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Blow up dynamics for the hyperbolic vanishing mean curvature flow of surfaces asymptotic to the Simons cone

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Abstract. In this article, we establish the existence of a family of hypersurfaces $(\Gamma(t))_{0<t\leq T}$ that evolve by the vanishing mean curvature flow in Minkowski space and that, as t tends to 0, blow up towards a hypersurface that behaves like the Simons cone both near the origin and at infinity. This issue amounts to singularity formation for a second-order quasilinear wave equation. Our constructive approach consists in proving the existence of finite-time blow up solutions of this hyperbolic equation of the form $u(t,x) \sim t^{\nu+1} Q(x/t^{\nu+1})$, where Q is a stationary solution and ν an arbitrarily large positive irrational number. Our approach roughly follows that of Krieger, Schlag and Tataru [22–24]. However, in contrast to these works, the equation to be handled in this article is quasilinear. This brings about a number of difficulties to overcome.

Keywords. Vanishing mean curvature flow, quasilinear wave equation, Simons cones, blow up dynamics

Contents

1. Introduction . . . 3802
   1.1. Setting of the problem . . . 3802
   1.2. Statement of the main result . . . 3805
   1.3. Strategy of the proof . . . 3807
2. Analysis of the stationary solution . . . 3810
   2.1. Asymptotic behavior of the stationary solution . . . 3810
   2.2. Properties of the linearized operator of the quasilinear wave equation around the ground state . . . 3811

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3. Approximate solution in the inner region . . . 3812
3.1. General scheme of construction of the approximate solution in the inner region . . . 3812
3.2. Analysis of the functions V_k . . . 3814
3.3. Estimate of the approximate solution in the inner region . . . 3818
4. Approximate solution in the self-similar region . . . 3820
4.1. General scheme of construction of the approximate solution in the self-similar region 3820
4.2. Analysis of the vector functions $\mathcal{W}_k$ . . . 3827
4.3. Estimate of the approximate solution in the self-similar region . . . 3841
5. Approximate solution in the remote region . . . 3844
5.1. General scheme of construction of the approximate solution in the remote region . . . 3844
5.2. Analysis of the functions $\mathfrak{g}_k$ . . . 3846
5.3. Estimate of the approximate solution in the remote region . . . 3848
6. Approximate solution in the whole space . . . 3850
7. Proof of the blow up result . . . 3852
7.1. Key estimates . . . 3852
7.2. End of the proof . . . 3865
A. Derivation of the equation . . . 3865
B. Study of the linearized operator of the quasilinear wave equation around the ground state 3867
C. Proof of the local well-posedness result . . . 3870
D. Some simple ordinary differential equations results . . . 3880
D.1. Proof of Duhamel's formula (3.16) . . . 3880
D.2. Proof of Lemma 4.3 . . . 3882
References . . . 3885

1. Introduction

1.1. Setting of the problem

In this article we address the question of singularity formation for the hyperbolic vanishing mean curvature flow of surfaces that are asymptotic to Simons cones at infinity.

In [5], Bombieri, De Giorgi and Giusti proved that the Simons cone defined by

$$C_n = \{X = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x|^2 = |y|^2\} \tag{1.1}$$

is a globally minimizing surface if and only if $n \geq 4$. It is clear that the Simons cone has dimension $d = 2n - 1$, is invariant under the action of the group $O(n) \times O(n)$, where $O(n)$ is the orthogonal group of $\mathbb{R}^n$, and can be parametrized in the following way:

$$\mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to C_n \subset \mathbb{R}^{2n}, \qquad (\rho, \omega_1, \omega_2) \mapsto (\rho\omega_1, \rho\omega_2). \tag{1.2}$$

The Simons cones are linked to Bernstein's problem,[1] which can be stated as follows: if the graph of a $\mathcal{C}^2$ function u on $\mathbb{R}^{m-1}$ is a minimal surface in $\mathbb{R}^m$, does this imply that this graph is a hyperplane? This amounts to asking if the solution u to the equation

$$\sum_{j=1}^{m-1} \partial_{x_j} \left(\frac{u_{x_j}}{\sqrt{1 + |\nabla u|^2}} \right) = 0,$$

[1] This problem is named after Sergei Natanovich Bernstein, who solved it in the case of $m = 3$ in 1914.

known as the *minimal surface equation*, is linear. This is true in dimensions $m \le 8$, but false for $m \ge 9$. For further details on Bernstein's problem and related issues, we refer for instance to [1–3, 5, 8, 9, 27, 30, 34, 35] and the references therein.

From [5,38], it is known that for $n \ge 4$ the complement of the Simons cone (which has two connected components $|x| < |y|$ and $|y| < |x|$) is foliated by two families of smooth globally minimizing surfaces $(M_a)_{a>0}$ and $(\widetilde{M}_a)_{a>0}$, asymptotic to the Simons cone at infinity. These families are scale invariant: $M_a = aM$ and $\widetilde{M}_a = a\widetilde{M}$ with M and $\widetilde{M}$ admitting respectively the parametrizations

$$\mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \ni (\rho, \omega_1, \omega_2) \mapsto (\rho\omega_1, Q(\rho)\omega_2) \in \mathbb{R}^{2n}, \tag{1.3}$$

$$\mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \ni (\rho, \omega_1, \omega_2) \mapsto (Q(\rho)\omega_1, \rho\omega_2) \in \mathbb{R}^{2n}, \tag{1.4}$$

where Q is a positive radial function that belongs to $\mathcal{C}^\infty(\mathbb{R}^n)$ and satisfies $Q(0) = 1$, $Q(\rho) > \rho$ for any positive real number ρ, and

$$Q(\rho) = \rho + \frac{d_\alpha}{\rho^\alpha}(1 + o(1))$$

as ρ tends to infinity, with d_α some positive constant and

$$\alpha = -1 + \tfrac{1}{2}\left((2n-1) - \sqrt{(2n-1)^2 - 16(n-1)}\right).$$

Minimal surfaces asymptotic to the Simons cone at infinity also exist for $2 \le n \le 3$, but they are no longer minimizing, which will be a crucial property for our analysis.

The minimal surface equation in Riemannian geometry has a natural hyperbolic analogue in the Lorentzian framework. In particular, working in the Minkowski space $\mathbb{R}^{1,N}$ equipped with the standard metric $dg = -dt^2 + \sum_{j=1}^N dx_j^2$, and considering surfaces that, for fixed t, are graphs of functions u over $\mathbb{R}^{N-1}$, we find that u satisfies the equation

$$\partial_t\left(\frac{u_t}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\right) - \sum_{j=1}^{N-1} \partial_{x_j}\left(\frac{u_{x_j}}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\right) = 0.$$

There is a broad literature devoted to this model, including results both on the local well-posedness of the corresponding Cauchy problem and on the stability of certain stationary solutions (see for instance [6, 10, 13, 16, 19, 25, 26] and the references therein).

In this paper, we focus on the case of time-like surfaces in the Minkowski space $\mathbb{R}^{1,2n}$ that, for fixed t, can be parametrized in the following way:

$$\Gamma(t) : \mathbb{R}^n \times \mathbb{S}^{n-1} \ni (x, \omega) \mapsto (x, u(t,x)\omega) \in \mathbb{R}^{2n}, \tag{1.5}$$

with some positive function u. This leads to the following quasilinear second-order wave equation (see Appendix A for the corresponding computations):

$$\partial_t\left(\frac{u_t}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\right) - \sum_{j=1}^{n} \partial_{x_j}\left(\frac{u_{x_j}}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\right) + \frac{n-1}{u\sqrt{1-(u_t)^2+|\nabla u|^2}} = 0,$$

which can also be rewritten as

$$\begin{aligned}(\mathrm{NW})u &:= (1+|\nabla u|^2)u_{tt} - (1-(u_t)^2+|\nabla u|^2)\Delta u \\ &+ \sum_{j,k=1}^{n} u_{x_j}u_{x_k}u_{x_jx_k} - 2u_t(\nabla u\cdot\nabla u_t) + \frac{n-1}{u}(1-(u_t)^2+|\nabla u|^2) = 0. \qquad (1.6)\end{aligned}$$

Note that this equation is invariant with respect to the isometries of the Minkowski space $\mathbb{R}^{1,n}$ and also invariant under the scaling transform

$$u_a(t,x) = au(t/a, x/a), \qquad (1.7)$$

in the sense that if u solves (1.6) then u_a is also a solution to (1.6). In the framework of Sobolev spaces[2], $\dot{H}^{(n+2)/2}(\mathbb{R}^n)$ is invariant under the scaling (1.7).

In this paper, we shall consider the case of $n = 4$ and assume that u is radial, which implies that for fixed t the surfaces we are considering are invariant under the action of the group $O(4)\times O(4)$. We can readily check that in this case the function u satisfies the following equation:

$$(1+u_\rho^2)u_{tt} - (1-u_t^2)u_{\rho\rho} - 2u_tu_\rho u_{\rho t} + 3(1+u_\rho^2-u_t^2)\left(\frac{1}{u}-\frac{u_\rho}{\rho}\right) = 0. \qquad (1.8)$$

Note that the Simons cone and the minimal surfaces M_a are stationary solutions of our model with $u(t,\rho) = \rho$ in the case of the Simons cone and $u(t,\rho) = Q_a(\rho)$, $Q_a(\rho) = aQ(\rho/a)$ in the case of M_a. Let us also emphasize that in the case of $n = 4$, we have

$$Q(\rho) = \rho + \frac{d_2}{\rho^2}(1+o(1)) \qquad (1.9)$$

as ρ tends to infinity, with d_2 some positive constant.

We shall be interested in time-like surfaces of the form (1.5) that are asymptotic to the Simons cone as $|x|\to\infty$. To handle this asymptotic behavior we introduce the spaces X_L, with L a sufficiently large integer, that we define as being the set of functions (u_0, u_1) such that $\nabla(u_0 - Q)$ and u_1 belong to $H^{L-1}(\mathbb{R}^4)$, and which satisfy

$$\inf u_0 > 0 \quad \text{and} \quad \inf(1+|\nabla u_0|^2-(u_1)^2) > 0. \qquad (1.10)$$

The Cauchy problem for the quasilinear wave equation (1.6) is locally well-posed in X_L provided that L is sufficiently large. More precisely one has the following theorem, the proof of which is given in Appendix C.

[2]Throughout this article, we shall denote by $H^s(\mathbb{R}^n)$ the non-homogeneous Sobolev space and by $\dot{H}^s(\mathbb{R}^n)$ the homogeneous Sobolev space. We refer to [4] and the references therein for all the necessary definitions and properties of these spaces.

Theorem 1.1. *Consider the Cauchy problem*

$$\begin{cases} (1.6)u = 0, \\ u_{|t=0} = u_0, \\ (\partial_t u)_{|t=0} = u_1. \end{cases} \tag{1.11}$$

Assume that the Cauchy data (u_0, u_1) *belongs to the space* X_L *with* $L > 4$*. Then there exists a unique maximal solution* u *of* (1.11) *on* $[0, T^*[$ *such that*

$$(u, \partial_t u) \in \mathcal{C}([0, T^*[, X_L). \tag{1.12}$$

Furthermore, if the maximal time T^* *of existence of such a solution is finite* (*we then say that the solution blows up*)*, then*

$$\limsup_{t \nearrow T^*} \left(\left\| \frac{1}{u(t,\cdot)} \right\|_{L^\infty} + \left\| \frac{1}{(1 + |\nabla u|^2 - (\partial_t u)^2)(t,\cdot)} \right\|_{L^\infty} + \sup_{|\gamma| \le 1} \left\| \partial_x^\gamma \nabla_{t,x} u \right\|_{L^\infty} \right) = \infty. \tag{1.13}$$

The question we would like to address in this paper is that of blow up, i.e., the description of possible singularities that smooth hypersurfaces may develop as they evolve by the Minkowski zero mean curvature flow. This amounts to investigating blow up dynamics for the corresponding quasilinear wave equations. There is by now a considerable literature dealing with the construction of type II blow up solutions for energy critical and energy supercritical semilinear heat, wave and Schrödinger type equations that become singular via a concentration of a stationary state profile (see for instance the articles [7, 11, 12, 15, 17, 18, 20–24, 28, 29, 31–33, 37, 38] and the references therein). In the hyperbolic setting, the first constructions of such blow up solutions go back to the seminal works of Krieger, Schlag and Tataru [22, 24] and Rodnianski and Sterbenz [33] on the energy critical wave equation in dimension 3 and on the energy critical wave map problem. The goal of the present paper is to extend the approach initiated by Krieger, Schlag and Tataru [22, 24] to the quasilinear setting under consideration in order to show that this blow up mechanism exists as well for the wave equation (1.6).

1.2. Statement of the main result

Our main result is given by the following theorem.

Theorem 1.2. *For any irrational number* $\nu > 1/2$ *and any positive real number* δ *sufficiently small, there exist a positive time* T *and a radial solution* u *to* (1.6) *such that*[3]

$$(u, \partial_t u) \in \mathcal{C}(]0, T], X_{L_0}) \quad \textit{with} \quad L_0 := 2M + 1,\ M = \left[\tfrac{3}{2}\nu + \tfrac{5}{4}\right], \tag{1.14}$$

[3] Throughout this paper, $[x]$ denotes the integer part of x.

and such that it blows up at $t = 0$ *by concentrating the soliton profile: there exist two radial functions* $\mathfrak{g}_0 \in \dot{H}^{s+1}(\mathbb{R}^4)$ *and* $\mathfrak{g}_1 \in \dot{H}^s(\mathbb{R}^4)$*, where* $0 \leq s < 3\nu + 2$*, such that*

$$u(t,x) = t^{\nu+1} Q(x/t^{\nu+1}) + \mathfrak{g}_0(x) + \eta(t,x),$$
$$u_t(t,x) = \mathfrak{g}_1(x) + \eta_1(t,x),$$

with

$$\|\nabla \eta(t,\cdot)\|_{H^2(\mathbb{R}^4)} + \|\eta_1(t,\cdot)\|_{H^2(\mathbb{R}^4)} \xrightarrow{t\to 0} 0.$$

Moreover, writing

$$u(t,x) = t^{\nu+1}(Q(x/t^{\nu+1}) + \zeta(t, x/t^{\nu+1})),$$
$$u_t(t,x) = \zeta_1(t, x/t^{\nu+1}),$$

we have

$$\|\nabla \zeta(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^4)} + \|\zeta_1(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^4)} \xrightarrow{t\to 0} 0, \quad \forall\, 2 < s \leq L_0 - 1.$$

Additionally, $\mathfrak{g}_0$*,* $\mathfrak{g}_1$ *are compactly supported, belong to* $\mathcal{C}^\infty(\mathbb{R}^4 \setminus \{0\})$ *and satisfy*

$$\|\nabla \mathfrak{g}_0\|_{H^s(\mathbb{R}^4)} + \|\mathfrak{g}_1\|_{H^s(\mathbb{R}^4)} \leq C_s \delta^{3\nu+2-s}, \quad \forall\, 0 \leq s < 3\nu + 2,$$
$$\mathfrak{g}_0(x) \sim \frac{d_2}{3\nu+4} |\sqrt{2}\, x|^{3\nu+1}, \quad \mathfrak{g}_1(x) \sim d_2 |\sqrt{2}\, x|^{3\nu}, \quad \text{as } x \to 0,$$

where d_2 *denotes the constant involved in* (1.9)*.*

Corollary 1.1. *There exists a family* $(\Gamma(t))_{0<t\leq T}$ *of hypersurfaces in* $\mathbb{R}^8$ *that evolve by the hyperbolic vanishing mean curvature flow, and that, as* t *tends to* 0*, blow up towards a hypersurface that behaves asymptotically like the Simons cone, both as* $x \to 0$ *and as* $|x| \to \infty$*. Moreover,*

$$t^{-(\nu+1)}\Gamma(t) \xrightarrow{t\to 0} M,$$

uniformly on compact sets, where M *denotes the hypersurface defined by* (1.3)*.*

Remark 1.1.

- Combining Theorem 1.2 with the asymptotics (1.9), we readily gather that the blow up solution u to (1.6) given by Theorem 1.2 satisfies
  (1) $\|\nabla(u(t,\cdot) - Q)\|_{L^\infty((0,T],\dot{H}^s(\mathbb{R}^4))} \lesssim 1$, $\forall\, 0 \leq s < 2$,
  (2) $\|\nabla(u(t,\cdot) - |x| - \mathfrak{g}_0)\|_{\dot{H}^s(\mathbb{R}^4)} \xrightarrow{t\to 0} 0$, $\forall\, 0 \leq s < 2$,
  (3) $\|\nabla(u(t,\cdot) - Q)\|_{\dot{H}^s(\mathbb{R}^4)} \xrightarrow{t\to 0} \infty$, $\forall\, 2 \leq s \leq L_0 - 1$.
- Theorem 1.2 gives the existence of blow up solutions with the prescribed pure power blow up rate $t^{-1-\nu}$ for any irrational $\nu > 1/2$, in the spirit of the original results of Krieger, Schlag and Tataru [22–24]. The blow up rate is related both to the singular

behavior of the solution near the light cone that determines its finite degree of regularity (1.14) (see Section 4 for the details) and to the asymptotics of the radiation part $\mathfrak{g}_0$, $\mathfrak{g}_1$ as $x \to 0$. Let us mention a recent work of Jendrej, Lawrie and Rodriguez [16] on the energy critical one-equivariant wave maps that provides an explicit link between the asymptotic behavior of the radiation near the origin and the blow up rate in the context of general one-bubble blow up solutions.

- The assumption that ν is irrational allows one to prevent the formation of additional logarithms in the construction of an approximate solution to (1.6), while the limitation to $\nu > 1/2$ is related to the local well-posedness result given by Theorem 1.1, which is not optimal.
- For the mean curvature flow which is a parabolic analogue of the flow we are considering in this paper, a similar result[4] was established by Velázquez [38] for all $n \geq 4$. We expect that our result also holds for any $n \geq 4$. However, we limit ourselves here to the case of $n = 4$ in order to avoid some additional technical difficulties related to a more complicated behavior of Q at infinity for $n \geq 5$.
- The mechanism of singularity formation described by Theorem 1.2 is not the only possible one for the model under consideration: (1.6) also admits finite-time blow up solutions of self-similar type (see Appendix A).

1.3. Strategy of the proof

We shall prove Theorem 1.2 by a two-step procedure, first building an arbitrary good approximate solution (Sections 3–5), and then completing it to an exact solution by means of energy estimates and compactness arguments (Section 7). As we shall see, the blow up result we establish in this article heavily relies on the asymptotic behavior of the soliton Q at infinity. Therefore we shall focus on its analysis in Section 2.

To build an appropriate approximate solution, we shall analyze separately the three regions that correspond to three different space scales: the inner region corresponding to $\rho/t \leq t^{\epsilon_1}$, the self-similar region where $\frac{1}{10}t^{\epsilon_1} \leq \rho/t \leq 10t^{-\epsilon_2}$, and the remote region defined by $\rho/t \geq t^{-\epsilon_2}$, where $0 < \epsilon_1 < \nu$ and $0 < \epsilon_2 < 1$ are two fixed positive real numbers. The inner region is the one where the blow up concentrates. In this region the solution will be constructed as a perturbation of the concentrating soliton profile $t^{\nu+1}Q(x/t^{\nu+1})$. In the self-similar region, the profile of the solution is determined uniquely by the matching conditions coming from the inner region, while in the remote region the profile remains essentially a free parameter of the construction.

Compared to the corresponding constructions in the semilinear setting, there are some modifications that we introduce in order to take care of the quasilinear character of (1.8). This concerns on the one hand the construction of the approximate solution in the self-similar region where we need to build up the solution at the same time as its characteristic set (see the discussion below), and on the other hand the energy estimates that we establish

[4] But with a countable family of blow up rates.

in Section 7, where we have to use the nonlinear energies tracing carefully the contribution of the quasilinear terms.

In Section 3, we investigate the equation in the inner region $\rho/t \leq t^{\epsilon_1}$. In this region, we shall look for an approximate solution as a power expansion in $t^{2\nu}$ of the form

$$u_{\rm in}^{(N)}(t,\rho) = t^{\nu+1} \sum_{k=0}^{N} t^{2\nu k} V_k(\rho/t^{\nu+1}), \tag{1.15}$$

where V_0 is the soliton Q, and the functions V_k, for $1 \leq k \leq N$, are obtained recursively, by solving a recurrent system

$$\begin{cases} \mathcal{L} V_k = F_k(V_0, \dots, V_{k-1}), \\ V_k(0) = 0 \quad \text{and} \quad V_k'(0) = 0, \end{cases}$$

where $\mathcal{L}$ is the operator defined by

$$\mathcal{L} = \partial_y^2 + \left(\frac{3}{y} + B_1\right)\partial_y + B_0 \tag{1.16}$$

with

$$\begin{cases} B_1(y) = 9\dfrac{Q_y^2}{y} - 6\dfrac{Q_y}{Q}, \\ B_0(y) = 3\dfrac{1+Q_y^2}{Q^2}. \end{cases} \tag{1.17}$$

As will be established in §3.2, these functions V_k grow at infinity as follows:

$$V_k(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n \geq 2-2(k-\ell)} d_{n,k,\ell}\, y^{-n}.$$

To obtain a meaningful approximate solution, we are then constrained to restrict the construction to the region $\rho/t \leq t^{\epsilon_1}$.

The aim of Section 4 is to extend the approximate solution built in Section 3 to the self-similar region $\frac{1}{10}t^{\epsilon_1} \leq \rho/t \leq 10t^{-\epsilon_2}$. Taking into account the matching conditions coming from the inner region, we seek this extension in the following form:

$$u(t,\rho) = \lambda(t)(z + W(t,z)), \qquad z = \rho/\lambda(t), \tag{1.18}$$

with

$$\begin{aligned} W(t,z) &= \sum_{k\geq 3} t^{\nu k} \sum_{\ell=0}^{\ell(k)} (\log t)^\ell\, w_{k,\ell}(z), \\ \lambda(t) &= t\Big(1 + \sum_{k\geq 3}\sum_{\ell=0}^{\ell(k)} \lambda_{k,\ell}\, t^{\nu k} (\log t)^\ell\Big), \end{aligned} \tag{1.19}$$

where $\ell(k) = [(k-3)/2]$.

Substituting (1.18) into (1.8), we get the following equation for W:

$$(2z^2 - 1 + A_0)\, W_{zz} + A_1 = 0, \tag{1.20}$$

where A_0 and A_1 can be explicitly expressed by means of W and $\lambda(t)$ as well as their derivatives.[5]

The introduction of the modified self-similar variable $z = \rho/\lambda(t)$ in (1.18) instead of the semilinear one ρ/t, is related to the quasilinear nature of (1.8): taking $\lambda(t) = t$ gives a recurrent system for $w_{k,\ell}$ leading to a loss of regularity at each step, which is not surprising since to propagate the regularity properly one has to follow the light cone of (1.8) that depends on the solution itself. Technically, this amounts to constructing W simultaneously with $\lambda(t)$ by requiring

$$A_{0|z=1/\sqrt{2}} = 0. \tag{1.21}$$

Substituting (1.19) into (1.20) and (1.21), we get a recurrent system that allows us to successively determine the coefficients $\lambda_{k,\ell}$ and the functions $w_{k,\ell}$ without loss of regularity. It will be shown in §4.2.1 that the functions $w_{k,\ell}$ obtained in this way are $\mathcal{C}^\infty$ away from the light cone $z = 1/\sqrt{2}$, and behave near $z = 1/\sqrt{2}$ like $(1/\sqrt{2} - z)^{3\nu+4}(1 + o(1))$ plus a regular function, which explains the finite degree of smoothness of our solutions. Furthermore, at infinity the functions $w_{k,\ell}$ have the following asymptotic behavior:

$$w_{k,\ell}(z) \sim c_{k,\ell}\, z^{k\nu+1}(\log z)^{(k-3)/2-\ell}.$$

This forces us to restrict the self-similar region to $\rho/t \lesssim t^{-\epsilon_2}$ with $0 < \epsilon_2 < 1$.

In Section 5, we construct an approximate solution $u_{\mathrm{out}}^{(N)}$, which extends the approximate solution built in Sections 3, 4 to the whole space. This is done by solving the quasilinear wave equation (1.8) by means of the following ansatz:

$$u_{\mathrm{out}}^{(N)}(t,\rho) = \rho + \mathfrak{g}_0(\rho) + t\,\mathfrak{g}_1(\rho) + \sum_{k=2}^{N} t^k \mathfrak{g}_k(\rho), \tag{1.22}$$

where the Cauchy data $(\cdot + \mathfrak{g}_0, \mathfrak{g}_1)$ are determined by the matching conditions coming from the self-similar region. Substituting (1.22) into (1.8), we get a recurrent relation

$$\mathfrak{g}_k = \mathscr{G}_k(\mathfrak{g}_j,\, j \le k-1),$$

which allows us to successively determine $\mathfrak{g}_k$ for $k \ge 2$. As will be seen in §5.2, the functions $\mathfrak{g}_k$, $k \ge 0$, are compactly supported and behave like $\rho^{1-k+3\nu}$ close to 0, which ensures that (1.22) provides a meaningful approximate solution in the remote region $\rho/t \ge t^{-\varepsilon_2}$.

The aim of Section 7 is to complete the approximate solution $u^{(N)}$ constructed in Sections 3– 5 to an exact solution. This is achieved by considering a sequence (u_n) of

[5] The computation of A_0 and A_1 will be performed in Section 4.

solutions of (1.8) with initial data $u_{n|t=t_n} = u^{(N)}(t_n)$, $(\partial_t u_n)_{|t=t_n} = (\partial_t u^{(N)})(t_n)$, where $t_n \searrow 0$. Performing energy estimates of the remainder $\varepsilon^{(N)} = u - u^{(N)}$, we obtain uniform control of this sequence on time intervals $[t_n, T]$, with $T > 0$ independent of n. This allows us to conclude the proof of Theorem 1.2 by passing to the limit $t_n \to 0$. The proof of the energy estimates for $\varepsilon^{(N)}$ relies heavily on the positivity property (2.12) of the operator $\mathscr{L}$, which will be proved in Appendix B and which is closely related to the minimality of the surfaces M_a and $\widetilde{M}_a$.

For the sake of simplicity, we shall omit in this text the dependence of all the functions on the parameter ν. All along this article, T and C will denote respectively a positive time and a constant that depend on several parameters, and that may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C.

2. Analysis of the stationary solution

2.1. Asymptotic behavior of the stationary solution

Our analysis in this paper is intimately connected to the behavior at infinity of the stationary solution to the quasilinear wave equation (1.8). In this subsection, we collect the properties of Q that we will use throughout this paper.

Lemma 2.1. *The Cauchy problem*

$$\begin{cases} -Q_{\rho\rho} + 3(1 + Q_\rho^2)\left(\dfrac{1}{Q} - \dfrac{Q_\rho}{\rho}\right) = 0, \\ Q(0) = 1 \quad \text{and} \quad Q_\rho(0) = 0, \end{cases} \tag{2.1}$$

has a unique solution[6] $Q \in \mathscr{C}^\infty(\mathbb{R}_+)$ *which satisfies the following properties:*

- *Q has an even Taylor expansion*[7] *at 0:*

$$Q(\rho) = \sum_{n \geq 0} \gamma_{2n} \rho^{2n} \tag{2.2}$$

  with some constants γ_{2n} such that $\gamma_0 = 1$,
- *Q enjoys the following bounds for any ρ in $\mathbb{R}_+$:*

$$Q(\rho) > \rho \quad \text{and} \quad Q''(\rho) > 0, \tag{2.3}$$

- *Q has the following asymptotic expansion as ρ tends to infinity:*

$$Q(\rho) = \rho + \sum_{n \geq 2} d_n \rho^{-n}, \tag{2.4}$$

  with some constants d_n such that $d_2 > 0$ and $d_4 = 0$.

[6] All along this paper, we identify radial functions on $\mathbb{R}^n$ with functions on $\mathbb{R}_+$.

[7] All the asymptotic expansions throughout this paper can be differentiated any number of times.

Proof. It is well-known (see for instance [5, 38] and the references therein) that the Cauchy problem (2.1) admits a unique solution Q in $\mathcal{C}^\infty(\mathbb{R}_+)$ satisfying (2.2), (2.3) and which behaves like

$$Q(\rho) = \rho + \frac{d_2}{\rho^2}(1 + o(1)) \quad \text{as } \rho \to \infty,$$

with $d_2 > 0$.

Writing $Q(\rho) = \rho\, v(\log \rho)$, we get the following equation for v:

$$-(v_{yy} + v_y) + 3(1 + (v + v_y)^2)(1/v - v - v_y) = 0. \tag{2.5}$$

Observe that the function $v \equiv 1$ solves (2.5) and the linearization of (2.5) around $v \equiv 1$is

$$-w_{yy} - 7w_y - 12w = 0. \tag{2.6}$$

The characteristic equation of (2.6) has two real distinct roots $r_1 = -3$ and $r_2 = -4$. This ensures that Q admits an asymptotic expansion of the form (2.4) as ρ tends to infinity, with $d_4 = 0$.

Finally, one can easily check that the formulae (2.2) and (2.4) can be differentiated to any order with respect to the variable ρ, which completes the proof of the lemma. ∎

2.2. *Properties of the linearized operator of the quasilinear wave equation around the ground state*

The blow up solution we construct in this paper is a small perturbation of the profile $t^{\nu+1}Q(\rho/t^{\nu+1})$, and thus the linearization of the quasilinear wave equation (1.8) around Q will play an important role. This linearized equation has the form

$$(1 + Q_\rho^2)w_{tt} - \mathcal{L}w = 0, \tag{2.7}$$

where $\mathcal{L}$ denotes the operator introduced in (1.16). We claim that the kernel of $\mathcal{L}$ contains the function $\Lambda Q := Q - \rho Q_\rho$, which is positive. Indeed, by Lemma 2.1, ΛQ tends to 0 at infinity and satisfies $(\Lambda Q)_\rho = -\rho Q_{\rho\rho}$. Recalling that $Q_{\rho\rho}(\rho) > 0$, we get the claim.

Setting $w = Hg$ with

$$H := \frac{(1 + Q_\rho^2)^{1/4}}{Q^{3/2}}, \tag{2.8}$$

one can rewrite the above equation (2.7) in the following way:

$$g_{tt} + \mathfrak{L}g = 0, \tag{2.9}$$

where

$$\mathfrak{L} = -q\Delta q + \mathcal{P}, \qquad q = \frac{1}{(1 + Q_\rho^2)^{1/2}}, \tag{2.10}$$

and $\mathcal{P}$ is a $\mathcal{C}^\infty_{\text{rad}}$ potential satisfying

$$\mathcal{P}(\rho) = -\frac{3}{8\rho^2}(1 + o(1)) \quad \text{as } \rho \to \infty. \tag{2.11}$$

The operator $\mathfrak{L}$ is self-adjoint on $L^2(\mathbb{R}^4)$ with domain $H^2(\mathbb{R}^4)$. Its spectral properties which are investigated in Appendix B rely on the asymptotic behavior $\mathscr{P}$ at infinity of the potential $\mathscr{P}$ given by (2.11). It follows from this spectral analysis that

$$(\mathfrak{L} f | f)_{L^2(\mathbb{R}^4)} \geq c \|\nabla f\|^2_{L^2(\mathbb{R}^4)}, \quad \forall f \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^4), \tag{2.12}$$

with some positive constant c.

3. Approximate solution in the inner region

3.1. General scheme of construction of the approximate solution in the inner region

In this section, we shall build, in the region $\rho/t \leq t^{\epsilon_1}$ (where $0 < \epsilon_1 < \nu$ is a fixed positive real number), a family of approximate solutions $u_{\mathrm{in}}^{(N)}$ to the quasilinear wave equation (1.8) as a perturbation of the profile $t^{\nu+1} Q(x/t^{\nu+1})$. Writing

$$u(t,\rho) = t^{\nu+1} V(t, \rho/t^{\nu+1}), \tag{3.1}$$

by straightforward computations we get

$$\begin{aligned}
u_\rho(t,\rho) &= V_y(t, \rho/t^{\nu+1}),\\
u_{\rho\rho}(t,\rho) &= \frac{1}{t^{\nu+1}} V_{yy}(t,\rho/t^{\nu+1}),\\
u_t(t,\rho) &= t^{\nu+1} V_t(t,\rho/t^{\nu+1}) + (\nu+1)\, t^\nu\, \Lambda V(t,\rho/t^{\nu+1}) =: t^\nu (\Gamma V)(t,\rho/t^{\nu+1}),\\
u_{t\rho}(t,\rho) &= t^{-1} (\Gamma V)_y (t,\rho/t^{\nu+1}),\\
t^{\nu+1} u_{tt}(t,\rho) &= t^{2\nu} [\Gamma^2 V - \Gamma V](t,\rho/t^{\nu+1}),
\end{aligned}$$

where we denote

$$\Gamma V := t\partial_t V + (\nu+1)\Lambda V \quad \text{with} \quad \Lambda V = V - yV_y \quad \text{and} \quad y = \rho/t^{\nu+1}. \tag{3.2}$$

Substituting (3.1) into (1.8) and multiplying by $t^{\nu+1}$, we get

$$\begin{aligned}
&(1+V_y^2) t^{2\nu} [\Gamma^2 V - \Gamma V] - (1 - t^{2\nu} (\Gamma V)^2) V_{yy}\\
&\qquad - 2t^{2\nu} V_y (\Gamma V)(\Gamma V)_y + 3(1 + V_y^2 - t^{2\nu}(\Gamma V)^2)\left(\frac{1}{V} - \frac{V_y}{y}\right) = 0.
\end{aligned} \tag{3.3}$$

Observe that the equation (3.3) multiplied by V/Q is polynomial of order 4 with respect to $(V, V_y, V_{yy}, \Gamma V, (\Gamma V)_y, \Gamma^2 V)$.

In what follows, we shall look for solutions V to (3.3) of the form

$$V(t,y) = \sum_{k\geq 0} t^{2\nu k} V_k(y) \tag{3.4}$$

with $V_0 = Q$, where Q is the stationary solution introduced in Lemma 2.1.

Substituting this ansatz into (3.3) multiplied by V/Q, we obtain the recurrent system

$$\mathcal{L}V_k = F_k(V_0, \dots, V_{k-1}), \tag{3.5}$$

where $k \geq 1$ and where F_k depends on V_j, $j = 0, \dots, k-1$, only.

Here $\mathcal{L}$ is defined by (1.16). The asymptotic formula (2.4) leads to the following asymptotic expansions of B_1 and B_0 as $y \to \infty$:

$$\begin{cases} B_1(y) = \dfrac{3}{y} + \displaystyle\sum_{n\geq 4} \beta_n y^{-n}, \\ B_0(y) = \dfrac{6}{y^2} + \displaystyle\sum_{n\geq 5} \alpha_n y^{-n}, \end{cases} \tag{3.6}$$

with some constants β_n and α_n that can be computed in terms of the coefficients d_n involved in (2.4).

Along the same lines, in view of (2.2) we find the following asymptotic formulae when y is close to 0:

$$\begin{cases} B_1(y) = \displaystyle\sum_{n\geq 0} a_{2n+1}\, y^{2n+1}, \\ B_0(y) = 3 + \displaystyle\sum_{n\geq 1} b_{2n}\, y^{2n}, \end{cases} \tag{3.7}$$

with some constants (a_{2n+1}) and (b_{2n}) that can be expressed in terms of the coefficients γ_{2n} appearing in (2.2).

The source term F_k can be split as

$$F_k = F_k^{(1)} + F_k^{(2)},$$

with $F_1^{(1)} = 0$, where $F_k^{(1)}$ comes from the expansion of the expression

$$-\frac{V}{Q} V_{yy} + 3(1 + V_y^2)\left(\frac{1}{Q} - \frac{V V_y}{yQ}\right),$$

while $F_k^{(2)}$ comes from the terms containing ΓV and $\Gamma^2 V$:

$$-\frac{V}{Q} V_{yy} + 3(1 + V_y^2)\left(\frac{1}{Q} - \frac{V V_y}{yQ}\right) = \sum_{k\geq 1} (-\mathcal{L}V_k + F_k^{(1)}) t^{2\nu k},$$

$$\frac{V}{Q}(1 + V_y^2)[\Gamma^2 V - \Gamma V] + \frac{V}{Q}(\Gamma V)^2 V_{yy} - 2\frac{V}{Q} V_y (\Gamma V)(\Gamma V)_y - 3(\Gamma V)^2\left(\frac{1}{Q} - \frac{V V_y}{yQ}\right)$$
$$= \sum_{k\geq 1} F_k^{(2)} t^{2\nu(k-1)}.$$

According to (3.4), this gives explicitly[8]

$$F_k^{(1)} = -\frac{1}{Q} \sum_{\substack{j_1+j_2=k \\ j_i \geq 1}} V_{j_1} \left((V_{j_2})_{yy} + 3 \frac{(V_{j_2})_y}{y} \right) - \frac{3}{yQ} \sum_{\substack{j_1+j_2+j_3+j_4=k \\ j_i \leq k-1}} (V_{j_1})_y (V_{j_2})_y (V_{j_3})_y V_{j_4} + \frac{3}{Q} \sum_{\substack{j_1+j_2=k \\ j_i \geq 1}} (V_{j_1})_y (V_{j_2})_y, \tag{3.8}$$

and

$$\begin{aligned} F_k^{(2)} = & \sum_{\substack{j_1+j_2+j_3+j_4=k-1 \\ j_i \geq 0}} \frac{V_{j_1}}{Q} (\Gamma_{j_2} V_{j_2})(\Gamma_{j_3} V_{j_3}) \left((V_{j_4})_{yy} + \frac{3(V_{j_4})_y}{y} \right) \\ & + \sum_{\substack{j_1+j_2=k-1 \\ j_i \geq 0}} \frac{V_{j_1}}{Q} (\Gamma_{j_2}^2 - \Gamma_{j_2}) V_{j_2} + \sum_{\substack{j_1+j_2+j_3+j_4=k-1 \\ j_i \geq 0}} \frac{V_{j_1} (V_{j_2})_y (V_{j_3})_y}{Q} (\Gamma_{j_4}^2 - \Gamma_{j_4}) V_{j_2} \\ & - 2 \sum_{\substack{j_1+j_2+j_3+j_4=k-1 \\ j_i \geq 0}} \frac{V_{j_1} (V_{j_2})_y}{Q} (\Gamma_{j_3} V_{j_3})(\Gamma_{j_4} V_{j_4})_y - \sum_{\substack{j_1+j_2=k-1 \\ j_i \geq 0}} \frac{3}{Q} (\Gamma_{j_1} V_{j_1})(\Gamma_{j_2} V_{j_2}), \end{aligned} \tag{3.9}$$

where $\Gamma_k = 2\nu k + (1+\nu)\Lambda$, so that

$$\Gamma(t^{2\nu k} V_k) = t^{2\nu k} \Gamma_k V_k. \tag{3.10}$$

We subject (3.5) to the initial conditions

$$V_k(0) = 0 \quad \text{and} \quad V_k'(0) = 0. \tag{3.11}$$

3.2. *Analysis of the functions V_k*

The goal of the present subsection is to prove the following result:

Lemma 3.1. *The recurrent system (3.5)–(3.11) has a unique solution $(V_k)_{k \geq 1}$ such that for any $k \geq 1$, the function V_k is in $\mathcal{C}^\infty(\mathbb{R}_+)$ and has the following asymptotic behavior:*

$$V_k(y) = \sum_{n \geq 1} c_{2n,k} y^{2n} \quad \text{as } y \sim 0, \tag{3.12}$$

$$V_k(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n \geq 2-2(k-\ell)} d_{n,k,\ell} y^{-n} \quad \text{as } y \sim \infty, \tag{3.13}$$

with

$$d_{-2(k-2),k,1} = 0. \tag{3.14}$$

[8] Here and below, we use the convention that the sum is null if it is over an empty set.

Proof. Let us first emphasize that by classical techniques of ordinary differential equations, for any regular function g, the solution to the Cauchy problem

$$\begin{cases} \mathcal{L} f = g, \\ f(0) = 0 \quad \text{and} \quad f'(0) = 0, \end{cases} \tag{3.15}$$

can be written in the form

$$f(y) = -(\Lambda Q)(y) \int_0^y \frac{(1 + (Q_r(r))^2)^{3/2}}{Q^3(r) r^3 (\Lambda Q)^2(r)} \int_0^r \frac{Q^3(s) s^3 (\Lambda Q)(s)}{(1 + (Q_s(s))^2)^{3/2}} g(s)\, ds\, dr \tag{3.16}$$

(see Appendix D).

Let us start by considering the case when $k = 1$. With notations (3.2) and in light of (3.8) and (3.9), we have

$$\begin{aligned} F_1(Q) = F_1^{(2)}(Q) &= (1 + Q_y^2)\big((1+\nu)^2 \Lambda^2 - (1+\nu)\Lambda\big) Q \\ &\quad - 2(1+\nu)^2 Q_y (\Lambda Q)(\Lambda Q)_y + (1+\nu)^2 (\Lambda Q)^2 \frac{Q_{yy}(Q_y)^2}{1 + Q_y^2}. \end{aligned} \tag{3.17}$$

According to (2.2), this implies that for y close to 0 the following asymptotic formula holds:

$$F_1(Q) = \sum_{n \geq 0} g_{2n,1} y^{2n}. \tag{3.18}$$

Moreover, in view of (2.4), we get the following expansion as $y \to \infty$:

$$F_1(Q) = \sum_{n \geq 2} c_{n,1,0} y^{-n}. \tag{3.19}$$

By Lemma 2.1 which asserts that[9] $d_{4,0,0} =: d_4 = 0$, we find that $c_{4,1,0} = 0$. Indeed, invoking (2.4) together with (3.17), we easily check that

$$c_{4,1,0} = 10(1+\nu)(4+5\nu) d_{4,0,0},$$

which implies that the coefficient $c_{4,1,0}$ is zero.

This ensures in view of the Duhamel formula (3.16) that for $k = 1$, the Cauchy problem (3.5), (3.11) has a unique solution V_1 in $\mathcal{C}^\infty(\mathbb{R}_+)$ admitting the asymptotic expansions (3.12) and (3.13) respectively close to 0 and at infinity.

Regarding the expansion coefficients $d_{n,1,\ell}$ of V_1 at infinity, we can find them by substituting

$$V_1(y) = \sum_{\ell=0}^{1} (\log y)^\ell \sum_{n \geq 2\ell} d_{n,1,\ell} y^{-n}$$

[9] In what follows, the coefficients d_p introduced in (2.4) will be denoted by $d_{p,0,0}$.

into (3.5) and taking into account (3.6) and (3.19). This gives rise to

$$\frac{2d_{1,1,0}}{y^3} - \sum_{n\geq 2}((2n+1)d_{n,1,1} - n(n+1)d_{n,1,0})y^{-n-2}$$
$$+\Big(\frac{6}{y}+\sum_{n\geq 4}\beta_n y^{-n}\Big)\Big(-\frac{d_{1,1,0}}{y^2}+\sum_{n\geq 2}(d_{n,1,1}-nd_{n,1,0})y^{-n-1}-\sum_{n\geq 2}nd_{n,1,1}(\log y)y^{-n-1}\Big)$$
$$+\Big(\frac{6}{y^2}+\sum_{n\geq 5}\alpha_n y^{-n}\Big)\Big(d_{0,1,0}+\frac{d_{1,1,0}}{y}+\sum_{n\geq 2}d_{n,1,0}y^{-n}+\sum_{n\geq 2}d_{n,1,1}(\log y)y^{-n}\Big)$$
$$+\sum_{n\geq 2}n(n+1)d_{n,1,1}(\log y)y^{-n-2} = \sum_{n\geq 2}c_{n,1,0}y^{-n}. \qquad (3.20)$$

In particular, the identification of the coefficient of y^{-4} in (3.20) gives

$$d_{2,1,1} = c_{4,1,0} = 0, \qquad (3.21)$$

which proves that (3.14) is fulfilled for $k = 1$.

Now using the fact that the coefficient of $(\log y)y^{-n-2}$ in (3.20) is null, we find that for any integer $n \geq 2$,

$$d_{n,1,1}(n^2-5n+6) + \sum_{\substack{k_1+k_2=n+2\\ k_1\geq 5,\, k_2\geq 2}} d_{k_2,1,1}\alpha_{k_1} - \sum_{\substack{k_1+k_2=n+1\\ k_1\geq 4,\, k_2\geq 2}} k_2 d_{k_2,1,1}\beta_{k_1} = 0.$$

Along the same lines, by computing the coefficients of y^{-n-2} we get

$$d_{n,1,0}(n^2-5n+6) + (5-2n)d_{n,1,1} + \sum_{\substack{k_1+k_2=n+1\\ k_1\geq 4,\, k_2\geq 2}} \beta_{k_1}(d_{k_2,1,1} - k_2 d_{k_2,1,0})$$
$$+ \sum_{\substack{k_1+k_2=n+2\\ k_1\geq 5,\, k_2\geq 2}} \alpha_{k_1} d_{k_2,1,0} = c_{n+2,1,0}.$$

This implies that all the coefficients $d_{n,1,\ell}$ can be determined successively in terms of the coefficients of $F_1(Q)$ involved in (3.19) and the coefficients $d_{2,1,0}$ and $d_{3,1,0}$ that are fixed by the initial data.

We next turn our attention to the general case of any index $k \geq 2$. To this end, we shall proceed by induction assuming that, for any integer $1 \leq j \leq k-1$, the Cauchy problem (3.5), (3.11) has a unique solution V_j in $\mathcal{C}^\infty(\mathbb{R}_+)$ satisfying formulae (3.12) and (3.13) as well as condition (3.14).

Invoking (3.8) together with (3.12) and (3.13), one can easily check that

$$F_k^{(1)}(V_0,\dots,V_{k-1})(y) = \sum_{n\geq 0} g_{2n,k}^{(1)} y^{2n} \quad \text{as } y \sim 0, \qquad (3.22)$$

$$F_k^{(1)}(V_0,\dots,V_{k-1})(y) = \sum_{\ell=0}^{k}(\log y)^\ell \sum_{n\geq 7-2(k-\ell)} c_{n,k,\ell}^{(1)} y^{-n} \quad \text{as } y \sim \infty. \qquad (3.23)$$

Similarly from (3.9), (3.12) and (3.13), we deduce that

$$F_k^{(2)}(V_0,\dots,V_{k-1})(y) = \sum_{n\geq 0} g_{2n,k}^{(2)} y^{2n} \quad \text{as } y \sim 0, \tag{3.24}$$

and

$$F_k^{(2)}(V_0,\dots,V_{k-1}) = (1+Q_y^2)(\Gamma_{k-1}^2 - \Gamma_{k-1})V_{k-1} + \widetilde{F}_k^{(2)}(V_0,\dots,V_{k-1}), \tag{3.25}$$

where $\widetilde{F}_k^{(2)}$ has the following expansion at infinity:

$$\widetilde{F}_k^{(2)}(V_0,\dots,V_{k-1})(y) = \sum_{\ell=0}^{k-1} (\log y)^\ell \sum_{n\geq 7-2(k-\ell)} \widetilde{c}_{n,k,\ell}^{(2)} y^{-n}. \tag{3.26}$$

Recall that by definition

$$\Gamma_{k-1} = 2\nu(k-1) + (1+\nu)\Lambda =: \alpha(\nu,k-1) + (1+\nu)\Lambda,$$

which by straightforward computations gives rise to[10]

$$(\Gamma_{k-1}^2 - \Gamma_{k-1})V_{k-1} = \alpha(\alpha-1)V_{k-1} + (1+\nu)(2\alpha-1)\Lambda V_{k-1} + (1+\nu)^2\Lambda^2 V_{k-1}.$$

Setting

$$\beta := \alpha(\alpha-1) + (1+\nu)(2\alpha-1) + (1+\nu)^2,$$

we easily gather that

$$(\Gamma_{k-1}^2 - \Gamma_{k-1})V_{k-1} = \beta V_{k-1} - (1+\nu)(2\alpha-1)y\partial_y V_{k-1} + (1+\nu)^2 y^2 \partial_y^2 V_{k-1}.$$

It follows from (2.4) and (3.13) that the following expansion holds at infinity:

$$(1+Q_y^2)(\Gamma_{k-1}^2 - \Gamma_{k-1})V_{k-1}(y) = \sum_{\ell=0}^{k-1} (\log y)^\ell \sum_{n\geq 2-2(k-1-\ell)} c_{n,k,\ell}^{(2)} y^{-n}, \tag{3.27}$$

where for any integer $0 \leq \ell \leq k-1$,

$$2c_{2-2(k-1-\ell),k,\ell}^{(2)} = (\beta + n(1+\nu)(2\alpha-1) + (1+\nu)^2 n(n+1)) d_{2-2(k-1-\ell),k-1,\ell}.$$

In view of the induction assumption (3.14) for the index $k-1$, we get

$$c_{2-2(k-2),k,1}^{(2)} = 0. \tag{3.28}$$

Combining (3.22) with (3.23), (3.24), (3.26), (3.27) and (3.28), we deduce that

$$F_k(V_0,\dots,V_{k-1}) = F_k^{(1)}(V_0,\dots,V_{k-1}) + F_k^{(2)}(V_0,\dots,V_{k-1})$$

[10] In order to make notations as light as possible, all along this proof we shall omit the dependence of the function α on the parameters ν and k.

admits the following asymptotic expansions:

$$F_k(V_0, \dots, V_{k-1})(y) = \sum_{n \geq 0} g_{2n,k} y^{2n} \quad \text{as } y \sim 0, \tag{3.29}$$

$$F_k(V_0, \dots, V_{k-1})(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n \geq 4-2(k-\ell)} c_{n,k,\ell} y^{-n} \quad \text{as } y \sim \infty, \tag{3.30}$$

with

$$c_{2-2(k-2),k,1} = 0. \tag{3.31}$$

Therefore the Duhamel formula (3.16) implies that the Cauchy problem (3.5), (3.11) admits a unique solution V_k in $\mathcal{C}^\infty(\mathbb{R}_+)$ satisfying the asymptotic formulae (3.12) and (3.13) respectively close to 0 and at infinity. As for V_1, we can determine all the coefficients $d_{n,k,\ell}$ in terms of F_k and $d_{2,k,0}$ and $d_{3,k,0}$ that are fixed by the initial data, by substituting the expansion

$$V_k(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n \geq 2-2(k-\ell)} d_{n,k,\ell} y^{-n}$$

into (3.5). In particular, for $0 \leq \ell \leq k-1$ and $n = 2 - 2(k-\ell)$ we get

$$(n^2 - 5n + 6) d_{n,k,\ell} = c_{n+2,k,\ell},$$

which by (3.31) ensures that

$$d_{-2(k-2),k,1} = 0$$

and proves (3.14). This concludes the proof of the lemma. ∎

3.3. *Estimate of the approximate solution in the inner region*

Under the above notations, for any integer $N \geq 2$ set

$$u_{\text{in}}^{(N)}(t, \rho) = t^{\nu+1} V_{\text{in}}^{(N)}(t, \rho/t^{\nu+1}) \quad \text{with} \quad V_{\text{in}}^{(N)}(t, y) = \sum_{k=0}^{N} t^{2\nu k} V_k(y). \tag{3.32}$$

Our aim in this subsection is to investigate the properties of $V_{\text{in}}^{(N)}$ in the inner region

$$\Omega_{\text{in}} := \{Y \in \mathbb{R}^4 : y = |Y| \leq t^{\epsilon_1 - \nu}\}. \tag{3.33}$$

Thanks to Lemma 3.1, we easily gather that $V_{\text{in}}^{(N)}$ satisfies the following L^∞ estimates on Ω_{in}:

Lemma 3.2. *For any multi-index α in $\mathbb{N}^4$ and any integer $\beta \leq |\alpha|$, there exist a positive constant $C_{\alpha,\beta}$ and a small positive time $T = T(\alpha, \beta, N)$ such that for all $0 < t \leq T$,*

$$\|\langle\cdot\rangle^{\beta}\nabla^{\alpha}(V_{\text{in}}^{(N)}(t,\cdot) - Q)\|_{L^{\infty}(\Omega_{\text{in}})} \leq C_{\alpha,\beta}t^{2\nu}, \tag{3.34}$$

$$\|\nabla^{\alpha}\partial_t V_{\text{in}}^{(N)}(t,\cdot)\|_{L^{\infty}(\Omega_{\text{in}})} \leq C_{\alpha}t^{2\nu-1}, \tag{3.35}$$

$$\|\langle\cdot\rangle^{\beta}\nabla^{\alpha}(\Gamma V_{\text{in}}^{(N)})(t,\cdot)\|_{L^{\infty}(\Omega_{\text{in}})} \leq C_{\alpha,\beta}, \tag{3.36}$$

$$\|\partial_t(\Gamma V_{\text{in}}^{(N)})(t,\cdot)\|_{L^{\infty}(\Omega_{\text{in}})} \leq Ct^{2\nu-1}, \tag{3.37}$$

$$\|\langle\cdot\rangle^{\beta}\nabla^{\alpha}((\Gamma^2 - \Gamma)V_{\text{in}}^{(N)})(t,\cdot)\|_{L^{\infty}(\Omega_{\text{in}})} \leq C_{\alpha,\beta}, \tag{3.38}$$

where as above $\Gamma = t\partial_t + (\nu + 1)\Lambda$.

Along the same lines, taking advantage of Lemma 3.1, we get the following L^2 estimates:

Lemma 3.3. *With the previous notations, for all $0 < t \leq T$ we have*

$$\|\nabla(V_{\text{in}}^{(N)}(t,\cdot) - Q)\|_{L^2(\Omega_{\text{in}})} \leq Ct^{\nu}, \tag{3.39}$$

$$\|\nabla^{\alpha}(V_{\text{in}}^{(N)}(t,\cdot) - Q)\|_{L^2(\Omega_{\text{in}})} \leq C_{\alpha}t^{2\nu}, \quad \forall|\alpha| \geq 2, \tag{3.40}$$

$$\|(\Gamma^{\ell}V_{\text{in}}^{(N)})(t,\cdot)\|_{L^2(\Omega_{\text{in}})} \leq C\log t, \quad \forall\ell = 1, 2, \tag{3.41}$$

$$\|\nabla^{\alpha}(\Gamma^{\ell}V_{\text{in}}^{(N)})(t,\cdot)\|_{L^2(\Omega_{\text{in}})} \leq C_{\alpha}, \quad \forall|\alpha| \geq 1, \forall\ell = 1, 2. \tag{3.42}$$

Remark 3.1. Denoting

$$\Omega_{\text{in}}^x := \{x \in \mathbb{R}^4 : |x| \leq t^{1+\epsilon_1}\},$$

and combining (3.32) with the above lemma, we infer that the radial function $u_{\text{in}}^{(N)}$ on Ω_{in}^x satisfies for all $0 < t \leq T$,

$$\|\nabla^{\alpha}(u_{\text{in}}^{(N)}(t,\cdot) - t^{\nu+1}Q(\cdot/t^{\nu+1}))\|_{L^2(\Omega_{\text{in}}^x)} \leq C_{\alpha}t^{\nu+(|\alpha|-3)(\nu+1)}, \quad \forall|\alpha| \geq 1, \tag{3.43}$$

$$\|\nabla^{\alpha}\partial_t u_{\text{in}}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\text{in}}^x)} \leq C_{\alpha}t^{\nu+(|\alpha|-3)(\nu+1)}, \quad \forall|\alpha| \geq 1, \tag{3.44}$$

$$\|\partial_t u_{\text{in}}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\text{in}}^x)} \leq Ct^{\nu-3(\nu+1)}\log t. \tag{3.45}$$

Let us end this section by estimating the remainder term

$$\mathcal{R}_{\text{in}}^{(N)} := (3.3)V_{\text{in}}^{(N)}.$$

Lemma 3.4. *For any multi-index α, there exist a positive constant $C_{\alpha,N}$ and a small positive time $T = T(\alpha, N)$ such that for all $0 < t \leq T$,*

$$\|\langle\cdot\rangle^{3/2}\nabla^{\alpha}\mathcal{R}_{\text{in}}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\text{in}})} \leq C_{\alpha,N}t^{2\nu+2N\epsilon_1-\frac{3}{2}(\nu-\epsilon_1)}. \tag{3.46}$$

Proof. In view of the computations carried out in §3.1, we have

$$\frac{V}{Q}\Big[(3.3)\Big(\sum_{k\geq 0} t^{2\nu k} V_k\Big)\Big] = \sum_{k\geq 1}(-\mathcal{L}V_k + F_k)t^{2\nu k}.$$

Thus recalling that $\frac{V}{Q}[(3.3)V]$ is a polynomial of order four and taking into account Lemma 3.1, we deduce that

$$\widetilde{\mathcal{R}}_{\mathrm{in}}^{(N)} = \frac{V_{\mathrm{in}}^{(N)}}{Q}\mathcal{R}_{\mathrm{in}}^{(N)} = \sum_{N+1\leq k\leq 4N} t^{2\nu k} G_k, \tag{3.47}$$

where G_k is defined, as the function F_k, by formulae (3.8) and (3.9), where we assume in addition that the indices j_i involved range from 0 to N.

This of course implies that the function G_k, $N+1 \leq k \leq 4N$, admits the following expansions, respectively close to 0 and at infinity:

$$G_k(y) = \sum_{n\geq 0} \widetilde{g}_{2n,k}\, y^{2n}, \tag{3.48}$$

$$G_k(y) = \sum_{\ell=0}^{k} (\log y)^{\ell} \sum_{n\geq 4-2(k-\ell)} \widetilde{c}_{n,k,\ell}\, y^{-n}, \tag{3.49}$$

with some constants $\widetilde{g}_{2n,k}$ and $\widetilde{c}_{n,k,\ell}$ that can be determined recursively in terms of the functions V_j with $j = 0, \ldots, N$.

Recalling that by definition

$$\mathcal{R}_{\mathrm{in}}^{(N)} = \frac{Q}{V_{\mathrm{in}}^{(N)}}\widetilde{\mathcal{R}}_{\mathrm{in}}^{(N)},$$

we deduce taking into account Lemma 3.2 and (3.33) that for any multi-index α, there exist a positive constant $C_{\alpha,N}$ and a positive time $T = T(\alpha, N)$ such that for any time $0 < t \leq T$, we have

$$\|\langle\cdot\rangle^{3/2}\nabla^{\alpha}\mathcal{R}_{\mathrm{in}}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\mathrm{in}})} \leq C_{\alpha,N}\, t^{2\nu+2N\epsilon_1-\frac{3}{2}(\nu-\epsilon_1)}.$$

This ends the proof of the lemma. ∎

4. Approximate solution in the self-similar region

4.1. *General scheme of construction of the approximate solution in the self-similar region*

Our aim in this section is to build, in the region $\frac{1}{10}t^{\epsilon_1} \leq \frac{\rho}{t} \leq 10t^{-\epsilon_2}$, an approximate solution $u_{\mathrm{ss}}^{(N)}$ to (1.8) that extends the approximate solution $u_{\mathrm{in}}^{(N)}$ constructed in the inner region $\rho/t \leq t^{\epsilon_1}$. Here $0 < \epsilon_2 < 1$ is fixed.

We shall look for this solution in the following form:

$$u(t,\rho) = \lambda(t)(z + W(t,z)) \quad \text{with} \quad z = \rho/\lambda(t), \tag{4.1}$$

where $\lambda(t)$ is a function that behaves like t for t close to 0 and that will be constructed at the same time as the profile W. In fact, $\lambda(t)$ will be given by an expression of the form

$$\lambda(t) = t\Big(1 + \sum_{k\geq 3}\sum_{\ell=0}^{\ell(k)} \lambda_{k,\ell} t^{\nu k}(\log t)^{\ell}\Big) \quad \text{with} \quad \ell(k) = [(k-3)/2]. \tag{4.2}$$

By straightforward computations, we find that

$$\begin{aligned} u_\rho(t,\rho) &= 1 + W_z(t,\rho/\lambda(t)),\\ u_{\rho\rho}(t,\rho) &= \lambda(t)^{-1} W_{zz}(t,\rho/\lambda(t)),\\ u_t(t,\rho) &= \lambda(t)\, W_t(t,\rho/\lambda(t)) + \lambda'(t)\Lambda W(t,\rho/\lambda(t)) =: W_1(t,\rho/\lambda(t)), \qquad (4.3)\\ u_{t\rho}(t,\rho) &= \lambda(t)^{-1}(\partial_z W_1)(t,\rho/\lambda(t)),\\ \lambda(t)u_{tt}(t,\rho) &= W_2(t,\rho/\lambda(t)), \qquad (4.4) \end{aligned}$$

with

$$\begin{aligned} W_2(t,z) &:= \lambda(t)\lambda''(t)\Lambda W + 2\lambda(t)\lambda'(t)\Lambda W_t + \lambda^2(t)\, W_{tt} + (\lambda'(t))^2 z^2 W_{zz}\\ &= z^2\, W_{zz}(t,z) + t^2 W_{tt}(t,z) + 2t\Lambda W_t(t,z) + \widetilde{W}_2(t,z), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \widetilde{W}_2(t,z) = ((\lambda'(t))^2 - 1)z^2 W_{zz} + (\lambda^2(t) - t^2)W_{tt} + 2(\lambda(t)\lambda'(t) - t)\Lambda W_t\\ + \lambda(t)\lambda''(t)\Lambda W, \end{aligned} \tag{4.6}$$

and where as above $\Lambda W = W - zW_z$.

Thus substituting (4.1) into (1.8) multiplied by $\lambda(t)$, we find that the function W solves the following equation:

$$\begin{aligned} (1 + (1+W_z)^2)W_2 - (1 - (W_1)^2)W_{zz} - 2(1+W_z)W_1(W_1)_z\\ - 3(1 + (1+W_z)^2 - (W_1)^2)\Big(\frac{\check{W}}{z^2} + \frac{W_z}{z}\Big) = 0, \end{aligned} \tag{4.7}$$

where $\check{W} = W/(1 + W/z)$. Introducing the notations

$$\begin{aligned} W_2' &:= W_2 - (\lambda')^2 z^2 W_{zz} = \lambda^2 W_{tt} + 2\lambda\lambda'\Lambda W_t + \lambda\lambda''\Lambda W,\\ W_3 &:= \lambda W_{tz}, \end{aligned} \tag{4.8}$$

we readily gather that (4.7) can be rewritten as

$$(2z^2 - 1 + A_0)W_{zz} + A_1 = 0 \tag{4.9}$$

with

$$A_0 = (\lambda' W + \lambda W_t)^2 + 2\lambda' z(\lambda' W + \lambda W_t) + 2((\lambda')^2 - 1)z^2,$$
$$A_1 = (1 + (1 + W_z)^2)W_2' - 2(1 + W_z)W_1 W_3 - 3\big(1 + (1 + W_z)^2 - (W_1)^2\big)\Big(\frac{\check{W}}{z^2} + \frac{W_z}{z}\Big). \tag{4.10}$$

Denoting by L the linear operator defined by

$$L = (2z^2 - 1)\partial_z^2 + 2t^2\partial_t^2 + 4t\Lambda\partial_t - 6z^{-1}\partial_z - 6z^{-2}, \tag{4.11}$$

we infer that (4.7) takes the form

$$\begin{aligned} LW &= -A_0 W_{zz} - [2W_z + (W_z)^2]W_2' - 2\widetilde{W}_2' + 2(1 + W_z)W_1 W_3 \\ &\quad - \frac{6}{z^3}\check{W} W + 3\big(2W_z + (W_z)^2 - (W_1)^2\big)\Big(\frac{\check{W}}{z^2} + \frac{W_z}{z}\Big), \end{aligned} \tag{4.12}$$

where

$$\widetilde{W}_2' := W_2' - t^2 W_{tt} - 2t\Lambda W_t = \widetilde{W}_2 - ((\lambda')^2 - 1)z^2 W_{zz}. \tag{4.13}$$

It will be useful later on to notice that with the above notations, (4.12) can also be rewritten as

$$\begin{aligned} LW &= -2\widetilde{W}_2 - [2W_z + (W_z)^2]W_2 - (W_1)^2 W_{zz} + 2(1 + W_z)W_1(W_1)_z \\ &\quad - \frac{6}{z^3}\check{W} W + 3(2W_z + (W_z)^2 - (W_1)^2)\Big(\frac{\check{W}}{z^2} + \frac{W_z}{z}\Big). \end{aligned} \tag{4.14}$$

The asymptotics of the solution (4.1) at the origin has to be consistent with that of (3.1) at infinity. To determine this asymptotics, we combine the expansion (4.2) with formula (3.13), which gives

$$u_{\mathrm{in}}(t,\rho) = \lambda(t)\Big(z + \sum_{k\geq 3} t^{\nu k} \sum_{\ell=0}^{\ell(k)} (\log t)^{\ell} \sum_{0\leq\alpha\leq(k-3)/2-\ell} (\log z)^{\alpha} \sum_{\beta\geq 1-k+2(\alpha+\ell)} c_{\alpha,\beta}^{k,\ell} z^{\beta}\Big) \tag{4.15}$$

as $z \to 0$, with the coefficients $c_{\alpha,\beta}^{k,\ell}$ admitting the representation

$$c_{\alpha,\beta}^{k,\ell} = c_{\alpha,\beta}^{k,\ell,0} + c_{\alpha,\beta}^{k,\ell,1},$$

where $c_{\alpha,\beta}^{k,\ell,0}$ are independent of λ and are given by

$$c_{\alpha,\beta}^{k,\ell,0} = \begin{cases} 0 & \text{if } \beta + k - 1 \text{ is odd,} \\ (-\nu)^{\ell}\binom{\alpha+\ell}{\alpha} d_{-\beta,\frac{\beta+k-1}{2},\alpha+\ell} & \text{if } \beta + k - 1 \text{ is even,} \end{cases}$$

and where the coefficients $c^{k,\ell,1}_{\alpha,\beta}$ depend only on $\lambda_{p,q}$ involved in (4.2) with $3 \le p \le k-3$ and are zero if $\beta + k - 1 - 2(\alpha+\ell) \le 2$ or if $k < 6$ or if $\ell > (k-6)/2$.

Let us point out that taking into account Lemma 2.1 together with (3.14), which respectively assert that $d_{4,0,0} = 0$ and $d_{4-2m,m,1} = 0$ for any integer $m \ge 1$, we infer that

$$c^{5,0}_{0,-4} = 0 \quad \text{and} \quad c^{5,1}_{0,\beta} = 0, \qquad \forall \beta. \tag{4.16}$$

Formula (4.15) makes us look for the approximate solution in the self-similar region in the form

$$u(t,\rho) = \rho + \lambda(t) W(t, \rho/\lambda(t)), \tag{4.17}$$

where

$$W(t,z) = \sum_{k\ge 3} t^{\nu k} \sum_{\ell=0}^{\ell(k)} (\log t)^{\ell}\, w_{k,\ell}(z). \tag{4.18}$$

To fix $\lambda(t)$, we require that the function A_0 defined by (4.10) satisfies

$$A_{0|z=1/\sqrt{2}} = 0. \tag{4.19}$$

One difficulty that we face in solving (4.12) is handling the degeneracy of the operator L defined by (4.11) on the light cone $z = 1/\sqrt{2}$. The condition (4.19) ensures that the coefficient of W_{zz} involved in the equation we deal with vanishes at $z = 1/\sqrt{2}$. This will enable us to determine successively the functions $w_{k,\ell}$ involved in (4.18) without loss of regularity at each step.

Invoking (4.2) together with (4.18), we infer that the functions W_1, W_2, $\widetilde{W}_2$, W_2', $\widetilde{W}_2'$, W_3, $\check{W}$ and A_0 defined above admit expansions of the same form as W. More precisely,

$$\begin{aligned}
W_i(t,z) &= \sum_{k\ge 3} t^{\nu k} \sum_{0\le \ell \le (k-3)/2} (\log t)^{\ell}\, w^i_{k,\ell}(z), \qquad i = 1,2,3,\\
\widetilde{W}_2(t,z) &= \sum_{k\ge 6} t^{\nu k} \sum_{0\le \ell \le (k-6)/2} (\log t)^{\ell}\, \widetilde{w}^2_{k,\ell}(z),\\
W_2'(t,z) &= \sum_{k\ge 3} t^{\nu k} \sum_{0\le \ell \le (k-3)/2} (\log t)^{\ell}\, w^{(2,')}_{k,\ell}(z),\\
\widetilde{W}_2'(t,z) &= \sum_{k\ge 6} t^{\nu k} \sum_{0\le \ell \le (k-6)/2} (\log t)^{\ell}\, \widetilde{w}^{(2,')}_{k,\ell}(z),\\
\check{W}(t,z) &= \sum_{k\ge 3} t^{\nu k} \sum_{0\le \ell \le (k-3)/2} (\log t)^{\ell}\, \check{w}_{k,\ell}(z),\\
A^0(t,z) &= \sum_{k\ge 3} t^{\nu k} \sum_{0\le \ell \le (k-3)/2} (\log t)^{\ell}\, A^0_{k,\ell}(z),
\end{aligned}$$

where $w^i_{k,\ell}$, $i = 1,2,3$, and $w^{(2,')}_{k,\ell}$ depend only on $w_{k',\ell'}$, $3 \le k' \le k$, and $\lambda_{k'',\ell''}$, $3 \le k'' \le k-3$, where $\widetilde{w}^2_{k,\ell}$ and $\widetilde{w}^{(2,')}_{k,\ell}$ depend on $w_{k',\ell'}$ and $\lambda_{k'',\ell''}$, $3 \le k', k'' \le k-3$, and $A^0_{k,\ell}$ on $w_{k',\ell'}$ and $\lambda_{k'',\ell''}$ with $3 \le k', k'' \le k$.

Observe also that[11]

$$\begin{aligned} w^1_{k,\ell} &= (\nu k + \Lambda) w_{k,\ell} + (\ell+1) w_{k,\ell+1} + \widetilde{w}^1_{k,\ell}, \\ \widetilde{w}^1_{k,\ell} &= \sum_{k_1+k_2=k,\, \ell_1+\ell_2=\ell} \lambda_{k_2,\ell_2}\big(\nu k_1\, w_{k_1,\ell_1} + (\ell_1+1) w_{k_1,\ell_1+1}\big) \\ &\quad + \sum_{k_1+k_2=k,\, \ell_1+\ell_2=\ell} [(1+\nu k_2)\lambda_{k_2,\ell_2} + (\ell_2+1)\lambda_{k_2,\ell_2+1}]\Lambda w_{k_1,\ell_1}, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} w^3_{k,\ell} &= \nu k \partial_z w_{k,\ell} + (\ell+1)\partial_z w_{k,\ell+1} + \widetilde{w}^3_{k,\ell}, \\ \widetilde{w}^3_{k,\ell} &= \sum_{k_1+k_2=k,\, \ell_1+\ell_2=\ell} \lambda_{k_2,\ell_2}\big(\nu k_1 \partial_z w_{k_1,\ell_1} + (\ell_1+1)\partial_z w_{k_1,\ell_1+1}\big). \end{aligned} \tag{4.21}$$

In addition, one has[12]

$$\begin{aligned} w^2_{k,\ell}(z) &= z^2\partial_z^2 w_{k,\ell} + \nu k(\nu k + 1 - 2z\partial_z) w_{k,\ell} \\ &\quad + (\ell+1)(2\nu k + 1 - 2z\partial_z) w_{k,\ell+1} + (\ell+1)(\ell+2) w_{k,\ell+2} + \widetilde{w}^2_{k,\ell}, \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} w^{(2,')}_{k,\ell} &= \nu k(\nu k + 1 - 2z\partial_z) w_{k,\ell} + (\ell+1)(2\nu k + 1 - 2z\partial_z) w_{k,\ell+1} \\ &\quad + (\ell+1)(\ell+2) w_{k,\ell+2} + \widetilde{w}^{(2,')}_{k,\ell}. \end{aligned} \tag{4.23}$$

Now substituting expansions (4.2) and (4.18) into (4.12) and (4.19), we deduce the following recurrent system for $(w_{k,\ell}, \lambda_{k,\ell})_{k\geq 3,\, 0\leq \ell\leq \ell(k)}$:

$$\begin{cases} \widetilde{\mathcal{L}}_k w_{k,\ell} = F_{k,\ell}, \quad 0 \leq \ell \leq \ell(k), \\ (1+\nu k)\lambda_{k,\ell} + (\ell+1)\lambda_{k,\ell+1} = -\big((1+\nu k) w_{k,\ell} + (\ell+1) w_{k,\ell+1}\big)_{|z=1/\sqrt{2}} + g_{k,\ell}. \end{cases} \tag{4.24}$$

Here $\widetilde{\mathcal{L}}_k$ is the operator

$$\widetilde{\mathcal{L}}_k = (2z^2-1)\partial_z^2 - \left(4z\nu k + \frac{6}{z}\right)\partial_z + 2\nu k(1+\nu k) - \frac{6}{z^2}, \tag{4.25}$$

and the source term $F_{k,\ell}$ can be divided into a linear part and a nonlinear part as follows:

$$F_{k,\ell} = F^{\mathrm{lin}}_{k,\ell} + F^{\mathrm{nl}}_{k,\ell}, \tag{4.26}$$

[11]With the convention all along this section that $\lambda_{k,\ell'} = 0$ and $w_{k,\ell'} \equiv 0$ if $k < 3$ or $\ell' > [(k-3)/2]$.

[12]One can give explicit expressions for $\widetilde{w}^2_{k,\ell}$ and $\widetilde{w}^{(2,')}_{k,\ell}$ of the same type as for $\widetilde{w}^1_{k,\ell}$ and $\widetilde{w}^3_{k,\ell}$, but for simplicity, we will not specify them.

where

$$F^{\mathrm{lin}}_{k,\ell} = -2(2\nu k+1)(\ell+1)w_{k,\ell+1} + 4z(\ell+1)(w_{k,\ell+1})_z - 2(\ell+1)(\ell+2)w_{k,\ell+2}, \tag{4.27}$$

and $F^{\mathrm{nl}}_{k,\ell}$ depends only on $w_{k',\ell'}$ and $\lambda_{k'',\ell''}$, for $3 \le k', k'' \le k-3$. Similarly, the coefficients $g_{k,\ell}$ depend only on the values of the functions $w_{k',\ell'}$ at $z = 1/\sqrt{2}$ and the coefficients $\lambda_{k'',\ell''}$ for $3 \le k', k'' \le k-3$ ($F^{\mathrm{nl}}_{k,\ell}$ and $g_{k,\ell}$ are identically null if $k < 6$ or $\ell > (k-6)/2$).

In other words, for any integer $k \ge 3$ the functions $(w_{k,\ell})_{0\le\ell\le\ell(k)}$ satisfy

$$\mathcal{S}_k \mathcal{W}_k = \mathcal{F}^{\mathrm{nl}}_k \tag{4.28}$$

where $\mathcal{S}_k$ denotes the matrix operator

$$\begin{pmatrix}
\tilde{\mathcal{L}}_k & \mathcal{A}_k(0)+\mathcal{B}(0,z)\partial_z & \mathcal{C}(0) & 0 & \dots & \dots & \dots \\
0 & \tilde{\mathcal{L}}_k & \mathcal{A}_k(1)+\mathcal{B}(1,z)\partial_z & \mathcal{C}(1) & 0 & \dots & \dots \\
\dots & 0 & \tilde{\mathcal{L}}_k & \dots & \dots & 0 & \dots \\
\dots & \dots & \dots & \tilde{\mathcal{L}}_k & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \tilde{\mathcal{L}}_k & \mathcal{A}_k(\ell(k)-1)+\mathcal{B}(\ell(k)-1,z)\partial_z \\
\dots & \dots & \dots & \dots & \dots & \dots & \tilde{\mathcal{L}}_k
\end{pmatrix}$$

with

$$\begin{aligned}
\mathcal{A}_k(\ell) &= -2(2\nu k+1)(\ell+1), \\
\mathcal{B}(\ell,z) &= 4z(\ell+1), \\
\mathcal{C}(\ell) &= -2(\ell+1)(\ell+2),
\end{aligned}$$

and

$$\mathcal{W}_k = \begin{pmatrix} w_{k,0} \\ \vdots \\ w_{k,\ell} \\ \vdots \\ w_{k,\ell(k)} \end{pmatrix}, \qquad \mathcal{F}^{\mathrm{nl}}_k = \begin{pmatrix} F^{\mathrm{nl}}_{k,0} \\ \vdots \\ F^{\mathrm{nl}}_{k,\ell} \\ \vdots \\ F^{\mathrm{nl}}_{k,\ell(k)} \end{pmatrix}.$$

Let us emphasize that we do not subject the above system to any Cauchy data as was the case for the system (3.5) corresponding to the inner region. In order to get a unique solution to (4.24), we shall take into account the matching conditions coming from the inner region: we require that

$$w_{k,\ell}(z) = \sum_{\substack{0\le\alpha\le(k-3)/2-\ell \\ \beta\ge 1-k+2(\alpha+\ell)}} c^{k,\ell}_{\alpha,\beta}(\log z)^\alpha z^\beta \quad \text{as } z \to 0, \tag{4.29}$$

where $c^{k,\ell}_{\alpha,\beta} = c^{k,\ell}_{\alpha,\beta}(\lambda)$ are given by (4.15).

In view of (4.12), one can write $F^{\rm nl}_{k,\ell}$ explicitly as follows:

$$F^{\rm nl}_{k,\ell} = F^{\rm nl,1}_{k,\ell} + F^{\rm nl,2}_{k,\ell} + F^{\rm nl,3}_{k,\ell} + F^{\rm nl,4}_{k,\ell}, \tag{4.30}$$

where

$$F^{\rm nl,1}_{k,\ell} = -2\widetilde{w}^{(2,')}_{k,\ell}, \tag{4.31}$$

$$\begin{aligned} F^{\rm nl,2}_{k,\ell} = &\sum_{\substack{j_1+j_2=k\\ \ell_1+\ell_2=\ell}} 6(w_{j_1,\ell_1})_z\left(\frac{1}{z}(w_{j_2,\ell_2})_z + \frac{1}{z^2}\check{w}_{j_2,\ell_2}\right) - 2(w_{j_1,\ell_1})_z w^{(2,')}_{j_2,\ell_2}\\ &+ \sum_{\substack{j_1+j_2=k\\ \ell_1+\ell_2=\ell}} 2\, w^1_{j_1,\ell_1} w^3_{j_2,\ell_2} - \frac{6}{z^3} w_{j_1,\ell_1}\check{w}_{j_2,\ell_2}, \end{aligned} \tag{4.32}$$

$$\begin{aligned} F^{\rm nl,3}_{k,\ell} = &\sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} 2w^1_{j_1,\ell_1} w^3_{j_2,\ell_2}(w_{j_3,\ell_3})_z - (w_{j_1,\ell_1})_z(w_{j_2,\ell_2})_z w^{(2,')}_{j_3,\ell_3}\\ &+ 3\sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} ((w_{j_1,\ell_1})_z(w_{j_2,\ell_2})_z - w^1_{j_1,\ell_1} w^1_{j_2,\ell_2})\\ &\qquad\qquad\times\left(\frac{\check{w}_{j_3,\ell_3}}{z^2} + \frac{(w_{j_3,\ell_3})_z}{z}\right), \end{aligned} \tag{4.33}$$

$$F^{\rm nl,4}_{k,\ell} = -\sum_{\substack{j_1+j_2=k\\ \ell_1+\ell_2=\ell}} A^0_{j_1,\ell_1}(w_{j_2,\ell_2})_{zz}. \tag{4.34}$$

For our purpose, it will be useful to point out that, according to (4.14),

$$F^{\rm nl}_{k,\ell} = \widetilde{F}^{\rm nl,1}_{k,\ell} + \widetilde{F}^{\rm nl,2}_{k,\ell} + \widetilde{F}^{\rm nl,3}_{k,\ell}, \tag{4.35}$$

where

$$\widetilde{F}^{\rm nl,1}_{k,\ell} = -2\widetilde{w}^2_{k,\ell}, \tag{4.36}$$

$$\begin{aligned} \widetilde{F}^{\rm nl,2}_{k,\ell} = &\sum_{\substack{j_1+j_2=k\\ \ell_1+\ell_2=\ell}} 6(w_{j_1,\ell_1})_z\left(\frac{1}{z}(w_{j_2,\ell_2})_z + \frac{1}{z^2}\check{w}_{j_2,\ell_2}\right) - 2(w_{j_1,\ell_1})_z w^2_{j_2,\ell_2}\\ &+ \sum_{\substack{j_1+j_2=k\\ \ell_1+\ell_2=\ell}} 2w^1_{j_1,\ell_1}(w^1_{j_2,\ell_2})_z - \frac{6}{z^3} w_{j_1,\ell_1}\check{w}_{j_2,\ell_2}, \end{aligned} \tag{4.37}$$

$$\widetilde{F}^{\mathrm{nl},3}_{k,\ell} = -\sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} (w_{j_1,\ell_1})_z (w_{j_2,\ell_2})_z w^2_{j_3,\ell_3} + w^1_{j_1,\ell_1} w^1_{j_2,\ell_2} (w_{j_3,\ell_3})_{zz}$$
$$+ 2 \sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} w^1_{j_1,\ell_1} (w^1_{j_2,\ell_2})_z (w_{j_3,\ell_3})_z$$
$$+ 3 \sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} ((w_{j_1,\ell_1})_z (w_{j_2,\ell_2})_z - w^1_{j_1,\ell_1} w^1_{j_2,\ell_2}) \left(\frac{\check{w}_{j_3,\ell_3}}{z^2} + \frac{(w_{j_3,\ell_3})_z}{z} \right). \tag{4.38}$$

4.2. *Analysis of the vector functions $\mathcal{W}_k$*

4.2.1. Study of the linear system $\mathcal{S}_k$. In order to determine successively $w_{k,\ell}$ and $\lambda_{k,\ell}$, let us start by investigating the homogeneous equation

$$\mathcal{S}_k X = 0. \tag{4.39}$$

We will prove the following lemma:

Lemma 4.1. *For j in $\{0, \ldots, \ell(k)\}$, define $(f^{j,\pm}_{k,\ell})_{0 \le \ell \le \ell(k)}$ by*

$$f^{j,\pm}_{k,\ell}(z) = \binom{j}{\ell} (\log|1/\sqrt{2} \pm z|)^{j-\ell} \frac{|1/\sqrt{2} \pm z|^{\alpha(\nu,k)}}{z^3}, \tag{4.40}$$
$$f^{j,\pm}_{k,\ell} = 0 \quad \text{for } j+1 \le \ell \le \ell(k),$$

where $\alpha(\nu,k) = \nu k + 4$, and denote

$$f^{j,\pm}_k = \begin{pmatrix} f^{j,\pm}_{k,0} \\ \vdots \\ f^{j,\pm}_{k,j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The vector functions $(f^{j,\pm}_k)_{0 \le j \le \ell(k)}$ constitute a basis of solutions to the homogeneous equation (4.39) on the intervals $]0, 1/\sqrt{2}[$ and $]1/\sqrt{2}, \infty[$.

Proof. Consider the linearization of (1.8) around ρ:

$$2v_{tt} - l_\rho v = 0, \tag{4.41}$$

where

$$l_\rho = \partial^2_\rho + 6\left(\frac{\partial_\rho}{\rho} + \frac{1}{\rho^2} \right). \tag{4.42}$$

Writing

$$v(t,\rho) = tw(t,z) \quad \text{with} \quad z = \rho/t,$$

we clearly get, with the notation (4.11),

$$Lw = 0.$$

Observe also that (4.41) is equivalent to

$$2(\rho^3 v)_{tt} - (\rho^3 v)_{\rho\rho} = 0. \tag{4.43}$$

Set

$$G(t,z) = t^{\nu k+1}(\log t + \log|1/\sqrt{2} \pm z|)^j \frac{|1/\sqrt{2} \pm z|^{\alpha(\nu,k)}}{z^3}.$$

Since

$$G(t,z) = (\log|t/\sqrt{2} \pm \rho|)^j \frac{|t/\sqrt{2} \pm \rho|^{\alpha(\nu,k)}}{\rho^3} = \frac{F(|t/\sqrt{2} \pm \rho|)}{\rho^3}$$

for some function F, we infer that $L(t^{-1}G) = 0$. This implies that

$$L\Big(t^{\nu k}(\log t + \log|1/\sqrt{2} \pm z|)^j \frac{|1/\sqrt{2} \pm z|^{\alpha(\nu,k)}}{z^3}\Big) = 0.$$

Since

$$t^{\nu k}(\log t + \log|1/\sqrt{2} \pm z|)^j \frac{|1/\sqrt{2} \pm z|^{\alpha(\nu,k)}}{z^3} = t^{\nu k}\sum_{\ell=0}^{j}(\log t)^\ell f_{k,\ell}^{j,\pm}(z), \tag{4.44}$$

we obtain the result, recalling that

$$L\Big(t^{\nu k}\sum_{\ell=0}^{j}(\log t)^\ell f_{k,\ell}^{j,\pm}(z)\Big) = 0 \iff \mathcal{S}_k f_k^{j,\pm} = 0. \qquad \blacksquare$$

Remark 4.1. Note that in view of the above lemma, the homogeneous equation $\widetilde{\mathcal{L}}_k f = 0$ has the following basis of solutions:

$$\begin{cases} f_{k,0}^{0,+}(z) = \dfrac{(1/\sqrt{2} + z)^{\alpha(\nu,k)}}{z^3}, \\ f_{k,0}^{0,-}(z) = \dfrac{|1/\sqrt{2} - z|^{\alpha(\nu,k)}}{z^3}. \end{cases} \tag{4.45}$$

Before concluding this section, let us collect some useful properties of the basis $(f_k^{j,\pm})_{0\le j\le \ell(k)}$ given above.

Lemma 4.2. *With the above notations, the following asymptotic expansions hold:*

$$[\nu k+\Lambda] f_{k,\ell}^{j,\pm}(z)+(\ell+1) f_{k,\ell+1}^{j,\pm}(z)=z^{\nu k} \sum_{0 \leq \alpha \leq j-\ell} \sum_{p \in \mathbb{N}} \gamma_{p,\alpha}^{k}(\log z)^{\alpha} z^{-p} \quad \text{as } z \rightarrow \infty, \tag{4.46}$$

$$\begin{aligned} {[z^2 \partial_z^2+\nu k(\nu k+1-2 z \partial_z)] f_{k,\ell}^{j,\pm}(z)+(\ell+1)\,[2 \nu k+1-2 z \partial_z] f_{k,\ell+1}^{j,\pm}(z)} & \\ +(\ell+1)(\ell+2) f_{k,\ell+2}^{j,\pm}(z)=z^{\nu k-1} \sum_{0 \leq \alpha \leq j-\ell} \sum_{p \in \mathbb{N}} \hat{\gamma}_{p,\alpha}^{k}(\log z)^{\alpha} z^{-p} & \quad \text{as } z \rightarrow \infty, \end{aligned} \tag{4.47}$$

for any integer $k \geq 3$ and any j, ℓ in $\{0, \ldots, \ell(k)\}$, for some constants $\gamma_{p,\alpha}^{k}$ and $\hat{\gamma}_{p,\alpha}^{k}$.

Proof. In view of (4.44), for large ρ we have

$$(\log(\rho \pm t/\sqrt{2}))^{j} \frac{(\rho \pm t/\sqrt{2})^{\alpha(\nu,k)}}{\rho^{3}}=t^{\nu k+1} \sum_{\ell=0}^{j}(\log t)^{\ell} f_{k,\ell}^{j,\pm}(\rho/t). \tag{4.48}$$

Therefore taking the derivative of the above identity with respect to t, we deduce that

$$\begin{aligned} \frac{1}{\sqrt{2}}\Big(j(\log(\rho \pm t/\sqrt{2}))^{j-1} \frac{(\rho \pm t/\sqrt{2})^{\alpha(\nu,k)-1}}{\rho^{3}} & \\ +\alpha(\nu,k)(\log(\rho \pm t/\sqrt{2}))^{j} \frac{(\rho \pm t/\sqrt{2})^{\alpha(\nu,k)-1}}{\rho^{3}}\Big) & \\ =t^{\nu k} \sum_{\ell=0}^{j}(\log t)^{\ell}((\nu k+\Lambda) f_{k,\ell}^{j,\pm}+(\ell+1) f_{k,\ell+1}^{j,\pm})(\rho/t). & \end{aligned} \tag{4.49}$$

Performing the change of variables $z=\rho/t$, we infer that

$$\begin{aligned} \frac{t^{\nu k}}{\sqrt{2}} \frac{(z \pm 1/\sqrt{2})^{\alpha(\nu,k)-1}}{z^{3}}\big(j(\log t+\log(z \pm 1/\sqrt{2}))^{j-1}+\alpha(\nu,k)(\log t+\log(z \pm 1/\sqrt{2}))^{j}\big) & \\ =t^{\nu k} \sum_{\ell=0}^{j}(\log t)^{\ell}\big((\nu k+\Lambda) f_{k,\ell}^{j,\pm}+(\ell+1) f_{k,\ell+1}^{j,\pm}\big)(z), & \end{aligned} \tag{4.50}$$

which concludes the proof of (4.46).

Along the same lines, taking the derivative with respect to t of (4.49) ensures (4.47), which ends the proof of the lemma. ∎

4.2.2. Study of the functions $w_{k,\ell}$. The goal of this subsection is to prove by induction that the system (4.24) has a solution satisfying the matching conditions (4.29).

For this purpose, let us start with the following useful lemma, which stems from standard techniques of ordinary differential equations. For the sake of completeness and the convenience of the reader, we outline its proof in Appendix D.

Lemma 4.3. *With the above notations,*[13] *the following properties hold:*

- *For any function g in $\mathcal{C}^\infty(\mathbb{R}_+^*)$, the equation $\widetilde{\mathcal{L}}_k f = g$ admits a unique solution f in $\mathcal{C}^\infty(\mathbb{R}_+^*)$ satisfying $f(1/\sqrt{2}) = 0$.*
- *For any function h in $\mathcal{C}^\infty(]0, 1/\sqrt{2}])$, any $\gamma > 0$, and any integer q, the equation*

$$\widetilde{\mathcal{L}}_k f(z) = (1/\sqrt{2} - z)^\gamma (\log(1/\sqrt{2} - z))^q h(z) \tag{4.51}$$

has a unique solution f of the form

$$f(z) = (1/\sqrt{2} - z)^{\gamma+1} \sum_{0 \le \ell \le q} (\log(1/\sqrt{2} - z))^\ell \, h_\ell(z),$$

where for all $0 \le \ell \le q$, the function h_ℓ is in $\mathcal{C}^\infty(]0, 1/\sqrt{2}])$, provided that the exponent γ satisfies

$$\nu k + 4 - \gamma \notin \mathbb{N}^* . \tag{4.52}$$

- *Let g be a function in $\mathcal{C}^\infty(]0, 1/\sqrt{2}[)$ with an asymptotic expansion at 0 of the form*

$$g(z) = (\log z)^{\alpha_0} \sum_{\beta \ge \beta_0} g_\beta z^{\beta-2},$$

for some integers α_0, β_0, then any solution f of the equation

$$\widetilde{\mathcal{L}}_k f = g \tag{4.53}$$

belongs to $\mathcal{C}^\infty(]0, 1/\sqrt{2}[)$ and has for z close to 0 an asymptotic expansion

$$f(z) = \sum_{\beta \ge -3} f_{0,\beta} z^\beta + \sum_{1 \le \alpha \le \alpha_0} \sum_{\beta \ge \beta_0} f_{\alpha,\beta} (\log z)^\alpha z^\beta$$

when $\beta_0 \ge -1$, and

$$f(z) = \sum_{\beta \ge \min(\beta_0, -3)} f_{0,\beta} z^\beta + \sum_{1 \le \alpha \le \alpha_0} \sum_{\beta \ge \beta_0} f_{\alpha,\beta} (\log z)^\alpha z^\beta + \sum_{\beta \ge \max(\beta_0, -3)} f_{\alpha_0+1,\beta} (\log z)^{\alpha_0+1} z^\beta ,$$

when $\beta_0 \le -2$.

[13] Again with the convention that the sum is null if it is over an empty set.

- *If* $g \in \mathcal{C}^\infty(]1/\sqrt{2},\infty[)$ *has at infinity an asymptotic expansion*

$$g(z) = \sum_{0\le\alpha\le\alpha_0} \sum_{p\in\mathbb{N}} \hat{g}_{\alpha,p}(\log z)^\alpha z^{A-p}$$

for some real $A < \nu k$ *and some integer* α_0*, then the equation*

$$\widetilde{\mathcal{L}}_k f = g \tag{4.54}$$

has a unique solution f *in* $\mathcal{C}^\infty(]1/\sqrt{2},\infty[)$ *such that*

$$f(z) = \sum_{0\le\alpha\le\alpha_0} \sum_{p\in\mathbb{N}} \hat{f}^k_{\alpha,p}(\log z)^\alpha z^{A-p} \quad \text{as } z\to\infty.$$

The key result of this subsection is the following proposition:

Proposition 4.1. *With the above notations, the following properties hold:*

(1) (Existence) *The system* (4.24) *a solution* $(w_{k,\ell},\lambda_{k,\ell})_{k\ge3,0\le\ell\le\ell(k)}$ *such that for any integer* $k\ge3$ *and any* $\ell\in\{0,\dots,\ell(k)\}$*, the function* $w_{k,\ell}$ *belongs to* $\mathcal{C}^{[\alpha(\nu,k)]}(\mathbb{R}^*_+)\cap\mathcal{C}^\infty(\mathbb{R}^*_+\setminus\{1/\sqrt{2}\})$ *and has the form*[14]

$$\begin{aligned} w_{k,\ell}(z) &= a^{\text{reg}}_{k,\ell}(z) \\ &+ (1/\sqrt{2}-z)^{k\nu+4} \sum_{0\le\alpha\le(k-3)/2-\ell} b^{\text{reg}}_{k,\ell,\alpha}(z)(\log(1/\sqrt{2}-z))^\alpha \chi_{]0,1/\sqrt{2}]}(z) \\ &+ \sum_{\substack{3\le\beta\le k-3\\ 0\le\alpha\le(k-6)/2-\ell}} b^{\text{reg}}_{k,\ell,\alpha,\beta}(z)(1/\sqrt{2}-z)^{\beta\nu+4}(\log(1/\sqrt{2}-z))^\alpha \chi_{]0,1/\sqrt{2}]}(z), \end{aligned} \tag{4.55}$$

where

$$\chi_{]0,1/\sqrt{2}]}(z) = \begin{cases} 1 & \text{for } z\le 1/\sqrt{2},\\ 0 & \text{for } z>1/\sqrt{2}. \end{cases}$$

In addition, the following asymptotics hold:

$$w_{k,\ell}(z) = \sum_{\substack{0\le\alpha\le(k-3)/2-\ell\\ \beta\ge1-k+2(\alpha+\ell)}} d^{k,\ell}_{\alpha,\beta}(\log z)^\alpha z^\beta \quad \text{as } z\to0, \tag{4.56}$$

with

$$d^{k,\ell}_{0,-2} = c^{k,\ell}_{0,-2}(\lambda), \qquad d^{k,\ell}_{0,-3} = c^{k,\ell}_{0,-3}(\lambda), \tag{4.57}$$

where $c^{k,\ell}_{0,\beta}(\lambda)$ *are the coefficients introduced in* (4.15).

[14] Here and below, the notation $^{\text{reg}}$ means that the corresponding function belongs to $\mathcal{C}^\infty(\mathbb{R}^*_+)$.

Moreover, for $z > 1/\sqrt{2}$, $w_{k,\ell}$ *can be split as*

$$w_{k,\ell}(z) = w^{\rm nl}_{k,\ell} + w^{\rm lin}_{k,\ell}, \tag{4.58}$$

where the nonlinear part $w^{\rm nl}_{k,\ell}$ *is null if* $k < 6$ *or* $\ell > (k-6)/2$, *and in all other cases, as* $z \to \infty$, *it has an asymptotic expansion*

$$w^{\rm nl}_{k,\ell}(z) = \sum_{\substack{3\le\beta\le k-3\\ 0\le\alpha\le(k-6)/2-\ell,\ p\in\mathbb{N}}} \hat{d}^{k,\ell}_{\alpha,\beta,p}(\log z)^\alpha z^{\beta\nu+1-p} + z^{\nu k+1}\sum_{0\le\alpha\le(k-3)/2-\ell,\ p\ge2} \hat{d}^{k,\ell}_{\alpha,k,p}(\log z)^\alpha z^{-p} \tag{4.59}$$

with some constants $\hat{d}^{k,\ell}_{\alpha,\beta,p}$, *and the linear part* $w^{\rm lin}_{k,\ell}$ *is given by*

$$w^{\rm lin}_{k,\ell}(z) = \sum_{0\le j\le\ell(k)} (\alpha_k^{j,+} f^{j,+}_{k,\ell} + \alpha_k^{j,-} f^{j,-}_{k,\ell}) \tag{4.60}$$

with some constants $\alpha_k^{j,\pm}$. *Here* $f_k^{j,\pm} = (f^{j,\pm}_{k,\ell})_{0\le j\le\ell(k)}$ *are the solutions of the homogeneous equation* (4.39) *defined in Lemma* 4.1.

(2) (Uniqueness) *Let* $(\lambda_{k,\ell})_{k\ge3,\,0\le\ell\le\ell(k)}$ *be fixed, and let* $(w^0_{k,\ell})_{3\le k\le M,\,0\le\ell\le\ell(k)}$ *and* $(w^1_{k,\ell})_{3\le k\le M,\,0\le\ell\le\ell(k)}$ *be two solutions of*

$$\widetilde{\mathcal{L}}_k w_{k,\ell} = F_{k,\ell}(\lambda; w), \quad 3\le k\le M, \tag{4.61}$$

defined and $\mathcal{C}^\infty$ *in a neighborhood of* 0, *with* $w^i_{5,1}\equiv 0$ *for* $i\in\{0,1\}$, *and which have an asymptotic expansion of the form* (4.56) *as* z *tends to* 0:

$$w^i_{k,\ell}(z) = \sum_{\substack{0\le\alpha\le(k-3)/2-\ell\\ \beta\ge1-k+2(\alpha+\ell)}} d^{k,\ell,i}_{\alpha,\beta}(\log z)^\alpha z^\beta.$$

If

$$d^{k,\ell,0}_{0,-2} = d^{k,\ell,1}_{0,-2}, \quad d^{k,\ell,0}_{0,-3} = d^{k,\ell,1}_{0,-3}, \tag{4.62}$$

then $w^0_{k,\ell} = w^1_{k,\ell}$ *for all* $3\le k\le M$ *and all* $0\le\ell\le\ell(k)$.

Similarly, if $(w^0_{k,\ell})_{3\le k\le M,\,0\le\ell\le\ell(k)}$ *and* $(w^1_{k,\ell})_{3\le k\le M,\,0\le\ell\le\ell(k)}$ *are two solutions of* (4.61) *defined and* $\mathcal{C}^\infty$ *around* $+\infty$, *with* $w^i_{5,1}\equiv0$ *for* $i\in\{0,1\}$, *and which satisfy, as* z *tends to infinity,*

$$w^i_{k,\ell} = \sum_{0\le j\le\ell(k)} \alpha_k^{j,+,i} f^{j,+}_{k,\ell} + \alpha_k^{j,-,i} f^{j,-}_{k,\ell} + \sum_{\substack{3\le\beta\le k-3\\ 0\le\alpha\le(k-6)/2-\ell,\ p\in\mathbb{N}}} \hat{d}^{k,\ell,i}_{\alpha,\beta,p}(\log z)^\alpha z^{\beta\nu+1-p} + z^{\nu k+1}\sum_{0\le\alpha\le(k-6)/2-\ell,\ p\ge2} \hat{d}^{k,\ell,i}_{\alpha,k,p}(\log z)^\alpha z^{-p}, \tag{4.63}$$

then

$$\alpha_k^{j,\pm,0} = \alpha_k^{j,\pm,1}, \quad \forall\, 3 \le k \le M \text{ and } 0 \le j \le \ell(k), \tag{4.64}$$

implies that

$$w_{k,\ell}^0 = w_{k,\ell}^1 \quad \text{for all } 3 \le k \le M \text{ and all } 0 \le \ell \le \ell(k).$$

Remark 4.2. By Lemma 4.1 and formulae (4.59) and (4.60), the functions $w_{k,\ell}$ have an asymptotic expansion

$$\begin{aligned} w_{k,\ell}(z) = & \sum_{\substack{3 \le \beta \le k-3 \\ 0 \le \alpha \le (k-6)/2-\ell,\ p\in\mathbb{N}}} w_{k,\ell,\alpha,\beta,p} (\log z)^\alpha z^{\beta\nu+1-p} \\ & + z^{k\nu+1} \sum_{\substack{0 \le \alpha \le (k-3)/2-\ell \\ p\in\mathbb{N}}} w_{k,\ell,\alpha,p} (\log z)^\alpha z^{-p} \quad \text{as } z \to \infty, \end{aligned} \tag{4.65}$$

for some constants $w_{k,\ell,\alpha,\beta,p}$ and $w_{k,\ell,\alpha,p}$.

Proof of Proposition 4.1. Let us start with the existence part of the proposition, and first consider the indices $k = 3, 4$ and 5.

In view of the computations carried out in §4.1 (see (4.16)), we have in this case $w_{5,1} = 0$ and

$$\widetilde{\mathcal{L}}_k w_{k,0} = 0, \quad k = 3, 4, 5. \tag{4.66}$$

In view of Remark 4.1, this implies that for $k = 3, 4, 5$,

$$w_{k,0} = \begin{cases} a_{0,+}^k f_{k,0}^{0,+}(z) + a_{0,-}^k f_{k,0}^{0,-}(z) & \text{for } z \le 1/\sqrt{2}, \\ a_{0,+}^k f_{k,0}^{0,+}(z) & \text{for } z > 1/\sqrt{2}, \end{cases} \tag{4.67}$$

where $a_{0,+}^3 = -a_{0,-}^3$ and where $\{f_{k,0}^{0,+}, f_{k,0}^{0,-}\}$ denotes the basis of solutions associated to the operator $\widetilde{\mathcal{L}}_k$ given by (4.45). The coefficients $a_{0,\pm}^k$ are determined by (4.57):

$$\begin{cases} 2(3\nu+4)(1/\sqrt{2})^{3\nu+3} a_{0,+}^3 = c_{0,-2}^{3,0}, \\ (1/\sqrt{2})^{\nu k+4}(a_{0,+}^k + a_{0,-}^k) = c_{0,-3}^{k,0}, \\ (\nu k+4)(1/\sqrt{2})^{\nu k+3}(a_{0,+}^k - a_{0,-}^k) = c_{0,-2}^{k,0}, \quad k = 4, 5. \end{cases} \tag{4.68}$$

Clearly the functions $w_{k,0}$, $k = 3, 4, 5$, satisfy properties (4.55)–(4.60).

Let us now consider the general case of any index $k \ge 6$. To this end, we shall proceed by induction, assuming that, for any integer $3 \le j \le k-1$ and all $0 \le \ell \le \ell(j)$, $(w_{j,\ell}, \lambda_{j,\ell})$ satisfies the conclusion of part (1) of Proposition 4.1.

The first step consists in establishing the following lemma:

Lemma 4.4. *Assume that* $(w_{j,\ell}, \lambda_{j,\ell})_{0\leq \ell\leq \ell(j)}$ *is a solution of the system* (4.24) *with* $3 \leq j \leq k-1$ *which satisfies* (4.55), (4.56), (4.58), (4.59) *and* (4.60). *Then*

$$\begin{aligned} F^{\mathrm{nl}}_{k,\ell}(z) &= f^{\mathrm{reg}}_{k,\ell}(z) \\ &\quad + (1/\sqrt{2}-z)^{k\nu+6} \sum_{0\leq\alpha\leq(k-6)/2-\ell} f^{\mathrm{reg}}_{k,\ell,\alpha}(z)(\log(1/\sqrt{2}-z))^{\alpha}\chi_{]0,1/\sqrt{2}]}(z) \\ &\quad + \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq(k-6)/2-\ell}} f^{\mathrm{reg}}_{k,\ell,\beta,\alpha}(z)(1/\sqrt{2}-z)^{\beta\nu+3}(\log(1/\sqrt{2}-z))^{\alpha}\chi_{]0,1/\sqrt{2}]}(z), \end{aligned} \tag{4.69}$$

and has the following asymptotic expansions:

$$F^{\mathrm{nl}}_{k,\ell}(z) = \sum_{\substack{0\leq\alpha\leq(k-6)/2-\ell\\ \beta\geq 1-k+2(\alpha+\ell)}} \widetilde{f}_{k,\ell,\alpha,\beta}(\log z)^{\alpha} z^{\beta-2} \quad \text{as } z\to 0, \tag{4.70}$$

$$\begin{aligned} F^{\mathrm{nl}}_{k,\ell}(z) &= z^{k\nu-1} \sum_{0\leq\alpha\leq(k-6)/2-\ell,\ p\in\mathbb{N}} \hat{f}_{k,\ell,\alpha,p}(\log z)^{\alpha} z^{-p} \\ &\quad + \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq(k-6)/2-\ell,\ p\in\mathbb{N}}} \hat{f}_{k,\ell,\alpha,\beta,p}(\log z)^{\alpha} z^{\nu\beta+1-p} \quad \text{as } z\to\infty, \end{aligned} \tag{4.71}$$

where the coefficients $\widetilde{f}_{k,\ell,\alpha,\beta}$ (*resp.* $\hat{f}_{k,\ell,\alpha,p}$ *and* $\hat{f}_{k,\ell,\alpha,\beta,p}$) *are uniquely determined in terms of the coefficients* $d^{j,\ell}_{\alpha,\beta}$ (*resp.* $w^{k,\ell}_{\alpha,\beta,p}$) *involved in* (4.56) (*resp.* (4.65)).

Proof. Let us first address the behavior of $F^{\mathrm{nl}}_{k,\ell}$ near $z=0$ and at infinity. To establish (4.70) and (4.71), we will use formulae (4.35)–(4.38), combining them with the corresponding asymptotics of $w^1_{j,\ell}$, $\widetilde{w}^1_{j,\ell}$, $w^2_{k,\ell}$, $\widetilde{w}^2_{j,\ell}$ and $\breve{w}_{j,\ell}$, which we start to describe now.

Consider $\widetilde{w}^1_{j,\ell}$. It follows from (4.20) that if $w_{j,\ell}$, $3\leq j\leq k-1$, satisfy (4.56) and (4.65), then for any $6\leq j\leq k+2$, $\widetilde{w}^1_{j,\ell}$ has the following asymptotic expansions:

$$\widetilde{w}^1_{j,\ell}(z) = \sum_{\substack{0\leq\alpha\leq(j-6)/2-\ell\\ \beta\geq 4-j+2(\alpha+\ell)}} \widetilde{w}^{1,0}_{j,\ell,\alpha,\beta}(\log z)^{\alpha} z^{\beta} \quad \text{as } z\to 0, \tag{4.72}$$

$$\widetilde{w}^1_{j,\ell}(z) = \sum_{\substack{0\leq\alpha\leq(j-6)/2-\ell\\ 3\leq\beta\leq j-3,\ p\geq 0}} \widetilde{w}^{1,\infty}_{j,\ell,\alpha,\beta,p}(\log z)^{\alpha} z^{\beta\nu+1-p} \quad \text{as } z\to\infty. \tag{4.73}$$

Combining (4.20) with (4.56) and (4.72), one can easily check that $w^1_{j,\ell}$ has the same asymptotic form as $w_{j,\ell}$ as z tends to 0:

$$w^1_{j,\ell}(z) = \sum_{\substack{0\leq\alpha\leq(j-3)/2-\ell\\ \beta\geq 1-j+2(\alpha+\ell)}} w^{1,0}_{j,\ell,\alpha,\beta}(\log z)^{\alpha} z^{\beta}. \tag{4.74}$$

Furthermore, invoking (4.20), (4.58), (4.59), (4.60), (4.73) and taking into account (4.46), one obtains, as $z \to \infty$,

$$w^1_{j,\ell}(z) = z^{j\nu} \sum_{\substack{0\le\alpha\le(j-3)/2-\ell \\ p\in\mathbb{N}}} w^{1,\infty}_{j,\ell,\alpha,j,p} (\log z)^\alpha z^{-p} + \sum_{\substack{0\le\alpha\le(j-6)/2-\ell \\ 3\le\beta\le j-3,\ p\in\mathbb{N}}} w^{1,\infty}_{j,\ell,\alpha,\beta,p} (\log z)^\alpha z^{\nu\beta+1-p} \tag{4.75}$$

for any integer $3 \le j \le k-1$.

The function $\widetilde{w}^2_{j,\ell}$ can be analyzed along the same lines as $\widetilde{w}^1_{j,\ell}$. In particular, using the definition (4.22), one can show that under the assumptions of Lemma 4.4, for any $6 \le j \le k+2$, $\widetilde{w}^2_{j,\ell}$ behaves in the same way as $\widetilde{w}^1_{j,\ell}$ when $z \to 0$ and $z \to \infty$:

$$\widetilde{w}^2_{j,\ell}(z) = \sum_{\substack{0\le\alpha\le(j-6)/2-\ell \\ \beta\ge 4-j+2(\alpha+\ell)}} \widetilde{w}^{2,0}_{j,\ell,\alpha,\beta} (\log z)^\alpha z^\beta \quad \text{as } z \to 0, \tag{4.76}$$

$$\widetilde{w}^2_{j,\ell}(z) = \sum_{\substack{0\le\alpha\le(j-6)/2-\ell \\ 3\le\beta\le j-3,\ p\ge 0}} \widetilde{w}^{2,\infty}_{j,\ell,\alpha,\beta,p} (\log z)^\alpha z^{\beta\nu+1-p} \quad \text{as } z \to \infty. \tag{4.77}$$

Combining (4.22) with (4.56), (4.58)–(4.60), (4.76) and (4.77), and taking into account (4.47), we deduce, as we have done for $w^1_{j,\ell}$, that $w^2_{j,\ell}$ has the same form as $w_{j,\ell}$, $w^1_{j,\ell}$ as $z \to 0$:

$$w^2_{j,\ell}(z) = \sum_{\substack{0\le\alpha\le(j-3)/2-\ell \\ \beta\ge 1-j+2(\alpha+\ell)}} w^{2,0}_{j,\ell,\alpha,\beta} (\log z)^\alpha z^\beta, \tag{4.78}$$

and as $z \to \infty$,

$$w^2_{j,\ell}(z) = z^{j\nu-1} \sum_{\substack{0\le\alpha\le(j-3)/2-\ell \\ p\in\mathbb{N}}} w^{2,\infty}_{j,\ell,\alpha,j,p} (\log z)^\alpha z^{-p} + \sum_{\substack{0\le\alpha\le(j-6)/2-\ell \\ 3\le\beta\le j-3,\ p\in\mathbb{N}}} w^{2,\infty}_{j,\ell,\alpha,\beta,p} (\log z)^\alpha z^{\nu\beta+1-p}, \tag{4.79}$$

for all $3 \le j \le k-1$.

Next we address $\check{w}_{j,\ell}$. Writing

$$\check{w}_{j,\ell} = \sum_{p\ge 1} \sum_{\substack{j_1+\cdots+j_p=j \\ \ell_1+\cdots+\ell_p=\ell}} (-1)^{p-1} z^{1-p} w_{j_1,\ell_1} \cdots w_{j_p,\ell_p}, \tag{4.80}$$

it is easy to check that if $w_{j,\ell}$, $3 \leq j \leq k-1$, satisfy (4.56) and (4.65), then the same is true for $\check{w}_{j,\ell}$, $3 \leq j \leq k-1$:

$$\check{w}_{j,\ell}(z) = \sum_{\substack{0\leq\alpha\leq(j-3)/2-\ell \\ \beta\geq 1-j+2(\alpha+\ell)}} \check{w}^{0}_{j,\ell,\alpha,\beta}(\log z)^{\alpha} z^{\beta} \quad \text{as } z \to 0, \tag{4.81}$$

$$\check{w}_{j,\ell}(z) = z^{j\nu+1} \sum_{\substack{0\leq\alpha\leq(j-3)/2-\ell \\ p\in\mathbb{N}}} \check{w}^{\infty}_{j,\ell,\alpha,j,p}(\log z)^{\alpha} z^{-p} + \sum_{\substack{0\leq\alpha\leq(j-6)/2-\ell \\ 3\leq\beta\leq j-3,\, p\in\mathbb{N}}} \check{w}^{\infty}_{j,\ell,\alpha,\beta,p}(\log z)^{\alpha} z^{\nu\beta+1-p} \tag{4.82}$$

as $z \to \infty$. Combining (4.30)–(4.34) with (4.74)–(4.79), (4.81) and (4.82), we obtain (4.70) and (4.71).

To end the proof of the lemma, it remains to establish (4.69). To this end, we will use the representations (4.30)–(4.34).

Let us start with $F^{\mathrm{nl},1}_{k,\ell}$ defined by (4.31). It stems from the definition of $\widetilde{w}^{(2,')}_{j,\ell}$ given by (4.23) that for any $6 \leq j \leq k+2$, $\widetilde{w}^{(2,')}_{j,\ell}$ has the form

$$f^{\mathrm{reg}}_{j,\ell}(z) + \sum_{\substack{3\leq\beta\leq j-3 \\ 0\leq\alpha\leq(j-6)/2-\ell}} (1/\sqrt{2}-z)^{\beta\nu+3}(\log(1/\sqrt{2}-z))^{\alpha} h^{\mathrm{reg}}_{\alpha,\beta}(z)\chi_{]0,1/\sqrt{2}]}(z), \tag{4.83}$$

which means that (4.69) holds for $F^{\mathrm{nl},1}_{k,\ell}$.

We next consider $F^{\mathrm{nl},i}_{k,\ell}$, $i = 2, 3$, defined by (4.32) and (4.33) respectively. In view of (4.20) and (4.21), we deduce that $\widetilde{w}^{1}_{j,\ell}$ and $\widetilde{w}^{3}_{j,\ell}$ have the form (4.83) for any $6 \leq j \leq k+2$, and therefore the functions $w^{1}_{j,\ell}$ and $w^{3}_{j,\ell}$ can be written in the following way:

$$f^{\mathrm{reg}}_{j,\ell}(z) + (1/\sqrt{2}-z)^{j\nu+3} \sum_{0\leq\alpha\leq(j-3)/2-\ell} (\log(1/\sqrt{2}-z))^{\alpha} h^{\mathrm{reg}}_{j,\ell,\alpha}(z)\chi_{]0,1/\sqrt{2}]}(z) + \sum_{\substack{3\leq\beta\leq j-3 \\ 0\leq\alpha\leq(j-6)/2-\ell}} (1/\sqrt{2}-z)^{\beta\nu+3}(\log(1/\sqrt{2}-z))^{\alpha} h^{\mathrm{reg}}_{j,\ell,\alpha,\beta}(z)\chi_{]0,1/\sqrt{2}]}(z). \tag{4.84}$$

Similarly, by (4.23) the same is true for $w^{(2,')}_{j,\ell}$. Finally using (4.80), one can easily check that the functions $\check{w}_{j,\ell}$, $3 \leq j \leq k-1$, are of the form (4.55), which can be viewed as a particular case of (4.84).

Since all the functions involved in (4.32) and (4.33) have the form (4.84), one easily deduces that $F^{\mathrm{nl},i}_{k,\ell}$, $i = 2, 3$, satisfy (4.69).

Now consider $F^{\mathrm{nl},4}_{k,\ell}$ given by (4.34). It follows from the definition of A_0 (see (4.10)) that, for all $3 \le j \le k-1$, the function $A^0_{j,\ell}$ admits a representation of the same form as $w_{j,\ell}$:

$$\begin{aligned} A^0_{j,\ell}(z) &= A^{\mathrm{reg}}_{j,\ell}(z) \\ &\quad + (1/\sqrt{2}-z)^{j\nu+4} \sum_{0\le\alpha\le(j-3)/2-\ell} (\log(1/\sqrt{2}-z))^{\alpha} A^{\mathrm{reg}}_{j,\ell,\alpha}(z)\chi_{]0,1/\sqrt{2}]}(z) \\ &\quad + \sum_{\substack{3\le\beta\le j-3 \\ 0\le\alpha\le(j-6)/2-\ell}} (1/\sqrt{2}-z)^{\beta\nu+4}(\log(1/\sqrt{2}-z))^{\alpha} A^{\mathrm{reg}}_{j,\ell,\alpha,\beta}(z)\chi_{]0,1/\sqrt{2}]}(z). \end{aligned} \tag{4.85}$$

Furthermore, by the required condition (4.19), the functions $A^0_{j,\ell}$, $3 \le j \le k-1$, vanish on $z = 1/\sqrt{2}$:

$$(A^0_{j,\ell})_{|z=1/\sqrt{2}} = 0, \tag{4.86}$$

which together with (4.34), (4.55) and (4.85) gives (4.69) for $F^{\mathrm{nl},4}_{k,\ell}$. ∎

The second step in the proof of Proposition 4.1 relies on the following lemma:

Lemma 4.5. *For $k \ge 6$, consider the nonhomogeneous equation*

$$\mathcal{S}_k X = \mathcal{F}^{\mathrm{nl}}_k, \tag{4.87}$$

where $\mathcal{S}_k$ is defined by (4.28) and $\mathcal{F}^{\mathrm{nl}}_k = (F^{\mathrm{nl}}_{k,\ell})_{0\le\ell\le\ell(k)}$. Then the following properties hold:

(1) *The system (4.87) has a unique solution $X^0 = (X_{0,\ell})_{0\le\ell\le\ell(k)}$ such that $X_{0,\ell} \equiv 0$ for any integer $\ell_1(k) < \ell \le \ell(k)$, where $\ell_1(k) = [(k-6)/2]$, and such that if $\ell \le \ell_1(k)$, then $X_{0,\ell}$ belongs to $\mathcal{C}^{[k\nu+4]}(\mathbb{R}^*_+) \cap \mathcal{C}^{\infty}(\mathbb{R}^*_+ \setminus \{1/\sqrt{2}\})$ and has the form*

$$\begin{aligned} &X_{0,\ell}(z) = X^{\mathrm{reg}}_{0,\ell}(z) \\ &+(1/\sqrt{2}-z)^{k\nu+7} \sum_{0\le\alpha\le(k-6)/2-\ell} (\log(1/\sqrt{2}-z))^{\alpha} X^{\mathrm{reg}}_{0,\ell,\alpha}(z)\chi_{]0,1/\sqrt{2}]}(z) \\ &+ \sum_{\substack{3\le\beta\le k-3 \\ 0\le\alpha\le(k-6)/2-\ell}} (1/\sqrt{2}-z)^{\beta\nu+4}(\log(1/\sqrt{2}-z))^{\alpha} X^{\mathrm{reg}}_{0,\ell,\beta,\alpha}(z)\chi_{]0,1/\sqrt{2}]}(z), \\ &X_{0,\ell}(1/\sqrt{2}) = 0. \end{aligned} \tag{4.88}$$

Moreover, it has an asymptotic expansion of the form (4.56) as $z \to 0$:

$$X_{0,\ell}(z) = \sum_{\substack{0\le\alpha\le(k-4)/2-\ell \\ \beta\ge1-k+2(\alpha+\ell)}} X_{0,\ell,\alpha,\beta}(\log z)^{\alpha} z^{\beta}. \tag{4.89}$$

(2) *The system* (4.87) *has a unique solution* $X^1 = (X_{1,\ell})_{0\leq \ell\leq \ell(k)}$ *such that* $X_{1,\ell} \equiv 0$ *for any integer* $\ell_1(k) < \ell \leq \ell(k)$, *and such that if* $\ell \leq \ell_1(k)$, *then* $X_{1,\ell}$ *belongs to* $\mathcal{C}^\infty(]1/\sqrt{2}, \infty[)$ *and has the following asymptotic behavior as* $z \to \infty$:

$$X_{1,\ell}(z) = z^{k\nu-1} \sum_{0\leq\alpha\leq(k-6)/2-\ell,\, p\in\mathbb{N}} X_{1,\ell,\alpha,p}(\log z)^\alpha z^{-p} + \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq(k-6)/2-\ell,\, p\in\mathbb{N}}} X_{1,\ell,\alpha,\beta,p}(\log z)^\alpha z^{\nu\beta+1-p}. \tag{4.90}$$

Proof. We use induction on ℓ. Since for any integer $k \geq 6$, we have

$$F^{\mathrm{nl}}_{k,\ell} \equiv 0, \quad \ell_1(k) < \ell \leq \ell(k),$$

we get

$$X_{0,\ell} \equiv 0, \quad \forall\, \ell_1(k) < \ell \leq \ell(k).$$

Consider now

$$\widetilde{\mathcal{L}}_k X_{0,\ell_1(k)} = F^{\mathrm{nl}}_{k,\ell_1(k)}. \tag{4.91}$$

Invoking formulae (4.69), (4.70) together with Lemma 4.3, we easily check that the above equation has a unique solution $X_{0,\ell_1(k)}$ that satisfies (4.88) and has an asymptotic expansion of the form (4.89) for z close to 0.

Let us assume now that for any integer $\ell < q \leq \ell_1(k)$, the equation

$$\widetilde{\mathcal{L}}_k X_{0,q} = F_{k,q}$$

has a unique solution $X_{0,q}$ satisfying (4.88) and (4.89). Then by (4.27), we find that

$$\begin{aligned} F^{\mathrm{lin}}_{k,\ell}(z) &= F^{\mathrm{reg}}_{k,\ell}(z) \\ &+ (1/\sqrt{2}-z)^{k\nu+6} \sum_{0\leq\alpha\leq(k-6)/2-\ell-1} (\log(1/\sqrt{2}-z))^\alpha\, F^{\mathrm{reg}}_{k,\ell,\alpha}(z)\chi_{]0,1/\sqrt{2}]}(z) \\ &+ \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq(k-6)/2-\ell-1}} (1/\sqrt{2}-z)^{\beta\nu+3}(\log(1/\sqrt{2}-z))^\alpha\, F^{\mathrm{reg}}_{k,\ell,\alpha,\beta}(z)\chi_{]0,1/\sqrt{2}]}(z), \end{aligned} \tag{4.92}$$

and behaves as follows as $z \to 0$:

$$F^{\mathrm{lin}}_{k,\ell}(z) = \sum_{\substack{0\leq\alpha\leq(k-6)/2-\ell\\ \beta\geq 5-k+2(\alpha+\ell)}} \widetilde{F}_{k,\ell,\alpha,\beta}(\log z)^\alpha z^{\beta-2}, \tag{4.93}$$

which implies that $F_{k,\ell} = F^{\mathrm{lin}}_{k,\ell} + F^{\mathrm{nl}}_{k,\ell}$ satisfies (4.69) and (4.70).

Therefore taking into account Lemma 4.3, we infer that the equation $\widetilde{\mathcal{L}}_k X_{0,\ell} = F_{k,\ell}$ has a unique solution $X_{0,\ell}$ that satisfies (4.88) and (4.89).

The proof of the second part of the lemma is also by induction on ℓ. First taking into account Lemma 4.3 together with (4.71), we infer that the equation

$$\widetilde{\mathcal{L}}_k X_{1,\ell_1(k)} = F^{\mathrm{nl}}_{k,\ell_1(k)}$$

has a unique solution $X_{1,\ell_1(k)} \in \mathcal{C}^\infty(]1/\sqrt{2},\infty[)$ of the form (4.90). Then assuming that for any integer $\ell < q \leq \ell_1(k)$, the equation

$$\widetilde{\mathcal{L}}_k X_{1,q} = F_{k,q}$$

has a unique solution $X_{1,q}$ satisfying (4.90), we deduce that $F^{\mathrm{lin}}_{k,\ell}$ defined by (4.27) has an expansion of the following form at infinity:

$$\begin{aligned} F^{\mathrm{lin}}_{k,\ell}(z) = & \sum_{\substack{3\leq\beta\leq k-3 \\ 0\leq\alpha\leq(k-6)/2-\ell-1,\ p\in\mathbb{N}}} \hat{F}_{k,\ell,\alpha,\beta,p}(\log z)^\alpha z^{\beta\nu+1-p} \\ & + z^{k\nu-1} \sum_{\substack{0\leq\alpha\leq(k-6)/2-\ell-1 \\ p\in\mathbb{N}}} \hat{F}^{k,\ell}_{k,\ell,\alpha,p}(\log z)^\alpha z^{-p}. \end{aligned} \tag{4.94}$$

Since $F_{k,\ell} = F^{\mathrm{lin}}_{k,\ell} + F^{\mathrm{nl}}_{k,\ell}$, it follows from (4.71), (4.94) and Lemma 4.3 that the equation $\widetilde{\mathcal{L}}_k X_{1,\ell} = F_{k,\ell}$ has a unique solution $X_{1,\ell}$ in $\mathcal{C}^\infty(]1/\sqrt{2},\infty[)$ with an asymptotic expansion of the form (4.90) as $z \to \infty$. This completes the proof of the lemma. ∎

We now return to the proof of Proposition 4.1. Taking advantage of Lemma 4.5(1), we get $W_k =: (w_{k,\ell})_{0\leq\ell\leq(k-3)/2}$ by setting

$$W_k = \begin{cases} X^0 + \sum\limits_{0\leq j\leq\ell(k)} (a^k_{j,+} f^{j,+}_k + a^k_{j,-} f^{j,-}_k) & \text{for } z \leq 1/\sqrt{2}, \\ X^0 + \sum\limits_{0\leq j\leq\ell(k)} a^k_{j,+} f^{j,+}_k & \text{for } z > 1/\sqrt{2}, \end{cases} \tag{4.95}$$

where $(f^{j,\pm}_k)_{0\leq j\leq\ell(k)}$ is the basis of solutions of $\mathcal{S}_k X = 0$ introduced in Lemma 4.1, X^0 is given by Lemma 4.5(1), and in view of (4.15), the coefficients $a^k_{j,\pm}$ are determined by the following relations:

$$\begin{cases} X_{0,\ell,0,-3} + \sum\limits_{\ell\leq j\leq\ell(k)} \mu^{j,\ell}_{k,0}(a^k_{j,+} + a^k_{j,-}) = c^{k,\ell}_{0,-3}, \\ X_{0,\ell,0,-2} + \sum\limits_{\ell\leq j\leq\ell(k)} \mu^{j,\ell}_{k,1}(a^k_{j,+} - a^k_{j,-}) = c^{k,\ell}_{0,-2}, \end{cases} \tag{4.96}$$

with $X_{0,\ell,0,-3}$, $X_{0,\ell,0,-2}$ and $c^{k,\ell}_{0,-3}$, $c^{k,\ell}_{0,-2}$ given by (4.89) and (4.15) respectively,[15] and with $\mu^{j,\ell}_{k,0}$ and $\mu^{j,\ell}_{k,1}$ defined by

$$f^{j,+}_{k,\ell}(z) = \frac{\mu^{j,\ell}_{k,0}}{z^3} + \frac{\mu^{j,\ell}_{k,1}}{z^2} + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \to 0. \tag{4.97}$$

[15] With the convention that $X_{0,\ell,0,\beta} = 0$ if $\ell > \ell_1(k)$ and $c^{k,\ell}_{0,-3} = 0$ if $\ell = (k-3)/2$.

By of (4.40), we easily deduce that

$$\begin{cases} \mu_{k,0}^{j,\ell} = (1/\sqrt{2})^{\alpha(\nu,k)} \binom{j}{\ell} (\log(1/\sqrt{2}))^{j-\ell} \\ \mu_{k,1}^{j,\ell} = \sqrt{2}\Big(\alpha(\nu,k) - \frac{j-\ell}{\log(\sqrt{2})}\Big)\, \mu_{k,0}^{j,\ell}. \end{cases} \tag{4.98}$$

By Lemma 4.5(2),

$$W_k = X^1 + \sum_{0\leq j\leq \ell(k)} (\alpha_k^{j,+} f_{k,\ell}^{j,+} + \alpha_k^{j,-} f_{k,\ell}^{j,-}),$$

with some coefficients $\alpha_k^{j,\pm}$, which concludes the proof of the first part of Proposition 4.1.

In order to establish part (2), we again proceed by induction. First, let us investigate the uniqueness of solutions to (4.61) near 0, and consider the indices $k = 3, 4$ and 5. By the computations carried out in §4.1 (see (4.16)), we have in this case $w_{k,1}^i = 0$ and

$$\widetilde{\mathcal{L}}_k w_{k,0}^i = 0,$$

which implies that

$$\widetilde{\mathcal{L}}_k (w_{k,0}^0 - w_{k,0}^1) = 0.$$

Invoking Remark 4.1 together with assumption (4.62), we easily gather that $w_{k,0}^0 = w_{k,0}^1$ in a neighborhood of 0, for $k = 3, 4$ and 5.

Let us assume now that under assumption (4.62), $w_{k,\ell}^0 = w_{k,\ell}^1$ for all $3 \leq k \leq k_0 - 1 \leq M - 1$ and all $0 \leq \ell \leq \ell(k)$. Since $F_{k_0,\ell}^{\rm nl}(\lambda; w)$ depends only on $w_{j,\ell}$, $j \leq k_0 - 3$, this ensures that

$$\mathcal{S}_{k_0}(\mathcal{W}_{k_0}^0 - \mathcal{W}_{k_0}^1) = 0, \tag{4.99}$$

where

$$\mathcal{W}_{k_0}^i = \begin{pmatrix} w_{k_0,0}^i \\ \vdots \\ w_{k_0,\ell}^i \\ \vdots \\ w_{k_0,\ell(k_0)}^i \end{pmatrix}. \tag{4.100}$$

In order to prove that $\mathcal{W}_{k_0}^0 = \mathcal{W}_{k_0}^1$, we shall proceed by induction on ℓ starting from $\ell(k_0)$. Taking into account (4.28) together with (4.101), we infer that

$$\widetilde{\mathcal{L}}_{k_0}(w_{k_0,\ell(k_0)}^0 - w_{k_0,\ell(k_0)}^1) = 0.$$

Thanks to Lemma 4.1 and (4.62), this implies that

$$w_{k_0,\ell(k_0)}^0 = w_{k_0,\ell(k_0)}^1.$$

Assume now that $w^0_{k_0,q} = w^1_{k_0,q}$ for any integer $\ell < q \le \ell(k_0)$. Then, in view of the definition of $\mathcal{S}_{k_0}$,

$$\widetilde{\mathcal{L}}_{k_0}(w^0_{k_0,\ell} - w^1_{k_0,\ell}) = 0,$$

which, due to Lemma 4.1 and (4.62), easily ensures that $w^0_{k_0,\ell} = w^1_{k_0,\ell}$. This completes the proof of the uniqueness of solutions to (4.61) near 0.

Second, let us investigate the uniqueness of solutions to (4.61) near $+\infty$. Again, we shall proceed by induction starting with the indices $k = 3, 4$ and 5. In this case, we have

$$\widetilde{\mathcal{L}}_k w^i_{k,0} = 0,$$

and the conclusion follows easily from (4.64). Now, assuming that under assumption (4.64), the uniqueness holds for any index $k \le k_0 - 1 \le M - 1$, let us consider the index k_0. Again, by the induction assumption, we have $\mathcal{S}_{k_0}(\mathcal{W}^0_{k_0} - \mathcal{W}^1_{k_0}) = 0$. This gives the result thanks to Lemma 4.1 and condition (4.64), which ends the proof of the proposition.

Remark 4.3. It is important to note that by the uniqueness established above,

$$d^{k,\ell}_{\alpha,\beta} = c^{k,\ell}_{\alpha,\beta}(\lambda), \qquad \forall k, \ell, \alpha, \beta. \tag{4.101}$$

4.3. Estimate of the approximate solution in the self-similar region

With the above notations, for any integer $N \ge 3$ set

$$\begin{aligned} V^{(N)}_{\rm ss}(t, y) &= y + \lambda^{(N)}(t) t^{-\nu-1} W^{(N)}_{\rm ss}\Big(t, y \frac{t^{\nu+1}}{\lambda^{(N)}(t)}\Big), \\ u^{(N)}_{\rm ss}(t, \rho) &= t^{\nu+1} V^{(N)}_{\rm ss}\Big(t, \frac{\rho}{t^{\nu+1}}\Big), \end{aligned} \tag{4.102}$$

with

$$\begin{aligned} W^{(N)}_{\rm ss}(t, z) &= \sum_{k=3}^{N} t^{\nu k} \sum_{\ell=0}^{\ell(k)} (\log t)^\ell \, w_{k,\ell}(z), \\ \lambda^{(N)}(t) &= t\Big(1 + \sum_{k=3}^{N} \sum_{\ell=0}^{\ell(k)} \lambda_{k,\ell} t^{\nu k} (\log t)^\ell\Big). \end{aligned} \tag{4.103}$$

The purpose of this subsection is first to estimate the radial function $V^{(N)}_{\rm ss}$ defined by (4.102) in the self-similar region

$$\Omega_{\rm ss} := \{Y \in \mathbb{R}^4 : t^{\epsilon_1 - \nu}/10 \le |Y| \le 10 t^{-\epsilon_2 - \nu}\}, \tag{4.104}$$

and second to study, for N sufficiently large, the remainder term.

Combining (4.15) with Lemma 4.1, we first get the following lemma:

Lemma 4.6. *There exist a positive constant C and a small positive time $T = T(N)$ such that for all $0 < t \leq T$,*

$$\|\langle\cdot\rangle^{|\alpha|-1}\nabla^\alpha(V_{\rm ss}^{(N)}(t,\cdot)-Q)\|_{L^\infty(\Omega_{\rm ss})} \leq C[t^{3(\nu-\epsilon_1)}+t^{3\nu(1-\epsilon_2)}], \quad \forall\, |\alpha|<3\nu+4, \tag{4.105}$$

$$\|\langle\cdot\rangle^{\beta}\nabla^\alpha(V_{\rm ss}^{(N)}(t,\cdot)-Q)\|_{L^\infty(\Omega_{\rm ss})} \leq C[t^{2\nu+2(\nu-\epsilon_1)}+t^{(\nu-\epsilon_1)(N+1)}+t^{\nu+1+(3\nu-1)(1-\epsilon_2)}], \quad \forall \beta\leq|\alpha|-2 \text{ and } 1\leq|\alpha|<3\nu+4. \tag{4.106}$$

In addition,

$$\|\partial_t V_{\rm ss}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq Ct^{-2-\nu}[t^{1+\nu+2(\nu-\epsilon_1)}+t^{(1+3\nu)(1-\epsilon_2)}], \tag{4.107}$$

$$\|\nabla^\alpha\partial_t V_{\rm ss}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq Ct^{-1}[t^{3(\nu-\epsilon_1)}+t^{3\nu(1-\epsilon_2)}], \quad \forall\, 1\leq|\alpha|<3\nu+3. \tag{4.108}$$

Moreover, for any multi-index α with $|\alpha|<3\nu+3$, the function[16] $V_{{\rm ss},1}^{(N)}(t,y) := (\partial_t u_{\rm ss}^{(N)})(t,\rho)$ *satisfies*

$$\begin{aligned}\|\langle\cdot\rangle^{\beta}\nabla^\alpha V_{{\rm ss},1}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} &\leq Ct^{\nu}[t^{3(\nu-\epsilon_1)}+t^{3\nu(1-\epsilon_2)}], \quad \forall \beta\leq|\alpha|-1,\\ \|\langle\cdot\rangle^{\alpha}\nabla^\alpha V_{{\rm ss},1}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} &\leq C[t^{3\nu-2\epsilon_1}+t^{3\nu(1-\epsilon_2)}],\end{aligned} \tag{4.109}$$

$$\|\partial_t V_{{\rm ss},1}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq Ct^{-1}[t^{3\nu-2\epsilon_1}+t^{3\nu(1-\epsilon_2)}]. \tag{4.110}$$

Finally, for any multi-index α of length $|\alpha|<3\nu+2$ and any integer $\beta\leq|\alpha|$, we have

$$\|\langle\cdot\rangle^{\beta}\nabla^\alpha V_{{\rm ss},2}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq C[t^{2\nu+2(\nu-\epsilon_1)}+t^{\nu+1+(3\nu-1)(1-\epsilon_2)}], \tag{4.111}$$

where $V_{{\rm ss},2}^{(N)}(t,y) = t^{\nu+1}(\partial_t^2 u_{\rm ss}^{(N)})(t,\rho)$.

In the spirit of Lemma 3.3, we also have the following result.

Lemma 4.7. *For all $0<t\leq T$,*

$$\|\nabla^\alpha(V_{\rm ss}^{(N)}(t,\cdot)-Q)\|_{L^2(\Omega_{\rm ss})} \leq C[t^{\nu|\alpha|-\epsilon_1(|\alpha|-2)}+t^{(\nu-\epsilon_1)(N+|\alpha|-3)}+t^{\nu|\alpha|-\epsilon_2(3\nu+3-|\alpha|)}], \quad \forall\, 1\leq|\alpha|<3\nu+4+1/2, \tag{4.112}$$

$$\|\nabla^\alpha(V_{{\rm ss},1}^{(N)})(t,\cdot)\|_{L^2(\Omega_{\rm ss})} \leq Ct^{\nu(|\alpha|+1)}[t^{-\epsilon_1|\alpha|}+t^{-\epsilon_2(3\nu+2-|\alpha|)}], \quad \forall\, 0\leq|\alpha|<3\nu+3+1/2, \tag{4.113}$$

$$\|\nabla^\alpha(V_{{\rm ss},2}^{(N)})(t,\cdot)\|_{L^2(\Omega_{\rm ss})} \leq Ct^{\nu(|\alpha|+2)}[t^{-\epsilon_1|\alpha|}+t^{-\epsilon_2(3\nu+1-|\alpha|)}(1+t^{3\nu-2\epsilon_2})], \quad \forall\, 0\leq|\alpha|<3\nu+2+1/2. \tag{4.114}$$

[16] We recall that $\rho = yt^{\nu+1}$.

Let us now consider the remainder

$$\mathcal{R}_{\rm ss}^{(N)}(t,y) := [(3.3)V_{\rm ss}^{(N)}](t,y).$$

Clearly,

$$\mathcal{R}_{\rm ss}^{(N)}(t,y) = \frac{t^{\nu+1}}{\lambda^{(N)}(t)}\widetilde{\mathcal{R}}_{\rm ss}^{(N)}\left(t, y\frac{t^{\nu+1}}{\lambda^{(N)}(t)}\right),$$

where

$$\widetilde{\mathcal{R}}_{\rm ss}^{(N)}(t,z) = [(4.7)W_{\rm ss}^{(N)}](t,z).$$

By construction,

$$\widetilde{\mathcal{R}}_{\rm ss}^{(N)}(t,z) = \sum_{\substack{k\geq N+1\\ \ell\leq (k-6)/2}} t^{\nu k}(\log t)^{\ell} r_{k,\ell}(z) \quad \text{with} \quad r_{k,\ell}(z) = F_{k,\ell}^{\rm nl}(W_{\rm ss}^{(N)},\lambda^{(N)}).$$

In view of the computations carried out in Section 4.1, we have

$$r_{k,\ell}(z) = r_{k,\ell}^{\rm reg}(z) + (1/\sqrt{2}-z)^{k\nu+6}\sum_{0\leq\alpha\leq (k-6)/2-\ell} r_{k,\ell,\alpha}^{\rm reg}(z)(\log(1/\sqrt{2}-z))^{\alpha}\chi_{]0,1/\sqrt{2}]}(z)$$

$$+ \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq (k-6)/2-\ell}} r_{k,\ell,\alpha,\beta}^{\rm reg}(z)(1/\sqrt{2}-z)^{\beta\nu+2}(\log(1/\sqrt{2}-z))^{\alpha}\chi_{]0,1/\sqrt{2}]}(z). \tag{4.115}$$

Furthermore, as $z\to 0$ and as $z\to\infty$, $r_{k,\ell}$ satisfies (4.70) and (4.71) respectively.

As a direct consequence of these properties, we obtain the following lemma:

Lemma 4.8. *There exist a small positive time $T = T(N)$ and a positive constant C_N such that for all $0 < t \leq T$,*

$$\|\langle\cdot\rangle^{3/2}\mathcal{R}_{\rm ss}^{(N)}(t,\cdot)\|_{H^{K_0}(\Omega_{\rm ss})} \leq C_N[t^{(\nu-\epsilon_1)(N-3/2)} + t^{\nu(1-\epsilon_2)(N+1)-\frac{5}{2}(\nu+1)}], \tag{4.116}$$

where $K_0 = [3\nu + 5/2]$.

Let us end this section by investigating $V_{\rm in}^{(N)} - V_{\rm ss}^{(N)}$ in the intersection of the inner and self-similar regions,

$$\Omega_{\rm in}\cap\Omega_{\rm ss} = \{Y\in\mathbb{R}^4 : t^{\epsilon_1-\nu}/10 \leq |Y| \leq t^{\epsilon_1-\nu}\}.$$

In view of (4.15), (4.55) and Remark 4.3, for any multi-index α and any integer m we have

$$\left|\partial_y^{\alpha}\partial_t^m(V_{\rm in}^{(N)} - V_{\rm ss}^{(N)})(t,y)\right| \lesssim t^{2\nu(N+1)-m}y^{2N-|\alpha|} + t^{-m}y^{-N-|\alpha|}$$

if $y\in\Omega_{\rm in}\cap\Omega_{\rm ss}$ and t is sufficiently small, which leads to the following result:

Lemma 4.9. *For any integer m and any multi-index α,*

$$\|\nabla^{\alpha}\partial_t^m(V_{\rm in}^{(N)} - V_{\rm ss}^{(N)})(t,\cdot)\|_{L^{\infty}(\Omega_{\rm in}\cap\Omega_{\rm ss})} \leq C_{N,\alpha,m}t^{-m+|\alpha|(\nu-\epsilon_1)}(t^{2\nu+2N\epsilon_1} + t^{N(\nu-\epsilon_1)}) \tag{4.117}$$

for all $0 < t \leq T = T(\alpha,m,N)$.

5. Approximate solution in the remote region

5.1. General scheme of construction of the approximate solution in the remote region

In the previous section, we have built, in the self-similar region, an approximate solution $u_{\rm ss}^{(N)}$ which extends the approximation solution $u_{\rm in}^{(N)}$ constructed in Section 3 in the inner region. Our goal here is to extend $u_{\rm ss}^{(N)}$ to the whole space.

Recall that the approximate solution $u_{\rm ss}^{(N)}$ built in Section 4 has the form

$$u_{\rm ss}^{(N)}(t,\rho) = \rho + \lambda^{(N)}(t) \sum_{k=3}^{N} t^{\nu k} \sum_{\ell=0}^{\ell(k)} (\log t)^{\ell}\, w_{k,\ell}\Big(\frac{\rho}{\lambda^{(N)}(t)}\Big),$$

where $\ell(k) = [(k-3)/2]$, and where $\lambda^{(N)}(t)$ is given by (4.103).

To achieve our goal, let us start by introducing the function

$$u^{{\rm lin},(N)}(t,\rho) := t \sum_{k=3}^{N} t^{\nu k} \sum_{\ell=0}^{\ell(k)} (\log t)^{\ell}\, w_{k,\ell}^{\rm lin}(\rho/t), \tag{5.1}$$

where $w_{k,\ell}^{\rm lin}$ denotes the linear part of the function $w_{k,\ell}$ given by (4.60).

The function $u^{{\rm lin},(N)}$ solves the Cauchy problem

$$\begin{cases} (2\partial_t^2 - l_\rho) u^{{\rm lin},(N)} = 0, \\ u^{{\rm lin},(N)}{}_{|t=0} = u_0^{{\rm lin},(N)}, \\ (\partial_t u^{{\rm lin},(N)})_{|t=0} = u_1^{{\rm lin},(N)}, \end{cases} \tag{5.2}$$

where l_ρ is defined by (4.42), and where

$$\begin{cases} u_0^{{\rm lin},(N)}(\rho) = \displaystyle\sum_{k=3}^{N} \sum_{\ell=0}^{\ell(k)} \mu_{k,\ell}^0\, \rho^{k\nu+1} (\log \rho)^{\ell}, \\ u_1^{{\rm lin},(N)}(\rho) = \displaystyle\sum_{k=3}^{N} \sum_{\ell=0}^{\ell(k)} \mu_{k,\ell}^1\, \rho^{k\nu} (\log \rho)^{\ell}, \end{cases} \tag{5.3}$$

with

$$\begin{cases} \mu_{k,\ell}^0 = \alpha_k^{\ell,+} + \alpha_k^{\ell,-}, \\ \mu_{k,\ell}^1 = \dfrac{1}{\sqrt{2}}\big((\nu k + 4)(\alpha_k^{\ell,+} - \alpha_k^{\ell,-}) + (\ell+1)(\alpha_k^{\ell+1,+} - \alpha_k^{\ell+1,-})\big), \end{cases} \tag{5.4}$$

$\alpha_k^{\ell,\pm}$ being the coefficients arising in (4.60). Here we again use the convention that $\alpha_k^{\ell+1,\pm} = 0$ if $\ell + 1 > (k-3)/2$.

Indeed, combining (4.60) with (5.1), we infer that

$$u^{\mathrm{lin},(N)}(t,\rho)=\sum_{k=3}^{N}t^{\nu k+1}\sum_{\ell=0}^{\ell(k)}(\log t)^{\ell}\sum_{0\le j\le (k-3)/2}\big(\alpha_k^{j,+}f_{k,\ell}^{j,+}(\rho/t)+\alpha_k^{j,-}f_{k,\ell}^{j,-}(\rho/t)\big).$$

Taking advantage of (4.44), this gives rise to

$$\begin{aligned}u^{\mathrm{lin},(N)}(t,\rho)&=\sum_{k=3}^{N}t^{\nu k+1}\sum_{0\le j\le (k-3)/2}\bigg(\alpha_k^{j,+}\big(\log t+\log(\rho/t+1/\sqrt{2})\big)^{j}\frac{\big(\frac{\rho}{t}+\frac{1}{\sqrt{2}}\big)^{\nu k+4}}{\big(\frac{\rho}{t}\big)^{3}}\\&\qquad+\alpha_k^{j,-}\big(\log t+\log(\rho/t-1/\sqrt{2})\big)^{j}\frac{\big(\frac{\rho}{t}-\frac{1}{\sqrt{2}}\big)^{\nu k+4}}{\big(\frac{\rho}{t}\big)^{3}}\bigg)\\&=\sum_{k=3}^{N}\sum_{0\le j\le (k-3)/2}\bigg(\alpha_k^{j,+}(\log(\rho+t/\sqrt{2}))^{j}\frac{\big(\rho+\frac{t}{\sqrt{2}}\big)^{\nu k+4}}{\rho^{3}}\\&\qquad+\alpha_k^{j,-}(\log(\rho-t/\sqrt{2}))^{j}\frac{\big(\rho-\frac{t}{\sqrt{2}}\big)^{\nu k+4}}{\rho^{3}}\bigg),\end{aligned}$$

which ensures the result.

Let now χ_0 be a radial smooth cutoff function on $\mathbb{R}^4$ equal to 1 on the unit ball centered at the origin and vanishing outside the ball of radius 2 centered at the origin, and consider, for a small positive real number δ, the compactly supported functions

$$\mathfrak{g}_0(\rho)=\chi_\delta(\rho)u_0^{\mathrm{lin},(N)}(\rho),\qquad \mathfrak{g}_1(\rho)=\chi_\delta(\rho)u_1^{\mathrm{lin},(N)}(\rho),\tag{5.5}$$

where $u_0^{\mathrm{lin},(N)}$ and $u_1^{\mathrm{lin},(N)}$ are the functions defined by (5.3), and where $\chi_\delta(\rho)=\chi_0(\rho/\delta)$.

Remark 5.1. Invoking (5.3) together with (5.5), we infer that there[17] exists $\delta_0(N)>0$ such that for any $0<\delta\le\delta_0(N)$ and any integer $m<3\nu+2$, the above functions $\mathfrak{g}_0$ and $\mathfrak{g}_1$ belong respectively to the Sobolev spaces $\dot H^{m+1}(\mathbb{R}^4)$ and $\dot H^{m}(\mathbb{R}^4)$, and satisfy

$$\|\mathfrak{g}_0\|_{\dot H^{m+1}(\mathbb{R}^4)}\le C\delta^{3\nu-m+2}\quad\text{and}\quad \|\mathfrak{g}_1\|_{\dot H^{m}(\mathbb{R}^4)}\le C\delta^{3\nu-m+2}.$$

We shall look for the solution in the remote region in the form

$$u_{\mathrm{out}}(t,\rho)=\rho+\mathfrak{g}_0(\rho)+t\mathfrak{g}_1(\rho)+\sum_{k\ge 2}t^{k}\mathfrak{g}_k(\rho).\tag{5.6}$$

To this end, we shall apply the lines of reasoning of Sections 3 and 4 and determine by induction the functions $\mathfrak{g}_k$, for $k\ge 2$, making use of the fact that u_{out} is a formal solution to the Cauchy problem

$$\begin{cases}(1.8)u_{\mathrm{out}}=0,\\ u_{\mathrm{out}|t=0}=\rho+\mathfrak{g}_0,\\ (\partial_t u_{\mathrm{out}})_{|t=0}=\mathfrak{g}_1.\end{cases}\tag{5.7}$$

[17] In what follows, the parameter $\delta_0(N)$ may vary from line to line.

For this purpose, we substitute (5.6) into (1.8), which by straightforward computations leads to the recurrent relation

$$\mathrm{g}_k = \frac{1}{k(k-1)(2+2(\mathrm{g}_0)_\rho + (\mathrm{g}_0)_\rho^2)} \mathcal{H}_k(\mathrm{g}_j, j \le k-1), \quad k \ge 2, \tag{5.8}$$

where the source term $\mathcal{H}_k$ has the form

$$\mathcal{H}_k = \mathcal{H}_k^{(1)} + \mathcal{H}_k^{(2)} + \mathcal{H}_k^{(3)} \tag{5.9}$$

with

$$\mathcal{H}_k^{(1)} = l_\rho \mathrm{g}_{k-2}, \tag{5.10}$$

$$\begin{aligned}\mathcal{H}_k^{(2)} &= -2 \sum_{\substack{k_1+k_2=k \\ k_2>0}} k_1(k_1-1-k_2)\mathrm{g}_{k_1}(\mathrm{g}_{k_2})_\rho \\ &\quad + 6 \sum_{k_1+k_2=k-2} \left(-\mathrm{g}_{k_1}\frac{\check{u}_{k_2}}{\rho^3} + (\mathrm{g}_{k_1})_\rho\left(\frac{\check{u}_{k_2}}{\rho^2} + \frac{(\mathrm{g}_{k_2})_\rho}{\rho}\right)\right),\end{aligned} \tag{5.11}$$

$$\begin{aligned}\mathcal{H}_k^{(3)} &= \sum_{k_1+k_2+k_3=k} k_1k_2\Bigg(-\mathrm{g}_{k_1}\mathrm{g}_{k_2}(\mathrm{g}_{k_3})_{\rho\rho} + 2\mathrm{g}_{k_1}(\mathrm{g}_{k_2})_\rho(\mathrm{g}_{k_3})_\rho \\ &\qquad\qquad - 3\mathrm{g}_{k_1}\mathrm{g}_{k_2}\left(\frac{\check{u}_{k_3}}{\rho^2} + \frac{(\mathrm{g}_{k_3})_\rho}{\rho}\right)\Bigg) \\ &\quad - \sum_{\substack{k_1+k_2+k_3=k \\ 2\le k_1<k}} k_1(k_1-1)\mathrm{g}_{k_1}(\mathrm{g}_{k_2})_\rho(\mathrm{g}_{k_3})_\rho \\ &\quad + 3 \sum_{k_1+k_2+k_3=k-2} (\mathrm{g}_{k_1})_\rho(\mathrm{g}_{k_2})_\rho\left(\frac{\check{u}_{k_3}}{\rho^2} + \frac{(\mathrm{g}_{k_3})_\rho}{\rho}\right),\end{aligned} \tag{5.12}$$

where $\check{u}_k$ is given by

$$\check{u} = \frac{u-\rho}{1+\frac{u-\rho}{\rho}} = \sum_{k\ge 0} t^k \check{u}_k. \tag{5.13}$$

Note that $\check{u}_k$ depends only on g_{k_i} with $k_i \le k$.

5.2. *Analysis of the functions* g_k

The aim of this section is to investigate the functions g_k defined above by (5.8)–(5.12). To this end, let us start by introducing the following definition.

Definition 5.1. Let $\mathcal{A}$ be the set of functions a in $\mathcal{C}^\infty(\mathbb{R}_+^*)$ supported in $\{0 < \rho \le 2\delta\}$, where δ is the positive parameter introduced in (5.5), and having for $\rho < \delta$ an absolutely convergent expansion

$$a(\rho) = \sum_{j\ge 3} \sum_{0\le \ell \le (j-3)/2} a_{j,\ell}\,\rho^{\nu j}(\log \rho)^\ell. \tag{5.14}$$

Remark 5.2. The function space $\mathcal{A}$ given by Definition 5.1 is an algebra, and for any function a in $\mathcal{A}$ and any integer m we have

$$\partial^m a \in \rho^{-m}\mathcal{A}. \tag{5.15}$$

Our aim now is to establish the following key result that describes the behavior of the functions $\mathfrak{g}_k$.

Lemma 5.1. *There exists $\delta_0(N) > 0$ such that for any $0 < \delta \le \delta_0(N)$,*

$$\mathfrak{g}_k \in \rho^{1-k}\mathcal{A}, \qquad \forall k \in \mathbb{N}\,.$$

Proof. First note that in view of (5.3) and (5.5), $\mathfrak{g}_0 \in \rho\mathcal{A}$ and $\mathfrak{g}_1 \in \mathcal{A}$ for any $\delta > 0$, and there exists $\delta_0(N) > 0$ such that

$$\frac{1}{1+(1+(\mathfrak{g}_0)_\rho)^2}\mathcal{A} \subset \mathcal{A}, \qquad \frac{1}{1+\mathfrak{g}_0/\rho}\mathcal{A} \subset \mathcal{A}, \tag{5.16}$$

for any $0 < \delta \le \delta_0(N)$.

Let us now show that for any $0 < \delta \le \delta_0(N)$, $\mathfrak{g}_k \in \rho^{1-k}\mathcal{A}$ for all $k \ge 2$. To this end, we shall proceed by induction assuming that, for any integer $j \le k-1$, the function $\mathfrak{g}_j$ belongs to $\rho^{1-j}\mathcal{A}$.

Recalling that

$$l_\rho v = v_{\rho\rho} + 6\Big(\frac{v}{\rho^2} + \frac{v_\rho}{\rho}\Big),$$

we infer, taking into account (5.15), that the function $\mathcal{H}_k^{(1)}$ given by (5.10) belongs to $\rho^{1-k}\mathcal{A}$.

Since $\check{u}_k$ is defined by

$$\check{u} = \frac{u-\rho}{1+\frac{u-\rho}{\rho}} = \sum_{k\ge 0} t^k \check{u}_k,$$

it readily follows from the induction assumption that $\check{u}_j \in \rho^{1-j}\mathcal{A}$ for any $j \le k-1$.

Combining the fact that $\mathcal{A}$ is an algebra with (5.15) and (5.16), we deduce that the function $\mathcal{H}_k^{(2)}$ defined by (5.11) belongs to $\rho^{1-k}\mathcal{A}$.

Along the same lines, taking into account (5.11), we readily gather that $\mathcal{H}_k^{(3)} \in \rho^{1-k}\mathcal{A}$. This concludes the proof of the result thanks to (5.8), (5.9) and (5.16). ■

Remark 5.3. Combining Definition 5.1 with Lemma 5.1, we infer that for any integer k, the function $\mathfrak{g}_k$ has an absolutely convergent expansion

$$\mathfrak{g}_k(\rho) = \rho^{1-k} \sum_{j\ge 3} \sum_{0\le \ell \le (j-3)/2} a^k_{j,\ell}\, \rho^{\nu j} (\log \rho)^\ell \tag{5.17}$$

for $\rho < \delta$, where

$$a^0_{j,\ell} = \mu^0_{j,\ell}, \quad a^1_{j,\ell} = \mu^1_{j,\ell} \text{ if } 3 \le j \le N, \quad a^0_{j,\ell} = a^1_{j,\ell} = 0 \text{ if } j \ge N+1. \tag{5.18}$$

5.3. Estimate of the approximate solution in the remote region

With the above notations, for any integer $N \geq 3$ set

$$\begin{aligned} u_{\text{out}}^{(N)}(t,\rho) &= \rho + \sum_{k=0}^{N} t^k g_k(\rho), \\ V_{\text{out}}^{(N)}(t,y) &= t^{-(\nu+1)} u_{\text{out}}^{(N)}(t, t^{\nu+1} y). \end{aligned} \tag{5.19}$$

Invoking Lemma 5.1, and recalling that for any integer k the function g_k is compactly supported in $\{0 \leq \rho \leq 2\delta\}$, we infer that $V_{\text{out}}^{(N)}$ defined by (5.19) satisfies the following L^∞ estimates in the remote region

$$\Omega_{\text{out}} := \{Y \in \mathbb{R}^4 : y = |Y| \geq t^{-\epsilon_2 - \nu}\}.$$

Lemma 5.2. *For any multi-index α, there exists $\delta_0(\alpha, N) > 0$ such that for any $0 < \delta \leq \delta_0(\alpha, N)$,*

$$\|\langle\cdot\rangle^{|\alpha|} \nabla^\alpha (V_{\text{out}}^{(N)}(t,\cdot) - Q)\|_{L^\infty(\Omega_{\text{out}})} \leq C_\alpha t^{-(\nu+1)} \delta^{3\nu+1}, \tag{5.20}$$

$$\|\langle\cdot\rangle^{|\alpha|-1} \nabla^\alpha (V_{\text{out}}^{(N)}(t,\cdot) - Q)\|_{L^\infty(\Omega_{\text{out}})} \leq C_\alpha \delta^{3\nu}, \tag{5.21}$$

$$\|\langle\cdot\rangle^{\beta} \nabla^\alpha (V_{\text{out}}^{(N)}(t,\cdot) - Q)\|_{L^\infty(\Omega_{\text{out}})} \leq C_{\alpha,\beta} (t^{3\nu(\nu+1)} + t^{\nu+1}), \quad \forall \beta \leq |\alpha| - 2, \tag{5.22}$$

$$\|\partial_t V_{\text{out}}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\text{out}})} \leq C t^{-(\nu+2)} \delta^{3\nu+1}, \tag{5.23}$$

$$\|\langle\cdot\rangle^{|\alpha|} \nabla^\alpha V_{\text{out},1}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\text{out}})} \leq C_\alpha \delta^{3\nu}, \tag{5.24}$$

$$\|\langle\cdot\rangle^{\beta} \nabla^\alpha V_{\text{out},1}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\text{out}})} \leq C_{\alpha,\beta} (t^{3\nu(\nu+1)} + t^{\nu+1}), \quad \forall \beta \leq |\alpha| - 1, \tag{5.25}$$

$$\|\partial_t V_{\text{out},1}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\text{out}})} \leq C t^{-1} \delta^{3\nu}, \tag{5.26}$$

$$\|\langle\cdot\rangle^{\beta} \nabla^\alpha V_{\text{out},2}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\text{out}})} \leq C_{\alpha,\beta} (t^{3\nu(\nu+1)} + \delta^{3\nu-1} t^{\nu+1}), \quad \forall \beta \leq |\alpha|, \tag{5.27}$$

for all $0 < t \leq T$ with $T = T(\alpha, \delta, N)$, where

$$V_{\text{out},1}^{(N)}(t,y) := (\partial_t u_{\text{out}}^{(N)})(t,\rho), \qquad V_{\text{out},2}^{(N)}(t,y) := t^{\nu+1} (\partial_t^2 u_{\text{out}}^{(N)})(t,\rho).$$

Moreover, for any multi-index α with $|\alpha| \geq 1$,

$$\|\nabla^\alpha \partial_t V_{\text{out}}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\text{out}})} \leq C_\alpha t^{-1} \delta^{3\nu} \tag{5.28}$$

for all $0 < t \leq T$.

Denote

$$\Omega_{\text{out}}^x := \{x \in \mathbb{R}^4 : |x| \geq t^{1-\epsilon_2}\}. \tag{5.29}$$

Lemma 5.3. *With the previous notations, for any* $0 < \delta \le \delta_0(\alpha, N)$ *and all* $0 < t \le T = T(\alpha, \delta, N)$*,*

$$\begin{aligned} &\big\| \nabla_x^\alpha \big(u_{\text{out}}^{(N)}(t,\cdot) - t^{\nu+1} Q(\cdot/t^{\nu+1}) - \mathfrak{g}_0 \big) \big\|_{L^2(\Omega_{\text{out}}^x)} \\ &\qquad\qquad \le C_\alpha t (1 + t^{(1-\epsilon_2)(3\nu+2-|\alpha|)}), \qquad \forall |\alpha| \ge 1, \quad (5.30) \\ &\big\| \nabla_x^\alpha (\partial_t^\ell u_{\text{out}}^{(N)}(t,\cdot) - \mathfrak{g}_\ell) \big\|_{L^2(\Omega_{\text{out}}^x)} \le C_\alpha t (1 + t^{(1-\epsilon_2)(3\nu+2-\ell-|\alpha|)}), \qquad \forall |\alpha| \ge 0, \quad (5.31) \end{aligned}$$

for $\ell = 1, 2$*.*

Remark 5.4. Combining (5.19) with Lemma 5.3, we infer that satisfies, for all $0 < t \le T$,

$$\begin{aligned} &\|\nabla^\alpha (V_{\text{out}}^{(N)}(t,\cdot) - Q)\|_{L^2(\Omega_{\text{out}})} \\ &\qquad \le C_\alpha t^{(|\alpha|-3)(\nu+1)} [\delta^{3\nu+3-|\alpha|} + t^{(1-\epsilon_2)(3\nu+3-|\alpha|)}], \qquad \forall |\alpha| \ge 1, \qquad (5.32) \\ &\|\nabla^\alpha V_{\text{out},\ell}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\text{out}})} \\ &\qquad \le C_\alpha t^{(|\alpha|-3+\ell)(\nu+1)} [\delta^{3\nu+3-\ell-|\alpha|} + t^{(1-\epsilon_2)(3\nu+3-\ell-|\alpha|)}], \qquad \forall |\alpha| \ge 0, \qquad (5.33) \end{aligned}$$

for $\ell = 1, 2$.

Let us now consider the remainder

$$\mathcal{R}_{\text{out}}^{(N)} := (3.3) V_{\text{out}}^{(N)}. \tag{5.34}$$

We have

$$\mathcal{R}_{\text{out}}^{(N)}(t,y) = t^{\nu+1} \widetilde{\mathcal{R}}_{\text{out}}^{(N)}(t, t^{\nu+1} y), \quad \text{where} \quad \widetilde{\mathcal{R}}_{\text{out}}^{(N)}(t,y) = [(1.8) u_{\text{out}}^{(N)}](t, t^{\nu+1} y).$$

It follows readily from the proof of Lemma 5.1 that

$$\big\| |\cdot|^{3/2} \nabla_x^\alpha \widetilde{\mathcal{R}}_{\text{out}}^{(N)}(t,\cdot) \big\|_{L^2(\Omega_{\text{out}}^x)} \le C_{\alpha,N} t^{N-1-(1-\epsilon_2)(|\alpha|+N-3\nu-7/2)} \tag{5.35}$$

for any $|\alpha| \ge 0$ provided that $N \ge 3\nu + 7/2$, which leads to the following lemma:

Lemma 5.4. *For any multi-index* α*,*

$$\|\langle \cdot \rangle^{3/2} \nabla^\alpha \mathcal{R}_{\text{out}}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\text{out}})} \le t^{\epsilon_2 N - \frac{5}{2}(\nu+1)} \tag{5.36}$$

for all $0 < t \le T = T(\alpha, \delta, N)$ *provided that* $N \ge 3\nu + 7/2$*.*

We next investigate $V_{\text{out}}^{(N)} - V_{\text{ss}}^{(N)}$ in $\Omega_{\text{out}} \cap \Omega_{\text{ss}}$. Assuming $\rho < \delta$, and rewriting $u_{\text{out}}^{(N)}$ in terms of the variable $z = \rho/\lambda^{(N)}(t)$, we get

$$\begin{aligned} &u_{\text{out}}^{(N)}(t,\rho) = \lambda^{(N)}(t) \\ &\times \Big[z + \sum_{k=3}^{N} \sum_{0 \le \ell \le (k-3)/2} t^{\nu k} (\log t)^\ell \Big(\sum_{\substack{3 \le \beta \le k-3 \\ 0 \le \alpha \le (k-6)/2 - \ell,\ p \ge 0}} w_{k,\ell,\alpha,\beta,p}^{\text{out}} (\log z)^\alpha z^{\nu\beta+1-p} \\ &\qquad\qquad + z^{\nu k+1} \sum_{0 \le \alpha \le (k-3)/2-\ell,\ p \ge 0} w_{k,\ell,\alpha,p}^{\text{out}} (\log z)^\alpha z^{-p} \Big) \Big], \end{aligned}$$

with some coefficients $w^{\rm out}_{k,\ell,\alpha,\beta,p}$, $w^{\rm out}_{k,\ell,\alpha,p}$ that can be expressed explicitly in terms of the coefficients $\lambda_{j,\ell}$ with $3 \le j \le N, 0 \le \ell \le \ell(j)$ and of the constants $a^k_{j,\ell}$ with $k \ge 0$, $j \ge 3$, $0 \le \ell \le \ell(j)$, introduced in Remark 5.3.

In particular,

$$w^{\rm out}_{k,\ell,\alpha,p} = \binom{\alpha+\ell}{\alpha} a^p_{k,\alpha+\ell} \tag{5.37}$$

for all $k \ge 3$, $\ell \le (k-3)/2$, $\alpha \le (k-3)/2 - \ell$, $p \ge 0$.

Combining (5.4), (5.18) with (5.37), we infer that

$$\begin{aligned}\sum_{0\le j\le \ell(k)} &(\alpha_k^{j,+} f^{j,+}_{k,\ell} + \alpha_k^{j,-} f^{j,-}_{k,\ell})\\ &= \sum_{\substack{0\le\alpha\le(j-3)/2-\ell\\ p=0,1}} z^{\nu k+1-p}(\log z)^\alpha w^{\rm out}_{k,\ell,\alpha,p} + \mathcal{O}(z^{\nu k-1}(\log z)^{\ell(j)-\ell})\end{aligned}$$

as $z \to \infty$, which by Proposition 4.1(2) (uniqueness near infinity) implies that

$$w^{\rm out}_{k,\ell,\alpha,\beta,p} = w_{k,\ell,\alpha,\beta,p}, \quad w^{\rm out}_{k,\ell,q,p} = w_{k,\ell,q,p}, \tag{5.38}$$

for any $3 \le k \le N, 0 \le \ell \le \ell(k), 0 \le \alpha \le (k-6)/2 - \ell, 0 \le q \le (k-3)/2 - \ell, 3 \le \beta \le k-3$, $p \ge 0$, where $w_{k,\ell,\alpha,\beta,p}$, $w_{k,\ell,q,p}$ are the coefficients involved in (4.65).

As a direct consequence of (5.38), we obtain

Lemma 5.5. *For any $\alpha \in \mathbb{N}^4$ and any integer m,*

$$\|\partial_t^m \nabla^\alpha (V^{(N)}_{\rm out} - V^{(N)}_{\rm ss})(t,\cdot)\|_{L^\infty(\Omega_{\rm out}\cap\Omega_{\rm ss})} \le t^{-m-\nu+|\alpha|(\nu+\epsilon_2)}(t^{\epsilon_2 N} + t^{-\epsilon_2+(1-\epsilon_2)\nu N}) \tag{5.39}$$

for all $0 < t < T = T(\alpha, m, N)$.

6. Approximate solution in the whole space

Let Θ be a radial function in $\mathcal{D}(\mathbb{R})$ satisfying

$$\Theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1/4,\\ 0 & \text{if } |\xi| \ge 1/2.\end{cases}$$

Set

$$\begin{aligned} V^{(N)}(t,y) &:= \Theta(yt^{\nu-\epsilon_1}) V^{(N)}_{\rm in}(t,y) + (\Theta(yt^{\nu+\epsilon_2}) - \Theta(yt^{\nu-\epsilon_1}))V^{(N)}_{\rm ss}(t,y)\\ &\quad + (1-\Theta(yt^{\nu+\epsilon_2}))V^{(N)}_{\rm out}(t,y), \\ u^{(N)}(t,\rho) &:= t^{\nu+1} V^{(N)}(t, \rho/t^{\nu+1}).\end{aligned} \tag{6.1}$$

Combining Lemmas 3.2, 4.6 and 5.2 with Lemmas 4.9 and 5.5, we infer that for N sufficiently large there exists a positive parameter $\delta_0(N)$ such that for any $0 < \delta \le \delta_0(N)$ there exists a positive time $T = T(\delta, N)$ so that the approximate solution $V^{(N)}$ defined by (6.1) satisfies the following L^∞ estimates:

Lemma 6.1. *The following estimates hold for $V^{(N)}$, for all $0 < t \le T$:*

$$\|\langle\cdot\rangle^{|\alpha|-1}\nabla^\alpha (V^{(N)} - Q)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le \delta^{3\nu}, \qquad \forall\, 0 \le |\alpha| < 3\nu + 4, \tag{6.2}$$

$$\|\langle\cdot\rangle^{\beta}\nabla^\alpha (V^{(N)} - Q)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le Ct^{\nu}, \qquad \forall\, 1 \le |\alpha| < 3\nu + 4 \text{ and } \beta \le |\alpha| - 2, \tag{6.3}$$

$$\left\|\nabla^\alpha \frac{y}{|y|^2}\cdot \nabla (V^{(N)} - Q)(t,\cdot)\right\|_{L^\infty(\mathbb{R}^4)} \le Ct^{\nu}, \qquad \forall\, 0 \le |\alpha| < 3\nu + 3. \tag{6.4}$$

Moreover, the time derivative of $V^{(N)}$ satisfies

$$\|\partial_t V^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le Ct^{-2-\nu}\delta^{3\nu+1}, \tag{6.5}$$

$$\|\nabla^\alpha\partial_t V^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le Ct^{-1}\delta^{3\nu}, \qquad \forall\, 1 \le |\alpha| < 3\nu + 3. \tag{6.6}$$

In addition, for any multi-index α with $|\alpha| < 3\nu + 3$, the function $V_1^{(N)}(t, y) := (\partial_t u^{(N)})(t,\rho)$ and its time derivative satisfy

$$\|\langle\cdot\rangle^{|\alpha|}\nabla^\alpha V_1^{(N)}\|_{L^\infty(\mathbb{R}^4)} \le C\delta^{3\nu}, \tag{6.7}$$

$$\|\langle\cdot\rangle^{\beta}\nabla^\alpha V_1^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le Ct^{\nu}, \qquad \forall \beta \le |\alpha| - 1, \tag{6.8}$$

$$\|\partial_t V_1^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le Ct^{-1}\delta^{3\nu}. \tag{6.9}$$

Finally, for any multi-index α with $|\alpha| < 3\nu + 2$ and any integer $\beta \le |\alpha|$,

$$\|\langle\cdot\rangle^{\beta}\nabla^\alpha V_2^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le Ct^{\nu}, \tag{6.10}$$

where $V_2^{(N)}(t, y) := t^{\nu+1}(\partial_t^2 u^{(N)})(t,\rho)$.

Along the same lines, taking advantage of Lemmas 3.3, 4.7 and 5.3, we get the following L^2 estimates, as before for N sufficiently large, $0 < \delta \le \delta_0(N)$ and $0 < t \le T(\delta, N)$:

Lemma 6.2. *For any $1 \le |\alpha| < 3\nu + 3$,*

$$\|\nabla^\alpha (u^{(N)}(t,\cdot) - t^{\nu+1}Q(\cdot/t^{\nu+1}) - \mathfrak{g}_0)\|_{L^2((\mathbb{R}^4)} \le C(t + t^{(1-\epsilon_2)(3\nu+3-|\alpha|)} + t^{3+5\nu-|\alpha|(1+\nu)}), \tag{6.11}$$

and for any $0 \le |\alpha| < 3\nu + 2$,

$$\|\nabla^\alpha (u_t^{(N)}(t,\cdot) - \mathfrak{g}_1)\|_{L^2(\mathbb{R}^4)} \le C(t + t^{(1-\epsilon_2)(3\nu+2-|\alpha|)} + t^{2+3\nu-|\alpha|(1+\nu)}). \tag{6.12}$$

Moreover,

$$\|\nabla^\alpha (V^{(N)}(t,\cdot) - Q)\|_{L^2(\mathbb{R}^4)} \le Ct^{2\nu}, \qquad \forall\, 3\nu + 3 < |\alpha| < 3\nu + 4 + 1/2, \tag{6.13}$$

$$\|\nabla^\alpha V_1^{(N)}(t,\cdot)\|_{L^2(\mathbb{R}^4)} \le Ct^{\nu}, \qquad \forall\, 3\nu + 2 < |\alpha| < 3\nu + 3 + 1/2, \tag{6.14}$$

$$\|\nabla^\alpha V_2^{(N)}(t,\cdot)\|_{L^2(\mathbb{R}^4)} \le Ct^{2\nu}, \qquad \forall\, 3\nu + 1 < |\alpha| < 3\nu + 2 + 1/2. \tag{6.15}$$

Remark 6.1. Lemma 6.2 implies that

$$\|\nabla^\alpha (V^{(N)}(t,\cdot) - Q)\|_{L^2(\mathbb{R}^4)} \le C(t^{2\nu} + t^{(1+\nu)(|\alpha|-3)}\delta^{3\nu+3-|\alpha|}), \quad \forall\, 1 \le |\alpha| < 3\nu + 4 + 1/2, \tag{6.16}$$

$$\|\nabla^\alpha V_1^{(N)}(t,\cdot)\|_{L^2(\mathbb{R}^4)} \le C(t^{\nu} + t^{(1+\nu)(|\alpha|-2)}\delta^{3\nu+2-|\alpha|}), \quad \forall\, 0 \le |\alpha| < 3\nu + 3 + 1/2, \tag{6.17}$$

and

$$\|\nabla^\alpha (u^{(N)}(t,\cdot) - t^{\nu+1}Q(\cdot/t^{\nu+1}) - \mathrm{g}_0)\|_{L^2(\mathbb{R}^4)} \xrightarrow{t\to 0} 0, \quad \forall\, 1 \le |\alpha| < 3 + \frac{2\nu}{\nu+1},$$

$$\|\nabla^\alpha (u_t^{(N)}(t,\cdot) - \mathrm{g}_1)\|_{L^2(\mathbb{R}^4)} \xrightarrow{t\to 0} 0, \quad \forall\, 0 \le |\alpha| < 2 + \frac{\nu}{\nu+1}.$$

Finally, if we denote

$$\mathcal{R}^{(N)} := (3.3)V^{(N)},$$

then invoking Lemmas 3.4, 4.8, 4.9, 5.4 and 5.5, we get the following result:

Lemma 6.3. *There exist $N_0 \in \mathbb{N}$ and $\kappa > 0$ such that*

$$\|\langle\cdot\rangle^{3/2}\mathcal{R}^{(N)}(t,\cdot)\|_{H^{K_0}(\mathbb{R}^4)} \le t^{\kappa N+\nu} \tag{6.18}$$

for all $N \ge N_0$ and $0 < t \le T(\delta, N)$, where $K_0 = [3\nu + 5/2]$ was introduced in Lemma 4.8.

Relabeling N, one can always assume that the approximate solutions $u^{(N)}$ are defined and satisfy Lemmas 6.1-6.2 for any integer $N \ge 1$, and that (6.18) holds with $\kappa = 1$ for all $N \ge 1$.

7. Proof of the blow up result

7.1. Key estimates

The approximate solutions $u^{(N)}$ constructed in the previous sections satisfy, for any integer $N \ge 1$,

$$(\nabla(u^{(N)} - Q), \partial_t u^{(N)}) \in \mathcal{C}(]0,T], H^{K_0+1}(\mathbb{R}^4)),$$

with[18] some $T = T(\delta, N) > 0$. Furthermore, by (6.2) and (6.8), there are positive constants c_0 and c_1 such that

$$u^{(N)}(t,\cdot) \ge c_0 t^{\nu+1}, \tag{7.1}$$

$$(1 + |\nabla u^{(N)}|^2 - (\partial_t u^{(N)})^2)(t,\cdot) \ge c_1, \tag{7.2}$$

[18] In what follows, δ is assumed to be less than $\delta_0(N)$, which may vary from line to line.

for any integer $N \geq 1$, and all t in $]0, T]$. This ensures that

$$(u^{(N)}(t,\cdot), \partial_t u^{(N)}(t,\cdot)) \in X_{K_0+2}, \qquad \forall t \in]0, T].$$

The goal of this subsection is to finish the proof of Theorem 1.2 by showing that for N sufficiently large, the approximate solution $u^{(N)}$ can be completed to an exact solution u to (1.8) that satisfies

$$\|\langle\cdot\rangle^{3/2}\partial_t(u-u^{(N)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} + \|\langle\cdot\rangle^{3/2}\nabla(u-u^{(N)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} \leq t^{N/2}, \qquad \forall t \in]0, T], \tag{7.3}$$

for some positive $T = T(\delta, N)$ and[19] $L_0 = 2M + 1$ with $M = [K_0/2]$. Note that since $\nu > 1/2$, we have $M \geq 2$, and thus $L_0 \geq 5$.

The mechanism for achieving this will rely on the following crucial result:

Proposition 7.1. *There is $N_0 \in \mathbb{N}$ such that for any integer $N \geq N_0$, there exists a small positive time $T = T(\delta, N)$ such that, for any time $0 < t_1 \leq T$, the Cauchy problem*

$$(\mathrm{NW})^{(N)} \begin{cases} (1.6)u = 0, \\ u_{|t=t_1} = u^{(N)}(t_1,\cdot), \\ (\partial_t u)_{|t=t_1} = \partial_t u^{(N)}(t_1,\cdot), \end{cases} \tag{7.4}$$

has a unique solution u on the interval $[t_1, T]$ which satisfies

$$\|\langle\cdot\rangle^{3/2}\partial_t(u-u^{(N)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} + \|\langle\cdot\rangle^{3/2}\nabla(u-u^{(N)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} \leq t^{N/2} \tag{7.5}$$

for all $t \in]t_1, T]$.

Proof. As mentioned above, for any t_1 sufficiently small, the initial data $(u^{(N)}(t_1,\cdot), \partial_t u^{(N)}(t_1,\cdot))$ belongs to X_{K_0+2}, and thus satisfies the hypothesis of Theorem 1.1. By construction $u^{(N)}(t,\rho) - \rho$ is compactly supported. Thus, to establish Proposition 7.1, it is enough to show that there exists $T = T(\delta, N) > 0$ such that the solution to the Cauchy problem (7.4) satisfies the estimate (7.5) for any time $t_1 \leq t \leq \min\{T(\delta, N), T^*\}$, where T^* is the maximal time of existence. This will be achieved in two steps:

(1) First writing $u(t,x) = t^{\nu+1}V(t,y)$, $V(t,y) = V^{(N)}(t,y) + \varepsilon^{(N)}(t,y)$, with $y = x/t^{\nu+1}$, $x \in \mathbb{R}^4$, we derive the equation satisfied by the remainder term $\varepsilon^{(N)}$. We next set

$$\varepsilon^{(N)}(t,y) = H(y)r^{(N)}(t,y),$$

where H is the function defined by (2.8), and rewrite the resulting equation in terms of $r^{(N)}$. As we will see later, the equation for $r^{(N)}$ involves the operator $\mathfrak{L}$ introduced in (2.10).

[19] We choose L_0 to be an odd integer to make the estimates we are dealing with easier.

(2) We deduce the desired result (inequalities (7.5)) from suitable energy estimates by making use of the behavior of the approximate solution $u^{(N)}$ described by Lemmas 6.1 and 6.2, and the spectral properties of the operator $\mathfrak{L}$, which turns out to be close to the Laplace operator.

In order to make notations as light as possible, we shall omit the dependence of the functions $\varepsilon^{(N)}$ and $r^{(N)}$ on N.

Denote

$$V_1(t,y) := a(t)V_t(t,y) + a'(t)\Lambda V(t,y) = u_t(t,x), \tag{7.6}$$

$$V_2(t,y) := a(t)(V_1)_t(t,y) - a'(t)(y\cdot\nabla V_1)(t,y) = t^{\nu+1}u_{tt}(t,x), \tag{7.7}$$

with $a(t) = t^{\nu+1}$ and $\Lambda V = V - y\cdot\nabla V$.

Multiplying the quasilinear wave equation (1.6) by $a(t)$ and rewriting it in terms of V, one gets

$$(1+|\nabla V|^2)V_2 - 2(\nabla V\cdot\nabla V_1)V_1 - (1-V_1^2+|\nabla V|^2)\Delta V + \sum_{j,k=1}^{4} V_{y_j}V_{y_k}\partial^2_{y_j y_k}V + \frac{3}{V}(1-V_1^2+|\nabla V|^2) = 0. \tag{7.8}$$

Thus recalling that the approximate solution $V^{(N)}$ satisfies (7.8) up to a remainder term $\mathcal{R}^{(N)}$, we infer that the function u solves (1.6) if and only if the remainder term ε satisfies the following equation:

$$(1+|\nabla V|^2)\varepsilon_2 - \mathcal{L}\varepsilon - 2V_1\nabla V\cdot\nabla\varepsilon_1 + (V_1^2-|\nabla\varepsilon|^2)\Delta\varepsilon + \sum_{j,k=1}^{4}\varepsilon_{y_j}\varepsilon_{y_k}\partial^2_{y_j y_k}\varepsilon + \mathcal{F} + \mathcal{R}^{(N)} = 0, \tag{7.9}$$

where

$$\varepsilon_2 = a(t)(\varepsilon_1)_t - a'(t)(y\cdot\nabla\varepsilon_1), \qquad \varepsilon_1 = a(t)\varepsilon_t + a'(t)\Lambda\varepsilon, \tag{7.10}$$

$\mathcal{L}$ is the linearized operator introduced in (1.16):

$$\mathcal{L}\varepsilon = \Delta\varepsilon + 3\left(\frac{(3y\cdot\nabla Q)\nabla Q}{|y|^2} - \frac{2\nabla Q}{Q}\right)\cdot\nabla\varepsilon + 3\frac{1+|\nabla Q|^2}{Q^2}\varepsilon,$$

and the term $\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{F} &= (|\nabla V|^2 - |\nabla V^{(N)}|^2)V_2^{(N)} - 2(V_1\nabla V - V_1^{(N)}\nabla V^{(N)})\cdot\nabla V_1^{(N)} \\ &\quad + (V_1^2 - (V_1^{(N)})^2)\Delta V^{(N)} - \frac{3}{VV^{(N)}}(V_1^2V^{(N)} - (V_1^{(N)})^2V) \\ &\quad + 3\left[\frac{1}{V^{(N)}}(|\nabla V|^2 - |\nabla V^{(N)}|^2) - \frac{2}{Q}\nabla Q\cdot\nabla\varepsilon\right] - 3\varepsilon\left[\frac{1+|\nabla V|^2}{VV^{(N)}} - \frac{1+|\nabla Q|^2}{Q^2}\right] \\ &\quad - 9(|\nabla V^{(N)}|^2 - |\nabla Q|^2)\frac{y\cdot\nabla\varepsilon}{|y|^2} - 9\frac{y\cdot\nabla V^{(N)}}{|y|^2}|\nabla\varepsilon|^2.\end{aligned}$$

Next, set

$$\varepsilon(t,y) = H(y)r(t,y) \tag{7.11}$$

with

$$H = \frac{(1+|\nabla Q|^2)^{1/4}}{Q^{3/2}}. \tag{7.12}$$

Let us emphasize that in view of Lemma 2.1, the function H enjoys the following property: for any $\alpha \in \mathbb{N}^4$, there exists a positive constant C_α such that for any y in $\mathbb{R}^4$,

$$\frac{1}{C_\alpha \langle y\rangle^{3/2+|\alpha|}} \leq |\nabla^\alpha H(y)| \leq \frac{C_\alpha}{\langle y\rangle^{3/2+|\alpha|}}. \tag{7.13}$$

Now in light of the definitions introduced in (7.10), we have

$$\varepsilon_1(t,y) = H(y)r_1(t,y), \qquad \varepsilon_2(t,y) = H(y)r_2(t,y),$$

where

$$\begin{aligned} r_1 &= ar_t + a'\Lambda r - a'\,\frac{y\cdot\nabla H}{H} r, \\ r_2 &= a(r_1)_t - a' y\cdot\nabla r_1 - a'\,\frac{y\cdot\nabla H}{H} r_1. \end{aligned} \tag{7.14}$$

Thus taking advantage of (7.9), we readily gather that the remainder term r given by (7.11) satisfies

$$\begin{aligned} &(1+|\nabla V|^2)r_2 + (1+|\nabla Q|^2)\,\mathfrak{L} r - \frac{2V_1}{H}\nabla V\cdot\nabla(Hr_1) + (V_1^2 - |\nabla(Hr)|^2)\Delta r \\ &\quad - \frac{2V_1}{H}\nabla V\cdot\nabla(Hr_1) + \frac{V_1^2 - |\nabla(Hr)|^2}{H}[\Delta, H]r + \sum_{j,k=1}^{4}(Hr)_{y_j}(Hr)_{y_k}\partial^2_{y_j y_k} r \\ &\quad + \sum_{j,k=1}^{4}\frac{(Hr)_{y_j}(Hr)_{y_k}}{H}[\partial^2_{y_j y_k}, H]r + \frac{\mathcal{F}}{H} + \frac{\mathcal{R}^{(N)}}{H} = 0, \end{aligned} \tag{7.15}$$

where $[A,B] = AB - BA$ denotes the commutator of the operators A and B, and where

$$\mathfrak{L} = -\frac{1}{H(1+|\nabla Q|^2)}\,\mathcal{L} H. \tag{7.16}$$

Let us recall that in view of (2.10),

$$\mathfrak{L} = -q\Delta q + \mathcal{P},$$

where $q = \frac{1}{(1+|\nabla Q|^2)^{1/2}}$ and $\mathcal{P}$ is a radial $\mathcal{C}^\infty$ function which satisfies

$$\mathcal{P} = -\frac{3}{8\rho^2}(1+o(1)) \quad \text{as } \rho \to \infty.$$

Dividing the equation at hand by $1+|\nabla V|^2$, we get

$$r_2+\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\mathfrak{L}r-\frac{2V_1}{1+|\nabla V|^2}\nabla V\cdot\nabla r_1+\frac{V_1^2-|\nabla(Hr)|^2}{1+|\nabla V|^2}\Delta r$$
$$+\frac{1}{1+|\nabla V|^2}\sum_{j,k=1}^{4}(Hr)_{y_j}(Hr)_{y_k}\partial^2_{y_jy_k}r+\widetilde{\mathcal{F}}+\widetilde{\mathcal{R}}^{(N)}=0, \tag{7.17}$$

with

$$\widetilde{\mathcal{R}}^{(N)}:=\frac{\mathcal{R}^{(N)}}{(1+|\nabla V|^2)H}, \tag{7.18}$$

$$\widetilde{\mathcal{F}}:=\frac{\mathcal{F}}{(1+|\nabla V|^2)H}-\frac{2V_1}{(1+|\nabla V|^2)H}\nabla V\cdot(\nabla H)r_1$$
$$+\frac{V_1^2-|\nabla(Hr)|^2}{(1+|\nabla V|^2)H}[\Delta,H]r+\sum_{j,k=1}^{4}\frac{(Hr)_{y_j}(Hr)_{y_k}}{(1+|\nabla V|^2)H}[\partial^2_{y_jy_k},H]r. \tag{7.19}$$

Observe that $\widetilde{\mathcal{F}}$ depends only on the remainder term r and its first derivatives. This achieves the goal of the first step.

The bound (7.5) will be proved by a bootstrap argument based on the following lemma, which we establish by combining the positivity property of the operator $\mathfrak{L}$ (see (2.12) and Appendix B) with the estimates obtained in Lemmas 6.1 and 6.2.

Lemma 7.1. *There is $N_0\in\mathbb{N}$ such that for any integer $N\geq N_0$, there exists $T=T(N,\delta)>0$ such that for any $t_1\in\,]0,T]$ and any $t_2\in[t_1,T]$ the following property holds. If for all $t\in[t_1,t_2]$,*

$$\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}+\|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\leq t^{2N}, \tag{7.20}$$

then

$$\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}+\|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\leq \frac{C}{N}t^{2N} \tag{7.21}$$

for any $t\in[t_1,t_2]$, where C is an absolute constant.

Proof of Lemma 7.1. In order to prove (7.21), let us start by applying the operator $\mathfrak{L}^M$ to (7.17). This gives

$$\mathfrak{L}^M r_2+\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\mathfrak{L}^{M+1}r-\frac{2V_1}{1+|\nabla V|^2}\nabla V\cdot\nabla\mathfrak{L}^M r_1+\frac{V_1^2-|\nabla\varepsilon|^2}{1+|\nabla V|^2}\Delta\mathfrak{L}^M r$$
$$+\frac{1}{1+|\nabla V|^2}\sum_{j,k=1}^{4}(Hr)_{y_j}(Hr)_{y_k}\partial^2_{y_jy_k}(\mathfrak{L}^M r)+\widetilde{\mathcal{F}}_M+\mathfrak{L}^M\widetilde{\mathcal{R}}^{(N)}=0 \tag{7.22}$$

with $\widetilde{\mathcal{F}}_M = \mathfrak{L}^M \widetilde{\mathcal{F}} + \mathcal{G}_M$, where

$$\mathcal{G}_M := \left[\mathfrak{L}^M, \frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\right]\mathfrak{L}r - 2\left[\mathfrak{L}^M, \frac{V_1}{1+|\nabla V|^2}\nabla V\cdot\nabla\right]r_1 + \left[\mathfrak{L}^M, \frac{V_1^2-|\nabla\varepsilon|^2}{1+|\nabla V|^2}\Delta\right]r + \sum_{j,k=1}^{4}\left[\mathfrak{L}^M, \frac{(Hr)_{y_j}(Hr)_{y_k}}{1+|\nabla V|^2}\partial^2_{y_j y_k}\right]r. \tag{7.23}$$

Now let us multiply (7.17) by $a^{-1}r_1$ and (7.22) by $a^{-1}\mathfrak{L}^M r_1$, and then integrate over $\mathbb{R}^4$. This easily gives rise to the identity

$$a^{-1}(t)\int_{\mathbb{R}^4}[r_1(7.17) + (\mathfrak{L}^M r_1)(7.22)](t,y)\,dy = 0. \tag{7.24}$$

Making use of (7.17) and (7.22), we deduce that (7.24) can be split as follows:

$$-a^{-1}(t)\int_{\mathbb{R}^4}[r_1(\widetilde{\mathcal{F}} + \widetilde{\mathcal{R}}^{(N)}) + (\mathfrak{L}^M r_1)(\widetilde{\mathcal{F}}_M + \mathfrak{L}^M\widetilde{\mathcal{R}}^{(N)})](t,y)\,dy = (I) + (II) + (III) + (IV),$$

with

$$(I) = a^{-1}(t)\int_{\mathbb{R}^4}(r_2 r_1 + \mathfrak{L}^M r_2\,\mathfrak{L}^M r_1)(t,y)\,dy,$$

$$(II) = a^{-1}(t)\int_{\mathbb{R}^4}\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}[(\mathfrak{L}r)r_1 + (\mathfrak{L}^{M+1}r)(\mathfrak{L}^M r_1)](t,y)\,dy,$$

$$(III) = -2a^{-1}(t)\int_{\mathbb{R}^4}\frac{V_1}{1+|\nabla V|^2}\nabla V\cdot[(\nabla r_1)r_1 + (\nabla\mathfrak{L}^M r_1)(\mathfrak{L}^M r_1)](t,y)\,dy,$$

$$(IV) = a^{-1}(t)\sum_{i,j=1}^{4}\int_{\mathbb{R}^4} g_{i,j}[\partial^2_{y_i y_j} r r_1 + (\partial^2_{y_i y_j}\mathfrak{L}^M r)(\mathfrak{L}^M r_1)](t,y)\,dy,$$

where for all $1 \le i, j \le 4$ the coefficients $g_{i,j}$ in the last integrand are defined by

$$g_{i,j} = \frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\delta_{i,j} + \frac{(Hr)_{y_i}(Hr)_{y_j}}{1+|\nabla V|^2}, \tag{7.25}$$

and obviously satisfy the symmetry relations $g_{i,j} = g_{j,i}$.

First, let us investigate the term (I). By definition

$$r_2 = a(r_1)_t - a'y\cdot\nabla r_1 - a'\frac{y\cdot\nabla H}{H}r_1,$$

and thus

$$\mathfrak{L}^M r_2 = a(\mathfrak{L}^M r_1)_t - a'y\cdot\nabla(\mathfrak{L}^M r_1) - a'X$$

with

$$X = [\mathfrak{L}^M, y\cdot\nabla]r_1 + \mathfrak{L}^M\frac{y\cdot\nabla H}{H}r_1.$$

We deduce that

$$(I) = \frac{1}{2}\frac{d}{dt}[\|r_1(t)\|^2_{L^2(\mathbb{R}^4)} + \|\mathfrak{L}^M r_1(t)\|^2_{L^2(\mathbb{R}^4)}]$$
$$-\frac{1+\nu}{t}\int_{\mathbb{R}^4}\Big[r_1 y\cdot\nabla r_1 + (\mathfrak{L}^M r_1)y\cdot\nabla(\mathfrak{L}^M r_1) + r_1\,\frac{y\cdot\nabla H}{H} r_1 + (\mathfrak{L}^M r_1)X\Big](t,y)\,dy.$$

Integrating by parts and taking into account that $\|X\|_{L^2(\mathbb{R}^4)} \lesssim \|r_1\|_{H^{L_0-1}(\mathbb{R}^4)}$, we find that

$$(I) = \frac{1}{2}\frac{d}{dt}[\|r_1(t)\|^2_{L^2(\mathbb{R}^4)} + \|\mathfrak{L}^M r_1(t)\|^2_{L^2(\mathbb{R}^4)}] + \frac{1}{t}\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big), \quad (7.26)$$

in the sense that (and all along this proof)

$$|\mathcal{O}(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)})| \lesssim \|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}.$$

Let us now estimate (II). First, we point out that it stems from the Hardy inequality and the asymptotic expansion (2.4) that for any f in $\dot{H}^1(\mathbb{R}^4)$,

$$\|\nabla(qf)\|_{L^2(\mathbb{R}^4)} \le C\|\nabla f\|_{L^2(\mathbb{R}^4)}. \quad (7.27)$$

Therefore performing an integration by parts, we get

$$(II) = \int_{\mathbb{R}^4}\nabla\Big(\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big)\cdot[qr_1\nabla(qr) + q\mathfrak{L}^M r_1\nabla(q\mathfrak{L}^M r)](t,y)\,dy$$
$$+\int_{\mathbb{R}^4}\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}[\nabla(qr_1)\cdot\nabla(qr)+\nabla(q\mathfrak{L}^M r_1)\cdot\nabla(q\mathfrak{L}^M r)+\mathcal{P}(rr_1+\mathfrak{L}^M r\mathfrak{L}^M r_1)](t,y)\,dy.$$

A straightforward computation gives

$$\nabla\Big(\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big) = \nabla\Big(1+\frac{|\nabla Q|^2-|\nabla V|^2}{1+|\nabla V|^2}\Big) = \nabla\Big(\frac{(\nabla Q-\nabla V)(\nabla Q+\nabla V)}{1+|\nabla V|^2}\Big).$$

We claim that there is a positive constant C such that for any $t\in[t_1,t_2]$ with $0<t_1\le t_2\le T$,

$$\Big\|\nabla\Big(\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \le Ct^\nu. \quad (7.28)$$

To see this, first observe that for any $t\in[t_1,t_2]$,[20]

$$\|\nabla^2(V-Q)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le Ct^\nu.$$

Indeed, by definition,

$$V = V^{(N)} + \varepsilon \quad \text{with} \quad \varepsilon = Hr,$$

[20]Here and below, we assume that $N>\nu$.

which gives the result, by applying the triangle inequality and making use of Lemma 6.1, the Hardy inequality, the estimates (6.3), (7.13) and the bootstrap assumption (7.20).

Along the same lines, we find that

$$\|\langle\cdot\rangle^{-1}\nabla(V-Q)(t,\cdot)\|_{L^\infty} \leq Ct^\nu, \quad \|\langle\cdot\rangle\nabla^2 V(t,\cdot)\|_{L^\infty} \leq C, \quad \|\nabla V(t,\cdot)\|_{L^\infty} \leq C,$$

which completes the proof of the claim (7.28).

We deduce that

$$(II) = \frac{1}{t}\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big)$$
$$+\int_{\mathbb{R}^4} \frac{1+|\nabla Q|^2}{1+|\nabla V|^2}[\nabla(qr_1)\cdot\nabla(qr)+\nabla(q\mathfrak{L}^M r_1)\cdot\nabla(q\mathfrak{L}^M r)+\mathcal{P}(rr_1+\mathfrak{L}^M r\mathfrak{L}^M r_1)](t,y)\,dy.$$

Moreover, remembering that

$$r_1 = ar_t + a'\Lambda r - a'\frac{y\cdot\nabla H}{H}r,$$

we obtain

$$\nabla(qr_1) = a\partial_t(\nabla qr) + a'\Lambda\nabla qr - a'\mathcal{Y}_0$$

with

$$\mathcal{Y}_0 = \nabla\left(q\frac{y\cdot\nabla H}{H}r\right) - [\nabla q,\Lambda]r.$$

Invoking (2.4) together with (7.13) and the Hardy inequality, we infer that

$$\|\mathcal{Y}_0\|_{L^2(\mathbb{R}^4)} \lesssim \|\nabla r\|_{L^2(\mathbb{R}^4)}. \tag{7.29}$$

Along the same lines, we readily gather that

$$\mathfrak{L}^M r_1 = a\partial_t(\mathfrak{L}^M r) + a'\Lambda\mathfrak{L}^M r - a'\mathcal{Y}_1,$$
$$\nabla q\mathfrak{L}^M r_1 = a\partial_t(\nabla q\mathfrak{L}^M r) + a'\Lambda\nabla q\mathfrak{L}^M r - a'\mathcal{Y}_2,$$

with

$$\mathcal{Y}_1 = \mathfrak{L}^M\frac{y\cdot\nabla H}{H}r - [\mathfrak{L}^M,\Lambda]r, \quad \mathcal{Y}_2 = -[\nabla q,\Lambda]\mathfrak{L}^M r + \nabla(q\mathcal{Y}_1)\,,$$

which clearly satisfy

$$\|\mathcal{Y}_1\|_{H^1(\mathbb{R}^4)} + \|\mathcal{Y}_2\|_{L^2(\mathbb{R}^4)} \lesssim \|\nabla r\|_{H^{L_0-1}(\mathbb{R}^4)}. \tag{7.30}$$

Taking advantage of (7.29) and (7.30), we infer that

$$(II) = \frac{d}{dt}\mathcal{E}_1(t) + (II)_1 + (II)_2 + \frac{1}{t}\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big)$$

with

$$\mathcal{E}_1(t) := \frac{1}{2}\int_{\mathbb{R}^4} \frac{1+|\nabla Q(y)|^2}{1+|\nabla V(t,y)|^2}[|\nabla(qr)|^2+|\nabla(q\mathfrak{L}^M r)|^2+\mathcal{P}(r^2+(\mathfrak{L}^M r)^2)](t,y)\,dy, \tag{7.31}$$

$$(II)_1 := -\frac{1}{2}\int_{\mathbb{R}^4} \partial_t\Big(\frac{1+|\nabla Q(y)|^2}{1+|\nabla V(t,y)|^2}\Big)[|\nabla(qr)|^2+|\nabla(q\mathfrak{L}^M r)|^2+\mathcal{P}(r^2+(\mathfrak{L}^M r)^2)](t,y)\,dy,$$

$$(II)_2 := \frac{1+\nu}{t}\int_{\mathbb{R}^4} \frac{1+|\nabla Q|^2}{1+|\nabla V|^2}[\nabla(qr)\cdot\Lambda\nabla(qr)+\nabla(q\mathfrak{L}^M r)\cdot\Lambda\nabla(q\mathfrak{L}^M r)](t,y)\,dy.$$

Again combining the bootstrap assumption (7.20) with (6.6), we claim that for any $t\in[t_1,t_2]$ with $0<t_1\le t_2\le T$,

$$\Big\|\partial_t\Big(\frac{1+|\nabla Q(y)|^2}{1+|\nabla V(t,y)|^2}\Big)(s,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \le Ct^{-1}. \tag{7.32}$$

It is obvious that (7.32) reduces to

$$\|\partial_t\nabla V(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le Ct^{-1}. \tag{7.33}$$

Now to establish (7.33), let us first recall that

$$V = V^{(N)}+\varepsilon \quad \text{with} \quad \varepsilon = Hr.$$

Applying the triangle inequality and invoking (6.6), we deduce that

$$\begin{aligned}\|\partial_t\nabla V(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} &\le \|\partial_t\nabla V^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} + \|\partial_t\nabla(Hr)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)}\\ &\le Ct^{-1}+\|\nabla(H\partial_t r)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)}.\end{aligned}$$

But in view of (7.14), we have

$$ar_t = ar_1 - a'\Lambda r + a'\,\frac{y\cdot\nabla H}{H}r,$$

which ends the proof of (7.33) thanks to the bootstrap assumption (7.20).

Consequently, we get

$$(II)_1 = \frac{1}{t}\mathcal{O}\big(\|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big). \tag{7.34}$$

To end the estimate of the second part, it remains to investigate the term $(II)_2$. For that purpose we perform an integration by parts, which implies that

$$(II)_2 = \frac{1+\nu}{2}\int_{\mathbb{R}^4}\Big((\nabla\cdot y)\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big)[|\nabla(qr)|^2+|\nabla(q\mathfrak{L}^M r)|^2](t,y)\,dy.$$

Taking into account Lemma 6.1 and the bootstrap assumption (7.20), this gives rise to

$$(II)_2 = \frac{1}{t}\mathcal{O}\big(\|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big)\cdot \tag{7.35}$$

In summary, we have

$$(II) = \frac{d}{dt}\mathcal{E}_1(t) + \frac{1}{t}\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big)\cdot \tag{7.36}$$

Moreover, it stems from the definition of the operator $\mathfrak{L}$ and the estimates (6.2), (7.20) that there is a positive constant C such that

$$\big|\mathcal{E}_1(t) - \tfrac{1}{2}\big(\mathfrak{L}r(t,\cdot)|r(t,\cdot)\big)_{L^2} - \tfrac{1}{2}\big(\mathfrak{L}^{M+1}r(t,\cdot)|\mathfrak{L}^M r(t,\cdot)\big)_{L^2}\big| \leq C\delta^{3\nu}\|\nabla r(t,\cdot)\|^2_{H^{L_0-1}}\cdot \tag{7.37}$$

Let us now estimate the third term (III). Integrating again by parts, we easily get

$$\begin{aligned}(III) &= -2a^{-1}(t)\int_{\mathbb{R}^4}\frac{V_1}{1+|\nabla V|^2}\nabla V\cdot[(\nabla r_1)r_1 + (\nabla\mathfrak{L}^M r_1)(\mathfrak{L}^M r_1)](t,y)\,dy\\ &= a^{-1}(t)\int_{\mathbb{R}^4}\partial_{y_j}\bigg(\frac{V_1}{1+|\nabla V|^2}\partial_{y_j}V\bigg)[(r_1)^2 + (\mathfrak{L}^M r_1)^2](t,y)\,dy\,\cdot\end{aligned}$$

Arguing as above, we infer that for any $t\in[t_1,t_2]$ with $0<t_1\leq t_2\leq T$,

$$\bigg\|\nabla\bigg(\frac{V_1}{1+|\nabla V|^2}\nabla V\bigg)(t,\cdot)\bigg\|_{L^\infty} \leq Ct^\nu. \tag{7.38}$$

This estimate is a direct consequence of the inequalities

$$\|\nabla V_1(t,\cdot)\|_{L^\infty}\leq Ct^\nu,\quad \|\langle\cdot\rangle^{-1}V_1(t,\cdot)\|_{L^\infty}\leq Ct^\nu,\quad \|\langle\cdot\rangle\nabla^2 V(t,\cdot)\|_{L^\infty}\leq C,$$

which readily follow from the bootstrap assumption (7.20) and Lemma 6.1. Hence

$$(III) = \frac{1}{t}\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big)\cdot \tag{7.39}$$

Finally, the last term (IV) can be dealt with along the same lines as (II). First, performing an integration by parts, we get

$$\begin{aligned}(IV) = &-\sum_{i,j=1}^4 a^{-1}(t)\int_{\mathbb{R}^4} g_{i,j}[(\partial_{y_i}r)(\partial_{y_j}r_1) + (\partial_{y_i}\mathfrak{L}^M r)(\partial_{y_j}\mathfrak{L}^M r_1)](t,y)\,dy\\ &-\sum_{i,j=1}^4 a^{-1}(t)\int_{\mathbb{R}^4}(\partial_{y_j}g_{i,j})[(\partial_{y_i}r)r_1 + (\partial_{y_i}\mathfrak{L}^M r)(\mathfrak{L}^M r_1)](t,y)\,dy,\end{aligned}$$

where the coefficients $g_{i,j}$ are defined by (7.25).

For any $t\in[t_1,t_2]$ with $0<t_1\leq t_2\leq T$, the functions $g_{i,j}$ for $1\leq i,j\leq 4$ satisfy

$$\|g_{i,j}(t)\|_{L^\infty(\mathbb{R}^4)}\leq C\delta^{6\nu}\quad\text{and}\quad\|\nabla g_{i,j}(t)\|_{L^\infty(\mathbb{R}^4)}\leq Ct^\nu. \tag{7.40}$$

Indeed, by definition

$$g_{i,j} = \frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\delta_{i,j} + \frac{(Hr)_{y_i}(Hr)_{y_j}}{1+|\nabla V|^2},$$

which leads to the result thanks to the estimates

$$\Big\|\Big(\frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq C\delta^{6\nu}, \qquad \Big\|\nabla\Big(\frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq Ct^{\nu},$$
$$\Big\|\nabla^\ell\Big(\frac{(Hr)_{y_j}(Hr)_{y_k}}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq Ct^{2N}, \quad \ell = 0,1,$$

fwhichthat can be proved in the same way as (7.28), making use of the bootstrap assumption (7.20) and Lemma 6.1.

Remembering that

$$r_1 = ar_t + a'\Lambda r - a'\,\frac{y\cdot\nabla H}{H}r,$$

we find that

$$\nabla(r_1) = a\partial_t(\nabla r) + a'\Lambda\nabla r - a'\,\widetilde{\mathcal{Y}}_0$$

with

$$\widetilde{\mathcal{Y}}_0 = \nabla\Big(\frac{y\cdot\nabla H}{H}r\Big) - [\nabla,\Lambda]r.$$

Along the same lines as for $\mathcal{Y}_0$, we have

$$\|\widetilde{\mathcal{Y}}_0\|_{L^2(\mathbb{R}^4)} \lesssim \|\nabla r\|_{L^2(\mathbb{R}^4)}. \tag{7.41}$$

Similarly, we easily check that

$$\nabla\mathfrak{L}^M r_1 = a\partial_t(\nabla\mathfrak{L}^M r) + a'\Lambda\nabla\mathfrak{L}^M r - a'\,\widetilde{\mathcal{Y}}_2,$$

with

$$\widetilde{\mathcal{Y}}_2 = -[\nabla,\Lambda]\mathfrak{L}^M r + \nabla(\mathcal{Y}_1),$$

which clearly satisfies

$$\|\widetilde{\mathcal{Y}}_2\|_{L^2(\mathbb{R}^4)} \lesssim \|\nabla r\|_{H^{L_0-1}(\mathbb{R}^4)}. \tag{7.42}$$

Therefore

$$\begin{aligned}(IV) = {} & \frac{d}{dt}\mathcal{E}_2(t) + \frac{1}{2}\sum_{i,j=1}^4 \int_{\mathbb{R}^4} (\partial_t g_{i,j})[(\partial_{y_i}r)(\partial_{y_j}r) + (\partial_{y_i}\mathfrak{L}^M r)(\partial_{y_j}\mathfrak{L}^M r)](t,y)\,dy \\ & - \frac{\nu+1}{t}\sum_{i,j=1}^4 \int_{\mathbb{R}^4} g_{i,j}[\partial_{y_i}r\Lambda\partial_{y_j}r + (\partial_{y_i}\mathfrak{L}^M r)(\Lambda\partial_{y_j}\mathfrak{L}^M r)](t,y)\,dy \\ & + \frac{1}{t}\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big)\end{aligned}$$

with

$$\mathcal{E}_2(t) = -\frac{1}{2}\sum_{i,j=1}^{4}\int_{\mathbb{R}^4} g_{i,j}[(\partial_{y_i} r)(\partial_{y_j} r) + (\partial_{y_i}\mathfrak{L}^M r)(\partial_{y_j}\mathfrak{L}^M r)](t,y)\,dy. \tag{7.43}$$

In view of the bootstrap assumption (7.20) and Lemma 6.1,

$$\|\langle\cdot\rangle\nabla g_{i,j}\|_{L^\infty(\mathbb{R}^4)} \le C, \tag{7.44}$$

which follows easily from the fact that for any $t \in [t_1, t_2]$,

$$\left\|\langle\cdot\rangle\nabla\left(\frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\right)(t,\cdot)\right\|_{L^\infty(\mathbb{R}^4)} \le C,$$
$$\left\|\langle\cdot\rangle\nabla\left(\frac{(Hr)_{y_j}(Hr)_{y_k}}{1+|\nabla V|^2}\right)(t,\cdot)\right\|_{L^\infty(\mathbb{R}^4)} \le Ct^{2N}.$$

An integration by parts thus gives rise to

$$(IV) = \frac{d}{dt}\mathcal{E}_2(t) + \frac{1}{2}\sum_{i,j=1}^{4}\int_{\mathbb{R}^4}(\partial_t g_{i,j})[(\partial_{y_i} r)(\partial_{y_j} r) + (\partial_{y_i}\mathfrak{L}^M r)(\partial_{y_j}\mathfrak{L}^M r)](t,y)\,dy$$
$$+ \frac{1}{t}\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big).$$

Now we claim that

$$\|\partial_t g_{i,j}\|_{L^\infty(\mathbb{R}^4)} \le Ct^{-1}. \tag{7.45}$$

This is again a consequence of the bootstrap assumption (7.20) and Lemma 6.1 which assert that there is a positive constant C such that for any $t \in [t_1, t_2]$,

$$\left\|\partial_t\left(\frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\right)(t,\cdot)\right\|_{L^\infty(\mathbb{R}^4)} \le Ct^{-1},$$
$$\left\|\partial_t\left(\frac{(Hr)_{y_j}(Hr)_{y_k}}{1+|\nabla V|^2}\right)(t,\cdot)\right\|_{L^\infty(\mathbb{R}^4)} \le Ct^{N}.$$

Therefore we obtain

$$(IV) = \frac{d}{dt}\mathcal{E}_2(t) + \frac{1}{t}\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big). \tag{7.46}$$

Observe also that by (7.40),

$$\big|\mathcal{E}_2(t)\big| \le C\delta^{6\nu}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big). \tag{7.47}$$

We finally address the terms $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}_M$ and $\widetilde{\mathcal{R}}^{(N)}$. Using the bootstrap assumption (7.20) and Lemmas 6.1 and 6.2 it is not difficult to show the following estimates.

Lemma 7.2. *There exists a positive constant C such that under assumption (7.20), for any $t \in [t_1, t_2]$ with $0 < t_1 \leq t_2 \leq T$,*

$$\|\widetilde{\mathcal{F}}(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} \leq Ct^{\nu}\big(\|\nabla r(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} + \|r_1(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}\big), \tag{7.48}$$

$$\|\widetilde{\mathcal{R}}^{(N)}(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} \leq Ct^{N+\nu}, \tag{7.49}$$

$$\|\widetilde{\mathcal{F}}_M(t,\cdot)\|_{L^2(\mathbb{R}^4)} \leq Ct^{\nu}\big(\|\nabla r(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} + \|r_1(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}\big). \tag{7.50}$$

We now combine the last lemma with the bootstrap assumption (7.20) and the above estimates (7.26), (7.37), (7.39), (7.47), (7.49) and (7.50) to get

$$\frac{d}{dt}\mathcal{E}(t) \leq Ct^{2N-1} \tag{7.51}$$

with

$$\mathcal{E}(t) = \frac{1}{2}[\|r_1(t)\|^2_{L^2(\mathbb{R}^4)} + \|\mathfrak{L}^M r_1(t)\|^2_{L^2(\mathbb{R}^4)}] + \mathcal{E}_1(t) + \mathcal{E}_2(t),$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are respectively given by (7.31) and (7.43).

It follows from (2.4), (2.12), (7.37) and (7.47) (see Remark B.1) that

$$\mathcal{E}(t) \geq C(\|r_1(t,\cdot)\|^2_{H^{L_0-1}} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}})$$

for some positive constant C provided that δ is taken sufficiently small. Therefore, integrating inequality (7.51) and taking into account that $r(t_1) = r_1(t_1) = 0$, we get

$$\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} \leq \frac{C}{N}t^{2N},$$

which completes the proof of Lemma 7.1. ■

Since by construction, we have

$$(u - u^{(N)})(t,x) = t^{\nu+1}(V - V^{(N)})(t, x/t^{\nu+1}) = t^{\nu+1}(Hr)(t, x/t^{\nu+1}),$$
$$\partial_t(u - u^{(N)})(t,x) = (V_1 - V_1^{(N)})(t, x/t^{\nu+1}) = (Hr_1)(t, x/t^{\nu+1}),$$

Proposition 7.1 follows readily from (7.13) and Lemma 7.1 by standard continuity arguments. ■

Remark 7.1. Combining Proposition 7.1 with the bounds (7.1) and (7.2), we find that for any $t \in [t_1, T]$, the solution to the Cauchy problem (7.4) satisfies

$$u(t,\cdot) \geq \tilde{c}_0 t^{\nu+1} \quad \text{and} \quad (1 + |\nabla u|^2 - (\partial_t u)^2)(t,\cdot) \geq \tilde{c}_1 \tag{7.52}$$

with some positive constants $\tilde{c}_0$ and $\tilde{c}_1$ provided that N is sufficiently large.

Furthermore, injecting the bounds (7.20) into (7.17) and taking into account Lemma 7.2, one obtains

$$\|\langle\cdot\rangle^{3/2}\partial_t^2(u - u^{(N)})(t,\cdot)\|_{H^{L_0-2}(\mathbb{R}^4)} \leq t^{N/2}, \qquad \forall t \in [t_1, T]. \tag{7.53}$$

7.2. End of the proof

We are now in a position to finish the proof of Theorem 1.2. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers in $]0,T]$ converging to 0, and consider the Cauchy problem

$$(\mathrm{NW})_{n,N}\ \begin{cases}(1.6)u = 0,\\ u_{|t=t_n} = u^{(N)}(t_n,\cdot),\\ (\partial_t u)_{|t=t_n} = (\partial_t u^{(N)})(t_n,\cdot).\end{cases}$$

In view of Proposition 7.1 and Remark 7.1, the following uniform local well-posedness result is straightforward:

Corollary 7.1. *There exists an integer N_0 such that for any $n\in\mathbb{N}$, the Cauchy problem $(\mathrm{NW})_{n,N_0}$ has a unique solution u_n on $[t_n,T]$ which satisfies*

$$\begin{aligned}&\|\langle\cdot\rangle^{3/2}\partial_t(u_n-u^{(N_0)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}\\&\qquad+\|\langle\cdot\rangle^{3/2}\nabla(u_n-u^{(N_0)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}\le t^{N_0/2},\quad \forall t\in[t_n,T].\end{aligned}\tag{7.54}$$

Furthermore,

$$u_n(t,x)\ge\tilde{c}_0t^{\nu+1},\quad 1+|\nabla u_n(t,x)|^2-(\partial_t u_n(t,x))^2\ge\tilde{c}_1,\quad\forall(t,x)\in[t_n,T]\times\mathbb{R}^4,\tag{7.55}$$

By the Ascoli theorem, the bounds (7.54) and (7.55) imply that there exists a solution u to the Cauchy problem (1.11) on $]0,T]$ satisfying $(u,\partial_t u)\in\mathcal{C}(]0,T],X_{L_0})$ and such that after passing to a subsequence, the sequence $((\nabla u_n,\partial_t u_n))_{n\in\mathbb{N}}$ converges to $(\nabla u,\partial_t u)$ in $C([T_1,T],H^{s-1}(\mathbb{R}^4))$ for any $T_1\in]0,T]$ and any $s<L_0$. Clearly the solution u satisfies

$$\|\partial_t(u-u^{(N_0)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}+\|\nabla(u-u^{(N_0)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}\le t^{N_0/2},\quad\forall t\in]0,T],$$
$$u(t,x)\ge\tilde{c}_0t^{\nu+1},\quad 1+|\nabla u(t,x)|^2-(\partial_t u(t,x))^2\ge\tilde{c}_1,\quad\forall(t,x)\in]0,T]\times\mathbb{R}^2.$$

Taking into account Lemma 6.2 and Remarks 5.1 and 6.1, this concludes the proof of Theorem 1.2.

Appendix A. Derivation of the equation

In this appendix, we carry out the derivation of the equation in the case of time-like surfaces with vanishing mean curvature that for fixed t are parametrized as follows:

$$\mathbb{R}^n\times\mathbb{S}^{n-1}\ni(x,\omega)\mapsto(x,u(t,x)\omega)\in\mathbb{R}^{2n},\tag{A.1}$$

with some positive function u. An elementary computation shows that in this case the volume density corresponding to the pull-back metric is given by

$$\mathcal{L}(u,u_t,\nabla u)=u^{n-1}\sqrt{1-(u_t)^2+|\nabla u|^2}.\tag{A.2}$$

Using the fact that the mean curvature is the first variation of the volume form, we can determine the equation of motion by formally considering the Euler–Lagrange equation associated to the density $\mathscr{L}$, which gives rise to[21]

$$\frac{\partial \mathscr{L}}{\partial u} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial \mathscr{L}}{\partial u_{x_j}} - \frac{\partial}{\partial t} \frac{\partial \mathscr{L}}{\partial u_t} = 0.$$

According to (A.2), this leads to

$$(n-1)u^{n-2}\sqrt{1-(u_t)^2+|\nabla u|^2} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{u^{n-1}u_{x_j}}{\sqrt{1-(u_t)^2+|\nabla u|^2}} + \frac{\partial}{\partial t} \frac{u^{n-1}u_t}{\sqrt{1-(u_t)^2+|\nabla u|^2}} = 0. \tag{A.3}$$

Therefore the quasilinear wave equation at hand takes the form

$$\partial_t\left(\frac{u_t}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\right) - \sum_{j=1}^{n} \partial_{x_j}\left(\frac{u_{x_j}}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\right) + \frac{n-1}{u\sqrt{1-(u_t)^2+|\nabla u|^2}} = 0. \tag{A.4}$$

Straightforward computations show that this can be rewritten as

$$u_{tt}(1+|\nabla u|^2) - \Delta u(1-(u_t)^2+|\nabla u|^2) + \sum_{j,k=1}^{n} u_{x_j}u_{x_k}u_{x_j x_k} - 2u_t(\nabla u \cdot \nabla u_t) + \frac{n-1}{u}(1-(u_t)^2+|\nabla u|^2) = 0, \tag{A.5}$$

which is (1.6).

Let us emphasize that in the particular case when $u(t,x) = R(t)$ with R a positive regular function, the above equation reduces to

$$RR'' + (n-1)(1-(R')^2) = 0. \tag{A.6}$$

Integrating this equation using the fact that $1-(R')^2R^{-2(n-1)}$ is constant, one readily gathers that if $R(0) > 0$ and $-1 < R'(0) < 0$, then there exists a finite positive time T such that $R(T) = 0$, and furthermore

$$R(t) = T - t + \mathcal{O}((T-t)^{2n-1}), \quad t \to T.$$

[21] These computations are justified by the invariance of our flow under the action of the group $\mathrm{I}_n \times O(n)$.

So, as t tends to T, the cylinders defined by

$$\mathbb{R}^n \times \mathbb{S}^{n-1} \ni (x, \omega) \mapsto (x, R(t)\omega) \in \mathbb{R}^{2n}$$

shrink to the $\mathbb{R}^n$ space.[22]

Appendix B. Study of the linearized operator of the quasilinear wave equation around the ground state

The aim of this section is to investigate the linearized operator $\mathcal{L}$ introduced in (1.16). To this end, let us consider the change of function

$$w(\rho) = H(\rho) f(\rho) \quad \text{with} \quad H = \frac{(1 + Q_\rho^2)^{1/4}}{Q^{3/2}}.$$

By easy computations, we deduce that

$$\mathcal{L} w = -H(1 + Q_\rho^2)\mathfrak{L} f \quad \text{with} \quad \mathfrak{L} = -q\Delta q + \mathcal{P},$$

where $q = 1/(1 + Q_\rho^2)^{1/2}$ and $\mathcal{P} = V^\flat/(1 + Q_\rho^2)$ with

$$V^\flat = \frac{-3(1 + Q_\rho^2)}{Q^2} + \frac{1}{2}(B_1)_\rho - \frac{1}{4}B_1^2 - \frac{3}{2}B_1\Big(-\frac{1 + Q_\rho^2}{\rho} + 2Q_\rho\Big(\frac{1}{Q} - \frac{Q_\rho}{\rho}\Big)\Big). \quad \text{(B.1)}$$

In view of Lemma 2.1, the potential $\mathcal{P}$ belongs to $\mathcal{C}^\infty_{\text{rad}}(\mathbb{R}^4)$ and satisfies

$$\mathcal{P} = -\frac{3}{8\rho^2}(1 + o(1)) \quad \text{as } \rho \to \infty. \quad \text{(B.2)}$$

The operator $\mathfrak{L}$ with domain $H^2(\mathbb{R}^4)$ is self-adjoint on $L^2(\mathbb{R}^4)$. The following positivity property of $\mathfrak{L}$ is at the heart of the analysis carried out in this article.

Lemma B.1. *There is a positive constant c such that for any function f in $\dot{H}^1_{\text{rad}}(\mathbb{R}^4)$,*

$$(\mathfrak{L} f | f)_{L^2(\mathbb{R}^4)} \geq c \, \|\nabla f\|^2_{L^2(\mathbb{R}^4)}. \quad \text{(B.3)}$$

Remark B.1. Taking into account (2.4), one easily deduces from (B.3) that for any integer m, there exists a positive constant c_m such that

$$(\mathfrak{L}^{m+1} f | f)_{L^2(\mathbb{R}^4)} + (\mathfrak{L} f | f)_{L^2(\mathbb{R}^4)} \geq c_m \, \|\nabla f\|^2_{H^m(\mathbb{R}^4)}, \quad \forall f \in \dot{H}^1_{\text{rad}}(\mathbb{R}^4) \cap \dot{H}^{m+1}(\mathbb{R}^4),$$

and

$$(\mathfrak{L}^{m+1} f | f)_{L^2(\mathbb{R}^4)} + (f | f)_{L^2(\mathbb{R}^4)} \geq c_m \|f\|^2_{H^m(\mathbb{R}^4)}, \quad \forall f \in H^{m+1}_{\text{rad}}(\mathbb{R}^4).$$

[22] Obviously, (A.6) has a trivial solution $T - t$ with T some positive constant. But to ensure that the surface is time-like, we must impose that $R'(t)^2 < 1$ for all t.

Proof of Lemma B.1. Recalling that the positive function $\Lambda Q = Q - \rho Q_\rho$ solves the homogeneous equation $\mathcal{L}w = 0$, we infer that the function

$$G := \frac{\Lambda Q}{H} \tag{B.4}$$

defines a positive solution to the homogeneous equation $\mathfrak{L} f = 0$. In a standard way this implies

$$\mathfrak{L} \geq 0. \tag{B.5}$$

In order to prove inequality (B.3), assume for contradiction that there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $\dot H^1_{\rm rad}(\mathbb{R}^4)$ satisfying $\|\nabla u_n\|_{L^2(\mathbb{R}^4)} = 1$ and

$$(\mathfrak{L}u_n|u_n)_{L^2(\mathbb{R}^4)} \xrightarrow{n\to\infty} 0. \tag{B.6}$$

Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $\dot H^1_{\rm rad}(\mathbb{R}^4)$, there is a function u in $\dot H^1_{\rm rad}(\mathbb{R}^4)$ such that, up to a subsequence (still denoted by u_n for simplicity),

$$u_n \xrightarrow{n\to\infty} u \quad \text{in } \dot H^1(\mathbb{R}^4). \tag{B.7}$$

We claim that $u \neq 0$ and $\mathfrak{L}u = 0$. Indeed,

$$(\mathfrak{L}u_n|u_n)_{L^2(\mathbb{R}^4)} = \int_{\mathbb{R}^4} |\nabla(qu_n)(x)|^2\,dx + \int_{\mathbb{R}^4} |(qu_n)(x)|^2 V^\flat(x)\,dx.$$

First, observe that there is a positive constant C such that, for any integer n,

$$\|\nabla(qu_n)\|_{L^2(\mathbb{R}^4)} > C. \tag{B.8}$$

Indeed,

$$\|\nabla u_n\|_{L^2(\mathbb{R}^4)} \leq \left\|\frac{1}{q}\nabla(qu_n)\right\|_{L^2(\mathbb{R}^4)} + \|\nabla(1/q)qu_n\|_{L^2(\mathbb{R}^4)},$$

which, in view of the Hardy inequality and Lemma 2.1, leads to (B.8).

Second, consider a smooth radial function θ valued in $[0, 1]$ and satisfying

$$\theta(x) = \begin{cases} 0 & \text{for } |x| \leq 1, \\ 1 & \text{for } |x| \geq 2, \end{cases}$$

and write

$$(\mathfrak{L}u_n|u_n)_{L^2(\mathbb{R}^4)} = \int_{\mathbb{R}^4} |\nabla(qu_n)(x)|^2\,dx - \frac{3}{4}\int_{\mathbb{R}^4} |(qu_n)(x)|^2 \frac{\theta(x)}{|x|^2}\,dx + \int_{\mathbb{R}^4} |(qu_n)(x)|^2 \tilde V(x)\,dx,$$

where

$$\tilde V(x) = V^\flat(x) + \frac{3}{4}\frac{\theta(x)}{|x|^2}.$$

Invoking formula (B.2), we infer that there is a positive constant δ such that

$$|\tilde{V}(x)| \lesssim \frac{1}{\langle x\rangle^{2+\delta}}.$$

Invoking the Rellich theorem and the Hardy inequality, we deduce that

$$\int_{\mathbb{R}^4} |(qu_n)(x)|^2 \tilde{V}(x)\,dx \xrightarrow{n\to\infty} \int_{\mathbb{R}^4} |(qu)(x)|^2 \tilde{V}(x)\,dx. \tag{B.9}$$

Now for any functions f and g in $\dot{H}^1(\mathbb{R}^4)$, denote

$$a(f,g) := \int_{\mathbb{R}^4} \nabla(qf)(x)\cdot\overline{\nabla(qg)(x)}\,dx - \frac{3}{4}\int_{\mathbb{R}^4} \frac{\theta(x)}{|x|^2}(qf)(x)\overline{(qg)(x)}\,dx.$$

Combining the Hardy inequality with Lemma 2.1, we easily gather that there exist positive constants $\alpha_0 < \alpha_1$ such that for any function f in $\dot{H}^1(\mathbb{R}^4)$, we have

$$\alpha_0\|\nabla f\|^2_{L^2(\mathbb{R}^4)} \le a(f,f) \le \alpha_1\|\nabla f\|^2_{L^2(\mathbb{R}^4)},$$

which ensures that $a(f,g)$ is a scalar product on $\dot{H}^1(\mathbb{R}^4)$ and that the norms $\sqrt{a(\cdot,\cdot)}$ and $\|\cdot\|_{\dot{H}^1(\mathbb{R}^4)}$ are equivalent.

Since $u_n \xrightarrow{n\to\infty} u$ in $\dot{H}^1(\mathbb{R}^4)$, we deduce that $a(u,u) \le \liminf_{n\to\infty} a(u_n,u_n)$, and thus

$$(\mathfrak{L}u|u)_{L^2(\mathbb{R}^4)} \le \liminf_{n\to\infty}(\mathfrak{L}u_n|u_n)_{L^2(\mathbb{R}^4)}.$$

Taking into account (B.5), (B.6) and (B.9), we obtain

$$(\mathfrak{L}u|u)_{L^2(\mathbb{R}^4)} = 0, \tag{B.10}$$

which, according to the fact that $\mathfrak{L}$ is positive, implies that $\mathfrak{L}u = 0$.

To end the proof of the claim, it remains to establish that $u \neq 0$. For that purpose, let us start by observing that by (B.6), (B.9) and (B.10),

$$\begin{aligned}\int_{\mathbb{R}^4} |\nabla(qu_n)(x)|^2\,dx - \frac{3}{4}\int_{\mathbb{R}^4} |(qu_n)(x)|^2\frac{\theta(x)}{|x|^2}\,dx&\\ \xrightarrow{n\to\infty} \int_{\mathbb{R}^4} |\nabla(qu)(x)|^2\,dx - \frac{3}{4}\int_{\mathbb{R}^4} |(qu_n)(x)|^2\frac{\theta(x)}{|x|^2}\,dx.&\end{aligned} \tag{B.11}$$

But in view of the Hardy inequality and the bound (B.8), we have

$$\begin{aligned}\int_{\mathbb{R}^4} |\nabla(qu_n)(x)|^2\,dx - \frac{3}{4}\int_{\mathbb{R}^4} |(qu_n)(x)|^2\frac{\theta(x)}{|x|^2}\,dx&\\ \ge \frac{1}{4}\int_{\mathbb{R}^4} |\nabla(qu_n)(x)|^2\,dx \ge \frac{C}{4}.&\end{aligned} \tag{B.12}$$

By passing to the limit, we obtain

$$\int_{\mathbb{R}^4} |\nabla(qu)(x)|^2\,dx - \frac{3}{4}\int_{\mathbb{R}^4} |(qu)(x)|^2\frac{\theta(x)}{|x|^2}\,dx \ge \frac{C}{4},$$

which proves that u is not null.

By construction the function u belongs to $\dot{H}^1_{\rm rad}(\mathbb{R}^4)$ and satisfies

$$-q\Delta qu + \mathscr{P}u = 0 \quad \text{with} \quad \mathscr{P} = -\frac{3}{8\rho^2}(1+o(1)) \quad \text{as } \rho \to \infty.$$

Therefore in view of the Hardy inequality, $\mathscr{P}u \in L^2_{\rm rad}(\mathbb{R}^4)$ and thus $q\Delta qu$ belongs to $L^2_{\rm rad}(\mathbb{R}^4)$, which ensures that $u \in \dot{H}^2_{\rm rad}(\mathbb{R}^4)$.

In the radial setting the homogeneous equation $\mathfrak{L}u = 0$ admits a basis of solutions $\{f_1, f_2\}$ given by[23]

$$f_1(\rho) = G(\rho), \qquad f_2(\rho) = G(\rho)\int_1^\rho \frac{(1+(Q_r(r))^2)^{3/2}}{Q^3(r)r^3(\Lambda Q)^2(r)}\,dr,$$

where G denotes the function defined by (B.4). By Lemma 2.1, one has

$$f_1(\rho) \sim 1, \qquad f_2(\rho) \sim 1/\rho^2,$$

near $\rho = 0$. Since $f_2 \notin \dot{H}^1_{\rm rad}(\mathbb{R}^4)$, we deduce that u is collinear to G. This yields a contradiction because in view of (2.4), the function G behaves like $1/\sqrt{\rho}$ when $\rho \to \infty$ and thus it does not belong to $\dot{H}^1_{\rm rad}(\mathbb{R}^4)$. This completes the proof of the lemma. ∎

Appendix C. Proof of the local well-posedness result

The aim of this appendix is to give an outline of the proof of Theorem 1.1. Since the subject is well known, we only indicate the main arguments. One can proceed in three steps:

(1) First, one proves that for some sufficiently small positive time

$$T = T\big(\|\nabla(u_0 - Q)\|_{H^{L-1}}, \|u_1\|_{H^{L-1}}, \inf u_0, \inf(1+|\nabla u_0|^2 - (u_1)^2)\big),$$

the Cauchy problem (1.11) has a solution u such that (u, u_t) belongs to the function space $\mathcal{C}([0,T], X_L)$, $u_t \in \mathcal{C}^1([0,T], H^{L-1})$, and for all t in $[0,T]$,

$$\|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}} + \|u_t(t,\cdot)\|_{H^{L-1}} \leq C\big(\|\nabla(u_0 - Q)\|_{H^{L-1}} + \|u_1\|_{H^{L-1}}\big)$$

for some positive constant

$$C = C\big(\|\nabla(u_0 - Q)\|_{H^{L-1}}, \|u_1\|_{H^{L-1}}, \inf u_0, \inf(1+|\nabla u_0|^2 - (u_1)^2)\big).$$

(2) Second, one shows the uniqueness of solutions by a continuity argument.

(3) Third, one establishes the blow up criterion (1.13).

[23] See Appendix D for the proof.

So let us consider the Cauchy problem (1.11) and assume that $\nabla(u_0 - Q)$ and u_1 belong to $H^{L-1}(\mathbb{R}^4)$, with L an integer strictly larger than 4, and that there exists $\varepsilon > 0$ such that

$$u_0 \geq 2\varepsilon \quad \text{and} \quad \frac{1-(u_1)^2+|\nabla u_0|^2}{1+|\nabla u_0|^2} \geq 2\varepsilon.$$

Defining, for $1 \leq i, j \leq 4$,

$$\begin{cases} a_{ij}(\nabla u, u_t) = \delta_{ij}\left(1 - \dfrac{(u_t)^2}{1+|\nabla u|^2}\right) - \dfrac{u_{x_j}u_{x_i}}{1+|\nabla u|^2}, \\ b_i(\nabla u, u_t) = \dfrac{2u_{x_i}u_t}{1+|\nabla u|^2}, \\ c(u, \nabla u, u_t) = -\dfrac{3(1-(u_t)^2+|\nabla u|^2)}{u(1+|\nabla u|^2)}, \end{cases} \tag{C.1}$$

we readily gather that (1.6) takes the form

$$u_{tt} - \sum_{i,j=1}^{4} a_{i,j}(\nabla u, u_t)u_{x_i x_j} - \sum_{i=1}^{4} b_i(\nabla u, u_t)u_{tx_i} - c(u, \nabla u, u_t) = 0. \tag{C.2}$$

To prove existence, we shall use an iterative scheme. To this end, introduce the sequence $(u^{(n)})_{n\in\mathbb{N}}$ defined by $u^{(0)} = Q$, which according to (2.1) satisfies

$$c(Q, \nabla Q, 0) + \sum_{i,j=1}^{4} a_{i,j}(\nabla Q, 0)Q_{x_i x_j} = 0,$$

and

$$(W)_{n+1} \begin{cases} u_{tt}^{(n+1)} - \displaystyle\sum_{i,j=1}^{4} a_{i,j}(\nabla u^{(n)}, u_t^{(n)})u_{x_i x_j}^{(n+1)} - \sum_{i=1}^{4} b_i(\nabla u^{(n)}, u_t^{(n)})u_{tx_i}^{(n+1)} \\ \qquad - c(u^{(n)}, \nabla u^{(n)}, u_t^{(n)}) = 0, \\ u^{(n+1)}{}_{|t=0} = u_0, \\ (\partial_t u^{(n+1)})_{|t=0} = u_1. \end{cases}$$

In order to investigate the sequence $(u^{(n)})_{n\in\mathbb{N}}$ defined above by induction, let us begin by proving that this sequence is well defined for any time t in some fixed interval $[0, T]$ which depends only on $\|\nabla(u_0 - Q)\|_{H^{L-1}}$, $\|u_1\|_{H^{L-1}}$ and ε. This will be deduced from the following result.

Proposition C.1. *Let u be such that $(u, u_t) \in \mathcal{C}([0, T], X_L)$, $u_{tt} \in \mathcal{C}([0, T], H^{L-2})$ for some integer $L > 4$ and some $0 < T \leq 1$. Assume that*

$$\|u_t\|_{L^\infty([0,T],H^{L-1})} + \|\nabla(u - Q)\|_{L^\infty([0,T],H^{L-1})} \leq A, \tag{C.3}$$

$$\|u_{tt}\|_{L^\infty([0,T],H^{L-2})} \leq A_1, \tag{C.4}$$

$$u(t, x) \geq \varepsilon, \quad \frac{1-(u_t(t,x))^2+|\nabla u(t,x)|^2}{1+|\nabla u(t,x)|^2} \geq \varepsilon, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d. \tag{C.5}$$

Consider the Cauchy problem

$$\begin{cases} \Phi_{tt} - \sum_{i,j=1}^{4} a_{i,j}(\nabla u, u_t)\Phi_{x_i x_j} - \sum_{i=1}^{4} b_i(\nabla u, u_t)\Phi_{t x_i} = c(u, \nabla u, u_t), \\ \Phi_{|t=0} = \Phi_0, \\ (\partial_t \Phi)_{|t=0} = \Phi_1, \end{cases} \tag{C.6}$$

assuming that $\nabla(\Phi_0 - Q)$ *and* Φ_1 *belong to* $H^{L-1}(\mathbb{R}^4)$. *Then the Cauchy problem* (C.6) *has a unique solution* Φ *on* $[0, T]$ *and the following energy inequalities hold:*

$$\begin{aligned} \|\Phi_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(\Phi(t,\cdot) - Q)\|_{H^{L-1}} &\leq C_\varepsilon \, e^{tC_{\varepsilon,A,A_1}} (\|\Phi_1\|_{H^{L-1}} + \|\nabla(\Phi_0 - Q)\|_{H^{L-1}}) \\ &\quad + C_{\varepsilon,A,A_1} \int_0^t \big(\|u_t(s,\cdot)\|_{H^{L-1}} + \|\nabla(u - Q)(s,\cdot)\|_{H^{L-1}}\big)\, ds, \end{aligned} \tag{C.7}$$

and

$$\begin{aligned} \|\Phi_{tt}(t,\cdot)\|_{H^{L-2}} \leq C_A(\|\Phi_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(\Phi(t,\cdot) - Q)\|_{H^{L-1}}) \\ + C_{\varepsilon,A}(\|u_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(u - Q)(t,\cdot)\|_{H^{L-1}}). \end{aligned} \tag{C.8}$$

Proof. Invoking hypothesis (C.5), we easily check that for any $\xi \in \mathbb{R}^4 \setminus \{0\}$, the characteristic polynomial of the wave equation (C.6),

$$\tau^2 - \tau \sum_{i=1}^{4} b_i(\nabla u, u_t)\xi_i - \sum_{i,j=1}^{4} a_{i,j}(\nabla u, u_t)\xi_i \xi_j \tag{C.9}$$

has two distinct real roots τ_1 and τ_2. Indeed, taking into account (C.1), we find that the discriminant of (C.9) is given by

$$\begin{aligned} \Delta &= \frac{4(u_t)^2}{(1+|\nabla u|^2)^2}\Big(\sum_{i=1}^{4} u_{x_i}\xi_i\Big)^2 + \frac{4(1-(u_t)^2+|\nabla u|^2)}{1+|\nabla u|^2}|\xi|^2 - \frac{4}{1+|\nabla u|^2}\Big(\sum_{i=1}^{4} u_{x_i}\xi_i\Big)^2 \\ &= \frac{4(1-(u_t)^2+|\nabla u|^2)}{(1+|\nabla u|^2)^2}|\xi|^2 + \frac{4(1-(u_t)^2+|\nabla u|^2)}{(1+|\nabla u|^2)^2}\Big(|\nabla u|^2|\xi|^2 - \Big(\sum_{i=1}^{4} u_{x_i}\xi_i\Big)^2\Big), \end{aligned}$$

which implies that

$$\Delta \geq \frac{4(1-(u_t)^2+|\nabla u|^2)}{(1+|\nabla u|^2)^2}|\xi|^2. \tag{C.10}$$

This ends the proof of the fact that the polynomial (C.9) has two distinct real roots τ_1 and τ_2, and ensures that (C.6) is strictly hyperbolic as long as

$$1 - (u_t)^2 + |\nabla u|^2 > 0,$$

and thus, in view of (C.5), on $[0, T] \times \mathbb{R}^4$.

Consider the function $\widetilde{\Phi} := \Phi - Q$. It satisfies

$$\begin{cases} \widetilde{\Phi}_{tt} - \sum_{i,j=1}^{4} a_{i,j}(\nabla u, u_t)\widetilde{\Phi}_{x_i x_j} - \sum_{i=1}^{4} b_i(\nabla u, u_t)\widetilde{\Phi}_{t x_i} = f(u, \nabla u, u_t), \\ \widetilde{\Phi}_{|t=0} = \Phi_0 - Q, \\ (\partial_t \widetilde{\Phi})_{|t=0} = \Phi_1, \end{cases} \tag{C.11}$$

with

$$f(u, \nabla u, u_t) = c(u, \nabla u, u_t) + \sum_{i,j=1}^{4} a_{i,j}(\nabla u, u_t) Q_{x_i x_j}. \tag{C.12}$$

First, note that the source term f belongs to $L^\infty([0,T], H^{L-1}(\mathbb{R}^4))$ and thus to $L^1([0,T], H^{L-1}(\mathbb{R}^4))$. Let us start by establishing that $f \in L^\infty([0,T], L^2(\mathbb{R}^4))$. Recalling that by (2.1), we have

$$c(Q, \nabla Q, 0) + \sum_{i,j=1}^{4} a_{i,j}(\nabla Q, 0) Q_{x_i x_j} = 0,$$

we deduce that f can be rewritten in the following way:

$$\begin{aligned} f &= c(u, \nabla u, u_t) - c(Q, \nabla Q, 0) + \sum_{i,j=1}^{4} (a_{i,j}(\nabla u, u_t) - a_{i,j}(\nabla Q, 0)) Q_{x_i x_j} \\ &= -3\Big(\frac{1}{u} - \frac{1}{Q}\Big) + \widetilde{f}, \end{aligned}$$

where

$$\widetilde{f} = c(u, \nabla u, u_t) + \frac{3}{u} + \sum_{i,j=1}^{4} \big(a_{i,j}(\nabla u, u_t) - a_{i,j}(\nabla Q, 0)\big) Q_{x_i x_j}. \tag{C.13}$$

Combining Lemma 2.1 with the hypotheses (C.3) and (C.5) we obtain, making use of Taylor's formula

$$|\widetilde{f}| \leq C_{\varepsilon,A}\big(|u_t| + |\nabla(u - Q)|\big),$$

which easily ensures that for all t in $[0,T]$, we have

$$\|\widetilde{f}(t,\cdot)\|_{L^2} \leq C_{\varepsilon,A}\big(\|u_t(t,\cdot)\|_{L^2} + \|\nabla(u - Q)(t,\cdot)\|_{L^2}\big). \tag{C.14}$$

Therefore we are reduced to the study of the term

$$-3\Big(\frac{1}{u} - \frac{1}{Q}\Big) = 3\,\frac{u - Q}{uQ}.$$

We claim that

$$\Big|\frac{1}{u} - \frac{1}{Q}\Big| \leq C_{\varepsilon,A}\,\frac{|u - Q|}{Q^2}. \tag{C.15}$$

On the one hand, according to estimate (C.3), the function $u - Q$ is bounded on $[0, T] \times \mathbb{R}^4$. Then writing

$$u = Q + (u - Q),$$

and recalling that the stationary solution Q behaves like ρ at infinity, we infer that there is a positive number $R_0 = R_0(A)$ such that for any $|x| \geq R_0$ and any t in $[0, T]$, we have

$$u(t, x) \geq Q(x)/2.$$

On the other hand, invoking (C.5) together with Lemma 2.1, we infer that there is a positive constant $C(\varepsilon, R_0)$ such that if $|x| \leq R_0$, then for all $0 \leq t \leq T$,

$$\frac{1}{u(t,x)} \leq \frac{C(\varepsilon, R_0)}{Q(x)}.$$

Now taking advantage of the Sobolev embedding $\dot{H}^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$, we deduce that

$$\begin{aligned}\Big\|\Big(\frac{1}{u} - \frac{1}{Q}\Big)(t,\cdot)\Big\|_{L^2} &\leq C_{\varepsilon,A}\, \|(u - Q)(t,\cdot)\|_{L^4} \Big\|\frac{1}{Q^2}\Big\|_{L^4} \\ &\leq C_{\varepsilon,A}\, \|\nabla(u - Q)(t,\cdot)\|_{L^2} \Big\|\frac{1}{Q^2}\Big\|_{L^4},\end{aligned}$$

which according to the fact that $1/Q(\rho) \lesssim 1/\langle\rho\rangle$ ensures that

$$\Big\|\Big(\frac{1}{u} - \frac{1}{Q}\Big)(t,\cdot)\Big\|_{L^2} \leq C_{\varepsilon,A}\, \|\nabla(u - Q)(t,\cdot)\|_{L^2}. \tag{C.16}$$

Together with (C.14), this implies that for all t in $[0, T]$,

$$\|f(t,\cdot)\|_{L^2} \leq C_{\varepsilon,A}\big(\|u_t(t,\cdot)\|_{L^2} + \|\nabla(u - Q)(t,\cdot)\|_{L^2}\big). \tag{C.17}$$

Thanks to the bound (C.3), this ends the proof that $f \in L^\infty([0, T], L^2(\mathbb{R}^4))$.

In order to establish that $f \in L^\infty([0, T], H^{L-1}(\mathbb{R}^4))$, first observe that by the assumption (C.3), the functions $(b_i(\nabla u, u_t))_{1\leq i\leq 4}$, $(a_{i,j}(\nabla u, u_t) - a_{i,j}(\nabla Q, 0))_{1\leq i,j\leq 4}$ as well as the function $c(u, \nabla u, u_t) + 3/u$ belong to $L^\infty([0, T], H^{L-1}(\mathbb{R}^4))$.

Thus taking advantage of Lemma 2.1 and recalling that $L > 4$, we find that $\widetilde{f}$ belongs to $L^\infty([0, T], H^{L-1}(\mathbb{R}^4))$ and satisfies the following estimate uniformly on $[0, T]$:

$$\|\widetilde{f}(t,\cdot)\|_{H^{L-1}} \leq C_{\varepsilon,A}\big(\|u_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(u - Q)(t,\cdot)\|_{H^{L-1}}\big).$$

Moreover, applying Leibniz's formula to the term $\frac{u-Q}{uQ}$ and taking into account (C.16), we infer that there is a positive constant $C_{\varepsilon,A}$ such that for all t in $[0, T]$, we have

$$\Big\|\Big(\frac{1}{u} - \frac{1}{Q}\Big)(t,\cdot)\Big\|_{H^{L-1}} \leq C_{\varepsilon,A}\, \|\nabla(u - Q)(t,\cdot)\|_{H^{L-1}}.$$

Combining the last two inequalities, we get

$$\|f(t,\cdot)\|_{H^{L-1}} \leq C_{\varepsilon,A}\big(\|u_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(u - Q)(t,\cdot)\|_{H^{L-1}}\big). \tag{C.18}$$

This concludes the proof of the fact that $f \in L^\infty([0, T], H^{L-1}(\mathbb{R}^4))$.

Finally, since the coefficients of equation (C.11) as well as their time and spatial derivatives are bounded on $[0,T]\times\mathbb{R}^4$, applying classical arguments we infer that the Cauchy problem (C.6) admits a unique solution on $[0,T]\times\mathbb{R}^4$.

To simplify the notations, in the rest of this proof we shall denote by $\mathcal{A}$ the matrix $(a_{i,j})_{1\le i,j\le 4}$ and by b the vector $(b_1,\dots,b_4)$, and omit the dependence of all the functions $a_{i,j}$ and b_i on $(\nabla u,u_t)$ and of the source term f on $(u,\nabla u,u_t)$.

Now to establish the energy inequality (C.7), we can proceed as follows. First we take the L^2-scalar product of (C.11) with $\widetilde{\Phi}_t-\frac{b}{2}\cdot\nabla\widetilde{\Phi}$, which gives rise to

$$\Big(\Big[\partial_t\Big(\widetilde{\Phi}_t-\frac{b}{2}\cdot\nabla\widetilde{\Phi}\Big)-\sum_{i,j=1}^{4}a_{i,j}\widetilde{\Phi}_{x_ix_j}-\frac{b}{2}\cdot\nabla\widetilde{\Phi}_t+\frac{b_t}{2}\cdot\nabla\widetilde{\Phi}\Big](t,\cdot)\Big|\Big(\widetilde{\Phi}_t-\frac{b}{2}\cdot\nabla\widetilde{\Phi}\Big)(t,\cdot)\Big)_{L^2}$$
$$=\Big(f(t,\cdot)\Big|\Big(\widetilde{\Phi}_t-\frac{b}{2}\cdot\nabla\widetilde{\Phi}\Big)(t,\cdot)\Big)_{L^2}.$$

Performing integrations by parts, we deduce that

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(\widetilde{\Phi})(t,\cdot)=I_0(t)+\Big(f(t,\cdot)\Big|\Big(\widetilde{\Phi}_t-\frac{b}{2}\cdot\nabla\widetilde{\Phi}\Big)(t,\cdot)\Big)_{L^2},\tag{C.19}$$

where

$$\begin{aligned}\mathcal{E}(\widetilde{\Phi})(t,\cdot):=\Big\|\Big(\widetilde{\Phi}_t-\frac{b}{2}\cdot\nabla\widetilde{\Phi}\Big)(t,\cdot)\Big\|^2_{L^2}+\sum_{i,j=1}^{4}\big(a_{i,j}\widetilde{\Phi}_{x_i}(t,\cdot)\big|\widetilde{\Phi}_{x_j}(t,\cdot)\big)_{L^2}\\+\Big\|\Big(\frac{b}{2}\cdot\nabla\widetilde{\Phi}\Big)(t,\cdot)\Big\|^2_{L^2},\end{aligned}\tag{C.20}$$

and where I_0 admits the estimate

$$|I_0(t)|\le a_0(t)\big(\|(\nabla\widetilde{\Phi})(t,\cdot)\|^2_{L^2}+\|\widetilde{\Phi}_t(t,\cdot)\|^2_{L^2}\big)\tag{C.21}$$

with

$$a_0(t)=\mathcal{T}\big(\|\mathcal{A}(t,\cdot)\|_{L^\infty},\|(\nabla_{t,x}\mathcal{A})(t,\cdot)\|_{L^\infty},\|b(t,\cdot)\|_{L^\infty},\|(\nabla_{t,x}b)(t,\cdot)\|_{L^\infty}\big),\tag{C.22}$$

$\mathcal{T}$ denoting a polynomial function of all its arguments.

By (C.3) and (C.4), we have

$$\|a_0\|_{L^\infty([0,T]\times\mathbb{R}^4)}\le C_{A,A_1}.\tag{C.23}$$

Observe also that thanks to (C.10), we have

$$\begin{aligned}\mathcal{E}(\widetilde{\Phi})(t,\cdot)&\le 4\big(\|(\nabla\widetilde{\Phi})(t,\cdot)\|^2_{H^{L-1}}+\|\widetilde{\Phi}_t(t,\cdot)\|^2_{H^{L-1}}\big),\\ \mathcal{E}(\widetilde{\Phi})(t,\cdot)&\ge\Big\|\Big(\widetilde{\Phi}_t-\frac{b}{2}\cdot\nabla\widetilde{\Phi}\Big)(t,\cdot)\Big\|^2_{L^2}+\varepsilon\|\nabla\widetilde{\Phi}(t,\cdot)\|^2_{L^2}.\end{aligned}\tag{C.24}$$

Now in order to estimate $\Phi_t(t,\cdot)$ and $\nabla(\Phi-Q)(t,\cdot)$ in H^{L-1}, we differentiate the nonlinear wave equation (C.11) with respect to the space variable up to order $L-1$. By straightforward computations we formally obtain, for any multi-index α of length $|\alpha|\leq L-1$,

$$\begin{cases}(\partial^\alpha\widetilde{\Phi})_{tt}-\displaystyle\sum_{i,j=1}^{4}a_{i,j}(\partial^\alpha\widetilde{\Phi})_{x_ix_j}-\sum_{i=1}^{4}b_i(\partial^\alpha\widetilde{\Phi})_{tx_i}=f_\alpha,\\ (\partial^\alpha\widetilde{\Phi})_{|t=0}=\partial^\alpha(\Phi_0-Q),\\ (\partial_t(\partial^\alpha\widetilde{\Phi}))_{|t=0}=\partial^\alpha\Phi_1,\end{cases}\tag{C.25}$$

with

$$f_\alpha=\partial^\alpha f+\widetilde{f_\alpha},$$

where

$$\widetilde{f_\alpha}:=\sum_{i,j=1}^{4}\sum_{\beta<\alpha}\binom{\alpha}{\beta}(\partial^{\alpha-\beta}a_{i,j})(\partial^\beta\widetilde{\Phi})_{x_ix_j}+\sum_{i=1}^{4}\sum_{\beta<\alpha}\binom{\alpha}{\beta}(\partial^{\alpha-\beta}b_i)(\partial^\beta\widetilde{\Phi})_{tx_i}.\tag{C.26}$$

Then taking the L^2-scalar product of (C.25) with $(\partial^\alpha\widetilde{\Phi})_t-\frac{b}{2}\cdot\nabla(\partial^\alpha\widetilde{\Phi})$ and applying the same line of reasoning as above, we get

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(\partial^\alpha\widetilde{\Phi})(t,\cdot)=I_\alpha(t)+\Big(f_\alpha(t,\cdot)\Big|\Big[(\partial^\alpha\widetilde{\Phi})_t-\frac{b}{2}\cdot\nabla(\partial^\alpha\widetilde{\Phi})\Big](t,\cdot)\Big)_{L^2},\tag{C.27}$$

where

$$|I_\alpha(t)|\leq a_0(t)\big(\|\nabla(\partial^\alpha\widetilde{\Phi})(t,\cdot)\|^2_{L^2}+\|(\partial^\alpha\widetilde{\Phi})_t(t,\cdot)\|^2_{L^2}\big).\tag{C.28}$$

Now since the functions $(\nabla a_{i,j})_{1\leq i,j\leq 4}$ and $(\nabla b_i)_{1\leq i\leq 4}$ belong to the Sobolev space $H^{L-2}(\mathbb{R}^4)$, the function $\widetilde{f_\alpha}$ belongs to $L^2(\mathbb{R}^4)$ and satisfies, uniformly on $[0,T]$,

$$\|\widetilde{f_\alpha}(t,\cdot)\|_{L^2}\leq C_A[\|\nabla\widetilde{\Phi}(t,\cdot)\|_{H^{|\alpha|}}+\|\widetilde{\Phi}_t(t,\cdot)\|_{H^{|\alpha|}}].\tag{C.29}$$

Therefore taking into account (C.18) we get, for any $|\alpha|\leq L-1$,

$$\begin{aligned}\Big|\Big(f_\alpha(t,\cdot)\Big|\Big[(\partial^\alpha\widetilde{\Phi})_t-\frac{b}{2}\cdot\nabla(\partial^\alpha\widetilde{\Phi})\Big](t,\cdot)\Big)_{L^2}\Big|&\leq C_{\varepsilon,A}\big[\|\nabla\widetilde{\Phi}(t,\cdot)\|^2_{H^{|\alpha|}}+\|\widetilde{\Phi}_t(t,\cdot)\|^2_{H^{|\alpha|}}\\&+\big(\|\nabla\widetilde{\Phi}(t,\cdot)\|_{H^{|\alpha|}}+\|\widetilde{\Phi}_t(t,\cdot)\|_{H^{|\alpha|}}\big)\big(\|u_t(t,\cdot)\|_{H^{L-1}}+\|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}}\big)\big].\end{aligned}\tag{C.30}$$

Combining (C.19), (C.21), (C.23), (C.27), (C.28) and (C.30), we easily gather that

$$\begin{aligned}\Big|\sum_{|\alpha|\leq L-1}\frac{d}{dt}\mathcal{E}(\partial^\alpha\widetilde{\Phi})(t,\cdot)\Big|&\leq C_{\varepsilon,A,A_1}\big[\|\nabla\widetilde{\Phi}(t,\cdot)\|^2_{H^{L-1}}+\|\widetilde{\Phi}_t(t,\cdot)\|^2_{H^{L-1}}\\&+(\|\nabla\widetilde{\Phi}(t,\cdot)\|_{H^{L-1}}+\|\widetilde{\Phi}_t(t,\cdot)\|_{H^{L-1}})(\|u_t(t,\cdot)\|_{H^{L-1}}+\|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}})\big].\end{aligned}\tag{C.31}$$

Applying the Gronwall lemma and taking into account (C.24), we deduce that for all t in $[0, T]$,

$$\|\nabla\widetilde{\Phi}(t,\cdot)\|_{H^{L-1}} + \|\widetilde{\Phi}_t(t,\cdot)\|_{H^{L-1}} \leq C_\varepsilon e^{tC_{\varepsilon,A,A_1}}(\|\Phi_1\|_{H^{L-1}} + \|\nabla(\Phi_0 - Q)\|_{H^{L-1}}) + C_{\varepsilon,A,A_1}\int_0^t \big(\|u_s(s,\cdot)\|_{H^{L-1}} + \|\nabla(u-Q)(s,\cdot)\|_{H^{L-1}}\big)\, ds. \tag{C.32}$$

To complete the proof of the energy estimates, it remains to estimate $\|\Phi_{tt}(t,\cdot)\|_{H^{L-2}(\mathbb{R}^4)}$. To this end, we make use of equation (C.11) which implies that

$$\Phi_{tt} = \sum_{i,j=1}^{4} a_{i,j}(\nabla u, u_t)\widetilde{\Phi}_{x_i x_j} + \sum_{i=1}^{4} b_i(\nabla u, u_t)\widetilde{\Phi}_{t x_i} + f(u, \nabla u, u_t).$$

This ensures the result according to (C.3) and (C.18). ■

Let us now return to the proof of Theorem 1.1. The first step can be deduced from Proposition C.1 by a standard argument that can be found for instance in the monographs [4,14,36]. The key point consists in proving that the sequence $(u^{(n)})_{n\in\mathbb{N}}$ defined by $(\mathrm{W})_n$ (see page 3871) is uniformly bounded, in the sense that there exist a small positive time

$$T = T\big(\|u_1\|_{H^{L-1}}, \|\nabla(u_0 - Q)\|_{H^{L-1}}, \varepsilon\big), \tag{C.33}$$

and a positive constant $C = C(\|u_1\|_{H^{L-1}}, \|\nabla(u_0 - Q)\|_{H^{L-1}}, \varepsilon)$ such that for any integer n and any time t in $[0, T]$, we have

$$\|u_t^{(n)}(t,\cdot)\|_{H^{L-1}} + \|\nabla(u^{(n)} - Q)(t,\cdot)\|_{H^{L-1}} + \|u_{tt}^{(n)}(t,\cdot)\|_{H^{L-2}} \leq C, \tag{C.34}$$

$$u^{(n)}(t,\cdot) \geq \varepsilon \quad \text{and} \quad (1 - (u_t^{(n)})^2 + |\nabla u^{(n)}|^2)(t,\cdot) \geq \varepsilon. \tag{C.35}$$

In order to establish the uniform estimate (C.34), set

$$A = 2C_\varepsilon\big(\|u_1\|_{H^{L-1}} + \|\nabla(u_0 - Q)\|_{H^{L-1}}\big), \tag{C.36}$$

$$A_1 = (C_A + C_{\varepsilon,A})A, \tag{C.37}$$

where C_ε, C_A and $C_{\varepsilon,A}$ are the constants introduced in (C.7)–(C.8).

We claim that there exists a positive time $T \leq 1$ of the form (C.33) such that for any integer $n \geq 0$ the following property holds. If for all $t \in [0, T]$, we have

$$\|u_t^{(n)}(t,\cdot)\|_{H^{L-1}} + \|\nabla(u^{(n)} - Q)(t,\cdot)\|_{H^{L-1}} \leq A, \tag{C.38}$$

$$\|u_{tt}^{(n)}(t,\cdot)\|_{H^{L-2}} \leq A_1, \tag{C.39}$$

$$u^{(n)}(t,x) \geq \varepsilon, \quad \frac{1 - (u_t^{(n)}(t,x))^2 + |\nabla u^{(n)}(t,x)|^2}{1 + |\nabla u^{(n)}(t,x)|^2} \geq \varepsilon, \quad \forall (t,x) \in [0,T]\times\mathbb{R}^d, \tag{C.40}$$

then the same bounds remain true for $u^{(n+1)}$.

Indeed, by the energy estimate (C.7) in Proposition C.1, $u^{(n+1)}$ satisfies, for all $t \in [0, T]$,

$$\|u_t^{(n+1)}(t,\cdot)\|_{H^{L-1}} + \|\nabla(u^{(n+1)} - Q)(t,\cdot)\|_{H^{L-1}} \\ \leq C_\varepsilon e^{tC_{\varepsilon,A,A_1}}(\|u_1\|_{H^{L-1}} + \|\nabla(u_0 - Q)\|_{H^{L-1}}) + AC_{\varepsilon,A,A_1}t,$$

which implies that there exists $T(A,\varepsilon) > 0$ such that if $t \leq T(A,\varepsilon)$, then

$$\|u_t^{(n+1)}(t,\cdot)\|_{H^{L-1}} + \|\nabla(u^{(n+1)} - Q)(t,\cdot)\|_{H^{L-1}} \\ \leq 2C_\varepsilon\big(\|u_1\|_{H^{L-1}} + \|\nabla(u_0 - Q)\|_{H^{L-1}}\big) = A.$$

Invoking then (C.8) we get, for all $t \leq T(A,\varepsilon)$,

$$\|u_{tt}^{(n+1)}(t,\cdot)\|_{H^{L-2}} \leq (C_A + C_{\varepsilon,A})A = A_1.$$

This ends the proof that $u^{(n+1)}$ satisfies (C.38) and (C.39).

Finally, (C.40) results directly from the following straightforward estimates:

$$\|u^{(n+1)}(t,\cdot) - u_0\|_{L^\infty(\mathbb{R}^4)} \leq \int_0^t \|\partial_s u^{(n+1)}(s,\cdot)\|_{L^\infty(\mathbb{R}^4)}\, ds \lesssim At,$$

$$\|(\partial_t u^{(n+1)})(t,\cdot) - u_1\|_{L^\infty(\mathbb{R}^4)} \leq \int_0^t \|\partial_s^2 u^{(n+1)}(s,\cdot)\|_{L^\infty(\mathbb{R}^4)}\, ds \lesssim A_1 t,$$

$$\|(\nabla u^{(n+1)})(t,\cdot) - \nabla u_0\|_{L^\infty(\mathbb{R}^4)} \leq \int_0^t \|(\partial_s \nabla u^{(n+1)})(s,\cdot)\|_{L^\infty(\mathbb{R}^4)}\, ds \lesssim At,$$

which implies (C.40) provided that $T = T(A, A_1, \varepsilon)$ is chosen sufficiently small. This completes the proof of the claim.

To end the proof of the local well-posedness for the Cauchy problem (1.11), it suffices to establish that the sequences $(\partial_t u^{(n)})_{n\in\mathbb{N}}$ and $(\nabla(u^{(n)} - Q))_{n\in\mathbb{N}}$ are Cauchy sequences in $L^\infty([0,T], H^{L-2}(\mathbb{R}^4))$. By a standard argument, this fact follows easily from (C.34). Indeed, setting $w^{(n+1)} := u^{(n+1)} - u^{(n)}$, we readily gather that for all $n \geq 0$,

$$\begin{cases} w_{tt}^{(n+1)} - \displaystyle\sum_{i,j=1}^{4} a_{i,j}(\nabla u^{(n)}, u_t^{(n)}) w_{x_i x_j}^{(n+1)} - \sum_{i=1}^{4} b_i(\nabla u^{(n)}, u_t^{(n)}) w_{t x_i}^{(n+1)} = g^{(n)}, \\ w^{(n+1)}{}_{|t=0} = 0, \\ (\partial_t w^{(n+1)})_{|t=0} = 0, \end{cases}$$

where

$$g^{(n)} = \sum_{i,j=1}^{4} (a_{i,j}(\nabla u^{(n)}, u_t^{(n)}) - a_{i,j}(\nabla u^{(n-1)}, u_t^{(n-1)})) u_{x_i x_j}^{(n)} \\ + \sum_{i=1}^{4} (b_i(\nabla u^{(n)}, u_t^{(n)}) - b_i(\nabla u^{(n-1)}, u_t^{(n-1)})) u_{t x_i}^{(n)} \\ + c(u^{(n)}, \nabla u^{(n)}, u_t^{(n)}) - c(u^{(n-1)}, \nabla u^{(n-1)}, u_t^{(n-1)}).$$

Since by construction we have, for any (t, x) in $[0, T] \times \mathbb{R}^d$,

$$u^{(n)}(t,x) \geq \varepsilon \quad \text{and} \quad \frac{1-(u_t^{(n)}(t,x))^2 + |\nabla u^{(n)}(t,x)|^2}{1+|\nabla u^{(n)}(t,x)|^2} \geq \varepsilon,$$

arguing in a similar way to the proof of Proposition C.1 we obtain

$$\begin{aligned}&\|w_t^{(n+1)}(t,\cdot)\|_{L^\infty([0,T],H^{L-2})} + \|\nabla w^{(n+1)}(t,\cdot)\|_{L^\infty([0,T],H^{L-2})}\\&\qquad \leq CT(\|w_t^{(n)}(t,\cdot)\|_{L^\infty([0,T],H^{L-2})} + \|\nabla w^{(n)}(t,\cdot)\|_{L^\infty([0,T],H^{L-2})}).\end{aligned}$$

This ensures the result provided that T is small enough and completes the proof of the first step.

Let us now address the second step, and establish the uniqueness of solutions to the Cauchy problem (1.11). For this purpose, we shall prove the following continuation criterion which easily gives the result:

Lemma C.1. *Let u and v be two solutions of the Cauchy problem* (1.11) *respectively associated to the initial data (u_0, u_1) and (v_0, v_1) in X_s, such that (u, u_t) and (v, v_t) are in $\mathcal{C}([0, T], X_s)$ and u_t and v_t belong to $\mathcal{C}^1([0, T], H^{s-1})$ for some $s > 4$. Then there is a positive constant C such that, for all t in $[0, T]$,*

$$\begin{aligned}&\|(u-v)_t(t,\cdot)\|_{L^2(\mathbb{R}^4)} + \|\nabla(u-v)(t,\cdot)\|_{L^2(\mathbb{R}^4)}\\&\qquad \leq C\big(\|u_1-v_1\|_{L^2(\mathbb{R}^4)} + \|\nabla(u_0-v_0)\|_{L^2(\mathbb{R}^4)}\big).\end{aligned}$$

Proof. By straightforward computations, the function $w := u - v$ solves the Cauchy problem

$$\begin{cases} w_{tt} - \sum_{i,j=1}^{4} a_{i,j}(\nabla u, u_t) w_{x_i x_j} - \sum_{i=1}^{4} b_i(\nabla u, u_t) w_{t x_i} = g, \\ w_{|t=0} = u_0 - v_0, \\ (\partial_t w)_{|t=0} = u_1 - v_1, \end{cases} \tag{C.41}$$

where

$$\begin{aligned} g = &\sum_{i,j=1}^{4} \big(a_{i,j}(\nabla u, u_t) - a_{i,j}(\nabla v, v_t)\big) v_{x_i x_j}\\ &+ \sum_{i=1}^{4} \big(b_i(\nabla u, u_t) - b_i(\nabla v, v_t)\big) v_{t x_i} + c(u, \nabla u, u_t) - c(v, \nabla v, v_t).\end{aligned}$$

Therefore, taking the L^2-scalar product of (C.41) with $w_t - \frac{b}{2} \cdot \nabla w$ we get, as in the proof of Proposition C.1, the energy inequality

$$\begin{aligned}&\|w_t(t,\cdot)\|_{L^2(\mathbb{R}^4)} + \|\nabla w(t,\cdot)\|_{L^2(\mathbb{R}^4)}\\&\quad \leq C\bigg(\|u_1-v_1\|_{L^2(\mathbb{R}^4)} + \|\nabla(u_0-v_0)\|_{L^2(\mathbb{R}^4)} + \int_0^t \|g(s,\cdot)\|_{L^2(\mathbb{R}^4)}\,ds\bigg).\end{aligned}$$

As before, by straightforward computations we have

$$\|g(s,\cdot)\|_{L^2(\mathbb{R}^4)} \leq C\big(\|w_t(t,\cdot)\|_{L^2(\mathbb{R}^4)} + \|\nabla w(t,\cdot)\|_{L^2(\mathbb{R}^4)}\big),$$

which easily completes the proof of the continuation criterion. ■

Finally the blow up criterion (1.13) results by standard arguments from the fact that if

$$\limsup_{t\nearrow T}\bigg(\Big\|\frac{1}{u(t,\cdot)}\Big\|_{L^\infty} + \Big\|\frac{1}{(1+|\nabla u|^2-(\partial_t u)^2)(t,\cdot)}\Big\|_{L^\infty} + \sup_{|\gamma|\leq 1}\|\partial_x^\gamma \nabla_{t,x} u\|_{L^\infty}\bigg) < \infty,$$

then the solution to the Cauchy problem (1.11) can be extended beyond T. This ends the proof of Theorem 1.1.

Appendix D. Some simple ordinary differential equations results

D.1. Proof of Duhamel's formula (3.16)

The formula results from the following lemma:

Lemma D.1. *With the previous notations, the homogeneous equation*

$$\mathcal{L} f = 0 \tag{D.1}$$

has a basis of solutions $\{\mathrm{e}_1, \mathrm{e}_2\}$ *given by*

$$\begin{cases} \mathrm{e}_1(y) = (\Lambda Q)(y), \\ \mathrm{e}_2(y) = (\Lambda Q)(y) \displaystyle\int_1^y \frac{(1+(Q_r(r))^2)^{3/2}}{Q^3(r) r^3 (\Lambda Q)^2(r)}\, dr. \end{cases} \tag{D.2}$$

Moreover, for any regular function g, the solution to the Cauchy problem

$$\begin{cases} \mathcal{L} f = g, \\ f(0) = 0 \quad \text{and} \quad f'(0) = 0, \end{cases} \tag{D.3}$$

can be written in the form

$$f(y) = -(\Lambda Q)(y) \int_0^y \frac{(1+(Q_r(r))^2)^{3/2}}{Q^3(r) r^3 (\Lambda Q)^2(r)} \int_0^r \frac{Q^3(s) s^3 (\Lambda Q)(s)}{(1+(Q_s(s))^2)^{3/2}} g(s)\, ds\, dr.$$

Proof. It has already been mentioned that $\mathrm{e}_1 := \Lambda Q$ is a positive solution of the homogeneous equation $\mathcal{L} f = 0$.

In order to obtain e_2 it is convenient to remove the first order derivative in $\mathcal{L}$ by setting $f = \hat{H} \hat{f}$, where

$$2\frac{(\hat{H})_y}{\hat{H}} = -\Big(\frac{3}{y} + B_1\Big) \tag{D.4}$$

with B_1 defined by (1.17). Then

$$\mathcal{L} f = g \iff \hat{\mathcal{L}} \hat{f} = \hat{g}$$

with $g = \hat{H} \hat{g}$ and

$$\hat{\mathcal{L}} = \partial_y^2 + \hat{\mathcal{P}},$$

where

$$\hat{\mathcal{P}} = B_0 + \Big(\frac{3}{y} + B_1\Big)\frac{(\hat{H})_y}{\hat{H}} + \frac{(\hat{H})_{yy}}{\hat{H}}.$$

Since for any two solutions $\hat{f}_1$ and $\hat{f}_2$ of the homogeneous equation

$$\hat{\mathcal{L}} \hat{f} = 0, \tag{D.5}$$

the Wronskian $W(\hat{f}_1, \hat{f}_2)$ is constant, the functions $\hat{\mathrm{e}}_1, \hat{\mathrm{e}}_2$ defined by

$$\hat{\mathrm{e}}_1 = \hat{H}^{-1}\mathrm{e}_1, \quad \hat{\mathrm{e}}_2(y) = \hat{\mathrm{e}}_1(y)\int_1^y \frac{ds}{\hat{\mathrm{e}}_1^2(s)} \tag{D.6}$$

constitute a fundamental system of solutions to (D.5). Since

$$B_1(y) = \frac{9Q_y^2}{y} - \frac{6Q_y}{Q},$$

in view of (2.1) we get

$$B_1(y) = 3\Big(\frac{Q_y}{Q} - \frac{Q_{yy}Q_y}{1+Q_y^2}\Big).$$

Therefore

$$\frac{3}{y} + B_1(y) = 3\Big(\log\Big(\frac{yQ}{(1+Q_y^2)^{1/2}}\Big)\Big)_y,$$

and thus taking into account (D.4), one can choose

$$\hat{H}(y) = \frac{(1+(Q_y)^2(y))^{3/4}}{(yQ(y))^{3/2}}. \tag{D.7}$$

This completes the proof of (D.2).

To end the proof of the lemma, it remains to establish Duhamel's formula (3.16). For this purpose, let us start by noticing that since by construction $W(\hat{\mathrm{e}}_1, \hat{\mathrm{e}}_2) = 1$, the solution of (D.3) can be written in the form

$$f(y) = \int_0^y \frac{\mathrm{e}_1(y)\mathrm{e}_2(s) - \mathrm{e}_1(s)\mathrm{e}_2(y)}{\hat{H}^2(s)} g(s)\, ds.$$

In view of (D.2), we deduce that

$$f(y) = \mathrm{e}_1(y) \int_0^y \left(\mathrm{e}_1(s) \int_1^s \frac{\hat{H}^2(s')\,ds'}{\mathrm{e}_1^2(s')} - \mathrm{e}_1(s) \int_1^y \frac{\hat{H}^2(s')\,ds'}{\mathrm{e}_1^2(s')} \right) \frac{g(s)}{\hat{H}^2(s)}\,ds$$
$$= -\mathrm{e}_1(y) \int_0^y \frac{\mathrm{e}_1(s) g(s)}{\hat{H}^2(s)} \int_s^y \frac{\hat{H}^2(s')\,ds'}{\mathrm{e}_1^2(s')}\,ds.$$

Finally, performing an integration by parts, we readily gather that

$$f(y) = -\mathrm{e}_1(y) \int_0^y \frac{\hat{H}^2(s)}{\mathrm{e}_1^2(s)} \int_0^s \frac{\mathrm{e}_1(s') g(s')}{\hat{H}^2(s')}\,ds'\,ds,$$

which ends the proof of the lemma by (D.7). ∎

D.2. Proof of Lemma 4.3

To prove the first item, let us, for g in $\mathcal{C}^\infty(\mathbb{R}_+^*)$, look for the solution f of the inhomogeneous equation

$$\begin{cases} \widetilde{\mathcal{L}}_k f = (2z^2 - 1)\partial_z^2 f - \left(\dfrac{6}{z} + 4z\nu k\right)\partial_z f - \left(\dfrac{6}{z^2} - 2\nu k(1+\nu k)\right) f = g, \\ f(1/\sqrt{2}) = 0, \end{cases}$$

in the form

$$f = f^{(0)} + f^{(1)} \quad \text{with} \quad f^{(0)}(z) := \sum_{m=1}^{N+1} \alpha_m (z - 1/\sqrt{2})^m,$$

where $N := [k\nu] + 3$ and where the coefficients α_m for $1 \le m \le N+1$ are uniquely determined by the requirement that the function

$$\widetilde{g} := g - \widetilde{\mathcal{L}}_k f^{(0)}$$

satisfies

$$\widetilde{g}^{(\ell)}(1/\sqrt{2}) = 0, \qquad \forall \ell \in \{0, \dots, N\}. \tag{D.8}$$

Then $f^{(1)}$ has to satisfy

$$\begin{cases} \widetilde{\mathcal{L}}_k f^{(1)} = \widetilde{g}, \\ f^{(1)}(1/\sqrt{2}) = 0, \quad f^{(1)} \in \mathcal{C}^\infty(\mathbb{R}_+^*), \end{cases}$$

and can be recovered by the Duhamel formula:

$$f^{(1)}(z) = \int_{1/\sqrt{2}}^z \frac{\widetilde{g}(s)}{2s^2 - 1} \frac{1}{\mathcal{W}(f_{k,0}^{0,+}, f_{k,0}^{0,-})(s)} \left(f_{k,0}^{0,-}(z) f_{k,0}^{0,+}(s) - f_{k,0}^{0,+}(z) f_{k,0}^{0,-}(s) \right) ds,$$

where

$$\mathcal{W}(f_{k,0}^{0,+}, f_{k,0}^{0,-}) := f_{k,0}^{0,+}(f_{k,0}^{0,-})_z - f_{k,0}^{0,-}(f_{k,0}^{0,+})_z$$

denotes the Wronskian of the basis $\{f_{k,0}^{0,+}, f_{k,0}^{0,-}\}$ defined by (4.45). By straightforward computations, we have

$$\mathcal{W}(f_{k,0}^{0,+}, f_{k,0}^{0,-})(z) = \frac{\sqrt{2}\,\alpha(\nu,k)\,\mathrm{sgn}(z-1/2)|z^2-1/2|^{\alpha(\nu,k)-1}}{z^6},$$

which implies that

$$f^{(1)}(z) = \frac{1}{2\sqrt{2}\,\alpha(\nu,k)} \int_{1/\sqrt{2}}^{z} s^3 \widetilde{g}(s)\Big(\frac{f_{k,0}^{0,-}(z)}{|s-1/\sqrt{2}|^{\alpha(\nu,k)}} - \frac{f_{k,0}^{0,+}(z)}{(s+1/\sqrt{2})^{\alpha(\nu,k)}}\Big)\,ds. \tag{D.9}$$

The uniqueness follows immediately from Remark 4.1.

Now we turn our attention to the second item. Our task here is to solve uniquely (4.51) in $\mathcal{C}^\infty(]0, 1/\sqrt{2}])$ under condition (4.52). Let us start with the case $q = 0$ and look for a solution f to the equation

$$\widetilde{\mathcal{L}}_k f(z) = (1/\sqrt{2} - z)^\gamma h(z)$$

in the form

$$f = f^{(0)} + f^{(1)}$$

with

$$f^{(0)}(z) := (1/\sqrt{2} - z)^{\gamma+1} \sum_{m=0}^{N} c_m (1/\sqrt{2} - z)^m,$$

where again $N = [k\nu] + 3$. Due to (4.52), the coefficients c_m for $0 \le m \le N$ can be fixed so that

$$\widetilde{\mathcal{L}}_k f^{(1)}(z) = (1/\sqrt{2} - z)^\gamma \widetilde{h}(z), \tag{D.10}$$

where $\widetilde{h}$ is a function in $\mathcal{C}^\infty(]0, 1/\sqrt{2}])$ that satisfies

$$\widetilde{h}^{(\ell)}(1/\sqrt{2}) = 0, \quad \forall \ell \in \{0, \dots, N\}.$$

But any solution to (D.10) is of the form

$$\frac{1}{2\sqrt{2}\,\alpha(\nu,k)} \int_{1/\sqrt{2}}^{z} s^3 (1/\sqrt{2} - s)^\gamma \widetilde{h}(s)\Big(\frac{f_{k,0}^{0,-}(z)}{(1/\sqrt{2}-s)^{\alpha(\nu,k)}} - \frac{f_{k,0}^{0,+}(z)}{(s+1/\sqrt{2})^{\alpha(\nu,k)}}\Big)\,ds + a_k^+ f_{k,0}^{0,+}(z) + a_k^- f_{k,0}^{0,-}(z)$$

for some constants a_k^+ and a_k^-. Invoking the fact that we look for solutions in $\mathcal{C}^\infty(]0, 1/\sqrt{2}])$ vanishing at $z = 1/\sqrt{2}$, we end up with the result, in the case $q = 0$, by taking $a_k^+ = a_k^- = 0$.

To establish the result for any integer $q \geq 1$, we shall proceed by induction assuming that under condition (4.52), for any integer $1 \leq j \leq q-1$, the inhomogeneous equation

$$\widetilde{\mathcal{L}}_k f(z) = (1/\sqrt{2}-z)^\gamma (\log(1/\sqrt{2}-z))^j h(z)$$

has a unique solution f of the form

$$f(z) = (1/\sqrt{2}-z)^{\gamma+1} \sum_{0\leq \ell \leq j} (\log(1/\sqrt{2}-z))^\ell h_\ell(z),$$

where for all $0 \leq \ell \leq j$, h_ℓ is in $\mathcal{C}^\infty(]0,1/\sqrt{2}])$. Then we look for a solution f to

$$\widetilde{\mathcal{L}}_k f(z) = (1/\sqrt{2}-z)^\gamma (\log(1/\sqrt{2}-z))^q h(z)$$

of the form

$$f(z) = (\log(1/\sqrt{2}-z))^q \widetilde{f}(z) + f^{(1)}(z), \tag{D.11}$$

where

$$\widetilde{\mathcal{L}}_k \widetilde{f}(z) = (1/\sqrt{2}-z)^\gamma h(z).$$

Thanks to the above computations, this implies that

$$\widetilde{f}(z) = (1/\sqrt{2}-z)^{\gamma+1} h_q(z),$$

where h_q belongs to $\mathcal{C}^\infty(]0,1/\sqrt{2}])$.

Since in view of (D.11),

$$\widetilde{\mathcal{L}}_k f^{(1)}(z) = (1/\sqrt{2}-z)^\gamma \sum_{0\leq \ell \leq q-1} (\log(1/\sqrt{2}-z))^\ell \widetilde{h}_\ell(z),$$

with $\widetilde{h}_\ell \in \mathcal{C}^\infty(]0,1/\sqrt{2}])$, this completes the proof of the second item by the induction assumption.

Let us now establish the third item. To this end, for $g \in \mathcal{C}^\infty(]0,1/\sqrt{2}[)$ with an asymptotic expansion at 0 of the form

$$g(z) = (\log z)^{\alpha_0} \sum_{\beta \geq \beta_0} g_\beta z^{\beta-2}$$

with some integers α_0 and β_0, we investigate the nonhomogeneous equation $\widetilde{\mathcal{L}}_k f = g$. Fixing some z_0 in $]0,1/\sqrt{2}[$ and invoking Duhamel's formula, we readily gather that for all z in $]0,1/\sqrt{2}[$, we have

$$f(z) = \frac{1}{2\sqrt{2}\,\alpha(\nu,k)} \int_{z_0}^{z} s^3 g(s) \Big(\frac{f_{k,0}^{0,-}(z)}{(1/\sqrt{2}-s)^{\alpha(\nu,k)}} - \frac{f_{k,0}^{0,+}(z)}{(s+1/\sqrt{2})^{\alpha(\nu,k)}} \Big) ds \\ + a_k^+ f_{k,0}^{0,+}(z) + a_k^- f_{k,0}^{0,-}(z)$$

for some constants a_k^+ and a_k^-.

Taking into account (4.45), we infer that any solution to $\widetilde{\mathcal{L}}_k f = g$ has, for z close to 0, an asymptotic expansion

$$f(z) = \sum_{\beta \geq -3} f_{0,\beta} z^\beta + \sum_{1 \leq \alpha \leq \alpha_0} \sum_{\beta \geq \beta_0} f_{\alpha,\beta} (\log z)^\alpha z^\beta$$

when $\beta_0 \geq -1$, and

$$f(z) = \sum_{\beta \geq \min(\beta_0,-3)} f_{0,\beta} z^\beta + \sum_{1 \leq \alpha \leq \alpha_0} \sum_{\beta \geq \beta_0} f_{\alpha,\beta} (\log z)^\alpha z^\beta + (\log z)^{\alpha_0+1} \sum_{\beta \geq \max(\beta_0,-3)} f_{\alpha,\beta} z^\beta$$

when $\beta_0 \leq -2$. This completes the proof of the third item.

To end the proof of the lemma, it remains to establish the fourth item. Applying Duhamel's formula, we get

$$f(z) = -\frac{1}{2\sqrt{2}\,\alpha(\nu,k)} \int_z^\infty s^3 g(s) \left(\frac{f_{k,0}^{0,-}(z)}{(s - 1/\sqrt{2})^{\alpha(\nu,k)}} - \frac{f_{k,0}^{0,+}(z)}{(s + 1/\sqrt{2})^{\alpha(\nu,k)}} \right) ds + a_k^+ f_{k,0}^{0,+}(z) + a_k^- f_{k,0}^{0,-}(z)$$

for some constants a_k^+ and a_k^-. Since $A < \nu k$, the unique solution to (4.54) that has at infinity an asymptotic expansion

$$f(z) = \sum_{0 \leq \alpha \leq \alpha_0} \sum_{p \in \mathbb{N}} \hat{f}_{\alpha,p}^k (\log z)^\alpha \, z^{A-p}$$

is given by the above formula with $a_k^+ = a_k^- = 0$. This ends the proof of the lemma.

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