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On the Moy–Prasad filtration

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Abstract. Let K be a maximal unramified extension of a non-archimedean local field with arbitrary residual characteristic p. Let G be a reductive group over K which splits over a tamely ramified extension of K. We show that the associated Moy–Prasad filtration representations are in a certain sense independent of p. We also establish descriptions of these representations in terms of explicit Weyl modules and as representations occurring in a generalized Vinberg–Levy theory.

As an application, we provide necessary and sufficient conditions for the existence of stable vectors in Moy–Prasad filtration representations, which extend earlier results by Reeder and Yu (which required p to be large) and by Romano and the present author (which required G to be absolutely simple and split). This yields new supercuspidal representations.

We also treat reductive groups G that are not necessarily split over a tamely ramified field extension.

Keywords. Moy–Prasad filtration, reductive group schemes, stable vectors, supercuspidal representations, Weyl modules

Contents

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1. Introduction

The introduction of Moy–Prasad filtrations in the 1990s revolutionized the study of the representation theory of p-adic groups. As one example, their introduction enabled a construction of supercuspidal representations – the building blocks in the representation theory of p-adic groups – that is exhaustive for large primes p under certain tameness assumptions. However, while this and similar advances are remarkable, the restrictions on the prime p are unsatisfying. Given their critical role, we expect that a better understanding of the Moy–Prasad filtrations will be a key ingredient for future progress. To that end, we introduce a "global" model for the Moy–Prasad filtration quotients. This allows us to compare the Moy–Prasad filtrations for different primes p and to deduce results for *all* primes p that were previously only known for large primes. Our global model also enables us to express the Moy–Prasad filtration quotients in terms of more traditional, well studied concepts, e.g. as explicit Weyl modules or in terms of a generalized Vinberg–Levy theory. As an application, we exhibit new supercuspidal representations for non-split p-adic groups, including non-tame groups.

To explain the content and background of the paper in more detail, let us introduce some notation. Let k be a non-archimedean local field with residual characteristic $p > 0$. Let K be a maximal unramified extension of k and identify its residue field with $\overline{\mathbb{F}}_p$. Let G be a (connected) reductive group over K. In [2, 3], Bruhat and Tits defined a building $\mathcal{B}(G, K)$ associated to G. For each point x in $\mathcal{B}(G, K)$, they constructed a bounded subgroup G_x of $G(K)$, called a parahoric subgroup. In [14, 15], Moy and Prasad defined a filtration of these parahoric subgroups by smaller subgroups

$$G_x = G_{x,0} \triangleright G_{x,r_1} \triangleright G_{x,r_2} \triangleright \cdots ,$$

where $0 < r_1 < r_2 < \cdots$ are real numbers depending on x. For simplicity, we assume that $r_1, r_2, \ldots$ are rational numbers. The quotient $G_{x,0}/G_{x,r_1}$ can be identified with the $\overline{\mathbb{F}}_p$-points of a reductive group $\mathbf{G}_x$, and $G_{x,r_i}/G_{x,r_{i+1}}$ $(i > 0)$ can be identified with an $\overline{\mathbb{F}}_p$-vector space $\mathbf{V}_{x,r_i}$ on which $\mathbf{G}_x$ acts.

Results about Moy–Prasad filtrations. We show for a large class of reductive groups G, which we call good groups (see Definition 3.1.1), that Moy–Prasad filtrations are in a certain sense (made precise below) independent of the residue field characteristic p. The class of good groups contains reductive groups that split over a tamely ramified field extension (which is the class that many authors restrict to), as well as simply connected and adjoint semisimple groups, and products and restriction of scalars along finite separable (not necessarily tamely ramified) field extensions of any of these. The restriction to this (large) subclass of reductive groups is necessary as the main result (Theorem 3.4.1) fails in general (see Remark 3.4.2). Given a good reductive group G over K, where K is a maximal unramified extension of k as above, a point x of the Bruhat–Tits building $\mathcal{B}(G, K)$ as above, and an arbitrary prime q coprime to a certain integer N that depends on the splitting field of G (N is coprime to p, for details see Definition 3.1.1), we construct a finite extension K_q of $\mathbb{Q}_q^{\text{ur}}$, a reductive group G_q over K_q and a point x_q in $\mathcal{B}(G_q, K_q)$. To these data, one can attach a Moy–Prasad filtration as above. The corresponding reductive quotient $\mathbf{G}_{x_q}$ is a reductive group over $\overline{\mathbb{F}}_q$ that acts on the quotients $\mathbf{V}_{x_q,r_i}$, which are identified with $\overline{\mathbb{F}}_q$-vector spaces. For a given positive integer i, we show in Theorem 3.4.1 that there exists a split reductive group scheme $\mathcal{H}$ over $\overline{\mathbb{Z}}[1/N]$ acting on a free $\overline{\mathbb{Z}}[1/N]$-module $\mathcal{V}$ such that the special fiber of this representation over $\overline{\mathbb{F}}_q$ is the above constructed Moy–Prasad filtration representations of $\mathbf{G}_{x_q}$ on $\mathbf{V}_{x_q,r_i}$ for all q coprime to N, and the special fiber over $\overline{\mathbb{F}}_p$ is the Moy–Prasad filtration representations of $\mathbf{G}_x$ on $\mathbf{V}_{x,r_i}$. This allows us to compare the Moy–Prasad filtration representations for different primes.

We also give a new description of the Moy–Prasad filtration representations, i.e. of $\mathbf{G}_x$ acting on $\mathbf{V}_{x,r_i}$, for reductive groups that split over a tamely ramified field extension of K. Let m be the order of x (see §3.2 for the definition of "order"). We define an action of the group scheme $\boldsymbol{\mu}_m$ of m-th roots of unity on a reductive group $\mathcal{G}_{\overline{\mathbb{F}}_p}$ over $\overline{\mathbb{F}}_p$, and denote by $\mathcal{G}_{\overline{\mathbb{F}}_p}^{\boldsymbol{\mu}_m,0}$ the identity component of the fixed-point group scheme. In addition, we define a related action of $\boldsymbol{\mu}_m$ on the Lie algebra $\operatorname{Lie}(\mathcal{G}_{\overline{\mathbb{F}}_p})$, which yields a decomposition $\operatorname{Lie}(\mathcal{G}_{\overline{\mathbb{F}}_p})(\overline{\mathbb{F}}_p) = \bigoplus_{i=1}^m \operatorname{Lie}(\mathcal{G}_{\overline{\mathbb{F}}_p})_i(\overline{\mathbb{F}}_p)$. Then we prove that the action of $\mathbf{G}_x$ on $\mathbf{V}_{x,r_i}$ corresponds to the action of $\mathcal{G}_{\overline{\mathbb{F}}_p}^{\boldsymbol{\mu}_m,0}$ on one of the graded pieces $\operatorname{Lie}(\mathcal{G})_j(\overline{\mathbb{F}}_p)$ of the Lie algebra of $\mathcal{G}_{\overline{\mathbb{F}}_p}$. This was previously known by [20] for sufficiently large primes p, and representations of the latter kind have been studied by Vinberg [24] in characteristic zero and generalized to positive characteristic coprime to m by Levy [13]. To be precise, in this paper we even prove a global version of the above mentioned result. See Theorem 4.1.1 for details. We also show that the same statement holds true for all good reductive groups after base change of $\mathcal{H}$ and $\mathcal{V}$ to $\overline{\mathbb{Q}}$ (see Corollary 4.2.1).

Moreover, the global version of the Moy–Prasad filtration representations given by Theorem 3.4.1 allows us to describe the representations occurring in the Moy–Prasad filtrations of good reductive groups explicitly in terms of Weyl modules; see Section 6 for precise formulas.

An application to supercuspidal representations. Suppose G is defined over k. In 1998, Adler [1] used the Moy–Prasad filtrations to construct supercuspidal representations of $G(k)$, and Yu [25] generalized his construction three years later. If G splits over a tamely ramified extension of k and p does not divide the order of the Weyl group of G, then

Yu's construction yields all supercuspidal representations [6, 12]. However, it is known that the construction does not give rise to all supercuspidal representations for small primes p.

In 2014, Reeder and Yu [20] gave a new construction of supercuspidal representations of smallest positive depth, which they called epipelagic representations. A vector in the dual $\check{\mathbf{V}}_{x,r_1} = (G_{x,r_1}/G_{x,r_2})^\vee$ of the first Moy–Prasad filtration quotient is called stable (in the sense of geometric invariant theory) if its orbit under $\mathbf{G}_x$ is closed and its stabilizer in $\mathbf{G}_x$ is finite. The only input for the new construction of supercuspidal representations in [20] is such a stable vector. Assuming that G is a semisimple group that splits over a tamely ramified field extension, Reeder and Yu gave a necessary and sufficient criterion for the existence of stable vectors for sufficiently large primes p. In [7], Romano and the present author removed the assumption on the prime p for absolutely simple split reductive groups G, which yielded new supercuspidal representations for split groups.

One application of our results on Moy–Prasad filtrations is a criterion for the existence of stable vectors for all primes p for a much larger class of semisimple groups (see Corollary 5.2.2). As a consequence we obtain new supercuspidal representations for a class of non-split p-adic reductive groups, including non-tame groups. Note that the assumption that the reductive group is semisimple is not crucial. One can easily generalize the construction of Reeder and Yu to non-semisimple reductive groups by considering the center. Similarly, we prove in Theorem 5.1.1 that the existence of semistable vectors is independent of the residue field characteristic. Semistable vectors play an important role when moving from epipelagic representations to representations of higher depth.

Structure of the paper. In Section 2, we first recall the Moy–Prasad filtration of G, and then in §2.5 we introduce a Chevalley system for the reductive quotient that will be used for the construction of the reductive group scheme $\mathcal{H}$ that appears in Theorem 3.4.1. In §2.6, we construct an inclusion of the Moy–Prasad filtration representation of G into that of G_F for a sufficiently large field extension F of K that will allow us to define the action of $\mathcal{H}$ on $\mathcal{V}$ in Theorem 3.4.1. Afterwards, in Section 3, we move from a previously fixed residue field characteristic p to other residue field characteristics q. More precisely, we first introduce the notion of a good group and define $K_q/\mathbb{Q}_q^{\mathrm{ur}}$, G_q over K_q, and $x_q \in \mathcal{B}(G_q, K_q)$. In §3.4, we prove our first main theorem, Theorem 3.4.1. Section 4 is devoted to giving a different description of the Moy–Prasad filtration representations and their global version as generalized Vinberg–Levy representations (Theorem 4.1.1). In Section 5, we use the results of the previous sections to show that the existence of (semi)stable vectors is independent of the residue characteristic. This leads to new supercuspidal representations. We conclude the paper by giving a description of the Moy–Prasad filtration representations in term of Weyl modules in Section 6.

Conventions and notation. If M is a free module over some ring A, and if there is no danger of confusion, then we denote the associated scheme whose functor of points is $B \mapsto M \otimes_A B$ for any A-algebra B also by M. In addition, if G and T are schemes over a scheme S, then we may abbreviate the base change $G \times_S T$ by G_T; and if $T = \operatorname{Spec} A$ for some ring A, then we may also write G_A instead of G_T.

When we talk about the identity component of a smooth group scheme G of finite presentation, we mean the unique open subgroup scheme whose fibers are the connected components of the respective fibers of the original scheme that contains the identity. The identity component of G will be denoted by G^0. If G is a group scheme defined over a ring R, then $\mathrm{Lie}(G)$ denotes the corresponding Lie algebra functor over R, and if $f : G \to H$ is a map between group schemes over R, then we write $\mathrm{Lie}(f)$ for the corresponding induced map $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$.

Throughout the paper, we require reductive groups to be connected.

For each prime number q, we fix an algebraic closure $\overline{\mathbb{Q}}_q$ of $\mathbb{Q}_q$ and an algebraic closure $\overline{\mathbb{F}_q((t))}$ of $\mathbb{F}_q((t))$. All algebraic field extensions of $\mathbb{Q}_q$ and $\mathbb{F}_q((t))$ are assumed to be contained in $\overline{\mathbb{Q}}_q$ and $\overline{\mathbb{F}_q((t))}$, respectively. We then denote by $\mathbb{Q}_q^{\mathrm{ur}}$ the maximal unramified extension of $\mathbb{Q}_q$ (inside $\overline{\mathbb{Q}}_q$), and by $\mathbb{F}_q((t))^{\mathrm{ur}}$ the maximal unramified extension of $\mathbb{F}_q((t))$. For any field extension F of $\mathbb{Q}_q$ (or of $\mathbb{F}_q((t))$), we denote by F^{tame} its maximal tamely ramified field extension. Similarly, we fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and we denote by $\overline{\mathbb{Z}}$ the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$ and by $\overline{\mathbb{Z}}_q$ the integral closure of $\mathbb{Z}_q$ in $\overline{\mathbb{Q}}_q$.

In addition, we will use the following notation throughout the paper: p denotes a fixed prime number, k is a non-archimedean local field (of arbitrary characteristic) with residual characteristic p, and K is the maximal unramified extension of k contained in the fixed algebraic closure above. We write $\mathcal{O}$ for the ring of integers of K, $\mathrm{v} : K \to \mathbb{Z} \cup \{\infty\}$ for the valuation on K with image $\mathbb{Z} \cup \{\infty\}$, and ϖ for a uniformizer of K. We denote by v also the unique extension of the valuation v to a discrete valuation on a finite field extension of K. Let G be a reductive group over K, and let E denote a splitting field of G, i.e., E is a minimal field extension of K such that G_E is split. Note that all reductive groups over K are quasi-split and hence E is unique. Let e be the degree of E over K, $\mathcal{O}_E$ the ring of integers of E, and ϖ_E a uniformizer of E. Without loss of generality, we assume that ϖ is chosen to equal ϖ_E^e modulo $\varpi_E^{e+1}\mathcal{O}_E$. We denote the (absolute) root datum of G by $R(G)$, and its root system by $\Phi = \Phi(G)$. We fix a point x in the (reduced) Bruhat–Tits building $\mathcal{B}(G, K)$ of G, denote by S a maximal split torus of G such that x is contained in the apartment $\mathcal{A}(S, K)$ associated to S, and let T be the centralizer of S, which is a maximal torus of G. Moreover, we fix a Borel subgroup B of G containing T, which yields a choice of simple roots Δ and positive roots Φ^+ in Φ. In addition, we denote by $\Phi_K = \Phi_K(G)$ the restricted root system of G, i.e., the restrictions of the roots in Φ from T to S. For $a \in \Phi_K$, we denote its preimage in Φ by Φ_a.

Moreover, to help the reader, we will adhere to the convention of labeling roots in Φ by Greek letters: $\alpha, \beta, \dots$, and roots in Φ_K by Latin letters: $a, b, \dots$.

2. Parahoric subgroups and Moy–Prasad filtration

In order to talk about the Moy–Prasad filtration, we will first recall the structure of the root groups following [3, Section 4]. For more details and proofs we refer to *loc. cit.*

2.1. Chevalley–Steinberg system

For $\alpha \in \Phi$, we denote by U_α^E the root subgroup of G_E corresponding to α. Note that $\mathrm{Gal}(E/K)$ acts on Φ. We denote by E_α the fixed subfield of E of the stabilizer $\mathrm{Stab}_{\mathrm{Gal}(E/K)}(\alpha)$ of α in $\mathrm{Gal}(E/K)$. In order to parametrize the root groups of G over K, we fix a *Chevalley–Steinberg system* $\{x_\alpha^E : \mathbb{G}_a \to U_\alpha^E\}_{\alpha\in\Phi}$ of G with respect to T, i.e. a Chevalley system $\{x_\alpha^E : \mathbb{G}_a \to U_\alpha^E\}_{\alpha\in\Phi}$ of G_E (see Remark 2.1.1) satisfying the following additional properties for all roots $\alpha \in \Phi$:

(i) The isomorphism $x_\alpha^E : \mathbb{G}_a \to U_\alpha^E$ is defined over E_α.

(ii) If the restriction $a \in \Phi_K$ of α to S is not divisible, i.e. $a/2 \notin \Phi_K$, then

$$x_{\gamma(\alpha)}^E = \gamma \circ x_\alpha^E \circ \gamma^{-1} \quad \text{for all } \gamma \in \mathrm{Gal}(E/K).$$

(iii) If the restriction $a \in \Phi_K$ of α to S is divisible, then there exist $\beta, \beta' \in \Phi$ restricting to $a/2$ such that $E_\beta = E_{\beta'}$ is a quadratic extension of E_α, and

$$x_{\gamma(\alpha)}^E = \gamma \circ x_\alpha^E \circ \gamma^{-1} \circ \epsilon \quad \text{for all } \gamma \in \mathrm{Gal}(E/E_\alpha),$$

where $\epsilon \in \{\pm 1\}$ is 1 if and only if γ induces the identity on E_β.

According to [3, 4.1.3], which is based on [22], such a Chevalley–Steinberg system does exist. It is a generalization of a Chevalley system to non-split groups and it will allow us to define a valuation of root groups in §2.2 even if the group G is non-split.

Remark 2.1.1. We follow the conventions resulting from [8, XXIII Définition 6.1], so we do not add the requirement of Bruhat and Tits that for each root α, x_α^E and $x_{-\alpha}^E$ are associated, i.e. $x_\alpha^E(1)x_{-\alpha}^E(1)x_\alpha^E(1)$ is contained in the normalizer of T. However, there exists $\epsilon_{\alpha,\alpha} \in \{1, -1\}$ such that

$$m_\alpha := x_\alpha^E(1)x_{-\alpha}^E(\epsilon_{\alpha,\alpha})x_\alpha^E(1)$$

is contained in the normalizer of T. Moreover, $\mathrm{Ad}(m_\alpha)(\mathrm{Lie}(x_\alpha^E)(1)) = \epsilon_{\alpha,\alpha}\,\mathrm{Lie}(x_{-\alpha}^E)(1)$.

Definition 2.1.2. For $\alpha, \beta \in \Phi$, we define $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ by

$$\mathrm{Ad}(m_\alpha)(\mathrm{Lie}(x_\beta)(1)) = \epsilon_{\alpha,\beta}\,\mathrm{Lie}(x_{s_\alpha(\beta)})(1),$$

where s_α denotes the reflection in the Weyl group W of $\Phi(G)$ corresponding to α. The integers $\epsilon_{\alpha,\beta}$ for α and β in Φ are called the *signs* of the Chevalley–Steinberg system $\{x_\alpha^E\}_{\alpha\in\Phi}$.

2.2. Parametrization and valuation of root groups

In this section, we associate a parametrization and a valuation to each root group of G.

Let $a \in \Phi_K = \Phi_K(G)$, and let U_a be the corresponding *root subgroup* of G, i.e., the connected unipotent (closed) subgroup of G normalized by S whose Lie algebra is the sum of the root spaces corresponding to the roots that are a positive integral multiple of a.

Let G_a be the subgroup of G generated by U_a and U_{-a}, and let $\pi : G^a \to G_a$ be a simply connected cover. Note that π induces an isomorphism between a root group U_+ of G^a and U_a. We call U_+ the *positive root group* of G^a. In order to describe the root group U_a, we distinguish two cases.

Case 1: The root $a \in \Phi_K$ is neither divisible nor multipliable, i.e. $a/2$ and $2a$ are both not in Φ_K.

Let $\alpha \in \Phi_a$ be a root that equals a when restricted to S. Then G^a is isomorphic to the Weil restriction $\mathrm{Res}_{E_\alpha/K} \mathrm{SL}_2$ of SL_2 over E_α to K, and $U_a \simeq \mathrm{Res}_{E_\alpha/K} U_\alpha^E$, where U_α^E is the root group of G_E corresponding to α as above. Note that $(U_a)_E$ is the product $\prod_{\beta\in\Phi_a} U_\beta^E$. Using the E_α-isomorphism $x_\alpha^E : \mathbb{G}_\mathrm{a} \to U_\alpha^E$, we obtain a K-isomorphism

$$x_a := \mathrm{Res}_{E_\alpha/K}\, x_\alpha^E : \mathrm{Res}_{E_\alpha/K}\, \mathbb{G}_\mathrm{a} \to \mathrm{Res}_{E_\alpha/K}\, U_\alpha^E \xrightarrow{\simeq} U_a,$$

which we call a *parametrization* of U_a. Note that for $u \in \mathrm{Res}_{E_\alpha/K}\, \mathbb{G}_\mathrm{a}(K) = E_\alpha$, we have

$$x_a(u) = \prod_{\beta\in\Phi_a} x_\beta^E(u_\beta) \quad \text{with} \quad u_{\gamma(\alpha)} = \gamma(u) \text{ for } \gamma \in \mathrm{Gal}(E/K).$$

This allows us to define the *valuation* $\varphi_a : U_a(K) \to \frac{1}{[E_\alpha:K]}\mathbb{Z} \cup \{\infty\}$ of $U_a(K)$ by

$$\varphi_a(x_a(u)) = \mathrm{v}(u).$$

Case 2: The root $a \in \Phi_K$ is divisible or multipliable, i.e. $a/2$ or $2a$ is in Φ_K.

We assume that a is multipliable and describe U_a and U_{2a}. Let $\alpha, \widetilde{\alpha} \in \Phi_a$ be such that $\alpha + \widetilde{\alpha}$ is a root in Φ. Then G^a is isomorphic to $\mathrm{Res}_{E_{\alpha+\widetilde{\alpha}}/K} \mathrm{SU}_3$, where SU_3 is the special unitary group over $E_{\alpha+\widetilde{\alpha}}$ defined by the hermitian form $(x, y, z) \mapsto \sigma(x)z + \sigma(y)y + \sigma(z)x$ on E_α^3 with σ the non-trivial element in $\mathrm{Gal}(E_\alpha/E_{\alpha+\widetilde{\alpha}})$. Hence, in order to parametrize U_a, we first parametrize the positive root group U_+ of SU_3. To simplify notation, write $L = E_\alpha = E_{\widetilde{\alpha}}$ and $L_2 = E_{\alpha+\widetilde{\alpha}}$. Following [3], we define the subset $H_0(L, L_2)$ of $L \times L$ by

$$H_0(L, L_2) = \{(u, v) \in L \times L \mid v + \sigma(v) = \sigma(u)u\}.$$

Viewing $L \times L$ as a four-dimensional vector space over L_2, and considering the corresponding scheme over L_2 (as described in “Conventions and notation” in Section 1), we can view $H_0(L, L_2)$ as a closed subscheme of $L \times L$ over L_2, which we will again denote by $H_0(L, L_2)$. Then there exists an L_2-isomorphism $\mu : H_0(L, L_2) \to U_+$ given by

$$(u, v) \mapsto \begin{pmatrix} 1 & -\sigma(u) & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix},$$

where σ is induced by the non-trivial element in $\mathrm{Gal}(L/L_2)$. Using this isomorphism, we can transfer the group structure of U_+ to $H_0(L, L_2)$ and thereby turn the latter into a group scheme over L_2. Let us denote the restriction of scalars $\mathrm{Res}_{L_2/K} H_0(L, L_2)$

of $H_0(L, L_2)$ from $E_{\alpha+\widetilde{\alpha}} = L_2$ to K by $H(L, L_2)$. Then, by identifying G^a with $\mathrm{Res}_{E_{\alpha+\widetilde{\alpha}}/K} \mathrm{SU}_3$, we obtain an isomorphism

$$x_a := \pi \circ \mathrm{Res}_{E_{\alpha+\widetilde{\alpha}}/K} \mu : H(L, L_2) \xrightarrow{\simeq} U_a,$$

which we call the *parametrization* of U_a. We can describe the isomorphism x_a on K-points as follows. Let $[\Phi_a]$ be a set of representatives in Φ_a of the orbits of the action of $\mathrm{Gal}(E_\alpha/E_{\alpha+\widetilde{\alpha}}) = \langle\sigma\rangle$ on Φ_a. We will choose the sets of representatives for Φ_a and Φ_{-a} such that $[\Phi_a]$ and $-[\Phi_{-a}]$ are disjoint. For $\beta \in [\Phi_a]$, choose $\gamma \in \mathrm{Gal}(E/K)$ such that $\beta = \gamma(\alpha)$ and set $\widetilde{\beta} = \gamma(\widetilde{\alpha})$ and $u_\beta = \gamma(u)$ for every $u \in L$. By replacing some $x^E_{\beta+\widetilde{\beta}}$ by $x^E_{\beta+\widetilde{\beta}} \circ (-1)$ if necessary, we ensure that

$$x^E_{\beta+\widetilde{\beta}} = \mathrm{Inn}(m^{-1}_{\widetilde{\beta}}) \circ x^E_{\beta}$$

(where $m_{\widetilde{\beta}}$ is defined as in Remark 2.1.1).[1] Moreover, we choose the identification of G^a with $\mathrm{Res}_{E_{\alpha+\widetilde{\alpha}}/K} \mathrm{SU}_3$ so that its restriction to the positive root group arises from the restriction of scalars of the identification that satisfies

$$\pi\left(\begin{pmatrix} 1 & -w & v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}\right) = x^E_\alpha(u) x^E_{\alpha+\widetilde{\alpha}}(v) x^E_{\widetilde{\alpha}}(w).$$

Then for $(u, v) \in H_0(L, L_2) = H(L, L_2)(K) \subset L \times L$ we have

$$x_a(u, v) = \prod_{\beta\in[\Phi_a]} x^E_\beta(u_\beta) x^E_{\beta+\widetilde{\beta}}(-v_\beta) x^E_{\widetilde{\beta}}(\sigma(u)_\beta). \tag{1}$$

The root group U_{2a} corresponding to $2a$ is the subgroup of U_a given by the image of $x_a(0, v)$. Hence $U_{2a}(K)$ is identified with the group of elements in E_α of trace zero with respect to the quadratic extension $E_\alpha/E_{\alpha+\widetilde{\alpha}}$, which we denote by E^0_α.

Using the parametrization x_a, we define the *valuation* φ_a of $U_a(K)$ and φ_{2a} of $U_{2a}(K)$ by

$$\varphi_a(x_a(u, v)) = \tfrac{1}{2}\mathrm{v}(v), \qquad \varphi_{2a}(x_a(0, v)) = \mathrm{v}(v).$$

Remark 2.2.1. (i) Note that $v + \sigma(v) = \sigma(u)u$ implies $\frac{1}{2}\mathrm{v}(v) \le \mathrm{v}(u)$.

(ii) The valuation of the root groups U_a can alternatively be defined for all roots $a \in \Phi_K$ as follows. Let $u \in U_a(K)$, and write $u = \prod_{\alpha\in\Phi_a\cup\Phi_{2a}} u_\alpha$ with $u_\alpha \in U_\alpha(E)$. Then

$$\varphi_a(u) = \inf\Big(\inf_{\alpha\in\Phi_a} \varphi^E_\alpha(u_\alpha), \inf_{\alpha\in\Phi_{2a}} \tfrac{1}{2}\varphi^E_\alpha(u_\alpha)\Big),$$

where $\varphi^E_\alpha(x_\alpha(v)) = \mathrm{v}(v)$. The equivalence of the definitions is an easy exercise (see also [3, 4.2.2]).

[1] Note that our choice of x^E_β or $x^E_{\beta+\widetilde{\beta}}$ for negative roots $\beta, \widetilde{\beta}$ deviates from Bruhat and Tits. It allows us a more uniform construction of the root group parametrizations that does not require us to distinguish between positive and negative roots, but that coincides with the ones defined by Bruhat and Tits [3].

2.3. *Affine roots*

Recall that the apartment $\mathcal{A} = \mathcal{A}(S, K)$ corresponding to the maximal split torus S of G is an affine space under the $\mathbb{R}$-subspace of $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by the coroots of G, where $X_*(S) = \mathrm{Hom}_K(\mathbb{G}_{\mathrm{m}}, S)$. The apartment $\mathcal{A}$ can be defined as corresponding to all valuations of $(T(K), (U_a(K))_{a\in\Phi_K})$ in the sense of [2, §6.2] that are equipolent to the one constructed in §2.2, i.e., families of maps $(\widetilde{\varphi}_a : U_a(K) \to \mathbb{R} \cup \{\infty\})_{a\in\Phi_K}$ such that there exists $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying $\widetilde{\varphi}_a(u) = \varphi_a(u) + a(v)$ for all $u \in U_a(K)$, for all $a \in \Phi_K$. In particular, the valuation defined in §2.2 corresponds to a (special) point in $\mathcal{A}$ that we denote by x_0. Then the set of affine roots Ψ_K on $\mathcal{A}$ consists of the affine functions on $\mathcal{A}$ given by

$$\Psi_K = \Psi_K(\mathcal{A}) = \{y \mapsto a(y - x_0) + \gamma' \mid a \in \Phi_K,\ \gamma' \in \Gamma'_a\},$$

where

$$\Gamma'_a = \{\varphi_a(u) \mid u \in U_a - \{1\},\ \varphi_a(u) = \sup \varphi_a(uU_{2a})\}.$$

It will turn out to be handy to introduce a more explicit description of Γ'_a. In order to do so, consider a multipliable root a and $\alpha \in \Phi_a$, and define

$$\begin{aligned}(E_\alpha)^0 &= \{u \in E_\alpha \mid \mathrm{Tr}_{E_\alpha/E_{\alpha+\tilde{\alpha}}}(u) = 0\},\\ (E_\alpha)^1 &= \{u \in E_\alpha \mid \mathrm{Tr}_{E_\alpha/E_{\alpha+\tilde{\alpha}}}(u) = 1\},\\ (E_\alpha)^1_{\max} &= \big\{u \in (E_\alpha)^1 \mid \mathrm{v}(u) = \sup\{\mathrm{v}(v) \mid v \in (E_\alpha)^1\}\big\}.\end{aligned}$$

Then, by [3, 4.2.20, 4.2.21], the set $(E_\alpha)^1_{\max}$ is non-empty, and, with λ any element of $(E_\alpha)^1_{\max}$ and a still being multipliable, we have

$$\Gamma'_a = \tfrac{1}{2}\mathrm{v}(\lambda) + \mathrm{v}(E_\alpha - \{0\}), \tag{2}$$

$$\Gamma'_{2a} = \mathrm{v}((E_\alpha)^0 - \{0\}) = \mathrm{v}(E_\alpha - \{0\}) - 2 \cdot \Gamma'_a. \tag{3}$$

For a being neither multipliable nor divisible and $\alpha \in \Phi_a$, we have

$$\Gamma'_a = \mathrm{v}(E_\alpha - \{0\}). \tag{4}$$

Remark 2.3.1. Note that if the residue field characteristic p is not 2, then $\frac{1}{2} \in (E_\alpha)^1_{\max}$ for a a multipliable root and $\alpha \in \Phi_a$, and hence $\Gamma'_a = \mathrm{v}(E_\alpha - \{0\})$. If the residue field characteristic is $p = 2$, then $\mathrm{v}(\lambda) < 0$ for $\lambda \in (E_\alpha)^1_{\max}$.

2.4. *Moy–Prasad filtration*

Bruhat and Tits [2, 3] associated to each point x in the (reduced) Bruhat–Tits building $\mathcal{B}(G, K)$ a parahoric group scheme over $\mathcal{O}$, which we denote by $\mathbb{P}_x$, whose generic fiber is isomorphic to G. We will quickly recall the filtration of $G_x := \mathbb{P}_x(\mathcal{O})$ introduced by Moy and Prasad in [14, 15] and thereby specify our convention for the parameter involved.

Define $T_0 = T(K) \cap \mathbb{P}_x(\mathcal{O})$. Then T_0 is a subgroup of finite index in the maximal bounded subgroup $\{t \in T(K) \mid \mathrm{v}(\chi(t)) = 0\ \forall \chi \in X^*(T) = \mathrm{Hom}_{\overline{K}}(T, \mathbb{G}_{\mathrm{m}})\}$ of $T(K)$. Note that this index equals 1 if G is split.

For every positive real number r, we define

$$T_r = \{t \in T_0 \mid \mathrm{v}(\chi(t) - 1) \geq r \text{ for all } \chi \in X^*(T) = \mathrm{Hom}_{\overline{K}}(T, \mathbb{G}_{\mathrm{m}})\}.$$

For every affine root $\psi \in \Psi_K$, we denote by $\dot{\psi}$ its gradient and define the subgroup U_ψ of $U_{\dot{\psi}}(K)$ by

$$U_\psi = \{u \in U_{\dot{\psi}}(K) \mid \varphi_{\dot{\psi}}(u) \geq \psi(x_0)\}.$$

Then the *Moy–Prasad filtration* subgroups of G_x are given by

$$G_{x,r} = \langle T_r, U_\psi \mid \psi \in \Psi_K,\ \psi(x) \geq r\rangle \quad \text{for } r \geq 0,$$

and we set

$$G_{x,r+} = \bigcup_{s>r} G_{x,s}.$$

The quotient $G_x/G_{x,0+}$ can be identified with the $\overline{\mathbb{F}}_p$-points of the reductive quotient of the special fiber $\mathbb{P}_x \times_{\mathcal{O}} \overline{\mathbb{F}}_p$ of the parahoric group scheme $\mathbb{P}_x$, which we denote by $\mathbf{G}_x$. From [3, Corollaire 4.6.12] we deduce the following lemma.

Lemma 2.4.1 ([3]). *Let $R_K(G) = (X_K = X^*(S), \Phi_K, \check{X}_K = X_*(S), \check{\Phi}_K)$ be the restricted root datum of G. Then the root datum $R(\mathbf{G}_x)$ of $\mathbf{G}_x$ is canonically identified with $(X_K, \Phi', \check{X}_K, \check{\Phi}')$ where*

$$\Phi' = \{a \in \Phi \mid a(x - x_0) \in \Gamma'_a\} \quad \textit{and} \quad \check{\Phi}' = \{\check{a} \in \check{\Phi} \mid a(x - x_0) \in \Gamma'_a\}.$$

We can define a filtration of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)(K)$ similar to the filtration of G_x. In order to do so, we denote the $\mathcal{O}$-lattice $\mathrm{Lie}(\mathbb{P}_x)$ of $\mathfrak{g}$ by $\mathfrak{p}_x$. Define $\mathfrak{u}_{a,x} = \mathfrak{p}_x \cap \mathfrak{u}_a$ for $a \in \Phi_K$ and $\mathfrak{t} = \mathrm{Lie}(T)(K)$, where $\mathfrak{u}_a$ is the subspace of $\mathfrak{g}$ on which $\mathfrak{t}$ acts via $\mathrm{Lie}(a)$.

We define the *Moy–Prasad filtration* of the Lie algebra $\mathfrak{t}$ for $r \in \mathbb{R}$ to be

$$\mathfrak{t}_r = \{X \in \mathfrak{t} \mid \mathrm{v}(\mathrm{Lie}(\chi)(X)) \geq r \text{ for all } \chi \in X^*(T)\}. \tag{5}$$

For every root $a \in \Phi_K$, we define the Moy–Prasad filtration of $\mathfrak{u}_a$ as follows. Let $\psi_{a,x}$ be the smallest affine root with gradient a such that $\psi_{a,x}(x) \geq 0$. For every $\psi \in \Psi_K$ with gradient a, we let $n_{\psi,x} = e_\alpha(\psi - \psi_{a,x})$, where $e_\alpha = [E_\alpha : K]$ for some root $\alpha \in \Phi_a$ that restricts to a. Note that $n_{\psi,x}$ is an integer. Choosing a uniformizer $\varpi_\alpha \in E_\alpha$ and viewing $\mathfrak{p}_a$ inside $\mathrm{Lie}(G)(E_\alpha)$ we set[2]

$$\mathfrak{u}_\psi = (\varpi_\alpha^{n_{\psi,x}} \mathcal{O}_{E_\alpha} \mathfrak{p}_{a,x}) \cap \mathfrak{g}.$$

Then the *Moy–Prasad filtration* of the Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g}_{x,r} = \langle \mathfrak{t}_r, \mathfrak{u}_\psi \mid \psi(x) \geq r\rangle \quad \text{for } r \in \mathbb{R}.$$

In general, the quotient $G_{x,r}/G_{x,r+}$ is not isomorphic to $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ for $r > 0$. However, it turns out that we can identify them (as $\overline{\mathbb{F}}_p$-vector spaces) under the following assumption.

[2]Note that $\mathfrak{u}_\psi$ does not depend on the choice of x inside $\mathcal{A}$.

Assumption 2.4.2. *The maximal* (*maximally split*) *torus T of G becomes an induced torus over a tamely ramified extension.*

Recall that the torus T is called *induced* if it is a product of separable Weil restrictions of $\mathbb{G}_{\mathrm{m}}$, i.e. $T \simeq \prod_{i=1}^{N} \mathrm{Res}_{K_i/K} \mathbb{G}_{\mathrm{m}}$ for some integer N and finite separable field extensions K_i/K, $1 \leq i \leq N$.

For the rest of Section 2, we impose Assumption 2.4.2.

Remark 2.4.3. Assumption 2.4.2 holds, for example, if G is either adjoint or simply connected semisimple, or if G splits over a tamely ramified extension.

For $r \in \mathbb{R}$, we denote the quotient $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ ($\simeq G_{x,r}/G_{x,r+}$ for $r > 0$) by $\mathbf{V}_{x,r}$. The adjoint action of $G_{x,0}$ on $\mathfrak{g}_{x,r}$ (or, equivalently, the conjugation action of $G_{x,0}$ on $G_{x,r}$ for $r > 0$) induces an action of the algebraic group $\mathbf{G}_x$ on the quotients $\mathbf{V}_{x,r}$.

2.5. Chevalley system for the reductive quotient

In this section we construct a Chevalley system for the reductive quotient $\mathbf{G}_x$ by reduction of the root group parametrizations given in §2.2. Let $\mathbf{U}_a$ denote the root group of $\mathbf{G}_x$ corresponding to the root $a \in \Phi(\mathbf{G}_x) \subset \Phi_K(G)$. We denote by $\mathcal{O}_{\mathbb{Q}_p^{\mathrm{ur}}}$ the ring of integers in $\mathbb{Q}_p^{\mathrm{ur}}$. If K is an extension of $\mathbb{Q}_p^{\mathrm{ur}}$, we let $\chi : \overline{\mathbb{F}}_p \to \mathcal{O}_{\mathbb{Q}_p^{\mathrm{ur}}}$ be the Teichmüller lift, i.e. the unique multiplicative section of the surjection $\mathcal{O}_{\mathbb{Q}_p^{\mathrm{ur}}} \twoheadrightarrow \overline{\mathbb{F}}_p$. If K is an extension of $\mathbb{F}_p((t))^{\mathrm{ur}} = \varinjlim_{n\in\mathbb{N}} \mathbb{F}_{p^n}((t))$, we let $\chi : \overline{\mathbb{F}}_p = \varinjlim_{n\in\mathbb{N}} \mathbb{F}_{p^n} \to \varinjlim_{n\in\mathbb{N}} \mathbb{F}_{p^n}[[t]]$ be the usual inclusion.

Lemma 2.5.1. *Let $\lambda = \lambda_a \in (E_\alpha)^1_{\max}$ for some $\alpha \in \Phi_a$, and write $\lambda = \lambda_0 \cdot \varpi_E^{\mathrm{v}(\lambda)e} \cdot \epsilon_0$ with $\lambda_0 \in \chi(\overline{\mathbb{F}}_p)$ and $\epsilon_0 \in 1 + \varpi_E \mathcal{O}_E$; e.g., take $\lambda_0\epsilon_0 = \lambda = 1/2$ if $p \neq 2$. Consider the map $\overline{\mathbb{F}}_p \to G_{x,0}$ given by*

$$u \mapsto \begin{cases} x_a\big(\sqrt{1/\lambda_0}\,\chi(u)\varpi_E^{s}\epsilon_1,\ \chi(u)\varpi_E^{s}\epsilon_1\sigma(\chi(u)\varpi_E^{s}\epsilon_1)\cdot\varpi_E^{\mathrm{v}(\lambda)e}\epsilon_0\big) & \text{if } a \text{ is multipliable},\\ x_a(0, \chi(u)\cdot\varpi_E^{-2a(x-x_0)\cdot e}\epsilon_2) & \text{if } a \text{ is divisible},\\ x_a(\chi(u)\cdot\varpi_E^{-a(x-x_0)\cdot e}\epsilon_3) & \text{otherwise},\end{cases}$$

where $s = -(a(x-x_0) + \mathrm{v}(\lambda)/2)\cdot e$, and $\epsilon_1, \epsilon_2, \epsilon_3 \in 1 + \varpi_E\mathcal{O}_E$ such that $\sqrt{1/\lambda_0}\,\chi(u)\varpi_E^{s}\epsilon_1$, $\chi(u)\varpi_E^{-2a(x-x_0)\cdot e}\epsilon_2$ and $\chi(u)\varpi_E^{-a(x-x_0)\cdot e}\epsilon_3$ are contained in E_α, and $\sqrt{1/\lambda_0} \in \chi(\overline{\mathbb{F}}_p)$ with $\sqrt{1/\lambda_0}^{\,2} = 1/\lambda_0$.

Then the composition of this map with the quotient map $G_{x,0} \twoheadrightarrow G_{x,0}/G_{x,0+}$ yields a root group parametrization $\overline{x}_a : \mathbb{G}_{\mathrm{a}} \to \mathbf{U}_a \subset \mathbf{G}_x$ (where $\mathbb{G}_{\mathrm{a}}$ is defined over $\overline{\mathbb{F}}_p$). Moreover, the root group parametrizations $\{\overline{x}_a\}_{a\in\Phi(\mathbf{G}_x)}$ form a Chevalley system for $\mathbf{G}_x$.

We remark that Gopal Prasad pointed out to us that a similar Chevalley system construction can be found in [18, 2.19, 2.20].

Proof of Lemma 2.5.1. Note first that since $a \in \Phi(\mathbf{G}_x)$, we have $a(x - x_0) \in \Gamma'_a$ by Lemma 2.4.1. Suppose a is multipliable. Then $\mathbf{U}_a(\overline{\mathbb{F}}_p)$ is the image of

$$\mathrm{Im} := \left\{ x_a(U, V) \;\middle|\; (U, V) \in H_0(E_\alpha, E_{\alpha+\tilde{\alpha}}),\ \tfrac{1}{2}\mathrm{v}(V) = -a(x - x_0) \right\}$$

in $G_{x,0}/G_{x,0+}$. Set

$$U(u) = \sqrt{1/\lambda_0}\,\chi(u) \cdot \varpi_E^{-(a(x-x_0)+\mathrm{v}(\lambda)/2)\cdot e} \epsilon_1$$

and

$$V(u) = \chi(u)\varpi_E^s \epsilon_1 \sigma(\chi(u)\varpi_E^s \epsilon_1) \cdot \varpi_E^{\mathrm{v}(\lambda) e} \epsilon_0.$$

Then $V(u) + \sigma(V(u)) = U(u)\sigma(U(u))$, i.e. $(U(u), V(u))$ is in $H_0(E_\alpha, E_{\alpha+\tilde{\alpha}})$, and $\mathrm{v}(V(u)) = -2a(x - x_0)$. Moreover, every element in Im is of the form $(U(u), V(u) + v_0)$ for $u \in \overline{\mathbb{F}}_p$ and some element $v_0 \in (E_\alpha)^0$ with $\mathrm{v}(v_0) > -2a(x - x_0)$, because $2a(x - x) \notin \mathrm{v}((E_\alpha)^0)$ (by equation (3), §2.3). Note that the images of $x_a(U(u), V(u) + v_0)$ and $x_a(U(u), V(u))$ in $G_{x,0}/G_{x,0+}$ agree. Thus, by the definition of x_a, we obtain an isomorphism of group schemes $\overline{x}_a : \mathbb{G}_\mathrm{a} \to \mathbf{U}_a$. Similarly, one can check that $\overline{x}_a$ yields an isomorphism $\mathbb{G}_\mathrm{a} \to \mathbf{U}_a$ for a not multipliable.

In order to show that $\{\overline{x}_a\}_{a \in \Phi(\mathbf{G}_x)}$ is a Chevalley system, suppose for the moment that a and b in $\Phi(\mathbf{G}_x)$ are neither multipliable nor divisible, and $\Phi_a = \{\alpha\}$ and $\Phi_b = \{\beta\}$ each contain only one root. Let $\check{\alpha}$ be the coroot of the root α, and denote by s_α the reflection in the Weyl group W of G corresponding to α. Then, using [4, Cor. 5.1.9.2], we obtain

$$\begin{aligned}
&\mathrm{Ad}\big(x_\alpha^E(\varpi_E^{-\alpha(x-x_0)e}) x_{-\alpha}^E(\epsilon_{\alpha,\alpha}\varpi_E^{-(-\alpha)(x-x_0)e}) x_\alpha^E(\varpi_E^{-\alpha(x-x_0)e})\big) \\
&\qquad\qquad\qquad\qquad \big(\mathrm{Lie}(x_\beta^E)(\varpi_E^{-\beta(x-x_0)e})\big) \\
&\quad = \mathrm{Ad}\big(\check{\alpha}(\varpi_E^{-\alpha(x-x_0)e})\big)\, \mathrm{Ad}\big(x_\alpha^E(1) x_{-\alpha}^E(\epsilon_{\alpha,\alpha}) x_\alpha^E(1)\big)\big(\varpi_E^{-\beta(x-x_0)e}\, \mathrm{Lie}(x_\beta^E)(1)\big) \\
&\quad = \mathrm{Ad}\big(\check{\alpha}(\varpi_E^{-\alpha(x-x_0)e})\big)\big(\epsilon_{\alpha,\beta}\varpi_E^{-\beta(x-x_0)e}\, \mathrm{Lie}(x_{s_\alpha(\beta)}^E)(1)\big) \\
&\quad = (s_\alpha(\beta))(\check{\alpha}(\varpi_E^{-\alpha(x-x_0)e}))\epsilon_{\alpha,\beta}\varpi_E^{-\beta(x-x_0)e}\, \mathrm{Lie}(x_{s_\alpha(\beta)}^E)(1) \\
&\quad = \varpi_E^{\langle\check{\alpha}, s_\alpha(\beta)\rangle(-\alpha(x-x_0))e} \epsilon_{\alpha,\beta}\varpi_E^{-\beta(x-x_0)e}\, \mathrm{Lie}(x_{s_\alpha(\beta)}^E)(1) \\
&\quad = \varpi_E^{\langle\check{\alpha}, \beta\rangle\alpha(x-x_0)e - \beta(x-x_0)e} \epsilon_{\alpha,\beta}\, \mathrm{Lie}(x_{s_\alpha(\beta)}^E)(1) \\
&\quad = \epsilon_{\alpha,\beta}\, \mathrm{Lie}(x_{s_\alpha(\beta)}^E)(\varpi_E^{-(s_\alpha(\beta))(x-x_0)e}).
\end{aligned}$$

This implies (assuming $\epsilon_3 = 1$, otherwise it is an easy exercise to add in the required constants) that for $\overline{m}_a := \overline{x}_a(1)\overline{x}_{-a}(\epsilon_{a,a})\overline{x}_a(1)$ with $\epsilon_{a,a} = \epsilon_{\alpha,\alpha}$ we have

$$\mathrm{Ad}(\overline{m}_a)(\mathrm{Lie}(\overline{x}_b)(1)) = \mathrm{Ad}\big(\overline{x}_a(1)\overline{x}_{-a}(\epsilon_{a,a})\overline{x}_a(1)\big)(\mathrm{Lie}(\overline{x}_b)(1)) = \epsilon_{\alpha,\beta}\, \mathrm{Lie}(\overline{x}_{s_a(b)})(1).$$

We obtain a similar result even if Φ_a and Φ_b are not singletons by the requirement that $\{x_\alpha^E\}_{\alpha\in\Phi}$ is a Chevalley–Steinberg system, i.e. compatible with the Galois action as described in Section 2. Similarly, we can extend the result that $\mathrm{Ad}(\overline{m}_a)(\mathrm{Lie}(\overline{x}_b)(1)) = \pm\,\mathrm{Lie}(\overline{x}_{s_a(b)})(1)$ to all non-multipliable roots $a, b \in \Phi(\mathbf{G}_x) \subset \Phi_K$.

Suppose now that $a \in \Phi(\mathbf{G}_x) \subset \Phi_K$ is multipliable, and let $\alpha \in \Phi_a$ and $\widetilde{\alpha} = \sigma(\alpha) \in \Phi_a$ as above. Following [3, 4.1.11], we define for $(U, V) \in H_0(E_\alpha, E_{\alpha+\widetilde{\alpha}})$

$$m_a(U, V) = x_a(UV^{-1}, \sigma(V^{-1}))x_{-a}(\epsilon_{\alpha,\alpha}U, V)x_a(U\sigma(V^{-1}), \sigma(V^{-1})).$$

Then Bruhat and Tits show in *loc. cit.* that $m_a(U, V)$ is in the normalizer of the maximal torus T and

$$m_a(U, V) = m_{a,1}\widetilde{a}(V) \quad \text{and} \quad x_{-a}(\epsilon_{\alpha,\alpha}U, V) = m_{a,1}x_a(U, V)m_{a,1}^{-1}, \tag{6}$$

where

$$\begin{aligned} m_{a,1} &= \pi \circ \mathrm{Res}_{E_{\alpha+\widetilde{\alpha}}/K} \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ \widetilde{a}(V) &= \pi \circ \mathrm{Res}_{E_{\alpha+\widetilde{\alpha}}/K} \begin{pmatrix} V & 0 & 0 \\ 0 & V^{-1}\sigma(V) & 0 \\ 0 & 0 & \sigma(V^{-1}) \end{pmatrix}. \end{aligned} \tag{7}$$

Note that

$$\begin{aligned} m_a\big(\sqrt{1/\lambda_0}\,&(-\varpi_E)^{(a(x-x_0)-\mathrm{v}(\lambda)/2)e}\epsilon_1, \\ &\varpi_E^{(a(x-x_0)-\mathrm{v}(\lambda)/2)e}\epsilon_1\sigma(\varpi_E^{(a(x-x_0)-\mathrm{v}(\lambda)/2)e}\epsilon_1)\varpi_E^{\mathrm{v}(\lambda)e}\epsilon_0\big) \in G_{x,0}, \end{aligned}$$

and denote its image in $G_{x,0}/G_{x,0+}$ by $\overline{m}_a$. Using the fact that $\mathrm{v}(\lambda) = 0$ if $p \neq 2$, and

$$\begin{aligned} \sigma(\varpi_E^{(a(x-x_0)-\mathrm{v}(\lambda)/2)e}\epsilon_1) &\equiv \pm\varpi_E^{(a(x-x_0)-\mathrm{v}(\lambda)/2)e}\epsilon_1 \\ &\equiv \varpi_E^{(a(x-x_0)-\mathrm{v}(\lambda)/2)e}\epsilon_1 \bmod \varpi_E^{(a(x-x_0)-\mathrm{v}(\lambda)/2)e+1} \end{aligned}$$

if $p = 2$, as well as the compatibility with Galois action properties of a Chevalley–Steinberg system, we obtain

$$\overline{m}_a = \overline{x}_a(1)\overline{x}_{-a}(\epsilon_{a,a})\overline{x}_a(1) \quad \text{with} \quad \epsilon_{a,a} = \epsilon_{\alpha,\alpha}(-1)^{(a(x-x_0)-\mathrm{v}(\lambda)/2)e}.$$

Moreover, an easy calculation using (6) and (7) shows that

$$\overline{x}_{-a}(\epsilon_{a,a}u) = \overline{m}_a x_a(u)\overline{m}_a^{-1}$$

for all $u \in \overline{\mathbb{F}}_p$. In other words,

$$\mathrm{Ad}(\overline{m}_a)(\mathrm{Lie}(\overline{x}_a)(1)) = \epsilon_{a,a}\,\mathrm{Lie}(\overline{x}_{-a})(1),$$

as desired. We obtain analogous results for $\overline{m}_{-a}$ being defined as above with "a" replaced by "$-a$". Moreover, $\overline{m}_a = \overline{m}_{-a}$, and hence $\mathrm{Ad}(\overline{m}_{-a})(\mathrm{Lie}(\overline{x}_a)(1)) = \epsilon_{a,a}\,\mathrm{Lie}(\overline{x}_{-a})(1)$.

In order to show that $\{\overline{x}_a\}_{a\in\Phi(\mathbf{G}_x)}$ forms a Chevalley system, it remains to check that

$$\mathrm{Ad}(\overline{m}_a)(\mathrm{Lie}(\overline{x}_b)(1)) = \pm\,\mathrm{Lie}(\overline{x}_{s_a(b)})(1) \tag{8}$$

for $a, b \in \Phi(\mathbf{G}_x)$ with $a \neq \pm b$ and either a or b multipliable. Note that if x_a and x_{-a} commute with x_b, then the statement is trivial. Note also that if b is multipliable and $\beta \in \Phi_b$, then β lies in the span of the roots of a connected component of the Dynkin diagram $\mathrm{Dyn}(G)$ of $\Phi(G)$ of type A_{2n} for some positive integer n. Hence, for some $\alpha \in \Phi_a$, α and β lie in the span of the roots of such a connected component. Moreover, by the compatibility of the Chevalley–Steinberg system $\{x_\alpha^E\}_{\alpha\in\Phi}$ with the Galois action, it suffices to restrict to the case where $\mathrm{Dyn}(G)$ is of type A_{2n} with simple roots labeled by $\alpha_n, \alpha_{n-1}, \dots, \alpha_1, \beta_1, \beta_2, \dots, \beta_n$ as in Figure 1, and the K-structure of G arises from

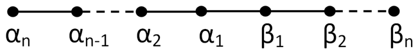

Fig. 1. Dynkin diagram of type A_{2n}

the unique outer automorphism of A_{2n} of order 2 that sends α_i to β_i. If a root in $\Phi_K(G)$ is multipliable, then it is the image of $\pm(\alpha_1 + \cdots + \alpha_s)$ in Φ_K for some $1 \leq s \leq n$. In particular, the positive multipliable roots are orthogonal to each other, by which we mean that $\langle \check{a}, b\rangle = 0$ for two distinct positive multipliable roots a and b. Equation (8) can now be verified by simple matrix calculations in SL_{2n+1}.

2.6. *Moy–Prasad filtration and field extensions*

Let F be a field extension of K of degree $d = [F : K]$ with ring of integers $\mathcal{O}_F$. Then there exists a $G(K)$-equivariant injection of the Bruhat–Tits building $\mathcal{B}(G, K)$ of G over K into the Bruhat–Tits building $\mathcal{B}(G_F, F)$ of $G_F = G \times_K F$ over F. We denote the image of the point $x \in \mathcal{B}(G, K)$ in $\mathcal{B}(G_F, F)$ by x as well. Using the definitions introduced in §2.4, but for notational convenience still with the valuation v (instead of replacing it by the normalized valuation $d \cdot \mathrm{v}$), we can define a Moy–Prasad filtration of $G(F)$ and $\mathfrak{g}_F$ at x, which we denote by $G^F_{x,r} (r \geq 0)$ and $\mathfrak{g}^F_{x,r} (r \in \mathbb{R})$, as well as its quotients $\mathbf{V}^F_{x,r} (r \in \mathbb{R})$ and the reductive quotient $\mathbf{G}^F_x$.

Suppose now that G_F is split, and that $\Gamma'_a \subset \mathrm{v}(F)$ for all restricted roots $a \in \Phi_K(G)$. This holds, for example, if F is an even-degree extension of the splitting field E. Then, using Remark 2.2.1(i) and the definition of the Moy–Prasad filtration, the inclusion $G(K) \hookrightarrow G(F)$ maps $G_{x,r}$ into $G^F_{x,r}$. Furthermore, recalling that for split tori $\widetilde{T}$ the subgroup $\widetilde{T}_0$ is the maximal bounded subgroup of the (rational points of) $\widetilde{T}$ and using the assumption that $\Gamma'_a \subset \mathrm{v}(F)$ for all restricted roots $a \in \Phi_K(G)$, we observe that this map induces an injection

$$\iota_{K,F} : G_{x,0}/G_{x,0+} \hookrightarrow G^F_{x,0}/G^F_{x,0+}, \tag{9}$$

which yields a map of algebraic groups $\mathbf{G}_x \to \mathbf{G}^F_x$, also denoted by $\iota_{K,F}$. If $p \neq 2$ or d is odd, then $\iota_{K,F}$ is a closed immersion.

To discuss a similar result for higher-depth quotients, we denote by Φ_K^{mul} the set of multipliable roots in Φ_K and by Φ_K^{nm} the set of non-multipliable roots in Φ_K.

Lemma 2.6.1. *Let F be as above, i.e. G_F is split and $\Gamma'_a \subset \mathrm{v}(F)$ for all restricted roots $a \in \Phi_K(G)$. Then, for every $r \in \mathbb{R}$, there exists an injection*

$$\iota_{K,F,r} : \mathbf{V}_{x,r} = \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+} \hookrightarrow \mathfrak{g}^F_{x,r}/\mathfrak{g}^F_{x,r+} = \mathbf{V}^F_{x,r}$$

such that $\iota_{K,F}(\mathbf{G}_x)$ preserves $\iota_{K,F,r}(\mathbf{V}_{x,r})$ under the action described in §2.4. Moreover, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{G}_x \times \mathbf{V}_{x,r} & \longrightarrow & \mathbf{V}_{x,r} \\ {\scriptstyle \iota_{K,F}\times\iota_{K,F,r}}\downarrow & & \downarrow{\scriptstyle \iota_{K,F,r}} \\ \mathbf{G}^F_x \times \mathbf{V}^F_{x,r} & \longrightarrow & \mathbf{V}^F_{x,r} \end{array} \tag{10}$$

unless $p = 2$ and there exists $a \in \Phi^{\mathrm{mul}}_K$ with $a(x - x_0) \in \Gamma'_a$ such that $a(x - x_0) - r \in \Gamma'_a$ or such that there exists $b \in \Phi^{\mathrm{nm}}_K$ with $b(x - x_0) - r \in \Gamma'_b$ and $\langle \check{a}, b\rangle \neq 0$.

Proof. For $p \neq 2$, let $\iota_{K,F,r}$ be induced by the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_F = \mathfrak{g} \otimes_K F$. This map is well defined, and it is easy to see that it is injective on $((\mathfrak{t} \cap \mathfrak{g}_{x,r}) + \mathfrak{g}_{x,r+})/\mathfrak{g}_{x,r+}$ and on $((\mathfrak{u}_a \cap \mathfrak{g}_{x,r}) + \mathfrak{g}_{x,r+})/\mathfrak{g}_{x,r+}$ for $a \in \Phi_K$ non-multipliable. Suppose a is multipliable. If $r - a(x - x_0) \in \Gamma'_a$, i.e. there exists an affine root $\psi : y \mapsto a(y - x_0) + \gamma'$ with $\psi(x) = r$, and $\varphi_a(x_a(u, v)) = \psi(x_0) = r - a(x - x_0) \in \Gamma'_a$, then $\mathrm{v}(u) = \frac{1}{2}\mathrm{v}(v) = r - a(x - x_0)$. This follows from the trace of $1/2$ being 1, hence $v - \frac{1}{2}\sigma(u)u$ is traceless and therefore has valuation outside $2\Gamma'_a$, while $\mathrm{v}(v) \in 2\Gamma'_a$. Hence the image of $\mathfrak{u}_a \cap \mathfrak{g}_{x,r}$ in $\mathbf{V}^F_{x,r}$ is non-vanishing if it is non-trivial in $\mathbf{V}_{x,r}$, i.e. if $r - a(x - x_0) \in \Gamma'_a$. Moreover, diagram (10) commutes.

In the case $p = 2$, if $a \in \Phi_K$ is multipliable and $r - a(x - x_0) \in \Gamma'_a$ and $\varphi_a(x_a(u, v)) = r - a(x - x_0)$, then $\mathrm{v}(u) = r - a(x - x_0) - \frac{1}{2}\mathrm{v}(\lambda_\alpha)$ for $\lambda_\alpha \in (E_\alpha)^1_{\max}$ by a reasoning analogous to that above. However, recall from Remark 2.3.1 that $\mathrm{v}(\lambda_\alpha) < 0$ for $p = 2$. Let ϖ_F be a uniformizer of F such that $\varpi_F^{d/e} = \varpi_F^{[F:E]} \equiv \varpi_E \bmod \varpi_F^{[F:E]+1}$ and let ϖ_α be a uniformizer of E_α with

$$\varpi_\alpha \equiv \varpi_F^{[F:E_\alpha]} = \varpi_F^{d/e_\alpha} \bmod \varpi_F^{[F:E_\alpha]+1}.$$

This allows us to define $\iota_{K,F,r}$ as follows. We define the linear morphism $i_{K,F,r} : \mathfrak{g} \hookrightarrow \mathfrak{g}_F$ to be the usual inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_F = \mathfrak{g} \otimes_K F$ on $\mathfrak{t} \oplus \bigoplus_{a\in\Phi^{\mathrm{nm}}_K} \mathfrak{u}_a$ and to be the linear map from $\bigoplus_{a\in\Phi^{\mathrm{mul}}_K} \mathfrak{u}_a$ onto $(\bigoplus_{a\in\Phi^{\mathrm{mul}}_K} \mathfrak{u}_a \otimes_K \varpi_F^{d\mathrm{v}(\lambda_\alpha)/2} K) \subset \mathfrak{g}_F$ on $\bigoplus_{a\in\Phi^{\mathrm{mul}}_K} \mathfrak{u}_a$ such that

$$\begin{aligned} &i_{K,F,r}\bigl(\mathrm{Lie}(x_a)(\varpi_\alpha^{(r-a(x-x_0)-\mathrm{v}(\lambda_\alpha)/2)e_\alpha}, 0)\bigr) \\ &\qquad\qquad = \mathrm{Lie}(x_a)\bigl(\varpi_\alpha^{(r-a(x-x_0)-\mathrm{v}(\lambda_\alpha)/2)e_\alpha} \otimes \varpi_F^{d\mathrm{v}(\lambda_\alpha)/2}, 0\bigr), \end{aligned}$$

where $\alpha \in \Phi_a$ for $a \in \Phi^{\mathrm{mul}}_K$. By restricting $i_{K,F,r}$ to $\mathfrak{g}_{x,r}$ and passing to the quotient, we obtain an injection $\iota_{K,F,r}$ of $\mathbf{V}_{x,r}$ into $\mathbf{V}^F_{x,r}$.

In order to prove that $\iota_{K,F}(\mathbf{G}_x)$ preserves $\iota_{K,F,r}(\mathbf{V}_{x,r})$ for $p=2$, it suffices to show that $\iota_{K,F}(\mathbf{G}_x)$ stabilizes the subspace

$$V' = \iota_{K,F,r}\Big(\overline{\mathfrak{g}_{x,r} \cap \bigoplus_{a\in\Phi_K^{\mathrm{mul}}} \mathfrak{u}_a}\Big),$$

where the overline denotes the image in $\mathbf{V}_{x,r}$.

First suppose that the Dynkin diagram $\mathrm{Dyn}(G)$ of $\Phi(G)$ is of type A_{2n} with simple roots labeled by $\alpha_n, \alpha_{n-1}, \dots, \alpha_2, \alpha_1, \beta_1, \beta_2, \dots, \beta_n$ as in Figure 1, and that the K-structure of G arises from the unique outer automorphism of A_{2n} of order 2 that sends α_i to β_i. If $a \in \Phi_K(G)$ is multipliable, then a is the image of $\pm(\alpha_1 + \dots + \alpha_s)$ for some $1 \le s \le n$. Suppose, without loss of generality, that a is the image of $\alpha_1 + \dots + \alpha_s$. Consider the action of the image of $\overline{x}_b$ in $\mathbf{G}_x^F$ for b the image of $-(\alpha_1+\dots+\alpha_t)$ for some $1 \le t \le n$. Note that $\iota_{K,F}(\overline{x_b(H_0(E_{-(\alpha_1+\dots+\alpha_t)}, K)) \cap G_{x,0}})$ is the image of $\overline{x^E_{-(\alpha_1+\dots+\alpha_t+\beta_1+\dots+\beta_t)}(E) \cap G^E_{x,0}}$ in $G^E_{x,0}/G^E_{x,0+}$. Hence the orbit of $\iota_{K,F}(\overline{x_b(H_0(E_{-(\alpha_1+\dots+\alpha_t)}, K)) \cap G_{x,0}})$ on $\iota_{K,F,r}(\overline{\mathfrak{g}_{x,r} \cap \mathfrak{u}_a})$ is contained in

$$\overline{(\mathfrak{g} \otimes_K \varpi_F^{d\mathrm{v}(\lambda_\alpha)/2} K) \cap \mathfrak{g}^F_{x,r} \cap (\mathfrak{g}^F_{\alpha_1+\dots+\alpha_s} \oplus \mathfrak{g}^F_{\beta_1+\dots+\beta_s} \oplus \mathfrak{g}^F_{-(\beta_1+\dots+\beta_t)} \oplus \mathfrak{g}^F_{-(\alpha_1+\dots+\alpha_t)})} \subset V'.$$

(Note that the last two summands can be deleted unless $s=t$.) Thus V' is preserved under the action of the image of $\overline{x}_b$ in $\mathbf{G}_x^F$. Similarly (but more easily) one can check that the action of the image of $\overline{x}_b$ in $\mathbf{G}_x^F$ for all other $b \in \Phi(\mathbf{G}_x)$ preserves V', and the same is true for the image of $T \cap G_{x,0}$ in $\mathbf{G}_x^F$. Hence $\iota_{K,F}(\mathbf{G}_x)$ stabilizes V'.

The case of a general group G follows by using the observation that if $a \in \Phi_K$ is multipliable, then each $\alpha \in \Phi_a$ is spanned by the roots of a connected component of the Dynkin diagram $\mathrm{Dyn}(G)$ of $\Phi(G)$ that is of type A_{2n}, together with the observation that the above explanation also works for $\mathrm{Dyn}(G)$ being a union of Dynkin diagrams of A_{2n} that are permuted transitively by the action of the absolute Galois group of K. Thus V' is preserved under the action of $\iota_{K,F}(\mathbf{G}_x)$.

In order to show that $\iota_{K,F,r}$ is compatible with the action of $\mathbf{G}_x$ as in diagram (10) for $p=2$, it remains to prove that $\mathbf{G}_x$ preserves $\overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \bigoplus_{a\in\Phi_K^{\mathrm{nm}}} \mathfrak{u}_a)}$. We consider the action on $\overline{\mathfrak{g}_{x,r} \cap \mathfrak{t}}$ and $\overline{\mathfrak{g}_{x,r} \cap \bigoplus_{a\in\Phi_K^{\mathrm{nm}}} \mathfrak{u}_a}$ separately.

We begin with the former, which is obviously preserved under the action of the image of $T \cap G_{x,0}$ in $\mathbf{G}_x$. So consider the action of the image of $\overline{x}_b$ in $\mathbf{G}_x$ for some $b \in \Phi(\mathbf{G}_x) \subset \Phi_K$. If b is non-multipliable in Φ_K, then the image of the action lands in $\overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \mathfrak{u}_b)} \subset \overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \bigoplus_{a\in\Phi_K^{\mathrm{nm}}} \mathfrak{u}_a)}$. If $b \in \Phi_K^{\mathrm{mul}}$, then the image of the action is contained in $\overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \mathfrak{u}_b \oplus \mathfrak{u}_{2b})}$. However, by the assumption in our lemma, we have $b(x-x_0) - r \notin \Gamma'_b$ and hence $\overline{\mathfrak{g}_{x,r} \cap \mathfrak{u}_b} = \{0\}$. Therefore the image of the action of $\overline{x}_b$ on $\overline{\mathfrak{g}_{x,r} \cap \mathfrak{t}}$ is contained in $\overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \mathfrak{u}_{2b})} \subset \overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \bigoplus_{a\in\Phi_K^{\mathrm{nm}}} \mathfrak{u}_a)}$.

It remains to consider the action of $\mathbf{G}_x$ on $\overline{\mathfrak{g}_{x,r} \cap \bigoplus_{a\in\Phi_K^{\mathrm{nm}}} \mathfrak{u}_a}$. Note that the image of $T \cap G_{x,0}$ in $\mathbf{G}_x$ preserves $\overline{\mathfrak{g}_{x,r} \cap \bigoplus_{a\in\Phi_K^{\mathrm{nm}}} \mathfrak{u}_a}$. Thus it remains to consider the action of $\overline{x}_b(\mathbb{G}_m)$ for some $b \in \Phi(\mathbf{G}_x) \subset \Phi_K$, and we may restrict to the case that $\mathrm{Dyn}(G)$ is of type A_{2n} with non-trivial Galois action as above. Let $b \in \Phi_K^{\mathrm{mul}}$, and assume without loss of generality that b is the image of $\alpha_1 + \cdots + \alpha_s$ for some $1 \leq s \leq n$. Let $a \in \Phi_K^{\mathrm{nm}}$ with $\overline{\mathfrak{g}_{x,r} \cap \mathfrak{u}_a} \not\simeq \{0\}$, i.e. $a(x - x_0) - r \in \Gamma'_a$. The assumption of the lemma implies that $\langle \check{b}, a \rangle = 0$. Hence a is the image of $\pm(\alpha_{s'} + \cdots + \alpha_{t'})$ for some $1 < s' < t' \leq n$ with $s' \neq s + 1 \neq t'$, or of $\pm(\alpha_1 + \cdots + \alpha_{t'} + \beta_1 + \cdots + \beta_{s'})$ for some $1 \leq s', t' \leq n$ with $s' \neq s \neq t'$ and $s' \neq t'$. In all cases, $\overline{x}_b(\mathbb{G}_m)$ acts trivially on $\overline{\mathfrak{g}_{x,r} \cap \mathfrak{u}_a}$, and therefore $\overline{x}_b(\mathbb{G}_m)$ preserves $\overline{\mathfrak{g}_{x,r} \cap \bigoplus_{a \in \Phi_K^{\mathrm{nm}}} \mathfrak{u}_a}$. Similarly, i.e. using what non-multiple roots in the A_{2n} case look like, we observe that if $b \in \Phi_K^{\mathrm{nm}}$, then $\overline{x}_b$ maps $\overline{\mathfrak{g}_{x,r} \cap \bigoplus_{a \in \Phi_K^{\mathrm{nm}}} \mathfrak{u}_a}$ to $\overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \bigoplus_{a \in \Phi_K^{\mathrm{nm}}} \mathfrak{u}_a)}$.

Hence diagram (10) commutes in the case $p = 2$ if there does not exist $a \in \Phi_K^{\mathrm{mul}}$ with $a(x - x_0) \in \Gamma'_a$ such that $a(x - x_0) - r \in \Gamma'_a$ or such that there exists $b \in \Phi_K^{\mathrm{nm}}$ with $b(x - x_0) - r \in \Gamma'_b$ and $\langle \check{a}, b \rangle \neq 0$. ∎

In what follows, we might abuse notation and identify $\mathbf{V}_{x,r}$ with its image in $\mathbf{V}_{x,r}^F$ under $\iota_{K,F}$.

3. Moy–Prasad filtration for different residual characteristics

In this section we compare the Moy–Prasad filtration quotients for groups over non-archimedean local fields of different residue field characteristics. In order to do so, we first introduce in Definition 3.1.1 the class of reductive groups that we are going to work with. We then show in Proposition 3.1.4 that this class contains reductive groups that split over a tamely ramified extension, i.e. those groups considered in [20], but also general simply connected and adjoint semisimple groups, among others. The restriction to this (large) class of reductive groups is necessary as the main result (Theorem 3.4.1) about the comparison of Moy–Prasad filtrations for different residue field characteristics does not hold true for some reductive groups that are not good groups (see Remark 3.4.2).

3.1. Definition and properties of good groups

Definition 3.1.1. We say that a reductive group G over K, whose splitting field is denoted by E, is *good* if there exist

- an action of a finite cyclic group $\Gamma = \langle \gamma \rangle$ on the root datum $R(G) = (X, \Phi, \check{X}, \check{\Phi})$ preserving the simple roots Δ,
- an element u generating the cyclic group $\mathrm{Gal}(E \cap K^{\mathrm{tame}}/K)$ and whose order $|\mathrm{Gal}(E \cap K^{\mathrm{tame}}/K)|$ is divisible by N where (throughout the remainder of the paper) we will write $|\Gamma| = p^s \cdot N$ for integers s and N with $(N, p) = 1$

such that the following two conditions are satisfied:

(i) The orbits of $\mathrm{Gal}(E/K)$ and Γ on Φ coincide, and, for every root $\alpha \in \Phi$, there exists $u_{1,\alpha} \in \mathrm{Gal}(E/K)$ such that

$$\gamma(\alpha) = u_{1,\alpha}(\alpha) \quad \text{and} \quad u_{1,\alpha}|_{E\cap K^{\mathrm{tame}}} = u.$$

(ii) There exists a basis $\mathcal{B}$ of X stabilized by $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$ and $\langle \gamma^N \rangle$ on which the $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$-orbits and $\langle \gamma^N \rangle$-orbits agree, and such that for any $B \in \mathcal{B}$, there exists an element $v_{1,B} \in \mathrm{Gal}(E/K)$ satisfying

$$\gamma(B) = v_{1,B}(B) \quad \text{and} \quad v_{1,B}|_{E\cap K^{\mathrm{tame}}} = u.$$

Remark 3.1.2. Note that condition (i) of Definition 3.1.1 is equivalent to the condition

(i′) The orbits of $\mathrm{Gal}(E/K)$ on Φ coincide with the orbits of Γ on Φ, and there exist representatives $C_1, \dots, C_n$ of the orbits of Γ on the connected components of the Dynkin diagram of $\Phi(G)$ satisfying the following. Denote by Φ_i the roots in Φ that are linear combinations of roots corresponding to C_i ($1 \le i \le n$). Then for every root $\alpha \in \Phi_1 \cup \dots \cup \Phi_n$ and $1 \le t_1 \le p^s N$, there exists $u_{t_1,\alpha} \in \mathrm{Gal}(E/K)$ such that

$$(\gamma)^{t_1}(\alpha) = u_{t_1,\alpha}\alpha \quad \text{and} \quad u_{t_1,\alpha}|_{E\cap K^{\mathrm{tame}}} = u^{t_1}.$$

Condition (ii) of Definition 3.1.1 is equivalent to the condition

(ii′) There exists a basis $\mathcal{B}$ of X stabilized by $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$ and by $\langle \gamma^N \rangle$ on which the $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$-orbits and $\langle \gamma^N \rangle$-orbits agree, and such that there exist representatives $\{B_1, \dots, B_{n'}\}$ for these orbits on $\mathcal{B}$, and elements $v_{t_1,i} \in \mathrm{Gal}(E/K)$ for all $1 \le t_1 \le p^s N$ and $1 \le i \le n'$ satisfying

$$(\gamma)^{t_1}(B_i) = v_{t_1,i}(B_i) \quad \text{and} \quad v_{t_1,i}|_{E\cap K^{\mathrm{tame}}} = u^{t_1}.$$

Before showing in Proposition 3.1.4 that a large class of reductive groups is good, we prove a lemma that shows some more properties of good groups.

Lemma 3.1.3. *We assume that G is a good group, use the notation introduced in Definition 3.1.1 and Remark 3.1.2, and denote by E_t the tamely ramified Galois extension of K of degree N contained in E. Then the following statements hold.*

(a) *The basis $\mathcal{B}$ of X given in property (ii) is stabilized by $\mathrm{Gal}(E/E_t)$ and the $\mathrm{Gal}(E/E_t)$-orbits and $\langle \gamma^N \rangle$-orbits on $\mathcal{B}$ agree.*

(b) *G satisfies Assumption 2.4.2; more precisely, $T \times_K E_t$ is induced.*

(c) *We have $X^{\gamma^N} = X^{\mathrm{Gal}(E/E_t)}$. Moreover, the action of u on $X^{\mathrm{Gal}(E/E_t)}$ agrees with the action of γ on $X^{\gamma^N} = X^{\mathrm{Gal}(E/E_t)}$, so $X^{\mathrm{Gal}(E/K)} = X^{\Gamma}$.*

Proof. To show part (a), consider a representative B_i for a $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$-orbit on $\mathcal{B}$ as in Remark 3.1.2. By property (ii′) there exists $v_{p^sN,i} \in \mathrm{Gal}(E/K)$ such that $v_{p^sN}(B_i) = (\gamma)^{p^sN}(B_i) = B_i$ and $v_{p^sN,i}|_{E\cap K^{\mathrm{tame}}} = u^{p^sN}$. Choose $u_0 \in \mathrm{Gal}(E/K)$ such that $u_0|_{E\cap K^{\mathrm{tame}}} = u$. Then we can write $v_{p^sN,i} = v \cdot u_0^{p^sN}$ for some v in $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$, and $u_0^{p^sN}(B_i) = v^{-1}(B_i)$ is contained in the $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$-orbit of B_i. Note that the elements $u_0^{p^sNt_2}$ for $1 \le t_2 \le [(E \cap K^{\mathrm{tame}}) : E_t]$ are in

$\mathrm{Gal}(E/E_t)$ and form a set of representatives for $\mathrm{Gal}(E/E_t)/\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$, and hence $\mathrm{Gal}(E/E_t)(B_i) = \mathrm{Gal}(E/E \cap K^{\mathrm{tame}})(B_i)$. Thus $\mathcal{B}$ is stabilized by $\mathrm{Gal}(E/E_t)$ and the $\mathrm{Gal}(E/E_t)$-orbits on $\mathcal{B}$ coincide with the $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$-orbits, which coincide with the $\langle \gamma^N \rangle$-orbits. This proves part (a).

Part (b) follows from (a) by the definition of an induced torus.

In order to show part (c), note that $X^{\mathrm{Gal}(E/E_t)}$ is spanned (over $\mathbb{Z}$) by

$$\Bigl\{ \sum_{B \in \mathrm{Gal}(E/E_t)(B_i)} B \Bigr\}_{1 \le i \le n'} = \Bigl\{ \sum_{B \in \langle \gamma^N \rangle (B_i)} B \Bigr\}_{1 \le i \le n'}.$$

The $\mathbb{Z}$-span of the latter equals X^{γ^N}, which implies $X^{\gamma^N} = X^{\mathrm{Gal}(E/E_t)}$. Using Definition 3.1.1(ii) and the observation that $u|_{E_t}$ is a generator of $\mathrm{Gal}(E_t/K)$, we conclude that the action of u on $X^{\mathrm{Gal}(E/E_t)}$ agrees with the action of γ on $X^{\gamma^N} = X^{\mathrm{Gal}(E/E_t)}$ and that

$$X^{\mathrm{Gal}(E/K)} = \bigl(X^{\mathrm{Gal}(E/E_t)}\bigr)^{\mathrm{Gal}(E_t/K)} = \bigl(X^{\gamma^N}\bigr)^{\gamma} = X^{\Gamma}. \qquad \blacksquare$$

Proposition 3.1.4. *Examples of good groups include*

(a) *reductive groups that split over a tamely ramified field extension of K,*

(b) *simply connected or adjoint (semisimple) groups,*

(c) *products of good groups,*

(d) *groups that are the restriction of scalars of good groups along finite separable field extensions.*

Proof. (a) follows by taking $\Gamma = \mathrm{Gal}(E/K)$ and $u = \gamma$.

(b) can be deduced from (c) and (d) (whose proofs do not depend on (b)) as follows. If G is a simply connected or adjoint group then G is the direct product of restrictions of scalars of simply connected or adjoint absolutely simple groups. Hence by (c) and (d) it suffices to show that if G is a simply connected or adjoint absolutely simple group, then G is good. Recall that these groups are classified by choosing the attribute "simply connected" or "adjoint" and giving a connected finite Dynkin diagram together with an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ on it. We distinguish two possible cases.

Case 1: G splits over a cyclic field extension E of K. Then take $\Gamma = \mathrm{Gal}(E/K)$ and $u = \gamma$ or $u = 1$ according as the field extension is tamely ramified or wildly ramified, and choose $\mathcal{B}$ to be the set of simple roots of G, if G is adjoint, and the set of fundamental weights dual to the simple coroots of G (i.e. those weights pairing with one simple coroot to 1, and with all others to 0), if G is simply connected.

Case 2: G does not split over a cyclic field extension. Then G has to be of type D_4 and split over a field extension E of K of degree 6 with $\mathrm{Gal}(E/K) \simeq S_3$, where S_3 is the symmetric group on three letters. In this case we observe (using that G is simply connected or adjoint) that the orbits of the action of $\mathrm{Gal}(E/K)$ on X are the same as the orbits of a subgroup $\mathbb{Z}/3\mathbb{Z} \subset \mathrm{Gal}(E/K) \simeq S_3$. Moreover, as S_3 does not contain a normal subgroup of order 2, i.e. there does not exist a tamely ramified Galois extension of K of degree 3, this case can only occur if $p = 3$, and we can choose $\Gamma = \mathbb{Z}/3\mathbb{Z}$, u the nontrivial element in $\mathrm{Gal}(E \cap K^{\mathrm{tame}}/K) \simeq \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{B}$ as in Case 1 to see that G is good.

(c) In order to show part (c), suppose that $G_1, \dots, G_k$ are good groups with splitting fields $E_1, \dots, E_k$ and corresponding cyclic groups $\Gamma_1 = \langle \gamma_1 \rangle, \dots, \Gamma_k = \langle \gamma_k \rangle$ and generators $u_i \in \mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K)$, $1 \le i \le k$. Let $G = G_1 \times \cdots \times G_k$. Then G splits over the composition field E of $E_1, \dots, E_k$, and $|\mathrm{Gal}(E \cap K^{\mathrm{tame}}/K)|$ is the least common multiple of $|\mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K)|$, $1 \le i \le k$. Choose a generator u of the group $\mathrm{Gal}(E \cap K^{\mathrm{tame}}/K)$. For $i \in [1, k]$, the image of u in $\mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K)$ equals $u_i^{r_i}$ for some integer r_i coprime to $|\mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K)|$, which we assume to be coprime to p by adding $|\mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K)|$ if necessary. Hence $(\gamma_i)^{r_i}$ is a generator of Γ_i, and we define $\gamma = (\gamma_1)^{r_1} \times \cdots \times (\gamma_k)^{r_k}$ and $\Gamma = \langle \gamma \rangle$. Note that the order $|\Gamma| = p^s N$ of Γ is the least common multiple of $|\Gamma_i|$, $1 \le i \le k$, and hence N divides $|\mathrm{Gal}(E \cap K^{\mathrm{tame}}/K)|$. By 3.1.1(i), if $\alpha \in \Phi(G_i)$ then there exists $\overline{u}_{1,\alpha} \in \mathrm{Gal}(E_i/K)$ such that

$$\gamma(\alpha) = (\gamma_i)^{r_i}(\alpha) = \overline{u}_{1,\alpha}\alpha \quad \text{with} \quad \overline{u}_{1,\alpha} \equiv u_i^{r_i} \equiv u \text{ in } \mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K).$$

Let $u_{1,\alpha}$ be a preimage of $\overline{u}_{1,\alpha}$ in $\mathrm{Gal}(E/K)$. Using the equality

$$\begin{aligned}|\mathrm{Gal}(E/E \cap E_i^{\mathrm{tame}})|\,|\mathrm{Gal}(E \cap E_i^{\mathrm{tame}}/E_i)|\,|\mathrm{Gal}(E_i/K)| \\ &= |\mathrm{Gal}(E/K)| \\ &= |\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})|\,|\mathrm{Gal}(E \cap K^{\mathrm{tame}}/E_i \cap K^{\mathrm{tame}})|\,|\mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K)|,\end{aligned}$$

by considering the factors prime to p we obtain

$$|\mathrm{Gal}(E \cap E_i^{\mathrm{tame}}/E_i)| = |\mathrm{Gal}(E \cap K^{\mathrm{tame}}/E_i \cap K^{\mathrm{tame}})|.$$

Moreover, the kernel of $\mathrm{Gal}(E \cap E_i^{\mathrm{tame}}/E_i) \to \mathrm{Gal}(E \cap K^{\mathrm{tame}}/E_i \cap K^{\mathrm{tame}})$, where the map arises from restriction to $E \cap K^{\mathrm{tame}}$, has order a power of p, hence is trivial; so we deduce that the map is an isomorphism. Thus we can choose an element $u_0 \in \mathrm{Gal}(E/E_i) \subset \mathrm{Gal}(E/K)$ such that $u_0|_{E \cap K^{\mathrm{tame}}} = u^{|\mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K)|}$, because $u^{|\mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K)|} \in \mathrm{Gal}(E \cap K^{\mathrm{tame}}/E_i \cap K^{\mathrm{tame}})$. Since $u_{1,\alpha}|_{E_i \cap K^{\mathrm{tame}}} = u|_{E_i \cap K^{\mathrm{tame}}}$ and $u^{|\mathrm{Gal}(E_i \cap K^{\mathrm{tame}}/K)|}$ is a generator of $\mathrm{Gal}(E \cap K^{\mathrm{tame}}/E_i \cap K^{\mathrm{tame}})$, by multiplying $u_{1,\alpha}$ with powers of $u_0 \in \mathrm{Gal}(E/E_i)$ if necessary we can ensure that $u_{1,\alpha}|_{E \cap K^{\mathrm{tame}}} = u$. As $\mathrm{Gal}(E/E_i)$ fixes α, we also have $\gamma(\alpha) = u_{1,\alpha}(\alpha)$, and we conclude that G satisfies property (i) of Definition 3.1.1 for all $\alpha \in \Phi(G) = \coprod_{i=1}^k \Phi(G_i)$.

Choosing $\mathcal{B}$ to be the union of the bases $\mathcal{B}_i$ corresponding to the good groups G_i (by viewing X_i embedded into $X := X_1 \times \cdots \times X_k$), we conclude similarly that G satisfies property (ii). This proves that G is a good group and finishes part (c).

(d) Let $G = \mathrm{Res}_{F/K} \widetilde{G}$ for $\widetilde{G}$ a good group over F, $K \subset F \subset E$. Then there exists a corresponding $\mathrm{Gal}(E/K)$-stable decomposition $X = \bigoplus_{i=1}^d X_i$, where $d = [F : K]$, together with a decomposition of Φ as a disjoint union $\coprod_{1 \le i \le f} \widetilde{\Phi}_i$ such that $\mathrm{Gal}(E/K)$ acts transitively on the set of subspaces X_i with $\mathrm{Stab}_{\mathrm{Gal}(E/K)}(X_i) \simeq \mathrm{Gal}(E/F)$, and $(X_i, \widetilde{\Phi}_i, \check{X}_i, \check{\widetilde{\Phi}}_i)$ is isomorphic to the root datum $R(\widetilde{G})$ of $\widetilde{G}$ for $1 \le i \le f$. We suppose without loss of generality that the fixed field of $\mathrm{Stab}_{\mathrm{Gal}(E/K)}(X_1)$ is F, i.e. $\mathrm{Stab}_{\mathrm{Gal}(E/K)}(X_1) = \mathrm{Gal}(E/F)$, and we write $d = d_p \cdot d_{p'}$, where d_p is a power of p and $d_{p'}$ is coprime to p. As $\widetilde{G}$ is good, there exist a cyclic group $\widetilde{\Gamma} = \langle\widetilde{\gamma}\rangle$ acting on $(X_1, \widetilde{\Phi}_1, \Delta_1)$ and a generator $\widetilde{u}$ of $\mathrm{Gal}(E \cap F^{\mathrm{tame}}/F)$ satisfying the conditions in Definition 3.1.1. Fix a splitting $\mathrm{Gal}(E \cap F^{\mathrm{tame}}/F) \hookrightarrow \mathrm{Gal}(E/F)$, and let $\widetilde{u}_0$ be the image of $\widetilde{u}$ under the composition $\mathrm{Gal}(E \cap F^{\mathrm{tame}}/F) \hookrightarrow \mathrm{Gal}(E/F) \hookrightarrow \mathrm{Gal}(E/K)$. Note that we have a commutative diagram (where $N' = |\mathrm{Gal}(E \cap F^{\mathrm{tame}}/F)|$)

$$\begin{array}{ccccc}
\mathrm{Gal}(E\cap F^{\mathrm{tame}}/F) & \hookrightarrow & \mathrm{Gal}(E/F) & \hookrightarrow & \mathrm{Gal}(E/K) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathbb{Z}/N'\mathbb{Z} & \hookrightarrow & \mathbb{Z}/N'\mathbb{Z} \ltimes \mathrm{Gal}(E/E\cap F^{\mathrm{tame}}) & \hookrightarrow & \mathbb{Z}/(N'd_{p'})\mathbb{Z} \ltimes \mathrm{Gal}(E/E\cap K^{\mathrm{tame}})
\end{array}$$

Hence we can choose $u_0 \in \mathrm{Gal}(E/K)$ such that

$$u_0^d|_{E\cap K^{\mathrm{tame}}} = \widetilde{u}_0|_{E\cap K^{\mathrm{tame}}},$$

and $u := u_0|_{E\cap K^{\mathrm{tame}}}$ is a generator of $\mathrm{Gal}(E \cap K^{\mathrm{tame}}/K)$ (because $d = d_p d_{p'}$ with d_p invertible in $\mathbb{Z}/(N'd_{p'})\mathbb{Z}$). After renumbering the subspaces X_i for $i > 1$ if necessary, we can choose elements $\gamma_{t_2 d_{p'}} \in \mathrm{Gal}(E/K)$ with

$$\gamma_{t_2 d_{p'}}|_{E\cap K^{\mathrm{tame}}} = u_0|_{E\cap K^{\mathrm{tame}}} = u$$

for $1 \le t_2 \le d_p$ such that if we set $\gamma_{t_1 + t_2 d_{p'}} = u_0$ for $1 \le t_1 < d_{p'}, 0 \le t_2 < d_p$ then $\gamma_i(X_i) = X_{i+1}$ for $1 \le i < d$ and $\gamma_d(X_d) = X_1$. By multiplying γ_d by an element in $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$ if necessary, we can assume that $\gamma_d \circ \gamma_{d-1} \circ \cdots \circ \gamma_1 = \widetilde{u}_0$. Define $\gamma \in \mathrm{Aut}(R(G), \Delta)$ by

$$X = \bigoplus_{i=1}^{d} X_i \ni (x_1, \ldots, x_d) \mapsto (\widetilde{\gamma} \circ \widetilde{u}_0^{-1} \circ \gamma_d x_d, \gamma_1 x_1, \gamma_2 x_2, \ldots, \gamma_{d-1} x_{d-1}).$$

Then the cyclic group $\Gamma = \langle\gamma\rangle$ preserves Δ, and we claim that Γ and u satisfy the conditions for G in Definition 3.1.1.

Property (i) of Definition 3.1.1 is satisfied by the construction of γ.

To check property (ii), let $\widetilde{\mathcal{B}}$ be a basis of $X_1 \subset X$ stabilized by $\mathrm{Gal}(E/E \cap F^{\mathrm{tame}})$ with a set of representatives $\{\widetilde{B}_1, \ldots, \widetilde{B}_{\widetilde{n}'}\}$ and $\widetilde{v}_{t_1,i} \in \mathrm{Gal}(E/F)$ with $(\widetilde{\gamma})^{t_1}(B_i) = \widetilde{v}_{t_1,i}(B_i)$ $(1 \le t_1 \le p^s N/d)$ satisfying all conditions of Remark 3.1.2(ii′) for $\widetilde{G}$. For $1 \le i \le \widetilde{n}'$ and $1 \le j \le d_{p'}$, define

$$B_{(i-1)d_{p'}+j} = u_0^{j-1}(\widetilde{B}_i) = \gamma_{j-1} \circ \cdots \circ \gamma_1(\widetilde{B}_i).$$

Note that $\langle\gamma^N\rangle(X_1) = \coprod_{0\le i<d_p} X_{1+id_{p'}}$, and hence, with $n' = \widetilde{n}' \cdot d_{p'}$, the set

$$\mathcal{B} = \bigcup_{1\le i\le n'} \langle\gamma^N\rangle(\{B_i\})$$

forms a basis of X (because γ^N has order d_p). We will show that $\mathcal{B}$ satisfies property (ii′) of Remark 3.1.2 with the set of orbit representatives $\{B_i\}_{1\le i\le n'}$ (and hence satisfies (ii) of Definition 3.1.1).

For $1\le t\le p^sN$, $1\le i\le \widetilde{n}'$, $1\le j\le d_{p'}$, we define $v_{t,(i-1)d_{p'}+j}\in \mathrm{Gal}(E/K)$ by

$$v_{t,(i-1)d_{p'}+j} = \begin{cases} \gamma_{j-1+t}\circ\cdots\circ\gamma_j & \text{if } j+t\le d,\\ \gamma_{t_2}\circ\cdots\circ\gamma_1\circ\widetilde{v}_{t_1,i}\circ\gamma_1^{-1}\circ\cdots\circ\gamma_{j-1}^{-1} & \text{if } j+t>d, t=dt_1+t_2-j+1.\end{cases}$$

Then using $(\gamma)^d|_{X_1}=\widetilde{\gamma}$ and $\widetilde{\gamma}^{t_1}(\widetilde{B}_i)=\widetilde{v}_{t_1,i}(\widetilde{B}_i)\in X_1$, we obtain

$$(\gamma)^t(B_i)=v_{t,i}(B_i)\quad \text{for all } 1\le t\le p^sN \text{ and } 1\le i\le n'.$$

Moreover, since

$$\widetilde{v}_{t_1,i}|_{E\cap F^{\mathrm{tame}}}=\widetilde{u}^{t_1} \implies \widetilde{v}_{t_1,i}|_{E\cap K^{\mathrm{tame}}}=\widetilde{u}_0^{t_1}|_{E\cap K^{\mathrm{tame}}}=u_0^{dt_1}|_{E\cap K^{\mathrm{tame}}}=u^{dt_1}$$

and $\gamma_k|_{E\cap K^{\mathrm{tame}}}=u$ for all $1\le k<d$ by definition, we obtain

$$v_{t,i}|_{E\cap K^{\mathrm{tame}}}=u^t\quad \text{for all } 1\le t\le p^sN \text{ and } 1\le i\le n'. \tag{11}$$

This shows that the action of $(\gamma)^{t_1}$ on B_i for $1\le t_1\le p^sN$ and $1\le i\le n'$ is as required by (ii′) of Remark 3.1.2. It remains to show that $\mathcal{B}$ is $\mathrm{Gal}(E/E\cap K^{\mathrm{tame}})$-stable and that the $\mathrm{Gal}(E/E\cap K^{\mathrm{tame}})$-orbits coincide with the $\langle\gamma^N\rangle$-orbits.

In order to do so, note that (11) implies in particular that for $1\le t_2\le d_p$, we have $v_{Nt_2,i}|_{E\cap K^{\mathrm{tame}}}\equiv u^{Nt_2}$, and hence $v_{Nt_2,i}\in\mathrm{Gal}(E/E_t)$ and

$$\langle\gamma^N\rangle(B_i)\subset \mathrm{Gal}(E/E_t)(B_i), \tag{12}$$

where E_t is the tamely ramified degree N field extension of K inside E. Let us denote by $\widetilde{E}_t$ the tamely ramified Galois extension of F of degree $N/d_{p'}$ contained in E. Note that E_t is the maximal tamely ramified subextension of $\widetilde{E}_t$ over K, and $[\widetilde{E}_t:E_t]=d_p$. As $\widetilde{G}$ is good, we deduce from Definition 3.1.1(ii) and Lemma 3.1.3(a) that

$$\langle\gamma^{Nd_p}\rangle(B_i)=\langle\widetilde{\gamma}^{N/d'_p}\rangle(B_i)=\mathrm{Gal}(E/E\cap F^{\mathrm{tame}})(B_i)=\mathrm{Gal}(E/\widetilde{E}_t)(B_i).$$

Using $\langle\gamma^N\rangle(X_1)=\coprod_{0\le i<d_p}X_{1+id_{p'}}$ and the inclusion (12), we find that

$$\begin{aligned}|\mathrm{Gal}(E/E_t)(B_i)|\ge|\langle\gamma^N\rangle(B_i)|&=d_p\cdot|\langle\gamma^{Nd_p}\rangle(B_i)|\\ &=d_p\cdot|\mathrm{Gal}(E/\widetilde{E}_t)(B_i)|\ge|\mathrm{Gal}(E/E_t)(B_i)|,\end{aligned}$$

which implies that $\langle\gamma^N\rangle(B_i)=\mathrm{Gal}(E/E_t)(B_i)\supset\mathrm{Gal}(E/E\cap K^{\mathrm{tame}})(B_i)$. In order to show that $\langle\gamma^N\rangle(B_i)=\mathrm{Gal}(E/E\cap K^{\mathrm{tame}})(B_i)$, we observe that $\mathrm{Gal}(E/E\cap F^{\mathrm{tame}})$ is a subgroup of $\mathrm{Gal}(E/E\cap K^{\mathrm{tame}})$ of index d_p coprime to the index $N/d_{p'}$ of $\mathrm{Gal}(E/E\cap F^{\mathrm{tame}})$ inside $\mathrm{Gal}(E/F)$. Therefore $\mathrm{Gal}(E/E\cap K^{\mathrm{tame}})\cap\mathrm{Gal}(E/F)=\mathrm{Gal}(E/E\cap F^{\mathrm{tame}})$ inside $\mathrm{Gal}(E/K)$. As $\mathrm{Gal}(E/F)$ is the stabilizer of X_1 in $\mathrm{Gal}(E/K)$, we deduce that there exist d_p representatives in $\mathrm{Gal}(E/E\cap K^{\mathrm{tame}})$ of the d_p classes in

$\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})/\,\mathrm{Gal}(E/E \cap F^{\mathrm{tame}})$ mapping X_1 to d_p distinct components X_i of X. In particular, we obtain

$$|\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})(B_i)| \geq d_p |\mathrm{Gal}(E/E \cap F^{\mathrm{tame}})(B_i)| = d_p |\langle \gamma^{N d_p} \rangle (B_i)| = |\langle \gamma^N \rangle (B_i)|,$$

and hence the $\mathrm{Gal}(E/E \cap K^{\mathrm{tame}})$-orbits on $\mathcal{B}$ agree with the $\langle \gamma^N \rangle$-orbits on $\mathcal{B}$. This finishes the proof that property (ii′) of Remark 3.1.2 and hence (ii) of Definition 3.1.1 is satisfied for our choice of Γ and u, and hence G is good. ■

From now on we assume that our group G is good.

3.2. *Construction of G_q*

In this section we define reductive groups G_q over non-archimedean local fields with arbitrary positive residue field characteristic q whose Moy–Prasad filtration quotients are in a certain way (made precise in Theorem 3.4.1) the "same" as those of the given good group G over K.

For the rest of the paper, assume $x \in \mathcal{B}(G, K)$ is a rational point of order m. Here *rational* means that $\psi(x)$ is in $\mathbb{Q}$ for all affine roots $\psi \in \Psi_K$, and the *order of a point in the Bruhat–Tits building* of x is defined to be the smallest positive integer m such that $\psi(x) \in \frac{1}{m}\mathbb{Z}$ for all affine roots $\psi \in \Psi_K$.

Fix a prime number q, and let Γ be the finite cyclic group acting on $R(G)$ as in Definition 3.1.1. Let F be a Galois extension of K containing E such that

- the set of valuations Γ'_a (defined in §2.3) is contained in $\mathrm{v}(F)$ for all $a \in \Phi_K$,
- $M := [F : K]$ is divisible by the order $p^s N$ of the group Γ,
- M is divisible by the order m of the point $x \in \mathcal{B}(G, K)$.

This implies that the image of x in $\mathcal{B}(G_F, F)$ is hyperspecial, and F satisfies all assumptions made in §2.6 in order to define $\iota_{K,F}$ and $\iota_{K,F,r}$. For later use, denote by ϖ_F a uniformizer of F such that $\varpi_F^{[F:E]} \equiv \varpi_E \bmod \varpi_F^{[F:E]+1}$, and let $\mathcal{O}_F$ be the ring of integers of F.

Let K_q be the splitting field of $x^M - 1$ over $\mathbb{Q}_q^{\mathrm{ur}}$, with ring of integers $\mathcal{O}_q$ and uniformizer ϖ_q. Let $F_q = K_q[x]/(x^M - \varpi_q)$ with uniformizer ϖ_{F_q} satisfying $\varpi_{F_q}^M = \varpi_q$ and with ring of integers $\mathcal{O}_{F_q}$. Recall that every reductive group over K_q is quasi-split, and there is a one-to-one correspondence between (quasi-split) reductive groups over K_q with root datum $R(G)$ and equivalence classes in $\mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{Q}}_q/K_q), \mathrm{Aut}(R(G), \Delta))/\text{conjugation by}\ \mathrm{Aut}(R(G), \Delta)$, where $\mathrm{Aut}(R(G), \Delta)$ denotes the group of automorphisms of the root datum $R(G)$ that stabilize Δ. Thus we can define a reductive group G_q over K_q by requiring that G_q has root datum $R(G)$ and that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_q/K_q)$ on $R(G)$ defining the K_q-structure factors through $\mathrm{Gal}(F_q/K_q)$ and is given by

$$\mathrm{Gal}(F_q/K_q) \simeq \mathbb{Z}/M\mathbb{Z} \xrightarrow{1 \mapsto \gamma} \Gamma \to \mathrm{Aut}(R(G), \Delta),$$

where the last map is the action of Γ on $R(G)$ as in Definition 3.1.1. This means that G_q is already split over $E_q := K_q[x]/(x^{p^s N} - \varpi_q)$. Note that by construction, Definition 3.1.1 and Lemma 3.1.3, the restricted root data of G_q and G agree:

$$R_{K_q}(G_q) = R_K(G),$$

and for all $\alpha \in \Phi = \Phi(G) = \Phi(G_q)$ we have

$$|\mathrm{Gal}(E/K) \cdot \alpha| = |\mathrm{Gal}(F_q/K_q) \cdot \alpha|. \tag{13}$$

All objects introduced in Section 2 can also be constructed for G_q, and we will denote them by the same letter(s), but with a G_q in parentheses to specify the group; e.g., we write $\Gamma'_a(G_q)$.

3.3. Construction of x_q

In order to compare the Moy–Prasad filtration quotients of G_q with those of G at x, we need to specify a point x_q in the Bruhat–Tits building $\mathcal{B}(G_q, K_q)$ of G_q. To do so, choose a maximal split torus S_q in G_q with centralizer denoted by T_q, and fix a Chevalley–Steinberg system $\{x_\alpha^{F_q}\}_{\alpha\in\Phi}$ for G_q with respect to T_q. For later use, we choose the Chevalley–Steinberg system to have signs $\epsilon_{\alpha,\beta}$ as in Definition 2.1.2, i.e.

$$m_\alpha^{F_q} := x_\alpha^{F_q}(1)x_{-\alpha}^{F_q}(\epsilon_{\alpha,\alpha})x_\alpha^{F_q}(1) \in N_{G_q}(T_q)(F_q),$$

where $N_{G_q}(T_q)$ denotes the normalizer of T_q in G_q, and

$$\mathrm{Ad}(m_\alpha^{F_q})(\mathrm{Lie}(x_\beta^{F_q})(1)) = \epsilon_{\alpha,\beta}\,\mathrm{Lie}(x_{s_\alpha(\beta)}^{F_q})(1).$$

That we can choose the same signs for both Chevalley–Steinberg systems follows from the property that the $\mathrm{Gal}(E/K)$-orbits on Φ agree with the $\mathrm{Gal}(E_q/K_q)$-orbits on Φ, which allows us to construct both Chevalley–Steinberg systems as the base change of the same Chevalley system of a split reductive group $\mathcal{G}$ over $\mathbb{Z}$ with root datum $R(G)$.

We provide a sketch of how this can be done: Start with a pinning $\{X_\alpha\}_{\alpha\in\Delta}$ for $\mathcal{G}$ (we omit the choice of a maximal split torus and a Borel subgroup from the notation in this sketch). Under appropriately chosen isomorphisms between $\mathcal{G}_E$ and G_E and between $\mathcal{G}_E$ and $(G_q)_{E_q}$ this pinning provides after base change a $\mathrm{Gal}(E/K)$-stable pinning of G_E and a $\mathrm{Gal}(E_q/K_q)$-stable pinning of $(G_q)_{E_q}$. By [8, XXIII, proof of Proposition 6.2] we can extend the pinning $\{X_\alpha\}_{\alpha\in\Delta}$ to a Chevalley system $\{X_\alpha\}_{\alpha\in\Phi}$ by choosing for every root $\alpha \in \Phi \setminus \Delta$ simple roots $\alpha_1, \dots, \alpha_j$ and α_Δ such that $\alpha = s_{\alpha_1} \dots s_{\alpha_j}(\alpha_\Delta)$, and defining $X_\alpha := \mathrm{Ad}(m_{\alpha_1} \dots m_{\alpha_j})X_{\alpha_\Delta}$. (Note that from these basis elements for the Lie algebra subspaces we can obtain isomorphisms from $\mathbb{G}_a$ to the corresponding root groups.) In order to obtain a Chevalley–Steinberg system after base change, we need to choose the α_i's compatible with the Galois action. This can be done by choosing for every $\mathrm{Gal}(E/K)$-orbit in $\Phi \setminus \Delta$ a representative α. Then for every representative α write $\alpha = s_{\alpha_1} \dots s_{\alpha_j}(\alpha_\Delta)$ and for every $\alpha' \in \mathrm{Gal}(E/K).\alpha$, pick $\sigma_{E,\alpha'} \in \mathrm{Gal}(E/K)$ (with $\sigma_{E,\alpha}$ the identity) such that $\alpha' = \sigma_{E,\alpha'}(\alpha)$. Set $X_{\alpha'} := \mathrm{Ad}(m_{s_{\sigma_{E,\alpha'}(\alpha_1)}} \dots m_{s_{\sigma_{E,\alpha'}(\alpha_j)}})X_{\sigma_{E,\alpha'}(\alpha_\Delta)}$.

Since the $\mathrm{Gal}(E/K)$-orbits on Φ agree with the $\mathrm{Gal}(E_q/K_q)$-orbits on Φ and since $\alpha, \alpha_1, \ldots, \alpha_j$ and α_Δ are all linear combinations of simple roots corresponding to the same connected component of the Dynkin diagram, we can choose $\sigma_{E,\alpha'}$ such that there exist $\sigma_{E_q,\alpha'} \in \mathrm{Gal}(E_q/K_q)$ with $\sigma_{E,\alpha'}(\alpha_i) = \sigma_{E_q,\alpha'}(\alpha_i)$ for $i \in \{1, \ldots, j\} \cup \{\Delta\}$. It remains to observe that the resulting Chevalley systems after base change satisfy all the properties of a Chevalley–Steinberg system. By construction, it suffices to consider the case of a connected Dynkin diagram. In this case the required properties can be shown using the commutation relations of a Chevalley system; see the proof of [22, 3.2 Lemma] for all cases except for D_4, and the case of D_4 can be easily worked out by hand; see also [22, §§10, 11].

Using the valuation constructed in §2.2 attached to the Chevalley–Steinberg system $\{x_\alpha^{F_q}\}_{\alpha\in\Phi}$, we obtain a point $x_{0,q}$ in the apartment $\mathcal{A}_q$ of $\mathcal{B}(G_q, K_q)$ corresponding to S_q. Fixing an isomorphism $f_{S,q} : X_*(S) \to X_*(S_q)$ that identifies $R_K(G)$ with $R_{K_q}(G_q)$, we define an isomorphism of affine spaces $f_{\mathcal{A},q} : \mathcal{A} \to \mathcal{A}_q$ by

$$f_{\mathcal{A},q}(y) = x_{0,q} + f_{S,q}(y - x_0) - \frac{1}{4} \sum_{a\in\Phi_K^{+,\mathrm{mul}}} \mathrm{v}(\lambda_a) \cdot \check{a}, \tag{14}$$

where $\Phi_K^{+,\mathrm{mul}}$ are the positive multipliable roots in Φ_K, $\lambda_a \in (E_\alpha)^1_{\max}(G)$ for some $\alpha \in \Phi_a$, and $\check{a}$ is the coroot of a, so we have $\check{a}(a) = 2$. We define $x_q := f_{\mathcal{A},q}(x)$.

Lemma 3.3.1. *The isomorphism $f_{\mathcal{A},q} : \mathcal{A} \to \mathcal{A}_q$ induces a bijection of affine roots $\Psi_{K_q}(\mathcal{A}_q) \to \Psi_K(\mathcal{A})$, $\psi \mapsto \psi \circ f_{\mathcal{A},q}$. Moreover, for all $a \in \Phi_K$ and $r \in \mathbb{R}$ we have $r - a(x - x_0) \in \Gamma'_a(G)$ if and only if $r - a(x_q - x_{0,q}) \in \Gamma'_a(G_q)$.*

Proof. As the set of affine roots for G on $\mathcal{A}$ (and analogously for G_q on $\mathcal{A}_q$) is

$$\Psi_K = \Psi_K(\mathcal{A}) = \{y \mapsto a(y - x_0) + \gamma' \mid a \in \Phi_K,\ \gamma' \in \Gamma'_a\},$$

we need to show that, for every $a \in \Phi_K = \Phi_K(G) = \Phi_{K_q}(G_q)$,

$$\Gamma'_a(G) = \Gamma'_a(G_q) - \frac{1}{4} \sum_{b\in\Phi_K^{+,\mathrm{mul}}} \mathrm{v}(\lambda_b) \cdot \check{b}(a). \tag{15}$$

Let us fix $a \in \Phi_K$ and $\alpha \in \Phi_a \subset \Phi = \Phi(G) = \Phi(G_q)$. Recall that $E_\alpha(G)$ is the fixed subfield of E under the action of $\mathrm{Stab}_{\mathrm{Gal}(E/K)}(\alpha)$. Using (13), we obtain

$$\begin{aligned}[E_\alpha(G) : K] &= \frac{|\mathrm{Gal}(E/K)|}{|\mathrm{Stab}_{\mathrm{Gal}(E/K)}(\alpha)|} = |\mathrm{Gal}(E/K)\cdot\alpha| = |\mathrm{Gal}(F_q/K_q)\cdot\alpha| \\ &= \frac{|\mathrm{Gal}(F_q/K_q)|}{|\mathrm{Stab}_{\mathrm{Gal}(F_q/K_q)}(\alpha)|} = [E_\alpha(G_q) : K_q],\end{aligned}$$

and hence

$$\mathrm{v}(E_\alpha(G) - \{0\}) = [E_\alpha(G)/K]^{-1} \cdot \mathbb{Z} = [E_\alpha(G_q)/K_q]^{-1} \cdot \mathbb{Z} = \mathrm{v}(E_\alpha(G_q) - \{0\}). \tag{16}$$

Note that the Dynkin diagram $\mathrm{Dyn}(G)$ of $\Phi(G)$ is a disjoint union of irreducible Dynkin diagrams, and if a is a multipliable root, then α is contained in the span of the simple roots of a Dynkin diagram of type A_{2n}. Thus by (16) and the description of Γ'_a as in (4) (§2.3), the equality (15) holds for α in the span of simple roots of an irreducible Dynkin diagram of any type other than A_{2n}, $n \in \mathbb{Z}_{>0}$, or in the span of an irreducible Dynkin diagram of type A_{2n} whose $2n$ simple roots lie in $2n$ distinct Galois orbits. It therefore remains to prove the lemma in the case of $\mathrm{Dyn}(G)$ being a disjoint union of finitely many A_{2n} whose simple roots form n orbits under the action of $\mathrm{Gal}(E/K)$. An easy calculation (see the proof of Lemma 2.6.1 for details) shows that, in this case, the positive multipliable roots of Φ_K form an orthogonal basis for the subspace of $X^*(S) \otimes \mathbb{R}$ generated by Φ_K, where by "orthogonal" we mean that $\check{b}(a) = 0$ if a and b are distinct positive multipliable roots, and that, if $b \in \Phi_K$ and $b = \sum_{a \in \Phi_K^{+,\mathrm{mul}}} \kappa_a a$ is not multipliable, then $\sum_{a \in \Phi_K^{+,\mathrm{mul}}} \kappa_a \in 2 \cdot \mathbb{Z}$. Moreover, by the definition of K_q and F_q, it is easy to check that for $\lambda_q \in (E_\alpha)^1_{\max}(G_q)$, we have $\mathrm{v}(\lambda_q) \in 2 \cdot \mathrm{v}(E_\alpha - \{0\})$. Thus using the description of Γ'_a as in (2) and (3) (§2.3), we see that the desired equation (15) holds.

The second claim of the lemma follows by combining (15) and the definition of x_q using the map in (14). ■

Note that Lemma 3.3.1 implies in particular that x_q is also a rational point of order m. Let us denote the reductive quotient of G_q at x_q by $\mathbf{G}_{x_q}$; the corresponding Moy–Prasad filtration groups by $G_{x_q,r}$, $r \geq 0$; the Lie algebra filtration by $\mathfrak{g}_{x_q,r}$, $r \in \mathbb{R}$; and the filtration quotients of the Lie algebra by $\mathbf{V}_{x_q,r}$, $r \in \mathbb{R}$. Then using Lemma 2.4.1, we obtain the following corollary to Lemma 3.3.1.

Corollary 3.3.2. *The root data $R(\mathbf{G}_x)$ and $R(\mathbf{G}_{x_q})$ are isomorphic.*

3.4. Global Moy–Prasad filtration representation

Since $R(\mathbf{G}_x) = R(\mathbf{G}_{x_q})$ (Corollary 3.3.2), we can define a split reductive group scheme $\mathcal{H}$ over $\mathbb{Z}$ by requiring that $R(\mathcal{H}) = R(\mathbf{G}_x)$, and then $\mathcal{H}_{\overline{\mathbb{F}}_p} \simeq \mathbf{G}_x$ and $\mathcal{H}_{\overline{\mathbb{F}}_q} \simeq \mathbf{G}_{x_q}$; i.e., we can define the reductive quotient "globally". In this section we show that we can globally define not only the reductive quotient, but also its action on the Moy–Prasad filtration quotients. More precisely, we will prove the following theorem, where N is as in Definition 3.1.1, i.e., in particular, N is coprime to p.

Theorem 3.4.1. *Let r be a real number, and keep the notation §3.2 and §3.3, so G is a good reductive group over K and x a rational point of $\mathcal{B}(G, K)$. Then there exists a split reductive group scheme $\mathcal{H}$ over $\overline{\mathbb{Z}}[1/N]$ acting on a free $\overline{\mathbb{Z}}[1/N]$-module $\mathcal{V}$ satisfying the following. For every prime q coprime to N, there exist isomorphisms $\mathcal{H}_{\overline{\mathbb{F}}_q} \simeq \mathbf{G}_{x_q}$ and $\mathcal{V}_{\overline{\mathbb{F}}_q} \simeq \mathbf{V}_{x_q,r}$ such that the induced representation of $\mathcal{H}_{\overline{\mathbb{F}}_q}$ on $\mathcal{V}_{\overline{\mathbb{F}}_q}$ corresponds to the usual adjoint representation of $\mathbf{G}_{x_q}$ on $\mathbf{V}_{x_q,r}$. Moreover, there are isomorphisms $\mathcal{H}_{\overline{\mathbb{F}}_p} \simeq \mathbf{G}_x$ and $\mathcal{V}_{\overline{\mathbb{F}}_p} \simeq \mathbf{V}_{x,r}$ such that the induced representation of $\mathcal{H}_{\overline{\mathbb{F}}_p}$ on $\mathcal{V}_{\overline{\mathbb{F}}_p}$ is the usual adjoint representation of $\mathbf{G}_x$ on $\mathbf{V}_{x,r}$. In other words, we have commutative* *diagrams*

$$\begin{array}{ccc} \mathcal{H}_{\overline{\mathbb{F}}_p} \times \mathcal{V}_{\overline{\mathbb{F}}_p} & \longrightarrow & \mathcal{V}_{\overline{\mathbb{F}}_p} \\ {\scriptstyle \simeq\times\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \mathbf{G}_x \times \mathbf{V}_{x,r} & \longrightarrow & \mathbf{V}_{x,r} \end{array} \qquad \begin{array}{ccc} \mathcal{H}_{\overline{\mathbb{F}}_q} \times \mathcal{V}_{\overline{\mathbb{F}}_q} & \longrightarrow & \mathcal{V}_{\overline{\mathbb{F}}_q} \\ {\scriptstyle \simeq\times\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \mathbf{G}_{x_q} \times \mathbf{V}_{x_q,r} & \longrightarrow & \mathbf{V}_{x_q,r} \end{array}$$

Remark 3.4.2. A reductive group G_q over K_q and a point $x_q \in \mathcal{B}(G_q, K_q)$ satisfying the conditions of the above theorem fail to exist for some reductive groups G that are not good groups. For example, let K be a maximal unramified extension of $\mathbb{Q}_2$, $E = K(\sqrt{-1})$, and G the corresponding norm 1 torus, i.e. the kernel of the norm map from $\mathrm{Res}_{E/K}\,\mathbb{G}_\mathrm{m}$ to $\mathbb{G}_\mathrm{m}$. Then $\mathcal{B}(G, K)$ consists of only one point x, the reductive quotient $\mathbf{G}_x$ is trivial, and

$$\mathbf{V}_{x,r} \simeq \begin{cases} \overline{\mathbb{F}}_p & \text{if } r \in \mathbb{Z}, \\ \{0\} & \text{if } r \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$$

However, for $q > 2$, there does not exist a reductive group G_q over a finite extension K_q of $\mathbb{Q}_q^{\mathrm{ur}}$ and $x_q \in \mathcal{B}(G_q, K_q)$ so that the above theorem holds. Here is a sketch of the argument: Assume such a group G_q exists. Since the reductive quotient is trivial, G_q has to be anisotropic. Since $\sum_{s \le r < s+1} \dim \mathbf{V}_{x_q,r} = \sum_{s \le r < s+1} \dim \mathbf{V}_{x,r} = 1$ for any $s \in \mathbb{R}$, the group G_q has to be a one-dimensional torus, hence G_q has to be the norm 1 torus of a quadratic extension E_q of K_q. However, this implies

$$\mathbf{V}_{x_q,r} \simeq \begin{cases} \overline{\mathbb{F}}_q & \text{if } r \in 1/2 + \mathbb{Z}, \\ \{0\} & \text{if } r \in \mathbb{R} \setminus (1/2 + \mathbb{Z}). \end{cases}$$

We prove the theorem in two steps. In §3.4.1 we construct a morphism from $\mathcal{H}$ to an auxiliary split reductive group scheme $\mathcal{G}$, and in §3.4.2 we construct $\mathcal{V}$ (largely) inside the Lie algebra of $\mathcal{G}$ and use the adjoint action of $\mathcal{G}$ on its Lie algebra to define the action of $\mathcal{H}$ on $\mathcal{V}$.

3.4.1. Global reductive quotient. Let $\mathcal{G}$ be a split reductive group scheme over $\mathbb{Z}$ whose root datum is the root datum of G. In this section we construct a morphism $\iota : \mathcal{H} \to \mathcal{G}$ that lifts all the morphisms $\iota_{K,F} : G_{x,0}/G_{x,0+} \hookrightarrow G^F_{x,0}/G^F_{x,0+}$ and $\iota_{K_q,F_q} : G_{x_q,0}/G_{x_q,0+} \hookrightarrow G^{F_q}_{x_q,0}/G^{F_q}_{x_q,0+}$ defined in §2.6. In order to do so, let us first describe the image of $\iota_{K,F}$ more explicitly. In analogy to the root group parametrization x_a defined in §2.2, and using the notation from that section, we define for $a \in \Phi_K(G)$ multipliable the more general map $X_a : F \times F \to G(F)$ by

$$X_a(u, v) = \prod_{\beta \in [\Phi_a]} x^E_\beta(u_\beta)\, x^E_{\beta+\tilde{\beta}}(-v_\beta)\, x^E_{\tilde{\beta}}(\sigma(u)_\beta),$$

where σ denotes an element of $\mathrm{Gal}(F/E_{\alpha+\tilde{\alpha}})$ that projects to the non-trivial element of $\mathrm{Gal}(E_\alpha/E_{\alpha+\tilde{\alpha}})$ and where $u_\beta = \gamma(u)$ for some fixed choice of $\gamma \in \mathrm{Gal}(F/K)$ with $\gamma(\alpha) = \beta$. Note that $X_a|_{H_0(E_\alpha, E_{\alpha+\tilde{\alpha}})}$ $(\alpha \in \Phi_a)$ agrees with x_a. We then have the following lemma.

Lemma 3.4.3. *Let $\chi : \overline{\mathbb{F}}_p \to \mathcal{O}_{\mathbb{Q}_p^{\mathrm{ur}}}$ (if $\mathbb{Q}_p \subset F$) or $\chi : \overline{\mathbb{F}}_p \to \mathcal{O}_{\mathbb{F}_p((t))^{\mathrm{ur}}}$ (if $\mathbb{F}_p((t)) \subset F$) is the Teichmüller lift, and $\mathbf{U}_a$ the root group of $\mathbf{G}_x$ corresponding to the root $a \in \Phi(\mathbf{G}_x) \subset \Phi_K(G)$. Define $y_a : \overline{\mathbb{F}}_p \to G^F_{x,0}$ by letting $y_a(u)$ be*

$$\begin{cases} X_a(\sqrt{2}\,\chi(u)\cdot \varpi_F^{-a(x-x_0)\cdot M}, \chi(u)\varpi_F^{-a(x-x_0)\cdot M}\sigma(\chi(u)\varpi_F^{-a(x-x_0)\cdot M})) & \\ & \text{if } a \text{ is multipliable and } p \neq 2, \\ X_a(0, \chi(u)\sigma(\chi(u))\varpi_F^{-2a(x-x_0)\cdot M}) & \text{if } a \text{ is multipliable and } p = 2, \\ X_a(0, \chi(u)\cdot \varpi_F^{-2a(x-x_0)\cdot M}) & \text{if } a \text{ is divisible}, \\ x_a(\chi(u)\cdot \varpi_F^{-a(x-x_0)\cdot M}) & \text{otherwise.} \end{cases}$$

Then the composition $\overline{y}_a$ of y_a with the quotient map $G^F_{x,0} \twoheadrightarrow G^F_{x,0}/G^F_{x,0+}$ is isomorphic to $\iota_{K,F} \circ \overline{x}_a : \overline{\mathbb{F}}_p \to \iota_{K,F}(\mathbf{U}_a(\overline{\mathbb{F}}_p)) \subset \mathbf{G}^F_x(\overline{\mathbb{F}}_p)$.

Proof. If $p \neq 2$ or if a is not multipliable, the conclusion follows immediately from Lemma 2.5.1.

For $p = 2$, note that (using the notation from Lemma 2.5.1)

$$\mathrm{v}\big(\chi(u)\varpi_F^{s'}\sigma(\chi(u)\varpi_F^{s'})\cdot \varpi_F^{\mathrm{v}(\lambda)M}\big) < 2\mathrm{v}\big(\sqrt{1/\lambda_0}\,\chi(u)\varpi_F^{s'}\big),$$

where $s' = -(a(x - x_0) + \mathrm{v}(\lambda)/2)M$, because $\mathrm{v}(\lambda) < 0$. Moreover, $\sigma(\varpi_F) \equiv \varpi_F \bmod \varpi_F^2$ in $\varpi_F\mathcal{O}_F/\varpi_F^2\mathcal{O}_F$, and hence $\overline{y}_a(u) = \iota_{K,F}(\overline{x}_a(u))$ by Lemma 2.5.1. ∎

Remark 3.4.4. An analogous statement holds for $\mathbf{G}_{x_q}$. In what follows, we denote the root group parametrizations constructed for $\mathbf{G}_{x_q}$ analogously to Lemma 2.5.1 by $\overline{x^{K_q}}_a : \mathbb{G}_{\mathrm{a}} \to \mathbf{U}^{K_q}{}_a, a \in \Phi(\mathbf{G}_{x_q})$.

Recall that x is hyperspecial in $\mathcal{B}(G_F, F)$, and hence the reductive quotient $\mathbf{G}^F_x$ of G_F at x is a split reductive group over $\overline{\mathbb{F}}_p$ with root datum $R(\mathbf{G}^F_x) = R(G)$. The analogous statement holds for x_q. Thus $\mathcal{G}_{\overline{\mathbb{F}}_p}$ is isomorphic to $\mathbf{G}^F_x$, and $\mathcal{G}_{\overline{\mathbb{F}}_q}$ is isomorphic to $\mathbf{G}^{F_q}_{x_q}$. In order to construct explicit isomorphisms, let us fix a split maximal torus $\mathcal{T}$ of $\mathcal{G}$ and a Chevalley system $\{\chi_\alpha : \mathbb{G}_{\mathrm{a}} \xrightarrow{\simeq} \mathcal{U}_\alpha \subset \mathcal{G}\}_{\alpha\in\Phi(\mathcal{G})=\Phi}$ for $(\mathcal{G}, \mathcal{T})$ with signs equal to $\epsilon_{\alpha,\beta}$ as in Definition 2.1.2; i.e., the Chevalley system $\{\chi_\alpha\}_{\alpha\in\Phi}$ for $(\mathcal{G}, \mathcal{T})$ and the Chevalley–Steinberg system $\{x_\alpha\}_{\alpha\in\Phi}$ for (G, T) have the same signs. This is possible since we can construct the Chevalley system underlying the Chevalley–Steinberg system $\{x_\alpha\}_{\alpha\in\Phi}$ for (G, T) over $\mathbb{Z}$ as outlined in §3.3.

Moreover, the split maximal torus $T_F \subset G_F$ and the Chevalley system $\{x^F_\alpha = x^E_\alpha \times_E F\}_{\alpha\in\Phi}$ yield a split maximal torus $\mathbf{T}^F_x$ of $\mathbf{G}^F_x$ and a Chevalley system $\{\overline{x^F}_\alpha : \mathbb{G}_{\mathrm{a}} \xrightarrow{\simeq} \mathbf{U}^F_\alpha \subset \mathbf{G}^F_x\}_{\alpha\in\Phi}$ for $(\mathbf{G}^F_x, \mathbf{T}^F_x)$ with signs $\epsilon_{\alpha,\beta}$, where $\mathbf{U}^F_\alpha$ denotes the root subgroup of $\mathbf{G}^F_x$ corresponding to α. Similarly, we obtain a split maximal torus $\mathbf{T}^{F_q}_{x_q}$ of $\mathbf{G}^{F_q}_{x_q}$ and a Chevalley system $\{\overline{x^{F_q}}_\alpha : \mathbb{G}_{\mathrm{a}} \xrightarrow{\simeq} \mathbf{U}^{F_q}_\alpha \subset \mathbf{G}^{F_q}_{x_q}\}_{\alpha\in\Phi}$ for $(\mathbf{G}^{F_q}_{x_q}, \mathbf{T}^{F_q}_{x_q})$ with signs $\epsilon_{\alpha,\beta}$, where $\mathbf{U}^{F_q}_\alpha$ denotes the root subgroup of $\mathbf{G}^{F_q}_{x_q}$ corresponding to α. In addition, we denote by $\mathbf{T}_x$ and $\mathbf{T}_{x_q}$ the maximal split tori of $\mathbf{G}_x$ and $\mathbf{G}_{x_q}$ corresponding to S and S_q.

Moreover, we define constants $c_{\alpha,q} \in \mathcal{O}_{F_q}$ and $c_\alpha \in \mathcal{O}_F$ for $\alpha \in \Phi$ as follows. We choose $\gamma' \in \mathrm{Gal}(F/K)$ such that

$$\gamma'|_{E \cap K^{\mathrm{tame}}} = u$$

and $\zeta_G \in \mathcal{O}_K$ satisfying

$$\gamma'(\varpi_F) \equiv \zeta_G \varpi_F \bmod \varpi_F^2.$$

Similarly, let $\gamma_q \in \mathrm{Gal}(F_q/K_q) \simeq \mathbb{Z}/M\mathbb{Z}$ correspond to $1 \in \mathbb{Z}/M\mathbb{Z}$, i.e.

$$\gamma_q|_{E_q} = \gamma \in \mathrm{Gal}(E_q/K)$$

and $\zeta_{G_q} \in \mathcal{O}_{K_q}$ such that

$$\gamma_q(\varpi_{F_q}) = \zeta_{G_q} \varpi_{F_q}.$$

Let $C_1, \dots, C_n$ be the representatives for the action of $\Gamma = \langle \gamma \rangle$ on the connected components of $\mathrm{Dyn}(G)$ as given in Remark 3.1.2(i′), and recall that Φ_i denotes the roots that are linear combinations of simple roots corresponding to C_i. For $\alpha \in \Phi$ there exists a unique triple $(i, \alpha_i, e_q(\alpha))$ with $i \in [1, n]$, $\alpha_i \in \Phi_i$ and $e_q(\alpha)$ minimal in $\mathbb{Z}_{\geq 0}$ such that $\gamma_q^{e_q(\alpha)}(\alpha_i) = \alpha$. Note that $e_q(\alpha)$ is independent of the choice of the prime number q. We also write $e(\alpha) = e_q(\alpha)$. We define

$$c_{\alpha,q} := \zeta_{G_q}^{e(\alpha)\cdot\alpha_i(x_q - x_{0,q})\cdot M} = \zeta_{G_q}^{e(\alpha)\cdot\alpha(x_q - x_{0,q})\cdot M},$$
$$c_\alpha := \zeta_G^{e(\alpha)\cdot\alpha_i(x - x_0)\cdot M} = \zeta_G^{e(\alpha)\cdot\alpha(x - x_0)\cdot M}.$$

Note that $\alpha_i(x - x_0) \cdot M$ is an integer, as the order m of x divides M and $\Gamma'_a \subset \mathrm{v}(F) = \frac{1}{M}\mathbb{Z}$, where a is the image of α in Φ_K.

Finally, we denote by $\overline{\zeta_G}$ and $\overline{\zeta_{G_q}}$ the images of ζ_G and ζ_{G_q} and by $\overline{c}_\alpha$ and $\overline{c}_{\alpha,q}$ the images of c_α and $c_{\alpha,q}$ under the surjections $\mathcal{O}_F \twoheadrightarrow \overline{\mathbb{F}}_p$ and $\mathcal{O}_{F_q} \twoheadrightarrow \overline{\mathbb{F}}_q$, respectively.

Remark 3.4.5. The integer $e(\alpha)$ depends only on the connected component of $\mathrm{Dyn}(G)$ in whose span α lies.

The definitions of ζ_G, ζ_{G_q} and $e(\alpha)$ are chosen so that the following lemma holds.

Lemma 3.4.6. *We keep the notation above and let $r \in \mathbb{R}$.*

(i) *If $\widetilde{\gamma} \in \mathrm{Gal}(F_q/K_q)$ with $\widetilde{\gamma}(\alpha_i) = \alpha$ and $r' := r - \alpha(x_q - x_{0,q}) \in \Gamma'_a(G_q)$, then*

$$\widetilde{\gamma}(\varpi_{F_q}^{r'M}) \equiv \zeta_{G_q}^{e(\alpha)\cdot(r - \alpha(x_q - x_{0,q}))M} \varpi_{F_q}^{r'M} \bmod \varpi_{F_q}^{r'M+1}.$$

(ii) *If $\widetilde{\gamma} \in \mathrm{Gal}(F/K)$ with $\widetilde{\gamma}(\alpha_i) = \alpha$ and $r' := r - \alpha(x - x_0) \in \Gamma'_a(G)$, then*

$$\widetilde{\gamma}(\varpi_F^{r'M}) \equiv \zeta_G^{e(\alpha)\cdot(r - \alpha(x - x_0))M} \varpi_F^{r'M} \bmod \varpi_F^{r'M+1}.$$

Proof. If $\widetilde{\gamma} \in \mathrm{Gal}(F_q/K_q)$ with $\widetilde{\gamma}(\alpha_i) = \alpha$, then $\widetilde{\gamma} = \gamma_q^{e(\alpha)+z|\langle\gamma\rangle(\alpha_i)|}$ for some integer z. As $r' \in \Gamma'_a(G_q) = \frac{1}{|\langle\gamma\rangle(\alpha_i)|}\mathbb{Z}$, we have $\overline{\zeta_{G_q}}^{|\langle\gamma\rangle(\alpha_i)|r'M} = 1$ and

$$\widetilde{\gamma}(\varpi_{F_q}^{r'M}) \equiv \gamma_q^{e(\alpha)+z|\langle\gamma\rangle(\alpha_i)|}(\varpi_{F_q}^{r'M}) \equiv \zeta_{G_q}^{e(\alpha)r'M}\varpi_{F_q}^{r'M}$$
$$\equiv \zeta_{G_q}^{e(\alpha)\cdot(r-\alpha(x_q-x_{0,q}))M}\varpi_{F_q}^{r'M} \bmod \varpi_{F_q}^{r'M+1},$$

which shows part (i).

In order to prove (ii), let $\widetilde{\gamma} \in \mathrm{Gal}(F/K)$ with $\widetilde{\gamma}(\alpha_i) = \alpha$, and write $\widetilde{\gamma} = \gamma'^{\widetilde{e}}\widetilde{w}$ for some integer $\widetilde{e}$ and $\widetilde{w} \in \mathrm{Gal}(F/E \cap K^{\mathrm{tame}})$. By Definition 3.1.1(i) and the definition of $e(\alpha)$ there exists $w \in \mathrm{Gal}(F/E \cap K^{\mathrm{tame}})$ such that $\gamma'^{e(\alpha)}w(\alpha_i) = \alpha$, and hence $\widetilde{w}^{-1}\gamma'^{e(\alpha)-\widetilde{e}}w(\alpha_i) = \alpha_i$, and therefore $(\gamma)^{e(\alpha)-\widetilde{e}}(\alpha_i) \in \mathrm{Gal}(F/E \cap K^{\mathrm{tame}})(\alpha_i)$. On the other hand, as the Γ-orbits on Φ agree with the $\mathrm{Gal}(F/K)$-orbits on Φ and $X^{\gamma^N} = X^{\mathrm{Gal}(F/E\cap K^{\mathrm{tame}})}$ (by Definition 3.1.1(ii) and Lemma 3.1.3), the $\mathrm{Gal}(F/E \cap K^{\mathrm{tame}})$-orbits on $\mathrm{Gal}(F/K)(\alpha_i)$ coincide with the $\langle\gamma^N\rangle$-orbits, which are the same as the $\langle\gamma^{N_i}\rangle$-orbits, where N_i is coprime to p such that $|\mathrm{Gal}(F/K)(\alpha_i)| = p^{s_i}N_i$ for some integer s_i. Thus $e(\alpha) - \widetilde{e} \equiv 0 \bmod N_i$. Note that $\overline{\zeta_G}^{N_i r'M} = 1$ in $\overline{\mathbb{F}}_p$, because $r' \in \Gamma'_a(G) = \frac{1}{p^{s_i}N_i}\mathbb{Z}$ if $p \neq 2$ and $r' \in \Gamma'_a(G) \subset \frac{1}{2p^{s_i}N_i}\mathbb{Z}$ if $p = 2$. Moreover, for $g \in \mathrm{Gal}(F/E \cap K^{\mathrm{tame}})$, $g(\varpi_F) \equiv \varpi_F \bmod \varpi_F^2$ as all p-power roots of unity in $\overline{\mathbb{F}}_p$ are trivial. Hence

$$\widetilde{\gamma}(\varpi_F^{r'M}) \equiv \gamma'^{\widetilde{e}}(\varpi_F^{r'M}) \equiv \zeta_G^{\widetilde{e}\cdot r'M}\varpi_F^{r'M} \equiv \zeta_G^{e(\alpha)\cdot(r-\alpha(x_q-x_{0,q}))M}\varpi_F^{r'M} \bmod \varpi_F^{r'M+1},$$

which proves (ii). ∎

Now let $f_T : \mathbf{T}_x^F \to \mathcal{T}_{\overline{\mathbb{F}}_p}$ be an isomorphism that identifies the root data $R(\mathbf{G}_x^F)$ and $R(\mathcal{G})$. Then we can extend f_T as follows.

Lemma 3.4.7. *There exists an isomorphism $f : \mathbf{G}_x^F \to \mathcal{G}_{\overline{\mathbb{F}}_p}$ extending f_T such that for $\alpha \in \Phi$ and $u \in \mathbb{G}_a(\overline{\mathbb{F}}_p)$ we have*

$$f(\overline{x^F}_\alpha(u)) = \chi_\alpha(\overline{c}_\alpha \cdot u). \tag{17}$$

Proof. Note that there exists a unique isomorphism $f : \mathbf{G}_x^F \to \mathcal{G}_{\overline{\mathbb{F}}_p}$ extending f_T and satisfying (17) for all $\alpha \in \Delta$. So we need to show that this f satisfies (17) for all $\alpha \in \Phi$. In order to do so, it suffices to show that the root group parametrizations $\{\chi_{\alpha\overline{\mathbb{F}}_p} \circ \overline{c}_\alpha\}_{\alpha\in\Phi}$ form a Chevalley system of $(\mathcal{G}_{\overline{\mathbb{F}}_p}, \mathcal{T}_{\overline{\mathbb{F}}_p})$ whose signs $\epsilon'_{\alpha,\beta}$ are equal to $\epsilon_{\alpha,\beta}$ $(\alpha, \beta \in \Phi)$, i.e. to the signs of $\{\overline{x^F}_\alpha\}_{\alpha\in\Phi}$. If α and β are linear combinations of roots in different connected components of the Dynkin diagram of Φ, then $\epsilon'_{\alpha,\beta} = 1 = \epsilon_{\alpha,\beta}$. Thus suppose $\alpha, \beta \in \gamma'(\Phi_1)$, and hence also $s_\alpha(\beta) \in \gamma'(\Phi_1)$, for some $\gamma' \in \mathrm{Gal}(F/K)$. By Remark 3.4.5 this implies that

$$\overline{\zeta}_{\gamma'} := \overline{\zeta_G}^{e(\alpha)} = \overline{\zeta_G}^{e(\beta)} = \overline{\zeta_G}^{e(s_\alpha(\beta))}.$$

We obtain (using [4, Cor. 5.1.9.2] for the second equality)

$$
\begin{aligned}
&\mathrm{Ad}\big(x_\alpha(\bar{c}_\alpha)x_{-\alpha}(\epsilon_{\alpha,\alpha}\bar{c}_{-\alpha})x_\alpha(\bar{c}_\alpha)\big)\big(\mathrm{Lie}(x_{\beta_{\overline{\mathbb{F}}_p}}\circ\bar{c}_\beta)(1)\big)\\
&\quad= \mathrm{Ad}\big(x_\alpha(\overline{\zeta}_{\gamma'}^{\alpha(x-x_0)\cdot M})x_{-\alpha}(\epsilon_{\alpha,\alpha}\overline{\zeta}_{\gamma'}^{-\alpha(x-x_0)\cdot M})x_\alpha(\overline{\zeta}_{\gamma'}^{\alpha(x-x_0)\cdot M})\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\big(\overline{\zeta}_{\gamma'}^{\beta(x-x_0)\cdot M}\,\mathrm{Lie}(x_{\beta_{\overline{\mathbb{F}}_p}})(1)\big)\\
&\quad= \mathrm{Ad}\big(\check{\alpha}(\overline{\zeta}_{\gamma'}^{\alpha(x-x_0)\cdot M})\big)\,\mathrm{Ad}(x_\alpha(1)x_{-\alpha}(\epsilon_{\alpha,\alpha})x_\alpha(1))\big(\overline{\zeta}_{\gamma'}^{\beta(x-x_0)\cdot M}\,\mathrm{Lie}(x_{\beta_{\overline{\mathbb{F}}_p}})(1)\big)\\
&\quad= \overline{\zeta}_{\gamma'}^{\beta(x-x_0)\cdot M}\cdot\mathrm{Ad}\big(\check{\alpha}(\overline{\zeta}_{\gamma'}^{\alpha(x-x_0)\cdot M})\big)\big(\epsilon_{\alpha,\beta}\,\mathrm{Lie}(x_{s_\alpha(\beta)_{\overline{\mathbb{F}}_p}})(1)\big)\\
&\quad= \overline{\zeta}_{\gamma'}^{\beta(x-x_0)\cdot M}\cdot(s_\alpha(\beta))(\check{\alpha}(\overline{\zeta}_{\gamma'}^{\alpha(x-x_0)\cdot M}))\cdot\epsilon_{\alpha,\beta}\,\mathrm{Lie}(x_{s_\alpha(\beta)_{\overline{\mathbb{F}}_p}})(1)\\
&\quad= \overline{\zeta}_{\gamma'}^{\beta(x-x_0)\cdot M}\cdot\overline{\zeta}_{\gamma'}^{\langle\check{\alpha},s_\alpha(\beta)\rangle\cdot\alpha(x-x_0)\cdot M}\cdot\epsilon_{\alpha,\beta}\,\mathrm{Lie}(x_{s_\alpha(\beta)_{\overline{\mathbb{F}}_p}})(1)\\
&\quad= \overline{\zeta}_{\gamma'}^{(s_\alpha(\beta))(x-x_0)\cdot M}\,\epsilon_{\alpha,\beta}\,\mathrm{Lie}(x_{s_\alpha(\beta)_{\overline{\mathbb{F}}_p}})(1)\\
&\quad= \epsilon_{\alpha,\beta}\big(\mathrm{Lie}(x_{s_\alpha(\beta)_{\overline{\mathbb{F}}_p}}\circ\bar{c}_{s_\alpha(\beta)})(1)\big).
\end{aligned}
$$

Thus the signs of the Chevalley system $\{x_{\alpha_{\overline{\mathbb{F}}_p}}\circ\bar{c}_\alpha\}_{\alpha\in\Phi}$ are $\epsilon_{\alpha,\beta}$ as desired. ■

Similarly, for each prime q, let $f_{T,q}:\mathbf{T}_{x_q}^{F_q}\to\mathcal{T}_{\overline{\mathbb{F}}_q}$ be an isomorphism that identifies the root data $R(\mathbf{G}_{x_q}^F)$ and $R(\mathcal{G})$. Then we have the analogous statement.

Lemma 3.4.8. *There exists an isomorphism $f_q:\mathbf{G}_{x_q}^F\to\mathcal{G}_{\overline{\mathbb{F}}_q}$ extending $f_{T,q}$ such that for $\alpha\in\Phi$ and $u\in\mathbb{G}_a(\overline{\mathbb{F}}_q)$ we have*

$$f_q(\overline{x^{F_q}}_\alpha(u)) = x_\alpha(\bar{c}_{\alpha,q}\cdot u). \tag{18}$$

This allows us to define a map ι from $\mathcal{H}$ to $\mathcal{G}$ as follows.

Let $\mathcal{S}$ be a split maximal torus of $\mathcal{H}$. Then we have

$$X_*(\mathcal{S}) = X_*(\mathbf{T}_x) = X_*(S) = X_*(T)^{\mathrm{Gal}(F/K)} \hookrightarrow X_*(T) = X_*(\mathcal{T}),$$

where the first identification arises from $R(\mathcal{H}) = R(\mathbf{G}_x)$, the second from Lemma 2.4.1 and the fourth from $R(\mathcal{G}) = R(G)$. This yields a closed immersion $f_{\mathcal{S}}:\mathcal{S}\to\mathcal{T}$. Note that $f_{\mathcal{S}}$ also corresponds to the injection

$$X_*(\mathcal{S}) = X_*(\mathbf{T}_{x_q}) = X_*(S_q) = X_*(T_q)^{\mathrm{Gal}(F_q/K_q)} \hookrightarrow X_*(T_q) = X_*(\mathcal{T}),$$

and we have commutative diagrams

$$
\begin{array}{ccc}
\mathcal{S}_{\overline{\mathbb{F}}_p} & \xrightarrow{f_{\mathcal{S}}} & \mathcal{T}_{\overline{\mathbb{F}}_p}\\
\simeq\big| & & \big|\simeq\\
\mathbf{T}_x & \xrightarrow{\iota_{K,F}} & \mathbf{T}_x^F
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{S}_{\overline{\mathbb{F}}_q} & \xrightarrow{f_{\mathcal{S}}} & \mathcal{T}_{\overline{\mathbb{F}}_q}\\
\simeq\big| & & \big|\simeq\\
\mathbf{T}_{x_q} & \xrightarrow{\iota_{K_q,F_q}} & \mathbf{T}_{x_q}^{F_q}
\end{array}
$$

To define ι on root groups, let $\{x_{\mathcal{H}a}\}_{a \in \Phi(\mathcal{H}) = \Phi(\mathbf{G}_x)}$ be a Chevalley system for $(\mathcal{H}, \mathcal{S})$ such that there exists an isomorphism $\mathcal{H}_{\overline{\mathbb{F}}_q} \xrightarrow{\simeq} \mathbf{G}_{x_q}$ mapping $\mathcal{S}_{\overline{\mathbb{F}}_q}$ to $\mathbf{T}_{x_q}$ and identifying $(x_{\mathcal{H}a})_{\overline{\mathbb{F}}_q}$ with $\overline{x^{K_q}}_a$, or equivalently having the same signs as the Chevalley system $\{\overline{x^{K_q}}_a\}_{a \in \Phi_K}$, for some $q \neq 2$.

Moreover, note that for $a \in \Phi_K = \Phi(\mathcal{H})$, there exists a unique integer in $[1, n]$, denoted by $n(a)$, such that $\Phi_a \cap \Phi_{n(a)} \neq \emptyset$ (see Remark 3.1.2 for the definition of $\Phi_i, i \in [1, n]$). We label the elements in $\Phi_a \cap \Phi_{n(a)}$ by $\{\alpha_i\}_{1 \leq i \leq |\Phi_a \cap \Phi_{n(a)}|}$ so that they satisfy the following two properties:

- If a is a multipliable root, we assume that $\alpha_1 \in [\Phi_a]$, where $[\Phi_a]$ is as defined in §2.2. (Note that a priori we have either α_1 or α_2 in $[\Phi_a]$.)
- Let γ be the generator of Γ as in Definition 3.1.1. Then for all $a \in \Phi_K$ such that $|\Phi_a \cap \Phi_{n(a)}| = 3$, there exists a minimal integer $e'(a)$ such that $\gamma^{e'(a)}$ preserves and acts non-trivially on $\Phi_a \cap \Phi_{n(a)}$, and we require that $\gamma^{e'(a)}(\alpha_1) = \alpha_2$. (Note that this implies $\gamma^{e'(a)}(\alpha_2) = \alpha_3$.)

We may (and do) assume that $[\Phi_a]$ is chosen to be $\{\gamma^i(\alpha_1) \mid 0 \leq i \leq |\Phi_a| - 1\}$.

Definition/Proposition 3.4.9. *There exists a unique group scheme homomorphism* $\iota : \mathcal{H}_{\overline{\mathbb{Z}}} \to \mathcal{G}_{\overline{\mathbb{Z}}}$ *extending* $f_{\mathcal{S}}$ *such that for all* $\overline{\mathbb{Z}}$*-algebras* A, $a \in \Phi(\mathcal{H}) = \Phi_K$ *and* $u \in \mathbb{G}_a(A)$ *we have*

$$\iota(x_{\mathcal{H}a}(u)) = \prod_{i=1}^{|\Gamma/\Gamma_{n(a)}|} x_{\gamma^{(i-1)}(\alpha_1)}(\sqrt{2}u)\, x_{\gamma^{(i-1)}(\alpha_1+\alpha_2)}(-(-1)^{-a(x-x_0)M} u^2)\, x_{\gamma^{(i-1)}(\alpha_2)}((-1)^{-a(x-x_0)M} \sqrt{2}u) \tag{19}$$

if a is multipliable,

$$\iota(x_{\mathcal{H}a}(u)) = \prod_{i=1}^{|\Gamma/\Gamma_{n(a)}|} x_{\gamma^{(i-1)}(\alpha_1)}(-u) \quad \text{if } a \text{ is divisible,} \tag{20}$$

and

$$\iota(x_{\mathcal{H}a}(u)) = \prod_{i=1}^{|\Gamma/\Gamma_{n(a)}|} \prod_{j=1}^{|\Phi_a \cap \Phi_{n(a)}|} x_{\gamma^{(i-1)}(\alpha_j)}(\zeta_{|\Phi_a \cap \Phi_{n(a)}|}^{-a(x-x_0)M(j-1)} u) \quad \text{otherwise,} \tag{21}$$

where ζ_i *is a primitive* i*-th root of unity,* $i = 1, 2$ *or* 3*, and* $\Gamma_{n(a)} = \mathrm{Stab}_\Gamma(\Phi_{n(a)})$.

Moreover, there exist unique isomorphisms $f_{\mathcal{H}} : \mathbf{G}_x \xrightarrow{\simeq} \mathcal{H}_{\overline{\mathbb{F}}_p}$ *and* $f_{\mathcal{H},q} : \mathbf{G}_{x_q} \xrightarrow{\simeq} \mathcal{H}_{\overline{\mathbb{F}}_q}$*, for every prime q, such that we have commutative diagrams*

$$\begin{array}{ccc} \mathcal{H}_{\overline{\mathbb{F}}_p} & \xrightarrow{\iota} & \mathcal{G}_{\overline{\mathbb{F}}_p} \\ {\scriptstyle f_{\mathcal{H}}} \big\downarrow \simeq & & \simeq \big\uparrow {\scriptstyle f} \\ \mathbf{G}_x & \xrightarrow{\iota_{K,F}} & \mathbf{G}_x^F \end{array} \qquad \begin{array}{ccc} \mathcal{H}_{\overline{\mathbb{F}}_q} & \xrightarrow{\iota} & \mathcal{G}_{\overline{\mathbb{F}}_q} \\ {\scriptstyle f_{\mathcal{H},q}} \big\downarrow \simeq & & \simeq \big\uparrow {\scriptstyle f_q} \\ \mathbf{G}_{x_q} & \xrightarrow{\iota_{K_q,F_q}} & \mathbf{G}_{x_q}^{F_q} \end{array}$$

Proof. Combining Lemma 3.4.3 and Remark 3.4.4 with Lemmas 3.4.7 and 3.4.8, we observe from Remark 3.1.2(i′) and Lemma 3.4.6 that $f \circ \iota_{K,F} \circ \overline{x}_a$ and $f_q \circ \iota_{K_q,F_q} \circ \overline{x^{K_q}}_a$ are described by the (reduction of the) right hand side of the three equations in the definition/proposition for all primes q. As $\iota_{K_q,F_q} \circ \overline{x^{K_q}}_a$ (and $\iota_{K,F} \circ \overline{x}_a$) are isomorphisms from $\mathbb{G}_{\mathrm{a}}$ to $\iota_{K_q,F_q}(\mathbf{U}_a^{K_q})$ (and $\iota_{K,F}(\mathbf{U}_a)$) for $q \neq 2$ (and for $p \neq 2$), the signs of the Chevalley systems $\{\overline{x^{K_q}}_a\}_{a\in\Phi_K}$ coincide with those of $\{\overline{x}_a\}$ and of $\{\chi_{\mathcal{H}a}\}$ for all q. (Note that $1 = -1$ in characteristic 2, i.e. the previous statement is trivial in this case.) This implies for every prime q the existence of a unique isomorphism $f_{\mathcal{H},q} : \mathbf{G}_{x_q} \xrightarrow{\simeq} \mathcal{H}_{\overline{\mathbb{F}}_q}$ that identifies $\mathbf{T}_{x_q}$ with $\mathcal{S}_{\overline{\mathbb{F}}_q}$ and $\overline{x^{K_q}}_a$ with $(\chi_{\mathcal{H}a})_{\overline{\mathbb{F}}_q}$ for all $a \in \Phi_K$, and similarly for $\mathbf{G}_x$.

Note that the equations (19)–(21) in the definition/proposition define group scheme homomorphisms $f_a : \mathbb{G}_{\mathrm{a}} \to \mathcal{G}_{\overline{\mathbb{Z}}}$ over $\overline{\mathbb{Z}}$ for $a \in \Phi(\mathcal{H})$. The maps $\{f_a\}_{a\in\Delta(\mathcal{H})}$ and $f_{\mathcal{S}}$ together with the requirement that $\chi_{\mathcal{H}a}(1)\chi_{\mathcal{H}-a}(\epsilon_{a,a})\chi_{\mathcal{H}a}(1) \mapsto f_a(1)f_{-a}(\epsilon_{a,a})f_a(1)$ for $a \in \Delta(\mathcal{H})$ define by [8, XXIII, Theorem 3.5.1] a unique group scheme homomorphism $\iota : \mathcal{H}_{\overline{\mathbb{Z}}} \to \mathcal{G}_{\overline{\mathbb{Z}}}$. (The required relations asked for in [8, XXIII, Theorem 3.5.1] can be checked to be satisfied using the fact that they hold in $\overline{\mathbb{F}}_q$ for all primes q by the existence of ι_{K_q,F_q} (similar to the subsequent argument).)

It remains to check that (19)–(21) hold for $a \in \Phi - \Delta(\mathcal{H})$. For this, note that $\iota(\chi_{\mathcal{H}s_b(a)}(\epsilon_{b,a}u)) = (f_b(1)f_{-b}(\epsilon_{b,b})f_b(1))\iota(\chi_{\mathcal{H}a}(u))(f_b(1)f_{-b}(\epsilon_{b,b})f_b(1))^{-1}$ for $a \in \Phi$, $b \in \Delta(\mathcal{H})$, where $\{\epsilon_{a,b}\}_{a,b\in\Phi_K}$ are the signs of the Chevalley system $\{\chi_{\mathcal{H}a}\}_{a\in\Phi_K}$. For $a, b \in \Delta(\mathcal{H})$, the validity of the equations in the proposition for $s_b(a)$ for all $u \in \mathbb{G}_{\mathrm{a}}(A)$ is therefore equivalent to the vanishing of a finite number of polynomials with coefficients in $\overline{\mathbb{Z}}$. As the latter vanish mod q for all primes q, these polynomials vanish also over $\overline{\mathbb{Z}}$, and the equations are satisfied for $s_b(a)$ ($b, a \in \Delta(\mathcal{H})$), and hence by repeating the argument for all roots $a \in \Phi$. ■

Remark 3.4.10. The morphism ι can be defined over $\mathbb{Z}[x]/(x^3 - 1) = \mathbb{Z}[\zeta_3]$ or even over $\mathbb{Z}$ if none of the connected components of $\mathrm{Dyn}(G)$ is of type D_4 with vertices contained in only two orbits.

In order to provide a different construction of $\mathcal{H}$ in Section 4, we use the following lemma.

Lemma 3.4.11. *Let ι be as in Definition/Proposition 3.4.9. Then $\iota_{\overline{\mathbb{Q}}} : \mathcal{H}_{\overline{\mathbb{Q}}} \to \mathcal{G}_{\overline{\mathbb{Q}}}$ is a closed immersion.*

Proof. It suffices to show that the kernel of $\iota_{\overline{\mathbb{Q}}}$ is trivial [4, Proposition 1.1.1]. As $\overline{\mathbb{Q}}$ is of characteristic zero, the kernel of $\iota_{\overline{\mathbb{Q}}}$ (a group scheme of finite type) is smooth. Hence we only need to show that $\iota_{\overline{\mathbb{Q}}}$ is injective on $\overline{\mathbb{Q}}$-points. Let $g \in \mathcal{H}(\overline{\mathbb{Q}})$. Let $\dot{W}$ be a set of representatives of the Weyl group of $\mathcal{H}$ in the normalizer of $\mathcal{S}$. Without loss of generality, we assume that the elements of $\dot{W}$ are products of $\chi_{\mathcal{H}a}(1)\chi_{\mathcal{H}-a}(\epsilon_{a,a})\chi_{\mathcal{H}a}(1)$, $a \in \Delta(\mathcal{H})$, or the identity. Let U be the unipotent radical of the Borel subgroup corresponding to $\Delta(\mathcal{H})$, U^- the one of the opposite Borel corresponding to $-\Delta(\mathcal{H})$, and $U_w = U(\overline{\mathbb{Q}}) \cap wU^-(\overline{\mathbb{Q}})w^{-1}$. By the Bruhat decomposition, we can write g uniquely as u_1wtu_2 with $w \in \dot{W}$, $t \in \mathcal{S}(\overline{\mathbb{Q}})$, $u_1 \in U_w$ and $u_2 \in U(\overline{\mathbb{Q}})$. By uniqueness $1 = \iota(g) =$

$\iota(u_1)\iota(w)\iota(t)\iota(u_2)$ if and only if $1 = \iota(u_1) = \iota(w) = \iota(t) = \iota(u_2)$. Note that $\iota(w) = 1$ implies $w = 1$ by our choice of $\dot{W}$, and $\iota(t) = 1$ implies $t = 1$. Choosing an order of the positive roots of Φ_K^+, there is a unique way to write $u_2 = \prod_{a \in \Phi_K^+} \chi_{\mathcal{H}_a}(u_a)$ with $u_a \in \overline{\mathbb{Q}}$ for all $a \in \Phi_K^+$. By choosing a compatible ordering of the roots in Φ^+ and the uniqueness of writing $\iota(u_2) = \prod_{\alpha \in \Phi^+} \chi_\alpha(u'_\alpha)$ with $u'_\alpha \in \overline{\mathbb{Q}}$ together with the explicit description of ι on root groups given in Definition/Proposition 3.4.9, we conclude that $u_a = 0$ for all $a \in \Phi_K^+$, and hence $u_2 = 1$. Similarly, $u_1 = 1$, which shows that the map ι is injective as desired. ∎

3.4.2. Global Moy–Prasad filtration quotients. In this section we will also lift the injections $\iota_{K,F,r} : \mathbf{V}_{x,r} \to \mathbf{V}_{x,r}^F$ and $\iota_{K_q,F_q,r} : \mathbf{V}_{x_q,r} \to \mathbf{V}_{x_q,r}^{F_q}$ in such a way that we get a lift of the commutative diagram (10). Using these injections we view $\mathbf{V}_{x,r}$ as a subspace of $\mathbf{V}_{x,r}^F$, and $\mathbf{V}_{x_q,r}$ as a subspace of $\mathbf{V}_{x_q,r}^{F_q}$. We will afterwards modify the global action slightly to also accommodate the case where $p = 2$ and there exists $a \in \Phi_K^{\mathrm{mul}}$ with $a(x - x_0) \in \Gamma'_a$ such that $a(x - x_0) - r \in \Gamma'_a$ or such that there exists $b \in \Phi_K^{\mathrm{nm}}$ with $b(x - x_0) - r \in \Gamma'_b$ and $\langle \check{a}, b \rangle \neq 0$.

We begin with the construction of an integral model for $\mathbf{V}_{x_q,r}$. Fix $r \in \mathrm{v}(F) = \mathrm{v}(F_q)$ (otherwise the diagram (10) would be trivial) and let ζ_M be a primitive M-th root of unity in $\overline{\mathbb{Z}}$ compatible with ζ_3 in Proposition 3.4.9, i.e. if $3 \mid M$, then $\zeta_M^{M/3} = \zeta_3$. Let ϑ denote the composition of the action of γ on $\mathrm{Lie}(\mathcal{T})(\overline{\mathbb{Z}}[1/N])$ induced from its action on $R(\mathcal{G}) = R(G)$ (as given by Definition 3.1.1), and multiplication by ζ_M^{rM}, and define $\mathcal{V}_T$ to be the free $\overline{\mathbb{Z}}[1/N]$-submodule of $\mathrm{Lie}(\mathcal{T})(\overline{\mathbb{Z}}[1/N])$ fixed by ϑ.

Next consider $a \in \Phi_K$. We recall that $\Gamma_{n(a)}$ denotes the stabilizer of the component $C_{n(a)}$ of the Dynkin diagram $\mathrm{Dyn}(G)$ inside Γ, and set $\mathcal{X}_\alpha = \mathrm{Lie}(\chi_\alpha)(1) \in \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$ for $\alpha \in \Phi$. We define

$$Y_a = \sum_{i=1}^{|\Phi_a \cap \Phi_{n(a)}|} \sum_{j=1}^{|\Gamma/\Gamma_{n(a)}|} \zeta_M{}^{e(\gamma(\alpha_1))(j-1)rM} \zeta_{|\Phi_a \cap \Phi_{n(a)}|}^{(-a(x_q - x_{0,q})+r)|\Gamma/\Gamma_{n(a)}|\,|\Phi_a \cap \Phi_{n(a)}|(i-1)} \mathcal{X}_{\gamma^{(j-1)}(\alpha_i)} \tag{22}$$

(note that $\zeta_{|\Phi_a \cap \Phi_{n(a)}|}^{(-a(x_q - x_{0,q})+r)|\Gamma/\Gamma_{n(a)}||\Phi_a \cap \Phi_{n(a)}|(i-1)} \in \{1, -1, \zeta_3, \zeta_3^2\}$) and let $\widetilde{\mathcal{V}}$ be the free $\overline{\mathbb{Z}}[1/N]$-submodule of $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$ generated by $\mathcal{V}_T$ and Y_a for all $a \in \Phi_K$ with $r - a(x_q - x_{0,q}) \in \Gamma'_a(G_q)$, or equivalently $r - a(x - x_0) \in \Gamma'_a(G)$ by Lemma 3.3.1. Note that $\widetilde{\mathcal{V}}$ as a $\overline{\mathbb{Z}}[1/N]$-module is a direct summand of the free $\overline{\mathbb{Z}}[1/N]$-module $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$.

Also note that the $\mathbf{G}_x^F$ representation $\mathbf{V}_{x,r}^F$ is isomorphic to the adjoint representation of $\mathbf{G}_x^F$ on $\mathrm{Lie}(\mathbf{G}_x^F)$, and similarly the $\mathbf{G}_{x_q}^{F_q}$ representation $\mathbf{V}_{x_q,r}^{F_q}$ is isomorphic to the adjoint representation of $\mathbf{G}_{x_q}^{F_q}$ on $\mathrm{Lie}(\mathbf{G}_{x_q}^{F_q})$. Hence the isomorphisms $f : \mathbf{G}_x^F \xrightarrow{\simeq} \mathcal{G}_{\overline{\mathbb{F}}_p}$ and $f_q : \mathbf{G}_{x_q}^{F_q} \xrightarrow{\simeq} \mathcal{G}_{\overline{\mathbb{F}}_q}$ from Lemmas 3.4.7 and 3.4.8 yield the isomorphisms $df := \mathrm{Lie}(f) : \mathbf{V}_{x,r}^F \simeq \mathrm{Lie}(\mathbf{G}_x^F)(\overline{\mathbb{F}}_p) \xrightarrow{\simeq} \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{F}}_p)$ and $df_q := \mathrm{Lie}(f_q) : \mathbf{V}_{x_q,r}^{F_q} \xrightarrow{\simeq} \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{F}}_q)$.

Proposition 3.4.12. *The adjoint action of* $\mathcal{G}_{\overline{\mathbb{Z}}[1/N]}$ *on* $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$ *restricts to an action of* $\mathcal{H}_{\overline{\mathbb{Z}}[1/N]}$ *on* $\widetilde{\mathcal{V}}$.

Let q be coprime to N. Then $df(\mathbf{V}_{x,r}) = \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_p}$ and $df_q(\mathbf{V}_{x_q,r}) = \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_q}$. Moreover, the following diagrams commute:

$$\begin{array}{ccc} \mathcal{H}_{\overline{\mathbb{F}}_p} \times \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_p} & \longrightarrow & \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_p} \\ {\scriptstyle f_{\mathcal{H}}^{-1} \times df^{-1}} \big\downarrow \simeq & & \simeq \big\downarrow {\scriptstyle df^{-1}} \\ \mathbf{G}_x \times \mathbf{V}_{x,r} & \longrightarrow & \mathbf{V}_{x,r} \end{array} \qquad \begin{array}{ccc} \mathcal{H}_{\overline{\mathbb{F}}_q} \times \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_q} & \longrightarrow & \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_q} \\ {\scriptstyle f_{\mathcal{H},q}^{-1} \times df_q^{-1}} \big\downarrow \simeq & & \simeq \big\downarrow {\scriptstyle df_q^{-1}} \\ \mathbf{G}_{x_q} \times \mathbf{V}_{x_q,r} & \longrightarrow & \mathbf{V}_{x_q,r} \end{array}$$

unless p or q is 2 (for the left or right diagram, respectively) and there exists $a \in \Phi_K^{\mathrm{mul}}$ with $a(x - x_0) \in \Gamma'_a$ such that $a(x - x_0) - r \in \Gamma'_a$ or such that there exists $b \in \Phi_K^{\mathrm{nm}}$ with $b(x - x_0) - r \in \Gamma'_b$ and $\langle \check{a}, b\rangle \neq 0$.

Proof. We first show that $df_q(\mathbf{V}_{x_q,r}) = \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_q}$ for q coprime to N and $df(\mathbf{V}_{x,r}) = \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_p}$ by considering the intersection of $\widetilde{\mathcal{V}}$ with the subspaces $\bigoplus_{\alpha\in\Phi(G)} \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])_\alpha$ and $\mathrm{Lie}(\mathcal{T})(\mathbb{Z}[1/N])$ of $\mathrm{Lie}(\mathcal{G})(\mathbb{Z}[1/N])$ separately.

For $\alpha \in \Phi$, denote by Γ_α the stabilizer of α in Γ, and let $\overline{X}_\alpha = \mathrm{Lie}(\overline{x^{F_q}}_\alpha)(1)$, $n_a = |\Phi_a \cap \Phi_{n(a)}| \in \{1, 2, 3\}$ and $\zeta_\gamma := \overline{\zeta_{G_q}}^{e(\gamma(\alpha_1))} = \overline{\zeta_{G_q}}^{e(\gamma(\alpha_i))}$, $1 \le i \le n_a$.

The image of $(\widetilde{\mathcal{V}} \cap \bigoplus_{\alpha\in\Phi(G)} \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])_\alpha) \otimes_{\overline{\mathbb{Z}}[1/N]} \overline{\mathbb{F}}_q$ under df_q^{-1} is then spanned by

$$\begin{aligned} \overline{Y}_a &= \sum_{i=1}^{n_a} \sum_{j=1}^{|\Gamma/\Gamma_{n(a)}|} \zeta_\gamma{}^{(j-1)rM} \zeta_{n_a}^{(-a(x_q - x_{0,q})+r)|\Gamma/\Gamma_{n(a)}|n_a(i-1)} \overline{c}^{-1}_{\gamma^{(j-1)}(\alpha_i),q} \overline{X}_{\gamma^{(j-1)}(\alpha_i)} \\ &= \sum_{i=1}^{n_a} \sum_{j=1}^{|\Gamma/\Gamma_{n(a)}|} \zeta_\gamma{}^{(j-1)rM} \zeta_{n_a}^{(-a(x_q - x_{0,q})+r)|\Gamma/\Gamma_{n(a)}|n_a(i-1)} \zeta_\gamma^{-\alpha(x_q - x_{0,q})M(j-1)} \overline{X}_{\gamma^{(j-1)}(\alpha_i)} \\ &= \sum_{j=1}^{|\Gamma/\Gamma_\alpha|} \zeta_\gamma{}^{(j-1)(r - a(x_q - x_{0,q}))M} \overline{X}_{\gamma^{(j-1)}(\alpha_1)} = \sum_{j=1}^{|\Gamma/\Gamma_\alpha|} \gamma^{(j-1)}(\overline{X}_{\alpha_1}) \end{aligned}$$

for $a \in \Phi_K$ with $r - a(x_q - x_{0,q}) \in \Gamma'_a(G_q)$ (where ζ_M maps to $\overline{\zeta_{G_q}}$ under the surjection $\overline{\mathbb{Z}}[1/N] \twoheadrightarrow \overline{\mathbb{F}}_q$). Here the action of Γ on $\mathbf{V}_{x_q,r}^{F_q}$ is the one induced from the action on $\mathfrak{g}_{x_q,r}^{F_q}$. Thus by definition of the Moy–Prasad filtration and the inclusion $\iota_{F_q,K_q,r}$ constructed in the proof of Lemma 2.6.1 we obtain the equality

$$\begin{aligned} df_q^{-1}(\widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_q}) \cap \bigoplus_{\alpha\in\Phi} \mathrm{Lie}(\mathbf{G}_{x_q}^{F_q})(\overline{\mathbb{F}}_q)_\alpha &= df_q^{-1}\Big(\Big(\widetilde{\mathcal{V}} \cap \bigoplus_{\alpha\in\Phi} \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])_\alpha\Big) \otimes \overline{\mathbb{F}}_q\Big) \\ &= \mathbf{V}_{x_q,r} \cap \bigoplus_{\alpha\in\Phi} \mathrm{Lie}(\mathbf{G}_{x_q}^{F_q})(\overline{\mathbb{F}}_q)_\alpha \qquad (23) \end{aligned}$$

inside $\mathbf{V}_{x_q,r}^{F_q} \simeq \mathrm{Lie}(\mathbf{G}_{x_q}^{F_q})(\overline{\mathbb{F}}_q)$.

In order to show the analogous statement for $V_{x,r}$, we claim that

$$\zeta_{n_a}^{(-a(x_q-x_{0,q})+r)|\Gamma/\Gamma_{n(a)}|n_a(i-1)} = \zeta_{n_a}^{(-a(x-x_0)+r)|\Gamma/\Gamma_{n(a)}|n_a(i-1)}$$

in $\overline{\mathbb{F}}_p$. This is obviously true for $p \neq 2$ as $a(x-x_0) = a(x_q - x_{0,q})$ in this case. If $p = 2$, then $\zeta_2 = -1 = 1$ in $\overline{\mathbb{F}}_p$ and we only have to consider the case $n_a = |\Phi_a \cap \Phi_{n(a)}| = 3$. However, $n_a = 3$ implies that the corresponding component $C_{n(a)}$ of $\mathrm{Dyn}(G)$ is of type D_4, and hence $\check{b}(a) = 0$ for all multipliable roots $b \in \Phi_K^{+,\mathrm{mul}}$. Thus $a(x-x_0) = a(x_q - x_{0,q})$ by definition (see (14)), and the claim follows.

Let $\zeta_\gamma = \overline{\zeta_G}^{e(\gamma(\alpha_1))}$, $\overline{X}_\alpha = \mathrm{Lie}(\overline{x^F}_\alpha)(1)$, and use otherwise the same notation as above. Then there exists a set of representatives $[\mathrm{Gal}(F/K)/\mathrm{Stab}_{\mathrm{Gal}(F/K)}(\alpha)]$ of $\mathrm{Gal}(F/K)/\mathrm{Stab}_{\mathrm{Gal}(F/K)}(\alpha)$ such that the image of $(\widetilde{\mathcal{V}} \cap \bigoplus_{\alpha\in\Phi(G)} \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])_\alpha) \otimes_{\overline{\mathbb{Z}}[1/N]} \overline{\mathbb{F}}_p$ under df^{-1} is spanned by

$$\begin{aligned}
\overline{Y}_a &= \sum_{i=1}^{n_a} \sum_{j=1}^{|\Gamma/\Gamma_{n(a)}|} \zeta_\gamma^{(j-1)rM} \zeta_{n_a}^{(-a(x_q-x_{0,q})+r)|\Gamma/\Gamma_{n(a)}|n_a(i-1)} \overline{c}^{-1}_{\gamma^{(j-1)}(\alpha_i)} \overline{X}_{\gamma^{(j-1)}(\alpha_i)} \\
&= \sum_{i=1}^{n_a} \sum_{j=1}^{|\Gamma/\Gamma_{n(a)}|} \zeta_\gamma^{(j-1)rM} \zeta_{n_a}^{(-a(x-x_0)+r)|\Gamma/\Gamma_{n(a)}|n_a(i-1)} \zeta_\gamma^{-\alpha(x-x_0)M(j-1)} \overline{X}_{\gamma^{(j-1)}(\alpha_i)} \\
&= \sum_{j=1}^{|\Gamma/\Gamma_\alpha|} \zeta_\gamma^{(j-1)(r-a(x-x_0))M} \overline{X}_{\gamma^{(j-1)}(\alpha_1)} = \sum_{\gamma'\in[\mathrm{Gal}(F/K)/\mathrm{Stab}_{\mathrm{Gal}(F/K)}(\alpha)]} \gamma'(\overline{X}_{\alpha_1}),
\end{aligned}$$

where the last equality follows from Lemma 3.4.6. Thus we obtain

$$df^{-1}(\widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_p}) \cap \bigoplus_{\alpha\in\Phi} \mathrm{Lie}(\mathbf{G}_x^F)(\overline{\mathbb{F}}_p)_\alpha = \mathbf{V}_{x,r} \cap \bigoplus_{\alpha\in\Phi} \mathrm{Lie}(\mathbf{G}_x^F)(\overline{\mathbb{F}}_p)_\alpha \tag{24}$$

inside $\mathbf{V}_{x,r}^F \simeq \mathrm{Lie}(\mathbf{G}_x^F)(\overline{\mathbb{F}}_p)$.

Let us consider $\mathcal{V}_T$. From the definition of the Moy–Prasad filtration $\mathfrak{t}_{x,r}^{E_t}$ of the Lie algebra $\mathfrak{t}_{E_t}$ of the torus T_{E_t} together with Lemma 3.1.3 and the observation that all p-power roots of unity in $\overline{\mathbb{F}}_p$ are trivial, we deduce (by sending $\zeta_M \otimes 1$ to $\overline{\zeta_G}$ under the isomorphism $\overline{\mathbb{Z}}[1/N] \otimes_{\overline{\mathbb{Z}}[1/N]} \overline{\mathbb{F}}_p \simeq \overline{\mathbb{F}}_p$, as above) that

$$df(\iota_{E_t,F,r}(\mathfrak{t}_{x,r}^{E_t}/\mathfrak{t}_{x,r+}^{E_t})) = (\mathrm{Lie}(\mathcal{T})(\overline{\mathbb{Z}}[1/N]))^{\vartheta^N} \otimes_{\overline{\mathbb{Z}}[1/N]} \overline{\mathbb{F}}_p.$$

Moreover, by combining Propositions 4.6.1 and 4.6.2 from [26, §4.6], we have $\mathfrak{t}_{x,r} = (\mathfrak{t}_{x,r}^{E_t})^{\mathrm{Gal}(E_t/K)}$ as E_t is tamely ramified over K, and we deduce (using tameness of E_t/K) that

$$\begin{aligned}
df(\mathfrak{t}_{x,r}/\mathfrak{t}_{x,r+}) = df((\mathfrak{t}_{x,r}^{E_t}/\mathfrak{t}_{x,r}^{E_t})^{\mathrm{Gal}(E_t/K)}) &= \left((\mathrm{Lie}(\mathcal{T})(\overline{\mathbb{Z}}[1/N]))^{\vartheta^N} \otimes_{\overline{\mathbb{Z}}[1/N]} \overline{\mathbb{F}}_p\right)^{\vartheta} \\
&= (\mathrm{Lie}(\mathcal{T})(\overline{\mathbb{Z}}[1/N]))^{\vartheta} \otimes_{\overline{\mathbb{Z}}[1/N]} \overline{\mathbb{F}}_p = \mathcal{V}_T \otimes_{\overline{\mathbb{Z}}[1/N]} \overline{\mathbb{F}}_p. \qquad (25)
\end{aligned}$$

For q coprime to N, we denote by $E_{t,q}$ the tamely ramified extension of degree N of K_q. Then we obtain by the same reasoning (replacing E_t by $E_{t,q}$)

$$df_q(\mathrm{t}_{x_q,r}/\mathrm{t}_{x_q,r+}) = \mathcal{V}_T \otimes_{\overline{\mathbb{Z}}[1/N]} \overline{\mathbb{F}}_q. \tag{26}$$

Combining (24) and (25), and (23) and (26), we find for q coprime to N that

$$df(\mathbf{V}_{x,r}) = \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_p} \quad \text{and} \quad df_q(\mathbf{V}_{x_q,r}) = \widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_q}.$$

In order to show that the adjoint action of $\mathcal{G}_{\overline{\mathbb{Z}}[1/N]}$ on $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$ restricts to an action of $\mathcal{H}_{\overline{\mathbb{Z}}[1/N]}$ on $\widetilde{\mathcal{V}}$, we observe that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{G}_{\overline{\mathbb{F}}_q} \times \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])_{\overline{\mathbb{F}}_q} & \longrightarrow & \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])_{\overline{\mathbb{F}}_q} \\
{\scriptstyle f_q^{-1}\times df_q^{-1}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle df_q^{-1}} \\
\mathbf{G}_{x_q}^{F_q} \times \mathbf{V}_{x_q,r}^{F_q} & \longrightarrow & \mathbf{V}_{x_q,r}^{F_q}
\end{array}$$

Since $\iota_{K_q,F_q}(\mathbf{G}_{x_q})$ preserves $\mathbf{V}_{x_q,r}$ (Lemma 2.6.1), we deduce that the induced action of $\mathcal{H}_{\overline{\mathbb{F}}_q}$ on $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{F}}_q)$ preserves $\widetilde{\mathcal{V}}_{\overline{\mathbb{F}}_q}$ for all q coprime to N. Hence the induced action of $\mathcal{H}_{\overline{\mathbb{Z}}[1/N]}$ on $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$ preserves $\widetilde{\mathcal{V}}$, and by construction and Lemma 2.6.1 and Definition/Proposition 3.4.9 the diagrams in the proposition commute (assuming the condition in the proposition in characteristic 2). ∎

In order to also obtain commutative diagrams in the case when p or q is 2 and there exists $a \in \Phi_K^{\mathrm{mul}}$ with $a(x - x_0) \in \Gamma'_a$, we define the $\overline{\mathbb{Z}}[1/N]$-submodule $\mathcal{V}$ of $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$ to be generated by $\mathcal{V}_T$, $\mathcal{V}_{\mathrm{nm}}$ and $\mathcal{V}_{\mathrm{mul}}$, where $\mathcal{V}_{\mathrm{nm}}$ is the $\overline{\mathbb{Z}}[1/N]$-submodule generated by Y_a for all $a \in \Phi_K^{\mathrm{nm}}$ with $r - a(x - x_0) \in \Gamma'_a(G)$ and $\mathcal{V}_{\mathrm{mul}}$ is the $\overline{\mathbb{Z}}[1/N]$-submodule generated by $\sqrt{2}\, Y_a$ for all $a \in \Phi_K^{\mathrm{mul}}$ with $r - a(x - x_0) \in \Gamma'_a(G)$. Note that $\mathcal{V}$ is a finite index submodule of $\widetilde{\mathcal{V}}$, and the injection $\mathcal{V} \hookrightarrow \widetilde{\mathcal{V}}$ yields an isomorphism $\mathcal{V} \otimes \overline{\mathbb{Z}}[1/(2N)] \xrightarrow{\simeq} \widetilde{\mathcal{V}} \otimes \overline{\mathbb{Z}}[1/(2N)]$.

Lemma 3.4.13. *Let R be a $\overline{\mathbb{Z}}[1/N]$-algebra. Then the image of $\mathcal{V} \otimes R$ in $\widetilde{\mathcal{V}} \otimes R$ is preserved by the action of $\mathcal{H}(R)$.*

Proof. To simplify notation, we assume $R = \overline{\mathbb{Z}}[1/N]$, but the proof is the same for general R. We need to show that $\mathcal{H}(R)$ maps $\mathcal{V}_T \oplus \mathcal{V}_{\mathrm{nm}}$ to $\mathcal{V}_T \oplus \mathcal{V}_{\mathrm{nm}} \oplus \mathcal{V}_{\mathrm{mul}}$. Since $\mathcal{S}$ preserves $\mathcal{V}_T \oplus \mathcal{V}_{\mathrm{nm}}$ it suffices to consider the action of the root groups $\chi_{\mathcal{H}_a}(R)$ for $a \in \Phi(\mathcal{H}) = \Phi(\mathbf{G}_x) \subset \Phi_K(G)$. Let $a \in \Phi(\mathcal{H}) \subset \Phi_K(G)$. If $\alpha \in \Phi_a$ is not contained in the span of roots of a connected component of the Dynkin diagram $\mathrm{Dyn}(G)$ that is of type A_{2n} and on which $\mathrm{Gal}(E/K)$ acts non-trivially, then $\chi_{\mathcal{H}_a}(R)$ preserves $\mathcal{V}_T \oplus \mathcal{V}_{\mathrm{nm}}$. By the same reasoning as in the proof of Lemma 2.6.1, if a corresponds to a non-multipliable root in $\Phi_K(G)$, then $\chi_{\mathcal{H}_a}(R)$ preserves $\mathcal{V}_T \oplus \mathcal{V}_{\mathrm{nm}}$ as well. Thus assume a is multipliable. Hence, for $u \in \mathbb{G}_a(R)$, by Definition/Proposition 3.4.9 we have

$$\iota(x_{\mathcal{H}_a}(u)) = \prod_{i=1}^{|\Gamma/\Gamma_{n(a)}|} x_{\gamma^{(i-1)}(\alpha_1)}(\sqrt{2}u)\, x_{\gamma^{(i-1)}(\alpha_1+\alpha_2)}(-(-1)^{-a(x-x_0)M}u^2) \cdot x_{\gamma^{(i-1)}(\alpha_2)}((-1)^{-a(x-x_0)M}\sqrt{2}u).$$

Let $H \in \mathcal{V}_T$. Using $x_\alpha(u)(H) = H - \mathrm{Lie}(\alpha)(H)u\mathcal{X}_\alpha$ for all $\alpha \in \Phi$, we observe that $x_{\mathcal{H}_a}(u)(H) = \iota(x_{\mathcal{H}_a}(u))(H)$ is contained in $\mathcal{V}_T \oplus \mathcal{V}_{\mathrm{nm}} \oplus \mathcal{V}_{\mathrm{mul}}$.

It remains to consider the action of $x_{\mathcal{H}_a}(u)$ on Y_b for $b \in \Phi_K^{\mathrm{nm}}$ with $r - b(x - x_0) \in \Gamma_b'(G)$. Let us assume (without influence on the arguments to follow) that α_1 and α_2 above are the simple roots α_1 and β_1 of a Dynkin diagram of type A_{2n} as in Figure 1 (§2.5). Then $x_{\mathcal{H}_a}(u)(Y_b) = Y_b$ unless b is the restriction of $\alpha_2 + \cdots + \alpha_t$ or of $-(\beta_1 + \alpha_1 + \cdots + \alpha_t)$ for some $2 < t \le n$ using the notation from Figure 1. In both cases we observe using the explicit formulas for $\iota(x_{\mathcal{H}_a}(u))$ and Y_b that $x_{\mathcal{H}_a}(u)(Y_b) = \iota(x_{\mathcal{H}_a}(u))(Y_b)$ is contained in $\mathcal{V}_T \oplus \mathcal{V}_{\mathrm{nm}} \oplus \mathcal{V}_{\mathrm{mul}}$. ∎

The lemma allows us to define an action of $\mathcal{H}$ on $\mathcal{V}$ by requiring that if R is an $\overline{\mathbb{Z}}[1/N]$-algebra in which $2 \neq 0$, then the action of $\mathcal{H}(R)$ on $\mathcal{V}_R$ is the restriction of the action of $\mathcal{H}(R)$ on $\widetilde{\mathcal{V}}_R$. Note that if N is odd, then for $\overline{g} \in \mathcal{H}(\overline{\mathbb{F}}_2)$ and $\overline{v} \in \mathcal{V}(\overline{\mathbb{F}}_2)$ there exist $g \in \mathcal{H}(\overline{\mathbb{Z}}[1/N])$ whose image in $\mathcal{H}(\overline{\mathbb{F}}_2)$ is $\overline{g}$ (because this holds for the root groups and the torus) and $v \in \mathcal{V}(\overline{\mathbb{Z}}[1/N])$ whose image in $\mathcal{V}(\overline{\mathbb{F}}_2)$ is $\overline{v}$, and $\overline{g}.\overline{v}$ is the image of $g.v \in \mathcal{V}(\overline{\mathbb{Z}}[1/N])$ in $\mathcal{V}(\overline{\mathbb{F}}_2)$ (which is independent of the choice of g and v).

Notice that the action of $\mathcal{H}$ on $\mathcal{V} \otimes \overline{\mathbb{Z}}[1/2N]$ corresponds to the action of $\mathcal{H}$ on $\widetilde{\mathcal{V}} \otimes \overline{\mathbb{Z}}[1/2N]$ under the identification $\mathcal{V} \otimes \overline{\mathbb{Z}}[1/2N] \xrightarrow{\simeq} \widetilde{\mathcal{V}} \otimes \overline{\mathbb{Z}}[1/2N]$ above. In order to treat the special fiber over $\overline{\mathbb{F}}_2$, we define isomorphisms $f_{\mathcal{V}} : \mathbf{V}_{x,r} \to \mathcal{V}_{\overline{\mathbb{F}}_p}$ if $p = 2$ and $f_{\mathcal{V},2} : \mathbf{V}_{x_2,r} \to \mathcal{V}_{\overline{\mathbb{F}}_2}$ if $2 \nmid N$ as follows.

Let $p = 2$. Let $f_{\mathcal{V}}|_{\overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \bigoplus_{a \in \Phi_K^{\mathrm{nm}}} \mathfrak{u}_a)}} : \overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \bigoplus_{a \in \Phi_K^{\mathrm{nm}}} \mathfrak{u}_a)} \xrightarrow{\simeq} (\mathcal{V}_T)_{\overline{\mathbb{F}}_2} \oplus (\mathcal{V}_{\mathrm{nm}})_{\overline{\mathbb{F}}_2}$ be given by the restriction of df, and let $f_{\mathcal{V}}(\overline{Y}_a) = \overline{\sqrt{\lambda_0}\sqrt{2}\,Y_a}$ for $a \in \Phi_K^{\mathrm{mul}}$ with $r - a(x - x_0) \in \Gamma_a'(G)$, where λ_0 is as defined in Lemma 2.5.1, $\overline{\sqrt{\lambda_0}\sqrt{2}\,Y_a}$ denotes the image of $\sqrt{\lambda_0}\sqrt{2}\,Y_a \in \mathcal{V}$ under the surjection $\mathcal{V} \to \mathcal{V} \otimes \overline{\mathbb{F}}_2$, and $\overline{Y}_a$ is as introduced in the proof of Proposition 3.4.12, i.e. $\overline{Y}_a = \sum_{\gamma' \in [\mathrm{Gal}(F/K)/\mathrm{Stab}_{\mathrm{Gal}(F/K)}(\alpha)]} \gamma'(\overline{X}_{\alpha_1})$ with the above notation.

Define the isomorphism $f_{\mathcal{V},2} : \mathbf{V}_{x_2,r} \to \mathcal{V}_{\overline{\mathbb{F}}_2}$ analogously.

Proposition 3.4.14. *For q coprime to N, the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{H}_{\overline{\mathbb{F}}_p} \times \mathcal{V}_{\overline{\mathbb{F}}_p} & \longrightarrow & \mathcal{V}_{\overline{\mathbb{F}}_p} \\ {\scriptstyle f_{\mathcal{H}}^{-1} \times f_{\mathcal{V}}^{-1}} \downarrow \simeq & & \simeq \downarrow {\scriptstyle f_{\mathcal{V}}^{-1}} \\ \mathbf{G}_x \times \mathbf{V}_{x,r} & \longrightarrow & \mathbf{V}_{x,r} \end{array} \qquad \begin{array}{ccc} \mathcal{H}_{\overline{\mathbb{F}}_q} \times \mathcal{V}_{\overline{\mathbb{F}}_q} & \longrightarrow & \mathcal{V}_{\overline{\mathbb{F}}_q} \\ {\scriptstyle f_{\mathcal{H},q}^{-1} \times f_{\mathcal{V},q}^{-1}} \downarrow \simeq & & \simeq \downarrow {\scriptstyle f_{\mathcal{V},q}^{-1}} \\ \mathbf{G}_{x_q} \times \mathbf{V}_{x_q,r} & \longrightarrow & \mathbf{V}_{x_q,r} \end{array}$$

where $f_{\mathcal{V}} := df$ if $p \neq 2$ and $f_{\mathcal{V},q} := df_q$ for $q \neq 2$.

Proof. By Proposition 3.4.12 and the above observation that $\mathcal{V}\otimes\overline{\mathbb{Z}}[1/(2N)] \hookrightarrow \widetilde{\mathcal{V}}\otimes\overline{\mathbb{Z}}[1/(2N)]$ is an isomorphism of $\mathcal{H}_{\overline{\mathbb{Z}}[1/(2N)]}$-modules, the right diagram commutes for $q \neq 2$ and the left diagram commutes if $p \neq 2$.

Let us now consider the commutativity of the left diagram for $p = 2$; the commutativity of the right diagram for $q = 2$ follows from the same arguments.

By construction, the action of the maximal torus $\mathbf{T}_x$ on $\mathbf{V}_{x,r}$ corresponds to the action of $\mathcal{S}_{\overline{\mathbb{F}}_2}$ on $\mathcal{V}_{\overline{\mathbb{F}}_2}$, and it remains to consider the action the root groups $\mathbf{U}_a \subset \mathbf{G}_x$ for $a \in \Phi(\mathbf{G}_x) \subset \Phi_K$. We first consider the action on $\overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \bigoplus_{a\in\Phi_K^{\mathrm{nm}}} \mathfrak{u}_a)} \simeq (\mathcal{V}_T)_{\overline{\mathbb{F}}_2} \oplus (\mathcal{V}_{\mathrm{nm}})_{\overline{\mathbb{F}}}$. In the proofs of Lemmas 2.6.1 and 3.4.13, we have seen that if a is non-multipliable, then $\mathbf{U}_a = \overline{x}_a(\mathbb{G}_{\mathrm{m}})$ preserves $\overline{\mathfrak{g}_{x,r} \cap (\mathfrak{t} \oplus \bigoplus_{a\in\Phi_K^{\mathrm{nm}}} \mathfrak{u}_a)}$ and $\chi_{\mathcal{H}a}((\mathbb{G}_{\mathrm{m}})_{\overline{\mathbb{F}}_2})$ preserves $(\mathcal{V}_T)_{\overline{\mathbb{F}}_2} \oplus (\mathcal{V}_{\mathrm{nm}})_{\overline{\mathbb{F}}}$, and hence, by construction, the actions agree under the isomorphisms $f_{\mathcal{H}}|_{\mathbf{U}_a}$ and $f_{\mathcal{V}}|_{\overline{\mathfrak{g}_{x,r}\cap(\mathfrak{t}\oplus\bigoplus_{a\in\Phi_K^{\mathrm{nm}}}\mathfrak{u}_a)}}$.

So consider a multipliable, and let $\overline{u} \in \overline{\mathbb{F}}_2$. Then

$$\overline{x}_a(\overline{u})(\overline{X}) = \overline{x_a(\sqrt{1/\lambda_0}\,\chi(u)\varpi_E^s\epsilon_1,\ \chi(u)\varpi_E^s\epsilon_1\sigma(\chi(u)\varpi_E^s\epsilon_1)\cdot\varpi_E^{\mathrm{v}(\lambda)e}\epsilon_0)(X)}$$

for $X \in \mathfrak{g}_{x,r}$, where we use the notation from Lemma 2.5.1 and $\overline{?}$ denotes the image of ? in $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+} = \mathbf{V}_{x,r}$. On the other hand, if $u \in \overline{\mathbb{Z}}[1/N]$ maps to $\overline{u} \in \overline{\mathbb{F}}_2$, then

$$\chi_{\mathcal{H}a}(\overline{u})(\overline{X}) = \overline{\iota(\chi_{\mathcal{H}a}(u))(X)}$$

for $X \in \mathcal{V}$, where $\overline{?}$ denotes the image of ? in $\mathcal{V}_{\overline{\mathbb{F}}_2}$. Moreover, by Definition/Proposition 3.4.9,

$$\iota(\chi_{\mathcal{H}a}(u)) = \prod_{i=1}^{|\Gamma/\Gamma_{n(a)}|} \chi_{\gamma^{(i-1)}(\alpha_1)}(\sqrt{2}\,u)\chi_{\gamma^{(i-1)}(\alpha_1+\alpha_2)}(-(-1)^{-a(x-x_0)M}u^2) \cdot \chi_{\gamma^{(i-1)}(\alpha_2)}((-1)^{-a(x-x_0)M}\sqrt{2}\,u).$$

Using these equations and the description of x_a in (1), easy calculations show that $f_{\mathcal{V}}(\overline{x}_a(\overline{u})(H)) = \chi_{\mathcal{H}a}(\overline{u})(f_{\mathcal{V}}(H)) = f_{\mathcal{H}}(\overline{x}_a(\overline{u}))(f_{\mathcal{V}}(H))$ for $H \in \overline{\mathfrak{g}_{x,r} \cap \mathfrak{t}}$ and $f_{\mathcal{V}}(\overline{x}_a(\overline{u})(\overline{Y}_b)) = \chi_{\mathcal{H}a}(\overline{u})(\overline{Y_b}) = f_{\mathcal{H}}(\overline{x}_a(\overline{u}))(f_{\mathcal{V}}(\overline{Y}_b))$ for $b \in \Phi_K^{\mathrm{nm}}$ with $r - b(x-x_0) \in \Gamma_b'(G)$.

It remains to consider the action on $\overline{\mathfrak{g}_{x,r} \cap \bigoplus a \in \Phi_K^{\mathrm{mul}}\mathfrak{u}_a} \xrightarrow{\simeq} (\mathcal{V}_{\mathrm{mul}})_{\overline{\mathbb{F}}_2}$. By Lemma 2.6.1 (and the definition of $\iota_{K,F,r}$ in its proof) and the definition of $\mathcal{V}$ and $\mathcal{V}_{\mathrm{mul}}$, the groups $\mathbf{G}_x$ and $\mathcal{H}_{\overline{\mathbb{F}}_2}$ preserve $\overline{\mathfrak{g}_{x,r} \cap \bigoplus_{a\in\Phi_K^{\mathrm{mul}}}\mathfrak{u}_a}$ and $(\mathcal{V}_{\mathrm{mul}})_{\overline{\mathbb{F}}_2}$, respectively. Hence, by the underlying constructions, their action agrees under the isomorphisms $f_{\mathcal{H}}$ and $f_{\mathcal{V}}|_{\overline{\mathfrak{g}_{x,r}\cap\bigoplus_{a\in\Phi_K^{\mathrm{mul}}}\mathfrak{u}_a}}$. ■

Now Theorem 3.4.1 is an immediate consequence of Proposition 3.4.14.

4. Moy–Prasad filtration representations and global Vinberg–Levy theory

In this section we will give a different description of the reductive group scheme $\mathcal{H}$ and its action on $\mathcal{V}$ from Theorem 3.4.1 as a fixed-point group scheme of a larger split reductive scheme $\mathcal{G}$ acting on a graded piece of Lie $\mathcal{G}$ (see Theorem 4.1.1). This means we are in the setting of a global version of Vinberg–Levy theory and the special fibers correspond to (generalized) Vinberg–Levy representations for all primes q. In order to give such a description integrally (i.e. over $\overline{\mathbb{Z}}[1/N]$), we will specialize to reductive groups G that become split over a tamely-ramified field extension in §4.1. Afterwards, in §4.2, we will show that such a description holds over $\overline{\mathbb{Q}}$ for all good groups. This will also allow us to study the existence of (semi)stable vectors in Section 5.

4.1. The case of G splitting over a tamely ramified extension

Let S be a scheme. We denote by $\boldsymbol{\mu}_{M,S}$ the group scheme of M-th roots of unity over S. We will often omit S if it can be deduced from the context. Given an S-group scheme $\mathcal{G}$, we denote by $\underline{\mathrm{Aut}}_{\mathcal{G}/S}$ its automorphism functor, which sends an S-scheme S' to the group of automorphisms of $\mathcal{G}_{S'}$ in the category of S'-group schemes, and by $\mathrm{Aut}_{\mathcal{G}/S}$ its representing group scheme if it exists. We will often omit S if it can be deduced from the context. Given, in addition, a morphism $\theta : \boldsymbol{\mu}_{M,S} \to \mathrm{Aut}_{\mathcal{G}}$, we denote by $\mathcal{G}^{\theta}$ the scheme-theoretic fixed locus of $\mathcal{G}$ under the action of $\boldsymbol{\mu}_{M,S}$ via θ, if it exists, i.e. $\mathcal{G}^{\theta}$ represents the functor that sends an S-scheme S' to the elements of $\mathcal{G}(S')$ on which $\boldsymbol{\mu}_{M,S'}$ acts trivially. If $\mathcal{G}^{\theta}$ is a smooth group scheme over S of finite presentation, we denote by $\mathcal{G}^{\theta,0}$ its identity component. Similarly, if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module, we denote by $\underline{\mathrm{Aut}}_{\mathcal{F}/\mathcal{O}_S}$ its automorphism functor, and by $\mathrm{Aut}_{\mathcal{F}/\mathcal{O}_S}$ (or simply $\mathrm{Aut}_{\mathcal{F}}$) the group scheme representing $\underline{\mathrm{Aut}}_{\mathcal{F}/\mathcal{O}_S}$ if it exists.

Theorem 4.1.1. *Suppose that G is a reductive group over K that splits over a tamely ramified field extension E of degree e over K. Let $r = d/M$ for some non-negative integers $d < M$, and let $\mathcal{H}$ be the split reductive group scheme over $\overline{\mathbb{Z}}[1/e]$ acting on the free $\overline{\mathbb{Z}}[1/e]$-module $\mathcal{V}$ as provided by Theorem 3.4.1, i.e. such that the special fibers each correspond to the action of a reductive quotient on a Moy–Prasad filtration quotient. Then there exists a split reductive group scheme $\mathcal{G}$ defined over $\overline{\mathbb{Z}}[1/e]$ and morphisms*

$$\theta : \boldsymbol{\mu}_M \to \mathrm{Aut}_{\mathcal{G}} \quad \text{and} \quad d\theta : \boldsymbol{\mu}_M \to \mathrm{Aut}_{\mathrm{Lie}(\mathcal{G})}$$

that induce a $\mathbb{Z}/M\mathbb{Z}$-grading $\mathrm{Lie}(\mathcal{G}) = \bigoplus_{i=1}^{M} \mathrm{Lie}(\mathcal{G})_i$ such that $\mathcal{H}$ is isomorphic to $\mathcal{G}^{\theta,0}$, $\mathcal{V}$ is isomorphic to $\mathrm{Lie}(\mathcal{G})_{M-d}(\overline{\mathbb{Z}}[1/e])$ and the action of $\mathcal{H}$ on $\mathcal{V}$ corresponds to the restriction of the adjoint action of $\mathcal{G}$ on $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/e])$ via these isomorphisms.

In particular, this implies that for q coprime to e we have commutative diagrams

$$\begin{array}{ccc} \mathcal{G}^{\theta,0}_{\overline{\mathbb{F}}_p} \times \mathrm{Lie}(\mathcal{G})_{M-d}(\overline{\mathbb{F}}_p) & \to & \mathrm{Lie}(\mathcal{G})_{M-d}(\overline{\mathbb{F}}_p) \\ \simeq\times\simeq \downarrow & & \downarrow \simeq \\ \mathbf{G}_x \times \mathbf{V}_{x,r} & \longrightarrow & \mathbf{V}_{x,r} \end{array} \qquad \begin{array}{ccc} \mathcal{G}^{\theta,0}_{\overline{\mathbb{F}}_q} \times \mathrm{Lie}(\mathcal{G})_{M-d}(\overline{\mathbb{F}}_q) & \to & \mathrm{Lie}(\mathcal{G})_{M-d}(\overline{\mathbb{F}}_q) \\ \simeq\times\simeq \downarrow & & \downarrow \simeq \\ \mathbf{G}_{x_q} \times \mathbf{V}_{x_q,r} & \longrightarrow & \mathbf{V}_{x_q,r} \end{array} \tag{27}$$

Remark 4.1.2. If p is odd, not torsion for G and does not divide m, then, if we choose M to be m, the left diagram in (27) is proven to exist and commute in [20, Theorem 4.1]. The proof given in *loc. cit.* does not work for all primes p, because it relies crucially on the assumption that p does not divide m.

Proof of Theorem 4.1.1. Let e', f be integers such that $e \mid e'$, $M = e'f$, $\gcd(e', f) = 1$ and e' is minimal satisfying these properties. Let $E_{e'}$ be the splitting field of $x^{e'} - 1$ over E, and let $\mathcal{O}_{e'}$ be the ring of integers in $E_{e'}$.

We let $\mathcal{G}$ be a split reductive group scheme over $\mathcal{O}_{e'}[1/e] \subset \overline{\mathbb{Z}}[1/e]$ whose root datum $R(\mathcal{G})$ coincides with the root datum $R(G)$ of G, i.e. $\mathcal{G}$ is as defined in §3.4.1 base changed to $\mathcal{O}_{e'}[1/e]$, and $\mathcal{T}$ denotes a split maximal torus of $\mathcal{G}$. Let G_{ad} be the adjoint group of G and T' the subtorus of T that consists of the images of the coroots of G. We have the usual map $G \to G_{\mathrm{ad}}$, and we denote the image of T under this map by T_{ad}. Restricting the map to T' induces an injection $X_*(T') \hookrightarrow X_*(T_{\mathrm{ad}})$ that yields an isomorphism $X_*(T') \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\simeq} X_*(T_{\mathrm{ad}}) \otimes_{\mathbb{Z}} \mathbb{R}$, which we use to identify the two real vector spaces. This allows us to choose $\lambda \in X_*(T_{\mathrm{ad}}) \subset X_*(T') \otimes \mathbb{R} \subset X_*(T) \otimes \mathbb{R}$ such that $x = x_0 + (1/M)\lambda$. Note that then, using the identification of $X_*(T)$ with $X_*(T_q)$, we have $x_q = x_{0,q} + (1/M)\lambda$. We also denote by λ the corresponding element in $X_*(\mathcal{T}_{\mathrm{ad}}) \subset X_*(\mathcal{T}) \otimes \mathbb{R}$ under the identification of $X_*(T)$ with $X_*(\mathcal{T})$. Consider the action θ_λ of $\boldsymbol{\mu}_M$ on $\mathcal{G}$ given by composition of the closed immersion $\boldsymbol{\mu}_M \to \mathbb{G}_{\mathrm{m}}$ with λ and the adjoint action of $\mathcal{G}_{\mathrm{ad}}$ on $\mathcal{G}$, i.e.

$$\theta_\lambda : \boldsymbol{\mu}_M \to \mathbb{G}_{\mathrm{m}} \xrightarrow{\lambda} \mathcal{T}_{\mathrm{ad}} \hookrightarrow \mathcal{G}_{\mathrm{ad}} \xrightarrow{\mathrm{Ad}} \mathrm{Aut}_{\mathcal{G}}\,.$$

Let $\vartheta \in \mathrm{Aut}(R(G), \Delta)$ denote the action of $\gamma \in \Gamma \simeq \mathrm{Gal}(E/K)$ on $R(G)$ given in the Definition 3.1.1 of a good group, and denote by $\underline{\mathbb{Z}/e\mathbb{Z}}_{\mathcal{O}_{e'}[1/e]}$ the constant group scheme over $\operatorname{Spec} \mathcal{O}_{e'}[1/e]$ corresponding to the group $\mathbb{Z}/e\mathbb{Z}$. Using the Chevalley system $\{x_\alpha : \mathbb{G}_{\mathrm{a}} \to \mathcal{U}_\alpha \subset \mathcal{G}\}_{\alpha \in \Phi(\mathcal{G}) = \Phi}$ for $(\mathcal{G}, \mathcal{T})$ (defined in §3.4.1), the automorphism ϑ defines a morphism of $\operatorname{Spec} \mathcal{O}_{e'}[1/e]$-schemes $\underline{\mathbb{Z}/e\mathbb{Z}}_{\mathcal{O}_{e'}[1/e]} \to \mathrm{Aut}_{\mathcal{G}}$. Note that we have an isomorphism of $\operatorname{Spec} \mathcal{O}_{e'}[1/e]$-schemes $\boldsymbol{\mu}_{e'} \xrightarrow{\simeq} \underline{\mathbb{Z}/e'\mathbb{Z}}_{\mathcal{O}_{e'}[1/e]}$ that yields the following morphism, which we again denote by ϑ:

$$\vartheta : \boldsymbol{\mu}_{e'} \xrightarrow{\simeq} \underline{\mathbb{Z}/e'\mathbb{Z}}_{\mathcal{O}_{e'}[1/e]} \xrightarrow{\cdot \frac{e'}{e}} \underline{\mathbb{Z}/e\mathbb{Z}}_{\mathcal{O}_{e'}[1/e]} \to \mathrm{Aut}_{\mathcal{G}}\,.$$

Fix an isomorphism $\boldsymbol{\mu}_M \simeq \boldsymbol{\mu}_{e'} \times \boldsymbol{\mu}_f$. This yields a projection map $p_{M,e'} : \boldsymbol{\mu}_M \to \boldsymbol{\mu}_{e'}$ and allows us to define $\theta : \boldsymbol{\mu}_M \to \mathrm{Aut}_{\mathcal{G}}$ as follows:

$$\theta : \boldsymbol{\mu}_M \xrightarrow{\mathrm{diag}} \boldsymbol{\mu}_M \times \boldsymbol{\mu}_M \xrightarrow{p_{M,e'} \times \mathrm{Id}} \boldsymbol{\mu}_{e'} \times \boldsymbol{\mu}_M \xrightarrow{\vartheta \times \theta_\lambda} \mathrm{Aut}_{\mathcal{G}} \times \mathrm{Aut}_{\mathcal{G}} \xrightarrow{\text{mult.}} \mathrm{Aut}_{\mathcal{G}}\,.$$

By [5, Proposition A.8.10], the fixed-point locus of $\mathcal{G}$ under the action of θ is representable by a smooth closed $\mathcal{O}_{e'}[1/e]$-subscheme $\mathcal{G}^\theta$ of $\mathcal{G}$. Moreover, by [5, Proposition A.8.12], the fiber $\mathcal{G}^{\theta,0}_{\bar{s}}$ is a reductive group for all geometric points $\bar{s}$ of $\operatorname{Spec} \mathcal{O}_{e'}[1/e]$. Similarly, $\mathcal{T}^{\theta,0} = \mathcal{T}^{\vartheta,0}$ is a smooth closed subscheme of $\mathcal{T}$. Hence $\mathcal{T}^{\vartheta,0}$ is a split torus over $\operatorname{Spec} \mathcal{O}_{e'}[1/e]$.

Let us denote $\mathcal{G}^{\theta,0}$ by $\mathcal{H}'$. We claim that $\mathcal{T}^{\vartheta,0}$ is a maximal torus of $\mathcal{H}'$. In order to prove the claim for geometric fibers, we use a similar argument to one used in [10, Section 4]. Let q be an arbitrary prime number coprime to e, $\mathcal{B}$ the Borel subgroup of $\mathcal{G}$ corresponding to the positive roots, and $\mathcal{U}$ its unipotent radical. As $\mathcal{H}'_{\overline{\mathbb{F}}_q}$ is a closed subgroup of $\mathcal{G}_{\overline{\mathbb{F}}_q}$, $\mathcal{H}'_{\overline{\mathbb{F}}_q}/(\mathcal{B}_{\overline{\mathbb{F}}_q} \cap \mathcal{H}'_{\overline{\mathbb{F}}_q})$ is proper in $\mathcal{G}_{\overline{\mathbb{F}}_q}/\mathcal{B}_{\overline{\mathbb{F}}_q}$, hence is proper. Thus $\mathcal{B}_{\overline{\mathbb{F}}_q} \cap \mathcal{H}'_{\overline{\mathbb{F}}_q}$ is a solvable parabolic subgroup, i.e. a Borel subgroup, and $\mathcal{B}^{\theta,0}_{\overline{\mathbb{F}}_q} = \mathcal{B}_{\overline{\mathbb{F}}_q} \cap \mathcal{H}'_{\overline{\mathbb{F}}_q}$. According to [23, 8.2], $\mathcal{U}^{\theta}_{\overline{\mathbb{F}}_q}$ is connected, and hence $\mathcal{B}^{\theta,0}_{\overline{\mathbb{F}}_q} = \mathcal{T}^{\theta,0}_{\overline{\mathbb{F}}_q} \ltimes \mathcal{U}^{\theta}_{\overline{\mathbb{F}}_q}$. This means that $\mathcal{T}^{\vartheta,0}_{\overline{\mathbb{F}}_q} = \mathcal{T}^{\theta,0}_{\overline{\mathbb{F}}_q}$ is a maximal torus of $\mathcal{H}'_{\overline{\mathbb{F}}_q}$. Hence $\mathcal{T}^{\vartheta,0}_{\overline{s}}$ is a maximal torus in $\mathcal{H}'_{\overline{s}}$ for all geometric points $\overline{s}$ of Spec $\mathcal{O}_{e'}[1/e]$, because the locus of the former points is open. This means that $\mathcal{T}^{\vartheta,0}$ is a maximal torus of $\mathcal{H}'$.

In addition, Pic(Spec $\overline{\mathbb{Z}}[1/e]$) is trivial (by the principal ideal theorem), and hence the root spaces for $(\mathcal{G}_{\overline{\mathbb{Z}}[1/e]}, \mathcal{T}_{\overline{\mathbb{Z}}[1/e]})$ are free line bundles. Using the fact that Spec $\overline{\mathbb{Z}}[1/e]$ is connected, we conclude that $\mathcal{H}'_{\overline{\mathbb{Z}}[1/e]}$ is a split reductive group scheme.

If q is a large enough prime number, then by [20, Theorem 4.1] we have $\mathcal{H}'_{\overline{\mathbb{F}}_q} \simeq \mathbf{G}_{x_q}$. Hence $R(\mathcal{H}') = R(\mathcal{H}'_{\overline{\mathbb{F}}_p}) = R(\mathbf{G}_{x_q}) = R(\mathcal{H})$, and $\mathcal{H}'_{\overline{\mathbb{Z}}[1/e]}$ is (abstractly) isomorphic to $\mathcal{H}$ as desired.

In order to give a new construction of $\mathcal{V}$, let $d : \mathrm{Aut}_{\mathcal{G}} \to \mathrm{Aut}_{\mathrm{Lie}(\mathcal{G})}$ be the map defined as follows. For any $\mathcal{O}_{e'}[1/e]$-algebra R, and $g \in \mathrm{Aut}_{\mathcal{G}}(R)$, define $dg := \mathrm{Lie}(g) \in \mathrm{Aut}(\mathrm{Lie}(\mathcal{G})_R)$. Then the action $d\theta$ defines a $\mathbb{Z}/M\mathbb{Z}$-grading on Lie($\mathcal{G}$), which we write as $\mathrm{Lie}\,\mathcal{G} = \bigoplus_{i=1}^{M} (\mathrm{Lie}\,\mathcal{G})_i$.

We define $\mathcal{V}'$ to be the free $\mathcal{O}_{e'}[1/e]$-module $\mathrm{Lie}(\mathcal{G})_{M-d}(\mathcal{O}_{e'}[1/e])$, and the action of $\mathcal{H}' := \mathcal{G}^{\theta,0}$ on $\mathcal{V}'$ should be given by the restriction of the adjoint action of $\mathcal{G}$ on $\mathrm{Lie}(\mathcal{G})(\mathcal{O}_{e'}[1/e])$.

In order to show that the $\mathcal{H}$-representation on $\mathcal{V}$ corresponds to the $\mathcal{H}'_{\overline{\mathbb{Z}}[1/e]}$-representation on $\mathcal{V}'_{\overline{\mathbb{Z}}[1/e]}$, we observe that $\mathcal{V}'_{\overline{\mathbb{Z}}[1/e]}$ is the $M-d$ weight space of the action of $\vartheta \cdot \mathrm{Ad}(\lambda(\zeta_M))$ for some primitive M-th root of unity ζ_M in $\overline{\mathbb{Z}}[1/e]$. Using the notation introduced in §3.4.1 preceding Remark 3.4.5, we let $C_\alpha = \zeta_M^{e(\alpha)\cdot\alpha(x-x_0)\cdot M}$. By the same arguments as in the proof of Lemma 3.4.7, we see that there exists an automorphism h of $\mathcal{G}_{\overline{\mathbb{Z}}[1/e]}$ that preserves $\mathcal{T}_{\overline{\mathbb{Z}}[1/e]}$ and sends x_α to $x_\alpha \circ C_\alpha$ for all $\alpha \in \Phi$.

Let q be a large enough prime, to be more precise: odd, not torsion for G and not dividing M. Then we deduce from the arguments used in [20, Section 4] that we have commutative diagrams

$$\begin{array}{ccc} \mathcal{H}'_{\overline{\mathbb{F}}_q} & \overset{\subset}{\hookrightarrow} & \mathcal{G}_{\overline{\mathbb{F}}_q} \\ {\scriptstyle f_q^{-1}\circ h|_{\mathcal{H}'_{\overline{\mathbb{F}}_q}}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle f_q^{-1}\circ h} \\ \mathbf{G}_{x_q} & \overset{\iota_{K_q,F_q}}{\hookrightarrow} & \mathbf{G}_{x_q}^{F_q} \end{array} \qquad \begin{array}{ccc} \mathcal{V}'_{\overline{\mathbb{F}}_q} & \overset{\subset}{\hookrightarrow} & \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{F}}_q) \\ {\scriptstyle \mathrm{Lie}(f_q^{-1}\circ h)|_{\mathcal{V}'(\overline{\mathbb{F}}_q)}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \mathrm{Lie}(f_q^{-1}\circ h)} \\ \mathbf{V}_{x_q,r} & \overset{\iota_{K_q,F_q,r}}{\hookrightarrow} & \mathbf{V}_{x_q,r}^{F_q} \end{array} \tag{28}$$

Moreover, the diagram on the right hand side is compatible with the action by the groups of the diagram on the left hand side.

Recall that in §3.4 we constructed a map $\iota : \mathcal{H} \to \mathcal{G}_{\overline{\mathbb{Z}}[1/e]}$ and $\mathcal{V}$ as a free $\overline{\mathbb{Z}}[1/e]$-submodule of $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/e])$ (because if e is odd, then Φ_K does not contain multipliable roots, and hence the submodules $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ agree in all cases) such that we have the following commutative diagrams for all primes q coprime to e:

$$\begin{array}{ccc} \mathcal{H}_{\overline{\mathbb{F}}_q} & \xrightarrow{\iota} & \mathcal{G}_{\overline{\mathbb{F}}_q} \\ \simeq\big\downarrow & & \simeq\big\downarrow f_q^{-1} \\ \mathbf{G}_{x_q} & \xhookrightarrow{\iota_{K_q,F_q}} & \mathbf{G}_{x_q}^{F_q} \end{array} \qquad \begin{array}{ccc} \mathcal{V}_{\overline{\mathbb{F}}_q} & \xhookrightarrow{\subset} & \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{F}}_q) \\ \simeq\big\downarrow & & \simeq\big\downarrow \mathrm{Lie}(f_q^{-1}) \\ \mathbf{V}_{x_q,r} & \xhookrightarrow{\iota_{K_q,F_q,r}} & \mathbf{V}_{x_q,r}^{F_q} \end{array} \tag{29}$$

where the diagram on the right hand side is compatible with the action of the groups on the left hand side by Proposition 3.4.12. Note that ι_{K_q,F_q} is a closed immersion because either q is odd or e is odd (see §2.6).

Thus we conclude that $h^{-1}(\iota(\mathcal{H}_{\overline{\mathbb{F}}_q})) = \mathcal{H}'_{\overline{\mathbb{F}}_q}$ for large enough q.

Let q now be any prime coprime to e, and let $\overline{g} \in \mathcal{H}(\overline{\mathbb{F}}_q)$. As $\mathcal{H}(\overline{\mathbb{Z}}[1/e])$ surjects onto $\mathcal{H}(\overline{\mathbb{F}}_q)$ (because this holds for the root groups and the torus), we can choose $g \in \mathcal{H}(\overline{\mathbb{Z}}[1/e])$ whose image in $\mathcal{H}(\overline{\mathbb{F}}_q)$ is $\overline{g}$. By combining the diagrams (28) and (29), we see that the image of $h^{-1}\iota(g)$ in $\mathcal{G}(\overline{\mathbb{F}}_{q'})$ is actually contained in $\mathcal{H}'(\overline{\mathbb{F}}_{q'})$ for all sufficiently large primes q'. Hence $h^{-1}\iota(g) \in \mathcal{H}'(\overline{\mathbb{Z}}[1/e]) \subset \mathcal{G}(\overline{\mathbb{Z}}[1/e])$, and $h^{-1} \circ \iota(\mathcal{H}(\overline{\mathbb{F}}_q)) \subset \mathcal{H}'(\overline{\mathbb{F}}_q)$. Since we have observed that $\mathcal{H}'_{\overline{\mathbb{F}}_q}$ is abstractly isomorphic to $\mathbf{G}_{x_q} \simeq h^{-1} \circ f_q(\iota_{K_q,F_q}(\mathbf{G}_{x_q})) \simeq h^{-1} \circ \iota(\mathcal{H}_{\overline{\mathbb{F}}_q})$, we conclude that

$$h^{-1} \circ \iota(\mathcal{H}_{\overline{\mathbb{F}}_q}) = \mathcal{H}'_{\overline{\mathbb{F}}_q} \tag{30}$$

for all primes q coprime to e. The same arguments show that

$$h^{-1} \circ \iota(\mathcal{H}_{\overline{\mathbb{F}}_p}) = \mathcal{H}'_{\overline{\mathbb{F}}_p}. \tag{31}$$

Moreover, we claim that $h^{-1} \circ \iota(\mathcal{H}_{\overline{\mathbb{Q}}}) = \mathcal{H}'_{\overline{\mathbb{Q}}}$. In order to prove the claim, note that $(\boldsymbol{\mu}_M)_{\overline{\mathbb{Q}}} \simeq \underline{\mathbb{Z}/M\mathbb{Z}}_{\overline{\mathbb{Q}}}$, and hence the action of the group scheme $\boldsymbol{\mu}_M$ on $\mathcal{G}_{\overline{\mathbb{Q}}}$ corresponds to the action of the finite group $\mathbb{Z}/M\mathbb{Z}$ generated by $\vartheta \cdot \mathrm{Inn}(\lambda(\zeta_M))$. Therefore, by the construction of $\iota : \mathcal{H}_{\overline{\mathbb{Z}}[1/e]} \to \mathcal{G}_{\overline{\mathbb{Z}}[1/e]}$ (see Proposition 3.4.9) and the definition of $h : \mathcal{G}_{\overline{\mathbb{Z}}[1/e]} \to \mathcal{G}_{\overline{\mathbb{Z}}[1/e]}$, we see that $h^{-1} \circ \iota(\mathcal{H}(\overline{\mathbb{Q}})) \subset \mathcal{G}^{\theta}(\overline{\mathbb{Q}})$. As $\iota_{\overline{\mathbb{Q}}} : \mathcal{H}_{\overline{\mathbb{Q}}} \to \mathcal{G}_{\overline{\mathbb{Q}}}$ is a closed immersion by Lemma 3.4.11, $h^{-1} \circ \iota(\mathcal{H}_{\overline{\mathbb{Q}}}) \simeq \mathcal{H}_{\overline{\mathbb{Q}}} \simeq \mathcal{G}^{\theta,0}_{\overline{\mathbb{Q}}} = \mathcal{H}'_{\overline{\mathbb{Q}}}$, and we conclude that

$$h^{-1} \circ \iota(\mathcal{H}_{\overline{\mathbb{Q}}}) = \mathcal{H}'_{\overline{\mathbb{Q}}}. \tag{32}$$

Thus, as $\mathcal{H}'_{\overline{\mathbb{Z}}[1/e]}$ is smooth over $\mathrm{Spec}\,\overline{\mathbb{Z}}[1/e]$, hence reduced, we deduce from the Nullstellensatz that $h^{-1} \circ \iota : \mathcal{H} \to \mathcal{G}_{\overline{\mathbb{Z}}[1/e]}$ factors through the closed subscheme $\mathcal{H}'_{\overline{\mathbb{Z}}[1/e]}$ of $\mathcal{G}_{\overline{\mathbb{Z}}[1/e]}$, i.e. we may write $h^{-1} \circ \iota : \mathcal{H} \to \mathcal{H}'_{\overline{\mathbb{Z}}[1/e]}$. As we have proved that $(h^{-1} \circ \iota)_s : \mathcal{H}_s \to (\mathcal{H}'_{\overline{\mathbb{Z}}[1/e]})_s$ is an isomorphism for all $s \in \mathrm{Spec}\,\overline{\mathbb{Z}}[1/e]$ (see (30)–(32)), we conclude that by [9, 17.9.5] the morphism $h^{-1} \circ \iota : \mathcal{H} \to \mathcal{H}'_{\overline{\mathbb{Z}}[1/e]}$ is an isomorphism.

Moreover, as $\mathrm{Lie}(h)(\mathcal{V}'_{\overline{\mathbb{F}}_q}) = \mathcal{V}_{\overline{\mathbb{F}}_q}$ for large enough primes q, we deduce that $\mathrm{Lie}(h)$: $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/e]) \to \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/e])$ yields an isomorphism of the direct $\overline{\mathbb{Z}}[1/e]$-module summands $\mathcal{V}'_{\overline{\mathbb{Z}}[1/e]}$ and $\mathcal{V}$.

As the action of $\mathcal{H}$ on $\mathcal{V}$ was defined via the adjoint action of $\mathcal{G}_{\overline{\mathbb{Z}}[1/e]} \supset \iota(\mathcal{H})$ onto $\mathrm{Lie}(\mathcal{G}_{\overline{\mathbb{Z}}[1/e]})(\overline{\mathbb{Z}}[1/e]) \supset \mathcal{V}$, the isomorphisms

$$h^{-1} : \mathcal{H} \to \mathcal{H}'_{\overline{\mathbb{Z}}[1/e]} = \mathcal{G}^{\theta,0}_{\overline{\mathbb{Z}}[1/e]}$$

and

$$\mathrm{Lie}(h^{-1}) : \mathcal{V} \to \mathcal{V}'_{\overline{\mathbb{Z}}[1/e]} = \mathrm{Lie}(\mathcal{G}_{\overline{\mathbb{Z}}[1/e]})_{M-d}(\overline{\mathbb{Z}}[1/e])$$

map the action of $\mathcal{H}$ onto $\mathcal{V}$ to the action of $(\mathcal{G}_{\overline{\mathbb{Z}}[1/e]})^{\theta,0}$ on $\mathrm{Lie}(\mathcal{G}_{\overline{\mathbb{Z}}[1/e]})_{M-d}(\overline{\mathbb{Z}}[1/e])$ which arises from the restriction of the adjoint action of $\mathcal{G}_{\overline{\mathbb{Z}}[1/e]}$ on $\mathrm{Lie}(\mathcal{G}_{\overline{\mathbb{Z}}[1/e]})(\overline{\mathbb{Z}}[1/e])$. The commutative diagrams in the theorem now follow by applying Theorem 3.4.1.

Remark 4.1.3. Let $E_{e'}$ be as defined in the proof of Theorem 4.1.1. Denote by E_H the Hilbert class field of $E_{e'}$ and by $\mathcal{O}_H$ the ring of integers in E_H. Then the group schemes $\mathcal{H}$ and $\mathcal{G}$ and the action of $\mathcal{H}$ on $\mathcal{V}$ appearing in Theorem 4.1.1 can be defined over $\mathrm{Spec}\, \mathcal{O}_H[1/e]$.

4.2. Vinberg–Levy theory for all good groups

Even though the Moy–Prasad filtration representation of groups that do not split over a tamely ramified extension might not be described as in Vinberg–Levy theory, its lift to characteristic zero can be described using Vinberg theory, i.e. as the fixed-point subgroup of a finite order automorphism on a larger group acting on some eigenspace in the Lie algebra of the larger group. To be more precise, we have the following corollary of Theorem 4.1.1 combined with Theorem 3.4.1.

Corollary 4.2.1. *Let G be a good reductive group over K, $r = d/M$ for some non-negative integer $d < M$, and let the representation of $\mathcal{H}$ on $\mathcal{V}$ be as in Theorem 3.4.1. Then there exist a reductive group scheme $\mathcal{G}_{\overline{\mathbb{Q}}}$ over $\overline{\mathbb{Q}}$ and morphisms*

$$\theta : \boldsymbol{\mu}_M \to \mathrm{Aut}_{\mathcal{G}_{\overline{\mathbb{Q}}}/\overline{\mathbb{Q}}} \quad \textit{and} \quad d\theta : \boldsymbol{\mu}_M \to \mathrm{Aut}_{\mathrm{Lie}(\mathcal{G}_{\overline{\mathbb{Q}}})/\overline{\mathbb{Q}}}$$

such that $\mathcal{H}_{\overline{\mathbb{Q}}} \simeq \mathcal{G}^{\theta,0}_{\overline{\mathbb{Q}}}$ and $\mathcal{V}_{\overline{\mathbb{Q}}} \simeq \mathrm{Lie}(\mathcal{G}_{\overline{\mathbb{Q}}})_{M-d}(\overline{\mathbb{Q}})$, and the action of $\mathcal{H}_{\overline{\mathbb{Q}}}$ on $\mathcal{V}_{\overline{\mathbb{Q}}}$ corresponds via these isomorphisms to the restriction of the adjoint action of $\mathcal{G}_{\overline{\mathbb{Q}}}$ on $\mathrm{Lie}(\mathcal{G}_{\overline{\mathbb{Q}}})(\overline{\mathbb{Q}})$.

Proof. Let q be a prime larger than $p^s N$. Then, by construction, the representation over $\overline{\mathbb{Z}}[1/(p^s N)]$ associated to G_q via the proof of Theorem 3.4.1 agrees with the representation of $\mathcal{H}_{\overline{\mathbb{Z}}[1/(p^s N)]}$ on $\mathcal{V}_{\overline{\mathbb{Z}}[1/(p^s N)]}$. As G_q splits over a tamely ramified extension, Theorem 4.1.1 allows us to deduce the corollary. ∎

5. Semistable and stable vectors

In this section we apply our results of Sections 3 and 4 to prove that the existence of stable and semistable vectors in the Moy–Prasad filtration representations is independent of the characteristic of the residue field. Recall that a vector v in a vector space V over an algebraically closed field is *stable* under the action of a reductive group G_V on V if the orbit $G_V v$ is closed and the stabilizer $\mathrm{Stab}_{G_V}(v)$ of v in G_V is finite. A vector $v \in V$ is called *semistable* if the closure of the orbit $G_V v$ does not contain zero.

5.1. Semistable vectors

The global version of the Moy–Prasad filtration representation as provided by Theorem 3.4.1 allows us to show that the existence of semistable vectors is independent of the residual characteristic p of K as follows, where N is the integer coprime to p introduced in Definition 3.1.1.

Theorem 5.1.1. *We keep the notation used in Theorem 3.4.1, in particular G is a good reductive group over K and $x \in \mathcal{B}(G, K)$. Then the following are equivalent:*

(i) $\mathbf{V}_{x,r}$ *has semistable vectors under the action of* $\mathbf{G}_x$.

(ii) $\mathbf{V}_{x_q,r}$ *has semistable vectors under the action of* $\mathbf{G}_{x_q}$ *for some prime q coprime to N.*

(iii) $\mathbf{V}_{x_q,r}$ *has semistable vectors under the action of* $\mathbf{G}_{x_q}$ *for all primes q coprime to N.*

Proof. We first show that (ii) implies (i). Suppose that (ii) holds, i.e. $\mathbf{V}_{x_q,r}$ contains semistable vectors under $\mathbf{G}_{x_q}$ for some prime q coprime to N. This implies by [14, Proposition 4.3] that $\mathcal{V}_{\overline{\mathbb{Q}}_q}$ has semistable vectors under the action of $\mathcal{H}_{\overline{\mathbb{Q}}_q}$, where $\mathcal{H}$ and $\mathcal{V}$ are as in Theorem 3.4.1. By [17, p. 41] (based on [16, Definition 1.7 and Proposition 2.2]) this means that there exists an $\mathcal{H}_{\overline{\mathbb{Q}}_q}$-invariant non-constant homogeneous element P_q in $\mathrm{Sym}\, \check{\mathcal{V}}_{\overline{\mathbb{Q}}_q}$. Moreover, there exists $X \in \mathcal{V}_{\overline{\mathbb{Q}}} \subset \mathcal{V}_{\overline{\mathbb{Q}}_q}$ such that $P_q(X) \neq 0$, i.e. X is semistable in $\mathcal{V}_{\overline{\mathbb{Q}}_q}$ under the action of $\mathcal{H}_{\overline{\mathbb{Q}}_q}$. Hence $X \neq 0$ is also semistable in $\mathcal{V}_{\overline{\mathbb{Q}}}$ under the action of $\mathcal{H}_{\overline{\mathbb{Q}}}$, which implies $(\mathrm{Sym}\, \check{\mathcal{V}}_{\overline{\mathbb{Q}}})^{\mathcal{H}(\overline{\mathbb{Q}})} \neq \overline{\mathbb{Q}}$. Thus, there also exists an $\mathcal{H}(\overline{\mathbb{Z}})$-invariant non-constant homogeneous element P in $\mathrm{Sym}\, \check{\mathcal{V}}_{\overline{\mathbb{Z}}}$. As P is non-constant and homogeneous, we can assume without loss of generality that the image $\overline{P}$ of P in $\mathrm{Sym}\, \check{\mathcal{V}}_{\overline{\mathbb{Z}}} \otimes \overline{\mathbb{F}}_p \simeq \mathrm{Sym}\, \check{\mathcal{V}}_{\overline{\mathbb{F}}_p}$ is non-constant. Note that $\mathcal{H}(\overline{\mathbb{Z}})$ surjects onto $\mathcal{H}(\overline{\mathbb{F}}_p)$, which follows from the surjections on all root groups and the split maximal torus. Hence $\overline{P}$ is $\mathcal{H}(\overline{\mathbb{F}}_p) \simeq \mathbf{G}_x(\overline{\mathbb{F}}_p)$-invariant and there exists $\overline{X} \in \mathcal{V}_{\overline{\mathbb{F}}_p} \simeq \mathbf{V}_{x,r}$ such that $\overline{f}(\overline{X}) \neq 0$, i.e. $\overline{X}$ is semistable by [17, p. 41]. Thus (i) is true.

The same arguments show that if $\mathbf{G}_{x,r}$ has semistable vectors, then $\mathbf{G}_{x_q,r}$ has semistable vectors for all primes q coprime to N, i.e. (i) implies (iii). As (iii) implies (ii), we conclude that all three statements are equivalent. ∎

Note that the same holds for the linear duals $\check{\mathbf{V}}_{x,r}$ and $\check{\mathbf{V}}_{x_q,r}$ of $\mathbf{V}_{x,r}$ and $\mathbf{V}_{x_q,r}$ using $\check{\mathcal{V}}$ instead of $\mathcal{V}$ in the proof above:

Corollary 5.1.2. *We use the same notation as above. Then $\check{\mathbf{V}}_{x,r}$ has semistable vectors under the action of $\mathbf{G}_x$ if and only if $\check{\mathbf{V}}_{x_q,r}$ has semistable vectors under the action of $\mathbf{G}_{x_q}$ for some prime q coprime to N if and only if $\check{\mathbf{V}}_{x_q,r}$ has semistable vectors under the action of $\mathbf{G}_{x_q}$ for all primes q coprime to N.*

Remark 5.1.3. For semisimple groups G that split over a tamely ramified extension and sufficiently large residue field characteristic p, Reeder and Yu classified in [20, Theorem 8.3] those x for which $\check{\mathbf{V}}_{x,r}$ contains semistable vectors in terms of conditions that are independent of the prime p. Corollary 5.1.2 allows us to conclude that these prime independent conditions also classify points x such that $\mathbf{V}_{x,r}$ contains semistable vectors for all good semisimple groups G (without any restriction on the residue field characteristic). Note that the removal of the restriction on the residue field characteristic for absolutely simple split reductive groups G is also contained in a joint paper of the present author with Romano [7]. For this result, it suffices to construct $\mathcal{H}$ acting on $\mathcal{V}$ over $\overline{\mathbb{Z}}_p$.

5.2. Stable vectors

In this section we show a result analogous to the one of §5.1 for stable vectors. This allows us to generalize the criterion in [20] for the existence of stable vectors in the dual of the first Moy–Prasad filtration quotient to arbitrary residual characteristics p and all good semisimple groups, which in turn produces new supercuspidal representations.

Theorem 5.2.1. *We keep the notation used above, in particular G is a good reductive group over K and $x \in \mathcal{B}(G, K)$. Then the following are equivalent:*

(i) *$\mathbf{V}_{x,r}$ has stable vectors under the action of $\mathbf{G}_x$.*

(ii) *$\mathbf{V}_{x_q,r}$ has stable vectors under the action of $\mathbf{G}_{x_q}$ for some prime q coprime to N.*

(iii) *$\mathbf{V}_{x_q,r}$ has stable vectors under the action of $\mathbf{G}_{x_q}$ for all primes q coprime to N.*

Before we prove the theorem, we mention that part of the following proof appears as well in [7] in order to prove the result of Corollary 5.2.3 below in the case of G being absolutely simple and split.

Proof of Theorem 5.2.1. We suppose without loss of generality that $r = d/M$ for some non-negative integers $d < M$.

Assume that (ii) is satisfied, i.e. there exists a prime q coprime to N such that $\mathbf{V}_{x_q,r}$ contains stable vectors under the action of $\mathbf{G}_{x_q}$.

A slight variation of the proof by Moy and Prasad of [14, Proposition 4.3] (see [7, Lemma 2] for a detailed proof) shows that then $\mathcal{V}_{\overline{\mathbb{Q}}_q}$ contains stable vectors under $\mathcal{H}_{\overline{\mathbb{Q}}_q}$, where $\mathcal{H}$ and $\mathcal{V}$ are as in Theorem 3.4.1.

Recall that by Corollary 4.2.1, $\mathcal{H}_{\overline{\mathbb{Q}}} \simeq \mathcal{G}_{\overline{\mathbb{Q}}}^{\theta,0}$ and $\mathcal{V}_{\overline{\mathbb{Q}}} \simeq \mathrm{Lie}(\mathcal{G}_{\overline{\mathbb{Q}}})_{M-d}(\overline{\mathbb{Q}})$ such that the action of $\mathcal{H}_{\overline{\mathbb{Q}}}$ on $\mathcal{V}_{\overline{\mathbb{Q}}}$ corresponds via these isomorphisms to the restriction of the adjoint action of $\mathcal{G}_{\overline{\mathbb{Q}}}$ on $\mathrm{Lie}(\mathcal{G}_{\overline{\mathbb{Q}}})(\overline{\mathbb{Q}})$. Let ζ_M be a primitive M-th root of unity in $\overline{\mathbb{Q}}$, denote $\mathcal{G}_{\overline{\mathbb{Q}}}^{\theta(\zeta_M)^{M/(d,M)},0}$ by $\mathcal{G}'$, its Weyl group by W', and let ϑ be the action of $\theta(\zeta_M)$ on the root datum $R(\mathcal{G}'_{\overline{\mathbb{Q}}_q})$. Then by [19, Corollary 14], the existence of stable vectors in $\mathcal{V}_{\overline{\mathbb{Q}}_q}$ is equivalent to the action of $\theta(\zeta_M)$ on $\mathcal{G}'_{\overline{\mathbb{Q}}_q}$ (or, equivalently, on $\mathcal{G}'$) being principal and $M/(d, M)$ being the order of an elliptic $\mathbb{Z}$-regular element of $W'\vartheta$. Hence we conclude by the same equivalence for the prime p that there exist stable vectors in $\mathcal{V}_{\overline{\mathbb{Q}}_p}$ under the action of $\mathcal{H}_{\overline{\mathbb{Q}}_p}$.

Thus the set $(\mathcal{V}_{\overline{\mathbb{Q}}_p})_s$ of stable vectors in $\mathcal{V}_{\overline{\mathbb{Q}}_p}$ is non-empty and open (see [16, 1.4, p. 37]). Hence there exists a non-zero polynomial P in the space of global sections $\mathcal{O}_\mathcal{V}(\mathcal{V}_{\overline{\mathbb{Q}}_p}) = \mathcal{O}_\mathcal{V}(\mathcal{V}_{\overline{\mathbb{Z}}_p}) \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p \simeq \overline{\mathbb{Z}}_p[x_1, \dots, x_n] \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p = \overline{\mathbb{Q}}_p[x_1, \dots, x_n]$ such that the $\overline{\mathbb{Q}}_p$-points of the closed reduced subvariety $V(P)$ of $\mathcal{V}_{\overline{\mathbb{Q}}_p}$ defined by the vanishing locus of P contain $(\mathcal{V}_{\overline{\mathbb{Q}}_p} - (\mathcal{V}_{\overline{\mathbb{Q}}_p})_s) \ni 0$. We can assume without loss of generality that the coefficients of P are in $\overline{\mathbb{Z}}_p$, i.e. $P \in \mathcal{O}_\mathcal{V}(\mathcal{V}_{\overline{\mathbb{Z}}_p}) \subset \mathcal{O}_\mathcal{V}(\mathcal{V}_{\overline{\mathbb{Q}}_p})$, and that at least one coefficient of P has p-adic valuation zero. Let $\overline{P}$ be the image of P under the reduction map $\mathcal{O}_\mathcal{V}(\mathcal{V}_{\overline{\mathbb{Z}}_p}) \simeq \overline{\mathbb{Z}}_p[x_1, \dots, x_n] \to \mathcal{O}_\mathcal{V}(\mathcal{V}_{\overline{\mathbb{F}}_p}) \simeq \overline{\mathbb{F}}_p[x_1, \dots, x_n]$. Then $\overline{P}$ is not constant, because $P(0) = 0$, and there exists $\overline{X} \in \mathcal{V}_{\overline{\mathbb{F}}_p} \simeq \mathbf{V}_{x,r}$ such that $\overline{P}(\overline{X}) \neq 0$.

We claim that $\overline{X}$ is a stable vector under the action of $\mathbf{G}_x$. We will prove the claim using the Hilbert–Mumford criterion that states that a vector is stable if and only if it has positive and negative weights for every non-trivial one-parameter subgroup (see [17, p. 41] based on [16, Theorem 2.1]). Let $\overline{\lambda} : \mathbb{G}_\mathrm{m} \to \mathbf{G}_x \simeq \mathcal{H}_{\overline{\mathbb{F}}_p}$ be a non-trivial one-parameter subgroup. Then $\overline{\lambda}$ is defined over some finite extension of $\mathbb{F}_p$, and hence by [21, IX, Corollaire 7.3] there exists a lift $\lambda : \mathbb{G}_\mathrm{m} \to \mathcal{H}_{\overline{\mathbb{Z}}_p}$ of $\overline{\lambda}$. The composition of λ with the action of $\mathcal{H}_{\overline{\mathbb{Z}}_p}$ on $\mathcal{V}_{\overline{\mathbb{Z}}_p}$ yields an action of $\mathbb{G}_\mathrm{m}$ on $\mathcal{V}_{\overline{\mathbb{Z}}_p}$, and we obtain a weight decomposition $\mathcal{V}_{\overline{\mathbb{Z}}_p} = \bigoplus_{m \in \mathbb{Z}} \mathcal{V}_m$. Denote $\bigoplus_{m \in \mathbb{Z}_{>0}} \mathcal{V}_m$ by $\mathcal{V}_+$ and $\bigoplus_{m \in \mathbb{Z}_{<0}} \mathcal{V}_m$ by $\mathcal{V}_-$, i.e. $\mathcal{V}_{\overline{\mathbb{Z}}_p} = \mathcal{V}_- \oplus \mathcal{V}_0 \oplus \mathcal{V}_+$. Let $X \in \mathcal{V}_{\overline{\mathbb{Z}}_p}$ be a lift of $\overline{X}$, and write $X = X_- + X_0 + X_+$ with $X_- \in \mathcal{V}_-$, $X_0 \in \mathcal{V}_0$, $X_+ \in \mathcal{V}_+$. Note that the weight decomposition of $\mathcal{V}_{\overline{\mathbb{F}}_p}$ under the action of $\mathbb{G}_\mathrm{m}$ via the composition of $\overline{\lambda}$ with the action of $\mathcal{H}_{\overline{\mathbb{F}}_p}$ on $\mathcal{V}_{\overline{\mathbb{F}}_p}$ is the image of the decomposition $\mathcal{V}_- \oplus \mathcal{V}_0 \oplus \mathcal{V}_+$, i.e. $(\mathcal{V}_{\overline{\mathbb{F}}_p})_- = \bigoplus_{m \in \mathbb{Z}_{<0}} (\mathcal{V}_{\overline{\mathbb{F}}_p})_m = (\mathcal{V}_-)_{\overline{\mathbb{F}}_p}$, $(\mathcal{V}_{\overline{\mathbb{F}}_p})_0 = (\mathcal{V}_0)_{\overline{\mathbb{F}}_p}$ and $(\mathcal{V}_{\overline{\mathbb{F}}_p})_+ = \bigoplus_{m \in \mathbb{Z}_{>0}} (\mathcal{V}_{\overline{\mathbb{F}}_p})_m = (\mathcal{V}_+)_{\overline{\mathbb{F}}_p}$. Hence $\overline{X} = \overline{X}_- + \overline{X}_0 + \overline{X}_+$ (where an overline denotes the image after base change to $\overline{\mathbb{F}}_p$) has positive and negative weights with respect to $\overline{\lambda}$ if and only if $\mathrm{v}(X_-) = 0 = \mathrm{v}(X_+)$.

Suppose that $\mathrm{v}(X_-) > 0$. Then $P(X) \equiv P(X_0 + X_+)$ modulo the maximal ideal of $\overline{\mathbb{Z}}_p$. However, $X_0 + X_+$ is not a stable vector, because it has no negative weights with respect to the non-trivial one-parameter subgroup $\lambda \times_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$, which implies $P(X_0 + X_+) = 0$. Hence $\overline{P}(\overline{X}) = 0$, contradicting the choice of $\overline{X}$. The same contradiction arises if we assume that $\mathrm{v}(X_-) > 0$. Thus, $\overline{X}$ has positive and negative weights for every non-trivial one-parameter subgroup, i.e. $\overline{X}$ is stable by the Hilbert–Mumford criterion. Hence, statement (i) of the theorem holds.

The same arguments show that if $\mathbf{G}_{x,r}$ has stable vectors, then $\mathbf{G}_{x_q,r}$ has stable vectors for all q coprime to N, i.e. (i) implies (iii). As (iii) implies (ii), the three statements are equivalent.

As in the semistable case, the same proof works for the linear duals of the Moy–Prasad filtration quotients:

Corollary 5.2.2. *We use the same notation as above. Then $\check{\mathbf{V}}_{x,r}$ has stable vectors under the action of $\mathbf{G}_x$ if and only if $\check{\mathbf{V}}_{x_q,r}$ has stable vectors under the action of $\mathbf{G}_{x_q}$ for some prime q coprime to N if and only if $\check{\mathbf{V}}_{x_q,r}$ has stable vectors under the action of $\mathbf{G}_{x_q}$ for all primes q coprime to N.*

Denote by $r(x)$ the smallest positive real number such that $\mathbf{V}_{x,r(x)} \neq \{0\}$, and let $\check{\rho} = \frac{1}{2}\sum_{\alpha\in\Phi^+}\check{\alpha}$, where Φ^+ are the positive roots of $\Phi = \Phi(G)$ (with respect to the fixed Borel B). Then Corollary 5.2.2 allows us to classify the existence of stable vectors in $\check{\mathbf{V}}_{x,r(x)}$ for arbitrary primes p and good semisimple groups below. This generalizes the result of [20, Corollary 5.1] for large primes p and semisimple groups that split over tamely ramified extensions.

Corollary 5.2.3. *Let G be a good semisimple group and x a rational point of order m in $\mathcal{A}(S,K) \subset \mathcal{B}(G,K)$. Then $\check{\mathbf{V}}_{x,r(x)}$ contains stable vectors under $\mathbf{G}_x$ if and only if x is conjugate under the affine Weyl group W_{aff} of the restricted root system of G to $x_0 + \check{\rho}/m$, $r(x) = 1/m$ and there exists an elliptic $\mathbb{Z}$-regular element $w\gamma$ of order m in $W\gamma$, where W is the absolute Weil group of G and γ is the automorphism of $R(G)$ given in the definition of a good group* (*Definition* 3.1.1).

Proof. Note that by Lemma 3.3.1 the order of x_q is m, and by Theorem 3.4.1 we have $r(x_q) = r(x)$. Let q be sufficiently large, i.e. coprime to M, not torsion and odd. Then G_q is a semisimple group that splits over a tamely ramified extension, and we deduce from the proof of [20, Lemma 3.1] that $\check{\mathbf{V}}_{x_q,r(x_q)}$ can only admit stable vectors under $\mathbf{G}_{x_q}$ if x_q is a barycenter of some facet of $\mathcal{A}_q = \mathcal{A}(S_q, K_q)$, and hence $r(x_q) = 1/m$. Therefore, as q is chosen sufficiently large, we deduce from [20, Corollary 5.1] that $\check{\mathbf{V}}_{x_q,r(x_q)}$ has stable vectors if and only if x_q is conjugate under the affine Weyl group $W_{\mathrm{aff}q}$ of the restricted root system of G_q to $x_{0,q} + \check{\rho}/m$, $r(x) = 1/m$ and there exists an elliptic $\mathbb{Z}$-regular element $w\gamma$ of order m in $W\gamma$, because W is isomorphic to the absolute Weil group of G_q. Note that

$$x_q \sim_{W_{\mathrm{aff}q}} x_{0,q} + \check{\rho}/m \quad \text{if and only if} \quad x \sim_{W_{\mathrm{aff}}} x_0 + \frac{1}{4}\sum_{a\in\Phi_K^{+,\mathrm{mul}}} \mathrm{v}(\lambda_a)\cdot\check{a} + \frac{\check{\rho}}{m},$$

and $x_0 + \frac{1}{4}\sum_{a\in\Phi_K^{+,\mathrm{mul}}} \mathrm{v}(\lambda_a)\cdot\check{a} + \check{\rho}/m$ is conjugate to $x_0 + \check{\rho}/m$ under the extended affine Weyl group of the restricted root system of G. However, by checking the tables for all possible points x_q whose first Moy–Prasad filtration quotient $\check{\mathbf{V}}_{x_q,r(x_q)}$ admits stable vectors in [19] and [20], we observe that the latter conjugacy can be replaced by conjugacy under the (unextended) affine Weyl group. Hence using Corollary 5.2.2, we conclude that $\check{\mathbf{V}}_{x,r(x)}$ contains stable vectors under the action of $\mathbf{G}_x$ if and only if $x \sim_{W_{\mathrm{aff}}} x_0 + \check{\rho}/m$, $r(x) = 1/m$, and there exists an elliptic $\mathbb{Z}$-regular element of order m in $W\gamma$. ∎

Recall that k is a non-archimedean local field with maximal unramified extension K.

Corollary 5.2.4. *Let G be a good semisimple group, and suppose that G is defined over k. Assume that $W\gamma$ contains an elliptic $\mathbb{Z}$-regular element. Then using the construction of* [20, §2.5] *we obtain supercuspidal* (*epipelagic*) *representations of $G(k')$ for some finite unramified field extension k' of k.*

Proof. Let m be the order of an elliptic $\mathbb{Z}$-regular element of $W\gamma$, and $x = x_0 + \check{\rho}/m \in \mathcal{A}(S, K)$. By Corollary 5.2.3, $\check{\mathbf{V}}_{x,r(x)}$ contains stable vectors under the action of $\mathbf{G}_x$. Since x is fixed under the action of the Galois group $\mathrm{Gal}(K/k'')$ for some finite unramified extension k'' of k, the vector space $\check{\mathbf{V}}_{x,r(x)}$ is defined over the residue field $\mathfrak{f}''$ of k''. Hence there exists a finite unramified field extension k' of k with residue field $\mathfrak{f}'$ such that $\check{\mathbf{V}}_{x,r(x)}$ contains a stable vector defined over $\mathfrak{f}'$. Applying [20, Proposition 2.4] yields the desired result. ∎

6. Moy–Prasad filtration representations as Weyl modules

In this section we describe the Moy–Prasad filtration representations in terms of Weyl modules. Recall that for $\lambda \in X^*(\mathcal{S})$ a dominant weight, the *Weyl module* $V(\lambda)$ (over $\overline{\mathbb{Z}}[1/N]$) is given by

$$V(\lambda) = \mathrm{ind}_{\mathcal{B}_{\mathcal{H}}^-}^{\mathcal{H}}(-w_0\lambda)^\vee,$$

where $\mathcal{B}_{\mathcal{H}}$ is the Borel subgroup of $\mathcal{H}$ corresponding to $\Delta(\mathcal{H})$, $\mathcal{B}_{\mathcal{H}}^-$ is the opposite Borel subgroup corresponding to $-\Delta(\mathcal{H})$, w_0 is the longest element of the Weyl group of $\Phi(\mathcal{H})$, and $(\cdot)^\vee$ denotes the dual [11, II.8.9]. We define

$$\Phi_{x,r} = \{a \in \Phi_K \mid r - a(x - x_0) \in \Gamma'_a(G)\},$$
$$\Phi_{x,r}^{\max} = \{a \in \Phi_{x,r} \mid a + b \notin \Phi_{x,r} \text{ for all } b \in \Phi^+(\mathcal{H}) \subset \Phi_K\}.$$

6.1. The split case

If G is split over K, then

$$\Phi_{x,r}^{\max} = \{\alpha \in \Phi \mid r - \alpha(x - x_0) \in \mathbb{Z},\ \alpha + \beta \notin \Phi \text{ for all } \beta \in \Phi^+(\mathcal{H}) \subset \Phi\}.$$

Theorem 6.1.1. *Let G be a split reductive group over K, r a real number and x a rational point of $\mathcal{B}(G, K)$. Let $\mathcal{V}$ be the corresponding global Moy–Prasad filtration representation of the split reductive group scheme $\mathcal{H}$ over $\overline{\mathbb{Z}}$* (*Theorem* 3.4.1). *Then*

$$\mathcal{V} \simeq \begin{cases} \mathrm{Lie}(\mathcal{H})(\overline{\mathbb{Z}}) & \text{if } r \text{ is an integer,} \\ \bigoplus_{\lambda \in \Phi_{x,r}^{\max}} V(\lambda) & \text{otherwise.} \end{cases}$$

Proof. If r is an integer, then by Theorem 4.1.1 we have $\mathcal{V} \simeq \mathrm{Lie}(\mathcal{G})_M(\overline{\mathbb{Z}}) = \mathrm{Lie}(\mathcal{G}^\theta)(\overline{\mathbb{Z}}) = \mathrm{Lie}(\mathcal{H})(\overline{\mathbb{Z}})$.

Suppose r is not an integer. Then $\mathcal{V} \subset \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}})$ is spanned by $\mathcal{X}_\alpha = \mathrm{Lie}(\chi_\alpha)(1)$ for $\alpha \in \Phi_{x,r}$ (§3.4.2). Thus the weights in $\Phi_{x,r}^{\max}$ are the highest weights of the representation of $\mathcal{H}$ on $\mathcal{V}$, and we have $\mathcal{V}_{\overline{\mathbb{Q}}} \simeq \bigoplus_{\lambda \in \Phi_{x,r}^{\max}} V(\lambda)_{\overline{\mathbb{Q}}}$. In order to show that

$\mathcal{V} \simeq \bigoplus_{\lambda \in \Phi_{x,r}^{\max}} V(\lambda)$, it suffices by [11, II.8.3] to prove that $\{\mathcal{H}(\overline{\mathbb{Z}})(\mathcal{X}_\alpha)\}_{\alpha \in \Phi_{x,r}^{\max}}$ spans $\mathcal{V}$, i.e. $\langle \mathcal{H}(\overline{\mathbb{Z}})(\mathcal{X}_\alpha)\rangle_{\alpha \in \Phi_{x,r}^{\max}}$ contains $\mathcal{X}_\alpha$ for all $\alpha \in \Phi_{x,r}$. Let $\alpha \in \Phi_{x,r} \backslash \Phi_{x,r}^{\max}$. Then there exists $\beta \in \Phi^+(\mathcal{H})$ such that $\alpha + \beta \in \Phi$. Let $N_{\alpha,\beta} > 0$ be the maximal integer such that $\alpha + N_{\alpha,\beta}\beta \in \Phi$, and let $N_{\alpha,\beta}^-$ be the maximal integer such that $\alpha - N_{\alpha,\beta}^-\beta \in \Phi$. We claim that $\mathcal{X}_\alpha + N_{\alpha,\beta}\beta \in \langle \mathcal{H}(\overline{\mathbb{Z}})(\mathcal{X}_\alpha)\rangle_{\alpha \in \Phi_{x,r}^{\max}}$ implies that $\mathcal{X}_\alpha \in \langle \mathcal{H}(\overline{\mathbb{Z}})(\mathcal{X}_\alpha)\rangle_{\alpha \in \Phi_{x,r}^{\max}}$, which will imply the theorem by induction.

Suppose that $\mathcal{X}_\alpha + N_{\alpha,\beta}\beta \in \langle \mathcal{H}(\overline{\mathbb{Z}})(\mathcal{X}_\alpha)\rangle_{\alpha \in \Phi_{x,r}^{\max}}$. Note that $N_{\alpha,\beta} + N_{\alpha,\beta}^- \in \{1, 2, 3\}$, and recall that

$$x_{-\beta}(u)(\mathcal{X}_{\alpha+N_{\alpha,\beta}\beta}) = \sum_{i=0}^{N_{\alpha,\beta}+N_{\alpha,\beta}^-} m_{\alpha,\beta,i} u^i \mathcal{X}_{\alpha+(N_{\alpha,\beta}-i)\beta} \quad \text{with } m_{\alpha,\beta,i} \in \{\pm 1\}, \tag{33}$$

for $u \in \mathbb{G}_a(\overline{\mathbb{Z}})$. By varying $u \in \mathbb{G}_a(\overline{\mathbb{Z}})$ and taking linear combinations, we conclude that $\mathcal{X}_\alpha$ is in the $\overline{\mathbb{Z}}$-span of $\{\mathcal{H}(\overline{\mathbb{Z}})(\mathcal{X}_\alpha)\}_{\alpha \in \Phi_{x,r}^{\max}}$. ■

The following corollary follows immediately by combining Theorems 6.1.1 and 3.4.1.

Corollary 6.1.2. *Let G be a split reductive group over K, r a real number and x a rational point of $\mathcal{B}(G, K)$. Then the representation of $\mathbf{G}_x$ on $\mathbf{V}_{x,r}$ is given by*

$$\mathbf{V}_{x,r} \simeq \begin{cases} \mathrm{Lie}(\mathbf{G}_x)(\overline{\mathbb{F}}_p) & \text{if } r \text{ is an integer,} \\ \bigoplus_{\lambda \in \Phi_{x,r}^{\max}} V(\lambda)_{\overline{\mathbb{F}}_p} & \text{otherwise.} \end{cases}$$

Remark 6.1.3. Note that if p is sufficiently large, then $V(\lambda)_{\overline{\mathbb{F}}_p}$ is an irreducible representation of $\mathbf{G}_x$ of highest weight λ.

6.2. The general case

Let $a \in \Phi_{x,r}^{\max}$ and let $\mathcal{U}_{\mathcal{H}}$ be the unipotent radical of $\mathcal{B}_{\mathcal{H}}$. By Frobenius reciprocity, we have [11, proof of Lemma II.2.13a)]

$$\begin{aligned} \mathrm{Hom}_{\mathcal{H}}\big(V(a), \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])\big) &\simeq \mathrm{Hom}_{\mathcal{H}}\big(\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])^\vee, \mathrm{ind}_{\mathcal{B}_{\mathcal{H}}^-}^{\mathcal{H}}(-w_0 a)\big) \\ &\simeq \mathrm{Hom}_{\mathcal{B}_{\mathcal{H}}^-}\big(\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])^\vee, -w_0 a\big) \\ &\simeq \mathrm{Hom}_{\mathcal{B}_{\mathcal{H}}^-}\big(w_0 a, \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])\big) \simeq \big((\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N]))^{\mathcal{U}_{\mathcal{H}}}\big)_a. \end{aligned}$$

Using these isomorphisms, the element $Y_a \in ((\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N]))^{\mathcal{U}_{\mathcal{H}}})_a \subset \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$ yields a morphism $V(a) \to \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$ of representations of $\mathcal{H}$. This morphism is an injection, and we will identify $V(a)$ with its image in $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N])$.

Theorem 6.2.1. *Let G be a good reductive group over K, r a real number and x a rational point of $\mathcal{B}(G, K)$. Let*

$$N' = \begin{cases} 2N & \text{if } \Phi_K \text{ contains multipliable roots,} \\ N & \text{otherwise.} \end{cases}$$

Then

$$\mathcal{V}_{\overline{\mathbb{Z}}[1/N']} \simeq (\mathcal{V}_T)_{\overline{\mathbb{Z}}[1/N']} + \bigoplus_{\lambda \in \Phi_{x,r}^{\max}} V(\lambda)_{\overline{\mathbb{Z}}[1/N']} \subset \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N']) \tag{34}$$

as representations of $\mathcal{H}_{\overline{\mathbb{Z}}[1/N']}$.

Proof. By the definition of N' the subspace $\mathcal{V}_{\overline{\mathbb{Z}}[1/N']} \subset \mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N'])$ is spanned by $\mathcal{V}_T$ and Y_a for $a \in \Phi_{x,r}$ (§3.4.2). Thus, analogously to the argument in the proof of Theorem 6.1.1, it suffices to show that $\langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a \in \Phi_{x,r}^{\max}}$ contains Y_b for all $b \in \Phi_{x,r}$. Let $a \in \Phi_{x,r}^{\max} \backslash \Phi_{x,r}$ and $b \in \Phi^+(\mathcal{H})$ with $a + b \in \Phi_{x,r}$, and let $N_{a,b} > 0$ be the maximal integer such that $a + N_{a,b} b \in \Phi_{x,r}$. We need to show that $Y_{a+N_{a,b}b} \in \langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a \in \Phi_{x,r}^{\max}}$ implies $Y_a \in \langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a \in \Phi_{x,r}^{\max}}$. We assume $Y_{a+N_{a,b}b} \in \langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a \in \Phi_{x,r}^{\max}}$ and distinguish four cases.

Case 1: $a\mathbb{R} \neq b\mathbb{R}$ and b is not multipliable. In this case the result follows from the proof of the split case (Theorem 6.1.1) and equations (21) of §3.4.1 and (22) of §3.4.2 (if b is non-divisible) or equations (20) and (22) (if b is divisible).

Case 2: $a\mathbb{R} = b\mathbb{R}$ and b is not multipliable. In this case $a = -(a + N_{a,b} b)$, and the element s_b in the Weyl group of $\mathcal{H}$ corresponding to reflection in direction of b sends $Y_{a+N_{a,b}b}$ to $\pm Y_{-(a+N_{a,b}b)} = \pm Y_a$. Hence $Y_a \in \langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a \in \Phi_{x,r}^{\max}}$.

Case 3: $a\mathbb{R} \neq b\mathbb{R}$ and b is multipliable. By taking Galois orbits over different connected components and using equations (19) and (22), it suffices to consider the case that $\mathrm{Dyn}(G) = A_{2n}$ with non-trivial Galois action. We label the simple roots by $\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, \alpha_1, \beta_1, \beta_2, \ldots, \beta_n$ as in Figure 1 (§2.5). Then b is the image of $\alpha_1 + \cdots + \alpha_s$ for some $1 \leq s \leq n$, and as $\langle \check{b}, a + N_{a,b} b \rangle > 0$, the root $a + N_{a,b} b$ is the image of

$-(\alpha_{s+1} + \cdots + \alpha_{s_1})$	for some $s < s_1 \leq n$, or
$\alpha_{s_2} + \cdots + \alpha_s$	for some $1 < s_2 \leq s$, or
$\alpha_1 + \cdots + \alpha_s + \beta_1 + \cdots + \beta_{s_3}$	for some $1 \leq s_3 < s$ or $s < s_3 \leq n$.

To simplify notation, we will prove the claim for the case that b is the image of α_1 and $a + N_{a,b} b$ is the image of $-\alpha_2$. All the other cases are handled analogously. Combining equations (19), (22) and (33), and using the fact that $\mathcal{H}_{\overline{\mathbb{Z}}[1/N']}$ preserves the subspace $\mathcal{V}_{\overline{\mathbb{Z}}[1/N']}$ of $\mathrm{Lie}(\mathcal{G})(\overline{\mathbb{Z}}[1/N'])$, we obtain

$$\begin{aligned}
&x_{\mathcal{H}-b}(u)(Y_{a+N_{a,b}b}) \\
&\quad = \left(x_{-\beta_1}(\sqrt{2}\,u)x_{-(\alpha_1+\beta_1)}(-(-1)^{b(x-x_0)M}u^2)x_{-\alpha_1}((-1)^{b(x-x_0)M}\sqrt{2}\,u)\right) \\
&\qquad \left(X_{-\beta_2} + (-1)^{(-(a+N_{a,b}b)(x_q-x_{0,q})+r)\cdot 2} X_{-\alpha_2}\right) \\
&\quad = Y_{a+N_{a,b}b} + m'_{a,b,1}\sqrt{2}\,uY_{a+(N_{a,b}-1)b} + m'_{a,b,2}u^2 Y_{a+(N_{a,b}-2)b}
\end{aligned}$$

with $m'_{a,b,1}, m'_{a,b,2} \in \{\pm 1\}$, for all $u \in \mathbb{G}_a(\overline{\mathbb{Z}}[1/N'])$. Since $2 \mid N'$, taking $\overline{\mathbb{Z}}[1/N']$-linear combinations of

$$Y_{a+N_{a,b}b} + m'_{a,b,1}\sqrt{2}uY_{a+(N_{a,b}-1)b} + m'_{a,b,2}u^2Y_{a+(N_{a,b}-2)b}$$

for different u implies that $Y_{a+(N_{a,b}-1)b}$ and $Y_{a+(N_{a,b}-2)b}$ are contained in $\langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a\in\Phi^{\max}_{x,r}}$, so

$$Y_a \in \langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a\in\Phi^{\max}_{x,r}}.$$

Case 4: $a\mathbb{R} = b\mathbb{R}$ and b is multipliable. As in Case 3, we can restrict to the case that $\mathrm{Dyn}(G) = A_{2n}$, and we may assume that b is the image of α_1. Then $a + N_{a,b}b$ is the image of α_1 or of $\alpha_1 + \beta_1$. If $N^-_{a,b}$ denotes the largest integer such that $a - N^-_{a,b}b \in \Phi_{x,r}$, then $Y_{a-N^-_{a,b}b}$ is conjugate to $\pm Y_{a+N_{a,b}b}$ under the Weyl group. Hence

$$Y_{a-N^-_{a,b}b} \in \langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a\in\Phi^{\max}_{x,r}}.$$

If $a + N_{a,b}b$ is the image of α_1, then $N^-_{a,b} = 0$, and we are done. Thus, suppose that $a + N_{a,b}b$ is the image of $\alpha_1 + \beta_1$.

Recall that for $\alpha \in \Phi$ and $H_\alpha := \mathrm{Lie}(\check{\alpha})(1)$, we have [4, Corollary 5.1.12]

$$\begin{aligned} x_{-\alpha}(u)(\mathcal{X}_\alpha) &= \mathcal{X}_\alpha + \epsilon_{\alpha,\alpha}uH_{-\alpha} - \epsilon_{\alpha,\alpha}u^2\mathcal{X}_{-\alpha}, \\ x_{-\alpha}(u)(H) &= H + \mathrm{Lie}(\alpha)(H)u\mathcal{X}_{-\alpha} \end{aligned}$$

for all $u \in \mathbb{G}_a(\overline{\mathbb{Z}}[1/N'])$ and all $H \in \mathrm{Lie}(\mathcal{T})(\overline{\mathbb{Z}}[1/N'])$. Using these identities, we obtain

$$\begin{aligned} &x_{\mathcal{H}-b}(u)(Y_{a+N_{a,b}b}) \\ &\quad = (x_{-\beta_1}(\sqrt{2}u)x_{-\alpha_1-\beta_1}(-(-1)^{b(x-x_0)M}u^2)x_{-\alpha_1}((-1)^{b(x-x_0)M}\sqrt{2}u))(\mathcal{X}_{\alpha_1+\beta_1}) \\ &\quad = Y_{a+N_{a,b}b} + m''_{a,1}\sqrt{2}uY_{a+(N_{a,b}-1)b} + H + m''_{a,3}\sqrt{2}u^3Y_{a+(N_{a,b}-3)b} \\ &\qquad + m''_{a,4}u^4Y_{a+(N_{a,b}-4)b}, \end{aligned}$$

with $m''_{a,1}, m''_{a,3} \in \{\pm 1\}$, $m''_{a,4} \in \{\pm 1, \pm 3\}$ and $H \in \mathcal{V}_T$. As $Y_{a+(N_{a,b}-4)b} = Y_{a-N^-_{a,b}b}$ and H are in $\langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a\in\Phi^{\max}_{x,r}}$, and since $2 \mid N'$, we also see that $Y_{a+(N_{a,b}-1)b}$ and $Y_{a+(N_{a,b}-3)b}$ are contained in $\langle \mathcal{H}(\overline{\mathbb{Z}}[1/N'])(Y_a), \mathcal{V}_T \rangle_{a\in\Phi^{\max}_{x,r}}$. ■

Corollary 6.2.2. *Let G be a good reductive group, $r \notin \frac{1}{p^sN}\mathbb{Z}$ a real number, and x a rational point of $\mathcal{B}(G, K)$. Then*

$$\mathbf{V}_{x,r} \simeq \bigoplus_{\lambda\in\Phi^{\max}_{x,r}} V(\lambda)_{\overline{\mathbb{F}}_p}.$$

Proof. If $r \notin \frac{1}{p^sN}\mathbb{Z}$, then $\mathcal{V}_T = \{0\}$. Hence, if $p \neq 2$, the claim follows by combining Theorems 6.2.1 and 3.4.1. The proof in the case $p = 2$ is completely analogous to the proof of Theorem 6.2.1 using the fact that $\mathcal{V}$ is spanned by $\mathcal{V}_T$, Y_a for all $a \in \Phi^{\mathrm{nm}}_K$ with $r - a(x - x_0) \in \Gamma'_a(G)$ and $\sqrt{2}\,Y_a$ for all $a \in \Phi^{\mathrm{mul}}_K$ with $r - a(x - x_0) \in \Gamma'_a(G)$. ■

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Selected definitions

Selected notation

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