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Multiplicities of cohomological automorphic forms on GL_2 and mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

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Abstract. We prove a new upper bound for the dimension of the space of cohomological automorphic forms of fixed level and growing parallel weight on GL_2 over a number field which is not totally real, improving the one obtained in [Ann. of Math. (2) 175 (2012)]. The main tool of the proof is the mod p representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ as started by Barthel–Livné and Breuil, and developed by Paškūnas.

Keywords. Completed cohomology, mod p representations

Contents

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1. Introduction

Let F be a finite extension of $\mathbb{Q}$ of degree r, and let r_1 (resp. $2r_2$) be the number of real (resp. complex) embeddings. Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$, so that

$$\mathrm{GL}_2(F_\infty) \cong \mathrm{GL}_2(\mathbb{R})^{r_1} \times \mathrm{GL}_2(\mathbb{C})^{r_2}.$$

Let Z_∞ be the center of $\mathrm{GL}_2(F_\infty)$, let K_f be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ and let

$$X = \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})/K_f Z_\infty.$$

If $\mathbf{d} = (d_1, \ldots, d_{r_1+r_2})$ is an (r_1+r_2)-tuple of positive even integers, we let $S_{\mathbf{d}}(K_f)$ denote the space of cusp forms on X which are of cohomological type with weight $\mathbf{d}$.

In this paper, we are interested in understanding the asymptotic behavior of the dimension of $S_{\mathbf{d}}(K_f)$ when $\mathbf{d}$ varies and K_f is fixed. Define

$$\Delta(\mathbf{d}) = \prod_{1\le i\le r_1} d_i \times \prod_{r_1<i\le r_1+r_2} d_i^2.$$

When F is totally real and K_f is fixed, Shimizu [31] proved that[1]

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f) \sim C \cdot \Delta(\mathbf{d})$$

for some constant C independent of $\mathbf{d}$. However, if F is not totally real, the actual growth rate of $\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f)$ is still a mystery; see the discussion below when F is imaginary quadratic.

The main result of this paper is the following (see Theorem 6.1 for a slightly more general statement).

Theorem 1.1. *If F is not totally real and $\mathbf{d} = (d, \ldots, d)$ is a parallel weight with $d \ge 2$ even, then for any fixed K_f, we have*

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f) \ll_{\epsilon, K_f} d^{r-\frac{1}{2}+\epsilon}.$$

To compare our result with the previous ones, let us restrict to the case when F is imaginary quadratic. In [12], Finis, Grunewald and Tirao proved the bounds

$$d \ll \dim_{\mathbb{C}} S_{\mathbf{d}}(K_f) \ll \frac{d^2}{\ln d}, \quad \mathbf{d} = (d, d),$$

[1] Given $r \ge 1$ and two functions $f, g : \mathbb{N}^r \to \mathbb{N}$, we write $f \ll g$ for the usual notation $f = O(g)$, meaning that there exist $M, C > 0$ such that for all $\mathbf{d} \in \mathbb{N}^r$ with $\max\{d_i\} > M$, $f(\mathbf{d}) \le C g(\mathbf{d})$. We write $f \sim g$ if both $f \ll g$ and $g \ll f$ hold. In case the constants M, C depend on other inputs $*$, we write $f \sim_* g$ or $f \ll_* g$ to indicate this.

using base change and the trace formula respectively (the lower bound is conditional on K_f, see [12]). In [19], Marshall has improved the upper bound to be

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f) \ll_{\epsilon, K_f} d^{\frac{5}{3}+\epsilon} \tag{1.1}$$

while our Theorem 1.1 gives

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f) \ll_{\epsilon, K_f} d^{\frac{3}{2}+\epsilon},$$

hence a saving by a power $d^{\frac{1}{6}}$. It is worth to point out that such a power saving is quite rare for tempered automorphic forms. Indeed, purely analytic methods, such as the trace formula, only allow to strengthen the trivial bound by a power of $\ln d$, see [12,30]. We refer to the introduction of [19] for a discussion on this point and a collection of known results.

Finally, we mention that the experimental data of [12] (when F is imaginary quadratic) suggests that the actual growth rate of $\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f)$ is probably d. We hope to return to this problem in future work.

Let us first explain Marshall's proof of the bound (1.1). It consists of two main steps, the first of which is to convert the problem to bounding the dimension of certain group cohomology of Emerton's completed cohomology spaces H^j (with mod p coefficients) and the second one is to establish this bound. For the first step, he used the (generalized) Eichler–Shimura isomorphism, Shapiro's lemma and a fundamental spectral sequence due to Emerton. For the second, he actually proved a bound in a more general setting which applies typically to H^j. To make this precise, let us mention a key intermediate result in this step (stated in the simplest version). Let p be a prime number and define

$$K_1 = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}, \quad T_1(p^n) = \begin{pmatrix} 1+p\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}.$$

Let $Z_1 \cong 1+p\mathbb{Z}_p$ be the center of K_1. Also let $\mathbb{F}$ be a sufficiently large finite extension of $\mathbb{F}_p$. By a careful and involved analysis of the structure of finitely generated torsion modules over the Iwasawa algebra $\Lambda := \mathbb{F}[\![K_1/Z_1]\!]$, Marshall proved the following ([19, Proposition 5]): if Π is a smooth admissible $\mathbb{F}$-representation of K_1/Z_1 which is cotorsion[2], then for any $i \geq 0$,

$$\dim_{\mathbb{F}} H^i(T_1(p^n)/Z_1, \Pi) \ll p^{\frac{4n}{3}}. \tag{1.2}$$

Our proof of Theorem 1.1 follows closely the above strategy. Indeed, the first step is identical to Marshall's. Our main innovation is in the second step by improving the bound (1.2). The key observation is that Emerton's completed cohomology is not just a representation of K_1, but also a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, which largely narrows the possible shape of H^j. This fact was already observed in [19] and used *once*[3] when deriving (1.1) from (1.2). However, the mod p representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ developed by Barthel and Livné [2], Breuil [4] and Paškūnas [26,27], allows us to make the most of the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ and prove the following result (see Theorem 5.24).

[2]That is, the Pontryagin dual $\Pi^\vee := \mathrm{Hom}_{\mathbb{F}}(\Pi, \mathbb{F})$ is torsion as an $\mathbb{F}[\![K_1/Z_1]\!]$-module.

[3]We mean the trick of "change of groups", see Section 5.4.

Theorem 1.2. *Let Π be a smooth admissible $\mathbb{F}$-representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *on which Z_1 acts trivially. Assume that Π is cotorsion as a Λ-module. Then for any $i \geq 0$,*

$$\dim_{\mathbb{F}} H^i(T_1(p^n)/Z_1, \Pi) \ll np^n.$$

We obtain the above bound by using numerous results of the mod p representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$. First, the classification theorems of [2] and [4] allow us to control the dimension of invariants for irreducible π (i.e. when $i = 0$), in which case we prove

$$\dim_{\mathbb{F}} H^0(T_1(p^n)/Z_1, \pi) \ll n. \tag{1.3}$$

In fact, to do this we also need more refined structure theorems due to Morra [21, 22]. Second, the theory of Paškūnas [26] allows us to pass to general admissible cotorsion representations. To explain this, let us assume moreover that all the Jordan–Hölder factors of Π are isomorphic to a given supersingular irreducible representation π. In [26] Paškūnas studied the universal deformation of $\pi^\vee$ and showed that the universal deformation space (with mod p coefficients) is three-dimensional. We show that the admissibility and cotorsion conditions imposed on Π force that $\Pi^\vee$ is a deformation of $\pi^\vee$ over a one-dimensional space. Knowing this, the case $i = 0$ of Theorem 1.2 follows easily from (1.3).

To prove Theorem 1.2 for higher cohomology degrees and to generalize it to a finite product of $\mathrm{GL}_2(\mathbb{Q}_p)$ (which is essential for our application), we need to solve several complications caused by the additional requirement of carrying an action of $\mathrm{GL}_2(\mathbb{Q}_p)$. In [8, 19] the higher cohomology degree case is treated by the standard dimension-shifting argument, for which one needs to consider admissible representations Π which are not necessarily cotorsion, that is, the Pontryagin dual $\Pi^\vee$ has a positive rank over Λ. Using the bound in the torsion case, one is reduced to consider torsion-free $\Pi^\vee$. The usual argument (as in [19, Section 3.2]) uses the existence of morphisms $\Lambda^s \to \Pi^\vee$ and $\Pi^\vee \to \Lambda^s$ with torsion cokernels, where s is the Λ-rank of $\Pi^\vee$. However, these are only morphisms of Λ-modules, so the bound for torsion modules does not apply to these cokernels. This issue makes the cohomology of general torsion-free modules difficult to control. To solve this, we introduce a special class of (coadmissible) torsion modules, called *elementary* torsion modules, whose higher degree cohomologies are zero except in degree $i = 1$ and can be determined from its degree 0 cohomology, and we show that $\Pi^\vee$ has a resolution by elementary torsion modules. The proof uses a generalization of an important construction of Breuil and Paškūnas [6] for $\mathrm{GL}_2(\mathbb{Q}_p)$ to a finite product of $\mathrm{GL}_2(\mathbb{Q}_p)$, which we carry out in the appendix Section A.

Notation. Throughout the paper, we fix a prime p and a finite extension $\mathbb{F}$ over $\mathbb{F}_p$ taken to be sufficiently large.

2. Non-commutative Iwasawa algebras

Let G be a p-adic analytic group of dimension d and let G_0 be an open compact subgroup of G. We assume that G_0 is pro-p and uniform ([1, Section 2.4]). Let Λ be the *Iwasawa* *algebra* of G_0 over $\mathbb{F}$, namely

$$\Lambda := \mathbb{F}[\![G_0]\!] = \varprojlim \mathbb{F}[G_0/N],$$

where the inverse limit is taken over the open normal subgroups N of G_0. It is a Noetherian local integral domain ([1, Section 3.6]). A finitely generated (left) Λ-module is said to have *codimension* c if $\mathrm{Ext}^i_\Lambda(M,\Lambda) = 0$ for all $i < c$ and is non-zero for $i = c$; the codimension of the zero module is defined to be ∞. We denote the codimension by $j_\Lambda(M)$. If M is non-zero, then $j_\Lambda(M) \le d$. For our purpose, it is more convenient to use the *canonical dimension* of M defined by

$$\delta_\Lambda(M) := d - j_\Lambda(M).$$

If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated Λ-modules, then

$$\delta_\Lambda(M) = \max\{\delta_\Lambda(M'), \delta_\Lambda(M'')\}. \tag{2.1}$$

If M is a finitely generated Λ-module, we have the notion of *Gelfand–Kirillov* dimension of M, defined to be the growth rate of the function $\dim_{\mathbb{F}} M/J^nM$, where J denotes the maximal ideal of Λ. We have the following important fact ([1, Section 5.4]).

Theorem 2.1. *For all finitely generated Λ-modules M, the canonical dimension and the Gelfand–Kirillov dimension of M coincide.*

For $n \ge 0$, define inductively

$$G_{n+1} := \overline{G_n^p[G_n, G_0]}$$

which are normal subgroups of G_0; the decreasing chain $G_0 \supseteq G_1 \supseteq \cdots$ is called the *lower p-series* of G_0, see [1, Section 2.4]. As G_0 is uniform, we have $|G_n : G_{n+1}| = p^d$. With this notation, the utility of Theorem 2.1 is the following result (see [11, Proposition 2.17] and its proof).

Corollary 2.2. *Let M be a finitely generated Λ-module with $\delta_\Lambda(M) = c$. Then there are real numbers $a \ge b > 0$ such that*

$$bp^{cn} + O(p^{(c-1)n}) \le \dim_{\mathbb{F}} H_0(G_n, M) \le ap^{cn} + O(p^{(c-1)n}). \tag{2.2}$$

Moreover, we have a uniform lower bound $b \ge \frac{1}{c!}$.

Proposition 2.3. *Let M be a finitely generated Λ-module and let $\phi : M \to M$ be an endomorphism of Λ-modules. Assume that $\bigcap_{k\ge1} \phi^k(M) = 0$. Then one of the following holds:*

(i) *ϕ is nilpotent and $\delta_\Lambda(M) = \delta_\Lambda(M/\phi(M))$,*

(ii) *ϕ is not nilpotent and for $k_0 \gg 1$,*

$$\delta_\Lambda(M) = \max\{\delta_\Lambda(M/\phi(M)), \delta_\Lambda(\phi^{k_0}(M)/\phi^{k_0+1}(M)) + 1\}. \tag{2.3}$$

In any case, $\delta_\Lambda(M) \le \delta_\Lambda(M/\phi(M)) + 1$.

Remark 2.4. It would be more natural to impose the condition $\phi(M) \subset JM$. We consider the present one for the following reasons. On the one hand, in practice we do need to deal with ϕ such that $\bigcap_{k\geq 1} \phi^k(M) = 0$ but $\phi(M) \not\subseteq JM$. On the other hand, since M is finitely generated, M/JM is finite-dimensional over $\mathbb{F}$, hence the condition $\bigcap_{k\geq 1} \phi^k(M) = 0$ implies $\phi^k(M) \subset JM$ for $k \gg 1$.

Proof. We assume first that ϕ is nilpotent, say $\phi^{k_0} = 0$ for some $k_0 \geq 1$. Then M admits a finite filtration by $\phi^k(M)$ (for $0 \leq k \leq k_0$). Since each of the graded pieces is a quotient of $M/\phi(M)$, the assertion follows from (2.1).

Now assume that ϕ is not nilpotent, so by Lemma 2.5 below ϕ induces an injection $\phi^{k_0}(M) \to \phi^{k_0}(M)$ for some $k_0 \gg 1$. It is clear that the right-hand side of (2.3) does not depend on the choice of k_0. The above argument shows that

$$\delta_\Lambda(M/\phi^{k_0}(M)) = \delta_\Lambda(M/\phi(M)).$$

Hence, by (2.1) applied to the short exact sequence

$$0 \to \phi^{k_0}(M) \to M \to M/\phi^{k_0}(M) \to 0,$$

it suffices to show

$$\delta_\Lambda(\phi^{k_0}(M)) = \delta_\Lambda(\phi^{k_0}(M)/\phi^{k_0+1}(M)) + 1.$$

That is, by replacing M by $\phi^{k_0}(M)$, we may assume that ϕ is injective and need to show $\delta_\Lambda(M) = \delta_\Lambda(M/\phi(M)) + 1$. Under the assumption $\bigcap_{k\geq 1} \phi^k(M) = 0$, this follows from [13, Lemma A.15] using Remark 2.4. ∎

Lemma 2.5. *Let M be a finitely generated Λ-module and let $\phi : M \to M$ be an endomorphism of Λ-modules. Then one of the following holds:*

(i) *ϕ is nilpotent,*

(ii) *ϕ is not nilpotent and for $k_0 \gg 0$, ϕ induces an injection $\phi^{k_0}(M) \to \phi^{k_0}(M)$.*

Proof. Let $M[\phi^\infty] \subset M$ denote the submodule $\bigcup_{k\geq 1} \ker(\phi^k)$. Since Λ is Noetherian, it follows that $M[\phi^\infty]$ is finitely generated, so there exists $k_0 \gg 1$ such that

$$M[\phi^\infty] = M[\phi^{k_0}].$$

If $M = M[\phi^{k_0}]$, then ϕ is nilpotent; otherwise, ϕ is not nilpotent, and

$$\phi : \phi^{k_0}(M) \to \phi^{k_0}(M)$$

is injective. ∎

2.1. *Torsion vs torsion-free*

As recalled above, Λ is a Noetherian local integral domain. Let $\mathcal{L}$ be the skew field of fractions of Λ (see [1, Section 3.6]). If M is a finitely generated Λ-module, then $\mathcal{L} \otimes_\Lambda M$ is a finite-dimensional $\mathcal{L}$-vector space, and we define the rank of M to be the dimension of this vector space. It is clear that taking rank is additive in short exact sequences and that M has rank 0 if and only if M is torsion as a Λ-module.

Let $\mathcal{O} = W(\mathbb{F})$ denote the ring of Witt vectors with coefficients in $\mathbb{F}$. Similar to $\Lambda = \mathbb{F}[\![G_0]\!]$, we may form the Iwasawa algebras

$$\widetilde{\Lambda} := \mathcal{O}[\![G_0]\!] = \varprojlim_{N \lhd G_0 \text{ open}} \mathcal{O}[G_0/N], \quad \widetilde{\Lambda}_{\mathbb{Q}_p} = \widetilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

They are both integral domains. Let $\mathcal{L}_{\mathbb{Q}_p}$ be the skew field of fractions of $\widetilde{\Lambda}_{\mathbb{Q}_p}$. If M is a finitely generated module over $\widetilde{\Lambda}_{\mathbb{Q}_p}$, we define its rank as above and the analogous facts hold.

Recall the following simple fact, see [9, Lemma 1.17].

Lemma 2.6. *Let M be a finitely generated $\widetilde{\Lambda}$-module which is furthermore p-torsion free. Then $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a torsion $\widetilde{\Lambda}_{\mathbb{Q}_p}$-module if and only if $M \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is a torsion Λ-module.*

3. Mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

Notation. Let p be a prime with $p \geq 5$,[4] $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$, let Z be the center of G, let T be the diagonal torus, and let $B = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ be the upper Borel subgroup.

Let $\mathrm{Rep}_{\mathbb{F}}(G)$ denote the category of smooth $\mathbb{F}$-representations of G with a (fixed) central character, say $\zeta : Z \to \mathbb{F}^{\times}$. Let $\mathrm{Rep}^{\mathrm{l,fin}}_{\mathbb{F}}(G)$ denote the subcategory of $\mathrm{Rep}_{\mathbb{F}}(G)$ consisting of locally finite objects. Here an object $\Pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is said to be *locally finite* if for all $v \in \Pi$ the $\mathbb{F}[G]$-submodule generated by v is of finite length.

Let $\mathrm{Mod}^{\mathrm{pro}}_{\mathbb{F}}(G)$ be the category of compact left $\mathbb{F}[\![K]\!]$-modules with an action of $\mathbb{F}[G]$ such that the two actions coincide when restricted to $\mathbb{F}[K]$ and that Z acts via ζ^{-1}. It is anti-equivalent to $\mathrm{Rep}_{\mathbb{F}}(G)$ under Pontryagin dual $\Pi \mapsto \Pi^{\vee} := \mathrm{Hom}_{\mathbb{F}}(\Pi, \mathbb{F})$. Here, $\Pi^{\vee}$ is naturally a *right* $\mathbb{F}[G]$-module, but for convenience we view it as a *left* $\mathbb{F}[G]$-module using the canonical anti-automorphism $\mathbb{F}[G] \xrightarrow{\sim} \mathbb{F}[G]$ induced by $(g \mapsto g^{-1}) : G \to G$. Let $\mathfrak{C} = \mathfrak{C}(G)$ be the full subcategory of $\mathrm{Mod}^{\mathrm{pro}}_{\mathbb{F}}(G)$ anti-equivalent to $\mathrm{Rep}^{\mathrm{l,fin}}_{\mathbb{F}}(G)$.

An object $M \in \mathfrak{C}$ is called *coadmissible* if $M^{\vee}$ is admissible in the usual sense, i.e. $(M^{\vee})^H$ is finite-dimensional for any open subgroup H of G. This is equivalent to requiring M to be finitely generated over $\mathbb{F}[\![K]\!]$ (or equivalently, finitely generated over $\mathbb{F}[\![H]\!]$ for any open compact subgroup $H \subset K$).

If H is a closed subgroup of K, denote by $\mathrm{Rep}_{\mathbb{F}}(H)$ the category of smooth $\mathbb{F}$-representations of H on which the intersection $H \cap Z$ acts via the restriction of ζ. Let $\mathfrak{C}(H)$ be the dual category of $\mathrm{Rep}_{\mathbb{F}}(H)$ (note that $\mathrm{Rep}_{\mathbb{F}}(H)$ coincides with $\mathrm{Rep}^{\mathrm{l,fin}}_{\mathbb{F}}(H)$ as H is compact).

For $n \geq 1$, let

$$K_n = \begin{pmatrix} 1 + p^n\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1 + p^n\mathbb{Z}_p \end{pmatrix}.$$

[4] The assumption $p \geq 5$ is not always necessary, but for convenience we make this assumption throughout the paper.

Also let $Z_1 := K_1 \cap Z$. Since Z_1 is pro-p and ζ is smooth, the restriction of ζ to Z_1 is trivial, and any $\mathbb{F}$-representation of G (resp. K) with central character ζ can be viewed as a representation of G/Z_1 (resp. K/Z_1). Set

$$\Lambda := \mathbb{F}[\![K_1/Z_1]\!].$$

Since K_1/Z_1 is uniform (as $p > 2$) and pro-p, the results in Section 2 apply to Λ. Note that $\dim(K_1/Z_1) = 3$. To shorten the notation, we write $j(\cdot) = j_\Lambda(\cdot)$ and $\delta(\cdot) = \delta_\Lambda(\cdot)$.

If H is a closed subgroup of G and σ is a smooth representation of H, we denote by $\operatorname{Ind}_H^G \sigma$ the usual smooth induction. When H is moreover open, we let $\text{c-Ind}_H^G \sigma$ denote the compact induction, meaning the subspace of $\operatorname{Ind}_H^G \sigma$ consisting of functions whose support is compact modulo H.

Let $\omega : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ be the mod p cyclotomic character. If H is any group, we write $\mathbf{1}_H$ for the trivial representation of H (over $\mathbb{F}$).

3.1. Irreducible representations

The work of Barthel and Livné [2] shows that absolutely irreducible objects in $\operatorname{Rep}_{\mathbb{F}}(G)$ fall into four classes:

(1) one-dimensional representations $\chi \circ \det$, where $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ is a smooth character,

(2) (irreducible) principal series $\operatorname{Ind}_B^G \chi_1 \otimes \chi_2$ with $\chi_1 \neq \chi_2$,

(3) special series, i.e. twists of the Steinberg representation $\mathrm{Sp} := (\operatorname{Ind}_B^G \mathbf{1}_T)/\mathbf{1}_G$,

(4) supersingular representations, i.e. irreducible representations which are not isomorphic to subquotients of any parabolic induction.

For r with $0 \leq r \leq p-1$, let $\operatorname{Sym}^r \mathbb{F}^2$ denote the standard symmetric power representation of $\mathrm{GL}_2(\mathbb{F}_p)$. Up to a twist by $\det^m$ with $0 \leq m \leq p-2$, any absolutely irreducible $\mathbb{F}$-representation of $\mathrm{GL}_2(\mathbb{F}_p)$ is isomorphic to $\operatorname{Sym}^r \mathbb{F}^2$. Inflating to K and letting $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ act trivially, we may view $\operatorname{Sym}^r \mathbb{F}^2$ as a representation of KZ. Let

$$I(\operatorname{Sym}^r \mathbb{F}^2) := \text{c-Ind}_{KZ}^G \operatorname{Sym}^r \mathbb{F}^2$$

denote the compact induction to G. It is well known that $\operatorname{End}_G(I(\operatorname{Sym}^r \mathbb{F}^2))$ is isomorphic to $\mathbb{F}[T]$ for a certain Hecke operator T (see [2]). For $\lambda \in \mathbb{F}$ we define

$$\pi(r, \lambda) := I(\operatorname{Sym}^r \mathbb{F}^2)/(T - \lambda).$$

If $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ is a smooth character, then let $\pi(r, \lambda, \chi) := \pi(r, \lambda) \otimes \chi \circ \det$. In [2], Barthel and Livné showed that any supersingular representation of G is a *quotient* of $\pi(r, 0, \chi)$ for suitable (r, χ). Later on, Breuil proved that $\pi(r, 0, \chi)$ is itself irreducible (see [4]), hence completed the classification of irreducible objects in $\operatorname{Rep}_{\mathbb{F}}(G)$. We will refer to (r, λ, χ) as above as a *parameter triple*.

Recall the link between non-supersingular representations and compact inductions: if $\lambda \neq 0$ and $(r, \lambda) \neq (0, \pm 1)$, then

$$\pi(r, \lambda) \cong \operatorname{Ind}_B^G \mu_{\lambda^{-1}} \otimes \mu_\lambda \omega^r, \tag{3.1}$$

where $\mu_x : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ denotes the unramified character sending p to x. In case that $(r, \lambda) \in \{(0, \pm 1), (p-1, \pm 1)\}$, we have non-split exact sequences:

$$0 \to \mathrm{Sp} \otimes \mu_{\pm 1} \circ \det \to \pi(0, \pm 1) \to \mu_{\pm 1} \circ \det \to 0, \tag{3.2}$$

$$0 \to \mu_{\pm 1} \circ \det \to \pi(p-1, \pm 1) \to \mathrm{Sp} \otimes \mu_{\pm 1} \circ \det \to 0. \tag{3.3}$$

It is clear for non-supersingular representations and follows from [4] for supersingular representations that any absolutely irreducible $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is admissible. Therefore $\pi^\vee$ is coadmissible and it makes sense to talk about $\delta(\pi^\vee)$.

Theorem 3.1. *Let $\Pi \in \mathrm{Rep}_{\mathbb{F}}(G)$. If Π is of finite length, then Π is admissible and $\delta(\Pi^\vee) \leq 1$.*

Proof. The first assertion is clear. For the second, we may assume Π is absolutely irreducible. Corollary 2.2 allows us to translate the problem to computing the growth of $\dim_{\mathbb{F}} \Pi^{K_n}$. If Π is non-supersingular, then it is easy, see [22, Proposition 5.3] for a proof. If Π is supersingular, this is first done in [25, Theorem 1.2] and later in [22, Corollary 4.15]. ∎

Recall that a *block* in $\mathrm{Rep}_{\mathbb{F}}(G)$ is an equivalence class of absolutely irreducible objects in $\mathrm{Rep}_{\mathbb{F}}(G)$, where $\tau \sim \pi$ if and only if there exists a series of irreducible representations $\tau = \tau_0, \tau_1, \ldots, \tau_n = \pi$ such that $\mathrm{Ext}^1_G(\tau_i, \tau_{i+1}) \neq 0$ or $\mathrm{Ext}^1_G(\tau_{i+1}, \tau_i) \neq 0$ for each i.

Proposition 3.2. *The category $\mathrm{Rep}^{\mathrm{l,fin}}_{\mathbb{F}}(G)$ decomposes into a direct product of subcategories*

$$\mathrm{Rep}^{\mathrm{l,fin}}_{\mathbb{F}}(G) = \bigoplus_{\mathfrak{B}} \mathrm{Rep}^{\mathrm{l,fin}}_{\mathbb{F}}(G)^{\mathfrak{B}},$$

where the direct sum is taken over all the blocks $\mathfrak{B}$ and the objects of $\mathrm{Rep}^{\mathrm{l,fin}}_{\mathbb{F}}(G)^{\mathfrak{B}}$ are representations with all the irreducible subquotients lying in $\mathfrak{B}$. Correspondingly, we have a decomposition of categories $\mathfrak{C} = \prod_{\mathfrak{B}} \mathfrak{C}^{\mathfrak{B}}$, where $\mathfrak{C}^{\mathfrak{B}}$ denotes the dual category of $\mathrm{Rep}_{\mathbb{F}}(G)^{\mathfrak{B}}$.

Proof. See [26, Proposition 5.34]. ∎

The following theorem describes the blocks (when $p \geq 5$ as we are assuming).

Theorem 3.3. *Let $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be absolutely irreducible and let $\mathfrak{B}$ be the block in which π lies. Then one of the following holds:*

(I) *If π is supersingular, then $\mathfrak{B} = \{\pi\}$.*

(II) *If $\pi \cong \mathrm{Ind}_B^G \chi_1 \otimes \chi_2\omega^{-1}$ with $\chi_1\chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$, then*

$$\mathfrak{B} = \{\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\omega^{-1}, \mathrm{Ind}_B^G \chi_2 \otimes \chi_1\omega^{-1}\}.$$

(III) *If $\pi = \mathrm{Ind}_B^G \chi \otimes \chi\omega^{-1}$, then $\mathfrak{B} = \{\pi\}$.*

(IV) *Otherwise, $\mathfrak{B} = \{\chi \circ \det, \mathrm{Sp} \otimes \chi \circ \det, (\mathrm{Ind}_B^G \alpha) \otimes \chi \circ \det\}$, where $\alpha := \omega \otimes \omega^{-1}$.*

Proof. See [26, Proposition 5.42]. ∎

Convention. By [26, Lemma 5.10], any smooth irreducible $\overline{\mathbb{F}}_p$-representation of G with a central character is defined over a finite extension of $\mathbb{F}_p$. Theorem 3.3 then implies that for a given block $\mathfrak{B}$, there is a common finite field $\mathbb{F}$ such that irreducible objects in $\mathfrak{B}$ are absolutely irreducible. Hereafter, given a finite set of blocks, we take $\mathbb{F}$ to be sufficiently large such that irreducible objects in these blocks are absolutely irreducible.

3.2. Projective envelopes

Let $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be irreducible and $\mathfrak{B}$ the block in which π lies. Let $\mathrm{Inj}_G\,\pi$ be an injective envelope of π in $\mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G)$; the existence is guaranteed by [26, Corollary 2.3]. Let $P = P_{\pi^\vee} := (\mathrm{Inj}_G\,\pi)^\vee \in \mathfrak{C}$ and $E = E_{\pi^\vee} := \mathrm{End}_{\mathfrak{C}}(P)$. Then P is a projective envelope of $\pi^\vee$ in $\mathfrak{C}$ and is naturally a left E-module. Since P is indecomposable, Proposition 3.2 implies that (the dual of) every irreducible subquotient of P lies in $\mathfrak{B}$. Also, E is a local $\mathbb{F}$-algebra (with residue field $\mathbb{F}$). Paškūnas has computed E and showed in particular that E is commutative, except when $\mathfrak{B}$ is of type (III) listed in Theorem 3.3; in any case, we denote by $R = Z(E)$ the center of E.

Theorem 3.4 (Paškūnas). *Keep the above notation. The following statements hold.*

(i) *R is naturally isomorphic to the Bernstein center of $\mathfrak{C}^{\mathfrak{B}}$. In particular, R acts on any object in $\mathfrak{C}^{\mathfrak{B}}$ and any morphism in $\mathfrak{C}^{\mathfrak{B}}$ is R-equivariant.*

(ii) *If $\mathfrak{B}$ is not of type (IV), then R is a regular local $\mathbb{F}$-algebra of Krull dimension* 3. *If $\mathfrak{B}$ is of type (IV), then R is isomorphic to $\mathbb{F}[\![x,y,z,w]\!]/(xw-yz)$. In particular, R is Cohen–Macaulay of Krull dimension* 3.

(iii) *$E = R$ except for blocks of type (III) in which case E is a free R-module of rank* 4.

(iv) *If $\mathfrak{B}$ is not of type (IV), then P is flat over both E and R.*

Proof. (i) This is [26, Theorem 1.5].

(ii)–(iii) These are proved in [26]. Precisely, see [26, Proposition 6.3] for type (I), [26, Corollary 8.7] for type (II), [26, Section 9] for type (III) and [26, Corollary 10.78, Lemma 10.93] for type (IV).

(iv) The flatness of P over E follows from [26, Corollary 3.12], because the setting in [26, Corollary 3.12] is satisfied for blocks of type (I)–(III). The flatness of P over R for blocks of type (III) follows from this and (iii). ∎

Proposition 3.5. *The object $\mathbb{F} \otimes_E P$ (resp. $\mathbb{F} \otimes_R P$) has finite length in $\mathfrak{C}$ and*

$$\delta(\mathbb{F} \otimes_E P) = \delta(\mathbb{F} \otimes_R P) = 1.$$

Proof. Note that $\mathbb{F} \otimes_E P$ is characterized as the maximal quotient of P which contains $\pi^\vee$ with multiplicity one, see [26, Remark 1.13]. This object is denoted by Q in [26, Section 3] and can be described explicitly. If $\mathfrak{B}$ is of type (I) or (III), Q is just $\pi^\vee$. If $\mathfrak{B}$ is of type (II), it has length 2 by [27, Proposition 6.1, (34)]. If $\mathfrak{B}$ is of type (IV), the assertion follows from Proposition 3.26 below in Section 3.7 where the explicit structure of $\mathbb{F} \otimes_E P$ is determined. The assertion $\delta(\mathbb{F} \otimes_E P) = 1$ follows from the explicit description together with Theorem 3.1.

To see that $\mathbb{F} \otimes_R P$ has finite length, we may assume $\mathfrak{B}$ is of type (III), in which case E is a free R-module of rank 4. Since E is a local ring with residue field $\mathbb{F}$, any irreducible (right) E-module is isomorphic to $\mathbb{F}$, so $\mathbb{F} \otimes_R E$ has a filtration of finite length with graded pieces isomorphic to $\mathbb{F}$. The result then follows from the isomorphism $\mathbb{F} \otimes_R P \cong (\mathbb{F} \otimes_R E) \otimes_E P$; cf. the proof of [28, Corollary 4.2]. ∎

Remark 3.6. Note that Paškūnas gave a short proof of Proposition 3.5 without explicitly determining $\mathbb{F} \otimes_E P$, see [28, Lemma 5.8]. We keep this proof because knowing the explicit form might be of independent interest.

3.3. Serre weights

We keep the notation in the previous subsection. Let $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be irreducible. By a *Serre weight* of π we mean an isomorphism class of (absolutely) irreducible $\mathbb{F}$-representations of K, say σ, such that $\mathrm{Hom}_K(\sigma, \pi) \neq 0$. Denote by $\mathcal{D}(\pi)$ the set of Serre weights of π. The description of $\mathcal{D}(\pi)$ can be deduced from [2] and [4]; see [27, Remark 6.2] for a summary (of most cases).

Lemma 3.7. *If $\pi \neq \pi'$ are two objects in a block $\mathfrak{B}$, then $\mathcal{D}(\pi) \cap \mathcal{D}(\pi') = \emptyset$.*

Proof. The assertion is trivial if $\mathfrak{B}$ is of type (I) or (III). For type (II) or type (IV), it is a direct check (using the assumption $p \geq 5$), see [27, Remark 6.2]. ∎

As before, write $P = P_{\pi^\vee}$ and let $\mathfrak{B}$ be the block in which π lies.

Lemma 3.8. *If $\pi \notin \{\mathbf{1}_G, \mathrm{Sp}\}$ up to twist, then*

$$\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee) \neq 0 \iff \sigma \in \mathcal{D}(\pi).$$

If $\pi \in \{\mathbf{1}_G, \mathrm{Sp}\}$, then

$$\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee) \neq 0 \iff \sigma \in \{\mathrm{Sym}^0 \mathbb{F}^2, \mathrm{Sym}^{p-1} \mathbb{F}^2\}.$$

Proof. The assertion is trivial if $\mathfrak{B}$ is of type (I) or (III), because the block contains only one irreducible object. If $\mathfrak{B}$ is of type (II) or $\pi = \pi_\alpha$ up to twist, the assertion is proved in [27, Theorem 6.6].

Assume that $\pi = \mathbf{1}_G$. Since

$$\mathcal{D}(\mathbf{1}_G) = \{\mathrm{Sym}^0 \mathbb{F}^2\}, \quad \mathcal{D}(\mathrm{Sp}) = \{\mathrm{Sym}^{p-1} \mathbb{F}^2\}, \quad \mathcal{D}(\pi_\alpha) = \{\sigma'\},$$

where $\sigma' := \mathrm{Sym}^{p-3} \mathbb{F}^2 \otimes \det$ (see [27, Remark 6.2]), we only need to prove (after taking dual)

$$\mathrm{Hom}_K(\mathrm{Sym}^{p-1} \mathbb{F}^2, \mathrm{Inj}_G \mathbf{1}_G) \neq 0, \quad \mathrm{Hom}_K(\sigma', \mathrm{Inj}_G \mathbf{1}_G) = 0.$$

The first statement follows from (3.3); indeed, since the sequence (3.3) is non-split, there exists a G-equivariant embedding $\pi(p-1, 1) \hookrightarrow \mathrm{Inj}_G \mathbf{1}_G$ and we note that

$$\mathrm{Hom}_K(\mathrm{Sym}^{p-1} \mathbb{F}^2, \pi(p-1, 1)) \neq 0.$$

By Frobenius reciprocity, the second statement is equivalent to

$$\mathrm{Hom}_G(I(\sigma'), \mathrm{Inj}_G \mathbf{1}_G) = 0.$$

If ϕ is such a morphism, then since $\mathrm{Inj}_G\,\pi$ is locally finite, ϕ must factor through a certain quotient $I(\sigma')/f(T)$ for some non-zero polynomial $f(T) \in \mathbb{F}[T]$ (see [2, Theorem 19]). If ϕ were non-zero, then its image must contain π as its G-socle. But it follows from [2, Theorem 33 (2)] that $I(\sigma')/f(T)$ cannot have $\mathbf{1}_G$ as a subquotient. This contradiction allows to conclude. Finally, the case $\pi = \mathrm{Sp}$ can be treated similarly. ∎

Proposition 3.9. *The following statements hold.*

(i) *Let $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$ be irreducible. Whenever $\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee)^\vee$ is non-zero, it is a cyclic E-module and if J_σ denotes its annihilator then $E/J_\sigma \cong \mathbb{F}[\![S]\!]$ (where S is a formal variable). Moreover, if $\mathfrak{B}$ is not of type (I), then $J_\sigma \subset E$ is independent of σ.*

(ii) *Let $\widetilde{\sigma} = \bigoplus_\sigma \sigma$, where the sum is taken over all irreducible objects $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$ such that $\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee) \neq 0$. Then $\mathrm{Hom}_K^{\mathrm{cont}}(P, \widetilde{\sigma}^\vee)^\vee$ is a Cohen–Macaulay R-module of Krull dimension 1.*

Proof. (i) If $\mathfrak{B}$ is of type (I), the statement is proved in [27, Theorem 6.6, (38)]. Assume $\mathfrak{B}$ is not of type (I) and let (r, λ, χ) be a parameter triple of π. If $(r, \lambda) \neq (p-1, \pm 1)$, this is proved in [15, Proposition 2.9] via [15, Corollary 2.5], where an (unfortunate) assumption (**H**) is imposed. If $(r, \lambda) = (p-1, \pm 1)$, i.e. assumption (**H**) is not satisfied, the statements are still true and the proof can be adapted from the case of $(r, \lambda) = (0, \pm 1)$, see [15, Remark 2.6].

(ii) If $\mathfrak{B}$ is of type (I), it is a special case of [27, Lemma 2.33] via [27, Theorem 5.2]. If $\mathfrak{B}$ is of type (II) or (IV), then $E = R$ and the result is a weaker form of (i). If $\mathfrak{B}$ is of type (III), we need to show that the image of $R \hookrightarrow E \twoheadrightarrow \mathbb{F}[\![S]\!]$ contains a regular element, for which it is enough to show that the image contains a non-zero element in the maximal ideal of $\mathbb{F}[\![S]\!]$. But this is clear because $\mathbb{F}[\![S]\!]$ is finite over (the image of) R. Alternatively, one can apply Proposition 3.19 below where the image of R in $\mathbb{F}[\![S]\!]$ is determined. ∎

The following general result is extracted from the proof of [14, Theorem 3.5].

Proposition 3.10. *Let $\widetilde{P} \in \mathfrak{C}(K)$ and $f \in \mathrm{End}_{\mathfrak{C}(K)}(\widetilde{P})$. Assume that*

(i) *$\widetilde{P}$ is projective in $\mathfrak{C}(K)$,*

(ii) *for any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$, the induced morphism*

$$f_* : \mathrm{Hom}_K^{\mathrm{cont}}(\widetilde{P}, \sigma^\vee)^\vee \to \mathrm{Hom}_K^{\mathrm{cont}}(\widetilde{P}, \sigma^\vee)^\vee$$

is injective.

Then f is injective and $\widetilde{P}/f\widetilde{P}$ is projective in $\mathfrak{C}(K)$.

Proof. Consider the complex $P_\bullet$ of projective modules

$$0 \to P_1 \xrightarrow{f} P_0 \to 0,$$

where $P_0 = P_1 = \widetilde{P}$. Applying $\mathrm{Hom}_K^{\mathrm{cont}}(-, \sigma^\vee)$ to it, where $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$ is irreducible, we obtain a convergent spectral sequence

$$E_2^{ij} := \mathrm{Ext}^i_{K/Z_1}(H_j(P_\bullet), \sigma^\vee) \Rightarrow H^{i+j}(\mathrm{Hom}_K^{\mathrm{cont}}(P_\bullet, \sigma^\vee)),$$

which gives the following long exact sequence:

$$\begin{aligned}\mathrm{Ext}^1_{K/Z_1}(\mathrm{coker}(f),\sigma^\vee) \hookrightarrow H^1(\mathrm{Hom}^{\mathrm{cont}}_K(P_\bullet,\sigma^\vee)) \\ \to \mathrm{Hom}^{\mathrm{cont}}_K(\ker(f),\sigma^\vee) \to \mathrm{Ext}^2_{K/Z_1}(\mathrm{coker}(f),\sigma^\vee).\end{aligned}$$

The assumption (after taking dual) says that the morphism

$$\mathrm{Hom}^{\mathrm{cont}}_K(P_0,\sigma^\vee) \to \mathrm{Hom}^{\mathrm{cont}}_K(P_1,\sigma^\vee)$$

is surjective, i.e. $H^1(\mathrm{Hom}^{\mathrm{cont}}_K(P_\bullet,\sigma^\vee)) = 0$, which forces

$$\mathrm{Ext}^1_{K/Z_1}(\mathrm{coker}(f),\sigma^\vee)) = 0.$$

This being true for any irreducible object $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$, we deduce that $\mathrm{coker}(f)$ is projective in $\mathfrak{C}(K)$, proving the second assertion. As a consequence, $\mathrm{Ext}^2_{K/Z_1}(\mathrm{coker}(f),\sigma^\vee)$ vanishes, and so does $\mathrm{Hom}^{\mathrm{cont}}_K(\ker(f),\sigma^\vee)$. This being true for any σ, we deduce that $\ker(f) = 0$, i.e. f is injective. ∎

3.4. *Principal series and deformations*

Recall that T denotes the diagonal torus of G. If $\eta : T \to \mathbb{F}^\times$ is a smooth character, let $\pi_\eta := \mathrm{Ind}_B^G\,\eta$ which is possibly reducible. Let $\mathrm{Inj}_T\,\eta$ be an injective envelope of η in $\mathrm{Rep}_{\mathbb{F}}(T)$ (with central character) and set

$$\Pi_\eta := \mathrm{Ind}_B^G\,\mathrm{Inj}_T\,\eta.$$

Then Π_η is a locally finite smooth representation of G. It is easy to see that

$$\mathrm{soc}_G\,\Pi_\eta = \mathrm{soc}_G\,\pi_\eta,$$

which we denote by π. So there is a G-equivariant embedding $\Pi_\eta \hookrightarrow \mathrm{Inj}_G\,\pi$ and by [26, Proposition 7.1] the image does not depend on the choice of the embedding.

Proposition 3.11. *The representation Π_η is not admissible.*

Proof. Let (r,λ,χ) be a parameter triple such that $\pi_\eta \cong \pi(r,\lambda,\chi)$. This is always possible by [2, Theorem 30] and we have $(r,\lambda) \neq (0,\pm 1)$. Moreover, we may assume $\chi = \mathbf{1}$ up to twist. It suffices to prove that

$$\dim_{\mathbb{F}} \mathrm{Hom}_K(\mathrm{Sym}^r\,\mathbb{F}^2,\Pi_\eta) = +\infty, \tag{3.4}$$

which follows from [2, 3] as we explain below.

Recall that the Hecke algebra associated to $I(\mathrm{Sym}^r\,\mathbb{F}^2)$ is isomorphic to $\mathbb{F}[T]$. In [2, Section 6] is constructed an $\mathbb{F}[T,T^{-1}]$-linear morphism

$$P : I(\mathrm{Sym}^r\,\mathbb{F}^2) \otimes_{\mathbb{F}[T]} \mathbb{F}[T,T^{-1}] \to \mathrm{Ind}_B^G\,X_1 \otimes X_2,$$

where $X_i : \mathbb{Q}_p^\times \to (\mathbb{F}[T,T^{-1}])^\times$ are tamely ramified characters given by

$$X_1 \text{ unramified}, \quad X_1(p) = T^{-1}, \quad X_1X_2 = \omega^r.$$

Note that specializing to $T = \lambda$, we have $X_1 \equiv \mu_{\lambda^{-1}}$ so that

$$\mathrm{Ind}_B^G X_1 \otimes X_2 \pmod{T-\lambda} \cong \pi_\eta$$

by (3.1).

By [2, Theorem 25], P is an isomorphism if $r \neq 0$, hence induces an isomorphism for $n \geq 1$:

$$P_n : I(\mathrm{Sym}^r \mathbb{F}^2)/(T-\lambda)^n \cong \mathrm{Ind}_B^G X_1 \otimes X_2 \bmod (T-\lambda)^n.$$

If $r = 0$, then P is injective and we have an exact sequence ([3, Theorem 20])

$$0 \to I(\mathrm{Sym}^0 \mathbb{F}^2) \otimes_{\mathbb{F}[T]} \mathbb{F}[T, T^{-1}] \xrightarrow{P} \mathrm{Ind}_B^G X_1 \otimes X_2 \to \mathrm{Sp} \otimes \mathbb{F}[T, T^{-1}]/(T^{-2}-1) \to 0.$$

But since $\lambda \neq \pm 1$, $T-\lambda$ acts invertibly on the last term, hence P modulo $(T-\lambda)^n$ still induces an isomorphism for any $n \geq 1$:

$$P_n : I(\mathrm{Sym}^0 \mathbb{F}^2)/(T-\lambda)^n \cong \mathrm{Ind}_B^G X_1 \otimes X_2 \bmod (T-\lambda)^n.$$

As remarked above, the right-hand side is the parabolic induction of a deformation of η to $\mathbb{F}[T]/(T-\lambda)^n$, hence embeds in Π_η. This implies (3.4) as

$$\dim_{\mathbb{F}} \mathrm{Hom}_K(\mathrm{Sym}^r \mathbb{F}^2, I(\mathrm{Sym}^r \mathbb{F}^2)/(T-\lambda)^n) = n. \qquad \blacksquare$$

Remark 3.12. Keep the notation in the proof of Proposition 3.11. If $r = p-1$, then $\omega^{p-1} = \omega^0$, and the above proof shows that $\mathrm{Hom}_K(\mathrm{Sym}^0 \mathbb{F}^2, \Pi_\eta)$ is also infinite-dimensional.

Let $M_{\eta^\vee} := (\Pi_\eta)^\vee$ and $E_{\eta^\vee} = \mathrm{End}_{\mathfrak{C}}(M_{\eta^\vee})$.

Lemma 3.13. *The ring $E_{\eta^\vee}$ is isomorphic to $\mathbb{F}[\![x, y]\!]$ and $M_{\eta^\vee}$ is flat over $E_{\eta^\vee}$.*

Proof. By [26, Proposition 7.1], we have a natural isomorphism

$$E_{\eta^\vee} \cong \mathrm{End}_{\mathfrak{C}(T)}((\mathrm{Inj}_T\, \eta)^\vee)$$

and the latter ring is isomorphic to $\mathbb{F}[\![x, y]\!]$ by [26, Corollary 7.2].

By [26, Section 3.2], $(\mathrm{Inj}_T\, \eta)^\vee$ is isomorphic to the universal deformation of the T-representation $\eta^\vee$ (with fixed central character), with $E_{\eta^\vee}$ being the universal deformation ring. In particular, it is flat over $E_{\eta^\vee}$. The second assertion follows from this and the definition of $M_{\eta^\vee}$. $\blacksquare$

Recall that π denotes the G-socle of π_η. Let $P = P_{\pi^\vee}$.

Proposition 3.14. *Let $M \in \mathfrak{C}$ be a coadmissible quotient of $M_{\eta^\vee}$. Then $\delta(M) \leq 2$.*

Proof. Since M is coadmissible while $M_{\eta^\vee}$ is not by Proposition 3.11, the kernel of $M_{\eta^\vee} \twoheadrightarrow M$ is non-zero and not coadmissible; denote it by M'. We claim that

$$\mathrm{Hom}_{\mathfrak{C}}(M_{\eta^\vee}, M') \neq 0.$$

For this it suffices to prove $\mathrm{Hom}_{\mathfrak{C}}(P, M') \neq 0$, because any morphism $P \to M'$ must factor through $P \twoheadrightarrow M_{\eta^\vee} \to M'$, see [26, Proposition 7.1(iii)]. To this end, assume that $\mathrm{Hom}_{\mathfrak{C}}(P, M') = 0$ for a contradiction. Then $\pi^\vee$ does not occur as a subquotient in M'. This is impossible unless π_η is reducible, i.e. $\pi_\eta \cong \pi(p-1, 1)$ up to twist. Assuming it is the case, we have $\pi = \mathbf{1}_G$ and all irreducible subquotients of M' are isomorphic to $\mathrm{Sp}^\vee$, see (3.3). In particular, we obtain $\mathrm{Hom}_K^{\mathrm{cont}}(M', (\mathrm{Sym}^0 \mathbb{F}^2)^\vee) = 0$ (as the K-socle of Sp is isomorphic to $\mathrm{Sym}^{p-1} \mathbb{F}^2$). However, this would imply an isomorphism

$$\mathrm{Hom}_K^{\mathrm{cont}}(M_{\eta^\vee}, (\mathrm{Sym}^0 \mathbb{F}^2)^\vee) \cong \mathrm{Hom}_K^{\mathrm{cont}}(M, (\mathrm{Sym}^0 \mathbb{F}^2)^\vee).$$

Together with Remark 3.12 this contradicts the coadmissibility of M.

Via the embedding

$$\mathrm{Hom}_{\mathfrak{C}}(M_{\eta^\vee}, M') \hookrightarrow \mathrm{Hom}_{\mathfrak{C}}(M_{\eta^\vee}, M_{\eta^\vee}) = E_{\eta^\vee},$$

the claim implies the existence of a non-zero element $f \in E_{\eta^\vee}$ which annihilates M. Since $E_{\eta^\vee} \cong \mathbb{F}[\![x, y]\!]$ is a regular ring of dimension 2, we may find $g \in E_{\eta^\vee}$ such that f, g is a system of parameters of $E_{\eta^\vee}$. Then $E_{\eta^\vee}/(f, g)$ is finite-dimensional over $\mathbb{F}$, and consequently $M_{\eta^\vee}/(f, g)$ has finite length in $\mathfrak{C}$. Hence, $M/(f, g)M = M/gM$ also has finite length and Theorem 3.1 implies that $\delta(M/gM) \leq 1$. We then conclude by Proposition 2.3. ∎

The inclusion $\Pi_\eta \hookrightarrow \mathrm{Inj}_G\, \pi$ induces a surjection $P \twoheadrightarrow M_{\eta^\vee}$. By [26, Proposition 7.1] this induces a surjective ring morphism $E \twoheadrightarrow E_{\eta^\vee}$, via which $P \twoheadrightarrow M_{\eta^\vee}$ is a morphism of (left) E-modules.

Corollary 3.15. *For any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$, the natural surjective morphism*

$$\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee)^\vee \to \mathrm{Hom}_K^{\mathrm{cont}}(M_{\eta^\vee}, \sigma^\vee)^\vee \tag{3.5}$$

is an isomorphism of E-modules.

Proof. The quotient $P \twoheadrightarrow M_{\eta^\vee}$ is a morphism of E-modules, hence so is (3.5).

We may assume $\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee)$ is non-zero, so that it is isomorphic to $\mathbb{F}[\![S]\!]$ as an E-module by Proposition 3.9 (i). Hence, to prove the injectivity of the morphism (3.5), it suffices to prove that $\mathrm{Hom}_K^{\mathrm{cont}}(M_{\eta^\vee}, \sigma^\vee)$ is infinite-dimensional. This is already established in the proof of Proposition 3.11, together with Remark 3.12 and Lemma 3.8 if $\sigma \in \{\mathrm{Sym}^0 \mathbb{F}^2, \mathrm{Sym}^{p-1} \mathbb{F}^2\}$ up to twist. ∎

3.5. *Coadmissible quotients*

Keep the notation in the previous subsection. Let $M \in \mathfrak{C}$ be a coadmissible quotient of $P = P_{\pi^\vee}$. We set

$$\mathrm{m}(M) := \mathrm{Hom}_{\mathfrak{C}}(P, M)$$

which is a finitely generated right E-module. There is a natural morphism

$$\mathrm{ev} : \mathrm{m}(M) \otimes_E P \to M, \tag{3.6}$$

which is surjective by [26, Lemma 2.10]. Remark that we should have written $\mathrm{m}(M)\widehat{\otimes}_E P$ in (3.6), where $\widehat{\otimes}$ means taking completed tensor product. But since $\mathrm{m}(M)$ is finitely generated over E, the completed and usual tensor product coincide, see the discussion before [27, Lemma 2.1].

Let Ker be the kernel of (3.6). By [26, Lemma 2.9] we have

$$\mathrm{Hom}_{\mathfrak{C}}(P, \mathrm{m}(M) \otimes_E P) \cong \mathrm{m}(M),$$

so $\mathrm{Hom}_{\mathfrak{C}}(P, \mathrm{Ker}) = 0$ because P is projective in $\mathfrak{C}$. This implies that Ker does not admit $\pi^\vee$ as a subquotient, i.e. if π' is an irreducible subquotient of Ker, then $\pi' \in \mathfrak{B}$ and $\pi' \not\cong \pi$. In particular, if $\mathfrak{B}$ is of type (I) or (III) of Theorem 3.3, then $\mathrm{Ker} = 0$ and (3.6) is an isomorphism. In any case, we have the following fact.

Corollary 3.16. *If $\sigma \in \mathcal{D}(\pi)$, then (3.6) induces an isomorphism*

$$\mathrm{Hom}_K^{\mathrm{cont}}(\mathrm{m}(M) \otimes_E P, \sigma^\vee)^\vee \cong \mathrm{Hom}_K^{\mathrm{cont}}(M, \sigma^\vee)^\vee.$$

Proof. Using Lemma 3.7, the above argument shows that $\mathrm{Hom}_K^{\mathrm{cont}}(\mathrm{Ker}, \sigma^\vee)^\vee = 0$ for any $\sigma \in \mathcal{D}(\pi)$, giving the result. ■

Proposition 3.17. *Let $M \in \mathfrak{C}$ be a coadmissible quotient of $P = P_{\pi^\vee}$. The following statements hold.*

(i) *$\mathrm{m}(M) \otimes_E P$ is coadmissible.*

(ii) *If M is torsion as a Λ-module, then so is $\mathrm{m}(M) \otimes_E P$.*

Proof. Since $\mathrm{Ker} = 0$ if $\mathfrak{B}$ is of type (I) or (III), both the assertions are trivial in these cases. Thus, we assume that $\mathfrak{B}$ is of type (II) or (IV) in the rest.

(i) The coadmissibility of M is equivalent to that $\mathrm{Hom}_K^{\mathrm{cont}}(M, \sigma^\vee)$ is finite-dimensional over $\mathbb{F}$ for any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$, and we need to check this property for $\mathrm{m}(M) \otimes_E P$. By [27, Proposition 2.4][5] we have a natural isomorphism of finitely generated E-modules

$$\mathrm{Hom}_K^{\mathrm{cont}}(\mathrm{m}(M) \otimes_E P, \sigma^\vee)^\vee \cong \mathrm{m}(M) \otimes_E \mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee)^\vee, \tag{3.7}$$

hence we only need to consider those (irreducible) σ such that $\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee)^\vee \neq 0$. Choose any weight $\sigma' \in \mathcal{D}(\pi)$, which clearly implies $\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma'^\vee) \neq 0$. Then using (3.7), Proposition 3.9 (i) and Corollary 3.16, we obtain isomorphisms

$$\mathrm{Hom}_K^{\mathrm{cont}}(\mathrm{m}(M) \otimes_E P, \sigma^\vee)^\vee \cong \mathrm{Hom}_K^{\mathrm{cont}}(\mathrm{m}(M) \otimes_E P, \sigma'^\vee)^\vee \cong \mathrm{Hom}_K^{\mathrm{cont}}(M, \sigma'^\vee)^\vee.$$

Since M is coadmissible by assumption, all these spaces are finite-dimensional over $\mathbb{F}$.

(ii) It is equivalent to show that Ker is a torsion Λ-module (it is coadmissible as a consequence of (i)). Since the case of type (II) is similar and simpler, we assume in

[5] This result is stated for a commutative subring of E rather than for E itself which can be non-commutative, but the same proof goes through without change.

the rest that $\mathfrak{B}$ is of type (IV), so that $\mathfrak{B}$ consists of three irreducible objects and we let π_1, π_2 be the two other than π. Since Ker is coadmissible by (i) and does not admit $\pi^\vee$ as a subquotient, we can find $s_1, s_2 \geq 0$ and a surjection

$$P_{\pi_1^\vee}^{\oplus s_1} \oplus P_{\pi_2^\vee}^{\oplus s_2} \twoheadrightarrow \mathrm{Ker}.$$

Let Q_1 (resp. Q_2) be the maximal quotient of $P_{\pi_1^\vee}$ (resp. $P_{\pi_2^\vee}$) none of whose irreducible subquotients is isomorphic to $\pi^\vee$. Then the above surjection must factor through the map $Q_1^{\oplus s_1} \oplus Q_2^{\oplus s_2} \twoheadrightarrow \mathrm{Ker}$. Hence, it is enough to show that any coadmissible quotient of Q_1 (resp. Q_2) is torsion. This follows from the results in [26, Section 10] as we explain below. Up to twist we may assume $\mathfrak{B} = \{\mathbf{1}_G, \mathrm{Sp}, \pi_\alpha\}$.

Let us first assume $\pi = \pi_\alpha$, so that up to order $\pi_1 = \mathbf{1}_G$ and $\pi_2 = \mathrm{Sp}$. We have the following exact sequences:

$$0 \to P_{\pi_\alpha^\vee} \to P_{\mathbf{1}_G^\vee} \to M_{\mathbf{1}_T^\vee} \to 0$$

and

$$P_{\pi_\alpha^\vee}^{\oplus 2} \to P_{\mathrm{Sp}^\vee} \to M_{\mathbf{1}_T^\vee, 0} \to 0,$$

see [26, (234),(236)], where $M_{\mathbf{1}_T^\vee,0}$ is a submodule of $M_{\mathbf{1}_T^\vee}$ defined by [26, (233)], namely defined by the exact sequence

$$0 \to M_{\mathbf{1}_T^\vee,0} \to M_{\mathbf{1}_T^\vee} \to \mathbb{F} \to 0. \tag{3.8}$$

It is easy to see that $M_{\mathbf{1}_T^\vee}$ (resp. $M_{\mathbf{1}_T^\vee,0}$) does not admit $\pi_\alpha^\vee$ as a subquotient, hence Q_1 (resp. Q_2) is equal to $M_{\mathbf{1}_T^\vee}$ (resp. $M_{\mathbf{1}_T^\vee,0}$). Proposition 3.14 then implies that any coadmissible quotient of Q_i is a torsion Λ-module; remark that $M_{\mathbf{1}_T^\vee,0}$ is *not* a quotient of $M_{\mathbf{1}_T^\vee}$ so that Proposition 3.14 does not apply directly to coadmissible quotients of $M_{\mathbf{1}_T^\vee,0}$, but we can conclude using the exact sequence (3.8). Finally, a similar argument works in the case $\pi \in \{\mathbf{1}_G, \mathrm{Sp}\}$. ∎

Remark 3.18. Given a block $\mathfrak{B}$, let $P_\mathfrak{B} = \bigoplus_{\pi \in \mathfrak{B}} P_{\pi^\vee}$ and $E_\mathfrak{B} = \mathrm{End}_\mathfrak{C}(P_\mathfrak{B})$. It follows from [26, Lemma 2.9 and Lemma 2.10] that for any $M \in \mathfrak{C}^\mathfrak{B}$, the evaluation morphism $\mathrm{Hom}_\mathfrak{C}(P_\mathfrak{B}, M) \otimes_{E_\mathfrak{B}} P_\mathfrak{B} \to M$ is always an isomorphism. This may suggest that, even for a quotient of $P_{\pi^\vee}$ for some fixed $\pi \in \mathfrak{B}$, we should consider the $E_\mathfrak{B}$-module $\mathrm{Hom}_\mathfrak{C}(P_\mathfrak{B}, M)$ rather than the $E_{\pi^\vee}$-module $\mathrm{Hom}_\mathfrak{C}(P_{\pi^\vee}, M)$ (they are both finitely generated modules over $Z(E_\mathfrak{B}) = Z(E_{\pi^\vee})$). However, for our main application, we need to translate the Λ-torsionness of M into a commutative algebra statement, and Proposition 5.8 below shows that $\mathrm{Hom}_\mathfrak{C}(P_{\pi^\vee}, M)$ fits in with the need. The analogue of Proposition 5.8 for $\mathrm{Hom}_\mathfrak{C}(P_\mathfrak{B}, M)$ need not hold: take $\mathfrak{B} = \{\mathbf{1}_G, \mathrm{Sp}, \pi_\alpha\}$ and $M = \mathbf{1}_G^\vee$ so that $\delta(M) = 0$. (But see [23, Proposition 3.6.12, Remark 3.6.13] for one inequality.)

3.6. Blocks of type (III)

In this subsection, we assume $\mathfrak{B}$ is of type (III), that is, $\mathfrak{B} = \{\pi\}$ with $\pi \cong \mathrm{Ind}_B^G \eta$, where $\eta = \chi \otimes \chi\omega^{-1}$. After twisting, we assume $\chi = \mathbf{1}$ is trivial. Let $P = P_{\pi^\vee}$ and $E = \mathrm{End}_\mathfrak{C}(P)$. Then E is non-commutative, and is free over its center R of rank 4, see Theorem 3.4. Let $M_{\eta^\vee}$ and $E_{\eta^\vee}$ be as in Section 3.4; recall that $E_{\eta^\vee}$ is isomorphic to $\mathbb{F}[\![x,y]\!]$ and is identified with E^{ab} (the maximal abelian quotient of E) by the discussion before [26, Lemma 9.2]. On the other hand, it is known that $\mathcal{D}(\pi)$ consists of one single weight, i.e. $\sigma = \mathrm{Sym}^{p-2}\,\mathbb{F}^2$, see [27, Remark 6.2]. Hence we have surjective morphisms via (3.5) and Proposition 3.9:

$$E \twoheadrightarrow E^{\mathrm{ab}} \twoheadrightarrow \mathbb{F}[\![S]\!].$$

The goal of this subsection is to prove the following result.

Proposition 3.19. *The following statements hold.*

(i) *We may choose the variables x, y in such a way that the image of the composite morphism $R \hookrightarrow E \twoheadrightarrow E^{\mathrm{ab}} \cong \mathbb{F}[\![x,y]\!]$ is equal to $\mathbb{F}[\![x^2, xy, y^2]\!]$.*

(ii) *We may choose the variable S in such a way that the image of the composite morphism $R \hookrightarrow E \twoheadrightarrow \mathbb{F}[\![S]\!]$ is equal to $\mathbb{F}[\![S^2]\!]$.*

(iii) *Let $I = \mathrm{Ker}(R \to \mathbb{F}[\![S]\!])$ and $J = \mathrm{Ker}(E \twoheadrightarrow \mathbb{F}[\![S]\!])$. Then $J^4 \subset IE \subset J$.*

The proof of Proposition 3.19 relies on another explicitly constructed ring defined in [26, (145)], which we denote by E' (instead of R in [26, (145)]). We briefly recall the construction. Let $\mathcal{G}$ be the maximal pro-p quotient of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathcal{G}^{\mathrm{ab}}$ the maximal abelian quotient of $\mathcal{G}$. By local class field theory, we have

$$G_{\mathbb{Q}_p}^{\mathrm{ab}} \cong \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times \times \widehat{\mathbb{Z}},$$

where μ_{p^∞} is the group of p-power order roots of unity in $\overline{\mathbb{Q}}_p$ and $\mathbb{Q}_p^{\mathrm{ur}}$ is the maximal unramified extension of $\mathbb{Q}_p$. Since $\mathcal{G}^{\mathrm{ab}}$ is equal to the maximal pro-p quotient of $G_{\mathbb{Q}_p}^{\mathrm{ab}}$, we obtain

$$\mathcal{G}^{\mathrm{ab}} \cong (1 + p\mathbb{Z}_p) \times \mathbb{Z}_p.$$

We choose a pair of generators $\overline{\gamma}, \overline{\delta}$ of $\mathcal{G}^{\mathrm{ab}}$ such that $\overline{\gamma} \mapsto (1+p, 0)$ and $\overline{\delta} \mapsto (1,1)$. Then $\mathcal{G}$ is a free pro-p group generated by 2 elements γ, δ which lift respectively $\overline{\gamma}, \overline{\delta}$. See [29, Section 2] for details. Following [26, (145)], we let (note that in [26, (145)] the ring is defined over $\mathcal{O}$ and is denoted by R):

$$E' := \frac{\mathbb{F}[\![t_1, t_2, t_3]\!] \,\widehat{\otimes}_{\mathbb{F}}\, \mathbb{F}[\![\mathcal{G}]\!]}{J}$$

where J is a certain closed two-sided ideal generated by the relations listed in [26, (146) and (147)].

With the notation in [26, Section 9.2], we have the following facts:

(a) The natural morphism $\mathbb{F}[\![t_1, t_2, t_3]\!] \to E'$ is injective and identifies $\mathbb{F}[\![t_1, t_2, t_3]\!]$ with the center of E', denoted by R'. E' is a free R'-module of rank 4, and E' contains two elements

$$u := \gamma - 1 - t_1, \quad v := \delta - 1 - t_2$$

such that $\{1, u, v, t'\}$ is an R'-basis, where $t' := uv - vu$. See [26, Corollary 9.24, Corollary 9.25].

(b) E'^{ab} is isomorphic to $\mathbb{F}[\![\overline{u},\overline{v}]\!]$, where $\overline{u}$ (resp. $\overline{v}$) denotes the image of u (resp. v). The kernel of $E' \to E'^{\mathrm{ab}}$ is equal to $t'E'$. See Lemma [26, Lemma 9.3] and the proof of [26, Corollary 9.27].

(c) E' is equipped with an involution $*$ which satisfies $u^* = -u$, $v^* = -v$, $t'^* = -t'$ and $R' = \{\phi \in E' : \phi = \phi^*\}$. See [26, (161), Lemma 9.14].

(d) There exists a ring isomorphism $\varphi : E \cong E'^{\mathrm{op}}$ by [26, Corollary 9.27]. Moreover, it is compatible with Colmez's functor $\check{\mathbb{V}}$ (modified as in [26, Section 5.7]) in the following sense. As explained in [26, Section 9.1], $\check{\mathbb{V}}$ induces a natural transformation $\mathrm{Def}_{\pi^\vee} \to \mathrm{Def}_{\check{\mathbb{V}}(\pi^\vee)}$ between certain deformation functors of $\pi^\vee$ and of $\check{\mathbb{V}}(\pi^\vee)$, which are respectively pro-represented by E and $\mathbb{F}[\![\mathcal{G}]\!]^{\mathrm{op}}$, hence induces a ring morphism by Yoneda's lemma

$$\varphi_{\check{\mathbb{V}}} : \mathbb{F}[\![\mathcal{G}]\!]^{\mathrm{op}} \to E,$$

which is uniquely determined up to conjugation by $E^\times$. Here, we consider deformation problems with coefficients in finite local (possibly non-commutative) Artinian $\mathbb{F}$-algebras with residue field $\mathbb{F}$. Then the following diagram is commutative:

$$\begin{tikzcd} \mathbb{F}[\![\mathcal{G}]\!]^{\mathrm{op}} \arrow[r] \arrow[dr, "\varphi_{\check{\mathbb{V}}}"'] & E'^{\mathrm{op}} \arrow[d, "\varphi^{-1}", "\cong"'] \\ & E, \end{tikzcd} \tag{3.9}$$

where the upper horizontal morphism is the natural one.

In summary, we have a commutative diagram

$$\begin{tikzcd} R' \arrow[r] \arrow[d, "\cong"'] & E'^{\mathrm{op}} \arrow[r, two heads] \arrow[d, "\varphi^{-1}", "\cong"'] & E'^{\mathrm{ab}} \arrow[d, "\cong"'] & \\ R \arrow[r] & E \arrow[r, two heads] & E^{\mathrm{ab}} \arrow[r, two heads] & \mathbb{F}[\![S]\!]. \end{tikzcd} \tag{3.10}$$

Thus, to prove Proposition 3.19, we may work with R', E'^{op}, E'^{ab} instead of R, E, E^{ab}, via the isomorphism φ^{-1}.

Lemma 3.20. *The following statements hold.*

(i) *The element $\gamma \in \mathcal{G}$ is sent to* 1 *under the composite map*

$$\mathbb{F}[\![\mathcal{G}]\!]^{\mathrm{op}} \xrightarrow{\varphi_{\check{\mathbb{V}}}} E \twoheadrightarrow \mathbb{F}[\![S]\!]. \tag{3.11}$$

(ii) *The element $u \in E'$ is sent to* 0 *under the composite map $E'^{\mathrm{op}} \xrightarrow{\sim} E \twoheadrightarrow \mathbb{F}[\![S]\!]$.*

Proof. (i) Recall that P is flat over E and can be viewed as a deformation of $\pi^\vee$ over E, in the sense of [26, Section 3.1]. Consider $\mathbb{F}[\![S]\!] \otimes_E P$ and view it as a deformation of $\pi^\vee$ to $\mathbb{F}[\![S]\!]$. It is proved in [18, Lemma 1.5.9] and reformulated in [15, Proposition 2.9, Proposition 2.11] that

$$\check{\mathbb{V}}(\mathbb{F}[\![S]\!] \otimes_E P) \cong \mu_{S+1}^{-1},$$

where $\mu_{S+1} : G_{\mathbb{Q}_p} \to \mathbb{F}[\![S]\!]^\times$ is the unramified character sending geometric Frobenii to $S+1$. Here we have used the isomorphism $\pi \cong \pi(p-2,1)$, see (3.1). It is clear that μ_{S+1} factors through $G_{\mathbb{Q}_p} \twoheadrightarrow \mathscr{G}^{\mathrm{ab}}$ and $\mu_{S+1}(\overline{\gamma}) = 1$ by our choice of the element γ. The result follows from this because, when viewed as a deformation of $\check{\mathbb{V}}(\pi^\vee)$ to $\mathbb{F}[\![S]\!]$, $\check{\mathbb{V}}(\mathbb{F}[\![S]\!] \otimes_E P)$ is obtained from the universal deformation $\mathbb{F}[\![\mathscr{G}]\!]$ via the map (3.11). Here, $\mathbb{F}[\![\mathscr{G}]\!]^{\mathrm{op}}$ is viewed as the universal deformation ring via $\mathbb{F}[\![\mathscr{G}]\!]^{\mathrm{op}} \cong \mathrm{End}_{\mathbb{F}[\![\mathscr{G}]\!]}(\mathbb{F}[\![\mathscr{G}]\!])$, see [26, Section 3.2].

(ii) Since $u = \frac{\gamma - \gamma^{-1}}{2}$ (as is shown after [26, (160)]), the result follows from (i). ■

Denote by $\overline{R}'$ the image of $R' \hookrightarrow E' \twoheadrightarrow E'^{\mathrm{ab}} \cong \mathbb{F}[\![\overline{u}, \overline{v}]\!]$, and let $\mathfrak{m}_{\overline{R}'}$ be its maximal ideal.

Lemma 3.21. *One has $\overline{u}^2, \overline{v}^2, \overline{uv} \in \overline{R}'$.*

Proof. It is proved in [26, (159)][6] that $u^2, v^2 \in R'$, hence $\overline{u}^2, \overline{v}^2 \in \overline{R}'$. On the other hand, we know that $uv + vu \in R'$ by (c), hence $\overline{uv} \in \overline{R}'$ because 2 is invertible in $\overline{R}'$ (recall $p \geq 5$). ■

Lemma 3.22. *For any $(a, b) \in \mathbb{F}^2 \backslash \{(0,0)\}$, $a\overline{u} + b\overline{v} \notin \mathfrak{m}_{\overline{R}'} \mathbb{F}[\![\overline{u}, \overline{v}]\!]$.*

Proof. The condition

$$a\overline{u} + b\overline{v} \in \mathfrak{m}_{\overline{R}'} \cdot \mathbb{F}[\![\overline{u}, \overline{v}]\!]$$

is equivalent to the existence of $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in R'$ and $\phi \in \mathfrak{m}_{R'} E'$ such that (in E')

$$au + bv = \phi + t'(\lambda_1 + \lambda_2 u + \lambda_3 v + \lambda_4 t').$$

Taking the involution $*$, we obtain

$$\begin{aligned} -au - bv &= \phi^* + (\lambda_1 - \lambda_2 u - \lambda_3 v - \lambda_4 t')(-t') \\ &= \phi^* + t'(-\lambda_1 - \lambda_2 u - \lambda_3 v + \lambda_4 t'), \end{aligned}$$

where the first equality follows from (c) and the second from [26, (160)]. This implies, as 2 is invertible,

$$au + bv = t'(\lambda_1 + \lambda_2 u + \lambda_3 v) + \frac{\phi - \phi^*}{2}. \tag{3.12}$$

A computation using the relations established in the proof of [26, Lemma 9.18] gives

$$t'(\lambda_1 + \lambda_2 u + \lambda_3 v) = \mu_2 u + \mu_3 v + \lambda_1 t',$$

where $\mu_2, \mu_3 \in R'$ are given by

$$\begin{aligned} \mu_2 &= \lambda_2(uv + vu) + 2\lambda_3 v^2, \\ \mu_3 &= -\lambda_3(uv + vu) - 2\lambda_2 u^2. \end{aligned}$$

[6] There is a typo in the formula, namely we should get $u^2 = 2t_1 + t_1^2$ and $v^2 = 2t_2 + t_2^2$ from [26, (148)]. This does not affect the rest of [26, Section 9], because we still deduce that u^2, v^2 are central elements and only this fact is used later (see [26, Lemma 9.18]).

In particular, $\mu_2, \mu_3 \in \mathfrak{m}_{R'}$. On the other hand, since $\frac{\phi-\phi^*}{2} \in \mathfrak{m}_{R'} E'$, it can be written as $\lambda'_1 + \lambda'_2 u + \lambda'_3 v + \lambda'_4 t'$ with $\lambda'_i \in \mathfrak{m}_{R'}$. Since E' is free over R' with basis $\{1, u, v, t'\}$, (3.12) forces $a, b \in \mathfrak{m}_{R'}$, hence $a = b = 0$ and the result follows. ∎

Proof of Proposition 3.19. The diagram (3.10) shows that we may work with R', E'^{op}, E'^{ab} instead of R, E, E^{ab}, via the isomorphism φ^{-1}.

(i) By Lemmas 3.21 and 3.22, the ideal $\mathfrak{m}_{\overline{R}'}\mathbb{F}[\![\overline{u}, \overline{v}]\!]$ is equal to $(\overline{u}^2, \overline{v}^2, \overline{uv})$, which is an ideal minimally generated by 3 elements. As a consequence, the embedding dimension of $\overline{R}'$ is greater or equal to 3. But this embedding dimension is ≤ 3, because $\overline{R}'$ a quotient of R' whose embedding dimension is 3. Therefore, $\overline{R}'$ is exactly the subring of $\mathbb{F}[\![\overline{u}, \overline{v}]\!]$ topologically generated by $\overline{u}^2, \overline{v}^2, \overline{uv}$. This finishes the proof of (i).

(ii) It follows from (i) using Lemma 3.20; for example we may take S to be the image of v.

(iii) Let $I' = \mathrm{Ker}(R' \xrightarrow{\sim} R \to \mathbb{F}[\![S]\!])$ and $J' = \mathrm{Ker}(E'^{\mathrm{op}} \xrightarrow{\sim} E \twoheadrightarrow \mathbb{F}[\![S]\!])$. We need to show $J'^4 \subset I' E'^{\mathrm{op}}$ (note that I' is contained in the center of E'). On the one hand, we have $t' \in J'$ by (b) and $u \in J'$ by Lemma 3.20. Since $E'^{\mathrm{op}}/(t', u) \cong \mathbb{F}[\![\overline{v}]\!]$ and since the morphism $E'^{\mathrm{op}} \twoheadrightarrow \mathbb{F}[\![S]\!]$ is surjective, we must have $J' = (t', u)$. On the other hand, it is easy to see that $t'^2, u^2 \in I'$. Using the relation [26, (160)], one easily checks the desired inclusion. ∎

We note the following consequence of Proposition 3.19.

Corollary 3.23. *The kernel of $R \to \mathbb{F}[\![S]\!]$ is minimally generated by two elements.*

Proof. By Proposition 3.19 (ii), the image of $R \to \mathbb{F}[\![S]\!]$ is isomorphic to $\mathbb{F}[\![S^2]\!]$, which is a regular local ring. It is a standard fact that the kernel of a surjective local morphism between two regular local rings can be generated by a regular sequence of length equal to its height, see [20, Theorem 21.2 (ii)]. ∎

Remark 3.24. In general, given a height two prime ideal $\mathfrak{p}$ in a 3-dimensional regular local ring, e.g. $\mathbb{F}[\![t_1, t_2, t_3]\!]$, it is not clear whether $\mathfrak{p}$ is the radical of an ideal generated by two elements, or equivalently, whether there exists a reduction of $\mathfrak{p}$ with two generators ([16, Chapter 8]).

3.7. *Blocks of type* (IV)

In this subsection, we complement some results in the work of Paškūnas [26, 27] when $\mathfrak{B}$ is of type (IV). Proposition 3.26 in Section 3.7.1 was used in the proof of Proposition 3.5, but can be avoided as explained in Remark 3.6. The results in Section 3.7.2, except Lemma 3.27, will not be used in this paper, but might be found useful elsewhere.

The notation here is the same as in the previous subsections. In particular, the object $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is irreducible of type (IV), and $P_{\pi^\vee}$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}$ and $E_{\pi^\vee} = \mathrm{End}_{\mathfrak{C}}(P_{\pi^\vee})$. Note that the rings $E_{\pi^\vee}$ are naturally isomorphic (to $\mathbb{F}[\![x, y, z, w]\!]/(xw - yz)$) for any $\pi \in \mathfrak{B}$ (see [26, Section 10]), so the subscript will be omitted in the rest (while the one of $P_{\pi^\vee}$ will be kept). Up to twist, we may assume $\mathfrak{B} = \{\mathbf{1}_G, \mathrm{Sp}, \pi_\alpha\}$.

3.7.1. $\mathbb{F}\otimes_E P_{\pi^\vee}$. Our first aim is to determine $\mathbb{F}\otimes_E P_{\pi^\vee}$ for any object $\pi\in\mathfrak{B}$. For two objects $\pi_1,\pi_2\in \mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G)^{\mathfrak{B}}$ (in particular, Z acts trivially on them), we write following [26, Section 10]

$$e^1(\pi_1,\pi_2):=\dim_{\mathbb{F}}\mathrm{Ext}^1_{G/Z}(\pi_1,\pi_2).$$

For convenience of the reader, we recall the list of $e^1(\pi_1,\pi_2)$ for $\pi_1,\pi_2\in\mathfrak{B}$, see [26, Section 10.1]:

$$\begin{aligned} &e^1(\mathbf{1}_G,\mathbf{1}_G)=0, \quad e^1(\mathrm{Sp},\mathbf{1}_G)=1, \quad e^1(\pi_\alpha,\mathbf{1}_G)=1,\\ &e^1(\mathbf{1}_G,\mathrm{Sp})=2, \quad e^1(\mathrm{Sp},\mathrm{Sp})=0, \quad e^1(\pi_\alpha,\mathrm{Sp})=0,\\ &e^1(\mathbf{1}_G,\pi_\alpha)=0, \quad e^1(\mathrm{Sp},\pi_\alpha)=1, \quad e^1(\pi_\alpha,\pi_\alpha)=2.\end{aligned}$$

We deduce that there exists a unique (up to isomorphism) non-split extension

$$0\to\mathbf{1}_G\to\kappa\to\pi_\alpha\to 0. \tag{3.13}$$

Also, let τ_1 be the universal extension of $\mathbf{1}_G^{\oplus 2}$ by Sp, i.e. we have

$$0\to\mathrm{Sp}\to\tau_1\to\mathbf{1}_G^{\oplus 2}\to 0$$

with $\mathrm{soc}_G\,\tau_1=\mathrm{Sp}$.

Lemma 3.25. *We have*

$$e^1(\mathrm{Sp},\kappa)=2, \quad e^1(\pi_\alpha,\tau_1)=2, \quad e^1(\tau_1,\pi_\alpha)=1.$$

Proof. See [26, Lemma 10.18] for the first equality, [26, Lemma 10.12] for the second, [26, (187)] and the argument before it for the third. ∎

Proposition 3.26. *Let $\pi\in\mathfrak{B}$ and set $Q_{\pi^\vee}=\mathbb{F}\otimes_E P_{\pi^\vee}$. In the following statements, the existence of the extension is guaranteed by Lemma 3.25.*

(i) *If $\pi=\mathbf{1}_G$, then $Q_{\mathbf{1}_G^\vee}$ is isomorphic to the universal extension of $\kappa^\vee$ by $(\mathrm{Sp}^\vee)^{\oplus 2}$:*

$$0\to(\mathrm{Sp}^\vee)^{\oplus 2}\to Q_{\mathbf{1}_G^\vee}\to\kappa^\vee\to 0. \tag{3.14}$$

(ii) *If $\pi=\mathrm{Sp}$, then $Q_{\mathrm{Sp}^\vee}$ is isomorphic to the universal extension of $\tau_1^\vee$ by $(\pi_\alpha^\vee)^{\oplus 2}$:*

$$0\to(\pi_\alpha^\vee)^{\oplus 2}\to Q_{\mathrm{Sp}^\vee}\to\tau_1^\vee\to 0. \tag{3.15}$$

(iii) *If $\pi=\pi_\alpha$, then $Q_{\pi_\alpha^\vee}$ is isomorphic to the unique non-split extension of $\pi_\alpha^\vee$ by $\tau_1^\vee$:*

$$0\to\tau_1^\vee\to Q_{\pi_\alpha^\vee}\to\pi_\alpha^\vee\to 0. \tag{3.16}$$

Proof. Note that $Q_{\pi^\vee}$ is characterized as the maximal quotient of $P_{\pi^\vee}$ which contains $\pi^\vee$ with multiplicity one, see [26, Remark 1.13]. Write (in this proof) τ for the dual of the extension (3.14), (3.15), (3.16), respectively. It is clear that π occurs in τ with multiplicity one and to show $Q_{\pi^\vee}\cong\tau^\vee$ it suffices to check that if π' is irreducible such that $\mathrm{Ext}^1_{G/Z}(\pi',\tau)\neq 0$, then $\pi'\cong\pi$. Proposition 3.2 implies that we may assume $\pi'\in\mathfrak{B}$.

(i) We need to check

$$\mathrm{Ext}^1_{G/Z}(\mathrm{Sp}, \tau) = 0 = \mathrm{Ext}^1_{G/Z}(\pi_\alpha, \tau).$$

Since $e^1(\mathrm{Sp}, \mathrm{Sp}) = 0$, the first equality follows from the construction of the element τ. The second is clear since we have $e^1(\pi_\alpha, \mathrm{Sp}) = 0$ (see the formulae recalled above) and $e^1(\pi_\alpha, \kappa) = 0$ by [26, (194)].

(ii) This is proved in [15, Lemma 4.4, (19)].

(iii) This is proved in [27, Proposition 6.1, (35)]. ∎

3.7.2. $\mathrm{Tor}_i^E(\mathbb{F}, P_{\pi^\vee})$. Recall that if $\eta : T \to \mathbb{F}^\times$ is a smooth character, we let

$$\pi_\eta = \mathrm{Ind}_B^G \eta$$

and

$$\Pi_\eta = \mathrm{Ind}_B^G \mathrm{Inj}_T \eta, \quad M_{\eta^\vee} = (\Pi_\eta)^\vee, \quad E_{\eta^\vee} = \mathrm{End}_{\mathfrak{C}}(M_{\eta^\vee}),$$

where $\mathrm{Inj}_T \eta$ denotes an injective envelope of η in $\mathrm{Rep}_{\mathbb{F}}(T)$. In the rest of this subsection, we only consider $\eta \in \{\mathbf{1}_T, \alpha\}$. By [26, Proposition 7.1] there is a natural surjective ring homomorphism $q : E \twoheadrightarrow E_{\eta^\vee}$ induced by $P_{\pi_\eta^\vee} \twoheadrightarrow M_{\eta^\vee}$.

Lemma 3.27. *In the isomorphism $E \cong \mathbb{F}[[x, y, z, w]]/(xw - yz)$, we may choose the variables so that the kernel of $q : E \twoheadrightarrow E_{\eta^\vee}$ is equal to (z, w).*

Proof. First, via Colmez's functor we may identity E with the special fiber of a certain universal Galois pseudo-deformation ring over $\mathcal{O} := W(\mathbb{F})$, see [26, Theorem 10.71]. This ring is denoted by R^ψ in [26, Theorem 10.71] and we write $\overline{R}^\psi$ for its special fiber. Let $\mathfrak{r}$ denote the reducible locus of R^ψ (see [26, Corollary B.6] for its definition) and let $\bar{\mathfrak{r}}$ be its image in $\overline{R}^\psi$. Then by [26, Corollary B.5 and Corollary B.6], the ring $\overline{R}^\psi$ is isomorphic to $\mathbb{F}[[c_0, c_1, d_0, d_1]]/(c_0 d_1 + c_1 d_0)$ and $\bar{\mathfrak{r}} = (c_0, c_1)$. On the other hand, via the natural isomorphism $E \cong \overline{R}^\psi$, $\ker(q)$ is identified with $\bar{\mathfrak{r}}$ and $E_{\eta^\vee}$ with $\overline{R}^\psi/\bar{\mathfrak{r}}$, see [26, Lemma 10.80]. This gives the result up to a change of variables. Note that the choice we make is not the one in [26, Lemma 10.93]. ∎

Lemma 3.28. *We have*

$$\mathrm{Tor}_i^E(\mathbb{F}, M_{\eta^\vee}) \cong (\pi_\eta^\vee)^{\oplus 2} \quad \textit{for all } i \geq 1.$$

Proof. By Lemma 3.27, we have a periodic (infinite) resolution of $E_{\eta^\vee}$ by free E-modules

$$\cdots \longrightarrow E^{\oplus 2} \xrightarrow{d'} E^{\oplus 2} \xrightarrow{d} E^{\oplus 2} \xrightarrow{d'} E^{\oplus 2} \xrightarrow{d} E^{\oplus 2} \xrightarrow{(w,z)} E \longrightarrow E_{\eta^\vee} \longrightarrow 0,$$

where d is represented by the matrix $\left(\begin{smallmatrix} x & -z \\ -y & w \end{smallmatrix}\right)$, sending $\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} x & -z \\ -y & w \end{smallmatrix}\right)\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right)$; similarly d' is represented by the matrix $\left(\begin{smallmatrix} w & z \\ y & x \end{smallmatrix}\right)$. We deduce that

$$\mathrm{Tor}_i^E(\mathbb{F}, E_{\eta^\vee}) \cong \mathbb{F}^{\oplus 2} \quad \text{for all } i \geq 1.$$

Because $M_{\eta^\vee}$ is a flat $E_{\eta^\vee}$-module by Lemma 3.13, by flat base change we obtain

$$\mathrm{Tor}_i^E(\mathbb{F}, M_{\eta^\vee}) \cong \mathrm{Tor}_i^E(\mathbb{F}, E_{\eta^\vee}) \otimes_{E_{\eta^\vee}} M_{\eta^\vee} \cong (\pi_\eta^\vee)^{\oplus 2},$$

as required. ∎

Lemma 3.29. *For $i \geq 1$, we have* $\mathrm{Hom}_{\mathfrak{C}}(P, \mathrm{Tor}_i^E(\mathbb{F}, P)) = 0$.

Proof. Choose a resolution of $\mathbb{F}$ by finite free E-modules: $F_\bullet \to \mathbb{F} \to 0$. Then the homology of $F_\bullet \otimes_E P$ computes $\mathrm{Tor}_i^E(\mathbb{F}, P)$. It is clear that

$$\mathrm{Hom}_{\mathfrak{C}}(P, F_\bullet \otimes_E P) \cong F_\bullet.$$

Since $\mathrm{Hom}_{\mathfrak{C}}(P, -)$ is exact, this implies

$$\mathrm{Hom}_{\mathfrak{C}}(P, H_i(F_\bullet \otimes_E P)) \cong H_i(F_\bullet)$$

and the result follows. ■

Proposition 3.30. *For any $i \geq 1$, we have*

$$\mathrm{Tor}_i^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \cong (\mathrm{Sp}^\vee)^{\oplus 2}, \quad \mathrm{Tor}_i^E(\mathbb{F}, P_{\pi_\alpha^\vee}) = (\mathbf{1}_G^\vee)^{\oplus 2},$$

and there is a short exact sequence

$$0 \to (\pi_\alpha^\vee)^{\oplus 2} \to \mathrm{Tor}_i^E(\mathbb{F}, P_{\mathrm{Sp}^\vee}) \to (\mathbf{1}_G^\vee)^{\oplus 2} \to 0.$$

Remark 3.31. It is natural to ask if $\mathrm{Tor}_i^E(\mathbb{F}, P_{\mathrm{Sp}^\vee})$ is actually isomorphic to $(\kappa^\vee)^{\oplus 2}$, where κ is defined by (3.13).

Proof. We first observe the following facts:

(a) $\mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on $\mathrm{Tor}_i^E(\mathbb{F}, P_{\pi_\alpha^\vee})$ for $i \geq 1$. Indeed, [26, Corollary 10.43] states this for $i = 1$ but the proof works for all $i \geq 1$. This implies that $\mathrm{Tor}_i^E(\mathbb{F}, P_{\pi_\alpha^\vee})$ is isomorphic to a finite direct sum of $\mathbf{1}_G^\vee$ since $e^1(\mathbf{1}_G, \mathbf{1}_G) = 0$.

(b) $\mathbf{1}_G^\vee$ does not occur in $\mathrm{Tor}_i^E(\mathbb{F}, P_{\mathbf{1}_G^\vee})$ for $i \geq 1$; this is a special case of Lemma 3.29.

Recall the following exact sequences:

$$0 \to P_{\pi_\alpha^\vee} \to P_{\mathbf{1}_G^\vee} \to M_{\mathbf{1}_T^\vee} \to 0 \tag{3.17}$$

and

$$0 \to P_{\mathrm{Sp}^\vee} \to P_{\pi_\alpha^\vee} \to M_{\alpha^\vee} \to 0, \tag{3.18}$$

see [26, (234), (235)]. From (3.17) we obtain a long exact sequence

$$\cdots \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \to \mathrm{Tor}_1^E(\mathbb{F}, M_{\mathbf{1}_T^\vee}) \to Q_{\pi_\alpha^\vee} \to Q_{\mathbf{1}_G^\vee} \to \pi_{\mathbf{1}_T}^\vee \to 0.$$

From (a) and (b), we deduce that the morphisms

$$\mathrm{Tor}_i^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \to \mathrm{Tor}_i^E(\mathbb{F}, P_{\mathbf{1}_G^\vee})$$

are zero for $i \geq 1$, hence we obtain a long exact sequence

$$0 \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \to \mathrm{Tor}_1^E(\mathbb{F}, M_{\mathbf{1}_T^\vee}) \to Q_{\pi_\alpha^\vee} \to Q_{\mathbf{1}_G^\vee} \to \pi_{\mathbf{1}_T}^\vee \to 0, \tag{3.19}$$

and short exact sequences for $i \geq 2$,

$$0 \to \mathrm{Tor}_i^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \to \mathrm{Tor}_i^E(\mathbb{F}, M_{\mathbf{1}_T^\vee}) \to \mathrm{Tor}_{i-1}^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \to 0. \tag{3.20}$$

Using Proposition 3.26 and Lemma 3.28, (3.19) implies $\mathrm{Tor}_1^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \cong (\mathrm{Sp}^\vee)^{\oplus 2}$, while (3.20) and (a)–(b) imply for $i \geq 2$,

$$\mathrm{Tor}_{i-1}^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \cong (\mathbf{1}_G^\vee)^{\oplus 2}, \quad \mathrm{Tor}_i^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \cong (\mathrm{Sp}^\vee)^{\oplus 2}.$$

This proves the first two assertions.

Similarly, the sequence (3.18) induces

$$\cdots \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\mathrm{Sp}^\vee}) \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \to \mathrm{Tor}_1^E(\mathbb{F}, M_{\alpha^\vee}) \to Q_{\mathrm{Sp}^\vee} \to Q_{\pi_\alpha^\vee} \to \pi_\alpha^\vee \to 0.$$

By Lemma 3.28 and (a), we see that the morphisms $\mathrm{Tor}_i^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \to \mathrm{Tor}_i^E(\mathbb{F}, M_{\alpha^\vee})$ are zero for $i \geq 1$, hence obtain short exact sequences

$$0 \to \mathrm{Tor}_{i+1}^E(\mathbb{F}, M_{\alpha^\vee}) \to \mathrm{Tor}_i^E(\mathbb{F}, P_{\mathrm{Sp}^\vee}) \to \mathrm{Tor}_i^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \to 0.$$

Using Lemma 3.28 and what has been proved, we deduce the result for $\mathrm{Tor}_i^E(\mathbb{F}, P_{\mathrm{Sp}^\vee})$. ■

3.7.3. Miracle flatness. In this subsection we prove the following result, using the "miracle flatness" criterion in [13, Proposition A.30]. Recall that the block $\mathfrak{B}$ is of type (IV).

Proposition 3.32. *There exists a subring $R' \subset E$ which is a regular local $\mathbb{F}$-algebra of Krull dimension 3 such that E is finite free over R' and P is flat over R'.*

Proof. Choose a weight $\sigma \in \mathcal{D}(\pi)$. With the notation in Proposition 3.9 we have isomorphisms

$$\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee)^\vee \cong E/J_\sigma \cong \mathbb{F}[\![S]\!]. \tag{3.21}$$

Let $x_1 \in E$ be a lift of S. Then x_1 is a regular element in E (which is a domain). Since E is a Cohen–Macaulay ring of Krull dimension 3, we may extend x_1 to a regular sequence in E, say $\{x_1, x_2, x_3\}$. Then the subring $R' := \mathbb{F}[\![x_1, x_2, x_3]\!]$ is a regular local ring of Krull dimension 3 and E is finite over R'. Moreover, the Auslander–Buchsbaum formula implies that E is free over R'.

We are left to prove that P is flat over R'. As in [13, A.14], we consider

$$A = \Lambda \mathbin{\widehat{\otimes}}_{\mathbb{F}} \mathbb{F}[\![x_1]\!], \quad B = \Lambda \mathbin{\widehat{\otimes}}_{\mathbb{F}} R',$$

which are both Auslander regular rings (see [13, Definition A.2]). The natural inclusion $\mathbb{F}[\![x_1]\!] \subset R'$ induces an inclusion $A \subset B$. We may view P as a module over both A and B. The exact sequence

$$P \xrightarrow{\times x_1} P \longrightarrow P/x_1P \longrightarrow 0$$

induces a sequence

$$0 \longrightarrow \mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee)^\vee \xrightarrow{\times S} \mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma^\vee)^\vee \longrightarrow \mathrm{Hom}_K^{\mathrm{cont}}(P/x_1P, \sigma^\vee)^\vee \longrightarrow 0,$$

which is exact by (3.21). By Proposition 3.9, the above exact sequence still exists if we replace σ by any irreducible object $\sigma' \in \mathrm{Rep}_{\mathbb{F}}(K)$ with $\mathrm{Hom}_K^{\mathrm{cont}}(P, \sigma'^\vee)^\vee \neq 0$, and $\mathrm{Hom}_K^{\mathrm{cont}}(P/x_1P, \sigma'^\vee)^\vee$ is 1-dimensional over $\mathbb{F}$. Therefore, P/x_1P is coadmissible and

P is finitely generated as an A-module (resp. as a B-module) by Nakayama's lemma. On the other hand, Proposition 3.10 implies that x_1 is P-regular and P/x_1P is also projective in $\mathfrak{C}(K)$; here we have used the main result of [10] stating that P remains projective in $\mathfrak{C}(K)$.

In particular, we see that P/x_1P is a Cohen–Macaulay module over Λ. Since x_1 is both A- and P-regular, a standard argument (using [13, Lemma A.15] for example) shows that P is a Cohen–Macaulay A-module and

$$\delta_A(P) = \delta_\Lambda(P/x_1P) + 1 = 3 + 1 = 4.$$

By using [13, Corollary A.29], it follows that P is also a Cohen–Macaulay B-module with $\delta_B(P) = 4$. Since E is finite free over R', we have $\delta_\Lambda(\mathbb{F} \otimes_{R'} P) = 1$ by Proposition 3.5, so

$$j_B(P) = \dim B - \delta_B(P) = 2 = j_\Lambda(\mathbb{F} \otimes_{R'} P).$$

By [13, Proposition A.30], we deduce that P is flat over R'. ■

Remark 3.33. We construct an explicit subring R' of E as in Proposition 3.32 as follows. Choose the isomorphism $E \cong \mathbb{F}[\![x, y, z, w]\!]/(xw - yz)$ in such a way that the kernel of $q : E \twoheadrightarrow E_{\eta^\vee}$ is equal to (z, w), see Lemma 3.27. Clearly, the elements $x, y - z, w$ form a regular sequence in R; let

$$R' := \mathbb{F}[\![x, y - z, w]\!]. \tag{3.22}$$

Then the composite morphism $R' \hookrightarrow E \twoheadrightarrow E_{\eta^\vee}$ remains surjective. In particular, a suitable linear combination of $x, y - z, w$ serves as a lift of S. This proves that R' is one of the rings considered there.

4. Key computation

We keep the notation of Section 3. For $n \geq 1$, let

$$K_n = \begin{pmatrix} 1 + p^n\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1 + p^n\mathbb{Z}_p \end{pmatrix},$$

$$T_1(p^n) = \begin{pmatrix} 1 + p\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}.$$

Recall that $\Lambda := \mathbb{F}[\![K_1/Z_1]\!]$.

Theorem 4.1. *Let $\Pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be an object of finite length. Then*

$$\dim_{\mathbb{F}} \Pi^{T_1(p^n)} \ll n.$$

It is clear that we may assume Π is irreducible in Theorem 4.1. Further, by the recall in Section 3.1, up to twist it is enough to prove the following.

Theorem 4.2. *For any $0 \leq r \leq p - 1$ and $\lambda \in \mathbb{F}$, we have*

$$\dim_{\mathbb{F}} \pi(r, \lambda, 1)^{T_1(p^n)} \ll n.$$

4.1. Preparation

We need some preparation to prove Theorem 4.2. To begin with, we establish a double coset decomposition formula in K. Let

$$K_0(p^n) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1+p\mathbb{Z}_p \end{pmatrix}.$$

Lemma 4.3. *For any $n \geq 1$, we have*

$$|K_0(p^n)\backslash K/H| = (2n-1)(p-1)+2.$$

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K$. We have the following facts:

(i) If $A \in K_0(p)$, i.e. $c \in p\mathbb{Z}_p$, we have two subcases:

- if $c \in p^n\mathbb{Z}_p$, then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(p^n)$,
- if $c \in p\mathbb{Z}_p \backslash p^n\mathbb{Z}_p$ (so $n \geq 2$), write $c = up^k$ with $u \in \mathbb{Z}_p^\times$ and $1 \leq k \leq n-1$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^{-1}(ad-bc) & ubd^{-1} \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^k & [\lambda] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix},$$

where $\lambda := \overline{u^{-1}d} \in \mathbb{F}_p^\times$ and $t := \frac{u^{-1}d}{[\lambda]} \in 1+p\mathbb{Z}_p$.

We deduce that

$$K_0(p) = K_0(p^n) \cup \left(\bigcup_{1 \leq k \leq n-1,\, \lambda \in \mathbb{F}_p^\times} K_0(p^n) \begin{pmatrix} 1 & 0 \\ p^k & [\lambda] \end{pmatrix} H \right).$$

It is easy to check that this is a disjoint union, so the cardinality of $K_0(p^n)\backslash K_0(p)/H$ is $1+(n-1)(p-1)$.

(ii) If $A \notin K_0(p)$, i.e. $c \in \mathbb{Z}_p^\times$, we still have two subcases:

- if $d \in \mathbb{Z}_p^\times$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -[\lambda]d^{-1}(ad-bc) & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & [\lambda] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix},$$

where $\lambda := \overline{c^{-1}d} \in \mathbb{F}_p^\times$ and $t = \frac{c^{-1}d}{[\lambda]} \in 1+p\mathbb{Z}_p$,

- if $d \in p\mathbb{Z}_p$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix} \in K_0(p).$$

By combining (i) and (ii), the cardinality of $K_0(p^n)\backslash K/H$ is equal to

$$[1+(n-1)(p-1)] + [(p-1)+1+(n-1)(p-1)] = (2n-1)(p-1)+2. \quad \blacksquare$$

Proposition 4.4. *Let $n \geq 1$ and σ a smooth $\mathbb{F}$-representation of $K_0(p^n)$ of finite dimension d. If V is a quotient K-representation of $\mathrm{Ind}_{K_0(p^n)}^K \sigma$, then $\dim_{\mathbb{F}} V^H \leq 2dpn$.*

Proof. Let W be the corresponding kernel so that we have an exact sequence

$$0 \to W \to \mathrm{Ind}_{K_0(p^n)}^{K} \sigma \to V \to 0.$$

By taking H-invariants, it induces

$$0 \to W^H \to (\mathrm{Ind}_{K_0(p^n)}^{K} \sigma)^H \to V^H \xrightarrow{\partial} H^1(H, W),$$

hence an equality of dimensions

$$\dim_{\mathbb{F}} W^H + \dim_{\mathbb{F}} V^H = \dim_{\mathbb{F}} (\mathrm{Ind}_{K_0(p^n)}^{K} \sigma)^H + \dim_{\mathbb{F}} \mathrm{Im}(\partial). \tag{4.1}$$

Now note that $H \cong 1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$ is a pro-p group of cohomological dimension 1, so by Lemma 4.5 below we have

$$\dim_{\mathbb{F}} W^H = \dim_{\mathbb{F}} H^1(H, W) \geq \dim_{\mathbb{F}} \mathrm{Im}(\partial),$$

hence by (4.1),

$$\dim_{\mathbb{F}} V^H \leq \dim_{\mathbb{F}} (\mathrm{Ind}_{K_0(p^n)}^{K} \sigma)^H.$$

We are thus reduced to prove the proposition in the special case $V = \mathrm{Ind}_{K_0(p^n)}^{K} \sigma$. By using [2, Lemma 3], it is easy to see that any irreducible smooth $\mathbb{F}$-representation of $K_0(p^n)$ is one-dimensional, so there exists a filtration of σ by sub-representations, of length d, such that all graded pieces are one-dimensional. Hence, we may assume $d = 1$, in which case the result follows from Lemma 4.3. ∎

Lemma 4.5. *Let W be a finite-dimensional smooth $\mathbb{F}$-representation of $\mathbb{Z}_p$. Then*

$$\dim_{\mathbb{F}} H^1(\mathbb{Z}_p, W) = \dim_{\mathbb{F}} H^0(\mathbb{Z}_p, W).$$

Proof. This is clear if $\dim_{\mathbb{F}} W = 1$ because then W must be the trivial representation of $\mathbb{Z}_p$ so that $H^1(\mathbb{Z}_p, W) \cong \mathrm{Hom}(\mathbb{Z}_p, \mathbb{F})$ is of dimension 1. The general case is proved by induction on $\dim_{\mathbb{F}} W$ using the fact that $H^2(\mathbb{Z}_p, *) = 0$ and that W always contains a one-dimensional sub-representation. ∎

Remark 4.6. In the proof of Proposition 4.4, we crucially used the fact that H has cohomological dimension 1. This fact, very special to the group $\mathrm{GL}_2(\mathbb{Q}_p)$, is also used in [4] and [25] (but for the unipotent subgroup of $B(\mathbb{Z}_p)$).

4.2. Supersingular case

We give the proof of Theorem 4.2 when Π is supersingular, i.e. $\Pi = \pi(r, 0, 1)$ for some $0 \leq r \leq p - 1$. Since we have a G-equivariant isomorphism ([4, Theorem 1.3])

$$\pi(r, 0, 1) \cong \pi(p - 1 - r, 0, \omega^r),$$

we may assume $r > 0$ in the following.

Set $\sigma := \mathrm{Sym}^r \mathbb{F}^2$ and for $n \geq 1$ denote by σ_n the following representation of $K_0(p^n)$:

$$\sigma_n\left(\begin{pmatrix} a & b \\ p^n c & d \end{pmatrix}\right) := \sigma\left(\begin{pmatrix} d & c \\ p^n b & a \end{pmatrix}\right).$$

Let $R_0 := \sigma$ and $R_n := \mathrm{Ind}_{K_0(p^n)}^K \sigma_n$ for $n \geq 1$. It is easy to see that

$$\dim_{\mathbb{F}} R_0 = (r+1), \qquad \dim_{\mathbb{F}} R_n = (r+1)(p+1)p^{n-1} \quad \text{for all } n \geq 1. \tag{4.2}$$

Moreover, the following properties hold (see [5, Section 4]):

(i) $\text{c-Ind}_{KZ}^G \sigma|_K \cong \bigoplus_{n \geq 0} R_n$,

(ii) the Hecke operator $T|_{R_n} : R_n \to R_{n+1} \oplus R_{n-1}$ is the sum of a K-equivariant injection $T^+ : R_n \hookrightarrow R_{n+1}$ and (for $n \geq 1$) a K-equivariant surjection $T^- : R_n \twoheadrightarrow R_{n-1}$,

(iii) we have an isomorphism of K-representations

$$\begin{aligned}\pi(r,0,1) \cong &\Big(\varinjlim_{n \text{ even}} R_0 \oplus_{R_1} \oplus R_2 \oplus_{R_3} \oplus \cdots \oplus R_n \Big) \\ &\oplus \Big(\varinjlim_{n \text{ odd}} (R_1/R_0) \oplus_{R_2} R_3 \oplus_{R_4} \oplus \cdots \oplus R_n \Big).\end{aligned} \tag{4.3}$$

Denote by $\Pi_0 = \varinjlim_{n \text{ even}}$ and $\Pi_1 = \varinjlim_{n \text{ odd}}$ the two direct summands of Π in (4.3). For all $n \geq 0$, we let $\overline{R}_n$ be the image of $R_n \to \pi(r,0,1)$. Then $\overline{R}_n \subset \Pi_0$ if n is even, and $\overline{R}_n \subset \Pi_1$ if n is odd.

Lemma 4.7. *For all $n \geq 0$, we have $\overline{R}_n \subset \overline{R}_{n+2}$ and $\dim_{\mathbb{F}} \overline{R}_n = (r+1)p^n$.*

Proof. The inclusion $\overline{R}_n \subset \overline{R}_{n+2}$ follows from (ii) and (4.3). The dimension formula follows from (4.3) using (4.2). Precisely, (4.3) implies that if n is even, then by (4.2),

$$\begin{aligned}\dim_{\mathbb{F}} \overline{R}_n &= \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{F}} R_k \\ &= (r+1)\Big(1 + (p+1)\sum_{k=1}^{n}(-1)^k p^{k-1}\Big) \\ &= (r+1)p^n,\end{aligned}$$

and if n is odd, then similarly

$$\dim_{\mathbb{F}} \overline{R}_n = \sum_{k=0}^{n} (-1)^{k+1} \dim_{\mathbb{F}} R_k = (r+1)p^n,$$

which finishes the proof. ∎

At this point, we need the following result of Morra. Recall that $\Pi = \Pi_0 \oplus \Pi_1$ as K-representations.

Theorem 4.8. *Let $n \geq 1$. For $i \in \{0,1\}$, the dimension of K_n-invariants of Π_i satisfies*

$$\dim_{\mathbb{F}} \Pi_i^{K_n} \leq (p+1)p^{n-1} - 1.$$

Moreover, Π_i is nearly uniserial in the following sense: if W_1, W_2 are two K-stable subspaces of Π_i such that

$$\dim_{\mathbb{F}} W_2 - \dim_{\mathbb{F}} W_1 \geq p,$$

then $W_1 \subset W_2$.

Proof. See [22, Corollary 4.14, Corollary 4.15] for the dimension formula. Note that the formula in [22, Corollary 4.15] is for the dimension of $\Pi_0^{K_n} \oplus \Pi_1^{K_n}$, but one can deduce the dimension of $\Pi_i^{K_n}$ from [22, Corollary 4.14]. The second statement follows from [21, Theorem 1.1] which describes the K-socle filtration of Π_i. To explain this, fix $i \in \{0, 1\}$. By [21, Theorem 1.1], Π_i admits an increasing filtration $\mathrm{Fil}^k \Pi_i$, $k \geq 0$ such that

$$\mathrm{Fil}^0 \Pi_i = 0, \quad \mathrm{Fil}^1 \Pi_i = \mathrm{soc}_K \Pi_i$$

and

$$\mathrm{Fil}^{k+1} \Pi_i / \mathrm{Fil}^k \Pi_i \cong \mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi_k \quad \text{for all } k \geq 2,$$

for suitable characters $\chi_k : B(\mathbb{F}_p) \to \mathbb{F}^\times$. In particular, the graded pieces have dimension $p + 1$ except for the first. Moreover, the filtration satisfies the property that for any K-stable subspace $W \subset \Pi_i$ and any $k \leq k'$, the condition

$$\dim_{\mathbb{F}} \mathrm{Fil}^k \Pi_i \leq \dim_{\mathbb{F}} W \leq \dim_{\mathbb{F}} \mathrm{Fil}^{k'} \Pi_i$$

implies

$$\mathrm{Fil}^k \Pi_i \subset W \subset \mathrm{Fil}^{k'} \Pi_i.$$

Now, for the given W_1 let k_1 be the smallest index such that $W_1 \subset \mathrm{Fil}^{k_1} \Pi_i$; then

$$\dim_{\mathbb{F}} \mathrm{Fil}^{k_1} \Pi_i - \dim_{\mathbb{F}} W_1 \leq p.$$

The assumption then implies

$$\dim_{\mathbb{F}} \mathrm{Fil}^{k_1} \Pi_i \leq \dim_{\mathbb{F}} W_2$$

and that W_2 contains $\mathrm{Fil}^{k_1} \Pi_i$, proving the result. ∎

Corollary 4.9. *We have* $\Pi^{K_n} \subset \overline{R}_n \oplus \overline{R}_{n+1}$.

Proof. We have assumed $r \geq 1$, so by Lemma 4.7 we get for $n \geq 1$,

$$\dim_{\mathbb{F}} \overline{R}_n \geq 2p^n \geq ((p+1)p^{n-1} - 1) + p \geq \dim_{\mathbb{F}} \Pi_i^{K_n} + p.$$

By the nearly uniserial property of Π_i, this implies that $\Pi_0^{K_n} \subset \overline{R}_n$ if n is even, while $\Pi_1^{K_n} \subset \overline{R}_n$ if n is odd. Putting them together, we obtain the result. ∎

Proof of Theorem 4.2 when $\lambda = 0$. Since $T_1(p^n)$ contains K_n, we have an inclusion

$$\Pi^{T_1(p^n)} \subset \Pi^{K_n},$$

so Corollary 4.9 implies $\Pi^{T_1(p^n)} \subset \overline{R}_n \oplus \overline{R}_{n+1}$, hence

$$\Pi^{T_1(p^n)} \subset (\overline{R}_n)^{T_1(p^n)} \oplus (\overline{R}_{n+1})^{T_1(p^n)} \subset (\overline{R}_n)^H \oplus (\overline{R}_{n+1})^H.$$

Noting that $\dim_{\mathbb{F}} \sigma \leq p$, we obtain by Proposition 4.4 that

$$\dim_{\mathbb{F}} \Pi^{T_1(p^n)} \leq \dim_{\mathbb{F}} (\overline{R}_n)^H + \dim_{\mathbb{F}} (\overline{R}_{n+1})^H \leq 4p^2 n,$$

hence the result. ∎

4.3. Non-supersingular case

Assume from now on $\lambda \neq 0$. We define the subspaces R_n ($n \geq 0$) of c-$\mathrm{Ind}_{KZ}^{G}\sigma$ as above. We still have properties (i) and (ii) recalled there. The only difference, also the key difference, with the supersingular case is that the induced morphisms $R_n \to \pi(r, \lambda, 1)$ are all injective (because $\lambda \neq 0$). Moreover, if we write $\overline{R}_n$ for the image of R_n in $\pi(r, \lambda, 1)$, then $\overline{R}_n \subset \overline{R}_{n+1}$ and

$$\pi(r, \lambda, 1) = \varinjlim_{n \geq 0} \overline{R}_n.$$

Proposition 4.10. *Let $n \geq 1$. Then we have an inclusion $\pi(r, \lambda, 1)^{K_n} \subset \overline{R}_n$.*

Proof. By [21, Theorem 1.2], $\pi(r, \lambda, 1)$ satisfies a (nearly) uniserial property as in the supersingular case. Moreover, we have (see [22, Section 5])

$$\dim_{\mathbb{F}} \pi(r, \lambda, 1)^{K_n} = (p + 1)p^{n-1}$$

while

$$\dim_{\mathbb{F}} \overline{R}_n = \dim_{\mathbb{F}} R_n = (r + 1)(p + 1)p^{n-1}.$$

We then conclude as in the supersingular case. ∎

Proof of Theorem 4.2 when $\lambda \neq 0$. Since $T_1(p^n)$ contains K_n, it follows from Proposition 4.10 that

$$\pi(r, \lambda, 1)^{T_1(p^n)} \subset (\overline{R}_n)^{T_1(p^n)} \subset (\overline{R}_n)^{H}.$$

The result then follows from Proposition 4.4. ∎

5. Main results

For application in Section 6, we need to generalize Theorem 4.1 to higher cohomological degrees and to representations of a finite product of $\mathrm{GL}_2(\mathbb{Q}_p)$.

We let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$ and other subgroups of G are defined as in the previous section. Given $r \geq 1$, we let

$$\mathcal{G} = \prod_{i=1}^{r} G, \quad \mathcal{K} = \prod_{i=1}^{r} K, \quad \mathcal{K}_1 = \prod_{i=1}^{r} K_1, \quad \mathcal{Z}_1 = \prod_{i=1}^{r} Z_1$$

and

$$\Lambda = \mathbb{F}[\![\mathcal{K}_1/\mathcal{Z}_1]\!] \cong \widehat{\bigotimes_{i=1}^{r}} \mathbb{F}[\![K_1/Z_1]\!].$$

That is, $\mathcal{G}$ is a product of r copies of G, and so on. If $\mathbf{n} = (n_1, \ldots, n_r) \in (\mathbb{Z}_{\geq 1})^r$, let

$$\mathcal{T}_1(p^{\mathbf{n}}) = \prod_{i=1}^{r} T_1(p^{n_i}), \quad \mathcal{K}_{\mathbf{n}} = \prod_{i=1}^{r} K_{n_i}.$$

As in the case $r = 1$, let $\mathrm{Rep}_{\mathbb{F}}(\mathcal{G})$ denote the category of smooth $\mathbb{F}$-representations of $\mathcal{G}$ with a central character and let $\mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(\mathcal{G})$ be the subcategory consisting of locally finite objects. Let $\mathfrak{C}(\mathcal{G})$ be the dual category of $\mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(\mathcal{G})$ under Pontryagin dual and $\mathfrak{C}^{\mathrm{fg,tor}}(\mathcal{G})$ the subcategory of $\mathfrak{C}(\mathcal{G})$ consisting of coadmissible *torsion* objects.

5.1. Generalizations

5.1.1. Blocks. For $1 \le i \le r$, let $\pi_i \in \mathfrak{C}(G)$ be (absolutely) irreducible and let $\mathfrak{B}_i$ be the block in which π_i lies. Recall that π_i is admissible (see Section 3.1).

Lemma 5.1. *The following statements hold.*

(i) *The tensor product $\pi_1 \otimes \cdots \otimes \pi_r$ is an irreducible admissible representation of $\mathcal{G}$. Conversely, up to enlarge $\mathbb{F}$, each irreducible representation π in $\mathrm{Rep}_{\mathbb{F}}(\mathcal{G})$ is of this form; in particular, π is admissible.*

(ii) *Let $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ be as in* (i)*. Then $\delta_\Lambda(\pi^\vee)$ is equal to the cardinality of the index $i \in \{1, \ldots, r\}$ such that π_i is infinite-dimensional.*

(iii) *Let $\pi = \bigotimes_{i=1}^r \pi_i$ and $\pi' = \bigotimes_{i=1}^r \pi_i'$ be irreducible representations in $\mathrm{Rep}_{\mathbb{F}}(\mathcal{G})$. Then $\pi \sim \pi'$ (i.e. in the same block) if and only if $\pi_i \sim \pi_i'$ for all i.*

Proof. (i) The first part is standard. For the second, see [13, Lemma B.7] for a proof. Note that we only need to assume π carries a central character, then π is automatically admissible; the proof uses the classification of irreducible objects in $\mathrm{Rep}_{\mathbb{F}}(G)$ (cf. [2] and [4]).

(ii) It is a direct consequence of Theorem 3.1, using [13, Lemma A.11].

(iii) It follows from the fact that $\mathrm{Ext}^1_{\mathcal{G}}(\pi, \pi') \neq 0$ if and only if there exists an i with $1 \le i \le r$ such that $\mathrm{Ext}^1_G(\pi_i, \pi_i') \neq 0$ and $\pi_j \cong \pi_j'$ for all $j \neq i$; see [23, Lemma 3.4.10] for a proof. ■

Let $\mathfrak{B}$ be the block in which $\pi = \bigotimes_{i=1}^r \pi_i$ lies. As a consequence of Lemma 5.1, $\mathfrak{B}$ is equal to $\mathfrak{B}_1 \otimes \cdots \otimes \mathfrak{B}_r := \{\bigotimes_{i=1}^r \pi_i' : \pi_i' \in \mathfrak{B}_i\}$.

Let $P_{\pi_i^\vee}$ be a projective envelope of $\pi_i^\vee$ in $\mathfrak{C}(G)$ and set $E_{\pi_i^\vee} = \mathrm{End}_{\mathfrak{C}(G)}(P_{\pi_i^\vee})$. Write

$$P := \widehat{\bigotimes_{i=1}^{r}} P_{\pi_i^\vee}, \quad E := \widehat{\bigotimes_{i=1}^{r}} E_{\pi_i^\vee},$$

where $\widehat{\otimes}$ denotes the completed tensor product over $\mathbb{F}$ (see [13, Section B.1] for the definition and basic properties). For each i, let $R_{\pi_i^\vee}$ be the center of $E_{\pi_i^\vee}$ and set

$$R := \widehat{\bigotimes_{i=1}^{r}} R_{\pi_i^\vee}.$$

Then R is contained in the center of E.[7]

[7]It seems that R is exactly the center of E, see [23, Remark 3.4.12]; but we do not need this property.

Lemma 5.2. *The following statements hold.*

(i) *P is a projective envelope of $\widehat{\bigotimes}_{i=1}^{r}\pi_i^\vee$ in $\mathfrak{C}(\mathcal{G})$ and $\mathrm{End}_{\mathfrak{C}(\mathcal{G})}(P)\cong E$.*

(ii) *R is a Noetherian complete local $\mathbb{F}$-algebra, Cohen–Macaulay of Krull dimension $3r$. Moreover, E is finite free over R.*

(iii) *$\mathbb{F}\otimes_E P$ (resp. $\mathbb{F}\otimes_R P$) has finite length in $\mathfrak{C}(\mathcal{G})$ and*
$$\delta_\Lambda(\mathbb{F}\otimes_E P)=\delta_\Lambda(\mathbb{F}\otimes_R P)=r.$$

Proof. Statement (i) is proved in [13, Lemma B.8], (ii) follows from Theorem 3.4, and (iii) follows from Proposition 3.5 and [13, Lemma A.11]. ■

Lemma 5.3. *If $M\in\mathfrak{C}(\mathcal{G})$ has finite length, then $\delta_\Lambda(M)\le r$ and*
$$\dim_{\mathbb{F}} M_{\mathcal{I}_1(p^{\mathfrak{n}})}\ll\prod_{i=1}^{r}n_i.$$

Proof. We may assume M is irreducible, so that $M\cong\widehat{\bigotimes}_{i=1}^{r}\pi_i^\vee$ with each π_i irreducible. The result then follows from Lemma 5.1 (ii) and Theorem 4.1. ■

5.1.2. Coadmissibility. Similar to Lemma 5.1 (i), an irreducible representation of $\mathcal{K}$ is of the form $\sigma=\bigotimes_{i=1}^{r}\sigma_i$ with each $\sigma_i\in\mathrm{Rep}_{\mathbb{F}}(K)$ irreducible. We have the obvious notion of a *Serre weight* for $\pi=\bigotimes_{i=1}^{r}\pi_i$ as in Section 3.3; denote by $\mathcal{D}(\pi)$ the set of Serre weights of π. Clearly, $\sigma\in\mathcal{D}(\pi)$ if and only if $\sigma_i\in\mathcal{D}(\pi_i)$ for each i. The following lemma is a direct generalization of Lemma 3.7 and Proposition 3.9.

Lemma 5.4. *The following statements hold.*

(i) *If $\pi\neq\pi'$ are two objects in a block $\mathfrak{B}$, then $\mathcal{D}(\pi)\cap\mathcal{D}(\pi')=\emptyset$.*

(ii) *Let $\sigma\in\mathrm{Rep}_{\mathbb{F}}(\mathcal{K})$ be irreducible. Whenever non-zero, $\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(P,\sigma^\vee)^\vee$ is a cyclic E-module and if J_σ denotes its annihilator, then*
$$E/J_\sigma\cong\mathbb{F}[\![S_1,\dots,S_r]\!].$$
If $\sigma=\bigotimes_{i=1}^{r}\sigma_i$ and $\sigma'=\bigotimes_{i=1}^{r}\sigma_i'$ are two irreducible $\mathcal{K}$-representations such that $\sigma_i=\sigma_i'$ whenever π_i is supersingular (i.e. $\mathfrak{B}_i$ is of type (I)), then
$$\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(P,\sigma^\vee)^\vee\cong\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(P,\sigma'^\vee)^\vee$$
as E-modules when they are both non-zero.

(iii) *Let $\widetilde{\sigma}=\bigoplus_\sigma\sigma$, where the sum is taken over all irreducible objects $\pi\in\mathrm{Rep}_{\mathbb{F}}(K)$ such that $\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(P,\sigma^\vee)\neq 0$. Then $\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(P,\widetilde{\sigma}^\vee)^\vee$ is a Cohen–Macaulay R-module of Krull dimension r.*

Proof. The universal property of the completed tensor product, see the proof of [13, Lemma B.8], gives
$$\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}\left(\widehat{\bigotimes_{i=1}^{r}}\pi_i^\vee,\widehat{\bigotimes_{i=1}^{r}}\sigma_i^\vee\right)\cong\widehat{\bigotimes_{i=1}^{r}}\mathrm{Hom}_{K}^{\mathrm{cont}}(\pi_i^\vee,\sigma_i^\vee).$$
This proves (i) using Lemma 3.7; (ii) is proved in a similar way using Proposition 3.9 (i).

(iii) One checks that $\widetilde{\sigma} = \bigotimes_{i=1}^{r} \widetilde{\sigma}_i$, where $\widetilde{\sigma}_i$ is the direct sum of all irreducible σ_i such that $\mathrm{Hom}_K^{\mathrm{cont}}(P_{\pi_i^\vee}, \sigma_i^\vee) \neq 0$. Hence

$$\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(P, \widetilde{\sigma}^\vee)^\vee \cong \widehat{\bigotimes_{i=1}^{r}} \mathrm{Hom}_K^{\mathrm{cont}}(P_{\pi_i^\vee}, \widetilde{\sigma}_i^\vee)^\vee$$

as R-modules. The result follows from Proposition 3.9 (ii) which says that each component $\mathrm{Hom}_K^{\mathrm{cont}}(P_{\pi_i^\vee}, \widetilde{\sigma}_i^\vee)^\vee$ is a Cohen–Macaulay $R_{\pi_i^\vee}$-module of Krull dimension 1. ■

If $M \in \mathfrak{C}^{\mathfrak{B}}$, let

$$\mathrm{m}(M) := \mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P, M)$$

which is a compact right E-module. If $\tau \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{K})$ is of finite length, let

$$P(\tau) := \mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(P, \tau^\vee)^\vee$$

which is a finitely generated left E-module.

Proposition 5.5. *Let $M \in \mathfrak{C}(\mathcal{G})$ be a coadmissible quotient of P. Then $\mathrm{m}(M) \otimes_E P$ is coadmissible.*

Proof. The proof is similar to the proof of Proposition 3.17 (i). Let Ker be the kernel of the natural morphism

$$\mathrm{ev} : \mathrm{m}(M) \otimes_E P \twoheadrightarrow M,$$

which is surjective by [26, Lemma 2.10]. Using [26, Lemma 2.9] and the projectivity of P, we have $\mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P, \mathrm{Ker}) = 0$, that is, $\pi^\vee$ does not occur in Ker.

We need to show that $\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(\mathrm{m}(M) \otimes_E P, \sigma^\vee)$ is finite-dimensional for any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{K})$. By [27, Proposition 2.4], we have an isomorphism of finitely generated E-modules

$$\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(\mathrm{m}(M) \otimes_E P, \sigma^\vee)^\vee \cong \mathrm{m}(M) \otimes_E P(\sigma).$$

Hence, it suffices to consider those σ such that $P(\sigma) \neq 0$, or equivalently such that $\mathrm{Hom}_K^{\mathrm{cont}}(P_{\pi_i^\vee}, \sigma_i^\vee) \neq 0$ for all i if we write $\sigma = \bigotimes_{i=1}^{r} \sigma_i$. Note that this implies automatically $\sigma_i \in \mathcal{D}(\pi_i)$ if π_i is supersingular, i.e. $\mathfrak{B}_i$ is of type (I). We choose another weight $\sigma' = \bigotimes_{i=1}^{r} \sigma_i'$ as follows: if $\mathfrak{B}_i$ is of type (I), let $\sigma_i' := \sigma_i$; otherwise, let σ_i' be any weight in $\mathcal{D}(\pi_i)$. Then the assumption of Lemma 5.4 (ii) is satisfied and we obtain an isomorphism of E-modules $P(\sigma) \cong P(\sigma')$. Moreover, we have $\sigma' \in \mathcal{D}(\pi)$ by construction, hence $\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(\mathrm{Ker}, \sigma'^\vee) = 0$ by Lemma 5.4 (i) because $\pi^\vee$ does not occur as a subquotient in Ker. As in the proof of Proposition 3.17 (i), we obtain isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(\mathrm{m}(M) \otimes_E P, \sigma^\vee)^\vee &\cong \mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(\mathrm{m}(M) \otimes_E P, \sigma'^\vee)^\vee \\ &\cong \mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(M, \sigma'^\vee)^\vee \end{aligned}$$

from which the result follows. ■

Proposition 5.6. *Let m be a finitely generated right E-module such that $\mathrm{m} \otimes_E P$ is coadmissible. Then $\mathrm{m} \otimes_R P$ is also coadmissible.*

Proof. As in the proof of Proposition 5.5, we need to show that $\mathrm{m} \otimes_R P(\sigma)$ is finite-dimensional for any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{K})$. Let $I_\sigma \subset R$ (resp. $J_\sigma \subset E$) be the annihilator of $P(\sigma)$. Then we have an isomorphism $E/J_\sigma \cong P(\sigma)$, see Lemma 5.4 (ii). Since $P(\sigma)$ is finitely generated over R, a standard argument in commutative algebra shows that the Krull dimension of $\mathrm{m} \otimes_R P(\sigma)$ is equal to that of $\mathrm{m} \otimes_R R/I_\sigma$. Letting $J'_\sigma := I_\sigma E$ which is a two-sided ideal of E contained in J_σ, there is an isomorphism

$$\mathrm{m} \otimes_R R/I_\sigma \cong \mathrm{m} \otimes_E E/J'_\sigma,$$

so that we are left to compare E/J'_σ and E/J_σ. Since $\mathrm{m} \otimes_E E/J_\sigma$ is finite-dimensional over $\mathbb{F}$ by assumption, it suffices to show that $J_\sigma^n \subset J'_\sigma$ for some $n \geq 1$, because then

$$\delta_\Lambda(\mathrm{m} \otimes_E E/J_\sigma^n) \geq \delta_\Lambda(\mathrm{m} \otimes_E E/J'_\sigma) \geq \delta_\Lambda(\mathrm{m} \otimes_E E/J_\sigma),$$

and consequently (as E is Noetherian)

$$\delta_\Lambda(\mathrm{m} \otimes_E E/J_\sigma) = \delta_\Lambda(\mathrm{m} \otimes_E E/J'_\sigma).$$

Recall that E and R only differ at the indices i where $\mathfrak{B}_i$ is of type (III), in which case $J_{\sigma_i}^4 \subset J'_{\sigma_i}$ by Proposition 3.19 (iii). The result follows. ∎

5.1.3. Dimension formula. We introduce another ring which lies between R and E: let

$$E' := \Big(\widehat{\bigotimes_{i \notin \mathrm{IV}}} E_{\pi_i^\vee}\Big) \widehat{\otimes} \Big(\widehat{\bigotimes_{i \in \mathrm{IV}}} R'_{\pi_i^\vee}\Big),$$

where the subscript $i \in \mathrm{IV}$ (resp. $i \notin \mathrm{IV}$) indicates that $\mathfrak{B}_i$ is (resp. not) of type (IV), and $R'_{\pi_i^\vee}$ is one of the subrings of $E_{\pi_i^\vee}$ constructed in Proposition 3.32 (e.g. the one constructed in Remark 3.33). By Theorem 3.4 and Proposition 3.32, E is finite free over E' and P is flat over E'.

Lemma 5.7. *Let* m *be a finitely generated right* E*-module and assume that* $\mathrm{m} \otimes_E P$ *is coadmissible. Then* $\mathrm{m} \otimes_{E'} P$ *is also coadmissible and* $\delta_\Lambda(\mathrm{m} \otimes_{E'} P) = \delta_\Lambda(\mathrm{m} \otimes_E P)$.

Proof. The first assertion can be deduced from Proposition 5.6; the proof below gives another proof.

We claim that there exists an exact sequence of (E, E)-bimodules for some $n \geq 1$,

$$E^{\oplus n} \to E \otimes_{E'} E \to E \to 0, \tag{5.1}$$

where the second morphism sends $x \otimes y$ to xy. In fact, by the definition of E', we only need to construct such an exact sequence for E (resp. E') replaced by $\widehat{\bigotimes}_{i \in \mathrm{IV}} E_{\pi_i^\vee}$ (resp. $\widehat{\bigotimes}_{i \in \mathrm{IV}} R'_{\pi_i^\vee}$), and the sequence (5.1) can be obtained by tensoring it with $\widehat{\bigotimes}_{i \notin \mathrm{IV}} E_{\pi_i^\vee}$.

But since $\widehat{\bigotimes}_{i \in \mathrm{IV}} E_{\pi_i^\vee}$ and $\widehat{\bigotimes}_{i \in \mathrm{IV}} R'_{\pi_i^\vee}$ are commutative Noetherian rings, the claim is standard.

We have a natural isomorphism

$$\mathrm{m} \otimes_{E'} P \cong \mathrm{m} \otimes_E (E \otimes_{E'} E) \otimes_E P,$$

which together with (5.1) gives an exact sequence

$$(\mathrm{m} \otimes_E P)^{\oplus n} \to \mathrm{m} \otimes_{E'} P \to \mathrm{m} \otimes_E P \to 0.$$

The result follows from this. ∎

We prove the following dimension formula.

Proposition 5.8. *Let* m *be a non-zero finitely generated (right) E-module and assume that* $\mathrm{m} \otimes_E P$ *is coadmissible. Then we have an equality*

$$\delta_\Lambda(\mathrm{m} \otimes_E P) = \dim_R \mathrm{m} + r,$$

where $\dim_R \mathrm{m}$ *denotes the Krull dimension of* m *as an R-module.*

Proof. Let $d := \dim_R \mathrm{m}$. We first prove (for possibly zero m)

$$\delta_\Lambda(\mathrm{m} \otimes_E P) \le d + r. \tag{5.2}$$

Indeed, we may find a system of parameters (in the maximal ideal of R) for m, say $a_1, \dots, a_d$, so that $\dim_{\mathbb{F}} \mathrm{m}/(a_1, \dots, a_d)$ is finite. Thus, $\mathrm{m}/(a_1, \dots, a_d) \otimes_E P$ has finite length in $\mathfrak{C}(\mathcal{G})$ by Lemma 5.2 (iii), and has canonical dimension $\le r$ by Lemma 5.3. As $\mathrm{m}/(a_1, \dots, a_d) \otimes_E P \cong (\mathrm{m} \otimes_E P)/(a_1, \dots, a_d)$, we deduce inequality (5.2) from Proposition 2.3.

Assuming m non-zero, we prove the inequality $\delta_\Lambda(\mathrm{m} \otimes_E P) \ge d + r$ by induction on d. By Lemma 5.7, we are reduced to prove

$$\delta_\Lambda(\mathrm{m} \otimes_{E'} P) \ge \dim_R \mathrm{m} + r.$$

The advantage to work with E' is that P is flat over E'. If $d = 0$ (but m is non-zero), then m is finite-dimensional over $\mathbb{F}$, and the result is a consequence of Lemma 5.2 (iii) and Lemma 5.7. Assume $d \ge 1$ and the statement is true for all (non-zero) E-modules m' with $\dim_R \mathrm{m}' \le d - 1$. Since m is finitely generated over R, we may choose $x \in R$ such that

$$\dim_R(\mathrm{m}/x\mathrm{m}) = d - 1;$$

this implies that x is not nilpotent on m and $\dim_R(x^k\mathrm{m}/x^{k+1}\mathrm{m}) = d - 1$ for any $k \ge 0$ (otherwise we would have $\dim_R(x^k\mathrm{m}/x^{k+1}\mathrm{m}) \le d - 2$, hence $\dim_R(x^k\mathrm{m}) \le d - 1$ and also $\dim_R \mathrm{m} \le d - 1$). Moreover, since $x^k\mathrm{m}$ is non-zero, $x^k\mathrm{m}/x^{k+1}\mathrm{m}$ is also non-zero by Nakayama's lemma. The inductive hypothesis then implies

$$\delta_\Lambda\big((x^k\mathrm{m}/x^{k+1}\mathrm{m}) \otimes_{E'} P\big) \ge (d-1) + r.$$

Since P is flat over E', we have an isomorphism

$$(x^k\mathrm{m}/x^{k+1}\mathrm{m}) \otimes_{E'} P \cong x^k(\mathrm{m} \otimes_{E'} P)/x^{k+1}(\mathrm{m} \otimes_{E'} P),$$

and we conclude by Proposition 2.3 (ii) applied to $M = \mathrm{m} \otimes_{E'} P$. ∎

Corollary 5.9. *Let* m *be a finitely generated right E-module and assume that* $\mathrm{m} \otimes_E P$ *is coadmissible. Then* $\delta_\Lambda(\mathrm{m} \otimes_R P) = \delta_\Lambda(\mathrm{m} \otimes_E P)$.

Proof. We know that $\mathfrak{m} \otimes_R P$ is coadmissible by Proposition 5.6 and it is clear that $\delta_\Lambda(\mathfrak{m} \otimes_R P) \geq \delta_\Lambda(\mathfrak{m} \otimes_E P)$. The first part of the proof of Proposition 5.8 still works if we replace E by R, showing

$$\delta_\Lambda(\mathfrak{m} \otimes_R P) \leq \dim_R \mathfrak{m} + r.$$

The result then follows from Proposition 5.8. ∎

Corollary 5.10. *Let M be a coadmissible quotient of P. Then* $\dim_R \mathfrak{m}(M) \leq 2r$.

Proof. By Proposition 5.5, $\mathfrak{m}(M) \otimes_E P$ is also coadmissible, so

$$\delta_\Lambda(\mathfrak{m}(M) \otimes_E P) \leq \dim \Lambda = 3r.$$

The result then follows from Proposition 5.8. ∎

Proposition 5.11. *Let M be a coadmissible quotient of P. Then there exists a sequence $f_1, \dots, f_r \in \mathrm{Ann}_R(M)$ which is both R-regular and P-regular, such that $P/(f_1, \dots, f_r)$ is finite free over Λ.*

Proof. Since M is coadmissible, so is $\mathfrak{m}(M) \otimes_R P$ by Propositions 5.5 and 5.6. Moreover, as in these propositions, this implies that $\mathfrak{m}(M) \otimes_R P(\widetilde{\sigma})$ is finite-dimensional over $\mathbb{F}$, where $\widetilde{\sigma} = \bigoplus_\sigma \sigma$ is the sum of all irreducible objects $\pi \in \mathrm{Rep}_{\mathbb{F}}(K)$ such that $P(\sigma) \neq 0$. Because both $\mathfrak{m}(M)$ and $P(\widetilde{\sigma})$ are finitely generated R-modules, we deduce that $R/(\mathrm{Ann}_R(\mathfrak{m}(M))+\mathrm{Ann}_R(P(\widetilde{\sigma})))$ is an Artinian ring. Writing $\mathfrak{a} = \mathrm{Ann}_R(\mathfrak{m}(M))$ (which coincides with $\mathrm{Ann}_R(M)$ by Lemma 5.19 below), then $R/\mathfrak{a} \otimes_R P(\widetilde{\sigma})$ is also finite-dimensional over $\mathbb{F}$. Since $P(\widetilde{\sigma})$ is a Cohen–Macaulay R-module of Krull dimension r, we can find a regular sequence $f_1, \dots, f_r$ in $\mathfrak{a}$ for $P(\widetilde{\sigma})$, see [7, Theorem 2.1.2 (b)].

As a generalization of [10], it is proved in Corollary 5.18 below that $P|_{\mathcal{K}}$ remains projective in $\mathfrak{C}(\mathcal{K})$. Applying repeatedly Proposition 3.10, we obtain that $f_1, \dots, f_r$ is a regular sequence for P and $P/(f_1, \dots, f_r)$ is again projective in $\mathfrak{C}(\mathcal{K})$. Moreover, $\mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(P/(f_1, \dots, f_r), \widetilde{\sigma}^\vee)$ is finite-dimensional over $\mathbb{F}$, so $P/(f_1, \dots, f_r)$ is coadmissible by the choice of $\widetilde{\sigma}$.

We are left to check that $f_1, \dots, f_r$ is an R-regular sequence. Since the sequence is P-regular, it suffices to prove that $R/(f_1, \dots, f_i)$ acts faithfully on $P/(f_1, \dots, f_i)$ for all $1 \leq i \leq r$. For this, it suffices to prove that there is a natural isomorphism

$$E/(f_1, \dots, f_i) \cong \mathrm{End}_{\mathfrak{C}(\mathcal{G})}(P/(f_1, \dots, f_i)). \tag{5.3}$$

In fact, since $f_1, \dots, f_r$ lie in the center of E, the proof of [14, Lemma 7.11] shows that any morphism

$$P \to P/(f_1, \dots, f_{i-1})$$

factors through

$$P/(f_1, \dots, f_{i-1}) \to P/(f_1, \dots, f_{i-1}).$$

Since P is projective, the exact sequence

$$0 \longrightarrow P/(f_1, \dots, f_{i-1}) \xrightarrow{f_i} P/(f_1, \dots, f_{i-1}) \longrightarrow P/(f_1, \dots, f_i) \longrightarrow 0$$

induces an isomorphism

$$\mathrm{End}_{\mathfrak{C}(\mathcal{G})}(P/(f_1,\dots,f_{i-1}))/f_i \cong \mathrm{End}_{\mathfrak{C}(\mathcal{G})}(P/(f_1,\dots,f_i)).$$

An obvious induction then gives (5.3). ∎

5.1.4. Torsion Λ-modules. If $\eta \in \mathrm{Rep}_{\mathbb{F}}(T)$ is a smooth character, recall from Section 3.4 that $M_{\eta^\vee} = (\mathrm{Ind}_B^G \mathrm{Inj}_T \eta)^\vee$ and $E_{\eta^\vee} = \mathrm{End}_{\mathfrak{C}(G)}(M_{\eta^\vee})$.

Definition 5.12. Let Σ be a subset of $\{1,\dots,r\}$. Given a smooth character $\eta_i \in \mathrm{Rep}_{\mathbb{F}}(T)$ for each $i \in \Sigma$ and an irreducible $\pi_i \in \mathrm{Rep}_{\mathbb{F}}(G)$ for each $i \notin \Sigma$, we define $P(\underline{\eta},\underline{\pi},\Sigma) \in \mathfrak{C}(\mathcal{G})$ by

$$P(\underline{\eta},\underline{\pi},\Sigma) := \Big(\widehat{\bigotimes_{i\in\Sigma}} M_{\eta_i^\vee}\Big) \widehat{\otimes} \Big(\widehat{\bigotimes_{i\notin\Sigma}} P_{\pi_i^\vee}\Big).$$

Denote by E_Σ the endomorphism ring $\mathrm{End}_{\mathfrak{C}(\mathcal{G})}(P(\underline{\eta},\underline{\pi},\Sigma))$; then

$$E_\Sigma \cong \Big(\widehat{\bigotimes_{i\in\Sigma}} E_{\eta_i^\vee}\Big) \widehat{\otimes} \Big(\widehat{\bigotimes_{i\notin\Sigma}} E_{\pi_i^\vee}\Big)$$

is a quotient of $E = \widehat{\bigotimes}_{i=1}^r E_{\pi_i^\vee}$, where $\pi_i := \mathrm{soc}_G\, \pi_{\eta_i}$ if $i \in \Sigma$. We employ the rings constructed in Proposition 3.32 once again by setting

$$R_\Sigma := \Big(\widehat{\bigotimes_{i\in\Sigma}} E_{\eta_i^\vee}\Big) \widehat{\otimes} \Big(\widehat{\bigotimes_{i\notin\Sigma,\, i\notin \mathrm{IV}}} R_{\pi_i^\vee}\Big) \widehat{\otimes} \Big(\widehat{\bigotimes_{i\notin\Sigma,\, i\in \mathrm{IV}}} R'_{\pi_i^\vee}\Big), \tag{5.4}$$

where the notation is as in Section 5.1.3. Then R_Σ is a (commutative) regular local ring of Krull dimension $3r - |\Sigma|$. Clearly, R_Σ is contained in (the center of) E_Σ and E_Σ is finite free over R_Σ.

Let $\sigma \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{K})$ be an irreducible representation with $P(\sigma) \neq 0$. Corollary 3.15 implies an isomorphism

$$P(\sigma) \cong \mathrm{Hom}_{\mathcal{K}}^{\mathrm{cont}}(P(\underline{\eta},\underline{\pi},\Sigma), \sigma^\vee)^\vee,$$

so $P(\sigma)$ can be viewed as a module over E_Σ and R_Σ.

Lemma 5.13. *With the above notation,* $\mathrm{Ann}_{R_\Sigma}(P(\sigma))$ *can be generated by* $2r - |\Sigma|$ *elements.*

Proof. To simplify and uniform the notation, we write R_i for the ring at index i in (5.4) so that

$$R_\Sigma = \widehat{\bigotimes_{i=1}^{r}} R_i;$$

similarly write

$$E_\Sigma = \widehat{\bigotimes_{i=1}^{r}} E_i.$$

By Lemma 5.4 (ii) together with Corollary 3.15, $P(\sigma)$ is a cyclic E_Σ-module such that

$$E_\Sigma/\mathrm{Ann}_{E_\Sigma}(P(\sigma)) \cong \widehat{\bigotimes_{i=1}^{r}} \mathbb{F}[[S]] \cong \mathbb{F}[[S_1,\dots,S_r]].$$

For each i, the composite morphism $R_i \to E_i \to \mathbb{F}[[S]]$ is either surjective or equal to $\mathbb{F}[[S^2]]$; this is an easy check if $i \in \Sigma$ or $i \notin \mathrm{III}$, and follows from Proposition 3.19 (ii) if $i \in \mathrm{III}$ (and $i \notin \Sigma$). In all, the image of $R_\Sigma \hookrightarrow E_\Sigma \twoheadrightarrow \widehat{\bigotimes}_{i=1}^{r}\mathbb{F}[[S]]$ is a regular local ring of Krull dimension r. The result then follows from [20, Theorem 21.2 (ii)]. ∎

Lemma 5.14. *Let M be a coadmissible quotient of $P(\underline{\eta},\underline{\pi},\Sigma)$ and assume Σ is nonempty. Then M is a torsion Λ-module. In fact, $\delta_\Lambda(M) \le 3r - |\Sigma|$.*

Proof. The action of E on $\mathrm{m}(M)$ factors through E_Σ by [26, Proposition 7.1(iii)], hence $\mathrm{m}(M)$ can be viewed as an R_Σ-module. Since M is coadmissible, $\mathrm{m}(M) \otimes_{R_\Sigma} P(\sigma)$ is finite-dimensional, where σ is as before. As a consequence,

$$R_\Sigma/\big(\mathrm{Ann}_{R_\Sigma}(\mathrm{m}(M)) + \mathrm{Ann}_{R_\Sigma}(P(\sigma))\big)$$

is an Artinian ring. As in Proposition 5.11, we can find a sequence

$$f_1,\dots,f_r \in \mathrm{Ann}_{R_\Sigma}(\mathrm{m}(M))$$

which is regular for $P(\sigma)$. By Lemma 5.13, $\mathrm{Ann}_{R_\Sigma}(P(\sigma))$ is generated by $2r-|\Sigma|$ elements, say $g_1,\dots,g_{2r-|\Sigma|}$. Since R_Σ is Cohen–Macaulay of Krull dimension $3r-|\Sigma|$, the sequence $f_1,\dots,f_r,g_1,\dots,g_{2r-|\Sigma|}$ is R_Σ-regular. As a consequence,

$$\dim R_\Sigma/(f_1,\dots,f_r) = 2r - |\Sigma|.$$

But, $\mathrm{m}(M)$ is annihilated by $(f_1,\dots,f_r)$, hence

$$\dim \mathrm{m}(M) \le \dim R_\Sigma/(f_1,\dots,f_r) = 2r - |\Sigma|.$$

We conclude by Proposition 5.8. ∎

Proposition 5.15. *Let $M \in \mathfrak{C}(\mathcal{G})$ be a coadmissible quotient of P. If M is a torsion Λ-module, then so is $\mathrm{m}(M) \otimes_E P$.*

Proof. As in the proof of Proposition 3.17 (ii), it is enough to show the following:

Claim. *For i with $1 \le i \le r$ and $\pi'_i \in \mathfrak{B}_i$ distinct with π_i, let Q'_i be the maximal quotient of $P_{\pi_i'^\vee}$ none of whose irreducible subquotients is isomorphic to $\pi_i^\vee$. If M is a coadmissible quotient of*

$$Q'_i \mathbin{\widehat{\otimes}} \Big(\widehat{\bigotimes_{j\neq i}} P_{\pi_j'^\vee}\Big),$$

then M is a torsion Λ-module.

As in Proposition 3.17 (ii), we know that $\mathfrak{B}_i$ can only be of type (II) or (IV), and we may assume that M is a coadmissible quotient of $M_{\eta_i^\vee} \mathbin{\widehat{\otimes}} (\widehat{\bigotimes}_{j\neq i} P_{\pi_j'^\vee})$. The claim is then a special case of Lemma 5.14. ∎

Corollary 5.16. *Let $M \in \mathfrak{C}(\mathcal{G})$ be a coadmissible quotient of P. The following statements hold:*

(i) $\mathrm{m}(M)$ *has Krull dimension* $\leq 2r$,

(ii) M *is a torsion* Λ*-module if and only if* $\dim_R \mathrm{m}(M) \leq 2r - 1$.

Proof. (i) Since M is coadmissible, so is $\mathrm{m}(M) \otimes_E P$ by Proposition 5.5. Since we always have $\delta_\Lambda(M) \leq \delta_\Lambda(\mathrm{m}(M) \otimes_E P) \leq 3r$, the result follows from Proposition 5.8.

(ii) We have that M is torsion if and only if $\mathrm{m}(M) \otimes_E P$ is torsion by Proposition 5.15, if and only if $\delta_\Lambda(\mathrm{m}(M) \otimes_E P) \leq 3r - 1$, if and only if $\dim_R \mathrm{m}(M) \leq 2r - 1$ by Proposition 5.8. ∎

5.1.5. Breuil–Paškūnas construction. In this subsection we generalize the construction of Breuil and Paškūnas [6, Section 9] to our setting.

Proposition 5.17. *Let $M \in \mathfrak{C}(\mathcal{G})$ be coadmissible and let $\widetilde{\sigma}$ be the $\mathcal{K}$-cosocle of M. Then there exists a surjection in $\mathfrak{C}(\mathcal{G})$,*

$$\overline{P} \twoheadrightarrow M,$$

where $\overline{P}|_{\mathcal{K}}$ is isomorphic to a projective envelope of $\widetilde{\sigma}$ (with central character). In particular, $\overline{P}$ is finite free as a Λ-module.

Proof. The proof is given in Theorem A.2. ∎

The following result generalizes [10, Corollary 3.8].

Corollary 5.18. *If $\widetilde{P} \in \mathfrak{C}(\mathcal{G})$ is projective, then $\widetilde{P}$ is also projective in $\mathfrak{C}(\mathcal{K})$.*

Proof. The proof is identical to that of [10, Corollary 3.8], using Proposition 5.17 in place of [10, Theorem 3.4]. ∎

5.1.6. Finite free modules. If M is a quotient of $P = P_{\pi^\vee}$, then R acts (from left) on M and (from right) on $\mathrm{m}(M) := \mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P, M)$. We make explicit these actions. Let $\phi \in R$ and view it as an endomorphism $\phi : P \to P$. It induces an endomorphism $\overline{\phi} : M \to M$ (because R is contained in the Bernstein center of $\mathfrak{C}(\mathcal{G})^{\mathfrak{B}}$), and for any $\theta \in \mathrm{m}(M)$ the following diagram is commutative:

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\theta"'] & P \arrow[d, "\theta"] \\ M \arrow[r, "\overline{\phi}"] & M. \end{tikzcd} \tag{5.5}$$

The action of R on M is given by $(\phi, m) \mapsto \overline{\phi}(m)$, and the action on $\mathrm{m}(M)$ is given by

$$(\theta, \phi) \mapsto \theta \circ \phi = \overline{\phi} \circ \theta. \tag{5.6}$$

We have the following simple lemma.

Lemma 5.19. *We have* $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(\mathrm{m}(M))$.

Proof. Given $\phi \in R$, we have the following equivalences

$$\begin{aligned}\phi \in \mathrm{Ann}_R(M) &\iff \overline{\phi} = 0\\ &\overset{(*)}{\iff} \overline{\phi} \circ \theta = 0 \text{ for all } \theta \in \mathrm{m}(M)\\ &\overset{(5.5)}{\iff} \theta \circ \phi = 0 \text{ for all } \theta \in \mathrm{m}(M)\\ &\overset{(5.6)}{\iff} \phi \in \mathrm{Ann}_R(\mathrm{m}(M)),\end{aligned}$$

where $(*)$ holds because there exists at least one θ which is surjective, e.g. the natural quotient map $P \twoheadrightarrow M$. ∎

Let $\overline{P}$ be a quotient of P which is finite free as a Λ-module. Set

$$\overline{R} := R/\mathrm{Ann}_R(\mathrm{m}(\overline{P})) = R/\,\mathrm{Ann}_R(\overline{P}).$$

Proposition 5.20. *The following statements hold:*

(i) *$\overline{R}$ has Krull dimension $2r$. There exists a subring $A \subset \overline{R}$ which is formally smooth of dimension $2r$ such that $\overline{R}$ is finite over A.*

(ii) *$\overline{P}$ is flat over A.*

Proof. (i) The first assertion is a direct consequence of Corollary 5.16. The second one is proved as in [28, Corollary 4.2] by applying Cohen's structure theorem for Noetherian complete local rings (see [20, Theorem 29.4 (iii)]).

(ii) Because $\overline{P}$ is Cohen–Macaulay with $\delta_\Lambda(\overline{P}) = 3r$ (being finite free over Λ) and $\delta_\Lambda(\mathbb{F} \otimes_A \overline{P}) = \delta_\Lambda(\mathbb{F} \otimes_{\overline{R}} \overline{P}) = r$, the result follows from the "miracle flatness" criterion, see [13, Proposition A.30]. ∎

5.2. A lemma

In this subsection, we prove a lemma which can be viewed as an analogue of Proposition 2.3. These two results will allow us to relate the canonical dimension of a coadmissible module $M \in \mathfrak{C}(\mathcal{G})$ and the $\mathbb{F}$-dimension of $\mathcal{T}_1(p^{\mathbf{n}})$-coinvariants of M. It is where the quantity $\kappa(\mathbf{n}) = \max_i\{n_i\}$ comes.

Lemma 5.21. *Let M be a finitely generated Λ-module and $\phi \in \mathrm{End}_\Lambda(M)$. Assume that $\bigcap_{k\geq 1} \phi^k(M) = 0$.*

(i) *If ϕ is nilpotent, then $\dim_{\mathbb{F}} M_{\mathcal{T}_1(p^{\mathbf{n}})} \sim \dim_{\mathbb{F}}(M/\phi(M))_{\mathcal{T}_1(p^{\mathbf{n}})}$.*

(ii) *If ϕ is not nilpotent, then for some sufficiently large k_0 we have*

$$\dim_{\mathbb{F}} M_{\mathcal{T}_1(p^{\mathbf{n}})} \ll \dim_{\mathbb{F}}(M/\phi(M))_{\mathcal{T}_1(p^{\mathbf{n}})} + p^{\kappa(\mathbf{n})} \dim_{\mathbb{F}}(\phi^{k_0}(M)/\phi^{k_0+1}(M))_{\mathcal{T}_1(p^{\mathbf{n}})},$$

where $\kappa(\mathbf{n}) := \max_i\{n_i\}$.

In any case, we have

$$\dim_{\mathbb{F}} M_{\mathcal{T}_1(p^{\mathbf{n}})} \ll p^{\kappa(\mathbf{n})} \dim_{\mathbb{F}}(M/\phi(M))_{\mathcal{T}_1(p^{\mathbf{n}})}.$$

Proof. (i) We trivially have $\dim_{\mathbb{F}} M_{\mathfrak{I}_1(p^{\mathbf{n}})} \geq \dim_{\mathbb{F}}(M/\phi(M))_{\mathfrak{I}_1(p^{\mathbf{n}})}$. If ϕ is nilpotent, then M admits a finite filtration by $\phi^k(M)$, for $k \leq k_0$ where $k_0 \gg 1$ is such that $\phi^{k_0} = 0$. For any $k \geq 1$, ϕ^k induces a surjective morphism $M/\phi(M) \twoheadrightarrow \phi^k(M)/\phi^{k+1}(M)$, hence $\dim_{\mathbb{F}} M_{\mathfrak{I}_1(p^{\mathbf{n}})} \leq k_0 \cdot \dim_{\mathbb{F}}(M/\phi(M))_{\mathfrak{I}_1(p^{\mathbf{n}})}$, giving the result.

(ii) By Lemma 2.5, there exists $k_0 \gg 0$ such that $\phi : \phi^{k_0}(M) \to \phi^{k_0}(M)$ is injective. Using the short exact sequence

$$0 \to \phi^{k_0}(M) \to M \to M/\phi^{k_0}(M) \to 0$$

and applying (i) to $M/\phi^{k_0}(M)$, we are reduced to prove

$$\dim_{\mathbb{F}} \phi^{k_0}(M)_{\mathfrak{I}_1(p^{\mathbf{n}})} \ll p^{\kappa(\mathbf{n})} \dim_{\mathbb{F}}(\phi^{k_0}(M)/\phi^{k_0+1}(M))_{\mathfrak{I}_1(p^{\mathbf{n}})}.$$

That is, by replacing M by $\phi^{k_0}(M)$, we may assume ϕ is injective in the rest of the proof.

Set $Q := M/\phi(M)$ so that we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{\phi} M \longrightarrow Q \longrightarrow 0.$$

Let J denote the maximal ideal of Λ. By Remark 2.4, we may choose $k_1 \gg 1$ such that $\phi^{k_1}(M) \subset JM$. Replacing ϕ by ϕ^{k_1} and Q by $M/\phi^{k_1}(M)$, we may assume that $\phi(M) \subset JM$. Since ϕ is a Λ-morphism, we obtain inductively

$$\phi^k(J^s M) \subset J^{k+s} M \quad \text{for all } k, s \geq 1. \tag{5.7}$$

Letting $Q_k := M/\phi^k(M)$, the short exact sequence

$$0 \longrightarrow M \xrightarrow{\phi^k} M \longrightarrow Q_k \longrightarrow 0$$

then induces by modulo J^{k+1}:

$$M/JM \xrightarrow{\phi^k} M/J^{k+1}M \longrightarrow Q_k/J^{k+1}Q_k \longrightarrow 0,$$

where we have used the fact $\phi^k(JM) \subset J^{k+1}M$ by (5.7). If I is a right ideal of Λ containing J^{k+1} (and contained in J), then, by tensoring the above sequence with $(\Lambda/I)\otimes_\Lambda$, we obtain an exact sequence of $\mathbb{F}$-vector spaces:

$$M/JM \to M/IM \to Q_k/IQ_k \to 0.$$

Since Q_k is a successive extension of Q (k times), we obtain the following inequalities with $c_0 := \dim_{\mathbb{F}} M/JM$:

$$\dim_{\mathbb{F}} M/IM \leq \dim_{\mathbb{F}} Q_k/IQ_k + c_0 \leq k \dim_{\mathbb{F}} Q/IQ + c_0. \tag{5.8}$$

We specialize (5.8) to our situation. Let $I_{\mathbf{n}}$ denote the *right* ideal of Λ generated by the maximal ideal of $\mathbb{F}[\![\mathfrak{I}_1(p^{\mathbf{n}})/\mathfrak{Z}_1]\!]$. Then Lemma 5.22 below shows that $J^{3rp^{\kappa(\mathbf{n})-1}}$ is contained in $I_{\mathbf{n}}$ because $I_{\mathbf{n}}$ clearly contains $J_{\mathbf{n}}\Lambda$. Applying inequality (5.8) to $I = I_{\mathbf{n}}$ and $k = 3rp^{\kappa(\mathbf{n})-1} - 1$, we obtain

$$\dim_{\mathbb{F}} M_{\mathfrak{I}_1(p^{\mathbf{n}})} = \dim_{\mathbb{F}} M/I_{\mathbf{n}}M \leq (3rp^{\kappa(\mathbf{n})-1} - 1) \cdot \dim_{\mathbb{F}} Q/I_{\mathbf{n}}Q + c_0,$$

giving the result. ∎

Lemma 5.22. *Let J be the maximal ideal of Λ and let $J_{\mathbf{n}}$ be the maximal ideal of $\mathbb{F}[\![\mathcal{K}_{\mathbf{n}}/(\mathcal{K}_{\mathbf{n}} \cap \mathcal{Z}_1)]\!]$ (viewed as a sub-algebra of Λ). Then the right ideal $J_{\mathbf{n}}\Lambda$ is a two-sided ideal of Λ and satisfies $J^{3rp^{\kappa(\mathbf{n})-1}} \subset J_{\mathbf{n}}\Lambda$.*

Proof. The first assertion follows from that $\mathcal{K}_{\mathbf{n}}$ is a normal subgroup of $\mathcal{K}_1$.

We are left to prove the inclusion

$$J^{3rp^{\kappa(\mathbf{n})-1}} \subset J_{\mathbf{n}}\Lambda.$$

We first consider the case $r = 1$. Then $\Lambda = \mathbb{F}[\![K_1/Z_1]\!]$ is topologically generated by three elements, say z_1, z_2, z_3, such that every element of Λ can be uniquely expressed as a sum over multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$:

$$x = \sum_{\alpha} \lambda_\alpha z^\alpha, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}.$$

Moreover, $z^\alpha z^\beta = z^{\alpha+\beta}$ up to terms of degree $> |\alpha| + |\beta|$, which we refer to as the almost commutativity of Λ; see [19, Theorem 10]. The ideal J is simply topologically spanned by the set of monomials z^α with $|\alpha| > 0$. Similarly, $\mathbb{F}[\![K_n/(K_n \cap Z_1)]\!]$ is topologically generated by

$$z_1^{p^{n-1}}, z_2^{p^{n-1}}, z_3^{p^{n-1}},$$

and J_n is topologically spanned by the set of monomials $z^{p^{n-1}\cdot\alpha}$ with $|\alpha| > 0$. Hence, the ideal $J_n\Lambda$ is topologically spanned by monomials z^α with at least one of α_j greater than or equal to p^{n-1} (cf. the proof of [19, Lemma 12]). By the almost commutativity, we deduce the inclusion $J^{3p^{n-1}} \subset J_n\Lambda$. For general r, the proof is identical noting that J is topologically generated by $3r$ elements

$$\{z_{i,j} : 1 \leq i \leq r, 1 \leq j \leq 3\},$$

and $J_{\mathbf{n}}$ is topologically generated by $z_{i,j}^{p^{n_i-1}}$, hence

$$J^{\sum_{i=1}^r 3p^{n_i-1}} \subset J_{\mathbf{n}}\Lambda$$

which in particular implies the result. ∎

Remark 5.23. In the proof of Lemma 5.21, it is crucial that we are working with $T_1(p^n)$ instead of $K_1(p^{2n})$ (this group is defined in (5.12) below), although they are (up to finite order) conjugate to each other in $\mathrm{GL}_2(\mathbb{Q}_p)$. We have learnt this trick of "averaging" from [19] (used in a different manner there).

5.3. Main result

In this subsection, we prove the following theorem.

Theorem 5.24. *Let $M \in \mathfrak{C}^{\mathrm{fg,tor}}(\mathcal{G})$. Then for any $i \geq 0$,*

$$\dim_{\mathbb{F}} H_i(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M) \ll \kappa(\mathbf{n})^r p^{(2r-1)\kappa(\mathbf{n})}, \tag{5.9}$$

where $\kappa(\mathbf{n}) := \max_i\{n_i\}$.

Lemma 5.25. *In Theorem 5.24, we may assume that M has an irreducible $\mathcal{G}$-cosocle (hence indecomposable).*

Proof. Let S be the $\mathcal{G}$-cosocle of M. Since M is coadmissible, it follows that S decomposes as a finite direct sum $\bigoplus_{i=1}^{s} S_i$ with each S_i irreducible. For each i, let P_{S_i} be a projective envelope of S_i in $\mathfrak{C}(\mathcal{G})$. The projection $M \twoheadrightarrow S_1$ extends to a $\mathfrak{C}(\mathcal{G})$-equivariant morphism $\alpha_1 : P_{S_1} \to M$. It is clear that $\mathrm{coker}(\alpha_1)$ has $\mathcal{G}$-cosocle isomorphic to $\bigoplus_{i=2}^{s} S_i$ and $\mathrm{Im}(\alpha_1)$ has $\mathcal{G}$-cosocle S_1. Continuing this with $\mathrm{coker}(\alpha_1)$, we get a finite filtration of M such that each graded piece, say $\mathrm{gr}^i(M)$, has an irreducible $\mathcal{G}$-cosocle. Since M is torsion as a Λ-module if and only if each $\mathrm{gr}^i(M)$ is, we are reduced to prove (5.9) for all $\mathrm{gr}^i(M)$. ■

Let M be a quotient of $P_{\pi^\vee}$ for some irreducible $\pi \in \mathfrak{C}(\mathcal{G})$. Let $P = P_{\pi^\vee}$, E, R be as before.

Definition 5.26. We say that $M \in \mathfrak{C}^{\mathrm{fg,tor}}(\mathcal{G})$ is *elementary*[8] if there exists a short exact sequence in $\mathfrak{C}(\mathcal{G})$:

$$0 \to \overline{P} \xrightarrow{a} \overline{P} \to M \to 0, \tag{5.10}$$

where $\overline{P} \in \mathfrak{C}(\mathcal{G})$ is a quotient of P and is finite free as a Λ-module, and

$$a \in \overline{R} := R/\mathrm{Ann}(\mathrm{m}(\overline{P})).$$

Lemma 5.27. *Theorem 5.24 is true if M is an elementary quotient of P.*

Proof. Let $\overline{P}$ and $a \in \overline{R}$ be as in (5.10). Since $\overline{P}$ is a free Λ-module, taking homology of (5.10) we obtain

$$\dim_{\mathbb{F}} H_0(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M) = \dim_{\mathbb{F}} H_1(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M)$$

and

$$H_i(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M) = 0 \quad \text{for all } i \geq 2.$$

Hence, it suffices to prove (5.9) when $i = 0$. Since a is $\overline{P}$-regular and $\overline{R}$ acts faithfully on $\overline{P}$ (by Lemma 5.19), a is also $\overline{R}$-regular. Since $\overline{R}$ has Krull dimension $2r$ by Proposition 5.20, we may extend a to a system of parameters of $\overline{R}$, say $a_1 = a, a_2, \ldots, a_{2r}$. Then $\overline{R}/(a_1, \ldots, a_{2r})$ is finite-dimensional over $\mathbb{F}$ and so $M/(a_2, \ldots, a_{2r})$ has finite length in $\mathfrak{C}(\mathcal{G})$. By Lemma 5.3, we deduce that

$$\dim_{\mathbb{F}} H_0(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M/(a_2, \ldots, a_{2r})) \ll \kappa(\mathbf{n})^r,$$

and we conclude by repeatedly applying Lemma 5.21 to $M/(a_2, \ldots, a_{2r-1}), \ldots, M$. ■

Remark 5.28. Let M be an elementary quotient of P and let $\overline{P}, a \in \overline{R}$ be as in (5.10). Moreover, we assume $\mathbf{n} = (n, \ldots, n)$ is parallel. Since $\overline{P}$ is finite free over Λ, we have

$$\dim_{\mathbb{F}} \overline{P}_{\mathcal{T}_1(p^{\mathbf{n}})} \sim [\mathcal{K}_1 : \mathcal{T}_1(p^{\mathbf{n}})] \sim p^{2rn}.$$

[8]The notation is motivated by the corresponding one in commutative ring theory; see for instance [17, Section 11.6].

From Lemma 5.21 we deduce that

$$\dim_{\mathbb{F}} M_{\mathcal{T}_1(p^{\mathbf{n}})} \gg p^{(2r-1)n}.$$

This shows that the upper bound (5.9) is almost optimal.

Proof of Theorem 5.24. For each $i \geq 0$, denote by (C_i) the inequality

$$\dim_{\mathbb{F}} H_i(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M) \ll \kappa(\mathbf{n})^r p^{(2r-1)\kappa(\mathbf{n})}.$$

We will prove (C_i) for any $M \in \mathfrak{C}^{\mathrm{fg,\,tor}}(\mathcal{G})$ by induction on i.

First prove (C_0). By Lemma 5.25, we may assume $M \in \mathfrak{C}^{\mathrm{fg,\,tor}}(\mathcal{G})$ is a quotient of $P = P_{\pi^\vee}$ for some irreducible $\pi \in \mathfrak{C}(\mathcal{G})$. By Proposition 5.17, we may find $\overline{P} \in \mathfrak{C}(\mathcal{G})$, which is finite free as a Λ-module and has the same $\mathcal{K}$-cosocle as M, such that M is a quotient of $\overline{P}$. In particular, $\overline{P}$ has the same $\mathcal{G}$-cosocle as M and we may view $\overline{P}$ as a quotient of P.[9] Set $\overline{R} = R/\mathrm{Ann}(\mathrm{m}(\overline{P}))$. By Proposition 5.20, $\overline{R}$ has Krull dimension $2r$ and is finite over a formally smooth subring $A \cong \mathbb{F}[\![y_1, \ldots, y_{2r}]\!]$. We view $\mathrm{m}(M)$ as an A-module. Since $\mathrm{m}(M)$ has Krull dimension $< 2r$ by Corollary 5.16, there exists a non-zero $a \in A$ which annihilates $\mathrm{m}(M)$. In particular, we obtain a surjection

$$\overline{P}/a\overline{P} \twoheadrightarrow M. \tag{5.11}$$

Proposition 5.20 (ii) shows that $\overline{P}$ is flat over A, hence a is $\overline{P}$-regular and $\overline{P}/a\overline{P}$ is an elementary module. By Lemma 5.27, (C_0) holds for $\overline{P}/a\overline{P}$, hence also holds for M.

Now we assume (C_i) holds for *any* object in $\mathfrak{C}^{\mathrm{fg,\,tor}}(\mathcal{G})$, and prove (C_{i+1}) for (the fixed) M. Let M' be the kernel of (5.11). Taking homology we obtain an exact sequence

$$H_{i+1}(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, \overline{P}/a\overline{P}) \to H_{i+1}(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M) \to H_i(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M').$$

Since $M' \in \mathfrak{C}^{\mathrm{fg,\,tor}}(\mathcal{G})$, (C_i) holds for M' by inductive hypothesis and (C_{i+1}) holds for $\overline{P}/a\overline{P}$ by Lemma 5.27, we obtain that (C_{i+1}) holds for M. ∎

5.4. *Change of groups*

We keep the notation in the previous subsection. For $n \geq 1$, let

$$K_1(p^n) := K_1 \cap K_0(p^n) = \begin{pmatrix} 1 + p\mathbb{Z}_p & p\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}. \tag{5.12}$$

These groups are closely related to the group $T_1(p^n)$ in the sense that letting

$$D = \begin{pmatrix} 1 & 0 \\ 0 & p^{\lfloor \frac{n}{2} \rfloor} \end{pmatrix}$$

and $n' = \lfloor \frac{n}{2} \rfloor + 1$, we have

$$D^{-1}K_1(p^n)D < T_1(p^{n'}), \quad [T_1(p^{n'}) : D^{-1}K_1(p^n)D] \leq p, \quad \left| n' - \frac{n}{2} \right| \leq 1. \tag{5.13}$$

[9] Alternatively, we may apply Proposition 5.11 to obtain such an object $\overline{P}$ (but without cosocle condition).

On the other hand, using (essentially) the fact that the restriction of $\mathbb{F}[K_1/K_1(p^n)]$ to $\left(\begin{smallmatrix}1 & 0\\ p\mathbb{Z}_p & 1\end{smallmatrix}\right)$ is uniserial, which follows from the Iwahori decomposition for K_1 and the fact that $\left(\begin{smallmatrix}1 & 0\\ p\mathbb{Z}_p & 1\end{smallmatrix}\right)$ is pro-cyclic, Marshall [19, Corollary 14] proved the following interesting result.

Lemma 5.29. *Let $L \subset \mathbb{F}[K_1/K_1(p^n)]$ be a submodule of dimension d, and let the base p expansion of d be written as*

$$d = \sum_{i=1}^{l} p^{\alpha(i)},$$

where $\{\alpha(i)\}$ is a non-increasing sequence of non-negative integers. Then there exists a filtration

$$0 = L_0 \subset \cdots \subset L_l = L$$

of L by submodules L_i such that $L_i/L_{i-1} \cong \mathbb{F}[K_1/K_1(p^{\alpha(i)+1})]$.

If $\mathbf{n} = (n_1, \ldots, n_r) \in (\mathbb{Z}_{\geq 1})^r$, let

$$\mathcal{K}_1(p^{\mathbf{n}}) = \prod_{i=1}^{r} K_1(p^{n_i}).$$

Theorem 5.30. *Let $M \in \mathfrak{C}^{\mathrm{fg,\,tor}}(\mathcal{G})$ and let L be any sub-representation of $\mathbb{F}[\mathcal{K}_1/\mathcal{K}_1(p^{\mathbf{n}})]$ which factorizes as $\otimes_{i=1}^{r} L_i$ with $L_i \subset \mathbb{F}[K_1/K_1(p^{n_i})]$. Then for $i \geq 0$ we have*

$$\dim_{\mathbb{F}} H_i(\mathcal{K}_1/\mathcal{Z}_1, M \otimes L) \ll \kappa(\mathbf{n})^{2r} p^{(r-\frac{1}{2})\kappa(\mathbf{n})}.$$

In particular, if $\mathbf{n} = (n, \ldots, n)$ is parallel, then

$$\dim_{\mathbb{F}} H_i(\mathcal{K}_1/\mathcal{Z}_1, M \otimes L) \ll n^{2r} p^{(r-\frac{1}{2})n}.$$

Proof. The proof goes as that of [19, Lemma 19]. For the convenience of the reader, we briefly explain it. First, if $L = \mathbb{F}[\mathcal{K}_1/\mathcal{K}_1(p^{\mathbf{m}})]$ for some $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r$, we apply Shapiro's lemma to obtain

$$H_i(\mathcal{K}_1/\mathcal{Z}_1, M \otimes L) \cong H_i(\mathcal{K}_1(p^{\mathbf{m}})/\mathcal{Z}_1, M). \tag{5.14}$$

Using a suitable diagonal element of $\mathcal{G}$, precisely

$$D = \left(\begin{pmatrix}1 & 0\\ 0 & p^{\lfloor\frac{m_1}{2}\rfloor}\end{pmatrix}, \ldots, \begin{pmatrix}1 & 0\\ 0 & p^{\lfloor\frac{m_r}{2}\rfloor}\end{pmatrix}\right)$$

we obtain by (5.13) that for some $\mathbf{m}'$,

$$D^{-1}\mathcal{K}_1(p^{\mathbf{m}})D \leq \mathcal{T}_1(p^{\mathbf{m}'}), \quad [\mathcal{T}_1(p^{\mathbf{m}'}) : D^{-1}\mathcal{K}_1(p^{\mathbf{m}})D] \leq p^r, \quad \left|m_i' - \frac{m_i}{2}\right| \leq 1.$$

Since M carries a compatible action of $\mathcal{G}$, we have natural isomorphisms

$$H_i(\mathcal{K}_1(p^{\mathbf{m}})/\mathcal{Z}_1, M) \cong H_i(D^{-1}\mathcal{K}_1(p^{\mathbf{m}})D/\mathcal{Z}_1, M).$$

Hence we deduce from [19, Lemma 20] that

$$\dim_{\mathbb{F}} H_i(\mathcal{K}_1(p^{\mathbf{m}})/\mathcal{Z}_1, M) \le p^r \dim_{\mathbb{F}} H_i(\mathcal{T}_1(p^{\mathbf{m}'})/\mathcal{Z}_1, M)$$

and the result follows from Theorem 5.24.

For general L, Lemma 5.29 provides a finite filtration of L

$$0 = L_0 \subset L_1 \subset \cdots \subset L$$

such that every quotient L_i/L_{i-1} is isomorphic to $\mathbb{F}[\mathcal{K}_1/\mathcal{K}_1(p^{\mathbf{m}})]$ for some $\mathbf{m} \le \mathbf{n}$ and each isomorphism class of quotient occurs at most p^r times. We then deduce from the first case that

$$\begin{aligned}\dim_{\mathbb{F}} H_i(\mathcal{K}_1/\mathcal{Z}_1, M \otimes L) &\le p^r \sum_{\mathbf{m}\le\mathbf{n}} \dim_{\mathbb{F}} H_i(\mathcal{K}_1(p^{\mathbf{m}})/\mathcal{Z}_1, M)\\ &\ll \sum_{\mathbf{m}\le\mathbf{n}} \kappa(\mathbf{m})^r p^{(r-\frac{1}{2})\kappa(\mathbf{m})}\\ &\ll \kappa(\mathbf{n})^{2r} p^{(r-\frac{1}{2})\kappa(\mathbf{n})}.\end{aligned}$$

Here we have used the fact that the cardinality of the set $\{\mathbf{m} : \mathbf{m} \le \mathbf{n}\}$ is $\prod_{i=1}^r n_i$, hence bounded by $\kappa(\mathbf{n})^r$. ■

5.5. $\mathrm{GL}_2(\mathbb{Q}_p)$ *vs* $\mathrm{SL}_2(\mathbb{Q}_p)$

For the application in Section 6, we need to consider smooth admissible $\mathbb{F}$-representations of $\mathcal{G}' = \prod_{i=1}^r \mathrm{SL}_2(\mathbb{Q}_p)$ and their Pontryagin duals. The results above translate to this situation. To explain this, we give the proof of the following analog of Theorem 5.24. If H is a subgroup of $\mathcal{G}$, we denote by H' the intersection $H \cap \mathcal{G}'$.

Theorem 5.31. *Let $M' \in \mathfrak{C}^{\mathrm{fg,tor}}(\mathcal{G}')$. Then for all $i \ge 0$,*

$$\dim_{\mathbb{F}} H_i(\mathcal{T}_1(p^{\mathbf{n}})'/\mathcal{Z}_1', M') \ll \kappa(\mathbf{n})^r p^{(2r-1)\kappa(\mathbf{n})}.$$

Proof. After twisting we may assume the central character of M' is trivial. Then we may extend M' to be an object M^+ in $\mathfrak{C}^{\mathrm{fg,tor}}(\mathcal{G}^+)$ by letting $\mathcal{Z}$ act trivially, where $\mathcal{G}^+ := \mathcal{G}'\mathcal{Z}$. Note that $\mathcal{T}_1(p^{\mathbf{n}})$ is contained in $\mathcal{G}^+$.

It is easy to see that $\mathcal{G}^+$ is a normal subgroup of $\mathcal{G}$ of finite index. Let

$$M := \mathrm{Ind}_{\mathcal{G}^+}^{\mathcal{G}} M^+,$$

which is an object in $\mathfrak{C}^{\mathrm{fg,tor}}(\mathcal{G})$. We can apply Theorem 5.24 to M and obtain

$$\dim_{\mathbb{F}} H_i(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M)) \ll \kappa(\mathbf{n})^r p^{(2r-1)\kappa(\mathbf{n})}.$$

Since M^+ is a direct summand of $M|_{\mathcal{G}^+}$ by Mackey's theorem, we obtain

$$\dim_{\mathbb{F}} H_i(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M^+)) \ll \kappa(\mathbf{n})^r p^{(2r-1)\kappa(\mathbf{n})}.$$

Since $\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1 \cong \mathcal{T}_1(p^{\mathbf{n}})'/\mathcal{Z}_1'$, restricting M^+ to $\mathcal{G}'$ gives the result. ■

6. Application

Let F be a number field of degree r, and let r_1 (resp. $2r_2$) be the number of real (resp. non-real) places. Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$, so that $\mathrm{SL}_2(F_\infty) \cong \mathrm{SL}_2(\mathbb{R})^{r_1} \times \mathrm{SL}_2(\mathbb{C})^{r_2}$. Let K_∞ be the standard maximal compact subgroup of $\mathrm{SL}_2(F_\infty)$.

Let $\{\sigma_1, \dots, \sigma_r\}$ be the set of complex embeddings of the number field F and let $\mathbf{d} = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ be an r-tuple indexed by the σ_i such that $d_i = d_j$ when σ_i and σ_j are complex conjugate to each other. Let $W_{\mathbf{d}}$ be the representation of $\mathrm{SL}_2(F_\infty)$ obtained by forming the tensor product

$$\Big(\bigotimes_{\sigma_i \text{ real}} \mathrm{Sym}^{d_i} \mathbb{C}^2\Big) \bigotimes \Big(\bigotimes_{\{\sigma_i,\sigma_j\} \text{ complex}} \mathrm{Sym}^{d_i} \mathbb{C}^2 \otimes \overline{\mathrm{Sym}}^{d_j} \mathbb{C}^2\Big).$$

If $K_f \subset \mathrm{SL}_2(\mathbb{A}_f)$ is a compact open subgroup, we write

$$Y(K_f) := \mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})/K_f K_\infty,$$

and still use $W_{\mathbf{d}}$ to denote the local system on $Y(K_f)$ attached to $W_{\mathbf{d}}$.

Theorem 6.1. *If F is not totally real and $K_f \subset \mathrm{SL}_2(\mathbb{A}_f)$ is a compact open subgroup, then*

$$\dim_{\mathbb{C}} H_i(Y(K_f), W_{\mathbf{d}}) \ll_\epsilon \kappa(\mathbf{d})^{r-\frac{1}{2}+\epsilon},$$

where $\kappa(\mathbf{d}) = \max_i\{d_i\}$.

Proof. The proof follows closely the one presented in [19, Section 5]. We content ourselves with briefly explaining the main ingredients. Below we abuse the notation by letting the same letters to denote subgroups of SL_2 obtained by intersection from GL_2. Write $Y = Y(K_f)$ in the proof.

(1) Choose a rational prime $p \geq 5$ which splits completely in F. By [19, Lemma 18], there exists a p-adic local system $V_{\mathbf{d}}$ defined over $\mathcal{O} = W(\mathbb{F})$ such that

$$\dim_{\mathbb{C}} H_i(Y, W_{\mathbf{d}}) = \dim_{\mathcal{O}[\frac{1}{p}]} H_i(Y, V_{\mathbf{d}}).$$

For this we need to choose a bijection between the set of complex embeddings $F \hookrightarrow \mathbb{C}$ and p-adic embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$, as done in [19, Lemma 18].

(2) As is explained in [19, Section 5], by passing to an open subgroup we may assume that K_f has the form $\prod_v K_{f,v}$, with $K_{f,v} = K_1$ ($\subset \mathrm{SL}_2(\mathbb{Z}_p)$) for all $v \mid p$. To achieve this, we could first choose p such that $K_{f,v} = \mathrm{SL}_2(\mathbb{Z}_p)$, then pass to an open subgroup with $K_{f,v} = K_1$.

(3) Emerton's theory of completed homology gives a bound ([19, Section 5, (34) and (35)])

$$\dim_{\mathcal{O}[\frac{1}{p}]} H_q(Y, V_{\mathbf{d}}) \leq \sum_{i+j=q} \dim_{\mathcal{O}[\frac{1}{p}]} H_i(\mathcal{K}_1/\mathcal{Z}_1, \widetilde{H}_{j,\mathbb{Q}_p} \otimes V_{\mathbf{d}}),$$

where $\widetilde{H}_j$ is the j-th completed homology of Emerton with (trivial) coefficients in $\mathcal{O}$, and $\widetilde{H}_{j,\mathbb{Q}_p} = \widetilde{H}_j \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Note that $\widetilde{H}_j$ is a coadmissible module over $\mathcal{O}[[\mathcal{K}_1]]$ and carries a natural compatible action of $\prod_{i=1}^r \mathrm{SL}_2(\mathbb{Q}_p)$.

(4) Let $\mathbf{n} = (n_1, \ldots, n_r)$, where n_i is the smallest integer such that $p^{n_i-1} - 1 \geq d_i$ (resp. $p^{n_i-1} - 1 \geq \frac{d_i}{2}$) if σ_i is real (resp. complex). By [19, Lemma 17] we may choose lattice $\mathcal{V}_{d_i} \subset V_{d_i}$ such that $\mathcal{V}_{d_i}/p \subset \mathbb{F}[\![K_1/K_1(p^{n_i})]\!]$. Let $L_{\mathbf{d}}$ be the reduction mod p of $\bigotimes_{i=1}^r \mathcal{V}_{d_i}$.

(5) Let M_j be the reduction modulo p of the image of $\widetilde{H}_j \to \widetilde{H}_{j,\mathbb{Q}_p}$. We then have

$$\dim_{\mathcal{O}[\frac{1}{p}]} H_i(\mathcal{K}_1/\mathcal{Z}_1, \widetilde{H}_{j,\mathbb{Q}_p} \otimes V_{\mathbf{d}}) \leq \dim_{\mathbb{F}} H_i(\mathcal{K}_1/\mathcal{Z}_1, M_j \otimes L_{\mathbf{d}}).$$

(6) Because $\mathrm{SL}_2(\mathbb{C})$ does not admit discrete series, the assumption that F is not totally real implies that $\widetilde{H}_{j,\mathbb{Q}_p}$ is a torsion $\mathcal{O}[\![\mathcal{K}_1]\!] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module, see [8, Theorem 3.4]. So by Lemma 2.6, M_j is a torsion Λ-module. Therefore our Theorem 5.30 applies, via Theorem 5.31, and shows that

$$\dim_{\mathbb{F}} H_i(\mathcal{K}_1, M_j \otimes L_{\mathbf{d}}) \ll \kappa(\mathbf{n})^{2r} p^{(r-\frac{1}{2})\kappa(\mathbf{n})} \ll_\epsilon \kappa(\mathbf{d})^{r-\frac{1}{2}+\epsilon}. \qquad \blacksquare$$

Remark 6.2. Our Theorem 6.1 only gives interesting bound when all the d_i tend to infinity at a parallel rate, while [19, Theorem 1] allows a subset of the weights d_i to be fixed. Nonetheless, this already includes the most interesting cases: for example, when F is imaginary quadratic, we do have $d_1 = d_2$.

Next we deduce Theorem 1.1 in the introduction. We change slightly the notation. Let Z_∞ be the center of $\mathrm{GL}_2(F_\infty)$, K_f be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ and let

$$X = \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})/K_f Z_\infty.$$

If $\mathbf{d} = (d_1, \ldots, d_{r_1+r_2})$ is an $(r_1 + r_2)$-tuple of positive even integers, let $S_{\mathbf{d}}(K_f)$ denote the space of cusp forms on X which are of cohomological type with weight $\mathbf{d}$. Then using the Eichler–Shimura isomorphism, see [19, Section 2.1], Theorem 6.1 can be restated as follows.

Theorem 6.3. *If F is not totally real, then for any fixed K_f and $\mathbf{d} = (d_1, \ldots, d_{r_1+r_2})$ as above, we have*

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f) \ll_\epsilon \kappa(\mathbf{d})^{r-\frac{1}{2}+\epsilon}.$$

In particular, when $\mathbf{d} = (d, \ldots, d)$ is parallel, we obtain

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f) \ll_\epsilon d^{r-\frac{1}{2}+\epsilon}$$

which strengthens [19, Corollary 2] by a power $d^{\frac{1}{6}}$.

Appendix A. A generalization of Breuil–Paškūnas' construction

In this appendix, we generalize a construction of Breuil and Paškūnas in [6] for $\mathrm{GL}_2(F)$ to a finite product of $\mathrm{GL}_2(F)$, where F is a local field with finite residue field k of characteristic p. Let $\mathcal{O}$ be the ring of integers in F with ϖ a fixed uniformizer. We assume $p > 2$ for simplicity.

In [6, Section 9] (which is based on [24]), Breuil and Paškūnas have proven the following theorem, see [6, Corollary 9.11].

Theorem A.1. *Let π be an admissible representation of* $\mathrm{GL}_2(F)$ *such that* $\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)$ *acts trivially on π and $\widetilde{\sigma} := \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O})}\,\pi$. Then there exists an injection $\pi \hookrightarrow \Omega$, where Ω is a smooth representation of* $\mathrm{GL}_2(F)$ *such that $\Omega|_{\mathrm{GL}_2(\mathcal{O})} \cong \mathrm{Inj}_{\mathrm{GL}_2(\mathcal{O})}\,\widetilde{\sigma}$ (an injective envelope of $\widetilde{\sigma}$ in the category of smooth $\mathbb{F}$-representations of* $\mathrm{GL}_2(\mathcal{O})$*).*

Let $r \geq 1$ be an integer. Denote

$$\mathcal{G} = \prod_{i=1}^{r} \mathrm{GL}_2(F), \quad \mathcal{K} = \prod_{i=1}^{r} \mathrm{GL}_2(\mathcal{O}), \quad \mathcal{Z}_\varpi = \prod_{i=1}^{r} \varpi^{\mathbb{Z}}\mathrm{Id}.$$

The main result of this appendix is the following.

Theorem A.2. *Let π be an admissible representation of $\mathcal{G}$ such that $\mathcal{Z}_\varpi$ acts trivially on π and $\widetilde{\sigma} := \mathrm{soc}_{\mathcal{K}}\,\pi$. Then there exists an injection $\pi \hookrightarrow \Omega$, where Ω is a representation of $\mathcal{G}$ such that $\Omega|_{\mathcal{K}} \cong \mathrm{Inj}_{\mathcal{K}}\,\widetilde{\sigma}$.*

If moreover π admits a central character η, then we may require Ω to be an injective envelope of $\widetilde{\sigma}$ in the category of smooth representations of $\mathcal{K}$ with the central character $\eta|_{\mathcal{K}}$.

The proof of Theorem A.2 is an easy generalization of the original proof in [6, Section 9]. We only indicate the changes needed; for this we keep mostly the notation there. We define the following three subgroups of $\mathrm{GL}_2(F)$ (where $\mathfrak{p} = \varpi\mathcal{O}$ and $[\cdot]$ means Teichmüller lift):

$$I_1 = \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}, \quad I = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{pmatrix},$$
$$H = \left\{ \begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix} : \lambda, \mu \in k^\times \right\} \subset I.$$

Let $\mathcal{I}_1, \mathcal{I}, \mathcal{H}$ be respectively a product of r copies of I_1, I, H, viewed as subgroups of $\mathcal{G}$. Then $\mathcal{H}$ has order prime to p and provides a section for $\mathcal{I} \twoheadrightarrow \mathcal{I}/\mathcal{I}_1$. Let $\mathcal{R}_1$ denote the normalizer of $\mathcal{I}$ in $\mathcal{G}$, which as a group is generated by $\mathcal{I}$ and the elements $\{t_i : 1 \leq i \leq r\}$, where $t_i \in \mathcal{G}$ takes $\left(\begin{smallmatrix}0 & 1\\ \varpi & 0\end{smallmatrix}\right)$ at the index i and $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ at other indices. Note that $t_it_j = t_jt_i$ and $t_i^2 \in \mathcal{Z}_\varpi$. In other words,

$$\Delta := \langle t_i : 1 \leq i \leq r\rangle/\mathcal{Z}_\varpi \cong \prod_{i=1}^{r} \mathbb{Z}/2\mathbb{Z}.$$

It is clear that Δ normalizes $\mathcal{H}$ and there is an isomorphism of groups

$$\mathcal{R}_1/\mathcal{I}_1\mathcal{Z}_\varpi \cong \mathcal{H} \rtimes \Delta. \tag{A.1}$$

Lemma A.3. *Let τ be a smooth admissible representation of $\mathcal{R}_1$ on which $\mathcal{Z}_\varpi$ acts trivially. Let $\iota : \tau|_{\mathcal{I}} \hookrightarrow \mathrm{Inj}_{\mathcal{I}}(\tau|_{\mathcal{I}})$ be an injective envelope of $\tau|_{\mathcal{I}}$. Then there exists an action of $\mathcal{R}_1$ on $\mathrm{Inj}_{\mathcal{I}}(\tau|_{\mathcal{I}})$ such that ι is $\mathcal{R}_1$-equivariant.*

Proof. The proof is identical to that of [6, Lemma 9.5], using the (generalized) property (S) defined in [6, Definition 9.1] which holds as $p > 2$ (see [6, Proposition 9.2]). ■

Any $\mathbb{F}$-representation V of $\mathcal{H}$ is semi-simple and decomposes as $\bigoplus_\chi V_\chi$, where $\chi : \mathcal{H} \to \mathbb{F}^\times$ runs over all characters and V_χ denotes the χ-isotypic subspace. For a character $\chi : \mathcal{H} \to \mathbb{F}^\times$ and $t \in \mathcal{R}_1$, we let χ^t denotes the conjugate character

$$\chi^t(g) := \chi(t^{-1}gt);$$

this induces an action of Δ on the set of characters $\{\chi : \mathcal{H} \to \mathbb{F}^\times\}$. We write $\langle \Delta.\chi\rangle$ for the Δ-orbit generated by χ, that is, the set of characters (without multiplicities) $\{\chi^t : t \in \Delta\}$.

Lemma A.4. *Let V be a finite-dimensional $\mathbb{F}$-representation of $\mathcal{H}$ such that*

$$\dim_{\mathbb{F}} V_\chi = \dim_{\mathbb{F}} V_{\chi^t} \quad \textit{for all } t \in \Delta. \tag{A.2}$$

Then the action of $\mathcal{H}$ on V can be extended to an action of $\mathcal{H} \rtimes \Delta$.

Proof. We will construct an action of Δ on V by constructing inductively actions of $\Delta_s := \prod_{i=1}^s \mathbb{Z}/2\mathbb{Z}$ (so $\Delta_r = \Delta$). The case $s = 1$ is easy, see the proof of [6, Lemma 9.6]. Assume the action of Δ_{s-1} has been constructed. To construct the action of Δ_s amounts to defining an action of t_s which has order 2 and commutes with the given one of Δ_{s-1}. It is clear that V decomposes as a direct sum of subspaces each of which has the form $\bigoplus_{\chi' \in \langle \Delta_s.\chi\rangle} V_{\chi'}$ (i.e. with respect to the action of Δ_s), so it suffices to define the action of t_s on each summand. Fixing a character χ, we have two cases:

- If $\chi^{t_s} = \chi$, then $\langle \Delta_s.\chi\rangle = \langle \Delta_{s-1}.\chi\rangle$, and we let t_s acts trivially on $\bigoplus_{\chi' \in \langle \Delta_s.\chi\rangle} V_{\chi'}$.
- If $\chi^{t_s} \neq \chi$, then we have a disjoint union

  $$\langle \Delta_s.\chi\rangle = \langle \Delta_{s-1}.\chi\rangle \cup \langle \Delta_{s-1}.\chi^{t_s}\rangle.$$

  We choose an (arbitrary) $\mathbb{F}$-linear isomorphism $\phi_{\chi,\chi^{t_s}} : V_\chi \xrightarrow{\sim} V_{\chi^{t_s}}$ and set

  $$\phi_{\chi^{t_s},\chi} := \phi^{-1}_{\chi,\chi^{t_s}} : V_{\chi^{t_s}} \xrightarrow{\sim} V_\chi.$$

  For any $t \in \Delta_{s-1}$, we consider

  $$\begin{tikzcd} V_\chi \arrow[d, "\phi", "\sim"'] & V_{\chi^t} \arrow[l, "t^{-1}"', "\sim"] \arrow[d, dashed] \\ V_{\chi^{t_s}} \arrow[r, "t", "\sim"'] & V_{\chi^{t\cdot t_s}} \end{tikzcd}$$

  and define $\phi_{\chi^t,\chi^{t\cdot t_s}} : V_{\chi^t} \xrightarrow{\sim} V_{\chi^{t\cdot t_s}}$ to be the composition $t \circ \phi \circ t^{-1}$, respectively

  $$\phi_{\chi^{t\cdot t_s},\chi^t} := \phi^{-1}_{\chi^t,\chi^{t\cdot t_s}} : V_{\chi^{t\cdot t_s}} \xrightarrow{\sim} V_{\chi^t}.$$

  Clearly, putting them together uniquely determines a (compatible) action of t_s, hence of Δ_s.

This finishes the proof by induction. ∎

Lemma A.5. *Let σ be an irreducible $\mathbb{F}$-representation of $\mathcal{K}$ and $\mathrm{Inj}_{\mathcal{K}}\,\sigma$ an injective envelope of σ. Then $V := (\mathrm{Inj}_{\mathcal{K}}\,\sigma)^{\mathcal{I}_1}$ satisfies the condition (A.2).*

Proof. Any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{K})$ has the form $\bigotimes_{i=1}^{r} \sigma_i$ with each $\sigma_i \in \mathrm{Rep}_{\mathbb{F}}(K)$. We have

$$(\mathrm{Inj}_{\mathcal{K}}\, \sigma)^{\mathcal{I}_1} = \big(\mathrm{Inj}_{\prod_{i=1}^{r} \mathrm{GL}_2(k)}\, \sigma\big)^{\mathcal{I}_1} \cong \bigg(\bigotimes_{i=1}^{r} \mathrm{Inj}_{\mathrm{GL}_2(k)}\, \sigma_i\bigg)^{\mathcal{I}_1} = \bigotimes_{i=1}^{r} (\mathrm{Inj}_{\mathrm{GL}_2(k)}\, \sigma_i)^{I_1};$$

here we have used the isomorphism

$$\mathrm{Inj}_{\prod_{i=1}^{r} \mathrm{GL}_2(k)}\, \sigma \cong \bigotimes_{i=1}^{r} \mathrm{Inj}_{\mathrm{GL}_2(k)}\, \sigma_i .$$

The result then follows from a similar result in the case $r = 1$, see for instance the proof of [6, Lemma 9.6]. ■

Proof of Theorem A.2. Since π is admissible, we may take an injective envelope

$$\iota : \pi \hookrightarrow \Omega.$$

We will define an action of $\mathcal{R}_1$ on Ω which extends the given action of $\mathcal{I}$ on Ω and such that ι is $\mathcal{R}_1$-equivariant.

By Lemma A.5, $V := \Omega^{\mathcal{I}_1}$ satisfies condition (A.2). On the other hand, since π carries an action of $\mathcal{G}$, $W := \pi^{\mathcal{I}_1}$ also satisfies (A.2). So we may decompose $\mathcal{I}$-equivariantly V as $W \oplus W'$ with W' satisfying (A.2), hence a decomposition

$$\Omega|_{\mathcal{I}} = \mathrm{Inj}_{\mathcal{I}}\, W \oplus \mathrm{Inj}_{\mathcal{I}}\, W'$$

such that $\pi \subset \mathrm{Inj}_{\mathcal{I}}\, W$. Lemma A.4 allows us to define an action of Δ, hence an action of $\mathcal{R}_1$ on W' via (A.1). Then Lemma A.3 allows to extend the action of $\mathcal{R}_1$ on π (resp. on W') to the whole $\mathrm{Inj}_{\mathcal{I}}\, W$ (resp. $\mathrm{Inj}_{\mathcal{I}}\, W'$). Putting them together, we obtain an action of $\mathcal{R}_1$ on Ω which makes ι to be $\mathcal{R}_1$-equivariant. Finally, using the “amalgame” structure of $\mathrm{GL}_2(F)$ which generalizes to $\mathcal{G}$, the two actions of $\mathcal{K}$ and $\mathcal{R}_1$ on Ω glue to an action of $\mathcal{G}$ on Ω (such that $\mathcal{Z}_{\varpi}$ acts trivially), as in [24, Corollary 5.5.5]. Note that in [24], Corollary 5.5.5 is proved by passing to diagrams, but this can be circumvented because we can simply write down the gluing action of $\mathcal{G}$ using the ones of $\mathcal{K}$ and $\mathcal{R}_1$.

The last assertion is clear, by taking the sub-space of $\Omega_\eta \subset \Omega$ on which the center of $\mathcal{G}$ acts via η. ■

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