodesic between $\sigma(t)$ and $\pi \circ \sigma(t)$ in C_κ. By Lemma 30, there is a Lipschitz continuous monotone map $v_2 : \bar{D} \to Z$ extending $\pi \circ \sigma$. Concatenating v_0 and v_1 provides the required map.

By the claim, it is enough to show that a solution u to the Plateau problem of $(\Gamma_0, \hat{X})$ has at least the area of $\hat{f}|_{Y_0}$. For topological reasons, any continuous disc filling Γ_0 has to contain A_0 in its image. Therefore, it is enough to show that the part of u which maps to X has at least the area of f. Let $\epsilon > 0$ be a small quasi regular value of $d_X \circ u$. By Proposition 15, the corresponding fiber Π_ϵ decomposes as $\Pi_\epsilon = N_\epsilon \cup \bigcup_{k=1}^{\infty} \Gamma_k$, where N_ϵ has $\mathcal{H}^1$-measure zero and each $\Gamma_k \subset D$ is a rectifiable Jordan curve. Because u is minimizing, each $u_*[\Gamma_i]$ has to be nontrivial in the first homology group with integer coefficients, $H_1(E_\kappa)$. Since u is locally Lipschitz continuous, the decomposition can only contain a finite number of Jordan curves, say $\Gamma_1, \ldots, \Gamma_n$. Denote by Ω_i the Jordan domain associated to Γ_i. By Corollary 56, the different Ω_i are all disjoint. Then the sum $\sum_{i=1}^{n} u_*[\Gamma_i]$ is equal to $u_*[\partial \bar{D}]$ in $H_1(E_\kappa)$. Choose Lipschitz maps $v_i : \bar{D} \to \bar{D}$ which extend arc length parametrizations of Γ_i and such that $\operatorname{area}(u \circ v_i) \leq \operatorname{area}(u|_{\Omega_i}) + \frac{\epsilon}{n}$. Since each $u \circ v_i|_{\partial \bar{D}}$ maps onto $\Gamma_\epsilon := d_X^{-1}(\epsilon)$, we can construct a Lipschitz map $v : \bar{D} \to \hat{X}$ which fills Γ_ϵ and such that $\operatorname{area}(v) \leq \sum_{i=1}^{n} \operatorname{area}(u|_{\Omega_i}) + \epsilon$ and $v|_{\partial \bar{D}}$ represents a generator of $H_1(E_\kappa)$. By Lemma 18, we can adjust the boundary parametrization to obtain a new Lipschitz disc $\tilde{v} : \bar{D} \to \hat{X}$ with $\operatorname{area}(\tilde{v}) = \operatorname{area}(v)$ and such that $\tilde{v}|_{\partial \bar{D}}$ is uniformly ϵ-close to an arc length parametrization of Γ. By the minimality of f and Lemma 19, we obtain $\operatorname{area}(v) \geq \operatorname{area}(f) - C \cdot \epsilon$ with a uniform constant $C > 0$. We conclude

$$\operatorname{area}(u) \geq \mathcal{H}^2(A_0) + \operatorname{area}(f) - (C + 1)\epsilon.$$

This holds for every small quasi regular value ϵ and therefore finishes the proof. ∎

Lemma 79. *The extended space $\hat{Z}$ is a* CAT(0) *plane.*

Proof. Note that $\hat{Z}$ is a geodesic space which is homeomorphic to the plane. Moreover, $\hat{Z}$ contains Z as a closed convex subset since the nearest point projection $E \to \partial E$ is short.

By [36, Corollary 1.5] it is enough to show that any Jordan domain D_c in $\hat{Z}$ bounded by a Jordan curve c satisfies

$$\mathcal{H}^2(D_c) \leq \frac{\operatorname{length}(c)^2}{4\pi}.$$

Let c be a rectifiable Jordan curve in $\hat{Z}$ and denote by D_c the associated Jordan domain. By Lemma 79 we know that $\hat{f}|_{D_c}$ is an area minimizing filling of $\hat{f} \circ c$ in $\hat{X}$. But since $\hat{X}$ is CAT(0), the Euclidean isoperimetric inequality holds and therefore

$$\operatorname{area}(\hat{f}|_{D_c}) \leq \frac{\operatorname{length}(\hat{f} \circ c)^2}{4\pi}.$$

Since $\hat{f}$ is short and preserves area, the claim follows. ∎

Hence, from Lemma 78 and Lemma 79 we obtain:

Proposition 80. *Let* Γ *be a Jordan curve of finite total curvature* κ *in a* CAT(0) *space* X. *Denote by* $\hat{X} := X \cup_\Gamma E_\kappa$ *the* CAT(0) *space obtained from the funnel construction. (See Section 3.1.) Let* $f : Z \to X$ *be an intrinsic minimal disc filling* Γ. *Then* $\hat{Z} := Z \cup_f E_\kappa$ *is a* CAT(0) *plane and the map* f *extends canonically to a proper intrinsic minimal plane* $\hat{f} : \hat{Z} \to \hat{X}$. *Moreover,* $\hat{f}$ *has area growth* $\Theta^\infty(\hat{f})$ *equal to* $\frac{\kappa}{2\pi}$.

This proposition allows us to relate the total curvature of the boundary curve to the multiplicity of points via our key estimate Lemma 76.

5. The Fáry–Milnor Theorem

In this last part we will apply the above results on minimal discs filling curves of finite total curvature in order to obtain the general version of the Fáry–Milnor Theorem.

Theorem 81 (Fáry–Milnor). *Let* Γ *be a Jordan curve in a* CAT(0) *space* X. *If* $\kappa(\Gamma) \leq 4\pi$, *then either* Γ *bounds an embedded disc, or* $\kappa(\Gamma) = 4\pi$ *and* Γ *bounds an intrinsically flat geodesic cone. More precisely, there is a map from a convex subset of a Euclidean cone of cone angle equal to* 4π *which is a local isometric embedding away from the cone point and which fills* Γ.

Proof. Let $u : D \to X$ be a solution to the Plateau problem for (Γ, X) provided by Theorem 39. Denote by $f := \bar{u} : Z \to X$ the induced intrinsic minimal disc. Then, by Theorem 41, the map $\partial f : \partial Z \to \Gamma$ is an arc length preserving homeomorphism. Next, let $\hat{X} := X \cup_\Gamma E_\kappa$ be the CAT(0) space obtained from the funnel construction, see Section 3.1. By Proposition 80 we can extend f to an intrinsic minimal plane $\hat{f} : \hat{Z} \to \hat{X}$ with area growth $\Theta^\infty(\hat{f})$ equal to $\frac{\kappa}{2\pi}$. Then, for any point p in the image of $\hat{f}$ our key estimate Lemma 76 reads

$$\#\hat{f}^{-1}(p) \leq \frac{\kappa}{2\pi}. \tag{3}$$

Now if $\kappa < 4\pi$ holds, then $\hat{f}$ is injective. Hence Γ bounds the embedded disc $f(Z)$.

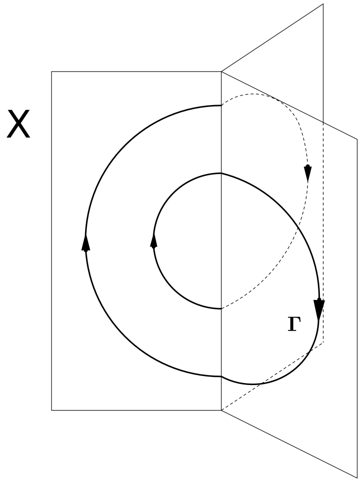
So we may assume that $\kappa = 4\pi$ and our intrinsic minimal disc f filling Γ is not embedded. Hence we find a point $p \in \mathrm{im}(\hat{f})$ where equality in (3) holds, i.e. which has exactly two inverse images x^+ and x^-. Moreover, by monotonicity (Proposition 62), we must have

$$A(r) = \frac{\Theta^\infty(\hat{f})}{2} r^2$$

for all $r > 0$.

Let m be the midpoint of x^+ and x^-. Set $r := \frac{|x^+, x^-|}{2}$. Then $\mathcal{H}^2(B_r(x^\pm)) = \pi r^2$ and $B_r(x^\pm) \subset \hat{Z}$ is a flat disc. Hence, by Corollary 48, the restriction $\hat{f}|_{B_r(x^\pm)}$ is an isometric embedding. Consequently, the geodesic x^+x^- is folded onto the geodesic $\hat{f}(m)p$. From Lemma 72 we conclude that $\mathcal{H}^1(\Sigma_m \hat{Z}) \geq 4\pi$. But the area growth of $\hat{Z}$ is equal to 2. Hence from Bishop–Gromov (Theorem 24) we conclude that $\hat{Z}$ is isometric to a Euclidean cone of cone angle 4π and the proof is complete. ∎

Example 1. Let X be the CAT(0) space which results from gluing two flat planes along a flat sector of angle $\alpha \leq \pi$. Then X contains a Jordan curve of total curvature equal to 4π but which does not bound an embedded disc. The picture shows an example in the case $\alpha = \pi$. Note that Γ, as shown in the picture, surrounds some points in X twice and therefore cannot bound an embedded disc.



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References

[1] Alberti, G., Bianchini, S., Crippa, G.: Structure of level sets and Sard-type properties of Lipschitz maps. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12**, 863–902 (2013) Zbl 1295.26016 MR 3184572

[2] Alexander, S. B., Bishop, R. L.: The Fary–Milnor theorem in Hadamard manifolds. Proc. Amer. Math. Soc. **126**, 3427–3436 (1998) Zbl 0911.57007 MR 1459103

[3] Alexander, S., Kapovitch, V., Petrunin, A.: Alexandrov geometry. http://anton-petrunin.github.io/book/all.pdf (2017)

[4] Alexandrow, A. D.: Über eine Verallgemeinerung der Riemannschen Geometrie. Schr. Forschungsinst. Math. **1**, 33–84 (1957) Zbl 0077.35702 MR 87119

[5] Aleksandrov, A. D., Zalgaller, V. A.: Intrinsic Geometry of Surfaces. Translated from the Russian by J. M. Danskin. Transl. Math. Monogr. 15, American Mathematical Society, Providence (1967) Zbl 0146.44103 MR 0216434

[6] Ambrosio, L., Kirchheim, B.: Currents in metric spaces. Acta Math. **185**, 1–80 (2000) Zbl 0984.49025 MR 1794185

[7] Ambrosio, L., Kirchheim, B.: Rectifiable sets in metric and Banach spaces. Math. Ann. **318**, 527–555 (2000) Zbl 0966.28002 MR 1800768

[8] Ballmann, W.: Lectures on spaces of nonpositive curvature. DMV Seminar 25, Birkhäuser, Basel (1995) Zbl 0834.53003 MR 1377265

[9] Ballmann, W.: *On the geometry of metric spaces.* Lecture notes, http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/sin40827.pdf (2004)

[10] Bridson, M. R., Haefliger, A.: Metric Spaces of Non-Positive Curvature. Grundlehren Math. Wiss. 319, Springer, Berlin (1999) Zbl 0988.53001 MR 1744486

[11] Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry. Grad.e Stud. Math. 33, American Mathematical Society, Providence (2001) Zbl 0981.51016 MR 1835418

[12] Burago, J. D.: On proportional approximation of a metric. Trudy Mat. Inst. Steklov. **76**, 120–123 (1965) Zbl 0142.20402 MR 0198387

[13] Burago, Y.: Bi-Lipschitz-equivalent Aleksandrov surfaces. II. Algebra i Analiz **16**, 28–52 (2004) Zbl 1092.53052 MR 2117448

[14] Choe, J., Gulliver, R.: Embedded minimal surfaces and total curvature of curves in a manifold. Math. Res. Lett. **10**, 343–362 (2003) Zbl 1055.53044 MR 1981908

[15] Daskalopoulos, G., Mese, C.: Monotonicity properties of harmonic maps into NPC spaces. J. Fixed Point Theory Appl. **11**, 225–243 (2012) Zbl 1273.58006 MR 3000669

[16] Ekholm, T., White, B., Wienholtz, D.: Embeddedness of minimal surfaces with total boundary curvature at most 4π. Ann. of Math. (2) **155**, 209–234 (2002) Zbl 1017.53013 MR 1888799

[17] Evans, L. C., Gariepy, R. F.: Measure Theory and Fine Properties of Functions. Stud. Adv. Math., CRC Press, Boca Raton (1992) Zbl 0804.28001 MR 1158660

[18] Fáry, I.: Sur la courbure totale d'une courbe gauche faisant un noeud. Bull. Soc. Math. France **77**, 128–138 (1949) Zbl 0037.23604 MR 33118

[19] Fuglede, B.: The Dirichlet problem for harmonic maps from Riemannian polyhedra to spaces of upper bounded curvature. Trans. Amer. Math. Soc. **357**, 757–792 (2005) Zbl 061.31008 MR 2095630

[20] Gromov, M., Schoen, R.: Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. Inst. Hautes Études Sci. Publ. Math. 165–246 (1992) Zbl 0896.58024 MR 1215595

[21] Guo, C.-Y., Wenger, S.: Area minimizing discs in locally non-compact metric spaces. Comm. Anal. Geom. **28**, 89–112 (2020) Zbl 07188062 MR 4077168

[22] Heinonen, J., Koskela, P., Shanmugalingam, N., Tyson, J. T.: Sobolev Spaces on Metric Measure Spaces. New Math. Monogr. 27, Cambridge University, Cambridge (2015) Zbl 1332.46001 MR 3363168

[23] Hruska, G. C., Kleiner, B.: Hadamard spaces with isolated flats. Geom. Topol. **9**, 1501–1538 (2005) Zbl 1087.20034 MR 2175151

[24] Jost, J.: Equilibrium maps between metric spaces. Calc. Var. Partial Differential Equations **2**, 173–204 (1994) Zbl 0798.58021 MR 1385525

[25] Kirchheim, B.: Rectifiable metric spaces: Local structure and regularity of the Hausdorff measure. Proc. Amer. Math. Soc. **121**, 113–123 (1994) Zbl 0806.28004 MR 1189747

[26] Kleiner, B., Leeb, B.: Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. Publ. Math. Inst. Hautes Études Sci. **86**, 115–197 (1997) Zbl 0910.53035 MR 1608566

[27] Korevaar, N. J., Schoen, R. M.: Sobolev spaces and harmonic maps for metric space targets. Comm. Anal. Geom. **1**, 561–659 (1993) Zbl 0862.58004 MR 1266480

[28] Kuwert, E.: Harmonic maps between flat surfaces with conical singularities. Math. Z. **221**, 421–436 (1996) Zbl 0871.58028 MR 1381590

[29] Lytchak, A.: Differentiation in metric spaces. Algebra i Analiz **16**, 128–161 (2004) Zbl 1099.53029 MR 2117451

[30] Lytchak, A.: Open map theorem for metric spaces. Algebra i Analiz **17**, 139–159 (2005) Zbl 1152.53033 MR 2167848

[31] Lytchak, A., Nagano, K.: Geodesically complete spaces with an upper curvature bound. Geom. Funct. Anal. **29**, 295–342 (2019) Zbl 1412.53058 MR 3925112

[32] Lytchak, A., Stadler, S.: Improvements of upper curvature bounds. Trans. Amer. Math. Soc. **373**, 7153–7166 (2020) Zbl 1453.53043 MR 4155203

[33] Lytchak, A., Wenger, S.: Area minimizing discs in metric spaces. Arch. Ration. Mech. Anal. **223**, 1123–1182 (2017) Zbl 1369.49057 MR 3594354

[34] Lytchak, A., Wenger, S.: Intrinsic structure of minimal discs in metric spaces. Geom. Topol. **22**, 591–644 (2018) Zbl 1378.49047 MR 3720351

[35] Lytchak, A., Wenger, S.: Isoperimetric characterization of upper curvature bounds. Acta Math. **221**, 159–202 (2018) Zbl 1412.53099 MR 3877021

[36] Lytchak, A., Wenger, S.: Canonical parameterizations of metric disks. Duke Math. J. **169**, 761–797 (2020) Zbl 1451.30118 MR 4073230

[37] Lytchak, A., Wenger, S., Young, R.: Dehn functions and Hölder extensions in asymptotic cones. J. Reine Angew. Math. **763**, 79–109 (2020) Zbl 1441.51005 MR 4104279

[38] Mese, C.: The curvature of minimal surfaces in singular spaces. Comm. Anal. Geom. **9**, 3–34 (2001) Zbl 0979.53068 MR 1807950

[39] Milnor, J. W.: On the total curvature of knots. Ann. of Math. (2) **52**, 248–257 (1950) Zbl 0037.38904 MR 37509

[40] Nagano, K.: A volume convergence theorem for Alexandrov spaces with curvature bounded above. Math. Z. **241**, 127–163 (2002) Zbl 1020.53026 MR 1930988

[41] Petrunin, A.: Metric minimizing surfaces. Electron. Res. Announc. Amer. Math. Soc. **5**, 47–54 (1999) Zbl 0916.53032 MR 1679453

[42] Petrunin, A.: Semiconcave functions in Alexandrov's geometry. In: Surveys in Differential Geometry. Vol. XI, Surv. Differ. Geom. 11, International Press, Somerville, 137–201 (2007) Zbl 1166.53001 MR 2408266

[43] Petrunin, A., Stadler, S.: Metric-minimizing surfaces revisited. Geom. Topol. **23**, 3111–3139 (2019) Zbl 1429.53060 MR 4039186

[44] Petrunin, A., Stadler, S.: Monotonicity of saddle maps. Geom. Dedicata **198**, 181–188 (2019) Zbl 1414.53066 MR 3933457

[45] Reshetnyak, Y. G.: Two-dimensional manifolds of bounded curvature. In: Geometry, IV, Encyclopaedia Math. Sci. 70, Springer, Berlin, 3–163 (1993) Zbl 0781.53050 MR 1263964

[46] Reshetnyak, Y. G.: Sobolev classes of functions with values in a metric space. II. Sibirsk. Mat. Zh. **45**, 855–870 (2004) Zbl 1085.46024 MR 2091651

[47] Tukia, P.: The planar Schönflies theorem for Lipschitz maps. Ann. Acad. Sci. Fenn. Ser. A I Math. **5**, 49–72 (1980) Zbl 0411.57015 MR 595177

[48] Wenger, S.: Spaces with almost Euclidean Dehn function. Math. Ann. **373**, 1177–1210 (2019) Zbl 1432.53061 MR 3953125