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The isometries of the space of Kähler metrics

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Abstract. Given a compact Kähler manifold, we prove that all global isometries of the space of Kähler metrics are induced by biholomorphisms and anti-biholomorphisms of the manifold. In particular, there exist no global symmetries for Mabuchi's metric. Moreover, we show that the Mabuchi completion does not even admit local symmetries. Closely related to these findings, we provide a large class of metric geodesic segments that cannot be extended at one end, exhibiting the first such examples in the literature.

Keywords. Kähler manifolds, Mabuchi metric, isometry group

1. The main results

Let (X, ω) be a compact connected Kähler manifold. Given a Kähler metric ω' cohomologuos to ω, by the $\partial\bar\partial$-lemma of Hodge theory there exists $u \in C^\infty(X)$ such that $\omega' := \omega + i\partial\bar\partial u$. Such a metric ω' is said to belong to the *space $\mathcal{H}$ of Kähler metrics*. By the above, up to a constant, one can identify $\mathcal{H}$ with the *space of Kähler potentials*:

$$\mathcal{H}_\omega := \{u \in C^\infty(X) : \omega + i\partial\bar\partial u > 0\}.$$

This space can be endowed with a natural infinite-dimensional L^2 type Riemannian metric [17, 25, 27]:

$$\langle \xi, \zeta \rangle_v := \frac{1}{V}\int_X \xi\zeta\omega_v^n, \qquad v \in \mathcal{H}_\omega,\ \xi, \zeta \in T_v\mathcal{H}_\omega \simeq C^\infty(X), \tag{1}$$

where $V = \int_X \omega^n$. Additionally, Donaldson and Semmes pointed out that $(\mathcal{H}_\omega, \langle\cdot,\cdot\rangle)$ can be thought of as a formal symmetric space [17, 28]:

$$\mathcal{H}_\omega \simeq \mathrm{Ham}_\omega^{\mathbb{C}}/\mathrm{Ham}_\omega, \tag{2}$$

where Ham_ω is the group of Hamiltonian symplectomorphisms of ω, and $\mathrm{Ham}_\omega^{\mathbb{C}}$ is its formal complexification. Though (2) is not quite precise, the underlying heuristic led to

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many advances in the understanding of the geometry of $\mathcal{H}_\omega$, as well as the formulation of stability conditions aiming to characterize existence of canonical metrics ([17, 18]; for an exposition see [29]).

Global L^2 isometries and symmetries of $\mathcal{H}_\omega$. For finite-dimensional Riemannian manifolds, the existence of a symmetric space structure arising as a quotient of Lie groups, as in (2), is equivalent to the existence of global symmetries at all points of the manifold [20]. Such maps are global involutive isometries reversing geodesics at a specific point. If such symmetries existed for $(\mathcal{H}_\omega, \langle\cdot,\cdot\rangle)$ it would perhaps allow one to make a precise sense of (2).

Recently a large class of local symmetries of $\mathcal{H}_\omega$ were constructed in [3], via complex Legendre transforms, having applications to interpolation of norms [2]. Moreover, it was shown in [24] that all local symmetries of $\mathcal{H}_\omega$ arise from the construction of [3]. Below we show that global symmetries actually do not exist, in particular the local symmetries cannot be extended to $\mathcal{H}_\omega$. This will follow from our characterization of the isometry group of $(\mathcal{H}_\omega, \langle\cdot,\cdot\rangle)$.

First we recall some terminology. Let $\mathcal{U}, \mathcal{V} \subset \mathcal{H}_\omega$ be open sets. We say that a map $F : \mathcal{U} \to \mathcal{V}$ is C^1, or (with slight abuse of terminology) differentiable, if $(F, F_*) : \mathcal{U} \times C^\infty(X) \to \mathcal{V} \times C^\infty(X)$ is continuous as a map of Fréchet spaces. Here F_* is the differential of F (see [21, p. 3] and references therein for more details). Moreover, $F : \mathcal{U} \to \mathcal{U}$ is a *differentiable L^2 symmetry at $\phi \in \mathcal{U}$* if $F^2 = \mathrm{Id}$, $F(\phi) = \phi$, $F_*|_\phi = -\mathrm{Id}$ and

$$\int_X |\xi|^2 \omega_v^n = \int_X |F_*\xi|^2 \omega_{F(v)}^n, \qquad v \in \mathcal{H}_\omega,\ \xi \in T_v\mathcal{H}_\omega. \tag{3}$$

If $F : \mathcal{U} \to \mathcal{V}$ is C^1, satisfies (3) and is bijective, then it is called a *differentiable L^2 isometry*. Due to infinite-dimensionality, it is not yet known if differentiable L^2 isometries are automatically smooth [23], hence the isometries we consider in this work are possibly more general than the ones in [3, 24].

A small class of global L^2 isometries has been previously known in the literature [23, p. 16]. One of them is the so called *Monge–Ampère flip* $\mathcal{I} : \mathcal{H}_\omega \to \mathcal{H}_\omega$, and is defined by the formula $\mathcal{I}(u) = u - 2I(u)$, where $I : \mathcal{H}_\omega \to \mathbb{R}$ is the Monge–Ampère energy,

$$I(u) = \frac{1}{V(n+1)} \sum_{j=0}^{n} \int_X u\omega^j \wedge \omega_u^{n-j}.$$

The map $\mathcal{I}$ is involutive and its name is inspired by the fact that it flips the sign of I. Indeed, $I(\mathcal{I}(u)) = -I(u)$.

We say that a biholomorphism $f : X \to X$ *preserves the Kähler class* $[\omega]$ if $[f^*\omega] = [\omega]$. Similarly, an anti-biholomorphism $g : X \to X$ *flips the Kähler class* $[\omega]$ if $[g^*\omega] = -[\omega]$. Such maps f and g induce a class of global L^2 isometries $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$, where at the level of Kähler metrics we have either $\omega_{F(u)} = f^*\omega_u$ or $\omega_{F(u)} = -g^*\omega_u$. We refer to Section 2.3 for the detailed construction.

In our first main result we point out that these maps and their compositions are the only global differentiable L^2 isometries:

Theorem 1.1. *Let $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ be a differentiable L^2 isometry. Then exactly one of the following holds:*

(i) *F is induced by a biholomorphism or an anti-biholomorphism $f : X \to X$ that preserves or flips $[\omega]$, respectively.*

(ii) *$F \circ \mathcal{I}$ is induced by a biholomorphism or an anti-biholomorphism $f : X \to X$ that preserves or flips $[\omega]$, respectively.*

The space $\mathcal{H}_\omega$ of potentials admits a Riemannian splitting $\mathcal{H}_\omega = \mathcal{H} \oplus \mathbb{R}$, via the Monge–Ampère energy I. As the fixed point set of $\mathcal{I}$ is exactly $\mathcal{H} = I^{-1}(0)$, we obtain the following corollary regarding isometries of $\mathcal{H}$:

Corollary 1.2. *Let $F : \mathcal{H} \to \mathcal{H}$ be a differentiable L^2 isometry. Then F is induced by a biholomorphism or an anti-biholomorphism $f : X \to X$ that preserves or flips $[\omega]$, respectively.*

The above results answer questions raised by Lempert regarding the extension property of local isometries [23, p. 3], though questions surrounding the isometry group of $(\mathcal{H}_\omega, \langle\cdot,\cdot\rangle)$ go back to early work of Semmes [27, 28].

Lastly, via the classification theorem of Lempert (recalled in Theorem 2.1), we will see that neither of the maps in the statement of Theorem 1.1 are symmetries, immediately giving the following non-existence result for differentiable L^2 symmetries:

Corollary 1.3. *There exists no differentiable L^2 symmetry $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ at any $\phi \in \mathcal{H}_\omega$.*

Non-existence of local L^2 symmetries on the completions. It was shown in [8] that (1) induces a path length metric space $(\mathcal{H}_\omega, d_2)$. We denote by $(\mathcal{E}^2_\omega, d_2)$ the d_2-metric completion of this space, which can be identified with a class of finite energy potentials [11].

By density, any differentiable L^2 isometry $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ extends to a unique metric d_2-isometry $F : \mathcal{E}^2_\omega \to \mathcal{E}^2_\omega$. The proof of Theorem 1.1 consists in showing that contradictions arise in this extension process, unless F is very special. With this and the above results in mind, one may hope that the isometry group of the metric space $(\mathcal{E}^2_\omega, d_2)$ could possibly admit elements beyond the ones that arise from the global differentiable L^2 isometries of $\mathcal{H}_\omega$. Though this may be true, we point out below that even local symmetries fail to exist in the context of the completion, further elaborating Corollary 1.3.

Before stating our result, we recall some facts about the d_2-geodesics of $\mathcal{E}^2$. For more details we refer to Section 2.2 and the recent survey [13]. Let $\mathcal{V} \subset \mathcal{E}^2_\omega$ be d_2-open with $\phi \in \mathcal{V} \cap \mathcal{H}_\omega$. Given a d_2-geodesic $[0,1] \ni t \mapsto \phi_t \in \mathcal{V}$ with $\phi_0 = \phi$, since $t \mapsto \phi_t(x)$ is t-convex for almost every $x \in X$, it is possible to introduce $\dot\phi_0 = \frac{d}{dt}\big|_{t=0}\phi_t$. Moreover, due to [10, Theorem 2], it follows that $\dot\phi_0 \in L^2(\omega^n_\phi)$.

Let $G : \mathcal{V} \to G(\mathcal{V}) \subset \mathcal{E}^2_\omega$ be an L^2 isometry, i.e., a bijective map satisfying $d_2(v_1, v_2) = d_2(G(v_1), G(v_2))$ for $v_1, v_2 \in \mathcal{V}$. It is clear that in this case $t \mapsto G(\phi_t)$ is also a d_2-geodesic. Furthermore, we say that G is a *metric L^2 symmetry at ϕ* if $G^2 = \mathrm{Id}$, $G(\phi) = \phi$ and $G(\dot\phi_0) = -\dot\phi_0$, i.e., G "reverses" d_2-geodesics at ϕ.

Unfortunately, metric L^2 symmetries actually do not exist, implying that the analog of [3, Theorem 1.2] does not hold in the context of the metric completion. This answers questions of Berndtsson and Rubinstein [26]:

Theorem 1.4. *Let $\mathcal{V} \subset \mathcal{E}^2_\omega$ be a d_2-open set and $\phi \in \mathcal{V} \cap \mathcal{H}_\omega$. There exists no metric L^2 symmetry $F : \mathcal{V} \to \mathcal{V}$ at ϕ.*

Given that $(\mathcal{E}^2_\omega, d_2)$ is CAT(0), the group of isometries of this metric space has special structure [6], as pointed out by B. McReynolds during the Ph.D. thesis defense of the present author. In light of the above result, we expect that the group of metric isometries can be characterized as in Theorem 1.1, though this remains an open question.

The extension property of geodesic segments. As an intermediate step in the proof of Theorem 1.4 we show that a large class of d_2-geodesic segments inside $\mathcal{E}^2_\omega$ cannot be extended at one of the endpoints. Previously no such examples were known.

Theorem 1.5. *Let $\phi_0 \in \mathcal{H}_\omega$ and $\phi_1 \in \mathcal{E}^2_\omega \setminus L^\infty$. Then the d_2-geodesic $t \mapsto \psi_t$ connecting these potentials cannot be extended to a d_2-geodesic $(-\varepsilon, 1] \ni t \mapsto \phi_t \in \mathcal{E}^2_\omega$ for any $\varepsilon > 0$.*

For finite-dimensional manifolds, topological and geodesical completeness are equivalent due to the classical Hopf–Rinow theorem. According to the above result, this is not the case for the completion $(\mathcal{E}^2_\omega, d_2)$, despite the fact that this space is non-positively curved [7, 11].

When restricting to toric metrics on a toric Kähler manifold, by means of the Legendre transform it is not hard to construct toric geodesic segments that do not extend to longer toric geodesic segments at one end. However it is not clear if one can extend these segments to longer non-toric geodesics.

It will be interesting to see if a similar non-extension property holds for the $C^{1,1}$-geodesics of Chen [8] and Chu–Tosatti–Weinkove [9], joining the potentials of $\mathcal{H}_\omega$.

Relation to the L^p geometry of $\mathcal{H}_\omega$. In [10] the author introduced a family of L^p Finsler metrics on $\mathcal{H}_\omega$ for any $p \geq 1$, generalizing (1):

$$\|\xi\|_{p,v} = \left(\frac{1}{V}\int_X |\xi|^p \omega_v^n\right)^{1/p}, \qquad v \in \mathcal{H}_\omega,\ \xi \in T_v\mathcal{H}_\omega.$$

These induce path length metric spaces $(\mathcal{H}_\omega, d_p)$, and in [10] the author computed the corresponding metric completions, which later found applications to existence of canonical metrics (for a survey see [13]). Though this more general context lacks the symmetric space interpretation, all of our above results can be considered in the L^p setting as well.

As the reader will be able to deduce from our arguments below, the L^p version of Theorem 1.4 holds for any $p > 1$. Our proof does not work when $p = 1$, since the class of finite energy geodesics may not be stable under isometries in this case (see [14, Theorem 1.2]). On the other hand, the L^p version of Theorem 1.5 does hold for all $p \geq 1$. Lastly, our argument for Theorem 1.1 would most likely go through in the L^p context in case one could obtain the analog of Theorem 2.1 for differentiable L^p isometries.

2. Preliminaries

For simplicity we assume throughout the paper that the Kähler metric ω satisfies the following volume normalization:

$$V = \int_X \omega^n = 1.$$

Using a dilation of ω this can always be achieved and represents no loss of generality.

2.1. The classification theorem of Lempert

In this subsection we recall the particulars of a result due to Lempert on the classification of local C^1 isometries on $\mathcal{H}_\omega$ [23, Theorem 1.1], tailored to our global setting:

Theorem 2.1. *Suppose that $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is a differentiable L^2 isometry. Then for $u \in \mathcal{H}_\omega$ there exists a unique C^∞ diffeomorphism $G_u : X \to X$ such that $G_u^* \omega_u = \pm \omega_{F(u)}$ and*

$$F_*(u)\xi = a\xi \circ G_u - b \int_X \xi \omega_u^n, \quad \xi \in T_u \mathcal{H}_\omega \simeq C^\infty(X), \tag{4}$$

where $a = 1$ or $a = -1$, and $b = 0$ or $b = 2a$.

In the particular case of the (local) L^2 symmetries constructed in [3], formula (4) is a consequence of [3, Theorem 5.1, Theorem 6.1, Proposition 7.1] with $a = -1$ and $b = 0$.

Remark 2.2. It follows from [23, proof of Theorem 1.1] that the integers a and b in the statement depend continuously on $u \in \mathcal{H}_\omega$, hence in our case they are independent of u, as $\mathcal{H}_\omega$ is connected. This was pointed out to us by L. Lempert [22].

In addition, both the anonymous referee and L. Lempert generously pointed out that one can also deduce this fact directly from the statement of Theorem 2.1. We provide the clever argument of the anonymous referee: if $\xi \in C^\infty(X)$, let $\xi_u \in T_u \mathcal{H}_\omega \approx C^\infty(X)$ be $\xi_u = \xi - \int_X \xi \omega_u^n$. Then ξ_u, ξ_u^2 depend continuously on u. We also have $aF_*(u)\xi_u = \xi_u \circ G_u$, hence

$$F_*(u)\xi_u^2 = a(\xi_u \circ G_u)^2 - b \int_X \xi_u^2 \omega_u^n = a(F_*(u)\xi_u)^2 - b \int_X \xi_u^2 \omega_u^n. \tag{5}$$

Taking differentials we get $d_X F_*(u)\xi_u^2 = a \cdot d_X (F_*(u)\xi_u)^2$. Here all quantities depend continuously on u, except perhaps a. Since F is a differentiable L^2 isometry, $F_*(u) : C^\infty(X) \to C^\infty(X)$ is injective and depends continuously on u. Since the space of constant functions of $C^\infty(X)$ is merely one-dimensional, there are many choices of ξ for which $d_X (F_*(u)\xi_u)^2$ is not identically zero in a neighborhood of u. Therefore a depends continuously on u. By (5), b also depends continuously on u (and by (4), the same holds for G_u, however this will not be needed later).

From the classification theorem we obtain the following simple monotonicity result:

Proposition 2.3. *Suppose that $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is a differentiable L^2 isometry with $b = 0$. Let $c \in \mathbb{R}$ and $u, v \in \mathcal{H}_\omega$ with $u \le v$.*

(i) *If $a = 1$ then $F(u) \le F(v)$ and $F(u + c) = F(u) + c$.*

(ii) *If $a = -1$ then $F(u) \ge F(v)$ and $F(u + c) = F(u) - c$.*

Proof. We only address (ii), as the proof of (i) is analogous. Let $[0, 1] \ni t \mapsto \gamma_t := v + t(u - v) \in \mathcal{H}_\omega$. Then $t \mapsto F(\gamma_t)$ is a C^1 curve connecting $F(v)$ and $F(u)$. Moreover, Theorem 2.1 implies that

$$F(u) - F(v) = \int_0^1 \frac{d}{dt} F(\gamma_t)\,dt = \int_0^1 -(u - v) \circ G_{\gamma_t}\,dt \ge 0.$$

The fact that $F(u + c) = F(u) - c$ follows after another application of Theorem 2.1 to the curve $[0, 1] \ni t \mapsto \eta_t := u + tc \in \mathcal{H}_\omega$. ∎

Corollary 2.4. *Suppose that $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is a differentiable L^2 isometry with $b = 0$. Then, in the language of Theorem 2.1 applied to F, we obtain $G_{u+c} = G_u$ for all $u \in \mathcal{H}_\omega$ and $c \in \mathbb{R}$.*

Proof. We only address the case $a = 1$, as the argument for $a = -1$ is identical. Let $\xi \in C^\infty(X)$. By Proposition 2.3(i) and Theorem 2.1 we have

$$\xi \circ G_{u+c} = F_*(u+c)\xi = \frac{d}{dt}\Big|_{t=0} F(u+t\xi+c) = \frac{d}{dt}\Big|_{t=0} F(u+t\xi) = F_*(u)\xi = \xi \circ G_u.$$

Since $\xi \in C^\infty(X)$ is arbitrary, we obtain $G_{u+c} = G_u$. ∎

2.2. *The complete metric space $(\mathcal{E}^2_\omega, d_2)$*

In this subsection we recall some aspects of the author's work related to the metric completion of $(\mathcal{H}_\omega, d_2)$. For details we refer to the survey [13].

As conjectured by V. Guedj [19], $\overline{(\mathcal{H}_\omega, d_2)}$ can be identified with $(\mathcal{E}^2_\omega, d_2)$, where $\mathcal{E}^2_\omega \subset \mathrm{PSH}(X, \omega)$ is an appropriate subset of ω-plurisubharmonic potentials [11, Theorem 1]. Moreover, $(\mathcal{E}^2_\omega, d_2)$ is a non-positively curved complete metric space, whose points can be joined by unique d_2-geodesics.

Given $u_0, u_1 \in \mathcal{E}^2_\omega$, the unique d_2-geodesic $[0, 1] \ni t \mapsto u_t \in \mathcal{E}^2_\omega$ connecting these points has special properties. To start, we recall that this curve arises as the following envelope:

$$u_t := \sup\{v_t \mid t \mapsto v_t \text{ is a subgeodesic}\}, \qquad t \in (0, 1). \tag{6}$$

Here a subgeodesic $(0, 1) \ni t \mapsto v_t \in \mathrm{PSH}(X, \omega)$ is a curve satisfying $\limsup_{t\to 0,1} v_t \le u_{0,1}$ and $u(s, x) := u_{\mathrm{Re}\, s}(x) \in \mathrm{PSH}(S \times X, \omega)$, where $S = \{0 < \mathrm{Re}\, s < 1\} \subset \mathbb{C}$.

It follows from (6) that $t \mapsto u_t(x)$, $t \in (0, 1)$, is convex for all $x \in X$ away from a set of measure zero. On the complement, $u_t(x) = -\infty$ for $t \in (0, 1)$. Moreover, due to [11, Corollary 7], we also have

$$\lim_{t\to 0} u_t(x) = u_0(x) \quad \text{and} \quad \lim_{t\to 1} u_t(x) = u_1(x) \tag{7}$$

for all $x \in X$ away from a set of measure zero. In the particular case when $u_0, u_1 \in \mathcal{H}_\omega$, the curve $t \mapsto u_t$ is $C^{1,1}$ on $[0,1] \times X$ [4, 8, 9].

We denote by $\mathcal{C}_\omega$ the set of continuous potentials in $\mathrm{PSH}(X,\omega)$. As pointed out previously, a differentiable L^2 isometry $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ induces a unique d_2-isometry $F : \mathcal{E}^2_\omega \to \mathcal{E}^2_\omega$, extending the original map (by density). Going forward, we do not distinguish F from its unique extension. Moreover, if F is an isometry with $b = 0$ (see Theorem 2.1), we point out that $\mathcal{C}_\omega$ is stable under extension:

Proposition 2.5. *Suppose $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is a differentiable L^2 isometry with $b = 0$. Then $F(\mathcal{C}_\omega) \subset \mathcal{C}_\omega$. More importantly,* $\sup_X \|u_j - u\| \to 0$ *implies* $\sup_X \|F(u_j) - F(u)\| \to 0$ *for any $u_j, u \in \mathcal{C}_\omega$.*

Proof. We only handle the case when $a = 1$, as in case $a = -1$ the proof is analogous. Since d_2-convergence implies pointwise a.e. convergence (see [10, Theorem 5]), Proposition 2.3(i) holds for the extension $F : \mathcal{E}^2_\omega \to \mathcal{E}^2_\omega$ and $u, v \in \mathcal{E}^2_\omega$ satisfying $u \le v$.

Let $u \in \mathcal{C}_\omega$. Then [5] implies existence of $u_k \in \mathcal{H}_\omega$ such that $u_k \searrow u$. In fact, according to Dini's lemma, the convergence is uniform. From Proposition 2.3 it follows that $\{F(u_k)\}_k \subset \mathcal{H}_\omega$ is decreasing. Due to uniform convergence, for any $\varepsilon > 0$ there exists k_0 such that $u \le u_k \le u + \varepsilon$ for $k \ge k_0$. Then Proposition 2.3 implies that $F(u) \le F(u_k) \le F(u) + \varepsilon$ for $k \ge k_0$. This shows that $F(u_k)$ converges to $F(u)$ uniformly, in particular $F(u) \in \mathcal{C}_\omega$.

Lastly, we can essentially repeat the above argument for continuous potentials u_j converging uniformly to u, deducing the last statement of the proposition. ∎

2.3. *Examples of differentiable L^2 isometries on $\mathcal{H}_\omega$*

In this subsection we describe three examples of global differentiable L^2 isometries on $\mathcal{H}_\omega$. Later we will argue that in fact all global isometries arise as compositions of these examples.

• First we take a closer look at the Monge–Ampère flip $\mathcal{I} : \mathcal{H}_\omega \to \mathcal{H}_\omega$, defined in Section 1, perhaps first introduced in [23]. Let $[0,1] \ni t \mapsto \gamma_t \in \mathcal{H}_\omega$ be a smooth curve. Since $\frac{d}{dt} I(\gamma_t) = \int_X \dot\gamma_t \omega^n_{\gamma_t}$, we obtain

$$\int_X \left(\frac{d}{dt}\mathcal{I}(\gamma_t)\right)^2 \omega^n_{\gamma_t} = \int_X \left(\dot\gamma_t - 2\int_X \dot\gamma_t \omega^n_{\gamma_t}\right)^2 \omega^n_{\gamma_t} = \int_X \dot\gamma_t^2 \omega^n_{\gamma_t},$$

hence $\mathcal{I}$ is indeed an involutive L^2 isometry, with $a = 1$ and $b = 2$ (see Theorem 2.1). This simple map has the following intriguing property, which will help us adjust the b parameter of arbitrary isometries without changing the a parameter:

Lemma 2.6. *Suppose that $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is a differentiable L^2 isometry. The a parameter of F and $F \circ \mathcal{I}$ is always the same. Regarding the b parameter the following hold:*

(i) *If $b = 0$ for F, then $b = 2a$ for $F \circ \mathcal{I}$.*

(ii) *If $b = 2a$ for F, then $b = 0$ for $F \circ \mathcal{I}$.*

Proof. Let $[0,1] \ni t \mapsto \gamma_t \in \mathcal{H}_\omega$ be a smooth curve. Then

$$\frac{d}{dt}F(\mathcal{J}(\gamma_t)) = F_*(\mathcal{J}_*\dot{\gamma}_t) = F_*\bigg(\dot{\gamma}_t - 2\int_X \dot{\gamma}_t \omega^n_{\gamma_t}\bigg).$$

If $a = 1$ and $b = 0$ for F, then $\frac{d}{dt}F(\mathcal{J}(\gamma_t)) = \dot{\gamma}_t \circ G_u - 2\int_X \dot{\gamma}_t\omega^n_{\gamma_t}$. If $a = -1$ and $b = 0$ for F, then $\frac{d}{dt}F(\mathcal{J}(\gamma_t)) = -\dot{\gamma}_t \circ G_u + 2\int_X \dot{\gamma}_t\omega^n_{\gamma_t}$, addressing (i).

In case $a = 1$ and $b = 2a$ for F, then $\frac{d}{dt}F(\mathcal{J}(\gamma_t)) = \dot{\gamma}_t \circ G_u$. Similarly, if $a = -1$ and $b = 2a$ for F, then $\frac{d}{dt}F(\mathcal{J}(\gamma_t)) = -\dot{\gamma}_t \circ G_u$, addressing (ii). ∎

• Now let $f : X \to X$ be a biholomorphism preserving the Kähler class $[\omega]$. Then f induces a map $L_f : \mathcal{H} \to \mathcal{H}$ via pullbacks: $\omega_{L_f(u)} := f^*\omega_u$, where we make the natural identification $\mathcal{H} \simeq I^{-1}(0)$. Using this identification it is possible to describe the action of F at the level of potentials in the following manner [15, Lemma 5.8]:

$$L_f(u) = L_f(0) + u \circ F, \quad u \in I^{-1}(0), \tag{8}$$

where $0 \in I^{-1}(0)$ is simply the zero Kähler potential. More importantly, L_f further extends to a map $L_f : \mathcal{H}_\omega \to \mathcal{H}_\omega$ in the following manner:

$$L_f(v) = L_f(v - I(v)) + I(v), \quad v \in \mathcal{H}_\omega.$$

We point out that L_f thus described gives a differentiable L^2 isometry of $\mathcal{H}_\omega$ with $a = 1$ and $b = 0$. To see this, let $[0,1] \ni t \mapsto \gamma_t \in \mathcal{H}_\omega$ be a smooth curve. Using (8) we can write

$$\frac{d}{dt}L_f(\gamma_t) = \frac{d}{dt}(\gamma_t \circ f - I(\gamma_t)) + \frac{d}{dt}I(\gamma_t) = \dot{\gamma}_t \circ f.$$

In the language of Theorem 2.1 applied to L_f, we have obtained $G_u = g$ for all $u \in \mathcal{H}_\omega$.

• Now let $g : X \to X$ be an anti-biholomorphism that flips the Kähler class $[\omega]$. By definition, such a map is a diffeomorphism satisfying $\frac{\partial g_j}{\partial z_k} = 0$ for all $j, k \in \{1, \dots, n\}$ in any choice of local coordinates. For example, the map $g(z) = \bar{z}$ is an anti-biholomorphism of the unit torus $\mathbb{C}/\mathbb{Z}[i]$ that flips that class of the flat Kähler metric.

Such a map g induces another map $N_g : \mathcal{H} \to \mathcal{H}$ via pullbacks: $\omega_{N_g(u)} := -g^*\omega_u$. Here we again use the identification $\mathcal{H} \simeq I^{-1}(0)$. Similar to (8), it is possible to describe the action of N_g at the level of potentials in the following manner:

$$N_g(u) = N_g(0) + u \circ g, \quad u \in I^{-1}(0). \tag{9}$$

To show this, we have to go through the proof of [15, Lemma 5.8] in the anti-holomorphic context. As a beginning remark, we notice that $g^*\partial\bar{\partial}v = -\partial\bar{\partial}v \circ g$ for all smooth functions v. With this in mind, we find that

$$\omega + i\partial\bar{\partial}(N_g(0) + u \circ g) = -g^*\omega - g^*i\partial\bar{\partial}u = -g^*\omega_u = \omega_{N_g(u)} = \omega + i\partial\bar{\partial}N_g(u).$$

In particular, $N_g(0) + u \circ g - N_g(u)$ is a constant. To show that this constant is zero, we only need to argue that $I(N_g(0) + u \circ g) = 0 = I(N_g(u))$. But this holds because of the following computation:

$$\begin{aligned} I(N_g(0) + u \circ g) &= I(N_g(0) + u \circ g) - I(N_g(0)) \\ &= \frac{1}{n+1} \sum_{j=0}^{n} \int_X (u \circ g) \omega^j_{N_g(0)+u\circ g} \wedge \omega^{n-j}_{N_g(0)} \\ &= \frac{\pm 1}{n+1} \sum_{j=0}^{n} \int_X (u \circ g) g^*(\omega^j_u \wedge \omega^{n-j}) \\ &= \frac{\pm 1}{n+1} \sum_{j=0}^{n} \int_X u \omega^j_u \wedge \omega^{n-j} = \pm I(u) = 0. \end{aligned}$$

As above, N_g extends to a map $N_g : \mathcal{H}_\omega \to \mathcal{H}_\omega$ in the following manner:

$$N_g(v) = N_g(v - I(v)) + I(v), \qquad v \in \mathcal{H}_\omega.$$

We point out that N_g thus described gives a differentiable L^2 isometry of $\mathcal{H}_\omega$ with $a = 1$ and $b = 0$. To see this, let $[0, 1] \ni t \mapsto \gamma_t \in \mathcal{H}_\omega$ be a smooth curve. Using (9) we can write

$$\frac{d}{dt} N_g(\gamma_t) = \frac{d}{dt}(\gamma_t \circ g - I(\gamma_t)) + \frac{d}{dt} I(\gamma_t) = \dot{\gamma}_t \circ g.$$

In the language of Theorem 2.1 applied to N_g, we have actually shown that $G_u = g$ for all $u \in \mathcal{H}_\omega$.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is split into two parts. First we show that there exist no global differentiable isometries with $a = -1$. Later we classify all global differentiable isometries with $a = 1$.

Before going into details, we recall the following simple lemma that will be used numerous times in our arguments:

Lemma 3.1 ([12, Lemma 3.1]). *Suppose that $u_0, u_1 \in \mathcal{C}_\omega$ and $[0, 1] \ni t \mapsto u_t \in \mathcal{E}^2_\omega$ is the d_2-geodesic connecting these potentials. Then*

$$\inf_X \dot{u}_0 = \inf_X (u_1 - u_0), \qquad \sup_X \dot{u}_0 = \sup_X (u_1 - u_0).$$

Proof. First we argue that $\inf_X \dot{u}_0 = \inf_X (u_1 - u_0)$. From (6) we obtain the estimate $u_t \geq u_0 + t \inf_X (u_1 - u_0)$ for $t \in [0, 1]$. In particular, $\dot{u}_0 \geq \inf_X (u_1 - u_0)$. Using t-convexity it follows that $u_t(y) = u_0(y) + t \inf_X (u_1 - u_0)$ for $y \in X$ such that $u_1(y) - u_0(y) = \inf_X (u_1 - u_0)$. This implies that $t \mapsto u_t(y)$ is linear, implying that $\inf_X \dot{u}_0 = \inf_X (u_1 - u_0)$.

For the second identity, we notice that t-convexity implies $\sup_X \dot{u}_0 \leq \sup_X (u_1 - u_0)$. In addition, (6) implies that $u_1 - (1-t)\sup_X (u_1 - u_0) \leq u_t$ for $t \in [0,1]$. Relying on t-convexity again, we obtain $\dot{u}_0(z) = u_1(z) - u_0(z) = \sup_X (u_1 - u_0)$ for $z \in X$ with $u_1(z) - u_0(z) = \sup_X (u_1 - u_0)$. Summarizing, we conclude that $\sup_X \dot{u}_0 = \sup_X (u_1 - u_0)$, as desired. ∎

3.1. Isometries with $a = -1$

We start with a lemma:

Lemma 3.2. *Suppose that $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is a differentiable L^2 isometry with $a = -1$ and $b = 0$. Let $\phi \in \mathcal{H}_\omega$ and $u \in \mathcal{H}_\omega$ with $u \leq \phi$. Then $F(u) \geq F(\phi)$ and*

$$\sup_X (F(u) - F(\phi)) = -\inf_X (u - \phi). \tag{10}$$

Proof. That $F(u) \geq F(\phi)$ follows from Proposition 2.3(ii). As pointed out in [23, p. 2], Theorem 2.1 implies that F is a d_p-isometry for any $p \geq 1$. This implies that $d_p(\phi, u) = d_p(F(\phi), F(u))$ for any $p \geq 1$.

Let $[0,1] \ni t \mapsto u_t, v_t \in \mathcal{H}_\omega^{1,1}$ be the $C^{1,1}$ geodesic connecting $u_0 := \phi$ to $u_1 := u$, respectively $v_0 := F(\phi)$ to $v_1 := F(u)$. By the comparison principle for weak geodesics (see for example [4, Proposition 2.2]) it follows that $v_t \geq F(\phi)$ and $u_t \leq \phi$ for any $t \in [0,1]$. In particular, $\dot{v}_0 \geq 0$ and $\dot{u}_0 \leq 0$.

Using [10, Theorem 1] we arrive at

$$\int_X |\dot{u}_0|^p \omega_\phi^n = d_p(\phi, u)^p = d_p(F(\phi), F(u))^p = \int_X |\dot{v}_0|^p \omega_{F(\phi)}^n, \qquad p \geq 1.$$

Raising to the $\frac{1}{p}$-th power and letting $p \to \infty$ gives

$$\sup_X \dot{v}_0 = -\inf_X \dot{u}_0. \tag{11}$$

From Lemma 3.1 we obtain $\inf_X \dot{u}_0 = \inf_X (u - \phi)$ and $\sup_X \dot{v}_0 = \sup_X (F(u) - F(\phi))$. Putting this together with (11), we obtain (10), as desired. ∎

Theorem 3.3. *There exists no differentiable L^2 isometry $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ with $a = -1$.*

We note that this result already implies Corollary 1.3.

Proof of Theorem 3.3. Due to Lemma 2.6, after possibly composing F with $\mathcal{I}$, we only need to worry about the case $a = -1$ and $b = 0$.

If $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is a differentiable L^2 isometry, it is also a d_2-isometry, hence it extends to a unique d_2-isometry $F : \mathcal{E}_\omega^2 \to \mathcal{E}_\omega^2$.

Let $\phi \in \mathcal{H}_\omega$. Let $u \in \mathcal{E}_\omega^2 \setminus L^\infty$ with $u \leq \phi - 1$, and choose $u_k \in \mathcal{H}_\omega$ such that $u_k \searrow u$ and $u_k \leq \phi$. Such a sequence can always be found [5].

Due to our choice of u we have $\inf_X (u_k - \phi) \searrow -\infty$. From Lemma 3.2 it follows that $\sup_X F(u_k) = \sup_X (F(u_k) - F(\phi)) \nearrow +\infty$. Since F is a d_2-isometry, $d_2(F(u_k), F(u)) = d_2(u, u_k) \to 0$. However, [10, Theorem 5(i)] implies $\sup_X F(u_k) \to \sup_X F(u) < +\infty$, a contradiction. ∎

3.2. Isometries with $a = 1$

To start, we point out an important relationship between d_2-geodesics and differentiable L^2 isometries with $a = 1$ and $b = 0$:

Proposition 3.4. *Suppose that $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is a differentiable L^2 isometry with $a = 1$ and $b = 0$. Let $[0,1] \ni t \mapsto u_t \in \mathcal{E}^2_\omega$ be the d_2-geodesic connecting $u_0 \in \mathcal{H}_\omega$ and $u_1 \in \mathcal{C}_\omega$. Then*

$$\dot{u}_0 \circ G_{u_0} = \dot{F(u_0)}. \tag{12}$$

Here and below $\dot{u}_0 := \frac{d}{dt}\big|_{t=0} F(u_t)$ and $\dot{F(u_0)} := \frac{d}{dt}\big|_{t=0} F(u_t)$ are the initial tangent vectors of the d_2-geodesics $t \mapsto u_t$ and $t \mapsto F(u_t)$, interpreted according to the discussion preceding Theorem 1.4.

Proof. There is a constant $c \in \mathbb{R}$ such that $u_0 > u_1 + c$. Since $F(u_t + tc) = F(u_t) + tc$ (Proposition 2.3(i)), we can assume without loss of generality that $u_0 > u_1$.

First, we show (12) in case $u_1 \in \mathcal{H}_\omega$. Let $[0,1] \ni t \mapsto u^\varepsilon_t \in \mathcal{H}_\omega$ be the smooth ε-geodesics of X. X. Chen, connecting u_0 and u_1 [8]. It is well known that $u^\varepsilon_t \nearrow u_t$ as $\varepsilon \to 0$, where $t \mapsto u_t$ is the $C^{1,1}$-geodesic joining u_0 and u_1. Due to Propositions 2.3 and 2.5, for the curves $t \mapsto F(u^\varepsilon_t), F(u_t)$ we find that $F(u^\varepsilon_t) \nearrow F(u_t)$. Since $t \mapsto F(u^\varepsilon_t)$ is a C^1 curve, via Theorem 2.1 we obtain

$$\dot{u}^\varepsilon_0 \circ G_{u_0} = \dot{F(u^\varepsilon_0)} \le \dot{F(u_0)} \le 0, \qquad \varepsilon > 0.$$

Taking the limit $\varepsilon \to 0$, since $u^\varepsilon \to_{C^{1,\alpha}} u$, we arrive at $\dot{u}_0 \circ G_{u_0} \le \dot{F(u_0)} \le 0$. By Theorem 2.1 we have $G^*_{u_0}\omega^n_{u_0} = \pm\omega^n_{F(u_0)}$. Using this and [8] (see also [10, Theorem 1]) we obtain

$$\begin{aligned}\int_X (\dot{u}_0 \circ G_{u_0})^2 \omega^n_{F(u_0)} &= \int_X \dot{u}_0^2 \omega^n_{u_0} = d_2(u_0,u_1)^2 = d_2(F(u_0),F(u_1))^2 \\ &= \int_X \dot{F(u_0)}^2 \omega^n_{F(u_0)}.\end{aligned}$$

Due to continuity we conclude that $\dot{u}_0 \circ G_{u_0} = \dot{F(u_0)}$, as desired.

Now we treat the general case. Let $u^k_1 \in \mathcal{H}_\omega$ for $k \in \mathbb{N}$ be such that $u_0 > u^k_1$ and $u^k_1 \searrow u_1 \in \mathcal{C}_\omega$. Also, by $[0,1] \ni t \mapsto u_t, u^k_t \in \mathcal{E}^2_\omega$ we denote the d_2-geodesics connecting u_0 to u_1, respectively u_0 to u^k_1. Since F is a d_2-isometry, we deduce that $[0,1] \ni t \mapsto F(u_t), F(u^k_t) \in \mathcal{E}^2_\omega$ are the d_2-geodesics connecting $F(u_0)$ to $F(u_1)$, respectively $F(u_0)$ and $F(u^k_1)$. Due to t-convexity, k-monotonicity and Proposition 2.3, we find that $\dot{u}^k_0 \searrow \dot{u}_0$ and $F(\dot{u}^k_0) \searrow F(\dot{u}_0)$. Letting $k \to \infty$ we arrive at the desired conclusion: $\dot{u}_0 \circ G_{u_0} = \lim_k(\dot{u}^k_0 \circ G_{u_0}) = \lim_k F(\dot{u}^k_0) = F(\dot{u}_0)$. ∎

This result together with Lemma 3.1 gives the following corollary, paralleling Lemma 3.2:

Corollary 3.5. *Suppose that $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is a differentiable L^2 isometry with $a = 1$ and $b = 0$. Suppose that $u, v \in \mathcal{C}_\omega$. Then $F(u), F(v) \in \mathcal{C}_\omega$ and*

$$\inf_X (F(u) - F(v)) = \inf_X (u - v). \tag{13}$$

By switching the roles of u and v, we obtain the above identity for the suprema as well.

Proof of Corollary 3.5. That $F(u), F(v) \in \mathcal{E}_\omega$ follows from Proposition 2.5. First we deal with the case when $u, v \in \mathcal{H}_\omega$. If $[0,1] \ni t \mapsto h_t \in \mathcal{H}_\omega$ is the $C^{1,1}$-geodesic connecting $h_0 := u$ and $h_1 := v$, then Lemma 3.1 gives

$$\inf_X (v-u) = \inf_X \dot{h}_0 \quad \text{and} \quad \inf_X (F(v) - F(u)) = \inf_X F(\dot{h}_0).$$

Putting this together with (12), we obtain $\inf_X (v-u) = \inf_X (F(v) - F(u))$, as desired.

When $u, v \in \mathcal{E}_\omega$, by [5] one can find $u^k, v^k \in \mathcal{H}_\omega$ such that $\sup_X |u^k - u| \to 0$ and $\sup_X |v^k - v| \to 0$. Then Proposition 2.5 implies that $\sup_X |F(u^k) - F(u)| \to 0$ and $\sup_X |F(v^k) - F(v)| \to 0$.

By uniform convergence we have

$$\inf_X (u^k - v^k) \to \inf_X (u-v) \quad \text{and} \quad \inf_X (F(u^k) - F(v^k)) \to \inf_X (F(u) - F(v)).$$

The conclusion follows after taking the k-limit of $\inf_X (u^k - v^k) = \inf_X (F(u^k) - F(v^k))$. ∎

To continue, we need an auxiliary construction. Fixing $x \in X$ and a small enough coordinate neighborhood $O_x \subset X$, we can find a function $\rho_x \in C^\infty(X)$ such that $\rho_x(y) = e^{-1/\|y-x\|^2}$ for all $y \in O_x$, and there exists $\beta > 0$ with $\beta \le \rho_x(y) \le 1$ for all $y \in X \setminus O_x$.

Proposition 3.6. *For $u \in \mathcal{H}_\omega$ and $x \in X$ there exists $\delta > 0$ such that $[0,1] \ni t \mapsto u_t := u + \delta(t + t^2/2)\rho_x \in \mathcal{H}_\omega$ is a subgeodesic.*

Proof. Let $U(s,y) = u_{\operatorname{Re} s}(y) \in C^\infty(S \times X)$, where $S = \{0 \le \operatorname{Re} z \le 1\} \subset \mathbb{C}$. It is clear that for small enough $\delta > 0$ we have $u_t \in \mathcal{H}_\omega$ for all $t \in [0,1]$. More precisely, there exists $\alpha > 0$ such that $\omega_{u_t} \ge \alpha\omega$ for all $t \in [0,1]$.

This implies that $\omega + i\partial_{S\times X}\bar{\partial}_{S\times X} U$ has at least n non-negative eigenvalues for all $(s,y) \in S \times X$ (relative to $\omega + i\partial\bar{\partial}|s|^2$). To conclude that $\omega + i\partial_{S\times X}\bar{\partial}_{S\times X} U \ge 0$ it is enough to show that the determinant of this Hermitian form is non-negative. This is equivalent to $\ddot{u}_t - \langle \partial \dot{u}_t, \bar{\partial}\dot{u}_t\rangle_{\omega_{u_t}} \ge 0$ on $[0,1] \times X$. To show this, we start with the following estimates:

$$\begin{aligned} \ddot{u}_t - \langle \partial \dot{u}_t, \bar{\partial}\dot{u}_t\rangle_{\omega_{u_t}} &= \delta\rho_x - \delta^2(1+t)^2 \langle \partial\rho_x, \bar{\partial}\rho_x\rangle_{\omega_{u_t}} \\ &\ge \delta\rho_x - \frac{\delta^2(1+t)^2}{\alpha}\langle \partial\rho_x, \bar{\partial}\rho_x\rangle_\omega. \end{aligned}$$

After possibly shrinking $\delta \in (0,1)$, we find that it is enough to conclude that the last expression is non-negative on the neighborhood O_x, where we know that $\rho_x(y) = e^{-1/\|y-x\|^2}$ for $y \in O_x$. In particular, on $O_x \setminus \{x\}$ we have

$$\langle \partial\rho_x, \bar{\partial}\rho_x\rangle_\omega / \rho_x \simeq e^{-1/\|y-x\|^2} \frac{1}{\|y-x\|^6},$$

which is uniformly bounded. In particular, after possibly further shrinking $\delta \in (0,1)$ we obtain

$$\ddot{u}_t - \langle \partial \dot{u}_t, \bar{\partial} \dot{u}_t \rangle_{\omega_{u_t}} \geq \delta \rho_x - \frac{\delta^2 (1+t)^2}{\alpha} \langle \partial \rho_x, \bar{\partial} \rho_x \rangle_\omega \geq 0,$$

as desired. ∎

Theorem 3.7. *Suppose that* $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ *is a differentiable* L^2 *isometry with* $a = 1$. *Then exactly one of the following holds:*

(i) F *is induced by a biholomorphism or an anti-biholomorphism* $f : X \to X$ *that preserves or flips the Kähler class* $[\omega]$, *respectively.*

(ii) $F \circ \mathcal{I}$ *is induced by a biholomorphism or an anti-biholomorphism* $f : X \to X$ *that preserves or flips the Kähler class* $[\omega]$, *respectively.*

Proof. Due to Lemma 2.6, after possibly composing F with $\mathcal{I}$, we only need to worry about the case $a = 1$ and $b = 0$. In this case we will show that F is induced by a biholomorphism or an anti-biholomorphism $f : X \to X$ that preserves or flips the Kähler class $[\omega]$.

In the language of Theorem 2.1 applied to F, the first step is to show that $G_u = G_v$ for all $u, v \in \mathcal{H}_\omega$.

We fix $x \in X$ and $u, v \in \mathcal{H}_\omega$. We will show that $G_u^{-1}(x) = G_v^{-1}(x)$. Since $G_{u+c} = G_u$ for any $c \in \mathbb{R}$ (Corollary 2.4), we can assume that $u(x) = v(x)$. First we prove that $G_u^{-1}(x) = G_v^{-1}(x)$ under the extra non-degeneracy condition $\nabla u(x) \neq \nabla v(x)$.

Let $\eta > 0$ be such that $w := \max(u, v) + \eta \rho_x \in \mathcal{E}_\omega$. From our setup it is clear that $w \geq \max(u, v)$, and the graphs of w, u and v only meet at x. Extending the isometry F to the metric completion, we see that Propositions 2.3 and 2.5 imply that $F(w) \geq \max(F(u), F(v))$, $F(w) \in \mathcal{E}_\omega$ and $F(u), F(v) \in \mathcal{H}_\omega$. Below we will show that $F(w)$ and $F(u)$ only meet at $G_u^{-1}(x)$, and moreover $F(w)$ and $F(v)$ only meet at $G_v^{-1}(x)$. Finally, we will show that the graphs of $F(w)$, $F(u)$ and $F(v)$ have to meet at some point of X, implying that $G_u^{-1}(x) = G_v^{-1}(x)$, as desired.

Let us denote by $[0,1] \ni t \mapsto u_t, v_t \in \mathcal{E}^2_\omega$ the d_2-geodesics joining $u_0 := u$ to $u_1 := w$, respectively $v_0 := v$ to $v_1 := w$. From Proposition 3.4 it follows that

$$\dot{F(u_0)} = \dot{u}_0 \circ G_u, \qquad \dot{F(v_0)} = \dot{v}_0 \circ G_v. \tag{14}$$

On account of (6) there exists a small enough $\delta > 0$ in the statement of Proposition 3.6 such that $u + \delta(t + t^2/2)\rho_x \leq u_t$ and $v + \delta(t + t^2/2)\rho_x \leq v_t$ for $t \in [0,1]$. Using this, t-convexity and (14), we obtain

$$F(w) - F(u) \geq \dot{F(u_0)} = \dot{u}_0 \circ G_u \geq \delta \rho_x \circ G_u,$$
$$F(w) - F(v) \geq \dot{F(v_0)} = \dot{v}_0 \circ G_v \geq \delta \rho_x \circ G_v.$$

Due to (13) these two estimates imply the existence of a unique $y \in X$ and a unique $z \in X$ such that

$$F(w)(y) - F(u)(y) = 0 \quad \text{and} \quad F(w)(z) - F(v)(z) = 0. \tag{15}$$

In fact, we need to have $y = G_u^{-1}(x)$ and $z = G_v^{-1}(x)$. In particular, the graphs of $F(w)$ and $F(u)$ only meet at y, and the graphs of $F(w)$ and $F(u)$ only meet at z.

In case $y \neq z$, uniqueness of y and z implies that $y \in \{F(u) > F(v)\}$ and $z \in \{F(v) > F(u)\}$ (recall that $F(w) \geq \max(F(u), F(v))$. This implies that the graphs of $F(w)$ and $\max(F(u), F(v))$ meet at only two points (y and z), away from the compact set $\{F(u) = F(v)\}$. Consequently, using classical Richberg approximation [16, Chapter I, Lemma 5.18], one can take a "regularized maximum" of $F(u)$ and $F(v)$ to obtain $\beta \in \mathcal{H}_\omega$ satisfying

$$F(w) \geq \beta \geq \max(F(u), F(v)).$$

Since $F : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is surjective, there exists a unique $\alpha \in \mathcal{H}_\omega$ such that $F(\alpha) = \beta$. Using (13) again, we obtain

$$\max(u, v) + \delta\rho_x = w \geq \alpha \geq \max(u, v).$$

Since $\nabla u(x) \neq \nabla v(x)$ and $w(x) = \alpha(x) = \max(u, v)(x)$, this contradicts the smoothness of α at x. Consequently, $G_u^{-1}(x) = y = z = G_v^{-1}(x)$, as desired.

In case $\nabla u(x) = \nabla v(x)$, one finds $q \in \mathcal{H}_\omega$ (via a small perturbation) such that $u(x) = v(x) = q(x)$ and $\nabla u(x) \neq \nabla q(x)$ along with $\nabla v(x) \neq \nabla q(x)$. By the above, $G_u^{-1}(x) = G_q^{-1}(x)$ and $G_v^{-1}(x) = G_q^{-1}(x)$, ultimately giving $G_u^{-1}(x) = G_v^{-1}(x)$ for any $u, v \in \mathcal{H}_\omega$.

Let $f := G_w$ for (any) $w \in \mathcal{H}_\omega$. Using Theorem 2.1, integration along the curve $t \mapsto tu$ gives

$$F(u) - F(0) = \int_0^1 (u \circ f)\, dt = u \circ f, \qquad u \in \mathcal{H}_\omega. \tag{16}$$

Returning to the statement of Theorem 2.1, we either have $f^*\omega_u = \omega_{F(u)}$ for all $u \in \mathcal{H}_\omega$, or $f^*\omega_u = -\omega_{F(u)}$ for all $u \in \mathcal{H}_\omega$.

Assuming that $f^*\omega_u = \omega_{F(u)}$, using (16) we arrive at the identity

$$f^*(i\partial\bar\partial u) = i\partial\bar\partial(u \circ f).$$

Since after a dilation all elements of $C^\infty(X)$ land in $\mathcal{H}_\omega$, we find that actually

$$f^*(i\partial\bar\partial v) = i\partial\bar\partial(v \circ f) \quad \text{ for all } f \in C^\infty(X).$$

According to the next lemma f has to be holomorphic, implying that $F = L_f$ (see Section 2.3).

In case $f^*\omega_u = -\omega_{F(u)}$, by a similar calculation we arrive at $f^*(i\partial\bar\partial v) = -i\partial\bar\partial(v \circ f)$ for all $v \in C^\infty(X)$. According to the next lemma f has to be anti-holomorphic, showing that $F = N_f$ (see Section 2.3), which finishes the proof. ∎

Lemma 3.8. *Suppose that $g : X \to X$ is a smooth map.*

(i) *If $i\partial\bar\partial(u \circ g) = g^*(i\partial\bar\partial u)$ for all $u \in C^\infty(X)$ then g is holomorphic.*

(ii) *If $i\partial\bar\partial(u \circ g) = -g^*(i\partial\bar\partial u)$ for all $u \in C^\infty(X)$ then g is anti-holomorphic.*

Proof. We only show (i) as the proof of (ii) is analogous. We start with the following computations expressed in local coordinates:

$$
\begin{aligned}
i\partial\bar\partial(u\circ g) &= i\frac{\partial^2(u\circ g)}{\partial z_j\partial\overline{z_k}}dz_j\wedge d\overline{z_k} = i\frac{\partial^2 u}{\partial z_a\partial\overline{z_b}}\left[\frac{\partial g_a}{\partial z_j}\frac{\partial\overline{g_b}}{\partial\overline{z_k}}+\frac{\partial g_a}{\partial\overline{z_k}}\frac{\partial\overline{g_b}}{\partial z_j}\right]dz_j\wedge d\overline{z_k}\\
&\quad + i\frac{\partial^2 u}{\partial z_a\partial z_b}\frac{\partial g_a}{\partial z_j}\frac{\partial g_b}{\partial\overline{z_k}}dz_j\wedge d\overline{z_k} + i\frac{\partial^2 u}{\partial\overline{z_a}\partial\overline{z_b}}\frac{\partial\overline{g_a}}{\partial z_j}\frac{\partial\overline{g_b}}{\partial\overline{z_k}}dz_j\wedge d\overline{z_k}\\
&\quad + i\frac{\partial u}{\partial z_b}\frac{\partial^2 g_b}{\partial z_j\partial\overline{z_k}}dz_j\wedge d\overline{z_k} + i\frac{\partial u}{\partial\overline{z_b}}\frac{\partial^2\overline{g_b}}{\partial z_j\partial\overline{z_k}}dz_j\wedge d\overline{z_k}.
\end{aligned}\tag{17}
$$

Knowing that $g^*(i\partial\bar\partial u)$ is a $(1,1)$ form we also have

$$
g^*(i\partial\bar\partial u) = i\frac{\partial^2 u}{\partial z_a\partial\overline{z_b}}\left[\frac{\partial g_a}{\partial z_j}\frac{\partial\overline{g_b}}{\partial\overline{z_k}}-\frac{\partial g_a}{\partial\overline{z_k}}\frac{\partial\overline{g_b}}{\partial z_j}\right]dz_j\wedge d\overline{z_k}.\tag{18}
$$

Clearly, it is enough to show that g is holomorphic in local coordinate charts. By linearity, $i\partial\bar\partial(u\circ g) = g^*(i\partial\bar\partial u)$ holds for complex valued smooth functions u.

Let $x\in X$, and pick u such that in a coordinate neighborhood of x we have $u(z)=z_b$ for some $b\in\{1,\dots,n\}$. Then $i\partial\bar\partial(u\circ g) = g^*(i\partial\bar\partial u)$ gives $\partial^2 g_b/\partial z_j\partial\overline{z_k}=0$ for all $j,k\in\{1,\dots,n\}$ at x. Similarly, after choosing $u(z)=\overline{z_b}$, $b\in\{1,\dots,n\}$ in a coordinate neighborhood of x, we obtain $\partial^2\overline{g_b}/\partial z_j\partial\overline{z_k}=0$ for all $j,k\in\{1,\dots,n\}$ at x. Since $x\in X$ was arbitrary, the terms in the last line of (17) vanish for any choice of u.

Repeating this reasoning for $u(z)=z_az_b$ and $u(z)=\bar z_a\bar z_b$, we conclude that the terms in the second line of (17) vanish as well, for any choice of u.

Revisiting the identity $i\partial\bar\partial(u\circ g) = g^*(i\partial\bar\partial u)$ one more time, after picking u such that $i\partial\bar\partial u$ is positive definite in a neighborhood of $x\in X$, we find that $\partial g_a/\partial\overline{z_j}=0$ for any $a,j\in\{1,\dots,n\}$ at x, implying that g is indeed holomorphic. ∎

4. Proofs of Theorems 1.4 and 1.5

We start with a lemma about the concatenation of geodesics in $\mathcal{E}^2_\omega$:

Lemma 4.1. *Suppose that* $[-1,0]\ni t\mapsto v_t\in\mathcal{E}^2_\omega$ *and* $[0,1]\ni t\mapsto u_t\in\mathcal{E}^2_\omega$ *are* d_2-*geodesics such that* $u_0=v_0\in\mathcal{H}_\omega$ *and* $\dot u_0=\dot v_0\in L^2(\omega^n)$. *Then* $[-1,1]\ni t\mapsto w_t\in\mathcal{E}^2_\omega$, *the concatenation of the curves* $t\mapsto u_t$ *and* $t\to v_t$, *is the* d_2-*geodesic joining* $v_{-1},u_1\in\mathcal{E}^2_\omega$.

Proof. By possibly changing the background metric, we can assume that $u_0=v_0=0$. From the L^2 version of [1, Lemma 3.4(ii)] (whose proof is identical to the L^1 version, presented in [1]) we obtain

$$
d_2(v_{-1},0)^2=\int_X|\dot u_0|^2\omega^n=\int_X|\dot v_0|^2\omega^n=d_2(0,u_1)^2.\tag{19}
$$

Next we point out that for any $a \in [-1, 0]$ and $b \in [0, 1]$ we have

$$d_2(v_a, u_b) = d_2(v_a, 0) + d_2(0, u_b). \tag{20}$$

Indeed, from the triangle inequality we see that $d_2(v_a, u_b) \le d_2(v_a, 0) + d_2(0, u_b)$. The reverse inequality follows from (19) and [14, Theorem 3.1]:

$$d_2(v_a, 0) + d_2(0, u_b) = \left(\int_X |(b-a)\dot{u}_0|^2 \omega^n \right)^{1/2} \le d_2(v_a, u_b).$$

To finish the proof, due to uniqueness of d_2-geodesic segments, we only need to show that for any $a, b \in [-1, 1]$ with $a < b$ we have

$$d_2(w_a, w_b) = \frac{b-a}{2} d_2(v_{-1}, u_1) = (b-a) d_2(0, u_1) = (b-a) d_2(v_{-1}, 0). \tag{21}$$

The last two identities follow from (20) and (19).

Regarding the first identity, since $t \mapsto u_t$ and $t \mapsto v_t$ are d_2-geodesics, we only need to treat the case $a \in [-1, 0]$ and $b \in [0, 1]$. Since $w_a = v_a$ and $w_b = v_b$, this follows from (20) and the last two identities of (21). ■

Proof of Theorem 1.5. By changing the background metric, we can assume without loss of generality that $\phi_0 = 0$. From (6) it follows that $t \mapsto \phi_t + Ct$ is a d_2-geodesic for any $C \in \mathbb{R}$. As a result, we can also assume that $\phi_1 \le 0$.

To derive a contradiction, let us further assume that there exists a d_2-geodesic $[-\varepsilon, 1] \ni t \mapsto \phi_t \in \mathcal{E}^2_\omega$, as described in the statement of the theorem.

First we show that $\phi_{-\varepsilon} \ge 0$. This is a simple consequence of t-convexity. By the results of [11] (see the discussion near (7)) there exists a set $Z \subset X$ of measure zero such that for all $x \in X \setminus Z$ the map $t \mapsto \phi_t(x)$ is convex, $\phi_0(x) = 0$, $\lim_{t \nearrow 1} \phi_t(x) = \phi_1(x) \le 0$, and $\lim_{t \searrow -\varepsilon} \phi_t(x) = \phi_{-\varepsilon}(x)$. Due to t-convexity, we find that $\phi_{-\varepsilon}(x) \ge 0$ away from Z. As $\phi_{-\varepsilon}$ is usc, we obtain $\phi_{-\varepsilon} \ge 0$.

Since $\phi_{-\varepsilon}$ is usc, it follows that $\sup_X \phi_{-\varepsilon} < +\infty$, i.e., $\phi_{-\varepsilon} \in L^\infty$. Using (6) for the d_2-geodesic joining $\phi_{-\varepsilon}$ and ϕ_0, it follows that

$$\phi_t \ge \phi_{-\varepsilon} - \frac{\varepsilon - t}{\varepsilon} \sup_X \phi_{-\varepsilon}, \qquad t \in [-\varepsilon, 0).$$

Since $(-\varepsilon, 1) \ni t \mapsto \phi_t(x)$ is t-convex for all $x \in X \setminus Z$, it follows that the above estimate extends to $t \in [-\varepsilon, 1]$, contradicting the assumption that $\phi_1 \in \mathcal{E}^2_\omega \setminus L^\infty$. ■

Proof of Theorem 1.4. We can assume without loss of generality that $\phi = 0$.

To derive a contradiction, we further assume that there exists a metric L^2 symmetry $F : \mathcal{V} \to \mathcal{V}$, as described in the statement of the theorem.

Since $\mathcal{V}$ is d_2-open, it follows that $0 \in B(0, \delta) \subset \mathcal{V}$ for some $\delta > 0$, where $B(0, \delta)$ is the d_2-ball of radius δ centered at 0. As F is a metric L^2 symmetry it follows that $F : B(0, \delta) \to B(0, \delta)$ is bijective.

Let $\psi_1 \in B(0,\delta)$ be such that $\psi_1 \in \mathcal{E}^2_\omega \setminus L^\infty$. One can find such ψ_1 as a consequence of [10, Theorem 3]. Let $[0,1] \ni t \mapsto \psi_t, F(\psi_t) \in B(0,\delta)$ be the d_2-geodesics connecting 0 to ψ_1, respectively 0 to $F(\psi_1)$.

Since F is a metric L^2 symmetry, by definition $\dot{\psi}_0 = -F(\dot{\psi}_0)$. Consequently, according to Lemma 4.1, the concatenation $[-1,1] \ni t \mapsto w_t \in B(0,\delta)$ of the curves $t \mapsto F(\psi_{-t})$ and $t \mapsto \psi_t$ is a d_2-geodesic. But then $t \mapsto w_t$ extends $t \mapsto \psi_t$ at $t=0$, contradicting Theorem 1.5. ∎

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