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On a class number formula of Hurwitz

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Abstract. In a little-known paper Hurwitz gave an infinite series representation of the class number for positive definite binary quadratic forms. In this paper we give a similar formula in the indefinite case. We also give a simple proof of Hurwitz’s formula and indicate some extensions.

Keywords. Binary quadratic forms, class numbers, Hurwitz

1. Introduction

Adolf Hurwitz made a number of important and influential contributions to the theory of binary quadratic forms. Yet his paper [4] on an infinite series representation of the class number in the positive definite case, which appeared in the Dirichlet-volume of Crelle’s Journal of 1905, has been essentially ignored. About the only references to this paper we found in the literature are in Dickson’s book [1, p. 167] and the more recent paper [8]. Perhaps one reason for this neglect is that Hurwitz gives a rather general treatment of certain projective integrals which, when applied in this special case, tends to obscure the basic mechanism behind the proof. Our main object here is to establish an indefinite version of Hurwitz’s formula and give a direct and uniform treatment of both cases. We also want to clarify the relation between these formulas and the much better known class number formulas of Dirichlet.

In another largely ignored paper [5], published after his death, Hurwitz further developed his method and applied it to get a formula for the class number of integral positive definite ternary quadratic forms. His general method deserves to be better known and we plan to give some different applications of it in future work.

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2. Dirichlet’s formulas

Before stating the Hurwitz formula and the indefinite version we will first set notation and recall Dirichlet’s formulas. A standard reference is Landau’s book [7]. For convenience we will generally use the notation from [2].

Consider the real binary quadratic form

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $\text{disc}(Q) = d = b^2 - 4ac$. Let $g = \pm \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{PSL}(2, \mathbb{R})$ act on Q by

$$Q \mapsto gQ(x, y) = Q(\delta x - \beta y, -\gamma x + \alpha y). \tag{2.1}$$

For d a fundamental discriminant let \mathcal{Q}_d be the set of all (necessarily primitive) integral binary quadratic forms $Q(x, y) = [a, b, c]$ of discriminant d that are positive definite when $d < 0$. Let $\Gamma \backslash \mathcal{Q}_d$ denote a set of representatives of all classes of discriminant d under the action of $\Gamma = \text{PSL}(2, \mathbb{Z})$. Let $h(d) = \#\Gamma \backslash \mathcal{Q}_d$ be the class number. Dirichlet’s class number formulas can be expressed in terms of the Dirichlet series

$$Z_d(s) = \sum_{\substack{a>0, 0\leq b<2a \\ b^2-4ac=d}} a^{-s}, \tag{2.2}$$

which is absolutely convergent for $\text{Re } s > 1$. Let $w_{-3} = 3, w_{-4} = 2$ and $w_d = 1$ for $d < -4$.

Theorem 1 (Dirichlet). *For any fundamental $d < 0$,*

$$\frac{1}{w_d} h(d) = \frac{\pi}{6} |d|^{1/2} \lim_{s \rightarrow 1^+} (s - 1) Z_d(s). \tag{2.3}$$

For $d > 0$ set $\epsilon_d = \frac{1}{2}(t + u\sqrt{d})$ with t, u the positive integers for which $t^2 - du^2 = 4$ where u is minimal. For $d > 0$ fundamental,

$$h(d) \log \epsilon_d = \frac{\pi^2}{6} d^{1/2} \lim_{s \rightarrow 1^+} (s - 1) Z_d(s). \tag{2.4}$$

Let $L(s, \chi_d) = \sum_{n \geq 1} \chi_d(n) n^{-s}$ be the Dirichlet L -series with $\chi_d(\cdot)$ the Kronecker symbol. Then by counting solutions to the quadratic congruence implicit in (2.2) we have

$$\zeta(2s) Z_d(s) = \zeta(s) L(s, \chi_d)$$

and so $\frac{\pi^2}{6} \text{res}_{s=1} Z_d(s) = L(1, \chi_d)$. This leads to the usual finite versions of Dirichlet’s formulas: when $d < 0$ we have

$$h(d) = -\frac{w_d}{2|d|} \sum_{1 \leq m < d} m \chi_d(m),$$

while when $d > 0$,

$$\epsilon_d^{h(d)} = \prod_{1 \leq m < d} \left(\sin \frac{\pi r}{d} \right)^{-\chi_d(m)}.$$

3. Hurwitz’s formula and the indefinite case

Hurwitz’s formula gives the class number when $d < 0$ in terms of an absolutely convergent analogue of the divergent series $Z_d(1)$ from (2.2). A nice feature is that approximations increase monotonically to their limit.

Theorem 2 (Hurwitz). *For $d < 0$ a fundamental discriminant,*

$$\frac{1}{w_d} h(d) = \frac{1}{12\pi} |d|^{3/2} \sum_{\substack{a>0 \\ b^2-4ac=d}} \frac{1}{a(a+b+c)c}.$$

Each term in the sum is positive and the sum converges.

In fact, this holds for any negative discriminant d when $\frac{1}{w_d} h(d)$ is replaced by the Hurwitz class number $H(-d)$, although Hurwitz only stated the formula for even d . The proofs immediately extend to include all negative discriminants. The convergence of this series is rather slow. Hurwitz also gave the equivalent formulation

$$\begin{aligned} & \sum_{\substack{a>0 \\ b^2-4ac=d}} \frac{1}{a(a+b+c)c} \\ &= |d|^{-1} \sum_{4rs=|d|} \frac{4}{r+s} + \sum_{n \geq 1} \frac{4}{n^2-d} \sum_{4rs=n^2-d} \left(\frac{1}{r+s+n} + \frac{1}{r+s-n} \right), \end{aligned} \tag{3.1}$$

where r, s are positive.

Actually Hurwitz gave a whole series of formulas for $h(d)$ consisting of sums of series of the same shape that individually converge faster. The nicest one is given by

$$\frac{1}{w_d} h(d) = \frac{1}{24\pi} |d|^{5/2} \sum_{\substack{a>0 \\ b^2-4ac=d}} \frac{1}{a^2(a+b+c)c^2}. \tag{3.2}$$

We will prove this and give others in §8.

Turning now to the indefinite case, we will show the following.

Theorem 3. *For $d > 0$ a fundamental discriminant,*

$$\begin{aligned} h(d) \log \epsilon_d &= d^{1/2} \sum_{\substack{[a,b,c] \text{ reduced} \\ b^2-4ac=d}} b^{-1} \\ &+ d^{3/2} \sum_{\substack{a,c,a+b+c>0 \\ b^2-4ac=d}} \frac{1}{3(b+2a)b(b+2c)}, \end{aligned} \tag{3.3}$$

where reduced means reduced in the sense of Zagier, meaning that $a, c > 0$ and $b > a + c$. The sum over reduced forms is finite and each term in the infinite sum is positive, the sum being convergent.

Note that it is no longer true that each individual factor $b + 2a, b, b + 2c$ is positive but their product is positive. As in the positive definite case, approximations increase monotonically as more terms are taken.

Consider the example $d = 5$. Taking $a \leq 100$ and $|b| \leq 100$ in the infinite sum yields the approximation 0.961098 to the correct value

$$h(5) \log \epsilon_5 = \log\left(\frac{3+\sqrt{5}}{2}\right) = 0.962424\dots, \tag{3.4}$$

while taking $a \leq 500$ and $|b| \leq 500$ gives 0.962282.

Similarly to (3.2), we can derive faster converging series at the expense of more complicated formulas. Here is the next case, to be proven at the end of §8:

$$\begin{aligned} h(d) \log \epsilon_d &= d^{1/2} \sum_{\substack{[a,b,c] \text{ reduced} \\ b^2-4ac=d}} \frac{b(4b+a) - c(4a+b)}{3b^3} \\ &+ d^{5/2} \sum_{\substack{a,c,a+b+c>0 \\ b^2-4ac=d}} \frac{3ab + 2ac + b^2}{3(b+2a)^2 b^2 (b+2c)^3}. \end{aligned} \tag{3.5}$$

Taking $a \leq 100$ and $|b| \leq 100$ in (3.5) gives the approximation 0.962405 to the value in (3.4) while taking $a \leq 500$ and $|b| \leq 500$ gives 0.962423.

As another example,

$$h(221) \log \epsilon_{221} = 4 \log\left(\frac{15+\sqrt{221}}{2}\right) = 10.8143\dots$$

is approximated by 10.8083 from (3.3) and by 10.8141 from (3.5) by taking $a \leq 2000$ and $|b| \leq 2000$.

Since the rational numbers given by the finite sums

$$R_1(d) = \sum_{\substack{[a,b,c] \text{ reduced} \\ b^2-4ac=d}} b^{-1} \quad \text{and} \quad R_2(d) = \sum_{\substack{[a,b,c] \text{ reduced} \\ b^2-4ac=d}} \frac{b(4b+a) - c(4a+b)}{3b^3}$$

give lower bounds for $L(1, \chi_d) = d^{-1/2}h(d) \log \epsilon_d$, it is obviously of interest (and no doubt extremely difficult) to estimate them from below. Numerically, the values of $R_j(d)$ for $j = 1, 2$ seem to account for at least a constant proportion of the value of $L(1, \chi_d)$, with the constant being larger for R_2 than for R_1 .

As will become clear, it is possible to give formulas of the same shape as those in (3.3) and (3.5) for the residue at $s = 1$ of an ideal class zeta function (and hence for $\log \epsilon_d$) by suitably restricting both the finite and infinite sums over $[a, b, c]$. Note that Kohnen and Zagier in [6, p. 223] gave the values of such zeta functions at $s = 1 - k$ for $k \in \{2, 3, 4, 5, 7\}$ as sums over reduced forms of certain polynomials in a, b, c .

4. Eisenstein series

It is instructive to sketch in some detail proofs of (2.3) and (2.4) of Theorem 1 that are prototypes for our proofs of Theorems 2 and 3. Define for $\tau \in \mathcal{H}$, the upper half-plane, and for $\text{Re}(s) > 1$ the Eisenstein series

$$E(\tau, s) = \sum_{g \in \Gamma_\infty \backslash \Gamma} (\text{Im } g\tau)^s.$$

Here Γ_∞ consists of the translations by integers in Γ . Let Δ be the (positive) hyperbolic Laplacian. Since $\Delta \text{Im } \tau = s(s - 1) \text{Im } \tau$ and Δ commutes with the usual linear fractional action $\tau \mapsto g\tau$, it follows that $E(\tau, s)$ is an eigenfunction of $-\Delta$:

$$-\Delta E(\tau, s) = s(1 - s)E(\tau, s).$$

It is a crucial result that $E(\tau, s)$ has a meromorphic continuation in s to \mathbb{C} and that it has a simple pole at $s = 1$ with residue that is constant. In fact

$$\text{res}_{s=1} E(\tau, s) = 3/\pi = \text{area } \Gamma \backslash \mathcal{H} \tag{4.1}$$

with respect to the usual invariant measure $d\mu(\tau)$.

To each real positive definite binary quadratic form $Q = [a, b, c]$ of discriminant d associate the point

$$\tau_Q = \frac{-b + \sqrt{d}}{2a} \in \mathcal{H}. \tag{4.2}$$

Note that for $\tau \mapsto g\tau$ the usual linear fractional action for $g \in \text{PSL}(2, \mathbb{R})$,

$$g\tau_Q = \tau_{gQ}. \tag{4.3}$$

Then it is straightforward to check using (2.1) that

$$E(\tau_Q, s) = (\sqrt{|d|}/2)^s \sum_{g \in \Gamma_\infty \backslash \Gamma} (gQ(1, 0))^{-s}. \tag{4.4}$$

Therefore for $d < 0$ we have

$$w_d (\sqrt{|d|}/2)^s Z_d(s) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_d} E(\tau_Q, s),$$

and (2.3) of Theorem 1 follows from this and (4.1).

The case $d > 0$ is more involved. Let S_Q be the oriented semi-circle defined by

$$a|\tau|^2 + b \text{Re } \tau + c = 0,$$

oriented counterclockwise if $a > 0$ and clockwise if $a < 0$. Given $z \in S_Q$ let C_Q be the directed arc on S_Q from z to the image of z under the canonical generator g_Q of the isotropy subgroup of Q , which is given by

$$g_Q z = \frac{(t/u + b)z + 2c}{-2az + t/u - b},$$

where t, u were defined in Theorem 1. We want to show that

$$d^{s/2} \frac{\Gamma(s/2)^2}{\Gamma(s)} Z_d(s) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_d} \int_{C_Q} E(\tau, s) d\tau_Q, \tag{4.5}$$

where $d\tau_Q = \frac{\sqrt{d} d\tau}{Q(\tau, 1)}$. Then the second formula of Theorem 1 follows by (4.1) and the fact that

$$\int_{C_Q} d\tau_Q = 2 \log \epsilon_d. \tag{4.6}$$

But (see e.g. [2, proof of Lemma 7])

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_d} \int_{C_Q} E(\tau, s) d\tau_Q = 2 \sum_{\substack{Q=[a,b,c] \\ a>0, 0 \leq b < 2a \\ b^2 - 4ac = d}} \int_{S_Q} (\text{Im } \tau)^s d\tau_Q. \tag{4.7}$$

Let $\tau = -\frac{b}{2a} + \frac{\sqrt{d}}{2a} e^{i\theta}$. Then using that $d\tau_Q = \frac{d\theta}{\sin \theta}$ we have

$$\begin{aligned} \int_{S_Q} (\text{Im } \tau)^s d\tau_Q &= \left(\frac{\sqrt{d}}{2a}\right)^s \int_0^\pi (\sin \theta)^{s-1} d\theta = \left(\frac{\sqrt{d}}{a}\right)^s \int_0^\infty u^{s-1} (1+u^2)^{-s} du \\ &= \frac{1}{2} d^{s/2} \frac{\Gamma(s/2)^2}{\Gamma(s)} a^{-s}, \end{aligned}$$

upon using the substitution $u = \tan \frac{\theta}{2}$ for which $\sin \theta = \frac{2u}{1+u^2}$ and $d\theta = \frac{2}{1+u^2} du$. This gives (4.5), hence (2.4) of Theorem 1.

5. Poincaré series

In this section we will prove Theorem 2 assuming the truth of Proposition 1 below, which we will prove in §7. This proposition asserts that a certain Poincaré series is in fact a constant and gives an analogue of the constant residue value of the Eisenstein series from (4.1). Set, for $\tau \in \mathcal{H}$ and $s_1, s_2, s_3 \in \mathbb{C}$,

$$H(\tau; s_1, s_2, s_3) = (\text{Im } \tau)^{s_1+s_2+s_3} |\tau|^{-2s_2} |\tau - 1|^{-2s_3}.$$

Lemma 1. *Suppose that $\sigma_j = \text{Re } s_j \geq 1$ for $j = 1, 2, 3$. Then the Poincaré series*

$$P(\tau) = P(\tau; s_1, s_2, s_3) = \sum_{g \in \Gamma} H(g\tau; s_1, s_2, s_3)$$

converges absolutely and uniformly for τ in any compact subset of \mathcal{H} . We have

$$P(g\tau) = P(\tau) \tag{5.1}$$

for any $g \in \Gamma$. Also, $P(\tau; s_1, s_2, s_3)$ is invariant under permutations of (s_1, s_2, s_3) and

$$P(-\bar{\tau}) = P(\tau). \tag{5.2}$$

Proof. Let

$$J(\tau; s_2, s_3) = \sum_{k \in \mathbb{Z}} |\tau + k|^{-2s_2} |\tau + k - 1|^{-2s_3}. \tag{5.3}$$

This sum converges to a continuous periodic function on \mathcal{H} . We can write

$$P(\tau; s_1, s_2, s_3) = \sum_{g \in \Gamma_\infty \setminus \Gamma} (\text{Im } g\tau)^{s_1+s_2+s_3} J(g\tau; s_2, s_3). \tag{5.4}$$

Suppose that for some constant $C > 1$ we have $\text{Im } \tau \leq C$. Separating the $k = 0$ and $k = 1$ terms from the sum in (5.3) shows that for such τ with $-1/2 \leq \text{Re } \tau \leq 1/2$ we have

$$J(\tau; s_2, s_3) \ll_C |\tau|^{-2\sigma}, \tag{5.5}$$

where $\sigma = \max(\sigma_2, \sigma_3)$. For $g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\text{Im } \tau \leq C$ we have

$$|g\tau|^{-1} = \left| \frac{c\tau + d}{a\tau + b} \right| \leq C(\text{Im } \tau)^{-1} |c\tau + d|. \tag{5.6}$$

This is trivial for $a = 0$ and follows from $|a\tau + b| \geq |\text{Im } \tau|$ otherwise. The sum (5.4) contains at most finitely many terms where $\text{Im } g\tau > C$. Also we may assume that $-1/2 \leq \text{Re } g\tau \leq 1/2$. Thus by (5.5) and (5.6) the sum in (5.4) is majorized by a constant multiple of

$$\sum_{(c,d)=1} |c\tau + d|^{-2\sigma_1 - 2\min(\sigma_2, \sigma_3)},$$

where the constant depends only on C . The claimed convergence follows from our assumption on s_1, s_2, s_3 . It is plain that this assumption can be weakened in various ways and we will still have convergence.

That $P(g\tau) = P(\tau)$ for all $g \in \Gamma$ is obvious and the symmetry of P in (s_1, s_2, s_3) follows from this and the easily verified identities

$$H\left(-\frac{1}{\tau}; s_1, s_2, s_3\right) = H(\tau; s_2, s_1, s_3) \quad \text{and} \quad H\left(\frac{-1}{\tau-1}; s_1, s_2, s_3\right) = H(\tau; s_2, s_3, s_1). \tag{5.7}$$

Now (5.2) also follows. ■

In analogy with (4.1) we have the following.

Proposition 1. *For $s_1 = s_2 = s_3 = 1$ the Poincaré series satisfies*

$$P(\tau; 1, 1, 1) = 3\pi/2. \tag{5.8}$$

Theorem 2 is an easy consequence of Proposition 1. First, we also have an analogue of (4.4) for our Poincaré series:

$$P(\tau_Q; s_1, s_2, s_3) = (\sqrt{|d|}/2)^{s_1+s_2+s_3} \sum_{g \in \Gamma} (gQ(1, 0))^{-s_1} (gQ(0, 1))^{-s_2} (gQ(1, 1))^{-s_3}. \tag{5.9}$$

A calculation using (5.9) and (2.1) gives

$$w_d(\sqrt{|d|}/2)^{s_1+s_2+s_3} \sum_{\substack{a>0 \\ b^2-4ac=d}} a^{-s_1}c^{-s_2}(a+b+c)^{-s_3} = \sum_{Q \in \Gamma \backslash \mathcal{Q}_d} P(\tau_Q; s_1, s_2, s_3). \tag{5.10}$$

Theorem 2 now follows from (5.10) and Proposition 1 by taking $s_1 = s_2 = s_3 = 1$, the convergence of the series following from that of $P(\tau; 1, 1, 1)$.

We remark that the fact that $P(\tau_Q; 1, 1, 1) = 3\pi/2$ for $P(\tau_Q; 1, 1, 1)$ in the form (5.9) was obtained by Hurwitz (see [4, (8), p. 200]).

6. Proof of Theorem 3

We turn now to the proof of Theorem 3, again assuming Proposition 1. As before, the case $d > 0$ is harder. Similarly to (4.7) we have

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_d} \int_{C_Q} P(\tau; s_1, s_2, s_3) d\tau_Q = 2 \sum_{\substack{Q=[a,b,c] \\ a>0 \\ b^2-4ac=d}} I_Q(s_1, s_2, s_3), \tag{6.1}$$

where

$$I_Q = I_Q(s_1, s_2, s_3) = \int_{S_Q} (\text{Im } \tau)^{s_1+s_2+s_3} |\tau|^{-2s_2} |\tau - 1|^{-2s_3} d\tau_Q.$$

The factor of 2 in (6.1) is due to the fact that the sum is restricted to $a > 0$. As before let $\tau = -\frac{b}{2a} + \frac{\sqrt{d}}{2a} e^{i\theta}$. Using that $d\tau_Q = \frac{d\theta}{\sin \theta}$, $|\tau|^2 = \frac{1}{(2a)^2} [b^2 + d - 2b\sqrt{d} \cos \theta]$ and

$$|\tau - 1|^2 = \frac{1}{(2a)^2} [(2a + b)^2 + d - 2(2a + b)\sqrt{d} \cos \theta]$$

we get

$$\begin{aligned} I_Q(s_1, s_2, s_3) &= \frac{d^{(s_1+s_2+s_3)/2}}{(2a)^{s_1-s_2-s_3}} \int_0^\pi \frac{(\sin \theta)^{s_1+s_2+s_3-1} d\theta}{(b^2 + d - 2b\sqrt{d} \cos \theta)^{s_2} ((2a + b)^2 + d - 2(2a + b)\sqrt{d} \cos \theta)^{s_3}}. \end{aligned}$$

Making the substitution $u = \tan \frac{\theta}{2}$ so that $\cos \theta = \frac{1-u^2}{1+u^2}$, $\sin \theta = \frac{2u}{1+u^2}$ and $d\theta = \frac{2}{1+u^2} du$ yields

$$\begin{aligned} I_Q &= \frac{4^{s_2+s_3} d^{(s_1+s_2+s_3)/2}}{a^{s_1-s_2-s_3}} \\ &\times \int_0^\infty u^{s_1+s_2+s_3-1} (1+u^2)^{-s_1} (A'^2 + A^2u^2)^{-s_2} (B'^2 + B^2u^2)^{-s_3} du, \tag{6.2} \end{aligned}$$

where

$$A = b + \sqrt{d}, A' = b - \sqrt{d}, B = b + 2a + \sqrt{d} \quad \text{and} \quad B' = b + 2a - \sqrt{d}. \quad (6.3)$$

We now restrict to $s_1 = s_2 = s_3 = 1$.

Lemma 2. For $A, A', B, B' \in \mathbb{R}$ we have

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty \frac{u^2 du}{(1 + u^2)(A'^2 + A^2u^2)(B'^2 + B^2u^2)} \\ = \frac{1}{(|A| + |A'|)(|B| + |B'|)(|AB'| + |A'B|)}. \end{aligned}$$

Proof. The integral can be evaluated easily using partial fractions. ■

We must split the evaluation of $I_Q = I_Q(1, 1, 1)$ in (6.2) into four cases depending on the signs of AA' and BB' . Denote by $I_Q^{\pm, \pm}$ the corresponding value of I_Q according to these signs. We have

$$\begin{aligned} I_Q^{+,+} &= \frac{d^{3/2}\pi}{2b(b + 2c)(b + 2a)}, & I_Q^{+,-} &= -\frac{d^{1/2}\pi}{2b}, \\ I_Q^{-,+} &= \frac{d^{1/2}\pi}{2(b + 2a)}, & I_Q^{-,-} &= -\frac{d^{1/2}\pi}{2(b + 2c)}. \end{aligned} \quad (6.4)$$

By (6.1), (6.4), Proposition 1 and (4.6) we have

$$h(d) \log \epsilon_d = \frac{1}{3}d^{3/2} \sum_{\substack{b^2 > d \\ (2a+b)^2 > d}} \frac{1}{b(2a + b)(b + 2c)} + d^{1/2}R_1(d), \quad (6.5)$$

where

$$3R_1(d) = - \sum_{\substack{b^2 > d \\ (2a+b)^2 < d}} \frac{1}{b} + \sum_{\substack{b^2 < d \\ (2a+b)^2 > d}} \frac{1}{2a + b} - \sum_{\substack{b^2 < d \\ (2a+b)^2 < d}} \frac{1}{b + 2c}. \quad (6.6)$$

All sums are over a, b, c with $a > 0$ and satisfying $b^2 - 4ac = d$. The positivity of the terms and the convergence of the infinite series follows from that of the Poincaré series and (6.1). The finite sum $R_1(d)$ may be simplified.

Lemma 3. For $R_1(d)$ in (6.6) and all discriminants $d > 0$ we have the identity

$$R_1(d) = \sum_{\substack{a > 0, c > 0 \\ b > a+c}} b^{-1},$$

the sum being over all Zagier reduced forms of discriminant d , which is a finite sum.

Proof. Since $a > 0$, elementary calculations give that $(2a - b)^2 < d$ if and only if $a + c < b$ and that $b^2 > d$ if and only if $c > 0$. Hence for the first sum in (6.6) we have

$$- \sum_{\substack{b^2 > d \\ (2a+b)^2 < d}} \frac{1}{b} = \sum_{\substack{b^2 > d \\ (2a-b)^2 < d}} \frac{1}{b} = \sum_{\substack{a > 0, c > 0 \\ b > a+c}} \frac{1}{b}.$$

By mapping $(a, b, c) \mapsto (a, 2a + b, a + b + c)$ we see that

$$\sum_{\substack{b^2 > d \\ (2a-b)^2 < d}} \frac{1}{b} = \sum_{\substack{b^2 < d \\ (2a+b)^2 > d}} \frac{1}{2a + b},$$

which is the second sum in (6.6). Mapping $(a, b, c) \mapsto (b - a - c, 2a - b, -a)$ in the third sum yields

$$\sum_{\substack{a > 0, b^2 < d \\ (2a+b)^2 < d}} \frac{-1}{b + 2c} = \sum_{\substack{a > 0, c > 0 \\ b > a+c}} \frac{1}{b},$$

which finishes the proof. ■

Theorem 3 now follows from (6.5) since the conditions in the infinite sum in (6.5) $a > 0, b^2 > d$ and $(2a + b)^2 > d$ are equivalent to $a, c, a + b + c > 0$.

7. Point-pair invariant

In this section we will prove Proposition 1 and hence finish the proofs of Theorems 2 and 3. Instead of following Hurwitz we will obtain it as a simple consequence of Selberg’s theory [9] of point-pair invariants.

Let $\delta(\tau, z)$ be the hyperbolic distance between $z, \tau \in \mathcal{H}$. It is well known that

$$\cosh \delta(\tau, z) = \frac{|\tau - z|^2}{2 \operatorname{Im}(\tau) \operatorname{Im}(z)} + 1. \tag{7.1}$$

Define

$$k(\tau, z) = (\cosh \delta(\tau, z))^{-3}, \tag{7.2}$$

which is a point-pair invariant in that $k(\tau, z) = k(g\tau, gz)$ for any $g \in \operatorname{PSL}(2, \mathbb{R})$. The associated Γ -invariant kernel is

$$K(\tau, z) = \sum_{g \in \Gamma} k(g\tau, z).$$

Recall from [9] that an eigenfunction ϕ of the Laplacian Δ is also an eigenfunction of the invariant integral operator

$$\phi \mapsto \int_{\Gamma \backslash \mathcal{H}} \phi(z) K(\tau, z) d\mu(z).$$

Therefore $\int_{\Gamma \backslash \mathcal{H}} K(\tau, z) d\mu(z) = c$ is constant since 1 is an eigenfunction of Δ and it is easy to compute that $c = \pi$. Proposition 1 follows from the next lemma.

Lemma 4. For $\tau \in \mathcal{H}$,

$$\int_{\Gamma \setminus \mathcal{H}} K(\tau, z) d\mu(z) = \frac{2}{3} P(\tau; 1, 1, 1).$$

Proof. Let $\mathcal{F}_1 = \{z = x + iy : 0 \leq x \leq 1/2 \text{ and } |z - 1| \geq 1\}$ be the hyperbolic triangle with vertices at $0, e^{\pi i/3}$ and ∞ . Note that \mathcal{F}_1 is obtained from the standard closed fundamental domain \mathcal{F} for Γ by mapping the left-hand half of \mathcal{F} to the hyperbolic triangle with corners at $0, i$ and $e^{\pi i/3}$ by using the inversion $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus \mathcal{F}_1 is also a closed fundamental domain for Γ . Let

$$\mathcal{F}_2 = \{z = x + iy : 0 \leq x \leq 1 \text{ and } |z - 1/2| \geq 1/2\}.$$

Then $\mathcal{F}_2 = \mathcal{F}_1 \cup ST^{-1}\mathcal{F}_1 \cup TS\mathcal{F}_1$, where $T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Any $\tau \in \mathcal{H}$ can be expressed uniquely as $\tau = \tau_Q$ for a positive definite real Q with $\text{disc}(Q) = 1$. For this Q we have

$$\begin{aligned} 3 \int_{\Gamma \setminus \mathcal{H}} K(\tau_Q, z) d\mu(z) &= 3 \int_{\mathcal{F}_1} K(\tau_Q, z) d\mu(z) \\ &= \int_{\mathcal{F}_2} K(\tau_Q, z) d\mu(z) \\ &= \int_{\mathcal{F}_2} \sum_{g \in \Gamma} k(g\tau_Q, z) d\mu(z) \\ &= \sum_{g \in \Gamma} \int_{\mathcal{F}_2} k(\tau_{gQ}, z) d\mu(z) \end{aligned} \tag{7.3}$$

by (4.3). A straightforward calculation using (4.2), (7.1) and (7.2) when $z = x + iy$ and $Q = [a, b, c]$ has $\text{disc}(Q) = 1$ gives

$$k(\tau_Q, z) = y^3 (c + bx + a(x^2 + y^2))^{-3}.$$

By an elementary substitution

$$\int_{\sqrt{1/4 - (x-1/2)^2}}^{\infty} \frac{y dy}{(c + bx + a(x^2 + y^2))^3} = \frac{1}{4a(x(a + b) + c)^2},$$

from which follows the evaluation

$$\int_{\mathcal{F}_2} k(\tau_Q, z) d\mu(z) = \frac{1}{4Q(1, 0)Q(0, 1)Q(1, 1)}. \tag{7.4}$$

We apply this in (7.3) and refer to (5.9) to finish the proof. ■

This completes the proofs of Theorems 2 and 3.

8. Variations and extensions

As we alluded to above, Hurwitz generalized Theorem 2. His general formula for any integer $m \geq 0$ can be written as

$$H(-d) = \frac{m+1}{6\pi} |d|^{m+3/2} \sum \frac{1}{a(a+b+c)c} \left(\frac{t_0 t_1}{a(a+b+c)} + \frac{t_1 t_2}{(a+b+c)c} + \frac{t_2 t_0}{ca} \right)^m, \tag{8.1}$$

where the sum is over $a > 0$ and $d = b^2 - 4ac$ as before. Here he uses symbolic notation so we must replace $t_0^{l_0} t_1^{l_1} t_2^{l_2}$ by

$$\frac{l_0! l_1! l_2!}{(l_0 + l_1 + l_2 + 2)!}$$

everywhere in the expansion (even when $m = 0$). We may group together any two monomials whose pattern of exponents can be permuted to one another since they yield the same sum over a, b, c (see the last statement of Lemma 1 and (5.10)). Taking $m = 1$ yields (3.2) since the monomials have exponent patterns $(1, 1, 0)$, $(0, 1, 1)$ and $(1, 0, 1)$. The next case $m = 2$ gives

$$\frac{1}{w_d} h(d) = \frac{1}{120\pi} |d|^{7/2} \left(\sum \frac{1}{a^3(a+b+c)^3c} + \sum \frac{1}{a^3(a+b+c)^2c^2} \right). \tag{8.2}$$

There are at least two ways to generalize our arguments to obtain (8.1). Both come down to applying the same technique to linear combinations of Poincaré series that are constant. One way these combinations can be obtained is by applying the Laplace operator repeatedly to $P(\tau, 1, 1, 1)$ and using the following readily established formula:

$$\begin{aligned} \Delta P(\tau; s_1, s_2, s_3) &= \sigma(\sigma - 1)P(\tau; s_1, s_2, s_3) - 4s_1s_2P(\tau; s_1 + 1, s_2 + 1, s_3) \\ &\quad - 4s_1s_3P(\tau; s_1 + 1, s_2, s_3 + 1) - 4s_2s_3P(\tau; s_1, s_2 + 1, s_3 + 1), \end{aligned}$$

where $\sigma = s_1 + s_2 + s_3$. By applying this together with Proposition 1 and using the symmetry of P in (s_1, s_2, s_3) from Lemma 1 we get

$$P(\tau; 2, 2, 1) = 3\pi/4, \tag{8.3}$$

from which (8.1) for $m = 1$ follows. An inductive argument gives (8.1).

Another way to get these combinations that is closer to Hurwitz’s method is to generalize (7.4) by using the point-pair invariant

$$k_m(\tau, z) = (\cosh \delta(\tau, z))^{-(2m+3)}$$

and the invariant kernel $K_m(\tau, z) = \sum_{g \in \Gamma} k_m(\tau, gz)$. We can show when $\text{disc}(Q) = 1$ that

$$\int_{\mathcal{F}_2} k_m(\tau_Q, z) d\mu(z) = \frac{T_m(a, b, c)}{(a(a+b+c)c)^{m+1}},$$

where $T_m(a, b, c)$ is a homogeneous polynomial with rational coefficients of degree m recursively given. Furthermore,

$$\int_{\Gamma \backslash \mathcal{X}} K_m(\tau, z) d\mu(z) = \frac{\pi}{m + 1},$$

and we may proceed as before.

Proof of formula (3.5)

The constant linear combinations of Poincaré series we get can be used in the indefinite case as well. Thus in order to prove formula (3.5) we apply the following analog of Lemma 2.

Lemma 5. For $A, A', B, B' \in \mathbb{R}$ we have

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty \frac{u^4 du}{(1 + u^2)(A'^2 + A^2u^2)^2(B'^2 + B^2u^2)^2} \\ = \frac{|A|(|B| + 2|B'|) + |A'|(|B'| + 2|B|)}{2(|A| + |A'|)^2(|B| + |B'|)^2(|AB'| + |A'B|)^3}. \end{aligned}$$

As before, putting the values from (6.3) into the formula of Lemma 5 and using (6.1), (4.6) and (8.3) shows that for $d > 0$ a fundamental discriminant,

$$h(d) \log \epsilon_d = d^{1/2} R_2(d) + \frac{1}{3} d^{5/2} \sum_{\substack{a>0, b^2>d \\ (2a+b)^2>d \\ b^2-4ac=d}} \frac{3ab + 2ac + b^2}{(b + 2a)^2 b^2 (b + 2c)^3},$$

where

$$3R_2(d) = \sum_{\substack{b^2>d \\ (2a-b)^2<d}} \frac{a + b}{b^2} + \sum_{\substack{b^2<d \\ (2a+b)^2>d}} \frac{3a + b}{(2a + b)^2} + \sum_{\substack{b^2<d \\ (2a+b)^2<d}} \frac{a(b + 6c) - b^2}{(b + 2c)^3}. \tag{8.4}$$

The infinite sum is absolutely convergent and an argument similar to that in the proof of Lemma 3 shows that the first two sums of (8.4) are equal. Then we make the change of variables $(a, b, c) \rightarrow (b - a - c, 2a - b, -a)$ in the third sum and add the three sums together to get

$$h(d) \log \epsilon_d = d^{1/2} \sum_{[a,b,c] \text{ reduced}} b^{-1} + d^{3/2} \sum_{\substack{a>0, b^2>d \\ (2a+b)^2>d \\ b^2-4ac=d}} \frac{1}{3(b + 2a)b(b + 2c)}.$$

As before the conditions $a > 0, b^2 > d$ and $(2a + b)^2 > d$ in the sum are equivalent to $a, c, a + b + c > 0$ and this yields the formula (3.5).

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