J. Eur. Math. Soc. 24, 37–135 (2022) DOI 10.4171/JEMS/1090

© 2021 European Mathematical Society
Published by EMS Press. This work is licensed under a CC BY 4.0 license.

Haozhao Li · Bing Wang

On Ilmanen's multiplicity-one conjecture for mean curvature flow with type-I mean curvature

Received December 8, 2018

Abstract. In this paper, we show that if the mean curvature of a closed smooth embedded mean curvature flow in $\mathbb{R}^3$ is of type-I, then the rescaled flow at the first finite singular time converges smoothly to a self-shrinker flow with multiplicity one. This result confirms Ilmanen's multiplicity-one conjecture under the assumption that the mean curvature is of type-I. As a corollary, we show that the mean curvature at the first singular time of a closed smooth embedded mean curvature flow in $\mathbb{R}^3$ is at least of type-I.

Keywords. Mean curvature flow, multiplicity-one conjecture, self-shrinker

Contents

Haozhao Li: Institute of Geometry and Physics, and Key Laboratory of Wu Wen-Tsun Mathematics, School of Mathematical Sciences, University of Science and Technology of China, No. 96 Jinzhai Road, Hefei, Anhui Province, 230026, China; hzli@ustc.edu.cn

Bing Wang: Institute of Geometry and Physics, and Key Laboratory of Wu Wen-Tsun Mathematics, School of Mathematical Sciences, University of Science and Technology of China, No. 96 Jinzhai Road, Hefei, Anhui Province, 230026, China; topspin@ustc.edu.cn

Mathematics Subject Classification (2020): Primary 53E10; Secondary 53C21, 35K93

1. Introduction

In this paper, we study finite time singularities of closed smooth embedded mean curvature flow in $\mathbb{R}^3$. A one-parameter family of hypersurfaces $\mathbf{x}(p,t) : \Sigma^n \to \mathbb{R}^{n+1}$ is called a mean curvature flow, if $\mathbf{x}$ satisfies the equation

$$\frac{\partial \mathbf{x}}{\partial t} = -H\mathbf{n}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1.1}$$

where H denotes the mean curvature of the hypersurface $\Sigma_t := \mathbf{x}(t)(\Sigma)$ and $\mathbf{n}$ denotes the outward unit normal of Σ_t. In the previous paper [46], we proved that the mean curvature of (1.1) must blow up at the first finite singular time for a closed smooth embedded mean curvature flow in $\mathbb{R}^3$. This paper can be viewed as a continuation of [46], and we will develop the techniques in [46] further to study the finite time singularities of mean curvature flow.

1.1. Singularities of mean curvature flow

The mean curvature flow with convexity conditions has been well studied during the past several decades. In [35], Huisken proved that if the initial hypersurface is uniformly convex, then after rescaling the mean curvature flow exists for all time and converges smoothly to a round sphere. When the initial hypersurface is mean-convex or two-convex, there are a number of estimates for the mean curvature flow (cf. Huisken and Sinestrari [37, 38], Haslhofer and Kleiner [33]), and these estimates are important to study the surgery of mean curvature flow (cf. Huisken and Sinestrari [39], Brendle and Huisken [8], Haslhofer and Kleiner [34]). Moreover, for mean curvature flow with mean convex initial hypersurfaces, B. White gave some structural properties of the singularities in [58, 59], and B. Andrews also showed a noncollapsing estimate in [1].

However, all these results rely on convexity conditions of initial hypersurfaces, and it is very difficult to study general cases. For the curve shortening flow in the plane, following the work Gage [28, 29] and Gage and Hamilton [30] on convex curves Grayson [31] proved that any embedded closed curve in the plane evolves to a convex curve and subsequently shrinks to a point, and Andrews and Bryan [2] gave a direct proof of Grayson's theorem without using the monotonicity formula or classification of singularities. In the higher dimensions, we know very little results without convexity conditions. Colding and Minicozzi studied the generic singularities of the mean curvature flow in [19, 22]. For the classification of self-shrinkers without convexity conditions, S. Brendle [7] proved that the round sphere is the only compact embedded self-shrinkers in $\mathbb{R}^3$ with genus 0, and L. Wang [55] showed that each end of a noncompact self-shrinker in $\mathbb{R}^3$ of finite topology is smoothly asymptotic to either a regular cone or a self-shrinking round cylinder. However, it still remains wide open to understand the behavior of mean curvature flow at the singular time in the general cases.

1.2. The multiplicity-one conjecture and the main theorems

To study the singularities of mean curvature flow without convexity conditions, Ilmanen proposed a series of conjectures in [40, 41]. Suppose that the mean curvature flow (1.1) reaches a singularity at (x_0, T) with $T < +\infty$. For any sequence $\{c_j\}$ with $c_j \to +\infty$, we rescale the flow (1.1) by

$$\Sigma_t^j := c_j(\Sigma_{T+c_j^{-2}t} - x_0), \quad t \in [-Tc_j^2, 0). \tag{1.2}$$

By Huisken's monotonicity formula [36] and Brakke's compactness theorem [6], a subsequence of Σ_t^j converges weakly to a limit flow $\mathcal{T}_t$, which is called a tangent flow at (x_0, T). In [40] Ilmanen showed that the tangent flow at the first singular time must be smooth for a smooth embedded mean curvature flow in $\mathbb{R}^3$, and he conjectured

Conjecture 1.1 (Ilmanen [40, 41], the multiplicity-one conjecture). For a smooth one-parameter family of closed embedded surfaces in $\mathbb{R}^3$ flowing by mean curvature, every tangent flow at the first singular time has multiplicity one.

Moreover, Ilmanen pointed out that the multiplicity-one conjecture implies a conjecture on the asymptotic structure of self-shrinkers in $\mathbb{R}^3$, and the latter conjecture has been confirmed recently by L. Wang [55]. If the initial hypersurface is mean convex or satisfies the Andrews condition, then the multiplicity-one conjecture holds (cf. White [58], Haslhofer and Kleiner [33], Andrews [1]). Recently, A. Sun [52] proved that the generic singularity of mean curvature flow of closed embedded surfaces in $\mathbb{R}^3$ modelled by closed self-shrinkers with multiplicity has multiplicity one. In general the multiplicity-one conjecture is still wide open. It is well known to experts that this conjecture holds if the second fundamental form A is of type-I. The main contribution of this paper is to confirm the multiplicity-one conjecture under the assumption that the mean curvature is of type-I, which is a much weaker condition.

To state our result, we first introduce some notations. A hypersurface $\mathbf{x} : \Sigma^n \to \mathbb{R}^{n+1}$ is called a self-shrinker if $\mathbf{x}$ satisfies the equation

$$H = \frac{1}{2}\langle \mathbf{x}, \mathbf{n} \rangle.$$

If Σ is a self-shrinker, then we call $\Sigma_t := \sqrt{-t}\,\Sigma$ $(t < 0)$ a self-shrinker flow.

The main theorem of this paper is the following result.

Theorem 1.2. *Let* $\mathbf{x}(t) : \Sigma^2 \to \mathbb{R}^3$ *(*$t \in [0, T)$*) a closed smooth embedded mean curvature flow with the first singular time* $T < +\infty$*. If the mean curvature satisfies*

$$\max_{\Sigma_t} |H|(p,t) \le \frac{\Lambda}{\sqrt{T-t}} \quad \text{for all } t \in [0,T), \tag{1.3}$$

for some $\Lambda > 0$*, then for any* $a, b \in \mathbb{R}$ *with* $-\infty < a < b < 0$ *and any sequence* $c_j \to +\infty$ *there exists a subsequence, still denoted by* $\{c_j\}$*, such that the flow* $\{\Sigma_t^j, a < t < b\}$ *defined by equation (1.2) converges smoothly to a self-shrinker flow with multiplicity one as* $j \to +\infty$*.*

It is not hard to see that Theorem 1.2 is equivalent to the following result.

Theorem 1.3. *Let* $\{(\Sigma^2, \mathbf{x}(t)), 0 \le t < +\infty\}$ *be a closed smooth embedded rescaled mean curvature flow*

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)^{\perp} = -\left(H - \frac{1}{2}\langle \mathbf{x}, \mathbf{n}\rangle\right)\mathbf{n} \tag{1.4}$$

satisfying

$$d(\Sigma_t, 0) \le D \quad \text{and} \quad \max_{\Sigma_t} |H(p,t)| \le \Lambda \tag{1.5}$$

for two constants $D, \Lambda > 0$*. Then for any sequence* $t_i \to +\infty$ *there exists a subsequence of* $\{\Sigma_{t_i+t}, -1 < t < 1\}$ *such that it converges in smooth topology to a complete smooth self-shrinker with multiplicity one as* $i \to +\infty$*.*

In [46], we showed Theorem 1.3 under the assumption that the mean curvature decays exponentially to zero. In this special case, the flow (1.4) converges smoothly to a plane passing through the origin with multiplicity one. Theorem 1.3 means that under the assumption that the mean curvature is bounded for all time the flow (1.4) also converges smoothly to a self-shrinker with multiplicity one. In fact, Theorem 1.3 is not stated with the optimal condition. Checking the proof carefully, one can see that the conclusion of Theorem 1.3 still holds under the assumption that the mean curvature is uniformly bounded on any ball for all time:

$$\max_{B_R(0)\cap\Sigma_t} |H|(p,t) \le C_R, \tag{1.6}$$

where C_R is a constant depending on R. Note that if the flow (1.4) converges smoothly to a self-shrinker with multiplicity one, condition (1.6) automatically holds by the self-shrinker equation. Thus, condition (1.6) is also necessary for the smooth convergence of the flow (1.4). Therefore, the solution of the multiplicity-one conjecture, i.e., Conjecture 1.1, is equivalent to the examination of (1.6), which will be an interesting subject of study in the near future.

The multiplicity-one conjecture is closely related to the extension problem of mean curvature flow. Huisken [35] proved that if the flow (1.1) develops a singularity at time $T < \infty$, then the second fundamental form will blow up at time T. A natural question is whether the mean curvature will blow up at the finite singular time of a mean curvature flow. Toward this question, A. Cooper [24] proved that $|A||H|$ must blow up at the singular time of the flow. In [44] Le and Sesum affirmatively answered this question under the assumption that the multiplicity-one conjecture holds, or the condition that the second fundamental form is of type-I at the singular time

$$\max_{\Sigma_t} |A| \leq \frac{C}{\sqrt{T-t}} \quad \text{for all } t \in [0, T). \tag{1.7}$$

Furthermore, Le and Sesum [45] proved that the mean curvature is at least of type-I for a mean curvature flow satisfying (1.7). Using Theorem 1.2, we can remove the type-I condition (1.7) of Le–Sesum's result as follows, which can also be viewed as an improvement of the extension theorem in [46].

Corollary 1.4. *If* $\mathbf{x}(t) : \Sigma^2 \to \mathbb{R}^3$ *($t \in [0, T)$) is a closed smooth embedded mean curvature flow with the first singular time $T < +\infty$, then there is a constant $\delta > 0$ such that*

$$\limsup_{t \to T} \sqrt{T-t} \max_{\Sigma_t} |H| \geq \delta \quad \textit{for all } t \in [0, T).$$

1.3. Outline of the proof

Now we sketch the proof of Theorem 1.3. Assume that the mean curvature satisfies the type-I condition (1.3) along the flow (1.1) and the first singular time $T < +\infty$. Then the mean curvature is uniform bounded along the rescaled flow (1.4). We have to show that the flow (1.4) converges smoothly to a self-shrinker with multiplicity one. The strategy is similar to [46], we first show a weak-compactness theorem and obtain the flow convergence is smooth away from a singular set. Then we use stability argument to remove the singular set. However, the technique here is much more involved. The proof consists of three steps:

Step 1. Convergence of the rescaled mean curvature flow with multiplicities. In this step, since the mean curvature is uniformly bounded along the flow, we have the short-time pseudolocality theorem and the energy concentration property, and we can follow the arguments in [46] to develop the weak compactness theory of mean curvature flow. However, compared with [46], since the mean curvature does not decay to zero, we have the following difficulties:

- No long time pseudolocality theorem.
- The space-time singularities in the limit do not move along straight lines.

Because of lacking these results, we face a number of new technical difficulties to show the L-stability of the limit self-shrinker. These difficulties force us to use analysis tools to study the asymptotical behavior of the solution of the limit parabolic equation near the singular set.

Step 2. Show that the multiplicity of the convergence is one for one subsequence. As in [46], it suffices to show that the limit self-shrinker is L-stable. By the convergence of the flow away from the singular set, if every limit has multiplicity greater than one, we can renormalize the “height-difference” function to obtain a positive solution of the equation

$$\frac{\partial w}{\partial t} = \Delta w - \frac{1}{2}\langle x, \nabla w\rangle + |A|^2 w + \frac{1}{2}w, \tag{1.8}$$

away from the singular set. To show the L-stability of the limit self-shrinker, we have to show the following two estimates:

- For each time, the asymptotical behavior of w is “good” near the singular set.
- Uniform L^1 norm of w independent of time.

By its construction, w is defined on any compact set away from the singular set and we have no estimates near the singular set by the geometric method. However, we found that w satisfies many good properties from the PDE point of view. In [42], Kan and Takahashi studied similar problem for some semilinear parabolic equations along time-dependent singularities in the Euclidean spaces. Kan and Takahashi showed their result for one time-dependent singularity, and the solution of the equation looks like $\log \frac{1}{r}$ in dimension 2, where r is the distance from any point x to the singularity. However, in our case the solution of (1.8) may have multiple singularities, and these singularities may coincide at one point. Thus, we cannot apply Kan–Takahashi’s result directly, and we need to develop their techniques to show that the solution w is in L^1 across the singularities and near the singular set the solution w roughly looks like

$$w(x,t) \sim \sum_{k=1}^{l} c_k(t) \log \frac{1}{\mathbf{r}_k(x,t)},$$

where $\mathbf{r}_k(x,t)$ denotes the intrinsic distance from a point x to a singularity $\xi_k(t)$ at time t. Here the constant c_i may depend on t. In general, the L^1 norm of w may tend to infinity as $t \to +\infty$. In order to show uniform L^1 norm of w, we refine the argument in [46] and also use the estimate of w near the singularities to choose a sequence of time slices $\{t_i\}$, and then we show that for such a special sequence the corresponding function w has uniform L^1 bound independent of t. Thus, for the special sequence t_i, the auxiliary function w satisfies the two desired estimates. Then we can follow the argument in [46] to show that the convergence of (1.4) is smooth and of multiplicity one, for the special sequence $\{t_i\}$.

Step 3. Show the multiplicity-one convergence for each subsequence. This step is a new difficulty beyond [46]. In [46], each limit, no matter what multiplicity it is, must be a flat plane passing through the origin. Therefore, up to rotation, different limits can be regarded as the same. By the monotonicity of the entropy, it is clear that if one limit is a multiplicity-one plane, then each limit must also be a multiplicity-one plane. However, in the current setting, each limit is only a self-shrinker and the limits may vary as the time sequences change. A priori, it is possible that one sequence converge to a multiplicity-one self-shrinker A, and the other sequence converge to a multiplicity two self-shrinker $B \neq A$. This possibility cannot be ruled out by only using the monotonicity of the entropy. To overcome this difficulty, we essentially use the smooth compactness theorem of self-shrinkers by Colding and Minicozzi [20]. Since the limit self-shrinkers form a compact set, we know that the local behavior of limit self-shrinkers are very close to that of planes on a fixed small scale. From this and the volume continuity, we derive an argument to show that the multiplicity is independent of the choice of subsequences. Therefore, every subsequence converges with multiplicity one.

It is interesting to know whether the above argument still works for the multiplicity-one conjecture without the mean curvature bound assumption (1.3). The main difficulty is the loss of pseudolocality result as in [46], since the points in the evolving surfaces may move drastically if the mean curvature is large. Furthermore, the loss of mean curvature bound also induces difficulties in applying PDE tools to analyze the singular set. However, as we discussed around (1.6), it is also logically possible to develop estimate (1.6) directly.

1.4. Relation with other geometric flows

It is interesting to compare the mean curvature flow with the Ricci flow. The extension problem for Ricci flow has been extensively studied recently. Corollary 1.4 has a cousin theorem in the Ricci flow. In [54, Theorem 1], it was shown that along the Ricci flow $\{(M, g(t)), 0 \le t < T\}$ with the singular time $T < +\infty$, we have

$$\max_M |\mathrm{Ric}|_{g(t)} \ge \frac{\delta}{\sqrt{T-t}}, \quad t \in [0, T), \tag{1.9}$$

which extends the famous Ricci extension theorem of N. Sesum [51]. Up to rescaling, the gap inequality (1.9) is equivalent to

$$\max_M |\mathrm{Ric}|_{g(t)} \ge \delta$$

along the rescaled Ricci flow solution

$$\partial_t g = -\mathrm{Ric} + g, \quad t \in [0, \infty). \tag{1.10}$$

Actually, we even believe that a gap for scalar curvature holds for a rescaled Ricci flow solution. In other words, along the rescaled Ricci flow (1.10) we should have

$$\max_M |R|_{g(t)} \ge \delta.$$

It is easy to see that the scalar extension conjecture of the Ricci flow will hold automatically if one can prove the above inequality along the rescaled Ricci flow (1.10), just like the extension theorem of mean curvature in [46] follows directly from Corollary 1.4.

The similarity between the regularity theory of rescaled mean curvature flow (1.4) and the rescaled Ricci flow (1.10) was noticed for a while. For example, such similarity was discussed in the introduction of [46]. Along the rescaled flows, the mean curvature bound condition (1.5) is comparable to the scalar curvature bound condition $|R| \le C$. Note that the Fano Kähler–Ricci flow provides many examples of the global solutions of the rescaled Ricci flow (1.10) and Perelman showed that $|R| \le C$ holds automatically. The boundedness of the scalar curvature is crucial to study the convergence of Kähler–Ricci flow to a limit flow (cf. [13, Theorem 1.5], with journal version [14] and [15]). For time-slice convergence, see Tian and Zhang [53], Bamler [3] and Chen and Wang [11] for example. Since (1.5) is the comparable condition of Perelman's estimates, we can view Theorem 1.2 as the analogue of the convergence results in the Fano Kähler–Ricci flow. However, we have to confess that we do not know any non-trivial examples satisfying the condition (1.5). By non-triviality we mean that the flow (1.4) is neither a self-shrinker nor convex. It will be very interesting to find out such examples.

The rescaled mean curvature flow can also be compared with the Calabi flow. In [9] E. Calabi studied the gradient flow of the L^2 norm of the scalar curvature among Kähler metrics in a fixed cohomology class on a compact Kähler manifold, which is now well known as the Calabi flow. X. X. Chen conjectured that the Calabi flow always exists globally for any initial smooth Kähler potential. Very recently, Chen and Cheng [10] proved that the Calabi flow exists as long as the scalar curvature is uniformly bounded. Therefore, to study the long time existence of Calabi flow, it is crucial to control the scalar curvature, which is similar to the mean curvature condition (1.5) for the rescaled mean curvature flow. Assuming the long time existence and the uniform boundedness of the scalar curvature, the current authors and K. Zheng showed the convergence of the Calabi flow in [47], just as Theorem 1.3 for rescaled mean curvature flow.

1.5. List of notations

In the following, we list the important notations in this paper.

- $d(x, y)$: the Euclidean distance from x to y. Defined in Definition 2.7.
- $B_r(p)$: the open ball in $\mathbb{R}^3$ centered at p with radius r. Defined in Definition 2.1.
- $d_g(x, y)$: the intrinsic distance of (Σ, g) from x to y. First appears in the beginning of Section 4.
- $\mathcal{B}_r(p)$: the intrinsic geodesic ball in (Σ, g) centered at p with radius r. Defined in Definition 2.1.
- $C_x(B_r(p) \cap \Sigma)$: the connected component of $B_r(p) \cap \Sigma$ containing $x \in \Sigma$. Defined in Definition 2.1.
- $\mathfrak{m}(x, t)$: the multiplicity at (x, t). Defined in (2.14).
- $\mathcal{S}$: the space-time singular set. Defined in Proposition 2.8.
- $\mathcal{S}_t = \{x \in \mathbb{R}^3 \mid (x, t) \in \mathcal{S}\}$: the singular set at time t. Defined in Proposition 2.8.
- $\xi(t)$: a Lipschitz singular curve in $\mathcal{S}$. First appears in Lemma 2.11.
- $\rho : \mathbb{R}^+ \to \mathbb{R}^+$: an increasing positive function. First appears in Definition 3.3.
- $|\Omega|$: the volume of a set $\Omega \subset \mathbb{R}^3$ with respect to the standard metric on $\mathbb{R}^3$. Defined in Lemma 3.10.
- $\Omega_{\epsilon,R}(t)$: a subset of the limit self-shrinker away from singularities. Defined in (3.23).
- $\mathcal{S}_I$: the union of the singular set on a time interval I. Defined in (3.24).
- u_i: the height difference function defined in (3.25).
- w_i: the normalized difference function defined in (3.27).
- d_H: the Hausdorff distance in the Euclidean space.

- $\mathbf{r}(x,t)$: the intrinsic distance function from x to the singular curve $\xi(t)$. Defined in (4.21).
- $\mathbf{r}_k(x,t)$: the intrinsic distance function from x to the singular curve $\xi_k(t)$. Defined in (3.88) and Theorem 4.2.
- $F_t^{(k)}(\delta)$ and $A_t^{(k)}(\delta,\rho)$: a subset around the singularities on the limit self-shrinker. Defined in (3.98) and (3.99).
- $\mathcal{M}_{k,m}(\rho,\Xi)$: a subset of a Riemannian manifold defined in Definition 4.1.
- $\Gamma_{\underline{t},\bar{t}}$: the union of space-time singular curves. Defined in (4.17) and (4.51).
- $Q_{r,\underline{t},\bar{t}}$ and $\hat{Q}_{r,\underline{t},\bar{t}}$: the neighborhood of the singular curves. Defined in (4.18) and (4.51).
- ϕ_ξ: cutoff functions around the singular curves. Defined in Definitions 4.7 and 4.12.
- I_ξ: a functional associated with a singular curve ξ. Defined in Definition 4.12.
- $\tilde{\mathbf{r}}(x,t)$ and $\tilde{v}(x,t)$: defined in Definition 4.12.
- $H(z)$: a cutoff function defined in Definition 4.7. Note that the function $H(z)$ is only used in Section 4. Since the mean curvature does not appear in Section 4, we keep the same notation $H(z)$ as in [42].

1.6. The organization

The organization of this paper is as follows. In Section 2 we recall some facts on the pseudolocality theorem and energy concentration property. Moreover, we will show the weak compactness of mean curvature flow under some geometric conditions and we show the multiplicity of the convergence is a constant. In Section 3 we show the rescaled mean curvature flow with bounded mean curvature converges smoothly to a self-shrinker with multiplicity one, under the assumption that the auxiliary function satisfies good growth properties at the singular set. In Section 4 we will show the estimates of the auxiliary function by developing Kan–Takahashi's argument. Finally, we finish the proof of Theorem 1.2 in Section 5. In the appendices, we include two versions of the parabolic Harnack inequality and give the full details on the calculation of the linearized equation of rescaled mean curvature flow.

2. Weak compactness of refined sequences

2.1. The pseudolocality theorem and energy concentration property

In this subsection, we recall some results in [46]. First, we have the following definition.

Definition 2.1. (1) We denote by $B_r(p)$ the ball in $\mathbb{R}^{n+1}$ centered at p with radius r with respect to the standard Euclidean metric, and $\mathcal{B}_r(p) \subset (M,g)$ the intrinsic geodesic ball on M centered at p with radius r with respect to the metric g.

(2) For any $r>0$, $p \in \mathbb{R}^{n+1}$ and $\Sigma^n \subset \mathbb{R}^{n+1}$, we denote by $C_x(B_r(p)\cap\Sigma)$ the connected component of $B_r(p)\cap\Sigma$ containing $x \in \Sigma$.

We first recall the following result of Chen and Yin [16].

Lemma 2.2 (cf. [16, Lemma 7.1]). *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be properly embedded in $B_{r_0}(x_0)$ for some $x_0 \in \Sigma$ with*

$$|A|(x) \le \frac{1}{r_0}, \quad x \in B_{r_0}(x_0) \cap \Sigma.$$

Let $\{x^1, \dots, x^{n+1}\}$ be the standard coordinates in $\mathbb{R}^{n+1}$. Assume that $x_0 = 0$ and the tangent plane of Σ at x_0 is $x^{n+1} = 0$. Then there is a map

$$u : \left\{ x' = (x^1, \dots, x^n) \;\middle|\; |x'| < \frac{r_0}{96} \right\} \to \mathbb{R}$$

with $u(0) = 0$ and $|\nabla u|(0) = 0$ such that the connected component containing x_0 of $\Sigma \cap \{(x', x^{n+1}) \in \mathbb{R}^{n+1} \mid |x'| < \frac{r_0}{96}\}$ can be written as a graph $\{(x', u(x')) \mid |x'| < \frac{r_0}{96}\}$ and

$$|\nabla u|(x') \le \frac{36}{r_0}|x'|.$$

Using Lemma 2.2, we show that the local area ratio of the surface is very close to 1.

Lemma 2.3 (cf. [46, Lemma 3.3]). *Suppose that $\Sigma^n \subset B_{r_0}(p) \subset \mathbb{R}^{n+1}$ is a hypersurface with $\partial\Sigma \subset \partial B_{r_0}(p)$ and*

$$\sup_{\Sigma} |A| \le \frac{1}{r_0}.$$

For any $\delta > 0$, there is a constant $\rho_0 = \rho_0(r_0, \delta)$ such that for any $r \in (0, \rho_0)$ and any $x \in B_{r_0/2}(p) \cap \Sigma$ we have

$$1 - \delta \le \frac{\mathrm{vol}_\Sigma(C_x(B_r(x) \cap \Sigma))}{\omega_n r^n} \le 1 + \delta. \tag{2.1}$$

Proof. By Lemma 2.2, for any $x \in B_{r_0/2}(p) \cap \Sigma$ the component $C_x(B_{\rho_0}(x) \cap \Sigma)$ with $\rho_0 = \frac{r_0}{192}$ can be written as a graph of a function u over the tangent plane at x, which we assume to be

$$P = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\},$$

with

$$|\nabla u|(x') \le \frac{72}{r_0}|x'|,$$

where $x' = (x_1, \dots, x_n)$. Let $r \in (0, \rho_0)$. Denote by Ω_r the projection of $C_x(B_r(x) \cap \Sigma)$ to the plane P. Then for any $x' \in \partial\Omega_r$ we have

$$u(x')^2 + |x'|^2 = r^2. \tag{2.2}$$

On the other hand, for any $x' \in \Omega_{\rho_0}$ we have the inequality

$$|u(x')| \le |u(0)| + \max_{t\in[0,1]} |\nabla u|(tx') \cdot |x'| \le \frac{72}{r_0}|x'|^2. \tag{2.3}$$

Note that (2.2) and (2.3) imply that for any $x' \in \partial\Omega_r$,

$$|x'|^2 \le r^2 = u(x')^2 + |x'|^2 \le |x'|^2\left(1 + \frac{5184}{r_0^2}\rho_0^2\right). \tag{2.4}$$

Thus, we have

$$\tilde{r} := \frac{r}{\sqrt{1 + \frac{5184}{r_0^2}\rho_0^2}} \le |x'| \le r \quad \text{for all } x' \in \partial\Omega_r,$$

which implies that

$$B_{\tilde{r}}(x) \cap P \subset \Omega_r \subset B_r(x) \cap P. \tag{2.5}$$

Thus, the volume ratio of $C_x(B_r(x) \cap \Sigma)$ is bounded from above

$$\begin{aligned} \frac{\mathrm{vol}_\Sigma(C_x(B_r(x) \cap \Sigma))}{\omega_n r^n} &= \frac{1}{\omega_n r^n}\int_{\Omega_r} \sqrt{1 + |\nabla u|^2}\, d\mu \\ &\le \frac{1}{\omega_n r^n}\int_{B_r(x)\cap P} \sqrt{1 + |\nabla u|^2}\, d\mu \\ &\le \sqrt{1 + \frac{5184}{r_0^2} r^2}, \end{aligned} \tag{2.6}$$

where we used (2.3) and (2.5). Moreover, the volume ratio of $C_x(B_r(x) \cap \Sigma)$ is bounded from below

$$\begin{aligned} \frac{\mathrm{vol}_\Sigma(C_x(B_r(x) \cap \Sigma))}{\omega_n r^n} &\ge \frac{1}{\omega_n r^n}\int_{B_{\tilde{r}}(x)\cap P} \sqrt{1 + |\nabla u|^2}\, d\mu \\ &\ge \frac{\tilde{r}^n}{r^n} \ge \left(1 + \frac{5184}{r_0^2}\rho_0^2\right)^{-\frac{n}{2}}. \end{aligned} \tag{2.7}$$

Combining (2.6) with (2.7), for any $\delta > 0$ we can choose $\rho_0 = \rho_0(n, \delta, r_0)$ further small such that (2.1) holds. The lemma is proved. ∎

Next we recall the two-sided pseudolocality theorem in [46]. If the initial hypersurface can be locally written as a graph of a single-valued function, then we have the pseudolocality type results of the mean curvature flow by Ecker and Huisken [26, 27], M. T. Wang [56], Chen and Yin [16] and Brendle and Huisken [8]. However, in our case we have to apply the pseudolocality theorem for the hypersurfaces which may converge with multiplicities. Thus, we use the boundedness of the mean curvature to get the two-sided pseudolocality theorem in [46].

Theorem 2.4 (Two-sided pseudolocality, cf. [46]). *For any $r_0 \in (0, 1], \Lambda, T > 0$, there exist $\eta = \eta(n, \Lambda), \epsilon = \epsilon(n, \Lambda) > 0$ satisfying*

$$\lim_{\Lambda \to 0} \eta(n, \Lambda) = \eta_0(n) > 0, \qquad \lim_{\Lambda \to 0} \epsilon(n, \Lambda) = \epsilon_0(n) > 0$$

and the following properties. Let $\{(\Sigma^n, \mathbf{x}(t)), -T \le t \le T\}$ be a closed smooth embedded mean curvature flow (1.1). Assume that

(1) *the second fundamental form satisfies $|A|(x, 0) \le \frac{1}{r_0}$ for any $x \in C_{p_0}(B_{r_0}(p_0) \cap \Sigma_0)$ where $p_0 = \mathbf{x}_0(p)$ for some $p \in \Sigma$,*

(2) *the mean curvature of $\{(\Sigma^n, \mathbf{x}_t), -T \le t \le T\}$ is bounded by Λ.*

Then for any (x,t) *satisfying*

$$x \in C_{p_t}(\Sigma_t \cap B_{\frac{1}{16}r_0}(p_0)), \quad t \in \left[-\frac{\eta r_0^2}{2(\Lambda+\Lambda^2)}, \frac{\eta r_0^2}{2(\Lambda+\Lambda^2)}\right] \cap [-T,T],$$

where $p_t = \mathbf{x}_t(p)$, *we have the estimate*

$$|A|(x,t) \le \frac{1}{\epsilon r_0}.$$

Using the pseudolocality theorem, we have the energy concentration property.

Lemma 2.5 (Energy concentration, cf. [46]). *For any* $\Lambda, K, T > 0$, *there exists a constant* $\epsilon(n,\Lambda,K,T) > 0$ *with the following property. Let* $\{(\Sigma^n, \mathbf{x}(t)), -T \le t \le T\}$ *be a closed smooth embedded mean curvature flow* (1.1). *Assume that*

$$\max_{\Sigma_t \times [-T,T]} |H|(p,t) \le \Lambda.$$

Then we have

$$\int_{\Sigma_0 \cap B_{Q^{-1}}(q)} |A|^n \, d\mu_0 \ge \epsilon(n,\Lambda,K,T)$$

whenever $q \in \Sigma_0$ *with* $Q := |A|(q,0) \ge K$.

A direct corollary of Lemma 2.5 is the following ϵ-regularity of the mean curvature flow, which can be viewed as a generalization of the result of Choi and Schoen [17].

Corollary 2.6 (ϵ-regularity, cf. [46]). *There exists* $\epsilon_0(n) > 0$ *satisfying the following property. Let* $\{(\Sigma^n, \mathbf{x}(t)), -1 \le t \le 1\}$ *be a closed smooth embedded mean curvature flow* (1.1). *Suppose that the mean curvature satisfies*

$$\max_{\Sigma_t \times [-1,1]} |H|(p,t) \le 1.$$

For any $q \in \Sigma_0$, *if*

$$\int_{\Sigma_0 \cap B_r(q)} |A|^n \, d\mu_0 \le \epsilon_0(n)$$

for some $r > 0$, *then we have*

$$\max_{B_{\frac{r}{2}}(q) \cap \Sigma_0} |A| \le \max\left\{1, \frac{2}{r}\right\}.$$

2.2. *Weak compactness*

As in [46], we use the pseudolocality theorem and the energy concentration property to develop the weak compactness of the mean curvature flow. Here we will replace the zero mean curvature condition in [46] by the boundedness of the mean curvature in the definition of refined sequences. By "refined" we mean that the sequence is taken after a point-selecting process such that many good properties already hold for the objects in this sequence. The name of refined sequence originates from [12].

Definition 2.7 (Refined sequences). Let $\{(\Sigma_i^2, \mathbf{x}_i(t)), -1 < t < 1\}$ be a one-parameter family of closed smooth embedded surfaces satisfying the mean curvature flow equation (1.1). It is called a refined sequence if the following properties are satisfied for every i:

(1) There exists a constant $D > 0$ such that $d(\Sigma_{i,t}, 0) \le D$ for any $t \in (-1, 1)$, where $d(\Sigma, 0)$ denotes the Euclidean distance from the point $0 \in \mathbb{R}^3$ to the surface $\Sigma \subset \mathbb{R}^3$ and $\Sigma_{i,t} = \mathbf{x}_i(t)(\Sigma_i)$.

(2) There is a uniform constant $\Lambda > 0$ such that

$$\max_{\Sigma_{i,t}\times(-1,1)} |H|(p,t) \le \Lambda. \tag{2.8}$$

(3) There exists an increasing positive function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $R > 0$,

$$\int_{\Sigma_{i,t}\cap B_R(0)} |A|^2 \, d\mu_{i,t} \le \rho(R) \quad \text{for all } t \in (-1,1). \tag{2.9}$$

(4) There is uniform $N > 0$ such that for all $r > 0$ and $p \in \mathbb{R}^3$ we have

$$\mathrm{Area}_{g_i(t)}(B_r(p) \cap \Sigma_{i,t}) \le N\pi r^2 \quad \text{for all } t \in (-1,1). \tag{2.10}$$

(5) There exist uniform constants $\bar{r}, \kappa > 0$ such that for any $r \in (0, \bar{r}]$ and any $p \in \Sigma_{i,t}$ we have

$$\mathrm{Area}_{g_i(t)}(B_r(p) \cap \Sigma_{i,t}) \ge \kappa r^2 \quad \text{for all } t \in (-1,1). \tag{2.11}$$

(6) There exists $T > 1$ such that

$$\lim_{i\to+\infty} \int_{-1}^{1} dt \int_{\Sigma_{i,t}} e^{-\frac{|\mathbf{x}_i|^2}{4(T-t)}} \left| H_i - \frac{1}{2(T-t)} \langle \mathbf{x}_i, \mathbf{n}\rangle \right|^2 d\mu_{i,t} = 0.$$

Following the arguments as in minimal surfaces (cf. White [57], or Colding and Minicozzi [18]), we have the weak compactness for mean curvature flow.

Proposition 2.8. *Let $\{(\Sigma_i^2, \mathbf{x}_i(t)), -1 < t < 1\}$ be a refined sequence. Then there exists a subsequence, still denoted by $\{(\Sigma_i^2, \mathbf{x}_i(t)), -1 < t < 1\}$, a smooth self-shrinker flow $\{(\Sigma_\infty, \mathbf{x}_\infty(t)), -1 < t < 1\}$ satisfying*

$$H = \frac{1}{2(T-t)} \langle \mathbf{x}_\infty, \mathbf{n}\rangle, \tag{2.12}$$

for some $T > 1$, and a space-time singular set $\mathcal{S} = \{(x,t) \mid t \in (-1,1),\ x \in \mathbb{R}^3\}$ satisfying the following properties:

(1) *The sequence $\{(\Sigma_i^2, \mathbf{x}_i(t)), -1 < t < 1\}$ converges locally smoothly, possibly with multiplicity at most N_0, to $\{(\Sigma_\infty, \mathbf{x}_\infty(t)), -1 < t < 1\}$ away from $\mathcal{S}$.*

(2) *For each time $t \in (-1,1)$ the singular set $\mathcal{S}_t = \{x \in \mathbb{R}^3 \mid (x,t) \in \mathcal{S}\}$ is locally finite in the sense that $\sharp\{\mathcal{S}_t \cap B_R(0)\}$ is uniformly bounded by a number depending only on $\rho(R)$.*

(3) *The sequence in* (1) *also converges in extrinsic Hausdorff distance.*

Proof. We first show that after taking a subsequence if necessary, $\Sigma_{i,0}$ converges locally smoothly to $\Sigma_{\infty,0}$ away from a locally finite set $\mathcal{S}_0$. To this end, fix large $R > 0$ and let $\Omega = B_R(0) \subset \mathbb{R}^3$. By property (1) in Definition 2.7, we have $\Sigma_{i,t} \cap \Omega \neq \emptyset$ for large $R > 0$ and any $t \in (-1, 1)$. For any $U \subset \Omega$, we define the measures ν_i by

$$\nu_i(U) = \int_{U \cap \Sigma_{i,0}} |A_i|^2 \, d\mu_{i,0} \leq \rho(R),$$

where we used (2.9) in the inequality. The general compactness of Radon measures implies that there is a subsequence, which we still denote by ν_i, converges weakly to a Radon measure ν with $\nu(\Omega) \leq \rho(R)$. We define the set

$$\mathcal{S}_0 = \{x \in \Omega \mid \nu(x) \geq \epsilon_0\},$$

where ϵ_0 is the constant in Corollary 2.6. It follows that the set $\mathcal{S}_0$ contains at most $\frac{\rho(R)}{\epsilon_0}$ points. Given any point $y \in \Omega \backslash \mathcal{S}_0$, there exists some $s > 0$ such that $B_{10s}(y) \subset \Omega$ and $\nu(B_{10s}(y)) < \epsilon_0$. Since $\nu_i \to \nu$, for i sufficiently large we have

$$\int_{B_{10s}(y) \cap \Sigma_{i,0}} |A_i|^2 \, d\mu_{i,0} < \epsilon_0.$$

Corollary 2.6 implies that for i sufficiently large we have the estimate

$$\max_{B_{5s}(y) \cap \Sigma_{i,0}} |A|(x, 0) \leq \max\left\{1, \frac{1}{5s}\right\}.$$

By Theorem 2.4 and (2.8) , there exists $\epsilon = \epsilon(s, n) > 0$ such that

$$\max_{B_{\epsilon r_0}(y) \cap \Sigma_{i,t}} |A|(x, t) \leq \frac{1}{\epsilon r_0} \quad \text{for all } t \in [-\epsilon r_0^2, \epsilon r_0^2], \tag{2.13}$$

where $r_0 = 5s$. Therefore, for large i we have all higher order estimates of the second fundamental form at any point in $\Sigma_{i,0} \backslash B_{2r_0}(\mathcal{S}_0)$, where $B_r(\mathcal{S}_0) = \{x \in \mathbb{R}^3 \mid d(x, \mathcal{S}_0) \leq r\}$. Using a diagonal sequence argument and taking $s \to 0$, we can show that a subsequence of $\Sigma_{i,0}$ converges in smooth topology, possibly with multiplicities, to a limit surface $\Sigma_{\infty,0}$ away from the singular set $\mathcal{S}_0$. Properties (2.10)–(2.11) imply that the multiplicity of the convergence is bounded by a constant N_0.

Note that by (2.13) the second fundamental form is uniformly bounded for any point $(x, t) \in (\Sigma_{i,t} \backslash B_{2r_0}(\mathcal{S}_0)) \times ([-\epsilon r_0^2, \epsilon r_0^2] \cap (-1, 1))$. By compactness of mean curvature flow (cf. [46, Theorem 2.6]), the flow $\{\Sigma_{i,t} \backslash B_{2r_0}(\mathcal{S}_0), t \in (-\epsilon r_0^2, \epsilon r_0^2) \cap (-1, 1)\}$ converges smoothly to a limit flow $\{\Sigma_{\infty,t} \backslash B_{2r_0}(\mathcal{S}_0), t \in (-\epsilon r_0^2, \epsilon r_0^2) \cap (-1, 1)\}$ and by property (6) in Definition 2.7 $\Sigma_{\infty,t} \backslash B_{2r_0}(\mathcal{S}_0)$ satisfies the self-shrinker equation (2.12) for $t \in (-\epsilon r_0^2, \epsilon r_0^2) \cap (-1, 1)\}$. We can also replace $t = 0$ by any other $t_0 \in (-1, 1)$ and the above argument still works for the time interval $(-\epsilon r_0^2 + t_0, \epsilon r_0^2 + t_0) \cap (-1, 1)$. Since $r_0 = 5s > 0$ is arbitrary small, by using a diagonal sequence argument and taking $s \to 0$ we have that $\{(\Sigma_i^2, \mathbf{x}_i(t)), -1 < t < 1\}$ converges locally smoothly to the flow $\{(\Sigma_\infty, \mathbf{x}_\infty(t)), -1 < t < 1\}$ away from $\mathcal{S}$ and $\Sigma_{\infty,t}$ satisfies equation (2.12). Note that Σ_∞ is a self-shrinker in $\mathbb{R}^3$ and it can be viewed as a minimal surface in $(\mathbb{R}^3, g_{ij})$ with

$$g_{ij} = e^{-\frac{|x|^2}{4}}\delta_{ij}.$$

Thus, we can follow the argument in minimal surfaces (cf. White [57], or Colding and Minicozzi [18]) to show that $\Sigma_{\infty,t} \cup \mathcal{S}_t$ is smooth and embedded and $\Sigma_{i,t}$ converges to $\Sigma_{\infty,t}$ in Hausdorff distance. The proposition is proved. ∎

As in [46], we show that the multiplicity in Proposition 2.8 is constant. To study the multiplicity, we define a function

$$\Theta(x,r,t) := \lim_{i\to+\infty} \frac{\mathrm{Area}_{g_i(t)}(\Sigma_{i,t} \cap B_r(x))}{\pi r^2} \quad \text{for all } (x,t) \in \Sigma_{\infty,t} \times (-1,1).$$

Then the multiplicity at $(x,t) \in \Sigma_{\infty,t} \times (-1,1)$ is defined by

$$\mathfrak{m}(x,t) := \lim_{r\to 0} \Theta(x,r,t). \tag{2.14}$$

It is clear that $\mathfrak{m}(x,t)$ is an integer. In the following result, we show that $\mathfrak{m}(x,t)$ is independent of x and t. Note that in [46, Lemma 3.14] we proved the same result under the assumption that the mean curvature decays exponentially to zero. The first two steps of the proof here are similar to that of [46] while the third step is different. We give all the details for completeness.

Lemma 2.9. *Under the assumption of Proposition 2.8, the function $\mathfrak{m}(x,t)$ is a constant integer on $\Sigma_{\infty,t} \times (-1,1)$. Namely, $\mathfrak{m}(x,t)$ is independent of x and t.*

Proof. The proof can be divided into three steps.

Step 1. For each $t \in (-1,1)$, $\mathfrak{m}(x,t)$ is constant on $\Sigma_{\infty,t}\backslash\mathcal{S}_t$. Fix $t_0 \in (-1,1)$, $R > 0$ and $x_0 \in (\Sigma_{\infty,t_0} \cap B_R(0))\backslash\mathcal{S}_{t_0}$. There exists $r_0 > 0$ such that for large i,

$$|A|(x,t_0) \le \frac{1}{r_0} \quad \text{for all } x \in B_{r_0}(x_0) \cap \Sigma_{\infty,t_0}. \tag{2.15}$$

By Lemma 2.2, we can assume r_0 small such that $B_{r_0}(x_0) \cap \Sigma_{\infty,t_0}$ can be written as a graph over the tangent plane of Σ_{∞,t_0} at x_0. Since x_0 is regular, we have $d(x_0, \mathcal{S}_{t_0}) > 0$. Let $r_1 = \frac{1}{4}\min\{r_0, d(x_0, \mathcal{S}_{t_0})\}$. For any $p \in B_{r_1}(x_0) \cap \Sigma_{i,t_0}$, we have $B_{r_1}(p) \subset B_{r_0}(x_0)$. Thus, (2.15) implies that for large i,

$$|A|(x,t_0) \le \frac{1}{r_1} \quad \text{for all } x \in B_{r_1}(p) \cap \Sigma_{i,t_0}.$$

By Lemma 2.3, for any $\delta > 0$ there exists $\rho_0 = \rho_0(r_1, \delta) \in (0, \frac{r_1}{200})$ such that for any $r \in (0, \rho_0)$ and any $p \in B_{r_1/2}(x_0) \cap \Sigma_{i,t_0}$ we have

$$1 - \delta \le \frac{\mathrm{Area}_{g_i(t_0)}(C_p(B_r(p) \cap \Sigma_{i,t_0}))}{\pi r^2} \le 1 + \delta. \tag{2.16}$$

Suppose that $B_{r_1}(x_0) \cap \Sigma_{i,t_0}$ has m_i connected components, where m_i is an integer bounded by a constant independent of i by Proposition 2.8. After taking a subsequence of $\{\Sigma_{i,t_0}\}$ if necessary, we can assume that m_i are the same integer denoted by m with $m \geq 1$. For any $x \in B_{r_1/2}(x_0) \cap \Sigma_{\infty,t_0}$, we denote by α_x the normal line passing through x of Σ_{∞,t_0}. Since each component of $B_{r_1}(x_0) \cap \Sigma_{i,t_0}$ converges to $B_{r_1}(x_0) \cap \Sigma_{\infty,t_0}$ smoothly and $B_{r_1}(x_0) \cap \Sigma_{\infty,t_0}$ is a graph over the tangent plane of Σ_{∞,t_0} at x_0, α_x intersects transversally each component of Σ_{i,t_0} at exactly one point. Suppose that

$$\alpha_x \cap (B_{r_1}(x_0) \cap \Sigma_{i,t_0}) = \{p_i^{(1)}, p_i^{(2)}, \ldots, p_i^{(m)}\}.$$

Then (2.16) implies that for any integer j with $1 \leq j \leq m$ and any $r \in (0, \rho_0)$,

$$1 - \delta \leq \frac{\mathrm{Area}_{g_i(t_0)}(C_{p_i^{(j)}}(B_r(p_i^{(j)}) \cap \Sigma_{i,t_0}))}{\pi r^2} \leq 1 + \delta. \tag{2.17}$$

After shrinking r_0 if necessary, we can assume that $B_r(x) \cap \Sigma_{\infty,t_0}$ has only one component for any $r \in (0, \frac{r_1}{2})$ and any $x \in B_{r_1/2}(x_0) \cap \Sigma_{\infty,t_0}$. Since for any $1 \leq j \leq m$ and $r \in (0, \rho_0)$ we have $p_i^{(j)} \to x$ and $C_{p_i^{(j)}}(B_r(p_i^{(j)}) \cap \Sigma_{i,t_0})$ converges smoothly to $B_r(x) \cap \Sigma_{\infty,t_0}$ as $i \to +\infty$, (2.17) implies that

$$m(1 - \delta) \leq \lim_{i \to +\infty} \frac{\mathrm{Area}_{g_i(t_0)}(B_r(x) \cap \Sigma_{i,t_0})}{\pi r^2} \leq m(1 + \delta).$$

In other words, for any $x \in B_{r_1/2}(x_0) \cap \Sigma_{\infty,t_0}$ and any $r \in (0, \rho_0)$ we have

$$m(1 - \delta) \leq \Theta(x, r, t_0) \leq m(1 + \delta). \tag{2.18}$$

Taking $r \to 0$ in (2.18), we have

$$\mathfrak{m}(x, t_0) = m \quad \text{for all } x \in B_{r_1/2}(x_0) \cap \Sigma_{\infty,t_0}.$$

By the connectedness of $\Sigma_{\infty,t_0} \backslash \mathcal{S}_{t_0}$, we know that $\mathfrak{m}(x, t_0)$ is constant on $\Sigma_{\infty,t_0} \backslash \mathcal{S}_{t_0}$.

Step 2. For each $t \in (-1, 1)$, $\mathfrak{m}(x, t)$ is constant on $\Sigma_{\infty,t}$. Fix $t_0 \in (-1, 1)$. It suffices to consider a singular point $p_0 \in \mathcal{S}_{t_0}$. Suppose that $B_r(p_0) \cap \Sigma_{\infty,t_0}$ has no other singular points except p_0 for any $r \in (0, r_0)$. Then all points in $(B_r(p_0) \backslash B_\epsilon(p_0)) \cap \Sigma_{\infty,t_0}$ are regular and $(B_r(p_0) \backslash B_\epsilon(p_0)) \cap \Sigma_{i,t_0}$ has m connected components. Thus, we have

$$\begin{aligned}
\mathrm{Area}_{g_i(t_0)}(\Sigma_{i,t_0} \cap B_r(p_0)) &\leq \mathrm{Area}_{g_i(t_0)}(\Sigma_{i,t_0} \cap (B_r(p_0) \backslash B_\epsilon(p_0))) \\
&\quad + \mathrm{Area}_{g_i(t_0)}(\Sigma_{i,t_0} \cap B_\epsilon(p_0)) \\
&\leq \mathrm{Area}_{g_i(t_0)}(\Sigma_{i,t_0} \cap (B_r(p_0) \backslash B_\epsilon(p_0))) + N\epsilon^2
\end{aligned} \tag{2.19}$$

and used (2.10) in the last inequality. Since each component of $\Sigma_{i,t_0} \cap (B_r(p_0) \backslash B_\epsilon(p_0))$ converges to $(B_r(p_0) \backslash B_\epsilon(p_0)) \cap \Sigma_\infty$ smoothly, we have

$$\begin{aligned}
&\lim_{i \to +\infty} \mathrm{Area}_{g_i(t_0)}(\Sigma_{i,t_0} \cap (B_r(p_0) \backslash B_\epsilon(p_0))) \\
&\quad = m\, \mathrm{Area}_{g_\infty(t_0)}(\Sigma_{\infty,t_0} \cap (B_r(p_0) \backslash B_\epsilon(p_0))).
\end{aligned} \tag{2.20}$$

Note that m is also the multiplicity at each regular point in Σ_{∞,t_0} by Step 1. Combining estimates (2.19) with (2.20), we have

$$\begin{aligned} &m\,\mathrm{Area}_{g_\infty(t_0)}(\Sigma_{\infty,t_0}\cap(B_r(p_0)\backslash B_\epsilon(p_0)))\\ &\quad\le \lim_{i\to+\infty}\mathrm{Area}_{g_i(t_0)}(\Sigma_{i,t_0}\cap B_r(p_0))\\ &\quad\le m\,\mathrm{Area}_{g_\infty(t_0)}(\Sigma_{\infty,t_0}\cap(B_r(p_0)\backslash B_\epsilon(p_0)))+N\epsilon^2. \end{aligned}\tag{2.21}$$

Taking $\epsilon\to 0$ in (2.21), we have

$$\lim_{i\to+\infty}\mathrm{Area}_{g_i(t_0)}(\Sigma_{i,t_0}\cap B_r(p_0))=m\,\mathrm{Area}_{g_\infty(t_0)}(\Sigma_{\infty,t_0}\cap B_r(p_0)).$$

Thus, we have

$$\begin{aligned} \mathfrak{m}(p_0,t_0)&=\lim_{r\to 0}\frac{\mathrm{Area}_{g_i(t_0)}(\Sigma_{i,t_0}\cap B_r(p_0))}{\pi r^2}\\ &=m\lim_{r\to 0}\frac{\mathrm{Area}_{g_\infty(t_0)}(\Sigma_{\infty,t_0}\cap B_r(p_0))}{\pi r^2}=m. \end{aligned}$$

This implies that the multiplicity of each singular point is the same as that of any regular point.

Step 3. The function $\mathfrak{m}(x,t)$ is constant in t. Fix any $t_0\in(-1,1)$, radius $R>0$ and $x_0\in(\Sigma_{\infty,t_0}\cap B_R(0))\backslash\mathcal{S}_{t_0}$. There exists $r_0>0$ such that for large i,

$$|A|(x,t_0)\le\frac{1}{r_0}\quad\text{for all } x\in B_{r_0}(x_0)\cap\Sigma_{\infty,t_0},\tag{2.22}$$

and for any radius $r\in(0,r_0)$ the surface $B_r(x_0)\cap\Sigma_{\infty,t_0}$ has only one component. Let $m_0=\mathfrak{m}(x_0,t_0)$ and $r_1=\frac{1}{4}\min\{r_0,d(x_0,\mathcal{S}_{t_0})\}>0$. For large i, $B_{r_1}(x_0)\cap\Sigma_{i,t_0}$ has m_0 connected components, which we denote by $\Omega_{i,1},\dots,\Omega_{i,m_0}$. Since for each integer $k\in[1,m_0]$ the component $\Omega_{i,k}$ converges smoothly to $\Sigma_{\infty,t_0}\cap B_{r_1}(x_0)$ as $i\to+\infty$, similar to Step 1, we can find $x_{i,k}\in\Omega_{i,k}$ such that $\lim_{i\to+\infty}d(x_{i,k},x_0)=0$. By the choice of r_1, we have

$$B_{r_1}(x_{i,k})\subset B_{r_0}(x_0).\tag{2.23}$$

Thus, (2.22) implies that for any integer $k\in[1,m_0]$ and large i,

$$|A|(x,t_0)\le\frac{1}{r_1}\quad\text{for all } x\in C_{x_{i,k}}(B_{r_1}(x_{i,k})\cap\Sigma_{i,t_0}).\tag{2.24}$$

By Lemma 2.3, for any $\delta>0$ there exists $\rho_0=\rho_0(r_1,\delta)\in(0,\frac{r_1}{2})$ such that for any $r\in(0,\rho_0)$ we have

$$1-\delta\le\frac{\mathrm{Area}_{g_{i,t_0}}(C_{x_{i,k}}(B_r(x_{i,k})\cap\Sigma_{i,t_0}))}{\pi r^2}\le 1+\delta.\tag{2.25}$$

Note that by (2.23) and the definition of $\Omega_{i,k}$, for any large i we have

$$C_{x_{i,k}}(B_{2\rho_0}(x_{i,k})\cap\Sigma_{i,t_0})\ne C_{x_{i,k'}}(B_{2\rho_0}(x_{i,k'})\cap\Sigma_{i,t_0})\quad\text{for all } k\ne k'.\tag{2.26}$$

Using (2.24) and the assumption that $\max_{\Sigma_{i,t}}|H|\le\Lambda$ by (2.8), Theorem 2.4 implies that there exists $\eta(\Lambda)$ and $\epsilon(\Lambda) > 0$ such that

$$|A|(x,t) \le \frac{1}{\epsilon r_1} \quad \text{for all } x \in C_{x_{i,k,t}}(B_{\frac{1}{16}r_1}(x_{i,k,t}) \cap \Sigma_{i,t}),\ t \in [t_0 - \eta r_1^2, t_0 + \eta r_1^2],$$

where $x_{i,k,t} = \mathbf{x}_t(\mathbf{x}_{t_0}^{-1}(x_{i,k}))$. Similar to (2.25), there exists $\rho_1 = \rho_1(r_1, \delta) \in (0, \frac{\rho_0}{10})$ such that for any $r \in (0, \rho_1)$ we have

$$1 - \delta \le \frac{\mathrm{Area}_{g_{i,t}}(C_{x_{i,k,t}}(B_r(x_{i,k,t}) \cap \Sigma_{i,t}))}{\pi r^2} \le 1 + \delta, \quad t \in [t_0 - \eta r_1^2, t_0 + \eta r_1^2]. \quad (2.27)$$

We show that we can choose ρ_1 and $\tau = \tau(r_0, \delta, \Lambda) \in (0, \eta r_1^2]$ small such that for any $k \ne k'$,

$$C_{x_{i,k,t}}(B_{\rho_1}(x_{i,k,t}) \cap \Sigma_{i,t}) \ne C_{x_{i,k',t}}(B_{\rho_1}(x_{i,k',t}) \cap \Sigma_{i,t}), \quad t \in [t_0 - \tau, t_0 + \tau]. \quad (2.28)$$

Suppose not, we can find $\tau_0 \in (0, \eta r_1^2]$, a continuous curve $\gamma_{\tau_0}(s)$ ($s \in [0,1]$) connecting $x_{i,k,t_0+\tau_0}$ and $x_{i,k',t_0+\tau_0}$ with

$$\gamma_{\tau_0} \subset B_{\rho_1}(x_{i,k,t_0+\tau_0}) \cap \Sigma_{i,t_0+\tau_0}, \quad \gamma_{\tau_0} \subset B_{\rho_1}(x_{i,k',t_0+\tau_0}) \cap \Sigma_{i,t_0+\tau_0}. \quad (2.29)$$

Let $\gamma_\tau = \mathbf{x}_{t_0+\tau}(\mathbf{x}_{t_0+\tau_0}^{-1}(\gamma_{\tau_0}))$. Then $\gamma_0(s)(s \in [0,1])$ is a curve connecting $x_{i,k}$ and $x_{i,k'}$. Since the mean curvature satisfies $\max_{\Sigma_{i,t}} |H| \le \Lambda$, we have

$$|\mathbf{x}(p,t) - \mathbf{x}(q,t)| \le |\mathbf{x}(p,s) - \mathbf{x}(q,s)| + 2\Lambda|t - s|. \quad (2.30)$$

For small τ_0, (2.30) with (2.29) implies that

$$\gamma_0 \subset B_{\rho_0}(x_{i,k}) \cap \Sigma_{i,t_0}, \quad \gamma_{\tau_0} \subset B_{\rho_0}(x_{i,k}) \cap \Sigma_{i,t_0},$$

which contradicts (2.26). Therefore, (2.28) holds.

Since $x_{i,k} \in B_{r_1}(x_0)$ and the mean curvature is uniformly bounded, it follows that the point $x_{i,k,t}$ lies in a bounded domain for any $t \in [t_0 - \tau, t_0 + \tau]$. Thus, for each integer $k \in [1, m_0]$ and any $t \in [t_0 - \tau, t_0 + \tau]$ a subsequence of $C_{x_{i,k,t}}(B_{\rho_1}(x_{i,k,t}) \cap \Sigma_{i,t})$ converges to $C_{x_t}(B_{\rho_1}(x_t) \cap \Sigma_{\infty,t})$ smoothly, where $x_t \in \Sigma_{\infty,t}$ is a limit point of $\{x_{i,k,t}\}_{i=1}^{\infty}$. Then (2.27) and (2.28) imply that for any $r \in (0, \rho_1)$ and $t \in [t_0 - \tau, t_0 + \tau]$ we have

$$\begin{aligned}\lim_{i \to +\infty} \frac{\mathrm{Area}_{g_i(t)}(B_r(x_{i,t}) \cap \Sigma_{i,t})}{\pi r^2} &\ge \lim_{i \to +\infty} \frac{\mathrm{Area}_{g_i(t)}(C_{x_{i,t}}(B_r(x_{i,t}) \cap \Sigma_{i,t}))}{\pi r^2} \\ &\ge m_0(1 - \delta).\end{aligned}$$

Thus, we have

$$\mathfrak{m}(x_t, t) \ge m_0 = \mathfrak{m}(x_0, t_0) \quad \text{for all } t \in [t_0 - \tau, t_0 + \tau]. \quad (2.31)$$

By Step 2, (2.31) implies that for any $x \in \Sigma_{\infty,t}$ and $y \in \Sigma_{\infty,t_0}$ we have

$$\mathfrak{m}(x,t) \ge \mathfrak{m}(y, t_0) \quad \text{for all } t \in [t_0 - \tau, t_0 + \tau].$$

Thus, the multiplicity $\mathfrak{m}(x,t)$ is a constant independent of x and t. The lemma is thus proved. ∎

To characterize the singular and regular points in $\Sigma_{\infty,t}$, we have the following result.

Lemma 2.10. *The same assumption as in Proposition 2.8. Fix any $t_0 \in (-1,1)$ and any $\delta, R > 0$.*

(1) *If $x_0 \in (\Sigma_{\infty,t_0} \cap B_R(0)) \backslash \mathcal{S}_{t_0}$ and $x_i \in \Sigma_{i,t_0}$ with $x_i \to x_0$, there exists a positive number $r' = r'(\delta, \Sigma_{\infty,t_0}, R, x_0, \mathcal{S}_{t_0})$ such that for any $r \in (0, r')$ we have*

$$1 - \delta \leq \lim_{i \to +\infty} \frac{\mathrm{Area}_{g_i(t_0)}(C_{x_i}(B_r(x_i) \cap \Sigma_{i,t_0}))}{\pi r^2} \leq 1 + \delta. \tag{2.32}$$

(2) *If $x_0 \in \mathcal{S}_{t_0} \cap B_R(0)$, there exist $r' = r'(\delta, \Sigma_{\infty,t_0}, R, \Lambda, x_0, \mathcal{S}_{t_0}) > 0$ and a sequence $x_i \in \Sigma_{i,t_0}$ with $x_i \to x_0$ such that for any $r \in (0, r')$ we have*

$$\lim_{i \to +\infty} \frac{\mathrm{Area}_{g_i(t_0)}(C_{x_i}(B_r(x_i) \cap \Sigma_{i,t_0}))}{\pi r^2} \geq 2(1 - \delta).$$

Proof. (1) Since Σ_{∞,t_0} is a smooth self-shrinker, there exists $r_0 = r_0(\Sigma_{\infty,t_0}, R) > 0$ such that for large i we have

$$|A|(x, t_0) \leq \frac{1}{r_0} \quad \text{for all } x \in B_{r_0}(x_0) \cap \Sigma_{\infty,t_0}.$$

Since $\mathcal{S}_{t_0}$ is locally finite and $x_0 \in (\Sigma_{\infty,t_0} \cap B_R(0)) \backslash \mathcal{S}_{t_0}$, the distance from x_0 to $\mathcal{S}_{t_0}$ satisfies $d(x_0, \mathcal{S}_{t_0}) > 0$. Let $r_1 = \frac{1}{2} \min\{r_0, d(x_0, \mathcal{S}_{t_0})\}$. Then for large i, we have

$$|A|(x, t_0) \leq \frac{1}{r_1} \quad \text{for all } x \in B_{r_1}(x_i) \cap \Sigma_{i,t_0}.$$

By Lemma 2.3, for any $\delta > 0$ there exists $r' = \rho(\delta, r_1) > 0$ such that for any $r \in (0, r')$ the area ratio of $C_{x_i}(B_r(x_i) \cap \Sigma_{i,t_0})$ is given by

$$1 - \delta \leq \frac{\mathrm{Area}_{g_{i,t_0}}(C_{x_i}(B_r(x_i) \cap \Sigma_{i,t_0}))}{\pi r^2} \leq 1 + \delta.$$

Thus, (2.32) holds.

(2) Let $x_0 \in \mathcal{S}_{t_0} \cap B_R(0)$ and $r_0 = r_0(\Sigma_{\infty,t_0}, R, x_0, \mathcal{S}_{t_0}) > 0$ such that the surface $\Sigma_{\infty,t_0} \cap B_{2r_0}(x_0)$ has only one component and no other singular points except x_0. Let

$$Q_i := \max_{B_{r_0}(x_0) \cap \Sigma_{i,t_0}} |A| \to +\infty.$$

Then Q_i is achieved by some point $x_i \in \overline{B_{r_0}(x_0)} \cap \Sigma_{i,t_0}$ with $x_i \to x_0$. As in Step 2 of the proof of Lemma 2.9, for any $r \in (0, r_0)$ we have

$$\lim_{i \to +\infty} \frac{\mathrm{Area}_{g_i(t_0)}(C_{x_i}(B_r(x_i) \cap \Sigma_{i,t_0}))}{\pi r^2} = m \frac{\mathrm{Area}_{g_\infty(t_0)}(B_r(x_0) \cap \Sigma_{\infty,t_0})}{\pi r^2}, \tag{2.33}$$

where m is a positive integer. Note that Lemma 2.3 implies that for any $\delta > 0$ there exists $r_0' = r_0'(\delta, \Sigma_{\infty,t_0}, R, x_0, \mathcal{S}_{t_0}) \in (0, r_0)$ such that

$$\frac{\mathrm{Area}_{g_\infty(t_0)}(B_r(x_0) \cap \Sigma_{\infty,t_0})}{\pi r^2} \leq 1 + \delta \quad \text{for all } r \in (0, r_0']. \tag{2.34}$$

Assume that $m = 1$. Then (2.34) and (2.33) imply that for large i,

$$\frac{\mathrm{Area}_{g_i(t_0)}(C_{x_i}(B_{r'}(x_i) \cap \Sigma_{i,t_0}))}{\pi r'^2} \le 1 + 2\delta \quad \text{for all } r \in (0, r'_0]. \tag{2.35}$$

We choose $r' = r'(\delta, \Sigma_{\infty,t_0}, R, \Lambda, x_0, \mathcal{S}_{t_0}) \in (0, r'_0)$ small such that

$$(1 + 2\delta)e^{\Lambda r'} \le 1 + 3\delta.$$

Since the mean curvature satisfies $\max_{\Sigma_{i,t}\times(-1,1)} |H| \le \Lambda$, by [46, Lemma 3.5] for any $r \in (0, r')$ we have

$$\begin{aligned} \frac{\mathrm{Area}_{g_i(t_0)}(C_{x_i}(B_r(x_i) \cap \Sigma_{i,t_0}))}{\pi r^2} &\le e^{\Lambda r'} \frac{\mathrm{Area}_{g_i(t_0)}(C_{x_i}(B_{r'}(x_i) \cap \Sigma_{i,t_0}))}{\pi r'^2} \\ &\le 1 + 3\delta, \end{aligned} \tag{2.36}$$

where we used (2.35). We rescale the surface by

$$\tilde{\Sigma}_{i,s} = Q_i(\Sigma_{i,t_0+Q_i^{-2}s} - x_i) \quad \text{for all } s \in (-(1+t_0)Q_i^2, (1-t_0)Q_i^2).$$

Then $\{\tilde{\Sigma}_{i,s}, -1 < s < 1\}$ is a sequence of mean curvature flow with

$$\max_{\tilde{\Sigma}_{i,s}\times(-1,1)} |H| \le Q_i^{-1}\Lambda \to 0.$$

By the choice of Q_i we have

$$\sup_{C_0(\tilde{\Sigma}_{i,0}\cap B_{\frac{1}{2}Q_i r_0}(0))} |A| \le 1.$$

By [46, Theorem 3.8], there exists a universal constant ϵ such that

$$\sup_{C_0(\tilde{\Sigma}_{i,s}\cap B_{\frac{1}{4}Q_i r_0}(0))} |A| \le \frac{1}{\epsilon} \quad \text{for all } s \in (-1, 1).$$

Thus, by the compactness of mean curvature flow (cf. [46, Theorem 2.6]) the surface $C_0(\tilde{\Sigma}_{i,0} \cap B_{Q_i r_0/2}(0))$ converges in smooth topology to a complete smooth minimal surface $\tilde{\Sigma}_\infty$ with

$$\sup_{\tilde{\Sigma}_\infty} |A| \le 1, \quad |A|(0) = 1. \tag{2.37}$$

Since (2.36) implies that

$$\frac{\mathrm{Area}_{\tilde{g}_i(0)}(C_0(B_r(0) \cap \tilde{\Sigma}_{i,0}))}{\pi r^2} \le 1 + 3\delta \quad \text{for all } r \in (0, Q_i r'),$$

we have

$$\frac{\mathrm{Area}_{\tilde{g}_\infty(0)}(C_0(B_r(0) \cap \tilde{\Sigma}_\infty))}{\pi r^2} \le 1 + 3\delta \quad \text{for all } r > 0.$$

By [46, Lemma 3.6], there exists a universal constant $\delta_0 > 0$ such that if we choose $\delta = \frac{\delta_0}{4}$, then $\tilde{\Sigma}_\infty$ must be a plane, which contradicts (2.37).

Therefore, $m \geq 2$ in (2.33) for any $r \in (0, r')$, where $r' = r'(\delta, \Sigma_{\infty,t_0}, R, \Lambda, x_0, \mathcal{S}_{t_0})$. By Lemma 2.3 we can find $r_1 = r_1(\delta, \Sigma_{\infty,t_0}, R, \Lambda, x_0, \mathcal{S}_{t_0}) \in (0, r')$ such that for any $r \in (0, r_1)$,

$$\frac{\mathrm{Area}_{g_\infty(t_0)}(B_r(x_0) \cap \Sigma_{\infty,t_0})}{\pi r^2} \geq 1 - \delta.$$

Since $m \geq 2$, for any $r \in (0, r_1)$ we have

$$\lim_{i\to+\infty} \frac{\mathrm{Area}_{g_i(t_0)}(C_{x_i}(B_r(x_i) \cap \Sigma_{i,t_0}))}{\pi r^2} = m \frac{\mathrm{Area}_{g_\infty(t_0)}(B_r(x_0) \cap \Sigma_{\infty,t_0})}{\pi r^2} \geq 2(1-\delta).$$

The lemma is proved. ∎

Using the boundedness of the mean curvature and Lemma 2.10, we show that the singular set $\mathcal{S}$ consists of locally finitely many Lipschitz curves.

Lemma 2.11. *Fix large $R > 0$. Under the assumption of Proposition 2.8, the singular set $\mathcal{S}$ is the union of locally finitely many space-time singular curves, i.e.,*

$$\mathcal{S} \cap (B_R(0) \times (-1,1)) = \bigcup_{k=1}^{l} \{(\xi_k(t), t) \mid t \in (-1,1),\ \xi_k(t) \in B_R(0) \cap \mathcal{S}_t\},$$

where $\mathcal{S}_t$ is defined in Proposition 2.8 and $\{\xi_k(t)\}_{k=1}^l$ are Λ'-Lipschitz curves, i.e.,

$$|\xi_k(t_1) - \xi_k(t_2)| \leq \Lambda'|t_1 - t_2| \quad \textit{for all } t_1, t_2 \in (-1,1).$$

Here Λ' depends only on the constant Λ in (2.8).

Proof. For any $t_1 \in (-1,1)$ and any point $p_{t_1} \in \mathcal{S}_{t_1} \cap B_R(0)$, we show that there exists a Lipschitz curve in $\mathcal{S}$ passing through p_{t_1}. Since p_{t_1} is singular, by Lemma 2.10 we can find a sequence of points $p_{i,t_1} \in \Sigma_{i,t_1}$ and $r' = r'(\Sigma_{\infty,t_1}, R, \Lambda, p_{t_1}, \mathcal{S}_{t_1}) > 0$ such that $p_{i,t_1} \to p_{t_1}$ and for any $r \in (0, r')$,

$$\lim_{i\to+\infty} \frac{\mathrm{Area}(C_{p_{i,t_1}}(B_r(p_{i,t_1}) \cap \Sigma_{i,t_1}))}{\pi r^2} \geq \frac{7}{4}. \tag{2.38}$$

We choose $\eta_0 > 0$ and $M_0 = 200$ such that

$$e^{-2\Lambda^2\eta_0}\left(1 + \frac{2}{M_0}\right)^{-2} \geq \frac{6}{7}, \quad M_0\Lambda\eta_0 < r'. \tag{2.39}$$

Let $t_2 \in (t_1 - \eta_0, t_1 + \eta_0) \cap (-1,1)$ and $r_1 = M_0\Lambda|t_2 - t_1|$. Then $r_1 < M_0\Lambda\eta_0 < r'$. By [46, Lemma 3.4] we have

$$\begin{aligned}
&\frac{\mathrm{Area}(C_{p_{i,t_2}}(B_{r_2}(p_{i,t_2}) \cap \Sigma_{i,t_2}))}{\pi {r_2}^2} \\
&\geq e^{-\Lambda^2|t_2-t_1|}\left(1 + \frac{2\Lambda}{r_1}|t_2 - t_1|\right)^{-2} \frac{\mathrm{Area}(C_{p_{i,t_1}}(B_{r_1}(p_{i,t_1}) \cap \Sigma_{i,t_1}))}{\pi r_1^2} \\
&\geq e^{-\Lambda^2\eta_0}\left(1 + \frac{2}{M_0}\right)^{-2} \frac{\mathrm{Area}(C_{p_{i,t_1}}(B_{r_1}(p_{i,t_1}) \cap \Sigma_{i,t_1}))}{\pi r_1^2},
\end{aligned} \tag{2.40}$$

where $r_2 = r_1 + 2\Lambda|t_2 - t_1|$. Combining (2.38)–(2.40), we have

$$\lim_{i\to+\infty} \frac{\mathrm{Area}(C_{p_{i,t_2}}(B_{r_2}(p_{i,t_2}) \cap \Sigma_{i,t_2}))}{\pi {r_2}^2} \geq \frac{3}{2}, \tag{2.41}$$

where $p_{i,t_2} = \mathbf{x}_{i,t_2}(\mathbf{x}_{i,t_1}^{-1}(p_{i,t_1}))$. Since the mean curvature is uniformly bounded along the flow, all points $\{p_{i,t_2}\}_{i=1}^{\infty}$ lie in a bounded ball centered at p_{t_1}. Thus, we can find a subsequence of $\{p_{i,t_2}\}_{i=1}^{\infty}$ such that it converges to a point, which we denoted by p_{t_2}. Since $C_{p_{i,t_2}}(B_{r_2}(p_{i,t_2}) \cap \Sigma_{i,t_2})$ converges locally smoothly to $B_{r_2}(p_{t_2}) \cap \Sigma_{\infty,t_2}$ away from singularities, by Step 2 of Lemma 2.9 we have

$$\lim_{i\to+\infty} \frac{\mathrm{Area}(C_{p_{i,t_2}}(B_{r_2}(p_{i,t_2}) \cap \Sigma_{i,t_2}))}{\pi {r_2}^2} = m\frac{\mathrm{Area}(B_{r_2}(p_{t_2}) \cap \Sigma_{\infty,t_2})}{\pi {r_2}^2}, \tag{2.42}$$

where $m \in \mathbb{N}$. Note that $B_R(0) \cap \Sigma_{\infty,t}$ has bounded geometry for any $t \in (-1, 1)$, we can find a uniform $r_2' > 0$ such that for any $(p, t) \in (\Sigma_{\infty,t} \cap B_R(0)) \times (-1, 1)$ and any $r \in (0, r_2')$,

$$\frac{\mathrm{Area}(B_r(p) \cap \Sigma_{\infty,t})}{\pi r^2} \leq \frac{5}{4}. \tag{2.43}$$

Moreover, we can choose η_0 small such that

$$r_2 = r_1 + 2\Lambda|t_2 - t_1| = (M_0 + 2)\Lambda|t_2 - t_1| \leq (M_0 + 2)\Lambda\eta_0 < r_2'. \tag{2.44}$$

Combining (2.41)–(2.44), we have $m \geq 2$ in (2.42). Thus, $C_{p_{i,t_2}}(B_{r_2}(p_{i,t_2}) \cap \Sigma_{i,t_2})$ converges locally smoothly to $B_{r_2}(p_{t_2}) \cap \Sigma_{\infty,t_2}$ with multiplicity $m \geq 2$. This implies that $B_{r_2}(p_{t_2}) \cap \Sigma_{\infty,t_2}$ contains a singular point, which we denoted by q_{t_2}. Here we used the fact that if $B_{r_2}(p_{t_2}) \cap \Sigma_{\infty,t_2}$ contains no singular points, then $C_{p_{i,t_2}}(B_{r_2}(p_{i,t_2}) \cap \Sigma_{i,t_2})$ will converge smoothly to $B_{r_2}(p_{t_2}) \cap \Sigma_{\infty,t_2}$ with multiplicity one.

Note that

$$|p_{i,t_1} - p_{i,t_2}| \leq \int_{t_1}^{t_2} |H|\, dt \leq \Lambda|t_1 - t_2|.$$

Taking the limit $i \to +\infty$ we have

$$|p_{t_1} - p_{t_2}| \leq \Lambda|t_1 - t_2|.$$

Thus, for any $t_2 \in (t_1 - \eta_0, t_1 + \eta_0) \cap (-1, 1)$ we have

$$|q_{t_2} - p_{t_1}| \leq |q_{t_2} - p_{t_2}| + |p_{t_2} - p_{t_1}| \leq r_2 + \Lambda|t_1 - t_2| \leq (M_0 + 3)\Lambda|t_1 - t_2|.$$

Therefore, p_{t_1} lies in a Λ'-Lipschitz curve in $\mathcal{S}$ with $\Lambda' = 203\Lambda$. Since for any $t \in (-1, 1)$ the set $\mathcal{S}_t$ is locally finite by Proposition 2.8, the singular curves are locally finite. The lemma is proved. ■

3. The rescaled mean curvature flow

In this section, we will show the smooth convergence of rescaled mean curvature flow under uniform mean curvature bound. As is pointed out in the introduction, we have no long-time pseudolocality of the flow and the singularities do not move along straight lines. When the multiplicity of the convergence is greater than one, in order to show the

L-stability of the limit self-shrinker we need an estimate on the asymptotical behavior of the positive solution near the singular set (cf. Lemma 3.21 and Lemma 3.28), and the proof of this estimate will be delayed to Section 4.

Theorem 3.1. *Let* $\{(\Sigma^2, \mathbf{x}(t)), 0 \le t < +\infty\}$ *be a closed smooth embedded rescaled mean curvature flow*

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)^{\perp} = -\left(H - \frac{1}{2}\langle \mathbf{x}, \mathbf{n}\rangle\right)\mathbf{n} \tag{3.1}$$

satisfying

$$d(\Sigma_t, 0) \le D \quad \text{and} \quad \max_{\Sigma_t} |H(p,t)| \le \Lambda \tag{3.2}$$

for two constants $D, \Lambda > 0$. *Then for any sequence* $t_i \to +\infty$ *there exists a subsequence of* $\{\Sigma_{t_i+t}, -1 < t < 1\}$ *such that it converges in smooth topology to a complete smooth self-shrinker with multiplicity one as* $i \to +\infty$.

We sketch the proof of Theorem 3.1. First, we show the weak compactness for any sequence of the rescaled mean curvature flow in Lemma 3.4. Suppose that the multiplicity is at least two. By using the decomposition of spaces (cf. Definition 3.5) we can select a special sequence $\{t_i\}$ in Lemma 3.13 for each $\epsilon > 0$. This special sequence is needed to control the upper bound of the function w_i away from the singular set by using the parabolic Harnack inequality (cf. Lemma 3.16). Then we can take the limit for the function w_i and obtain a positive function w with uniform bounds on any compact set away from the singular set (cf. Lemma 3.17). The function w satisfies the linearized mean curvature flow equation. To study the growth behavior of w near the singular set, we take a sequence of $\epsilon_i \to 0$ and for each ϵ_i we repeat the above process to get a sequence of functions $\{w_{i,k}\}_{k=1}^{\infty}$. After choosing a diagonal sequence and taking the limit, we get a function w with good growth estimates near the singular set (cf. Proposition 3.23) by assuming Theorem 4.2 in the next section. The bounds of w imply the L-stability of the limit self-shrinker (cf. Lemma 3.25), and this step also relies on Theorem 4.2. However, the limit self-shrinker is not L-stable by Colding–Minicozzi's theorem (cf. Theorem 3.7) and we obtain a contradiction.

3.1. Convergence away from singularities

We recall Ilmanen's local Gauss–Bonnet formula in [40] to control the L^2 norm of the second fundamental form. Let Σ be a smooth surface with smooth boundary $\partial\Sigma$. We denote by $e(\Sigma)$ the genus of Σ which is the genus of the closed surface obtained by capping off the boundary components of Σ by disks.

Lemma 3.2 (cf. Ilmanen [40]). *Let* $R > 1$ *and let* Σ *be a surface properly immersed in* $B_R(p)$. *Then for any* $\epsilon > 0$ *we have*

$$(1-\epsilon)\int_{\Sigma \cap B_1(p)} |A|^2 \, d\mu \le \int_{\Sigma \cap B_R} |H|^2 \, d\mu + 8\pi e(\Sigma \cap B_R(p)) + \frac{24\pi R^2}{\epsilon (R-1)^2} \sup_{r \in [1,R]} \frac{\mathrm{Area}(\Sigma \cap B_r(p))}{\pi r^2}.$$

For simplicity, we introduce the following definition.

Definition 3.3. Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing positive function. For any $N > 0$, we denote by $\mathcal{C}(N, \rho)$ the space of all smooth embedded self-shrinkers $\Sigma^2 \subset \mathbb{R}^3$ satisfying the properties that for any $r > 0$ and $p \in \Sigma$,

$$\int_{\Sigma \cap B_r(0)} |A|^2 \le \rho(r) \quad \text{and} \quad \text{Area}(B_r(p) \cap \Sigma) \le \pi N r^2.$$

We note that the space $\mathcal{C}(N, \rho)$ is compact in the smooth topology by Colding and Minicozzi [20], and the distance from the origin to any self-shrinker in $\mathbb{R}^3$ is at most 2 by avoidance principle (cf. [25, Corollary 3.6]). The total curvature bound in Definition 3.3 can also be derived from genus bound by exploiting Lemma 3.2.

The following result shows that the rescaled mean curvature flow converges locally smoothly to a self-shrinker away from singularities.

Lemma 3.4. *Under the assumption of Theorem 3.1, for any sequence $t_i \to +\infty$, there is a smooth self-shrinker $\Sigma_\infty \in \mathcal{C}(N, \rho)$ and a space-time set $\mathcal{S} \subset \Sigma_\infty \times \mathbb{R}$ satisfying the following properties:*

1. *For any $T > 1$, there is a subsequence, still denoted by $\{t_i\}$, such that the sequence $\{\Sigma_{t_i+t}, -T < t < T\}$ converges in smooth topology, possibly with multiplicities, to Σ_∞ away from $\mathcal{S}$.*
2. *For any $R > 0$, $\mathcal{S} \cap (B_R(0) \times (-T, T))$ consists of finite many σ-Lipschitz curves with Lipschitz constant σ depending only on Λ, T and R.*
3. *The convergence in part (1) is also in (extrinsic) Hausdorff distance.*
4. *The limit self-shrinker Σ_∞ is independent of the choice of T. In other words, for different T we can choose two different subsequences of $\{t_i\}$ such that the corresponding flows in part (1) have the same limit self-shrinker Σ_∞.*

Proof. We divide the proof into the following steps.

Step 1. The area ratio along the flow (3.1) is uniformly bounded from above. In fact, we rescale the flow (3.1) by

$$s = 1 - e^{-t}, \quad \hat{\Sigma}_s = \sqrt{1-s}\, \Sigma_{-\log(1-s)}$$

such that $\{\hat{\Sigma}_s, 0 \le s < 1\}$ is a mean curvature flow satisfying equation (1.1). By [19, Lemma 2.9] and [46, Lemma 2.3], we have that the area ratio of (3.1) is uniformly bounded from above.

Step 2. For any large radius R, the energy of $\Sigma_t \cap B_R(0)$ is uniformly bounded along the flow (3.1). In fact, by Lemma 3.2 we have

$$\begin{aligned}\int_{\Sigma_t \cap B_R(0)} |A|^2 \, d\mu_t &\le 2 \int_{\Sigma_t \cap B_{2R}(0)} |H|^2 \, d\mu_t + C(N, e(\Sigma)) \\ &\le 8\pi N \Lambda^2 R^2 + C(N, e(\Sigma)),\end{aligned} \tag{3.3}$$

where N denotes the upper bound of the area ratio. Therefore, for any $t > 0$ the energy of $\Sigma_t \cap B_R(0)$ is bounded by a constant $C(N, \Lambda, R, e(\Sigma))$.

Step 3. For each sequence $t_i \to +\infty$, we obtain a refined sequence converging to a limit self-shrinker. For any sequence $t_i \to +\infty$, we rescale the flow Σ_t by

$$s = 1 - e^{-(t-t_i)}, \quad \tilde{\Sigma}_{i,s} = \sqrt{1-s}\ \Sigma_{t_i - \log(1-s)} \tag{3.4}$$

such that for each i the flow $\{\tilde{\Sigma}_{i,s}, 1 - e^{t_i} \le s < 1\}$ is a mean curvature flow satisfying (1.1) with the following properties:

(a) For any small $\lambda > 0$, the mean curvature of $\tilde{\Sigma}_{i,s}$ satisfies

$$\max_{\tilde{\Sigma}_{i,s} \times [1-e^{t_i}, 1-\lambda]} |\tilde{H}_i|(p,s) \le \tilde{\Lambda} := \frac{\Lambda}{\sqrt{\lambda}}.$$

(b) For any large R, the energy of $\tilde{\Sigma}_{i,s} \cap B_R(0)$ is uniformly bounded.

(c) The area ratio is uniformly bounded from above.

(d) The area ratio is uniformly bounded from below.

(e) There exists a constant $D' > 0$ such that $d(\tilde{\Sigma}_{i,s}, 0) \le D'$ for any i.

(f) We have

$$\lim_{i \to +\infty} \int_{-T}^{1-\lambda} dt \int_{\tilde{\Sigma}_{i,s}} e^{-\frac{|\tilde{\mathbf{x}}_i|^2}{4(1-s)}} \left| \tilde{H}_i - \frac{1}{2(1-s)} \langle \tilde{\mathbf{x}}_i, \mathbf{n} \rangle \right|^2 d\tilde{\mu}_{i,s} = 0. \tag{3.5}$$

In fact, property (a) and (e) follow from the assumption (3.2), and property (b) follows from (3.3). Property (c) follows from Step 1, and property (d) follows from [46, Lemma 3.5]. To prove property (f), by Huisken's monotonicity formula along the rescaled mean curvature flow (3.1) we have

$$\frac{d}{dt} \int_{\Sigma_t} e^{-\frac{|\mathbf{x}|^2}{4}} d\mu_t = -\int_{\Sigma_t} e^{-\frac{|\mathbf{x}|^2}{4}} \left| H - \frac{1}{2} \langle \mathbf{x}, \mathbf{n} \rangle \right|^2 d\mu_t.$$

This implies that

$$\int_0^{\infty} dt \int_{\Sigma_t} e^{-\frac{|\mathbf{x}|^2}{4}} \left| H - \frac{1}{2} \langle \mathbf{x}, \mathbf{n} \rangle \right|^2 d\mu_t < +\infty.$$

Let $T, \lambda > 0$ with $-T < 1 - \lambda$. For any $t_i \to +\infty$, we have

$$\lim_{t_i \to +\infty} \int_{t_i - \log(1+T)}^{t_i - \log \lambda} dt \int_{\Sigma_t} e^{-\frac{|\mathbf{x}|^2}{4}} \left| H - \frac{1}{2} \langle \mathbf{x}, \mathbf{n} \rangle \right|^2 d\mu_t = 0. \tag{3.6}$$

Then (3.5) follows from equations (3.4) and (3.6). Therefore, by Definition 2.7 for any $T > 0$, small $\lambda > 0$ and any $s_0 \in [-T+1, -\lambda]$ the sequence $\{\tilde{\Sigma}_{i, s_0 + \tau}, -1 < \tau < 1\}$ is a refined sequence. By Proposition 2.8 and Lemma 2.11, we have that a subsequence of $\{\tilde{\Sigma}_{i,s}, -T < s < 1 - \lambda\}$ converges in smooth topology, possibly with multiplicities, to a self-shrinker flow $\{\tilde{\Sigma}_{\infty,s}, -T < s < 1 - \lambda\}$ away from a space-time, $\tilde{\Lambda}'$-Lipschitz singular set $\tilde{\mathcal{S}}$ with $\tilde{\Lambda}' = 203\tilde{\Lambda}$.

Step 4. Let $t' = t - t_i$ and $\Sigma_{i,t'} = \Sigma_{t_i + t'}$. Since the sequence $\{\tilde{\Sigma}_{i,s}, -T < s < 1 - \lambda\}$ converges locally smoothly to $\{\tilde{\Sigma}_{\infty,s}, -T < s < 1 - \lambda\}$ away from $\tilde{\mathcal{S}}$, by equation (3.4) the flow $\{\Sigma_{i,t'}, -\log(1+T) < t' < -\log\lambda\}$ also converges locally smoothly to a self-shrinker Σ_∞ satisfying

$$H - \frac{1}{2}\langle \mathbf{x}, \mathbf{n}\rangle = 0$$

away from a space-time singular set $\mathcal{S}$ with

$$\mathcal{S}_{t'} = \frac{1}{\sqrt{1-s}}\tilde{\mathcal{S}}_s.$$

Here $s = 1 - e^{-t'}$. Now we show the Lipschitz property of $\mathcal{S}$. By (3.4), for any curve $\xi(t')$ of $\mathcal{S}$, we can find a curve $\xi(s)$ of $\tilde{\mathcal{S}}$ such that

$$\tilde{\xi}(s) = \sqrt{1-s}\,\xi(t'), \quad t' = -\log(1-s).$$

Since $\tilde{\xi}(s)$ is $\tilde{\Lambda}'$-Lipschitz, we have

$$|\tilde{\xi}(s_1) - \tilde{\xi}(s_2)| \le \tilde{\Lambda}'|s_1 - s_2| \quad \text{for all } s_1, s_2 \in (-T, 1-\lambda),$$

which implies that

$$|e^{-\frac{t_1}{2}}\xi(t_1) - e^{-\frac{t_2}{2}}\xi(t_2)| \le \tilde{\Lambda}'|e^{-\frac{t_1}{2}} - e^{-\frac{t_2}{2}}|.$$

Suppose that $|\xi(t)| \le R$. For any t_1', t_2' with $|t_1' - t_2'| \le 1$ we have

$$\begin{aligned} |\xi(t_1') - \xi(t_2')| &= |\xi(t_1') - e^{\frac{t_1'-t_2'}{2}}\xi(t_2')| + |e^{\frac{t_1'-t_2'}{2}} - 1||\xi(t_2')| \\ &\le \tilde{\Lambda}'|1 - e^{\frac{t_1'-t_2'}{2}}| + |e^{\frac{t_1'-t_2'}{2}} - 1||\xi(t_2')| \\ &\le (\tilde{\Lambda}' + R)|t_1' - t_2'|, \end{aligned} \tag{3.7}$$

where we used the inequality

$$|e^x - 1| \le 2|x| \quad \text{for all } x \in [-1, 1].$$

Note that the Lipschitz constant in (3.7) is given by $\sigma = \tilde{\Lambda}' + R$. Thus, if we consider the convergence of $\{\Sigma_{t_i+t}, -T < t < T\}$ as in part (1), then $\mathcal{S} \cap (B_R(0) \times (-T, T))$ consists of Lipschitz curves with Lipschitz constant σ. The convergence is also in extrinsic Hausdorff distance by Proposition 2.8 and the limit self-shrinker is independent of the choice of T by the argument of [46, Claim 4.3]. The lemma is proved. ∎

3.2. Decomposition of spaces

In this subsection, we follow the argument in [46] to decompose the space and define an almost "monotone decreasing" quantity, which will be used to select time slices such that the limit self-shrinker is L-stable. First, we decompose the space as follows.

Definition 3.5 ([46]). Fix large $R > 0$ and small $\epsilon > 0$.

(1) We define the set $\mathbf{S} = \mathbf{S}(\Sigma_t, \epsilon, R) = \{y \in \Sigma_t \mid |y| < R, |A|(y,t) > \epsilon^{-1}\}$.

(2) The ball $B_R(0)$ can be decomposed into three parts as follows:

- the high curvature part **H**, which is defined by

$$\mathbf{H} = \mathbf{H}(\Sigma_t, \epsilon, R) = \left\{ x \in \mathbb{R}^3 \;\middle|\; |x| < R, d(x, \mathbf{S}) < \frac{\epsilon}{2} \right\},$$

- the thick part **TK**, which is defined by

$$\begin{aligned}\mathbf{TK} &= \mathbf{TK}(\Sigma_t, \epsilon, R)\\ &= \big\{ x \in \mathbb{R}^3 \;\big|\; |x| < R, \text{ there is a continuous curve } \gamma \subset B_R(0)\backslash(\mathbf{H} \cup \Sigma_t)\\ &\qquad \text{connecting } x \text{ and some } y \text{ with } B(y, \epsilon) \subset B_R(0)\backslash(\mathbf{H} \cup \Sigma_t)\big\},\end{aligned}$$

- the thin part **TN**, which is defined by $\mathbf{TN} = \mathbf{TN}(\Sigma_t, \epsilon, R) = B_R(0)\backslash(\mathbf{H} \cup \mathbf{TK})$.

As is pointed out in [46], the high curvature part **H** is the neighborhood of points with large second fundamental form, and the thin part **TN** is the domain between the top and bottom sheets. Moreover, the thick part **TK** is the union of path connected components of the domain "outside" the sheets. The readers are referred to [46] for more explanation on the definition.

As in [20], we define the L-stability of a self-shrinker.

Definition 3.6. For any $R > 0$, a complete smooth self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$ is called L-stable in the ball $B_R(0)$ if for any function $\varphi \in W_0^{1,2}(B_R(0))$, we have

$$\int_\Sigma -\varphi L_\Sigma \varphi \, e^{-\frac{|x|^2}{4}} \geq 0, \tag{3.8}$$

where L_Σ is the operator on Σ defined by

$$L_\Sigma = \Delta - \frac{1}{2}\langle x, \nabla(\cdot)\rangle + |A|^2 + \frac{1}{2}.$$

The subindex Σ in L_Σ will be omitted when it is clear in the context. We say Σ is not L-stable in the ball $B_R(0)$ if (3.8) does not hold for some $\varphi \in W_0^{1,2}(B_R(0))$. We call that Σ is L-stable in $\mathbb{R}^{n+1}$ if Σ is L-stable in the ball $B_R(0)$ of $\mathbb{R}^{n+1}$ for any $R > 0$.

Recall Colding–Minicozzi's result:

Theorem 3.7 (cf. [19,20]). *There are no L-stable smooth complete self-shrinkers without boundary and with polynomial volume growth in $\mathbb{R}^{n+1}$.*

As a corollary of Theorem 3.7, we have the following result.

Lemma 3.8. *Let $N > 0$ and let ρ be an increasing positive function. There exists a positive radius $R_0 = R_0(N, \rho)$ such that any self-shrinker $\Sigma \in \mathcal{C}(N, \rho)$ is not L-stable in the ball $B_{R_0}(0)$.*

Proof. For otherwise, we can find a sequence $R_i \to +\infty$ and self-shrinkers $\Sigma_i \in \mathcal{C}(N, \rho)$ such that Σ_i is L-stable in the ball $B_{R_i}(0)$. By smooth compactness of $\mathcal{C}(N, \rho)$ in [20], a subsequence of $\{\Sigma_i\}$ converges smoothly to a self-shrinker $\Sigma_\infty \in \mathcal{C}(N, \rho)$. By Theorem 3.7, Σ_∞ is not L-stable in a ball $B_{R_0}(0)$ for some $R_0 > 0$. This implies that there exists a smooth function $\varphi_\infty \in C_0^\infty(\Sigma_\infty \cap B_{R_0}(0))$ such that

$$\int_{\Sigma_\infty} -\varphi_\infty L_\Sigma \varphi_\infty e^{-\frac{|x|^2}{4}} < 0. \tag{3.9}$$

Since Σ_i converges smoothly to Σ_∞, we define the map $f_i : \Sigma_\infty \cap B_{R_0+1}(0) \to \Sigma_i$ by

$$f_i(x) = x + u_i(x)\mathbf{n}(x) \quad \text{for all } x \in \Sigma_\infty \cap B_{R_0+1}(0),$$

where $\mathbf{n}(x)$ denotes the normal vector field of Σ_∞ and $u_i(x)$ is the graph function of Σ_i over Σ_∞. Let $\Omega = \Sigma_\infty \cap B_{R_0+1}(0)$ and $\Omega_i = f_i(\Omega) \subset \Sigma_i$. We assume that i is large such that $\Omega_i \subset \Sigma_i \cap B_{R_0+2}(0)$. Note that f_i converges smoothly to the identity map on Ω as $i \to +\infty$ and for large i its inverse map $f_i^{-1} : \Omega_i \to \Omega$ exists and is also smooth. Moreover, f_i^{-1} also converges smoothly to the identity map on Ω as $i \to +\infty$. We define the function $\varphi_i := (f_i^{-1})^*\varphi_\infty \in C_0^\infty(\Omega_i)$ and we can extend φ_i to Σ_i such that φ is zero on $\Sigma_i \backslash \Omega_i$. Then by (3.9) the function $\varphi_i \in C_0^\infty(\Sigma_i)$ satisfies

$$\lim_{i\to+\infty} \int_{\Sigma_i} -\varphi_i L_{\Sigma_i} \varphi_i e^{-\frac{|x|^2}{4}} = \int_{\Sigma_\infty} -\varphi_\infty L_\Sigma \varphi_\infty e^{-\frac{|x|^2}{4}} < 0.$$

Thus, for large i we have

$$\int_{\Sigma_i} -\varphi_i L_{\Sigma_i} \varphi_i e^{-\frac{|x|^2}{4}} < 0. \tag{3.10}$$

Note that $\mathrm{Supp}(\varphi_i) \subset \Omega_i \subset \Sigma_i \cap B_{R_0+2}(0)$ for large i. Thus, inequality (3.10) contradicts our assumption that Σ_i is L-stable in the ball $B_{R_i}(0)$ and $R_i \to +\infty$. The lemma is proved. ∎

Lemma 3.9. *Let $R, N > 0$ and ρ an increasing positive function. For any $\Sigma \in \mathcal{C}(N, \rho)$ and $x \in \Sigma$, we define $r_\Sigma(x)$ the supreme of the radius r such that*

$$B_r(x + r\mathbf{n}(x)) \cap \Sigma = \emptyset, \quad B_r(x - r\mathbf{n}(x)) \cap \Sigma = \emptyset, \tag{3.11}$$

where $\mathbf{n}(x)$ denotes the normal vector of Σ at x. Then there exists $\epsilon_0(R, N, \rho) > 0$ such that for any $\Sigma \in \mathcal{C}(N, \rho)$ and $x \in \Sigma \cap B_R(0)$ we have

$$r_\Sigma(x) \geq \epsilon_0.$$

Proof. We divide the proof into several steps.

Step 1. For otherwise, we can find a sequence of $\Sigma_i \in \mathcal{C}(N, \rho)$ and $x_i \in \Sigma_i \cap B_R(0)$ with $\delta_i := r_{\Sigma_i}(x_i) \to 0$. By the smooth compactness of $\mathcal{C}(N, \rho)$, there is a subsequence of $\{\Sigma_i\}$ converging smoothly to a self-shrinker Σ_∞ in $\mathcal{C}(N, \rho)$. We assume that

$$x_i \to x_\infty \in \Sigma_\infty \cap B_{R+1}(0).$$

By the embeddedness of Σ_∞, we have $\delta := r_{\Sigma_\infty}(x_\infty) > 0$. Since Σ_∞ is smooth and embedded, there exists $r' > 0$ such that $B_{r'}(x_\infty) \cap \Sigma_\infty$ has only one component and

$$\inf_{y \in B_{r'}(x_\infty)\cap\Sigma_\infty} r_{\Sigma_\infty}(y) \geq \frac{\delta}{2}. \tag{3.12}$$

Moreover, we choose r' sufficiently small such that $B_{r'}(x_\infty) \cap \Sigma_\infty$ is almost flat by Lemma 2.2. Let

$$\Omega\left(r', \frac{\delta}{2}\right) := \bigcup_{y \in B_{r'}(x_\infty) \cap \Sigma_\infty} \left(B_{\frac{\delta}{2}}\left(y + \frac{\delta}{2}\mathbf{n}_{\Sigma_\infty}(y)\right) \cup B_{\frac{\delta}{2}}\left(y - \frac{\delta}{2}\mathbf{n}_{\Sigma_\infty}(y)\right)\right).$$

Then (3.12) implies that $\Omega(r', \frac{\delta}{2}) \cap \Sigma_\infty = \emptyset$. By the smooth convergence of Σ_i to Σ_∞, for large i we have

$$\Omega\left(\frac{r'}{2}, \frac{\delta}{4}\right) \cap (\Sigma_i \backslash B_{r'}(x_\infty)) = \emptyset. \tag{3.13}$$

By the construction of $\Omega(r', \frac{\delta}{2})$, we have

$$B_{\frac{\delta}{4}}(y) \subset \Omega\left(\frac{r'}{2}, \frac{\delta}{4}\right) \cup (\Sigma_\infty \cap B_{r'}(x_\infty)) \quad \text{for all } y \in B_{\frac{r'}{4}}(x_\infty) \cap \Sigma_\infty. \tag{3.14}$$

Step 2. Since $x_i \to x_\infty$, we can choose r' sufficiently small such that for all large i the projection of x_i to Σ_∞ lie in the ball $B_{r'/2}(x_\infty)$. This can be done since $B_{r'}(x_\infty) \cap \Sigma_\infty$ is almost flat. Denote by y_i the projection of x_i to Σ_∞ and we have $y_i \in B_{r'/2}(x_\infty) \cap \Sigma_\infty$. Let $s_i \in \mathbb{R}$ such that $y_i + s_i \mathbf{n}_{\Sigma_\infty}(y_i) = x_i$. Combining this with $x_i \to x_\infty$, we have

$$B_{2\delta_i}(x_i \pm 2\delta_i \mathbf{n}_{\Sigma_i}(x_i)) \subset B_{4\delta_i}(x_i) \subset B_{4\delta_i + |s_i|}(y_i). \tag{3.15}$$

On the other hand, $|s_i| \to 0$ and for large i we have

$$B_{4\delta_i + |s_i|}(y_i) \subset B_{\frac{\delta}{4}}(y_i). \tag{3.16}$$

Combining (3.14)–(3.16), we have

$$B_{2\delta_i}(x_i \pm 2\delta_i \mathbf{n}_{\Sigma_i}(x_i)) \subset \left(\Omega\left(\frac{r'}{2}, \frac{\delta}{4}\right) \cup (\Sigma_\infty \cap B_{r'}(x_\infty))\right). \tag{3.17}$$

Step 3. We show that

$$B_{2\delta_i}(x_i \pm 2\delta_i \mathbf{n}_{\Sigma_i}(x_i)) \cap \Sigma_i = \emptyset. \tag{3.18}$$

Let $\Sigma_i = \Sigma_i^{(1)} \cup \Sigma_i^{(2)}$, where $\Sigma_i^{(1)}$ and $\Sigma_i^{(1)}$ are defined by

$$\Sigma_i^{(1)} = \Sigma_i \cap B_{r'}(x_\infty), \quad \Sigma_i^{(2)} = \Sigma_i \backslash B_{r'}(x_\infty).$$

By the smooth convergence of Σ_i to Σ_∞ and the choice of r' such that $B_{r'}(x_\infty) \cap \Sigma_\infty$ is almost flat, we have that for large i, $B_{r'}(x_\infty) \cap \Sigma_i$ is also almost flat. Consequently, for large i we have

$$B_{2\delta_i}(x_i \pm 2\delta_i \mathbf{n}_{\Sigma_i}(x_i)) \cap \Sigma_i^{(1)} = \emptyset. \tag{3.19}$$

On the other hand, (3.13) and (3.17) imply that

$$\begin{aligned} B_{2\delta_i}(x_i \pm 2\delta_i \mathbf{n}_{\Sigma_i}(x_i)) \cap \Sigma_i^{(2)} &\subset \left(\Omega\left(\frac{r'}{2}, \frac{\delta}{4}\right) \cup (\Sigma_\infty \cap B_{r'}(x_\infty))\right) \cap \Sigma_i^{(2)} \\ &= \Omega\left(\frac{r'}{2}, \frac{\delta}{4}\right) \cap \Sigma_i^{(2)} = \emptyset, \end{aligned} \tag{3.20}$$

where we used that $\Sigma_i^{(2)} \cap B_{r'}(x_\infty) = \emptyset$. Thus, (3.18) follows from (3.19) and (3.20). Note that (3.18) contradicts the definition of $\delta_i = r_{\Sigma_i}(x_i)$. The lemma is proved. ∎

A direct corollary of Lemma 3.9 is the following result.

Lemma 3.10. *Let $R, N > 0$ and an increasing positive function ρ. Then there exists a constant $\epsilon_0(R, N, \rho) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ we have*

$$|\mathbf{TN}(\Sigma, \epsilon, R)| = 0 \quad \text{for all } \Sigma \in \mathcal{C}(N, \rho). \tag{3.21}$$

Here the notation $|\Omega|$ denotes the volume of Ω with respect to the standard metric on $\mathbb{R}^3$.

Proof. We choose ϵ_0 the same constant in Lemma 3.9. Thus, equation (3.21) follows from Lemma 3.9 and the definition of **TN**. ■

Using Lemma 3.10 we show that the quantity $|\mathbf{TN}|$ along the flow will tend to zero.

Lemma 3.11. *Fix $R, N > 0$, and an increasing positive function ρ. Under the assumption of Theorem 3.1, there exists a constant $\epsilon_0(R, N, \rho) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, we have*

$$\lim_{t\to\infty} |\mathbf{TN}(\Sigma_t, \epsilon, R)| = 0.$$

Proof. By Lemma 3.4, for any $t_i \to \infty$ there exists a subsequence, still denoted by $\{t_i\}$, such that it converges locally smoothly to a limit self-shrinker $\Sigma_\infty \in \mathcal{C}(N, \rho)$ away from the singular set $\mathcal{S}_0 \subset \mathbb{R}^3$. For any $\epsilon > 0$, by Definition 3.5 we have

$$\mathbf{TN}(\Sigma_{t_i}, \epsilon, R) \to \mathbf{TN}(\Sigma_\infty, \epsilon, R)\backslash B_{\frac{\epsilon}{2}}(\mathcal{S}_0),$$

where $B_\epsilon(\mathcal{S}_0) = \bigcup_{p\in\mathcal{S}_0} B_\epsilon(p)$. Therefore, by Lemma 3.10 we have

$$\lim_{t_i\to+\infty} |\mathbf{TN}(\Sigma_{t_i}, \epsilon, R)| \le \lim_{t_i\to+\infty} |\mathbf{TN}(\Sigma_\infty, \epsilon, R)| = 0,$$

where $\epsilon \in (0, \epsilon_0)$ and ϵ_0 is the constant in Lemma 3.10. The lemma is proved. ■

As in [46, Lemma 4.7], we have:

Lemma 3.12. *Fix $R > 0$ and $\tau \in (0, 1)$. Let $\{t_i\}$ be any sequence as in Lemma 3.4. If the multiplicity of the convergence in Lemma 3.4 is more than one, then for any $\epsilon > 0$, there exists $i_0 > 0$ such that for any $i \ge i_0$ we have*

$$\inf_{t\in[t_i-\tau, t_i]} |\mathbf{TN}(\Sigma_t, \epsilon, R)| > 0.$$

Proof. Since Σ_t is embedded and $\{\Sigma_{t_i+t}, -\tau \le t \le \tau\}$ converges locally smoothly to the limit self-shrinker Σ_∞, all components of $(\Sigma_t \cap B_R(0))\backslash\mathbf{H}(\epsilon, \Sigma_t, R)$ with $t \in [t_i - \tau, t_i]$ lie in the $\frac{\epsilon}{2}$-neighborhood of Σ_∞. By the definition of **TN**, for any $t \in [t_i - \tau, t_i]$ the quantity $\mathbf{TN}(\epsilon, \Sigma_t, R)$ is nonempty and we have $|\mathbf{TN}(\epsilon, \Sigma_t, R)| > 0$. ■

Using Lemmas 3.11 and 3.12, we have the following result as [46, Lemma 4.8].

Lemma 3.13. *Let $R, \epsilon, \tau > 0$ and $f(t, \epsilon) = \inf_{s\in[t-\tau,t]} |\mathbf{TN}(\Sigma_s, \epsilon, R)|$. For any $t_0 > 0$ and $l > 0$, we can find a sequence $\{t_i\}$ with $t_{i+1} > t_i + l$ such that for any $i \in \mathbb{N}$,*

$$\sup_{t\in[t_i, t_i+l]} f(t, \epsilon) \le 2f(t_i, \epsilon). \tag{3.22}$$

Proof. By Lemma 3.12, we can find $s_1 > t_0 + l$ with $f(s_1, \epsilon) > 0$. We search for time $t \in [s_1, s_1 + l]$ satisfying $f(t, \epsilon) > 2f(t_i, \epsilon)$. If no such time exists, then we set $t_1 = s_1$. Otherwise, we choose such a time and denote it by $s_1^{(1)}$. Then search the time interval $[s_1^{(1)}, s_1^{(1)} + l]$. Inductively, we search $[s_1^{(k)}, s_1^{(k)} + l]$. If we have

$$\sup_{t\in[s_1^{(k)}, s_1^{(k)}+l]} f(t, \epsilon) \le 2f(s_1^{(k)}, \epsilon),$$

then we denote

$$t_1 = s_1^{(k)}$$

and stop the searching process. Otherwise, choose a time $s_1^{(k+1)} \in [s_1^{(k)}, s_1^{(k)} + l]$ with more than doubled value and continue the process. Note that

$$f(s_1^{(k)}, \epsilon) \ge 2^k f(s_1, \epsilon) \to \infty \quad \text{as } k \to \infty.$$

Since $\lim_{t\to+\infty} f(t, \epsilon) = 0$ by Lemma 3.11, this process must stop in finite steps, and we can find k_1 such that

$$\sup_{t\in[s_1^{(k_1)}, s_1^{(k_1)}+l]} f(t, \epsilon) \le 2f(s_1^{(k_1)}, \epsilon).$$

We denote by $t_1 = s_1^{(k_1)}$. After we find t_1, set $s_2^{(0)} = t_1 + l + 1$ and continue the previous process to find time in $[s_2^{(0)}, s_2^{(0)} + l]$ such that $f(t, \epsilon) > 2f(s_2^{(0)}, \epsilon)$. Similarly, for some k we have

$$\sup_{[s_2^{(k)}, s_2^{(k)}+l]} f(t, \epsilon) \le 2f(s_2^{(k)}, \epsilon).$$

Then we define $t_2 = s_2^{(k)}$. Inductively, after we find t_l, we set $s_{l+1}^{(0)} = t_l + l + 1$. Then we start the process to search time in $[s_{l+1}^{(0)}, s_{l+1}^{(0)} + l]$ with $f(t, \epsilon) > 2f(s_{l+1}^{(0)}, \epsilon)$. This process is well defined. Repeating this process and we can find a sequence of times $\{t_i\}$ such that for any t_i inequality (3.22) holds. The lemma is proved. ∎

3.3. *Construction of auxiliary functions*

In this subsection, we construct functions which will be used to show the L-stability of the limit self-shrinker. We fix $R, T > 1$ in this section. For any sequence $t_i \to +\infty$, by Lemma 3.4 a subsequence of $\{\Sigma_{i,t}, -T < t < T\}$ converges in smooth topology to a self-shrinker Σ_∞ away from a locally finite, σ-Lipschitz singular set $\mathcal{S} \subset \mathbb{R}^3 \times (-T, T)$. We denote by $\mathcal{S}_t = \{x \in \mathbb{R}^3 \mid (x, t) \in \mathcal{S}\}$ the singular set in $\mathbb{R}^3$ at time t. By Lemma 2.9, we assume that the multiplicity of the convergence is a constant $N_0 \ge 2$. As in [46], we construct some functions as follows:

(1) Let $\epsilon > 0$ and large $R > 0$. We define

$$\Omega_{\epsilon,R}(t) = (\Sigma_\infty \cap B_R(0)) \backslash B_\epsilon(\mathcal{S}_t) \tag{3.23}$$

and for any time interval $I \subset (-T, T)$ we define

$$\Omega_{\epsilon,R}(I) = \bigcap_{t\in I} \Omega_{\epsilon,R}(t), \quad \mathcal{S}_I = \bigcup_{t\in I} \mathcal{S}_t. \tag{3.24}$$

For any $\epsilon > 0$, the surface $\Sigma_{i,t} \cap B_R(0)$ is a union of graphs over the set $\Omega_{\epsilon,R}(t)$ for large t_i and any $t \in (-T, T)$.

(2) Let $u_i^+(x,t)$ and $u_i^-(x,t)$ be the graph functions representing the top and bottom sheets (which we denote by $\Sigma_{i,t}^+$ and $\Sigma_{i,t}^-$, respectively) over $\Sigma_\infty \cap B_R(0)$. The readers are referred to [46] for the details on the construction of $u_i^+(x,t)$ and $u_i^-(x,t)$. By the convergence property of the flow $\{(\Sigma_{i,t}, \mathbf{x}_i(t)), -T < t < T\}$, for any $\epsilon > 0$ and large R there exists $i_0 > 0$ such that for any $i \geq i_0$ and any $t \in (-T,T)$ the functions $u_i^+(x,t)$ and $u_i^-(x,t)$ are well defined on $\Omega_{\epsilon,R}(t)$. By the calculation in Appendix C, the function

$$u_i(x,t) = u_i^+(x,t) - u_i^-(x,t), \tag{3.25}$$

which we call the *height difference function* of $\Sigma_{i,t}$ over Σ_∞, satisfies the equation

$$\frac{\partial u_i}{\partial t} = \Delta_0 u_i - \frac{1}{2}\langle x, \nabla u_i\rangle + |A|^2 u_i + \frac{u_i}{2} + a_i^{pq} u_{i,pq} + b_i^p u_{i,p} + c_i u_i \tag{3.26}$$

for any $(x,t) \in \Omega_{\epsilon,R}(I) \times I$. Here Δ_0 denotes the Laplacian operator on Σ_∞. The coefficients a_i^{pq}, b_i^p and c_i are small on $\Omega_{\epsilon,R}(I) \times I$ as t_i large and tend to zero as $t_i \to +\infty$.

(3) Fix a point $x_0 \in (\Sigma_\infty \cap B_R(0)) \backslash \mathcal{S}_1$. We choose a sequence of points

$$\{x_i\}_{i=1}^\infty \subset (\Sigma_\infty \backslash \mathcal{S}_1) \cap B_R(0) \quad \text{with } x_i \to x_0.$$

Then for sufficiently small $\epsilon > 0$ we have $x_0 \in \Omega_{\epsilon,R}(1)$ and $\{x_i\}_{i=1}^\infty \subset \Omega_{\epsilon,R}(1)$. For any $t \in (-T,T)$ and $x \in \Omega_{\epsilon,R}(t)$ we define the normalized height difference function

$$w_i(x,t) = \frac{u_i(x,t)}{u_i(x_i,1)}, \tag{3.27}$$

Then $w_i(x,t)$ is a positive function with $w_i(x_i,1) = 1$ and by (3.26) $w_i(x,t)$ satisfies the equation on $\Omega_{\epsilon,R}(I) \times (I)$ for any $I \subset (-T,T)$,

$$\frac{\partial w_i}{\partial t} = \Delta_0 w_i - \frac{1}{2}\langle x, \nabla w_i\rangle + |A|^2 w_i + \frac{w_i}{2} + a_i^{pq} w_{i,pq} + b_i^p w_{i,p} + c_i w_i. \tag{3.28}$$

Note that the construction of the function w_i is slightly different from that of [46]. In (3.27) we choose a sequence of points $\{x_i\} \subset \Sigma_\infty \backslash \mathcal{S}_1$ to normalize the function u_i, while in [46] we choose a fixed point x_0. The reason why we choose such a normalization is that we need inequality (3.51) in Lemma 3.19 below.

As in [46], we have the following result which implies that for large t_i the integral of u_i is comparable to the volume $|\mathbf{TN}|$.

Lemma 3.14 (cf. [46]). *Fix ϵ, R and T as above. For any sequence $\{t_i\}$ chosen in Lemma 3.4, there exists $t_T > 0$ such that for any $t \in (-T,T)$ and $t_i > t_T$ we have*

$$\frac{1}{2}\int_{\Omega_{\epsilon,R}(t)} u_i(x,t)\, d\mu_\infty \leq |\mathbf{TN}(\Sigma_{i,t}, \epsilon, R)| \leq 2\int_{\Omega_{\frac{\epsilon}{5},R}(t)} u_i(x,t)\, d\mu_\infty,$$

where $d\mu_\infty$ denotes the volume form of Σ_∞.

The proof of Lemma 3.14 is similar to that of [46, Lemma 4.13]. Note that the coefficients 2 and $\frac{1}{2}$ are chosen to absorb the error term caused by the second fundamental of Σ_∞.

Since w_i satisfies the parabolic equation (3.28), we have the following parabolic Harnack inequality by using Theorem A.5 in Appendix A.

Lemma 3.15. *For any $-T < a < s < t < b < T$, any $\epsilon > 0$, and any points $x \in \Omega_{\epsilon,R}(s)$ and $y \in \Omega_{\epsilon,R}(t)$, there exists a constant $C = C(\epsilon, R, s-a, t-s, \Sigma_\infty, \mathcal{S}_{[a,b]})$ such that*

$$w_i(x,s) \le C w_i(y,t).$$

Proof. We divide the proof into several steps.

Step 1. Since $\mathcal{S}_t \cap B_R(0)$ consists of finitely many points, we can choose sufficiently small $\delta_0(\Sigma_\infty, \mathcal{S}_{[a,b]}) > 0$ such that for any $s \in [a,b]$,

$$\Omega_{2\epsilon,R}(s) \subset \Omega_{\epsilon,R}(t), \quad \Omega_{\frac{1}{2}\epsilon,R+2}(s) \subset \Omega_{\frac{1}{5}\epsilon,R+2}(t), \quad t \in [s-\delta_0, s+\delta_0] \cap [a,b]. \tag{3.29}$$

Let N be a positive integer satisfying

$$N > \max\left\{\frac{5(b-a)}{\delta_0}, \frac{b-a}{s-a}, \frac{5(b-a)}{t-s}\right\}. \tag{3.30}$$

Set

$$\tau_k = a + \frac{b-a}{N}k \quad \text{for all } k \in \{0, 1, \ldots, N\}. \tag{3.31}$$

Then $\tau_0 = a$ and $\tau_N = b$. By (3.30) we have $s \ge \tau_1$. Note that (3.29) and (3.31) imply that for any $k = 1, 2, \ldots, N-1$ we have

$$\Omega_{\frac{1}{2}\epsilon,R+2}(\tau_k) \subset \Omega_{\frac{\epsilon}{5},R+2}(t) \quad \text{for all } t \in [\tau_{k-5}, \tau_{k+5}] \cap [a,b].$$

Step 2. Let

$$\Omega' := \Omega_{\epsilon,R}(\tau_k), \quad \Omega'' := \Omega_{\frac{2}{3}\epsilon,R+1}(\tau_k), \quad \Omega := \Omega_{\frac{1}{2}\epsilon,R+2}(\tau_k).$$

Then we have $\Omega' \subset \Omega'' \subset \Omega$. Clearly, Ω'' has a positive distance $\delta = \delta(\epsilon)$ away from the boundary of Ω. Sine $\bar{\Omega}'$ is compact, we can cover Ω' by finite many balls contained in Ω'' with radius $r = \frac{\epsilon}{100}$ and the number of these balls is bounded by a constant depending only on ϵ, R and Σ_∞. Since w_i satisfies the parabolic equation (3.28), applying Theorem A.5 in Appendix A for the function w_i, the domains $\Omega', \Omega'', \Omega$ and the interval $[\tau_{k-1}, \tau_{k+1}]$, we have

$$w_i(x,\tau_k) \le C w_i(y, \tau_{k+1}) \quad \text{for all } x, y \in \Omega_{\epsilon,R}(\tau_k), \tag{3.32}$$

where $C = C(\epsilon, R, b-a, N, \Sigma_\infty, \mathcal{S}_{[a,b]})$ is a constant independent of i. Moreover, since $S_{[a,b]} \cap B_R(0)$ consists of finitely many Lipschitz curves, there exists a sequence of points $\{z_k\}$ such that

$$z_k \in \Omega_{2\epsilon,R}([\tau_{k-1}, \tau_k]) \cap \Omega_{2\epsilon,R}([\tau_k, \tau_{k+1}]) \neq \emptyset. \tag{3.33}$$

Step 3. For $s, t \in (a,b)$ with $s < t$, there exist integers k_s and k_t such that $s \in [\tau_{k_s}, \tau_{k_s+1})$ and $t \in (\tau_{k_t}, \tau_{k_t+1}]$. Note that (3.30) implies

$$t - s \ge \frac{5(b-a)}{N}. \tag{3.34}$$

On the other hand, (3.31) implies that

$$t-s \le \tau_{k_t+1}-\tau_{k_s}=\frac{b-a}{N}(k_t+1-k_s). \tag{3.35}$$

Combining (3.35) with (3.34), we have

$$k_t-k_s \ge 4. \tag{3.36}$$

Thus, (3.32), (3.36)and (3.33) implies that

$$w_i(z_{k_s+2},\tau_{k_s+2}) \le C w_i(z_{k_s+3},\tau_{k_s+3}) \le \cdots \le C^N w_i(z_{k_t-1},\tau_{k_t-1}), \tag{3.37}$$

where $C=C(\epsilon,R,b-a,N,\Sigma_\infty,\mathcal{S}_{[a,b]})$.

Step 4. Set

$$\Omega'=\Omega_{\epsilon,R}(s),\quad \Omega''=\Omega_{\frac{2}{3}\epsilon,R+1}(s),\quad \Omega=\Omega_{\frac{1}{2}\epsilon,R+2}(s).$$

Then by (3.29) we have

$$\Omega'\subset\Omega''\subset\Omega=\Omega_{\frac{1}{2}\epsilon,R+2}(s)\subset\Omega_{\frac{1}{5}\epsilon,R+2}(s') \quad \text{for all } s'\in[\tau_{k_s-2},\tau_{k_s+2}],$$

where we used the fact that $[\tau_{k_s-2},\tau_{k_s+2}]\subset[s-\delta_0,s+\delta_0]$. Note that by (3.33) and (3.29), we have

$$z_{k_s+2}\in\Omega_{2\epsilon,R}(\tau_{k_s+2})\subset\Omega_{\epsilon,R}(s). \tag{3.38}$$

As in Step 2, Ω'' has a positive distance $\delta=\delta(\epsilon)$ from the boundary of Ω, and we can cover Ω' by finite many balls contained in Ω'' with radius $r=\frac{\epsilon}{100}$ and the number of these balls is bounded by a constant depending only on ϵ, R and Σ_∞. Applying Theorem A.5 for such Ω',Ω'',Ω and the interval $[\tau_{k_s-2},\tau_{k_s+2}]$ and using (3.38), we have

$$w_i(x,s)\le C w_i(z_{k_s+2},\tau_{k_s+2}) \quad \text{for all } x\in\Omega_{\epsilon,R}(s), \tag{3.39}$$

where $C=C(\epsilon,R,b-a,N,\Sigma_\infty,\mathcal{S}_{[a,b]})$.

Step 5. Set

$$\Omega'=\Omega_{\epsilon,R}(t),\quad \Omega''=\Omega_{\frac{2}{3}\epsilon,R+1}(t),\quad \Omega=\Omega_{\frac{1}{2}\epsilon,R+2}(t).$$

Then by (3.29) we have

$$\Omega'\subset\Omega''\subset\Omega=\Omega_{\frac{1}{2}\epsilon,R+2}(t)\subset\Omega_{\frac{1}{5}\epsilon,R+2}(t') \quad \text{for all } t'\in[\tau_{k_t-2},\tau_{k_t+2}],$$

where we used the fact that $[\tau_{k_t-2},\tau_{k_t+2}]\subset[t-\delta_0,t+\delta_0]$. Note that by (3.33) and (3.29), we have

$$z_{k_t-1}\in\Omega_{2\epsilon,R}(\tau_{k_t-1})\subset\Omega_{\epsilon,R}(t). \tag{3.40}$$

Applying Theorem A.5 as in Step 4 for such Ω',Ω'',Ω and the interval $[\tau_{k_t-2},\tau_{k_t+2}]$ and using (3.40), we have

$$w_i(z_{k_t-1},\tau_{k_t-1})\le C w_i(y,t) \quad \text{for all } y\in\Omega_{\epsilon,R}(t),$$

where $C=C(\epsilon,R,b-a,N,\Sigma_\infty,\mathcal{S}_{[a,b]})$. Combining this with (3.39) and (3.37), we have

$$w_i(x,s)\le C w_i(y,t) \quad \text{for all } x\in\Omega_{\epsilon,R}(s),\quad y\in\Omega_{\epsilon,R}(t),$$

where $C=C(\epsilon,R,b-a,N,\Sigma_\infty,\mathcal{S}_{[a,b]})$. The lemma is proved. ∎

For any fixed ϵ, R and T, the following result shows that we can find a sequence $\{t_i\}$ such that the functions w_i are uniformly bounded on a compact set away from singularities. Note that we have no estimates of w_i near the singularities.

Lemma 3.16. *Fix $\epsilon, \tau \in (0, \frac{1}{2})$ and R, T large. Let $\{t_i\}$ be the sequence from Lemma 3.13 for such ϵ, τ, R and $l = T$. For any time interval $I = [a, b] \subset [-1, T-2]$ and a compact set $K \subset\subset (\Sigma_\infty \cap B_R(0)) \backslash \mathcal{S}_I$, there exists a constants $C = C(K, \Sigma_\infty, \mathcal{S}_{[-2,b+2]}) > 0$ such that the function w_i defined by (3.27) satisfies*

$$0 < w_i(x,t) < C \quad \text{for all } (x,t) \in K \times I.$$

Moreover, if $a \in [2, T-2]$), there exists $C' = C'(K, \Sigma_\infty, \mathcal{S}_{[0,a+1]}) > 0$ independent of b such that

$$w_i(x,a) \geq C'.$$

Proof. By the assumption, we can assume that $K \subset \Omega_{\epsilon',R}(I)$ and $\{x_i\} \subset \Omega_{\epsilon',R}(1)$ for some $\epsilon' \in (0, \epsilon)$, where $\{x_i\}$ is the sequence in (3.27). Note that $w_i(x_i, 1) = 1$. We divide the rest of the proof into several steps.

Step 1. w_i is bounded on $K \times I$ for the time interval $I = [-1, \frac{1}{2}]$ and any K above. Applying Lemma 3.15 for $a = -2$ and $b = 2$ we have

$$\begin{aligned} w_i(x,t) &\leq C(\epsilon', R, \Sigma_\infty, \mathcal{S}_{[-2,2]}) w_i(x_i, 1) \\ &= C(\epsilon', R, \Sigma_\infty, \mathcal{S}_{[-2,2]}) \quad \text{for all } (x,t) \in K \times I. \end{aligned} \tag{3.41}$$

Step 2. w_i is bounded from above on $K \times I$ for any $I = [a, b] \subset (0, T-2)$ and K above. For any $t \in [a, b] \subset (0, T-2)$, we have $t' := t + 1 \in (1, T-1)$. By Lemma 3.14 and Lemma 3.13 for large i we have

$$\begin{aligned} \inf_{s \in [t'-\tau, t']} \int_{\Omega_{\epsilon,R}(s)} w_i(x,s)\, d\mu &\leq \frac{2}{u_i(x_i,1)} \inf_{s \in [t'-\tau, t']} |\mathbf{TN}(\Sigma_{t_i+s}, \epsilon, R)| \\ &\leq \frac{4}{u_i(x_i,1)} \inf_{s \in [-\tau, 0]} |\mathbf{TN}(\Sigma_{t_i+s}, \epsilon, R)| \\ &\leq 4 \inf_{s \in [-\tau, 0]} \int_{\Omega_{\frac{\epsilon}{5},R}(s)} w_i(x,s)\, d\mu. \end{aligned} \tag{3.42}$$

Moreover, by (3.41) we have

$$w_i(x,0) \leq C(\epsilon, R, \Sigma_\infty, \mathcal{S}_{[-2,2]}) \quad \text{for all } x \in \Omega_{\frac{\epsilon}{5},R}(0),$$

which implies that

$$\int_{\Omega_{\frac{\epsilon}{5},R}(0)} w_i(x,0) \leq C(\epsilon, R, \Sigma_\infty, \mathcal{S}_{[-2,2]}). \tag{3.43}$$

Combining (3.43) with (3.42), we have

$$\inf_{s \in [t'-\tau, t']} \int_{\Omega_{\epsilon,R}(s)} w_i(x,s)\, d\mu \leq C(\epsilon, R, \Sigma_\infty, \mathcal{S}_{[-2,2]}).$$

This implies that for any $t \in (1, T-1)$ there exists $s(t) \in [t-\tau, t]$ such that

$$\int_{\Omega_{\epsilon,R}(s(t))} w_i(x, s(t))\, d\mu \le C(\epsilon, R, \Sigma_\infty, \mathcal{S}_{[-2,2]}). \tag{3.44}$$

On the other hand, Lemma 3.15 implies that for any $x \in K \subset \Omega_{\epsilon',R}([a,b]), t \in [a,b]$, and $y \in \Omega_{\epsilon,R}(s(t+1)) \subset \Omega_{\epsilon',R}(s(t+1))$ we have

$$w_i(x,t) \le C(\epsilon', R, \Sigma_\infty, \mathcal{S}_{[a-1,b+2]}) w_i(y, s(t+1)), \tag{3.45}$$

where we used the fact that $\tau \in (0, \frac{1}{2})$ and

$$s(t+1) \ge t+1-\tau \ge t+\frac{1}{2}.$$

Integrating the right-hand side of (3.45) and using (3.44), we have

$$\begin{aligned} w_i(x,t) &\le C(\epsilon', R, \Sigma_\infty, \mathcal{S}_{[a-1,b+2]}) \int_{\Omega_{\epsilon,R}(s(t+1))} w_i(y, s(t+1)) \\ &\le C(\epsilon', R, \Sigma_\infty, \mathcal{S}_{[-2,b+2]}) \quad \text{for all } t \in [a,b]. \end{aligned}$$

Step 3. $w_i(x,t)$ is bounded from below on $K \times I$ for any $I = [a,b] \subset [2, T-2]$ and K above. By Lemma 3.15, for any $(x,t) \in K \times I$ we have

$$w_i(x,t) \ge C(\epsilon', R, \Sigma_\infty, \mathcal{S}_{[0,t+1]}) w_i(x_i, 1). \tag{3.46}$$

In particular, for $t = a$ the constant in (3.46) depends only on $\epsilon', R, \Sigma_\infty$ and $\mathcal{S}_{[0,a+1]}$. Thus, the lemma is proved. ∎

Lemma 3.17. *The same assumption as in Lemma 3.16. As $t_i \to +\infty$, we can take a subsequence of the functions $w_i(x,t)$ such that it converges in C^2 topology on any compact subset $K \subset\subset (\Sigma_\infty \cap B_R(0)) \backslash \mathcal{S}_I$, where $I = [a,b] \subset [-1, T-2]$, to a positive function $w(x,t)$ with $w(x_0, 1) = 1$ and satisfying*

$$\frac{\partial w}{\partial t} = \Delta_0 w + |A|^2 w - \frac{1}{2}\langle x, \nabla w\rangle + \frac{1}{2} w \quad \textit{for all } (x,t) \in K \times I. \tag{3.47}$$

Proof. Since w_i is positive by definition and w_i is uniformly bounded from above by Lemma 3.16, by the interior estimates of the parabolic equation we have the space-time $C^{2,\alpha}$ estimates of w_i (cf. [50, Theorem 4.9]), and the estimates are independent of i. Therefore, as $i \to +\infty$, the function w_i converges to a limit function w in C^2 topology on $K \times [a,b]$ with $w(x_0, 1) = 1$ and w is positive by the strong maximal principle. The lemma is proved. ∎

3.4. The auxiliary functions near the singular set

In this subsection, we show that there exists a refined sequence such that the limit auxiliary function has uniform estimates across the singular set. Recall that by Lemma 3.17 the function w is uniformly bounded on any compact set away from the singular set and w has no estimates near the singularities. In this subsection, we will use Lemma 3.13 repeatedly for a sequence $\{\epsilon_i\}$ decreasing to zero, and after taking a diagonal subsequence we can construct a auxiliary function which has uniform estimates across the singular set.

Lemma 3.18. *Let $R > 1, \tau \in (0, \frac{1}{2})$, and let $\{\epsilon_i\}$ be a sequence of positive numbers with $\epsilon_i \to 0$. For any $i \in \mathbb{N}$, there exists a sequence $\{t_{i,k}\}_{k=1}^{\infty}$ with $t_{i,k+1} > t_{i,k} + i$ satisfying the following properties:*

(1) *For any $k \in \mathbb{N}$,*

$$\sup_{s\in[0,i]} f(t_{i,k} + s, \epsilon_i) \le 2f(t_{i,k}, \epsilon_i), \tag{3.48}$$

where $f(t, \epsilon) = \inf_{s\in[t-\tau,t]} |\mathbf{TN}(\Sigma_s, \epsilon, R)|$.

(2) *For any $T > 0$, $\{\Sigma_{t_{i,k}+s}, -T < s < T\}$ converges locally smoothly to a self-shrinker $\Sigma_{i,\infty} \in \mathcal{C}(N, \rho)$ away from the space-time singular set $\mathcal{S}_i$ as $k \to +\infty$.*

(3) *For large k the surface $\Sigma_{t_{i,k}+s}$ can be written as a union of graphs over $\Sigma_{i,\infty}$ away from the singular set $\mathcal{S}_{i,s}$. We denote by $\tilde{u}^+_{i,k}(x, s), \tilde{u}^-_{i,k}(x, s)$ the graph functions of the top and bottom sheets of $\Sigma_{t_{i,k}+s}$ over $\tilde{\Omega}_{i,\epsilon,R}(s)$, where*

$$\tilde{\Omega}_{i,\epsilon,R}(s) = (\Sigma_{i,\infty} \cap B_R(0))\backslash B_\epsilon(\mathcal{S}_{i,s}).$$

Let $\tilde{u}_{i,k}(x, s) = \tilde{u}^+_{i,k}(x, s) - \tilde{u}^-_{i,k}(x, s)$ be the height difference function of $\Sigma_{t_{i,k}+s}$ over $\tilde{\Omega}_{i,\epsilon,R}(s)$. These functions are constructed as in Section 3.3. By Lemma 3.14, we can choose k_i large such that for any $k \ge k_i$ and $s \in (-T, T)$,

$$\frac{1}{2}\int_{\tilde{\Omega}_{i,\epsilon_i,R}(s)} \tilde{u}_{i,k}(x, s) \le |\mathbf{TN}(\Sigma_{t_{i,k}+s}, \epsilon_i, R)| \le 2\int_{\tilde{\Omega}_{i,\frac{\epsilon_i}{5},R}(s)} \tilde{u}_{i,k}(x, s). \tag{3.49}$$

(4) *By the smooth compactness of $\mathcal{C}(N, \rho)$ in [20], we assume that $\Sigma_{i,\infty}$ in item (2) converges smoothly to $\Sigma_\infty \in \mathcal{C}(N, \rho)$.*

(5) *For any $i \in \mathbb{N}$, there exists $k_i > 0$ satisfying the following property. For any $\{s_i\}_{i=1}^{\infty}$ with $s_i > k_i$, $\{\Sigma_{t_{i,s_i}+s}, -T < s < T\}$ converges locally smoothly to the same self-shrinker Σ_∞ as in item (4) away from the space-time singular set $\mathcal{S}_\infty$. Moreover, the singular set $\mathcal{S}_i$ in item (2) converges to $\mathcal{S}_\infty$ in Hausdorff distance.*

Proof. Applying Lemma 3.13 for ϵ_i and $l = i$, we have (3.48). Item (2) follows from Lemma 3.4, and item (3) follows from Lemma 3.14. It is clear that item (4) follows from Colding–Minicozzi's compactness theorem [20].

To prove item (5), we first note that the convergence in item (2) is also in Hausdorff distance by Lemma 3.4, for any i there exists $k_i > 0$ such that for any $k \ge k_i$ and any $s \in (-2, 2)$ we have

$$\begin{aligned} d_H\big(\Sigma_{t_{i,k}+s} \cap B_R(0), \Sigma_{i,\infty} \cap B_R(0)\big) &\le \frac{1}{i}, \\ d_H\big(\mathbf{S}(\Sigma_{t_{i,k}+s}, \epsilon_i, R), \mathcal{S}_{i,s} \cap B_R(0)\big) &\le \frac{1}{i}, \end{aligned} \tag{3.50}$$

where d_H denotes the Hausdorff distance. By item (4), we assume that $\Sigma_{i,\infty}$ converges smoothly to $\Sigma_\infty \in \mathcal{C}(N, \rho)$. By Lemma 3.4 for any sequence of times $\{s_i\}_{i=1}^{\infty}$ with

$s_i > k_i$ the surfaces $\{\Sigma_{t_{i,s_i}+s}, -T < s < T\}$ converge locally smoothly to a self-shrinker, which is denoted by $\hat{\Sigma}_\infty$, away from a singular set $\mathcal{S}_s \subset \Sigma_\infty$ as $i \to +\infty$. Moreover, as $i \to +\infty$,

$$\begin{aligned} & d_H\big(\hat{\Sigma}_\infty \cap B_R(0), \Sigma_\infty \cap B_R(0)\big) \\ & \quad \le d_H\big(\hat{\Sigma}_\infty \cap B_R(0), \Sigma_{t_{i,s_i}+s} \cap B_R(0)\big) + d_H\big(\Sigma_{t_{i,s_i}+s} \cap B_R(0), \Sigma_{i,\infty} \cap B_R(0)\big) \\ & \qquad + d_H\big(\Sigma_{i,\infty} \cap B_R(0), \Sigma_\infty \cap B_R(0)\big) \\ & \quad \le d_H\big(\hat{\Sigma}_\infty \cap B_R(0), \Sigma_{t_{i,s_i}+s} \cap B_R(0)\big) + \frac{1}{i} + d_H\big(\Sigma_{i,\infty} \cap B_R(0), \Sigma_\infty \cap B_R(0)\big) \\ & \quad \to 0, \end{aligned}$$

where we used (3.50). Thus, $\hat{\Sigma}_\infty$ coincides with Σ_∞. Moreover, since $\mathbf{S}(\Sigma_{t_{i,s_i}+s}, \epsilon_i, R)$ converges to $\mathcal{S}_s \cap B_R(0)$ as $i \to +\infty$, we have

$$\begin{aligned} & d_H\big(\mathcal{S}_{i,s} \cap B_R(0), \mathcal{S}_s \cap B_R(0)\big) \\ & \quad \le d_H\big(\mathbf{S}(\Sigma_{t_{i,s_i}+s}, \epsilon_i, R), \mathcal{S}_s \cap B_R(0)\big) + d_H\big(\mathbf{S}(\Sigma_{t_{i,s_i}+s}, \epsilon_i, R), \mathcal{S}_{i,s} \cap B_R(0)\big) \\ & \quad \le d_H\big(\mathbf{S}(\Sigma_{t_{i,s_i}+s}, \epsilon_i, R), \mathcal{S}_s \cap B_R(0)\big) + \frac{1}{i} \\ & \quad \to 0, \end{aligned}$$

where we used (3.50). Thus, $\mathcal{S}_{i,s} \cap B_R(0)$ converges to $\mathcal{S}_\infty \cap B_R(0)$ as $i \to +\infty$. The lemma is proved. ∎

Lemma 3.19. *Under the same assumptions as in Lemma 3.18, we can choose a point $x_0 \in (\Sigma_\infty \backslash \mathcal{S}_1) \cap B_R(0)$ and $\{x_{i,k}\} \subset (\Sigma_{i,\infty} \backslash \mathcal{S}_{i,1}) \cap B_R(0)$ satisfying the following properties:*

(1) *$x_{i,k} \to x_0$ as $i \to +\infty$ and $k \to +\infty$.*

(2) *For each i, there exists $k_i > 0$ such that for any $k \ge k_i$,*

$$\tilde{u}_{i,k}(x_{i,k}, 1) \le 2u_{i,k}(x_0, 1) \quad \text{for all } k \ge k_i. \tag{3.51}$$

Here $u_{i,k}$ denotes the height difference function of $\Sigma_{t_{i,k}+s}$ over Σ_∞.

Proof. Choose $x_0 \in (\Sigma_\infty \backslash \mathcal{S}_1) \cap B_R(0)$ and we denote by l_{x_0} the normal line of Σ_∞ passing through the point x_0. Then the set $\Sigma_{i,k} \cap l_{x_0}$ is nonempty for large i and k. Since $\Sigma_{i,k}$ can be viewed as a union of multiple graphs over $\Sigma_{i,\infty}$ away from singularities, we assume that l_{x_0} intersects with the bottom sheet of $\Sigma_{i,k}$ at the point $y_{i,k}$, and the projection of $y_{i,k}$ on $\Sigma_{i,\infty}$ is $x_{i,k} \in \Sigma_{i,\infty}$. We denote by $l_{x_{i,k}}$ the normal line of $\Sigma_{i,\infty}$ passing through the point $x_{i,k}$. Since $x_0 \notin \mathcal{S}_1$, we have $x_{i,k} \notin \mathcal{S}_{i,1}$ for large i and k. By the construction of $x_{i,k}$, it is clear that $x_{i,k}$ converges to x_0 as $i \to +\infty$ and $k \to +\infty$.

Fix $\theta_0 \in (0, \frac{\pi}{2})$. Since $\Sigma_{i,\infty}$ converges smoothly to Σ_∞, the angle between the two lines l_{x_0} and $l_{x_{i,k}}$ will lie in $[0, \theta_0)$ for large i and there is a uniform $r_0 > 0$ independent of i such that

$$|A|(x) \le \frac{1}{r_0} \quad \text{for all } x \in \Sigma_{i,\infty} \cap B_{r_0}(x_0).$$

We assume that Σ' is $\Sigma_{i,\infty}$, Σ'_{u_1} is the top sheet of $\Sigma_{i,k}$, Σ'_{u_2} is the bottom sheet of $\Sigma_{i,k}$ and P is the point $x_{i,k}$ as above. Then we apply Lemma 3.24 below for such Σ', Σ'_{u_1}, Σ'_{u_2} and the point P and we can get that the functions $\tilde{u}_{i,k}$ and $u_{i,k}$ satisfy (3.51) for large k. The lemma is proved. ∎

By Lemma 3.17 for each i the function $\tilde{w}_{i,k}$ converges in C^2 to the limit function $\tilde{w}_{i,\infty}$ on any $K \times I$ with $I \subset [-1, T-2]$ and $K \subset\subset (\Sigma_{i,\infty} \cap B_R(0)) \backslash \mathcal{S}_{i,I}$, and $x_{i,k} \to x_{i,0}$ as $k \to +\infty$. Moreover, $\tilde{w}_{i,\infty}(x_{i,0}, 1) = 1$. Note that $\tilde{w}_{i,\infty}$ satisfies equation (3.47), and the function

$$\hat{w}_i = \tilde{w}_{i,\infty} e^{-\frac{|x|^2}{8}}$$

satisfies the equation

$$\frac{\partial \hat{w}_i}{\partial t} = \Delta \hat{w}_i + \left(|A|^2 + \frac{3}{4} - \frac{1}{16}|x|^2\right)\hat{w}_i \quad \text{for all } (x,t) \in K \times I. \tag{3.52}$$

We would like to show that $\hat{w}_i$ satisfies the parabolic Harnack inequality with uniform constants independent of i. Note that here we need to use Theorem B.3 in Appendix B instead of Theorem A.5 in Appendix A. The reason is that $\hat{w}_i$ are functions defined on subdomains in $\Sigma_{i,\infty}$, which varies when i is different. The constants in the Harnack inequality of Theorem A.5 depend on the manifold $\Sigma_{i,\infty}$ and it is difficult to show that the constants are independent of i. However, we can use Theorem B.3 to avoid this difficulty since the constants can be explicitly written down by Theorem B.1. We note that Theorem B.3 cannot be used for equation (3.28) of w_i and we have to use Theorem A.5 in the proof of Lemma 3.15.

Lemma 3.20. *Let $\hat{w}_i = \tilde{w}_{i,\infty} e^{-\frac{|x|^2}{8}}$. For any $-T < a < s < t < b < T$, any $\epsilon > 0$, and any $x \in \Omega_{i,\epsilon,R}(s)$ and $y \in \Omega_{i,\epsilon,R}(t)$, there exists $C = C(\epsilon, R, s-a, t-s, \Sigma_\infty, \mathcal{S}_{[a,b]})$ independent of i such that*

$$\hat{w}_i(x,s) \le C \hat{w}_i(y,t).$$

Proof. The lemma follows from the combination of the proof of Lemma 3.15 and Theorem B.3. For the readers' convenience, we give the detailed proof here.

By Lemma 3.18, $\mathcal{S}_i$ converges to $\mathcal{S}$ in the Hausdorff topology. Since $\mathcal{S}_t \cap B_R(0)$ consists of finitely many points, we can choose $\delta_0(\Sigma_\infty, \mathcal{S}_{[a,b]}) > 0$ small such that for any $s \in [a,b]$,

$$\Omega_{\frac{3}{2}\epsilon,R}(s) \subset \Omega_{\epsilon,R}(t), \quad \Omega_{\frac{1}{3}\epsilon,R+2}(s) \subset \Omega_{\frac{1}{5}\epsilon,R+2}(t), \quad t \in [s-\delta_0, s+\delta_0] \cap [a,b]. \tag{3.53}$$

Thus, for large i we have

$$\Omega_{i,2\epsilon,R}(s) \subset \Omega_{i,\epsilon,R}(t), \quad \Omega_{i,\frac{1}{2}\epsilon,R+2}(s) \subset \Omega_{i,\frac{1}{5}\epsilon,R+2}(t), \quad t \in [s-\delta_0, s+\delta_0] \cap [a,b].$$

Let N be a positive integer satisfying

$$N > \max\left\{\frac{5(b-a)}{\delta_0}, \frac{b-a}{s-a}, \frac{5(b-a)}{t-s}\right\}. \tag{3.54}$$

Set

$$\tau_k = a + \frac{b-a}{N} k \quad \text{for all } k \in \{0, 1, \dots, N\}. \tag{3.55}$$

Then $\tau_0 = a$ and $\tau_N = b$. By (3.54) we have $s \geq \tau_1$. Note that (3.53) and (3.55) imply that for any $k = 1, 2, \ldots, N-1$ we have

$$\Omega_{i,\frac{1}{2}\epsilon,R+2}(\tau_k) \subset \Omega_{i,\frac{\epsilon}{5},R+2}(t) \quad \text{for all } t \in [\tau_{k-5}, \tau_{k+5}] \cap [a,b].$$

Let

$$\Omega_i' := \Omega_{i,\epsilon,R}(\tau_k), \quad \Omega_i'' := \Omega_{i,\frac{2}{3}\epsilon,R+1}(\tau_k), \quad \Omega_i := \Omega_{i,\frac{1}{2}\epsilon,R+2}(\tau_k).$$

Then we have $\Omega_i' \subset \Omega_i'' \subset \Omega_i$. By Lemma 3.18, $\Sigma_{i,\infty}$ converges smoothly to Σ_∞, $\mathcal{S}_i$ converges to $\mathcal{S}$, the domains $\Omega_i', \Omega_i'', \Omega_i$ converge to $\Omega', \Omega'', \Omega$, respectively, where

$$\Omega' := \Omega_{\epsilon,R}(\tau_k), \quad \Omega'' := \Omega_{\frac{2}{3}\epsilon,R+1}(\tau_k), \quad \Omega := \Omega_{\frac{1}{2}\epsilon,R+2}(\tau_k).$$

Note that $\hat{w}_i$ satisfies equation (3.52), which is exactly the same as equation (B.2) in the appendix B. Thus, we can apply Theorem B.3 in appendix B for the function w_i, the domains $\Omega_i', \Omega_i'', \Omega_i$ and the interval $[\tau_{k-1}, \tau_{k+1}]$ to obtain

$$\hat{w}_i(x,\tau_k) \leq C\hat{w}_i(y,\tau_{k+1}) \quad \text{for all } x, y \in \Omega_{i,\epsilon,R}(\tau_k), \tag{3.56}$$

where $C = C(\epsilon, R, b-a, N, \Sigma_\infty, \mathcal{S}_{[a,b]})$ is a constant independent of i. Moreover, there exists a sequence of points $\{z_k\}$ such that

$$z_k \in \Omega_{i,2\epsilon,R}([\tau_{k-1},\tau_k]) \cap \Omega_{i,2\epsilon,R}([\tau_k,\tau_{k+1}]) \neq \emptyset. \tag{3.57}$$

For $s,t \in (a,b)$ with $s < t$, there exist integers k_s and k_t such that $s \in [\tau_{k_s}, \tau_{k_s+1})$ and $t \in (\tau_{k_t}, \tau_{k_t+1}]$. Then we have

$$k_t - k_s \geq 4 \tag{3.58}$$

as in Lemma 3.15. Set

$$\Omega_i' = \Omega_{i,\epsilon,R}(s), \quad \Omega_i'' = \Omega_{i,\frac{2}{3}\epsilon,R+1}(s), \quad \Omega = \Omega_{i,\frac{1}{2}\epsilon,R+2}(s).$$

Applying Theorem B.3 for such sets $\Omega', \Omega'', \Omega$ and the interval $[\tau_{k_s-2}, \tau_{k_s+2}]$ as in Lemma 3.15, we have

$$\hat{w}_i(x,s) \leq C\hat{w}_i(z_{k_s+2},\tau_{k_s+2}) \quad \text{for all } x \in \Omega_{i,\epsilon,R}(s). \tag{3.59}$$

Moreover, (3.56) and (3.57) implies that

$$\hat{w}_i(z_{k_s+2},\tau_{k_s+2}) \leq C\hat{w}_i(z_{k_s+3},\tau_{k_s+3}) \leq \cdots \leq C^N\hat{w}_i(z_{k_t-1},\tau_{k_t-1}), \tag{3.60}$$

where we used (3.58). Similar to the proof of (3.59), we have

$$\hat{w}_i(z_{k_t-1},\tau_{k_t-1}) \leq C\hat{w}_i(y,t) \quad \text{for all } y \in \Omega_{i,\epsilon,R}(t). \tag{3.61}$$

Combining this with (3.59)–(3.61), we have

$$\hat{w}_i(x,s) \leq C\hat{w}_i(y,t) \quad \text{for all } x \in \Omega_{i,\epsilon,R}(s), \quad y \in \Omega_{i,\epsilon,R}(t). \tag{3.62}$$

The constants C in (3.59)–(3.62) depend on ϵ, R, $b-a$, N, Σ_∞ and $\mathcal{S}_{[a,b]}$. The lemma is proved. ∎

The next result shows that the normalized height difference function $\tilde{w}_{i,k}$ has uniformly L^1 estimate away from the singular set near $t = 0$, and the estimate does not depend on i. The proof of this result relies on the growth estimates of $\tilde{w}_{i,\infty}$ near the singular set, which is given in Theorem 4.2 in the next section.

Lemma 3.21. *Fix $\tau \in (0, \frac{1}{2})$. Under the same assumptions as in Lemma 3.18, for each i we can choose k_i sufficiently large such that for any $k \geq k_i$ the normalized height difference function*

$$\tilde{w}_{i,k}(x,s) = \frac{\tilde{u}_{i,k}(x,s)}{\tilde{u}_{i,k}(x_{i,k},1)},$$

where the points $\{x_{i,k}\}$ are chosen as in Lemma 3.19, satisfies the inequality

$$\inf_{s\in[-\tau,0]} \int_{\Sigma_{i,\infty}\cap\tilde{\Omega}_{i,\frac{\epsilon_i}{5},R}} \tilde{w}_{i,k}(x,s) \leq 2W_0. \tag{3.63}$$

Here W_0 is a constant independent of i.

Proof. Fix large $R > 0$. Since $\Sigma_{i,\infty}$ converges to Σ_∞ smoothly, there exist uniform constants $\rho_0, \Xi_0 > 0$ such that for any large i we have $B_R(0) \cap \Sigma_{i,\infty} \in \mathcal{M}_{k_0,2}(\rho_0, \Xi_0)$. Here the set $\mathcal{M}_{k_0,2}(\rho_0, \Xi_0)$ is defined in Definition 4.1. Note that by Lemma 3.17 for each i the function $\tilde{w}_{i,k}$ converges in C^2 to the limit function $\tilde{w}_{i,\infty}$ away from $\mathcal{S}_i$ and $x_{i,k} \to x_{i,0}$ as $k \to +\infty$. Applying Theorem 4.2 to the function

$$\hat{w}_i = \tilde{w}_{i,\infty} e^{-\frac{|x|^2}{8}},$$

we obtain that there exist uniform constants $C = C(\rho_0, \Xi_0, R)$ and $r_1(\rho_0, \Xi_0, R) > 0$ such that

$$\|\tilde{w}_{i,\infty}\|_{L^1((\Sigma_{i,\infty}\cap B_R(0))\times[-\frac{1}{2},0])} \leq C(R,\rho_0,\Xi_0)\|\tilde{w}_{i,\infty}\|_{L^1(K_i)}, \tag{3.64}$$

where K_i is a compact set defined by

$$K_i := \left\{(x,t) \in (\Sigma_{i,\infty} \cap B_{R+1}(0)) \times \left[-1, \frac{1}{2}\right] \,\middle|\, \min_{p\in\mathcal{S}_{i,t}\cap B_{R+1}(0)} d_{g_i}(x,p) \geq r_1\right\},$$

where d_{g_i} denotes the intrinsic distance function of $(\Sigma_{i,\infty}, g_i)$. For any $t \in (-T,T)$, we define

$$K_{i,r}(t) = \left\{x \in \Sigma_{i,\infty} \,\middle|\, \min_{p\in\mathcal{S}_{i,t}\cap B_{R+1}(0)} d_{g_i}(x,p) \geq r\right\}.$$

Since $(\Sigma_{i,\infty}, g_i)$ converges smoothly to $(\Sigma_\infty, g_\infty)$ and $\mathcal{S}_i$ converges to $\mathcal{S}_\infty$ by Lemma 3.18, for any $t \in (-T,T)$, $K_{i,r}(t)$ converges smoothly to a limit set, which we denote by $K_{\infty,r}(t) \subset \Sigma_\infty$. By part (5) of Lemma 3.18, $K_{\infty,r}(t) \cap \mathcal{S}_t = \emptyset$. Note that $K_{\infty,r}(t)$ is defined with respect to the metric g_∞ while $\Omega_{r,R}(t)$ is with respect to the Euclidean metric in $\mathbb{R}^3$. Let

$$r_1' := \frac{1}{2}\min\left\{d(x,p) \,\middle|\, x \in K_{\infty,r_1}(t),\ p \in \mathcal{S}_t,\ t \in [-2,2]\right\} > 0,$$

where $d(x,p)$ denotes the Euclidean distance in $\mathbb{R}^3$. Thus, we have

$$K_{\infty,r_1}(t) \subset \Omega_{r_1',R+1}(t) \subset \Sigma_\infty \quad \text{for all } t \in [-2,2].$$

Since $K_{i,r_1}(t)$ and $\tilde{\Omega}_{i,r_1',R+1}(t)$ converge to $K_{\infty,r_1}(t)$ and $\Omega_{r_1',R+1}(t)$, respectively, for each t, for large i we have

$$K_{i,r_1}(t) \subset \tilde{\Omega}_{i,\frac{1}{2}r_1',R+2}(t) \quad \text{for all } t \in [-2,2].$$

Applying Lemma 3.20 for $\tilde{\Omega}_{i,\frac{1}{2}r_1',R+2}(t)$ and $[-2,2]$, we have

$$\begin{aligned}\tilde{w}_{i,\infty}(x,t) &\le C(r_1', R, \Sigma_\infty, \mathcal{S}_{[-2,2]})\tilde{w}_{i,\infty}(x_{i,0},1)\\ &= C(r_1', R, \Sigma_\infty, \mathcal{S}_{[-2,2]}) \quad \text{for all } (x,t) \in K_i,\end{aligned} \tag{3.65}$$

where we used the fact that $\tilde{w}_{i,\infty}(x_{i,0},1) = 1$. Integrating both sides of (3.65) on K_i, we have

$$\begin{aligned}\|\tilde{w}_{i,\infty}\|_{L^1(K_i)} &\le C(r_1', R, \Sigma_\infty, \mathcal{S}_{[-2,2]})\,\text{Area}_{g_i}(\Sigma_{i,\infty} \cap B_{R+1}(0))\\ &\le C(r_1', R, \Sigma_\infty, \mathcal{S}_{[-2,2]}, N),\end{aligned} \tag{3.66}$$

where we used the upper bound of area ratio in Lemma 3.4 in the last inequality. Combining (3.64) with (3.66), we have

$$\|\tilde{w}_{i,\infty}\|_{L^1((\Sigma_{i,\infty}\cap B_R(0))\times[-\frac{1}{2},0])} \le C(R, \Sigma_\infty, \mathcal{S}_{[-2,2]}, N, \rho_0, \Xi_0). \tag{3.67}$$

Thus, the L^1 norm of $\tilde{w}_{i,\infty}$ is uniformly bounded. Since $\tilde{w}_{i,k}$ converges to $\tilde{w}_{i,\infty}$ on any compact set away from singularities as $k \to +\infty$, we can choose k_i large such that for any $k \ge k_i$,

$$\begin{aligned}\inf_{s\in[-\tau,0]} \int_{\Sigma_{i,\infty}\cap\tilde{\Omega}_{i,\frac{\epsilon_i}{5},R}} \tilde{w}_{i,k}(x,s) &\le 2 \inf_{s\in[-\tau,0]} \int_{\Sigma_{i,\infty}\cap\tilde{\Omega}_{i,\frac{\epsilon_i}{5},R}} \tilde{w}_{i,\infty}(x,s)\\ &\le \frac{2}{\tau}\int_{-\tau}^0 dt \int_{\Sigma_{i,\infty}\cap B_R(0)} \tilde{w}_{i,\infty}(x,t)\\ &\le C(R, \Sigma_\infty, \mathcal{S}_{[-2,2]}, N, \rho_0, \Xi_0, \tau),\end{aligned}$$

where we used inequality (3.67). Thus, inequality (3.63) is proved. ∎

Combining Lemma 3.18, Lemma 3.19 with Lemma 3.21, we have the following result.

Lemma 3.22. *Let $R > 0$ and $\tau \in (0, \frac{1}{2})$. There is a sequence of times $t_i \to \infty$, a self-shrinker $\Sigma_\infty \in \mathcal{C}(N,\rho)$, a locally finite singular set $\mathcal{S}$, and a constant W satisfying the following properties:*

(1) *For any $T > 1$, there exists a subsequence $\{t_{i_k}\}_{k=1}^\infty$ of $\{t_i\}$ such that the sequence $\{\Sigma_{t_{i_k}+s}, -T < s < T\}$ converges locally smoothly to $\Sigma_\infty \in \mathcal{C}(N,\rho)$ away from $\mathcal{S}$.*

(2) *Let $x_0 \in \Sigma_\infty \backslash \mathcal{S}_1$. We define the functions u_i as in (3.25) and w_i by*

$$w_i(x,t) = \frac{u_i(x,t)}{u_i(x_0,1)}.$$

For any $\epsilon > 0$ *and large* t_i*, we have the inequality*

$$\inf_{s\in[t-\tau,t]} \int_{\Omega_{\epsilon,R}(s)} w_i(x,s) \le W \quad \text{for all } t \in [2,T), \tag{3.68}$$

where W *is a constant independent of* ϵ*,* i *and* T*.*

(2) *For any* $I = [a,b] \subset [-1, T-2]$ *and* $K \subset\subset (\Sigma_\infty \cap B_R(0)) \backslash \mathcal{S}_I$*, there exists a constant* $C = C(\epsilon, K, \mathcal{S}_I, a, b)$ *such that*

$$0 < w_i(x,t) < C \quad \text{for all } (x,t) \in K \times I. \tag{3.69}$$

Moreover, if $a \in [2, T-2]$*, there exists* $C' = C'(K, \Sigma_\infty, \mathcal{S}_{[0,a+1]})$ *independent of* b *such that*

$$w_i(x,a) \ge C' \quad \text{for all } x \in K. \tag{3.70}$$

Proof. Fix a sequence of $\epsilon_i \to 0$. We choose $t_i = t_{i,k_i}$ with k_i large such that Lemma 3.19 and Lemma 3.21 hold. Note that $u_i(x,s) = u_{i,k_i}(x,s)$ is the height difference function of $\Sigma_{t_{i,k_i}+s}$ over Σ_∞. Then for any $T > 1$ the sequence $\{\Sigma_{t_i+s}, -T < s < T\}$ converges locally smoothly to $\Sigma_\infty \in \mathcal{C}(N,\rho)$ away from $\mathcal{S}$. Note that the limit self-shrinker Σ_∞ is independent of the choice of T by Lemma 3.4. For any $\epsilon > 0$, we have $\epsilon_i \in (0,\epsilon)$ for large i. Moreover, for large t_i we have

$$\begin{aligned}\inf_{s\in[t-\tau,t]} \int_{\Omega_{\epsilon,R}(s)} w_i(x,s) &\le \frac{2}{u_i(x_0,1)} \inf_{s\in[t-\tau,t]} |\mathbf{TN}(\Sigma_{t_i+s}, \epsilon_i, R)| \\ &\le \frac{4}{u_i(x_0,1)} \inf_{s\in[-\tau,0]} |\mathbf{TN}(\Sigma_{t_i+s}, \epsilon_i, R)|,\end{aligned}$$

where we used Lemma 3.14 in the first inequality and (3.48) in the second inequality. Note that (3.49) implies that

$$|\mathbf{TN}(\Sigma_{t_i+s}, \epsilon_i, R)| \le 2 \int_{\tilde{\Omega}_{i,\frac{\epsilon_i}{5},R}} \tilde{u}_{i,k_i}(x,s).$$

Thus, we have

$$\begin{aligned}\inf_{s\in[t-\tau,t]} \int_{\Omega_{\epsilon,R}(s)} w_i(x,s) &\le \frac{4\tilde{u}_{i,k_i}(x_{i,k_i},1)}{u_i(x_0,1)} \cdot \frac{1}{\tilde{u}_{i,k_i}(x_{i,k_i},1)} \inf_{s\in[-\tau,0]} |\mathbf{TN}(\Sigma_{t_i+s}, \epsilon_i, R)| \\ &\le \frac{8\tilde{u}_{i,k_i}(x_{i,k_i},1)}{u_i(x_0,1)} \cdot \frac{1}{\tilde{u}_{i,k_i}(x_{i,k_i},1)} \inf_{t\in[-\tau,0]} \int_{\tilde{\Omega}_{i,\frac{\epsilon_i}{5},R}} \tilde{u}_{i,k_i}(x,s) \\ &\le \frac{8\tilde{u}_{i,k_i}(x_{i,k_i},1)}{u_i(x_0,1)} \cdot \inf_{s\in[-\tau,0]} \int_{\tilde{\Omega}_{i,\frac{\epsilon_i}{5},R}} \tilde{w}_{i,k_i}(x,s) \\ &\le 16W_0 \cdot \frac{\tilde{u}_{i,k_i}(x_{i,k_i},1)}{u_i(x_0,1)} \le 32W_0,\end{aligned} \tag{3.71}$$

where we used (3.63) in the fourth inequality and (3.51) in the last inequality. As in the proof of Lemma 3.16, (3.71) implies a uniform upper bound of w_i on K, and we also have the lower bounds (3.69)–(3.70) of w_i. The lemma is proved. ∎

Proposition 3.23. *Under the same assumption as in Lemma 3.22, w_i converges in C^2 to a positive function $w(x,t)$ satisfying (3.47) on $(\Sigma_\infty \times (0,\infty))\backslash \mathcal{S}$ with $w(x_0,1) = 1$ and*

$$\inf_{s\in[t-\tau,t]} \int_{\Sigma_\infty\cap B_R(0)} w(x,s) \leq W \quad \text{for all } t \in [1,\infty). \tag{3.72}$$

Moreover, for any $a \in [2,\infty)$ there exists a constant $C = C(a, \Sigma_\infty, \mathcal{S}_{[1,a+1]}, K) > 0$ such that the function $w(x,t)$ satisfies

$$\int_{\Sigma_\infty\cap B_R(0)} w(x,a) \geq C. \tag{3.73}$$

Proof. For any $I \subset [1, T-2]$ and $K \subset\subset (\Sigma_\infty \cap B_R(0))\backslash \mathcal{S}_I$, by Lemma 3.22 and the interior estimates of the parabolic equations (cf. [50, Theorem 4.9]), we have the space-time $C^{2,\alpha}$ estimates of w_i on $K \times I$. Taking the limit $i \to +\infty$, w_i converges in C^2 to a limit function $w(x,t)$ on $K \times I$ with estimate (3.69)–(3.70). Moreover, (3.72) holds on I by (3.68) and (3.73) holds on $I \cap [2,\infty)$ by (3.70). Since Σ_∞ is independent of the choice of T and the estimates of w are independent of T, by taking $T \to +\infty$ we obtain a function, still denoted by w, on $(\Sigma_\infty \times (0,\infty))\backslash \mathcal{S}$ with estimates (3.72)–(3.73). The proposition is proved. ∎

The following result was used in the proof of Lemma 3.19.

Lemma 3.24. *Let $\Sigma \subset \mathbb{R}^3$ be a surface properly embedded in $B_{r_0}(x_0)$ with*

$$|A|(x) \leq \frac{1}{r_0}, \quad x \in B_{r_0}(x_0) \cap \Sigma.$$

Assume that Σ_{u_i} is the graph of a functions u_i over Σ for $i = 1,2$ and $\Sigma_{u_1} \cap \Sigma_{u_2} = \emptyset$. Let $P \in \Sigma$, l_P the normal of Σ at the point P, $G = l_P \cap \Sigma_{u_1}$ and $Q = l_P \cap \Sigma_{u_2}$. For any $\theta \in [0, \frac{\pi}{2})$, we denote by l_θ the line which passes through Q and has angle θ with the line l_P. Let $B = \Sigma_{u_1} \cap l_\theta$. Then there are two constants $\epsilon \in (0,1)$ and $\theta_0 > 0$ both depending only on r_0 such that if $\theta \in (0,\theta_0)$ and

$$\|u_1\|_{C^1(\Sigma\cap B_{r_0}(x_0))} + \|u_2\|_{C^1(\Sigma\cap B_{r_0}(x_0))} \leq \epsilon, \tag{3.74}$$

then we have

$$|GQ| \leq 2|BQ|.$$

Proof of Lemma 3.24. Without loss of generality, we assume that the tangent plane of Σ at P is the plane $\pi := \{(x_1, x_1, 0) \mid x_1, x_2 \in \mathbb{R}\}$ and the point P is the origin O of $\mathbb{R}^3$. Let $\hat{B}_{\delta_0}(0) = \{(x_1,x_2,x_3) \mid x_1^2 + x_2^2 < \delta_0^2\}$. By Lemma 2.2, there exists $\delta_0 = \delta_0(r_0) > 0$ such that $\Sigma \cap \hat{B}_{\delta_0}(0)$ can be written as a graph of a function f over the plane π,

$$\Sigma \cap \hat{B}_{\delta_0}(0) = \{(x_1, x_2, f(x_1,x_2)) \mid |x| < \delta_0\}, \tag{3.75}$$

where $x = (x_1,x_2)$, and the graph function f satisfies

$$f(0) = 0, \quad Df(0) = 0, \quad |\nabla f|(y) \leq C_0|y|. \tag{3.76}$$

Here the constant C_0 depends only on r_0. Note that the coordinates of G and Q are given by $G = (0, 0, u_1(0))$ and $Q = (0, 0, u_2(0))$, respectively. For the point $B \in \Sigma_{u_1} \cap l_\theta$, we define the point $E \in \Sigma$ the projection of B onto Σ, which means

$$\overrightarrow{OE} + u_1(E)\mathbf{n}(E) = \overrightarrow{OB}, \tag{3.77}$$

where $\mathbf{n}(E)$ is the unit normal vector of Σ at E.

We claim that there exist $\epsilon_0 = \epsilon_0(\delta_0) \in (0, 1)$ and $\theta_0 = \arctan 3 > 0$ such that if $\theta \in (0, \theta_0)$ and (3.74) holds for some $\epsilon \in (0, \epsilon_0)$, then $E \in \Sigma \cap \hat{B}_{\delta_0}(0)$. In fact, we assume that $\theta_0 = \arctan \frac{\delta_0}{4\epsilon}$. Then for any $\theta \in (0, \theta_0)$, we have

$$|\overrightarrow{OB}| \le 2\epsilon + \frac{\delta_0}{2}.$$

Combining this with (3.77), we have

$$|\overrightarrow{OE}| \le |\overrightarrow{OB}| + |u_1(E)| \le 3\epsilon + \frac{\delta_0}{2} \le \frac{3}{4}\delta_0,$$

where we choose $\epsilon \in (0, \frac{1}{12}\delta_0)$. Therefore, by (3.75) we have $E \in \Sigma \cap \hat{B}_{\delta_0}(0)$. The claim is proved.

Assume that $E = (y, f(y)) \in \Sigma \cap \hat{B}_\delta(0)$ with $y = (y_1, y_2)$ and $\delta \in (0, \delta_0)$. Note that the normal vector at E is given by

$$\mathbf{n}(E) = \frac{(-\partial_{y_1} f(y), -\partial_{y_2} f(y), 1)}{\sqrt{1 + |\nabla f(y)|^2}},$$

and by (3.77) the coordinates of $B = (B_1, B_2, B_3)$ are given by

$$B_1 = y_1 - \frac{u_1(y)\partial_{y_1} f(y)}{\sqrt{1 + |\nabla f(y)|^2}}, \tag{3.78}$$

$$B_2 = y_2 - \frac{u_1(y)\partial_{y_2} f(y)}{\sqrt{1 + |\nabla f(y)|^2}}, \tag{3.79}$$

$$B_3 = f(y) + \frac{u_1(y)}{\sqrt{1 + |\nabla f(y)|^2}}, \tag{3.80}$$

where we write $u_1(y) = u_1(y_1, y_2, f(y_1, y_2))$ for simplicity. Since

$$B_1^2 + B_2^2 = |QB|^2 \sin^2 \theta,$$

using (3.78)–(3.79) we have

$$y_1^2 + y_2^2 + \frac{u_1(y)^2 |\nabla f(y)|^2}{1 + |\nabla f(y)|^2} - 2\frac{u_1(y)\langle y, \nabla f(y)\rangle}{\sqrt{1 + |\nabla f(y)|^2}} = |QB|^2 \sin^2 \theta,$$

where $\langle y, \nabla f(y)\rangle = y_1 \partial_{y_1} f(y) + y_2 \partial_{y_2} f(y)$. Combining this with (3.76), we have

$$\begin{aligned} |QB|^2 \sin^2 \theta &\ge y_1^2 + y_2^2 - 2\frac{u_1(y)\langle y, \nabla f(y)\rangle}{\sqrt{1 + |\nabla f(y)|^2}} \\ &\ge (1 - 2C_0|u_1(y)|)(y_1^2 + y_2^2) \ge (1 - 2C_0\epsilon)(y_1^2 + y_2^2). \end{aligned}$$

Thus, if $\epsilon \in (0, \frac{1}{2C_0})$, we have

$$|y|^2 = y_1^2 + y_2^2 \le \frac{|QB|^2 \sin^2\theta}{1 - 2C_0\epsilon}. \tag{3.81}$$

Since l_θ has the angle θ with the line l_P, we assume that the unit direction vector of l_θ is $\vec{v} = (v_1, v_2, \cos\theta)$. Thus, we have

$$\begin{aligned}|QB| &= |\langle \overrightarrow{QB}, \vec{v}\rangle| \\ &= |B_1 v_1 + B_2 v_2 + (B_3 - u_2(0))\cos\theta| \\ &\ge |(B_3 - u_2(0))\cos\theta| - |B_1 v_1| - |B_2 v_2|. \end{aligned}\tag{3.82}$$

Note that by (3.76),

$$\begin{aligned}|B_3 - u_2(0)| &= \left| f(y) + \frac{u_1(y)}{\sqrt{1 + |\nabla f(y)|^2}} - u_2(0)\right| \\ &\ge |u_1(0) - u_2(0)| - |u_1(0) - u_1(y)| \\ &\quad - |u_1(y)| \cdot \left|\frac{1}{\sqrt{1 + |\nabla f(y)|^2}} - 1\right| - |f(y)| \\ &\ge |u_1(0) - u_2(0)| - \max_{B_\delta(0)} |\nabla u_1| \cdot |y| - C_0\Big(1 + \max_{B_\delta(0)} |u_1|\Big)|y|^2 \\ &\ge |u_1(0) - u_2(0)| - C_1|y|, \end{aligned}\tag{3.83}$$

where $C_1 = \epsilon + C_0(1+\epsilon)\delta_0$, and by (3.76), (3.78)–(3.79) we have

$$|B_1| \le (1 + C_0\epsilon)|y|, \quad |B_2| \le (1 + C_0\epsilon)|y|. \tag{3.84}$$

Combining (3.81)–(3.84), we have

$$\begin{aligned}|QB| &\ge |u_1(0) - u_2(0)|\cos\theta - C_1|y| \\ &\ge |u_1(0) - u_2(0)|\cos\theta - C_1\sin\theta|QB|. \end{aligned}$$

This implies that

$$\frac{|GQ|}{|QB|} = \frac{|u_1(0) - u_2(0)|}{|QB|} \le \frac{1 + C_1\sin\theta}{\cos\theta} \le 2$$

if we choose θ sufficiently small. Thus, the lemma is proved. ■

3.5. The L-stability of the limit self-shrinker

In this subsection, we show that the limit self-shrinker is L-stable. The rough idea is similar to that of [46], but the details are much more complicated. Compared with [46], the singularities here no longer move along straight lines, we cannot choose time large enough such that a given compact set does not contain the singularities (cf. [46, Lemma 4.13]). Therefore, we have to choose a cutoff function near the singularities and analyze the asymptotical behavior of the positive solution near the singular set. The analysis of the asymptotical behavior is very difficult and we delay the arguments in the next section.

The main result in this subsection is the following lemma.

Lemma 3.25. *Fix $R > 1$. Let $\{t_i\}$ be the sequence of times and $\Sigma_\infty \in \mathcal{C}(N, \rho)$ the self-shrinker in Lemma 3.22. Then we have*

$$-\int_{\Sigma_\infty} (\psi L\psi)e^{-\frac{|x|^2}{4}} \ge 0 \tag{3.85}$$

for any smooth function $\psi \in C_0^\infty(\Sigma_{\infty,R})$.

Let w be the function obtained in Proposition 3.23 and $v = \log w$. Then v is a function satisfying

$$\frac{\partial v}{\partial t} = \Delta_0 v + |A|^2 + \frac{1}{2} - \frac{1}{2}\langle x, \nabla v\rangle + |\nabla v|^2 \quad \text{for all } (x,t) \in (\Sigma_\infty \times (0,\infty))\backslash \mathcal{S}.$$

Let $I = [a,b] \subset (0,\infty)$. We assume that $\phi(x,t)$ is a function satisfying the properties that for any $t \in I$ we have

$$\phi(\cdot, t) \in W_0^{1,2}(\Sigma_{\infty,R}), \quad \overline{\mathrm{Supp}(\phi(\cdot,t))} \cap \mathcal{S}_t = \emptyset. \tag{3.86}$$

Then for any $t \in I$, we have

$$\begin{aligned} 0 &= \int_{\Sigma_\infty} \mathrm{div}\big(\phi^2 e^{-\frac{|x|^2}{4}} \nabla v\big) \\ &= \int_{\Sigma_\infty} \Big(2\phi\langle\nabla\phi, \nabla v\rangle + \Big(\frac{\partial v}{\partial t} - \frac{1}{2} - |A|^2 - |\nabla v|^2\Big)\phi^2\Big)e^{-\frac{|x|^2}{4}} \\ &\le \int_{\Sigma_\infty} \Big(|\nabla\phi|^2 - \frac{1}{2}\phi^2 - |A|^2\phi^2 + \frac{\partial v}{\partial t}\phi^2\Big)e^{-\frac{|x|^2}{4}}. \end{aligned}$$

This implies that for any $t \in I$,

$$\begin{aligned} -\int_{\Sigma_\infty} (\phi L\phi)e^{-\frac{|x|^2}{4}} &\ge -\int_{\Sigma_\infty} \frac{\partial v}{\partial t}\phi^2 e^{-\frac{|x|^2}{4}} \\ &= -\frac{d}{dt}\int_{\Sigma_\infty} v\phi^2 e^{-\frac{|x|^2}{4}} + \int_{\Sigma_\infty} 2v\phi\frac{\partial\phi}{\partial t}e^{-\frac{|x|^2}{4}}. \end{aligned}$$

Integrating both sides with respect to $t \in I$, we have

$$\begin{aligned} -\int_a^b \int_{\Sigma_\infty} (\phi L\phi)e^{-\frac{|x|^2}{4}} \ge \int_{\Sigma_\infty} v\phi^2 e^{-\frac{|x|^2}{4}}\Big|_{t=a} - \int_{\Sigma_\infty} v\phi^2 e^{-\frac{|x|^2}{4}}\Big|_{t=b} \\ + \int_a^b \int_{\Sigma_\infty} 2v\phi\frac{\partial\phi}{\partial t}e^{-\frac{|x|^2}{4}}. \end{aligned} \tag{3.87}$$

To get inequality (3.85), the main difficulty is to estimate the last term of (3.87). Using a cutoff function inspired by [46], we will see that the last term of (3.87) depends on the asymptotical behavior of w near the singular set.

We now construct the cutoff function near the singular set. Let $\{\xi_1(t), \xi_2(t), \ldots, \xi_l(t)\}$ ($t \in I$) be σ-Lipschitz curves on Σ_∞. We denote by

$$\Gamma_k = \{(\xi_k(t), t) \mid t \in I\} \subset \Sigma_\infty \times I, \quad \Gamma = \bigcup_{k=1}^{l} \Gamma_k.$$

Choose $0<\delta<\rho<1$. We define the function η on $\mathbb{R}$ by

$$\eta(s)=\begin{cases}\frac{\log\rho}{\log|s|}, & 0<|s|<\rho,\\ 1, & |s|\geq\rho\end{cases}$$

and the function $\beta(s)\in C^\infty(\mathbb{R})$ such that $\beta(s)=0$ for $|s|<\frac{\delta}{2}$, $\beta(s)=1$ for $|s|\geq\delta$, $0\leq\beta(s)\leq 1$ and $|\nabla\beta|\leq\frac{3}{\delta}$. We define the function $f_{\delta,\rho}$ on $\Sigma_\infty\times I$ by

$$f_{\delta,\rho}(x,t)=\prod_{k=1}^{l}\big(\eta(\mathbf{r}_k(x,t))\beta(\mathbf{r}_k(x,t))\big)\in W^{1,2}((\Sigma_\infty\times I)\backslash\Gamma),$$

where

$$\mathbf{r}_k(x,t)=d_g(x,\xi_k(t)). \tag{3.88}$$

For any $\psi(x)\in C_0^\infty(\Sigma_{\infty,R})$, we define

$$\phi(x,t)=\psi(x)f_{\delta,\rho}(x,t). \tag{3.89}$$

Then $\phi(x,t)$ satisfies the properties in (3.86). With loss of generality, we assume that $\sup_{\Sigma_\infty}|\psi|\leq 1$. Then we have:

Lemma 3.26. *For any small $\epsilon>0$ we have*

$$-\int_{\Sigma_\infty}(\phi L\phi)e^{-\frac{|x|^2}{4}}\leq-\int_{\Sigma_\infty}\psi L(\psi)e^{-\frac{|x|^2}{4}}+\Psi(\epsilon,\rho,\delta\mid\Sigma_{\infty,R}), \tag{3.90}$$

where Ψ depends on ρ,δ,ϵ and the geometry of $\Sigma_{\infty,R}$ and satisfies

$$\lim_{\epsilon\to0}\lim_{\rho\to0}\lim_{\delta\to0}\Psi(\epsilon,\rho,\delta\mid\Sigma_{\infty,R})=0. \tag{3.91}$$

Proof. Since the function $\phi(x,t)=\psi(x)f_{\delta,\rho}(x,t)$ satisfies

$$|\nabla\phi|^2\leq(1+\epsilon)f_{\delta,\rho}^2|\nabla\psi|^2+\left(1+\frac{1}{\epsilon}\right)\psi^2|\nabla f_{\delta,\rho}|^2,$$

we have

$$\begin{aligned}-\int_{\Sigma_\infty}\phi L(\phi)e^{-\frac{|x|^2}{4}}&=\int_{\Sigma_\infty}\left(|\nabla\phi|^2-\left(\frac{1}{2}+|A|^2\right)\phi^2\right)e^{-\frac{|x|^2}{4}}\\
&\leq\int_{\Sigma_\infty}\left(|\nabla\psi|^2-\left(\frac{1}{2}+|A|^2\right)\psi^2\right)e^{-\frac{|x|^2}{4}}\\
&\quad+\int_{\Sigma_\infty}\left((1+\epsilon)f_{\delta,\rho}^2-1\right)|\nabla\psi|^2e^{-\frac{|x|^2}{4}}\\
&\quad+\int_{\Sigma_\infty}\left(\frac{1}{2}+|A|^2\right)(1-f_{\delta,\rho}^2)\psi^2\,e^{-\frac{|x|^2}{4}}\\
&\quad+\left(1+\frac{1}{\epsilon}\right)\int_{\Sigma_\infty}\psi^2|\nabla f_{\delta,\rho}|^2\,e^{-\frac{|x|^2}{4}}\\
&=:I_0+I_1+I_2+I_3.\end{aligned} \tag{3.92}$$

Note that $|f_{\delta,\rho}| \le 1$ and $\lim_{\rho\to 0}\lim_{\delta\to 0} f_{\delta,\rho}(x,t) = 1$ for any $(x,t) \in (\Sigma \times I)\backslash\Gamma$. The Lebesgue dominated convergence theorem implies that

$$\lim_{\epsilon\to 0}\lim_{\rho\to 0}\lim_{\delta\to 0} I_1 = 0, \quad \lim_{\rho\to 0}\lim_{\delta\to 0} I_2 = 0. \tag{3.93}$$

We next estimate I_3. Let $f_k(x,t) = \eta(\mathbf{r}_k(x,t))\beta(\mathbf{r}_k(x,t))$. We define

$$\Xi_R := \inf\left\{\Xi > 0 \;\middle|\; \frac{1}{\Xi}\delta_{ij} \le g_{ij}(x) \le \Xi\delta_{ij} \text{ for all } x \in B_R(0)\cap\Sigma_\infty\right\},$$

where g_{ij} is the induced metric on Σ_∞. Note that

$$\int_{\Sigma_\infty} |\nabla f_k|^2 e^{-\frac{|x|^2}{4}} \le 2\int_{\Sigma_\infty} \left(\beta^2|\nabla\eta|^2 + \eta^2|\nabla\beta|^2\right) e^{-\frac{|x|^2}{4}}. \tag{3.94}$$

We estimate

$$\begin{aligned}\int_{\Sigma_\infty} \beta(\mathbf{r}_k)^2|\nabla(\eta(\mathbf{r}_k))|^2 e^{-\frac{|x|^2}{4}} &\le \int_{\frac{\delta}{2}\le \mathbf{r}_k\le\rho} (\eta'(\mathbf{r}_k))^2 \le C\int_{\frac{\delta}{2}}^{\rho} \frac{(\log\rho)^2}{s(\log s)^4}\, ds\\ &\le C\left(\frac{1}{|\log\rho|} + \frac{(\log\rho)^2}{|\log\frac{\delta}{2}|^3}\right),\end{aligned} \tag{3.95}$$

where C is a constant depending on the metric g. Moreover,

$$\begin{aligned}\int_{\Sigma_\infty} \eta^2|\nabla\beta|^2 e^{-\frac{|x|^2}{4}} &\le \int_{\frac{\delta}{2}\le\mathbf{r}_k\le\delta} \eta(\mathbf{r}_k)^2(\beta'(\mathbf{r}_k))^2|\nabla\mathbf{r}_k|^2\\ &\le C\int_{\frac{\delta}{2}}^{\delta} \frac{(\log\rho)^2}{(\log s)^2}\cdot\frac{4}{\delta^2}\cdot s\, ds\\ &\le C\int_{\frac{\delta}{2}}^{\delta}\frac{(\log\rho)^2}{s(\log s)^2}\, ds \le C\frac{(\log\rho)^2}{|\log\delta|},\end{aligned}$$

where C is a constant depending on the metric g. Combining this with (3.94) and (3.95), we have

$$\begin{aligned}\int_{\Sigma_\infty} |\nabla f_k|^2 e^{-\frac{|x|^2}{4}} &\le 2\int_{\Sigma_\infty}\left(\beta^2|\nabla\eta|^2 + \eta^2|\nabla\beta|^2\right) e^{-\frac{|x|^2}{4}}\\ &\le C\left(\frac{1}{|\log\rho|} + \frac{|\log\rho|^2}{|\log\delta|}\right).\end{aligned}$$

Since $|\psi| \le 1$ and $|f_k| \le 1$, we have

$$\begin{aligned}\int_{\Sigma_\infty} \psi^2|\nabla f_{\delta,\rho}|^2 e^{-\frac{|x|^2}{4}} &\le l\int_{\Sigma_\infty}\sum_{k=1}^{l}|\nabla f_k|^2 e^{-\frac{|x|^2}{4}}\\ &\le C(l,g)\left(\frac{1}{|\log\rho|} + \frac{|\log\rho|^2}{|\log\delta|}\right).\end{aligned}$$

Therefore, we have

$$\lim_{\rho\to 0}\lim_{\delta\to 0} I_3 = 0. \tag{3.96}$$

Combining (3.93)–(3.96) with (3.92), we have (3.90) and (3.91). ■

Lemma 3.27. *For the function ϕ defined by* (3.89), *we have*

$$\int_{\Sigma_\infty} 2v\phi \frac{\partial \phi}{\partial t} e^{-\frac{|x|^2}{4}} \geq -2\sigma(\log\rho)^2 \sum_{k=1}^{l} \Big(\int_{A_t^{(k)}(\frac{\delta}{2},\delta)\cap F_t^{(k)}(\frac{\delta}{2})} \frac{3|v|}{\delta|\log \mathbf{r}_k|^2} + \int_{A_t^{(k)}(\frac{\delta}{2},\rho)\cap F_t^{(k)}(\frac{\delta}{2})} \frac{|v|}{\mathbf{r}_k|\log \mathbf{r}_k|^3} \Big), \tag{3.97}$$

where $F_t^{(k)}(\delta)$ and $A_t^{(k)}(\delta,\rho)$ are defined by

$$F_t^{(k)}(\delta) = \bigcap_{i\neq k} \{x \in \Sigma_\infty \mid \mathbf{r}_i(x,t) \geq \delta\}, \tag{3.98}$$

$$A_t^{(k)}(\delta,\rho) = \{x \in \Sigma_\infty \mid \delta < \mathbf{r}_k(x,t) < \rho\}. \tag{3.99}$$

Proof. We use the same notations as in the proof of Lemma 3.26. Direct calculation shows that

$$\begin{aligned} \Big| \int_{\Sigma_\infty} 2v\phi \frac{\partial \phi}{\partial t} e^{-\frac{|x|^2}{4}} \Big| &\leq \sum_{k=1}^{l} \int_{\Sigma_\infty \cap F_t^{(k)}(\frac{\delta}{2})} 2|v| f_k \Big| \frac{\partial f_k}{\partial t} \Big| e^{-\frac{|x|^2}{4}} \\ &\leq \sum_{k=1}^{l} \int_{\Sigma_\infty \cap F_t^{(k)}(\frac{\delta}{2})} 2|v|\eta(\mathbf{r}_k)\beta(\mathbf{r}_k)\big(|\eta'(\mathbf{r}_k)|\beta(\mathbf{r}_k) \\ &\qquad + |\beta'(\mathbf{r}_k)|\eta(\mathbf{r}_k)\big) \Big| \frac{\partial \mathbf{r}_k}{\partial t} \Big| e^{-\frac{|x|^2}{4}}. \end{aligned} \tag{3.100}$$

Note that for a.e. $t \in I$, $|\frac{\partial \mathbf{r}_k}{\partial t}| \leq |\xi_k'(t)| \leq \sigma$, and we assumed that $\sup_{\Sigma_\infty} |\psi| \leq 1$. Therefore, using the definition of η and β we have

$$\begin{aligned} &\int_{\Sigma_\infty \cap F_t^{(k)}(\frac{\delta}{2})} 2|v|\eta(\mathbf{r}_k)\beta^2(\mathbf{r}_k)|\eta'(\mathbf{r}_k)| \cdot \Big| \frac{\partial \mathbf{r}_k}{\partial t} \Big| e^{-\frac{|x|^2}{4}} \\ &\leq 2\sigma \int_{\Sigma_\infty \cap F_t^{(k)}(\frac{\delta}{2})} \eta(\mathbf{r}_k)\beta(\mathbf{r}_k)^2|v| \cdot |\eta'(\mathbf{r}_k)| \\ &\leq 2\sigma \int_{A_t^{(k)}(\delta,\rho)\cap F_t^{(k)}(\frac{\delta}{2})} |v| \frac{(\log\rho)^2}{|\log \mathbf{r}_k|^3 \mathbf{r}_k}, \end{aligned} \tag{3.101}$$

$$\begin{aligned} &\int_{\Sigma_\infty \cap F_t^{(k)}(\frac{\delta}{2})} 2|v|\eta(\mathbf{r}_k)^2\beta(\mathbf{r}_k)|\beta'| \cdot \Big| \frac{\partial \mathbf{r}_k}{\partial t} \Big| e^{-\frac{|x|^2}{4}} \\ &\leq 2\sigma \int_{\Sigma_\infty \cap F_t^{(k)}(\frac{\delta}{2})} |v|\eta^2(\mathbf{r}_k)\beta(\mathbf{r}_k)|\beta'(\mathbf{r}_k)| \\ &\leq \frac{6\sigma}{\delta} \int_{A_t^{(k)}(\frac{\delta}{2},\delta)\cap F_t^{(k)}(\frac{\delta}{2})} |v| \Big(\frac{\log\rho}{\log \mathbf{r}_k} \Big)^2. \end{aligned} \tag{3.102}$$

Combining (3.100)–(3.102), we have (3.97). ■

Lemma 3.28. *For any $t > 0$, we have*

$$\lim_{\rho\to 0}\lim_{\delta\to 0}(\log\rho)^2\int_a^b dt\int_{A_t^{(k)}(\delta,\rho)\cap F_t^{(k)}(\frac{\delta}{2})}\frac{|v|}{\mathbf{r}_k|\log\mathbf{r}_k|^3}=0, \tag{3.103}$$

$$\lim_{\rho\to 0}\lim_{\delta\to 0}\frac{(\log\rho)^2}{\delta}\int_a^b dt\int_{A_t^{(k)}(\frac{\delta}{2},\delta)\cap F_t^{(k)}(\frac{\delta}{2})}\frac{|v|}{|\log\mathbf{r}_k|^2}=0. \tag{3.104}$$

Proof. Since $w(x,t)$ satisfies (3.47) away from the singular set, the function

$$f(x,t)=w(x,t)e^{-\frac{|x|^2}{8}}$$

satisfies the equation

$$\frac{\partial f}{\partial t}=\Delta f+\left(|A|^2+\frac{3}{4}-\frac{1}{16}|x|^2\right)f.$$

By Theorem 4.2, we have

$$\lim_{\rho\to 0}\lim_{\delta\to 0}\int_a^b dt\int_{A_t^{(k)}(\delta,\rho)\cap F_t^{(k)}(\frac{\delta}{2})}\frac{f}{\mathbf{r}_k|\log\mathbf{r}_k|}=0, \tag{3.105}$$

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_a^b dt\int_{A_t^{(k)}(\frac{\delta}{2},\delta)\cap F_t^{(k)}(\frac{\delta}{2})}f<+\infty. \tag{3.106}$$

Since near the singular curve $\xi(t)$, the function w is large and we have $v=\log w\le w$. Thus, (3.105)–(3.106) also hold for v, and this directly implies (3.103)–(3.104). The lemma is proved. ∎

Combining the above results, we can show Lemma 3.25.

Proof of Lemma 3.25. Combining Lemma 3.26, Lemma 3.27 with inequality (3.87), we have

$$\begin{aligned}
&-(b-a)\int_{\Sigma_\infty}\psi L(\psi)e^{-\frac{|x|^2}{4}}+\Psi(\epsilon,\rho,\delta\,|\,\Sigma_{\infty,R})(b-a)\\
&\ge\int_{\Sigma_\infty}v\phi^2e^{-\frac{|x|^2}{4}}\Big|_{t=a}-\int_{\Sigma_\infty}v\phi^2e^{-\frac{|x|^2}{4}}\Big|_{t=b}\\
&\quad-2\sigma(\log\rho)^2\sum_{k=1}^{l}\Bigg(\int_a^b\int_{A_t^{(k)}(\frac{\delta}{2},\delta)\cap F_t^{(k)}(\frac{\delta}{2})}\frac{3|v|}{\delta|\log\mathbf{r}_k|^2}\\
&\quad+\int_a^b\int_{A_t^{(k)}(\delta,\rho)\cap F_t^{(k)}(\frac{\delta}{2})}\frac{|v|}{\mathbf{r}_k|\log\mathbf{r}_k|^3}\Bigg).
\end{aligned}\tag{3.107}$$

Taking $\delta\to 0$ and next $\rho\to 0$, and then $\epsilon\to 0$ in (3.107), we get

$$-\int_{\Sigma_\infty}\psi L(\psi)e^{-\frac{|x|^2}{4}}\ge\frac{1}{b-a}\left(\int_{\Sigma_\infty}v\psi^2e^{-\frac{|x|^2}{4}}\Big|_{t=a}-\int_{\Sigma_\infty}v\psi^2e^{-\frac{|x|^2}{4}}\Big|_{t=b}\right). \tag{3.108}$$

Note that by Proposition 3.23, $w(x,t)$ is a function on $(\Sigma_\infty \times (0,\infty))\backslash \mathcal{S}$ with uniform estimates (3.72) and (3.73). Thus, there is a sequence $b_i \to +\infty$ such that

$$\int_{\Sigma_\infty} v\psi^2 e^{-\frac{|x|^2}{4}} \, d\mu_\infty\Big|_{t=b_i} \le \int_{\Sigma_\infty} w \, d\mu_\infty\Big|_{t=b_i} \le W$$

for a constant W. Therefore, by taking $b_i \to +\infty$ and $a = 2$ in (3.108) we get (3.85). The lemma is proved. ∎

3.6. *Proof of Theorem* 3.1

In this subsection, we show Theorem 3.1. First, using Lemma 3.9 and the compactness result of Colding and Minicozzi [20] we have the following result.

Lemma 3.29. *Let $R, N > 0$ and let ρ be an increasing positive function. For any $\delta > 0$, there exists a constant $\xi = \xi(R,N,\rho,\delta) > 0$ such that for any $\Sigma \in \mathcal{C}(N,\rho)$ and any $r \in (0,\xi]$ we have*

$$1-\delta \le \frac{\mathrm{Area}(\Sigma \cap B_r(x))}{\pi r^2} \le 1+\delta \quad \text{for all } x \in B_R(0)\cap\Sigma. \tag{3.109}$$

Proof. We show that there exists a positive constant $C_R = C(R,N,\rho)$ such that for all $\Sigma \in \mathcal{C}(N,\rho)$ we have $\sup_{\Sigma\cap B_{R+1}(0)} |A| \le C_R$. For otherwise, we can find a sequence $\Sigma_i \in \mathcal{C}(N,\rho)$ such that

$$\sup_{\Sigma_i\cap B_{R+1}(0)} |A| \to +\infty. \tag{3.110}$$

On the other hand, by the compactness theorem of Colding and Minicozzi [20], Σ_i converges smoothly to $\Sigma_\infty \in \mathcal{C}(N,\rho)$, which has bounded $|A|$ on any compact set. This contradicts (3.110).

As $\sup_{\Sigma\cap B_{R+1}(0)} |A|$ is uniformly bounded by C_R, estimate (3.109) follows directly from Lemma 2.3. The lemma is proved. ∎

Using the uniform upper bound of the area ratio and Lemma 3.4, we have the following result.

Lemma 3.30. *Under the same assumption as in Lemma 3.4, if $\{\Sigma_{t_i}\}$ converges locally smoothly to Σ_∞ with multiplicity $m \in \mathbb{N}$ away from a locally finite singular set $\mathcal{S}_0$, then for any $x_i \in \Sigma_{t_i}\cap B_R(0)$ with $x_i \to x_\infty \in \Sigma_\infty \cap B_{R+1}(0)$ and $r > 0$ we have*

$$\lim_{i\to+\infty} \mathrm{Area}(\Sigma_{t_i}\cap B_r(x_i)) = m\cdot \mathrm{Area}(\Sigma_\infty \cap B_r(x_\infty)). \tag{3.111}$$

Proof. Since $\mathcal{S}_0$ is locally finite, without loss of generality we assume that $B_r(x_\infty)\cap\Sigma_\infty$ consists of only one singular point y_∞. For any $\epsilon > 0$ by the smooth convergence of $\Sigma_{t_i}\cap(B_r(x_i)\backslash B_\epsilon(y_\infty))$ we have

$$\lim_{i\to+\infty} \mathrm{Area}\big(\Sigma_{t_i}\cap(B_r(x_i)\backslash B_\epsilon(y_\infty))\big) = m\cdot \mathrm{Area}\big(\Sigma_\infty\cap(B_r(x_\infty)\backslash B_\epsilon(y_\infty))\big). \tag{3.112}$$

Since the area ratio is uniformly bounded from above along the rescaled mean curvature flow, we have

$$\mathrm{Area}(\Sigma_{t_i} \cap B_\epsilon(y_\infty)) \le N\pi\epsilon^2 \to 0 \quad \text{as } \epsilon \to 0.$$

Taking $\epsilon \to 0$ in both sides of (3.112), we have (3.111). The lemma is proved. ∎

Combining Lemma 3.29, Lemma 3.30 with Lemma 2.9, we show that the area ratio is always close to an integer after a fixed time.

Lemma 3.31. *Fix large R and small $\delta_0 \in (0, \frac{1}{2})$. Under the same assumption as in Lemma 3.4, there exists $t_0 > 0$ such that for any $t > t_0$ we have*

$$m(1-2\delta_0) < \frac{\mathrm{Area}(\Sigma_t \cap B_\xi(x))}{\pi\xi^2} < m(1+2\delta_0) \quad \text{for all } x \in B_R(0) \cap \Sigma_t, \tag{3.113}$$

where m is a positive integer independent of x and t. Here $\xi = \xi(R+1, N, \rho, \delta_0)$ is the constants in Lemma 3.29 with N and ρ determined as in assumption (3.2) and Lemma 3.4.

Proof. We divide the proof into several steps.

Step 1. We show that there exists $t_0 > 0$ such that for any $t > t_0$, (3.113) holds for some integer $m(x,t)$, which may depend on x and t. For otherwise, there exist a sequence $t_i \to +\infty$ and $x_i \in B_R(0) \cap \Sigma_{t_i}$ such that

$$\left| \frac{\mathrm{Area}(\Sigma_{t_i} \cap B_\xi(x_i))}{\pi m\xi^2} - 1 \right| \ge 2\delta_0 \quad \text{for all } m \in \mathbb{N} \cap [1, N_0]. \tag{3.114}$$

By Proposition 2.8, by taking a subsequence if necessary we assume that Σ_{t_i} converges locally smoothly to a self-shrinker $\Sigma_\infty \in \mathcal{C}(N, \rho)$ with multiplicity $m_0 \in \mathbb{N}$ and

$$x_i \to x_\infty \in \Sigma_\infty \cap B_{R+1}(0).$$

By the convergence of $\{\Sigma_{t_i}\}$ and Lemma 3.30, we have

$$\lim_{i\to+\infty} \frac{\mathrm{Area}(\Sigma_{t_i} \cap B_\xi(x_i))}{\pi\xi^2} = m_0 \frac{\mathrm{Area}(\Sigma_\infty \cap B_\xi(x_\infty))}{\pi\xi^2}. \tag{3.115}$$

Lemma 3.29 implies that

$$1-\delta_0 \le \frac{\mathrm{Area}(\Sigma_\infty \cap B_\xi(x_\infty))}{\pi\xi^2} \le 1+\delta_0. \tag{3.116}$$

Combining (3.115) with (3.116), for large t_i we have

$$\left| \frac{\mathrm{Area}(\Sigma_{t_i} \cap B_\xi(x_i))}{\pi m_0\xi^2} - 1 \right| \le \frac{3}{2}\delta_0, \tag{3.117}$$

which contradicts (3.114).

Step 2. We show that $m(x,t)$ is independent of x and we can write $m(t)$ for short. For otherwise, we can find a sequence $t_i \to +\infty$ and $x_i, y_i \in \Sigma_{t_i}$ with $m(x_i, t_i) \ne m(y_i, t)$. Because $m(x,t) \in [1, N_0]$, by taking a subsequence if necessary we may assume that

$m(x_i, t_i) = m_1$ for all i. Thus, for any i we have

$$m(y_i, t_i) \neq m_1. \tag{3.118}$$

By Proposition 2.8, by taking a subsequence if necessary we assume that Σ_{t_i} converges locally smoothly to a self-shrinker $\Sigma_\infty \in \mathcal{C}(N, \rho)$ with multiplicity $m_0 \in \mathbb{N}$, and

$$x_i \to x_\infty, \quad y_i \to y_\infty, \quad x_\infty, y_\infty \in \Sigma_\infty \cap B_{R+1}(0).$$

By (3.117), we have $m(x_i, t_i) = m_0 = m(y_i, t_i)$, which contradicts (3.118).

Step 3. We show that $m(t)$ is independent of t. It suffices to show that for any $s \in (-\frac{1}{2}, \frac{1}{2})$, we have

$$m(t) = m(t + s).$$

For otherwise, we can find a sequence $t_i \to +\infty$ and $s_i \in (-\frac{1}{2}, \frac{1}{2})$ such that for all i,

$$m(t_i) \neq m(t_i + s_i). \tag{3.119}$$

We follow the same argument as in Step 2. Since $m(t_i)$ is uniformly bounded, by taking a subsequence if necessary we can assume that $m(t_i) = m_1$ for all i. By (3.119), for all i we have

$$m(t_i + s_i) \neq m_1. \tag{3.120}$$

Note that $m(t_i + s_i)$ is also bounded, we can assume that a subsequence of $\{m(t_i + s_i)\}$ converges to an integer m_2 with

$$m_2 \neq m_1 \tag{3.121}$$

by (3.120). By Proposition 2.8, by taking a subsequence if necessary we assume that $\{\Sigma_{t_i+s}, -1 < s < 1\}$ converges locally smoothly to a self-shrinker $\Sigma_\infty \in \mathcal{C}(N, \rho)$ with multiplicity $m_0 \in \mathbb{N}$. Inequality (3.117) implies that $m_0 = m_1$. Since the multiplicity m_0 of the convergence is independent of time by Lemma 2.9, we have $m_0 = m_2$. Thus, we have $m_1 = m_2$, which contradicts (3.121). ∎

Using Lemma 3.31 and the results in previous sections, we show Theorem 3.1.

Proof of Theorem 3.1. Fix large $R > R_0$, where R_0 is the constant chosen in Lemma 3.8, and we choose a sequence $t_i \to +\infty$ as in Lemma 3.22. Then there is a self-shrinker $\Sigma_\infty \in \mathcal{C}(N, \rho)$ such that for any $T > 1$ we can find a subsequence, still denoted by $\{t_i\}$, such that $\{\Sigma_{t_i+t}, -T < t < T\}$ converges in smooth topology, possibly with multiplicities at most N_0, to Σ_∞ away from a singular set $\mathcal{S}$. If the multiplicity of the convergence is greater than one, Lemma 3.25 shows that the limit self-shrinker Σ_∞ is L-stable in the ball $B_R(0)$. This contradicts Lemma 3.8. Therefore, the multiplicity is one and the convergence is smooth.

We next show that for any sequence of $s_i \to +\infty$ there exists a subsequence such that the multiplicity of the convergence is also one. For otherwise, there exists a sequence $s_i \to +\infty$ such that Σ_{s_i} converges locally smoothly to a self-shrinker $\Sigma'_\infty \in \mathcal{C}(N, \rho)$ with multiplicity $m' > 1$. By Lemma 3.31, there exists $t_0 > 0$ such that for any $t > t_0$ we have

$$m(1 - 2\delta_0) < \frac{\text{Area}(\Sigma_t \cap B_\xi(x))}{\pi\xi^2} < m(1 + 2\delta_0) \quad \text{for all } x \in B_R(0) \cap \Sigma_t, \tag{3.122}$$

where m is a positive integer independent of x and t. By taking $t = t_i \to +\infty$ in (3.122), we have $m = 1$. On the other hand, taking $t = s_i \to +\infty$ in inequality (3.122), we have $m = m' > 1$, which is a contradiction. Thus, the theorem is proved. ∎

4. Estimates near the singular set

In this section, we will study the asymptotical behavior of the function w near the singular set. These estimates are used in the proof of Lemma 3.21 and Lemma 3.28. In [42], Kan and Takahashi studied time-dependent singularities in semilinear parabolic equations along one singular curve. Here we develop Kan–Takahashi's techniques to estimate the solution when the singular sets consists of multiple singular curves.

First, we introduce the following notations. Throughout this section, we denote by $\mathcal{B}_r(p)$ the (intrinsic) geodesic ball centered at p in (M, g) and $d_g(x, y)$ the distance from x to y with respect to the metric g.

Definition 4.1. Let (M, g) be a complete Riemannian manifold of dimension m. For any $k \in \mathbb{N}$, $\rho, \Xi > 0$, we define $\mathcal{M}_{k,m}(\rho, \Xi)$ the set of all subsets $A \subset (M, g)$ such that

(1) for any $p \in A$, the harmonic radius at p satisfies $r_h(p) \geq \rho$,

(2) for any $p \in A$, the ball $\mathcal{B}_\rho(p)$ has harmonic coordinates $\{x_1, x_2, \dots, x_m\}$ such that the metric tensor g_{ij} in these coordinates satisfies

$$\Xi^{-1}\delta_{ij} \leq g_{ij} \leq \Xi\delta_{ij}, \quad \left|\frac{\partial^\alpha g_{ij}}{\partial x^\alpha}\right| \leq \Xi \quad \text{on } \mathcal{B}_\rho(p)$$

for any multi-index α with $1 \leq |\alpha| \leq k$.

The following theorem is the main result in this section, which gives the asymptotical behavior of a positive solution of a parabolic equation near a time-dependent singular set.

Theorem 4.2. *Let (Σ^2, g) be a two-dimensional complete surface and let $\{\xi_1, \xi_2, \dots, \xi_l\}$ with $\xi_k : [T_1, T_2] \to \Sigma$ be σ-Lipschitz curves in Σ. Assume that*

$$u(x, t) \in L^1_{\mathrm{loc}}\left((\Sigma \times (T_1, T_2)) \setminus \bigcup_{k=1}^{l} \Gamma_k\right)$$

is a nonnegative solution of the equation

$$\frac{\partial u}{\partial t} = \Delta_g u + c(x, t)u, \tag{4.1}$$

where $c(x, t) \in L^\infty_{\mathrm{loc}}(\Sigma \times [T_1, T_2])$ and $\Gamma_k = \{(\xi_k(t), t)\} \subset \Sigma \times [T_1, T_2]$. Assume that for any $k \in \{1, 2, \dots, l\}$ and any $t \in [T_1, T_2]$ the ball $\mathcal{B}_1(\xi_k(t))$ is in $\mathcal{M}_{k_0,2}(\rho_0, \Xi_0)$, where k_0 is an integer chosen as in Corollary 4.4. Then we have:

(1) *$u \in L^1_{\mathrm{loc}}(\Sigma \times (T_1, T_2))$. More precisely, for any $(t_1, t_2) \subset (T_1, T_2)$, there exists a constant $r_1 = r_1(\rho_0, \Xi_0, l, t_1, t_2, T_1, T_2) > 0$ such that*

$$\|u\|_{L^1(Q_{r_1,t_1,t_2})} \leq C\|u\|_{L^1(K)}, \tag{4.2}$$

where C is a constant depending on $\|c\|_{L^\infty(Q_{1,T_1,T_2})}, \rho_0, \Xi_0, \sigma, t_1, t_2, T_1, T_2$ and K is defined by $K = \overline{Q_{2r_1,T_1,T_2}} \backslash Q_{r_1,T_1,T_2}$. Here Q_{r,t_1,t_2} is defined by

$$Q_{r,t_1,t_2} = \bigcup_{k=1}^{l} \{(x,t) \in \Sigma \times \mathbb{R} \mid x \in \mathcal{B}_r(\xi_k(t)) \subset \Sigma,\ t \in (t_1,t_2)\}.$$

(2) *For any $(t_1,t_2) \subset (T_1,T_2)$, we have*

$$\lim_{R\to 0}\lim_{\delta\to 0}\int_{t_1}^{t_2} dt \int_{A_t^{(k)}(\frac{\delta}{2},R)\cap F_t^{(k)}(\frac{\delta}{2})} \frac{u}{\mathbf{r}_k|\log \mathbf{r}_k|}\, d\mathrm{vol} = 0, \tag{4.3}$$

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_1}^{t_2} dt \int_{A_t^{(k)}(\frac{\delta}{2},\delta)\cap F_t^{(k)}(\frac{\delta}{2})} u\, d\mathrm{vol} < +\infty, \tag{4.4}$$

where $A_t^{(k)}$ and $F_t^{(k)}$ are defined by (3.98)–(3.99) and $\mathbf{r}_k(x,t) = d(x,\xi_k(t))$.

We sketch the proof of Theorem 4.2. First, we show an asymptotical formula for the heat kernel on a Riemannian manifold in Theorem 4.3. Using this formula, we construct a special function $U_k(x,t)$ for each singular curve ξ_k and a measure ν, and show that $U_k(x,t)$ behaves like $\log 1/\mathbf{r}_k(x,t)$ when the point x is near ξ_k and ν is the Lebesgue measure in Lemma 4.5. Moreover, $U_k(x,t)$ satisfies the growth estimates (4.3)–(4.4) by Lemma 4.6, and we use $U_k(x,t)$ to construct a function v_k in Lemma 4.5, which satisfies the backward heat equation. The function v_k is important to construct some cutoff functions (cf. Definition 4.12). When the singular curves are disjoint, using these cutoff functions we can show (4.2) directly in Lemma 4.9. When the singular curves are not disjoint, we show the finiteness of a functional I and use the functional I to show the L^1 norm of u (4.2) in Lemma 4.13. By using the functional I, we get a positive linear functional μ_k for each singular curve ξ_k in Lemma 4.15, and by Lemma 4.16 μ_k is uniformly bounded even if the singular curves are not disjoint. Finally, we use μ_k to construct U_k and show that u is controlled by U_k. By the properties of U_k, we have that u satisfies the growth estimates (4.3)–(4.4).

4.1. Properties of the heat kernel

In this subsection, we will give the expansion of the heat kernel on Riemannian manifolds. Let (M,g) be a complete Riemannian manifold (without boundary) of dimension m. Suppose that $p(x,y,t)$ is the heat kernel. Then $p(x,y,t)$ has the following asymptotical formula (cf. [48, Theorem 11.1]):

$$p(x,y,t) \sim (4\pi t)^{-\frac{m}{2}} e^{-\frac{d_g^2(x,y)}{4t}}$$

as $t \to 0$ and $d_g(x,y) \to 0$. The next result gives more estimates on the asymptotical formula.

Theorem 4.3 (cf. [48, Theorem 11.1] or [5, Theorem 2.30]). *Let $\rho_0, \Xi_0 > 0$ and integers $m \geq 2, k \geq 0$. There exists an integer $k_0 = k_0(k)$ depending only on k satisfying the following property. Let (M,g) be a complete Riemannian manifold of dimension m and*

$x_0 \in M$ with $\mathcal{B}_{\rho_0}(x_0) \in \mathcal{M}_{k_0,m}(\rho_0, \Xi_0)$. There exists a sequence of smooth functions $\{u_i(x,y)\}$ with $u_0(x,x) = 1$ such that for any $x, y \in \mathcal{B}_{\rho_0/2}(x_0)$ and $t \in (0,1]$ we have

$$\left| p(x,y,t) - (4\pi t)^{-\frac{m}{2}} e^{-\frac{d_g^2(x,y)}{4t}} \sum_{i=0}^{k} u_i(x,y) t^i \right| \le C(\rho_0, \Xi_0, m) t^{k+1-\frac{m}{2}} \tag{4.5}$$

and

$$\left| \nabla_x p(x,y,t) - \nabla_x \left((4\pi t)^{-\frac{m}{2}} e^{-\frac{d_g^2(x,y)}{4t}} \sum_{i=0}^{k} u_i(x,y) t^i \right) \right| \le C(\rho_0, \Xi_0, m) t^{k-\frac{m}{2}}. \tag{4.6}$$

Proof. We follow the argument in [48, Theorem 11.1] to prove (4.5)–(4.6). Define the function

$$G(x,y,t) = (4\pi t)^{-\frac{m}{2}} e^{-\frac{d_g^2(x,y)}{4t}} \sum_{i=0}^{k} u_i(x,y) t^i.$$

Direct calculation shows that

$$\begin{aligned} \left(\Delta_y - \frac{\partial}{\partial t}\right) G = (4\pi t)^{-\frac{m}{2}} e^{-\frac{d_g^2(x,y)}{4t}} \times \Bigg(\left(-\frac{r\Delta_y r}{2} + \frac{m-1}{2} \right) \sum_{i=-1}^{k-1} u_{i+1} t^i \\ - r \sum_{i=-1}^{k-1} \langle \nabla r, \nabla u_{i+1} \rangle t^i + \sum_{i=0}^{k} (\Delta_y u_i) t^i - \sum_{i=0}^{k-1} (i+1) u_{i+1} t^i \Bigg). \end{aligned}$$

For fixed x and $y \in \mathcal{B}_{\rho_0/2}(x)$, there exists a sequence of function $\{u_i(x,y)\}$ satisfying

$$\begin{gathered} \left(\frac{r\Delta_y r}{2} - \frac{m-1}{2} \right) u_0 + r \langle \nabla r, \nabla u_0 \rangle = 0, \\ \left(\frac{r\Delta_y r}{2} - \frac{m-1}{2} \right) u_{i+1} + r \langle \nabla r, \nabla u_{i+1} \rangle + (i+1) u_{i+1} = \Delta_y u_i, \quad 0 \le i \le k-1. \end{gathered}$$

This implies that

$$\left(\Delta_y - \frac{\partial}{\partial t}\right) G = (4\pi t)^{-\frac{m}{2}} e^{-\frac{d_g^2(x,y)}{4t}} \Delta_y u_k t^k. \tag{4.7}$$

As in the proof of [48, Theorem 11.1], we have

$$\begin{aligned} u_0(x,y) &= C d_g(x,y)^{\frac{m-1}{2}} J^{-\frac{1}{2}}, \\ u_{i+1} &= d_g(x,y)^{\frac{m-3-2i}{2}} J^{-\frac{1}{2}} \int_0^r s^{\frac{2i+1-m}{2}} J^{\frac{1}{2}} \Delta u_i \, ds, \end{aligned}$$

where C is a constant such that $u_0(x,x) = 1$ and $J(y)$ is the area element of the sphere of radius $d_g(x,y)$ at the point y. There exists integer k_0 depending only on k such that under the assumption $\mathcal{B}_{\rho_0}(x_0) \in \mathcal{M}_{k_0,m}(\rho_0, \Xi_0)$, for any integer $i \in [0,k]$ we have

$$|u_i(x,y)| + |\nabla_y u_i(x,y)| + |\nabla_x \nabla_y \nabla_y u_i(x,y)| \le C(\rho_0, \Xi_0, m) \tag{4.8}$$

for all $x, y \in \mathcal{B}_{\rho_0/2}(x_0)$.

Let $\rho = \frac{\rho_0}{4}$. Now we choose a cutoff function $\eta(r)$ with $0 \le \eta \le 1$ such that $\eta(r) = 1$ when $r \le \rho$ and $\eta(r) = 0$ when $r \ge 2\rho$. Define

$$\chi(x, y) = \eta(d_g(x, y))$$

and

$$F(x, y, t) = \chi(x, y)G(x, y, t).$$

If $d_g(x, y) \le \rho$ and $t \le 1$, identity (4.7) gives that

$$\begin{aligned}\left|\left(\Delta_y - \frac{\partial}{\partial t}\right)F\right| &= \left|\left(\Delta_y - \frac{\partial}{\partial t}\right)G\right| \\ &\le C(\rho_0, \Xi_0)t^{k-\frac{m}{2}}e^{-\frac{d_g^2(x,y)}{4t}}\end{aligned}$$

and

$$\begin{aligned}\left|\left(\Delta_y - \frac{\partial}{\partial t}\right)\nabla_x F\right| &= \left|\left(\Delta_y - \frac{\partial}{\partial t}\right)\nabla_x G\right| \\ &= \left|(4\pi t)^{-\frac{m}{2}}t^k e^{-\frac{d_g^2(x,y)}{4t}}\left(-\frac{1}{2t}d\nabla_x d\Delta_y u_k + \nabla_x\Delta_y u_k\right)\right| \\ &\le C(\rho_0, \Xi_0)t^{k-1-\frac{m}{2}}e^{-\frac{d_g^2(x,y)}{4t}},\end{aligned}$$

where we used (4.8) in the last inequality. Similarly, for $\rho \le d_g(x, y) \le 2\rho$ we can also check that

$$\begin{aligned}\left|\left(\Delta_y - \frac{\partial}{\partial t}\right)F\right| &\le C(\rho_0, \Xi_0)t^{-\frac{m}{2}-1}e^{-\frac{d_g^2(x,y)}{4t}}, \\ \left|\left(\Delta_y - \frac{\partial}{\partial t}\right)\nabla_x F\right| &\le C(\rho_0, \Xi_0)t^{-2-\frac{m}{2}}e^{-\frac{d_g^2(x,y)}{4t}}.\end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned}&|F(x, y, t) - p(x, y, t)| \\ &\quad= \left|\int_0^t ds \int_M p(z, y, t-s)\left(\Delta_z - \frac{\partial}{\partial s}\right)F(x, z, s)\,dz\right| \\ &\quad\le C(\rho_0, \Xi_0)\int_0^t s^{k-\frac{m}{2}}ds\int_{\mathcal{B}_\rho(x)} p(z, y, s)\,dz \\ &\qquad\quad + C(\rho_0, \Xi_0)\int_0^t s^{-\frac{m}{2}-1}e^{-\frac{\rho^2}{4s}}ds\int_{\mathcal{B}_{2\rho}(x)\setminus\mathcal{B}_\rho(x)} p(z, y, s)\,dz \\ &\quad\le C(\rho_0, \Xi_0, m)t^{k+1-\frac{m}{2}},\end{aligned} \tag{4.9}$$

where we used the fact that

$$\int_M p(x, y, t)\,d\mathrm{vol}_y \le 1.$$

Thus, (4.9) gives (4.5). Similarly, we can show that

$$\begin{aligned}
&|\nabla_x F(x,y,t) - \nabla_x p(x,y,t)| \\
&= \left| \int_0^t ds \int_M p(z,y,t-s)\Big(\Delta_z - \frac{\partial}{\partial s}\Big)\nabla_x F(x,z,s)\, dz \right| \\
&\le C(\rho_0,\Xi_0)\int_0^t s^{k-\frac{m}{2}-1} ds \int_{\mathcal{B}_\rho(x)} p(z,y,s)\, dz \\
&\quad + C(\rho_0,\Xi_0)\int_0^t s^{-2-\frac{m}{2}} e^{-\frac{\rho^2}{4t}} ds \int_{\mathcal{B}_{2\rho}(x)\setminus \mathcal{B}_\rho(x)} p(z,y,s)\, dz \\
&\le C(\rho_0,\Xi_0,m) t^{k-\frac{m}{2}}. \qquad (4.10)
\end{aligned}$$

Thus, (4.10) implies (4.6). The theorem is proved. ■

As a corollary, we have the following result in dimension two.

Corollary 4.4. *Fix $\rho_0, \Xi_0 > 0$ and an integer $k_0 = k_0(0)$ chosen as in Theorem 4.3 for $k = 0$. Let (Σ^2, g) be a complete surface and $x_0 \in \Sigma$ with $\mathcal{B}_1(x_0) \in \mathcal{M}_{k_0,2}(\rho_0, \Xi_0)$, there exists a constant $C(\rho_0, \Xi_0) > 0$ such that for any $x, y \in \mathcal{B}_{\frac{\rho_0}{2}}(x_0)$ and $t \in (0, 1]$ we have*

$$p(x,y,t) \le \big(1 + C(\rho_0,\Xi_0) d_g(x,y)\big) p_0(x,y,t) + C(\rho_0,\Xi_0), \qquad (4.11)$$

$$p(x,y,t) \ge \big(1 - C(\rho_0,\Xi_0) d_g(x,y)\big) p_0(x,y,t) - C(\rho_0,\Xi_0), \qquad (4.12)$$

$$|\nabla_x p(x,y,t)| \le \big(1 + C(\rho_0,\Xi_0) d_g(x,y)\big)|\nabla_x p_0(x,y,t)| + \frac{C(\rho_0,\Xi_0)}{t}, \qquad (4.13)$$

$$|\nabla_x p(x,y,t)| \ge \big(1 - C(\rho_0,\Xi_0) d_g(x,y)\big)|\nabla_x p_0(x,y,t)| - \frac{C(\rho_0,\Xi_0)}{t}, \qquad (4.14)$$

where $p_0(x,y,t) = \frac{1}{4\pi t} e^{-\frac{d_g(x,y)^2}{4t}}$.

Proof. By (4.8), for any $x, y \in \mathcal{B}_{\rho_0/2}(x_0)$ we have

$$|u_0(x,y) - 1| \le \sup_{\mathcal{B}_{\rho_0/2}(x_0)} |\nabla_y u_0| \cdot d_g(x,y) \le C(\rho_0,\Xi_0)\, d_g(x,y). \qquad (4.15)$$

Applying Theorem 4.3 for $k = 0$ and using (4.15), we have (4.11)–(4.14). The corollary is proved. ■

4.2. *Properties of a solution with time-dependent singularities*

In this subsection, we follow the arguments in [42, Section 3] to discuss a solution of the linear equation on (Σ, g)

$$\frac{\partial f(x,t)}{\partial t} = \Delta f(x,t) + \delta_{\xi(t)} \otimes \nu, \qquad (4.16)$$

where Σ is a complete two-dimensional surface. Here we assume that $\xi : (\underline{T}, \bar{T}) \to \Sigma$ is a σ-Lipschitz curve with $-\infty < \underline{T} < \bar{T} < +\infty$, $\delta_{\xi(t)}$ denotes the Dirac function with the pole $\xi(t)$ and $\nu \in (C_0((\underline{T}, \bar{T})))'$. For $0 < r < +\infty$ and $\underline{T} \le \underline{t} < \bar{t} \le \bar{T}$, we set

$$\Gamma_{\underline{t},\bar{t}} = \{(\xi(t), t) \in \Sigma \times \mathbb{R} \mid t \in (\underline{t}, \bar{t})\}, \tag{4.17}$$

$$Q_{r,\underline{t},\bar{t}} = \{(x,t) \in \Sigma \times \mathbb{R} \mid x \in \mathcal{B}_r(\xi(t)) \subset \Sigma, t \in (\underline{t}, \bar{t})\}. \tag{4.18}$$

We say that $f(x,t)$ is a solution of (4.16) if for any $\varphi \in C_0^\infty(Q_{\infty,\underline{T},\bar{T}})$,

$$\int_{Q_{\infty,\underline{T},\bar{T}}} f(x,t)\Big(-\frac{\partial \varphi(x,t)}{\partial t} - \Delta\varphi(x,t)\Big)\, d\,\mathrm{vol}\, dt = \int_{\underline{T}}^{\bar{T}} \varphi(\xi(s), s)\, d\nu(s). \tag{4.19}$$

We define the function $U(x,t)$ by

$$U(x,t) = \int_{\underline{T}}^{t} p(x, \xi(s), t-s)\, d\nu(s), \tag{4.20}$$

where $p(x,y,t)$ is the heat kernel of (Σ, g). Then $U(x,t)$ satisfies (4.19) (cf. [42]). Moreover, we define

$$\Phi(x,y) = \frac{1}{2\pi}\log\frac{1}{d_g(x,y)}, \quad \mathbf{r}(x,t) = d_g(x, \xi(t)). \tag{4.21}$$

Following the argument in [42] and using Theorem 4.3, we have:

Lemma 4.5 (cf. [42]). *Let $\xi : (\underline{T}, \bar{T}) \to \Sigma$ be a σ-Lipschitz curve and $\underline{T} < \underline{t} < \bar{t} < \bar{T}$. Assume that for any $t \in (\underline{T}, \bar{T})$ the ball $\mathcal{B}_1(\xi(t))$ is in $\mathcal{M}_{k_0,2}(\rho_0, \Xi_0)$ as in Corollary 4.4. Then we have:*

(1) *Assume that ν is the Lebesgue measure. For any $\epsilon > 0$, there exists a positive constant $r_0 = r_0(\epsilon, \sigma, \Xi_0, \rho_0, \underline{T}, \bar{T})$ such that if $\mathbf{r}(x,t) \le r_0$ and $t \in (\underline{t}, \bar{t})$, then we have*

$$(1-\epsilon)\Phi(x, \xi(t)) \le U(x,t) \le (1+\epsilon)\Phi(x,\xi(t)), \tag{4.22}$$

$$(1-\epsilon)|\nabla\Phi(x,\xi(t))| \le |\nabla U(x,t)| \le (1+\epsilon)|\nabla\Phi(x,\xi(t))|. \tag{4.23}$$

(2) *For any $\gamma \in (\frac{1}{2}, 1)$, there exist a constant $r_0 = r_0(\rho_0, \Xi_0, \sigma, \underline{T}, \bar{T}, \gamma) \in (0,1)$ and a function $v \in C^\infty(\overline{Q_{1,\underline{t},\bar{t}}} \backslash \overline{\Gamma_{\underline{t},\bar{t}}})$ satisfying*

$$\frac{\partial v}{\partial t} + \Delta v = 0 \quad \text{in } Q_{1,\underline{t},\bar{t}} \backslash \Gamma_{\underline{t},\bar{t}} \tag{4.24}$$

such that for all $(x,t) \in Q_{r_0,\underline{t},\bar{t}} \backslash \Gamma_{\underline{t},\bar{t}}$ the following inequalities hold:

$$\gamma \log\frac{1}{\mathbf{r}(x,t)} \le v(x,t) \le \log\frac{1}{\mathbf{r}(x,t)}, \tag{4.25}$$

$$\gamma \mathbf{r}(x,t)^{-1} \le |\nabla v(x,t)| \le \mathbf{r}(x,t)^{-1}. \tag{4.26}$$

Proof. The proof is almost the same as that of [42, Proposition 3.1, Proposition 3.3 and Lemma 4.1], and we sketch some details here. For $r > 0$, $\beta > 0$ and $\delta > 0$, we define

$$S_\beta(r) = \pi^{-1}\int_0^\delta (4s)^{-\frac{\beta}{2}} e^{-\frac{r^2}{4s}}\, ds.$$

Since ξ is σ- Lipschitz continuous, we have

$$|\mathbf{r}(x,t)-\mathbf{r}(x,s)|\leq \sigma|t-s|.$$

Thus, for any $c>0$ we have

$$\mathbf{r}(x,s)^2\leq (1+c)\mathbf{r}(x,t)^2+\left(1+\frac{1}{c}\right)\sigma^2|t-s|^2.$$

This implies that

$$\mathbf{r}(x,s)^2\geq \frac{1}{1+c}\mathbf{r}(x,t)^2-\frac{1}{c}\sigma^2|t-s|^2. \tag{4.27}$$

Combining this with Corollary 4.4, we have

$$\begin{aligned}
&\int_{t-\delta}^{t} p(x,\xi(s),t-s)\,ds\\
&\leq \big(1+C(\rho_0,\Xi_0)(\mathbf{r}(x,t)+\sigma\delta)\big)e^{\frac{\sigma^2\delta}{4c}}\int_{t-\delta}^{t}\frac{1}{4\pi(t-s)}e^{-\frac{\mathbf{r}(x,t)^2}{4(1+c)(t-s)}}\,ds+C(\rho_0,\Xi_0,\delta)\\
&=\big(1+C(\rho_0,\Xi_0)(\mathbf{r}(x,t)+\sigma\delta)\big)e^{\frac{\sigma^2\delta}{4c}}S_2\left(\frac{\mathbf{r}(x,t)}{\sqrt{1+c}}\right)+C(\rho_0,\Xi_0,\delta).
\end{aligned}$$

Choosing the constant $c=\sqrt{\delta}$, we have

$$\begin{aligned}
U(x,t)&=\left(\int_{t-\delta}^{t}+\int_{\underline{T}}^{t-\delta}\right)p(x,\xi(s),t-s)ds\\
&\leq \big(1+C(\rho_0,\Xi_0)(\mathbf{r}(x,t)+\sigma\delta)\big)e^{\frac{\sigma^2\sqrt{\delta}}{4}}S_2\left(\frac{\mathbf{r}(x,t)}{\sqrt{1+\sqrt{\delta}}}\right)+C(\rho_0,\Xi_0,\delta,\underline{T},\bar{T}).
\end{aligned}$$

Note that

$$\lim_{r\to 0}\left(\log\frac{1}{r}\right)^{-1}S_2(r)=\frac{1}{2\pi}.$$

Therefore, for any $\epsilon>0$ there exists a constant $r_0=r_0(\epsilon,\sigma,\Xi_0,\rho_0,\underline{T},\bar{T})$ such that for any x with $\mathbf{r}(x,t)\leq r_0$ we have

$$\frac{U(x,t)}{\Phi(x,\xi(t))}\leq 1+\epsilon.$$

Similarly, we can show that

$$\frac{U(x,t)}{\Phi(x,\xi(t))}\geq 1-\epsilon$$

when $\mathbf{r}(x,t)$ is small. Thus, (4.22) is proved. Similarly, we can use (4.13) and (4.14) of Corollary 4.4 to estimate $|\nabla U|$.

To prove part (2), we denote by $U(x,t;\xi,\nu)$ the function (4.20) constructed by $\xi(t)$ and the measure ν. We define $\tilde{\xi}(t)=\xi(\underline{T}+\bar{T}-t)$ and let ν be the Lebesgue measure. Then the function $v(x,t)=kU(x,\underline{T}+\bar{T}-t;\tilde{\xi},\nu)$ satisfies the properties in part (2) by choosing some $k>0$. See [42, Lemma 4.1] for details. ∎

Using Corollary 4.4, we have the following result.

Lemma 4.6. *Let $\xi : (\underline{T}, \bar{T}) \to \Sigma$ be a σ-Lipschitz curve and $\underline{T} < \underline{t} < \bar{t} < \bar{T}$. Assume that for any $t \in (\underline{T}, \bar{T})$ the ball $\mathcal{B}_1(\xi(t))$ is in $\mathcal{M}_{k_0,2}(\rho_0, \Xi_0)$ as in Corollary 4.4. Let $\nu \in (C_0((\underline{T}, \bar{T})))'$ and $U(x,t)$ the function defined by (4.20). Then for $\underline{T} < t_1 < t_2 < \bar{T}$ we have*

$$\lim_{R\to 0}\lim_{\delta\to 0}\int_{t_1}^{t_2} dt \int_{A_t(\delta,R)} \frac{U(x,t)}{\mathbf{r}(x,t)|\log \mathbf{r}(x,t)|} = 0, \tag{4.28}$$

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_1}^{t_2} dt \int_{A_t(\frac{\delta}{2},\delta)} U(x,t) = 0, \tag{4.29}$$

where $A_t(\delta, R) = \{x \in \Sigma \mid \delta < \mathbf{r}(x,t) < R\}$.

Proof. We follow the arguments in the proof of [42, Proposition 3.3]. Without loss of generality, we can assume that the curve $\xi(s)$ ($s \in (\underline{T}, \bar{T})$) is contained in $\mathcal{B}_{\rho_0/2}(x_0)$ for some $x_0 \in \Sigma$ and $\mathcal{B}_{\rho_0}(x_0) \in \mathcal{M}_{k_0,2}(\rho_0, \Xi_0)$. Corollary 4.4 gives that for any $x \in \mathcal{B}_{\rho_0/2}(x_0)$ and $t \in (\underline{T}, \bar{T})$,

$$\begin{aligned} U(x,t) &\le \int_{\underline{T}}^{t} \big(C(\rho_0,\Xi_0)p_0(x,\xi(s),t-s) + C(\rho_0,\Xi_0)\big)\,d\nu \\ &= C(\rho_0,\Xi_0)U_0(x,t) + C(\rho_0,\Xi_0)(\bar{T}-\underline{T}), \end{aligned}$$

where U_0 is defined by

$$U_0(x,t) = \int_{\underline{T}}^{t} p_0(x,\xi(s),t-s)\,d\nu.$$

Thus, it suffices to show (4.28)–(4.29) for $U_0(x,t)$.

For $t \in (\underline{T}, \bar{T})$ with $|D\nu| < +\infty$ we write

$$\nu((s,t]) = D\nu(t)(t-s) - G(s), \quad T_1 < s < t,$$

where $G(s)$ satisfies $\lim_{s\to t^-}\frac{G(s)}{t-s} = 0$ for a.e. $t \in (\underline{T}, \bar{T})$. Let $\lambda \in (0, t-\underline{t})$. Note that U_0 can be written as

$$\begin{aligned} U_0(x,t) = &\int_{\underline{T}}^{t-\lambda} p_0(x,\xi(s),t-s)\,d\nu(s) + D\nu(t)\int_{t-\lambda}^{t} p_0(x,\xi(s),t-s)\,ds \\ &+ \int_{t-\lambda}^{t} p_0(x,\xi(s),t-s)\,dG(s) =: I_1 + I_2 + I_3. \end{aligned}$$

By Theorem 4.3 I_1 satisfies $I_1 \le \frac{1}{4\pi\lambda}\nu((\underline{T}, t-\lambda)) < +\infty$. Thus, we have

$$\begin{aligned} &\int_{t_1}^{t_2} dt \int_{A_t(\delta,R)} \frac{I_1(x,t)}{\mathbf{r}(x,t)|\log \mathbf{r}(x,t)|}\,d\mathrm{vol} \\ &\le \frac{1}{4\pi\lambda}\nu((\underline{T},\bar{T}))\int_{t_1}^{t_2} dt \int_{A_t(\delta,R)} \frac{1}{\mathbf{r}(x,t)|\log \mathbf{r}(x,t)|}\,d\mathrm{vol} \\ &= \frac{1}{2\lambda}(t_2-t_1)\nu((\underline{T},\bar{T}))\int_{\delta}^{R} \frac{1}{|\log \mathbf{r}|}\,d\mathbf{r} \\ &\le \frac{1}{2\lambda}(t_2-t_1)\nu((\underline{T},\bar{T}))(R-\delta), \end{aligned} \tag{4.30}$$

where we assumed that R is small such that $|\log \mathbf{r}| \geq 1$ for any $\mathbf{r} \in (0, R)$. Moreover, we have

$$\frac{1}{\delta}\int_{t_1}^{t_2} dt \int_{A_t(\frac{\delta}{2},\delta)} I_1(x,t)\, d\text{vol} \leq \frac{C_1}{\lambda}(t_2 - t_1)\nu((\underline{T}, \bar{T}))\delta, \tag{4.31}$$

where C_1 is a universal constant. Next, we estimate I_2. Using Corollary 4.4, (4.27) and integration by parts we have

$$\begin{aligned}
&\int_{t-\lambda}^{t} p_0(x, \xi(s), t-s)\, ds \\
&\leq \int_0^{\lambda} \frac{1}{4\pi\tau} e^{-\frac{\mathbf{r}(x,s)^2}{4\tau}}\, d\tau \leq e^{\frac{\sigma^2\lambda}{4c}} \int_0^{\lambda} \frac{1}{4\pi\tau} e^{-\frac{\mathbf{r}(x,t)^2}{4(1+c)\tau}}\, d\tau \\
&\leq \frac{1}{4\pi} e^{\frac{\sigma^2\lambda}{4c}} \left(\log \frac{4(1+c)\lambda}{\mathbf{r}(x,t)^2} e^{-\frac{r^2}{4(1+c)\lambda}} + \int_{\frac{\mathbf{r}(x,t)^2}{4(1+c)\lambda}}^{\infty} e^{-z} \log z\, dz \right) \\
&\leq C_2 |\log \mathbf{r}(x,t)| + C_2,
\end{aligned} \tag{4.32}$$

where we can choose $c = 1$ and C_2 is a constant depending on σ and λ. Therefore, we have

$$\begin{aligned}
&\int_{t_1}^{t_2} dt \int_{A_t(\delta,R)} \frac{I_2(x,t)}{\mathbf{r}(x,t)|\log \mathbf{r}(x,t)|}\, d\text{vol} \\
&\leq C_2 \int_{t_1}^{t_2} d\nu(t) \int_{\delta}^{R} \frac{1}{\mathbf{r}|\log \mathbf{r}|}(|\log \mathbf{r}| + 1)\, \mathbf{r}\, d\mathbf{r} \\
&\leq 2C_2 \cdot (R - \delta)\nu((t_1, t_2))
\end{aligned} \tag{4.33}$$

and

$$\frac{1}{\delta}\int_{t_1}^{t_2} dt \int_{\frac{\delta}{2} < \mathbf{r}(x,t) < \delta} I_2(x,t)\, d\text{vol} \leq C_2 \cdot (\delta + \delta|\log \delta|)\nu((t_1, t_2)). \tag{4.34}$$

Finally, we estimate the term I_3. Using inequality (4.27) for $c = 1$ and integration by parts, we have

$$\begin{aligned}
&\int_{t-\lambda}^{t} p_0(x, \xi(t), t-s) d|G(s)| \\
&\leq \frac{1}{4\pi} e^{\frac{\sigma^2\lambda}{4}} \int_{t-\lambda}^{t} \frac{1}{t-s} e^{-\frac{\mathbf{r}(x,t)^2}{8(t-s)}} \\
&\leq \frac{1}{4\pi} e^{\frac{\sigma^2\lambda}{4}} \left(-\frac{1}{\lambda} e^{-\frac{\mathbf{r}(x,t)^2}{8\lambda}} + \int_{t-\lambda}^{t} |G(s)| \left(\frac{1}{(t-s)^2} + \frac{\mathbf{r}(x,t)^2}{8(t-s)^3} \right) e^{-\frac{\mathbf{r}(x,t)^2}{8(t-s)}} \right) ds \\
&\leq \frac{1}{4\pi} e^{\frac{\sigma^2\lambda}{4}} \sup_{(t-\lambda,t)} \frac{|G(s)|}{t-s} \cdot \int_{t-\lambda}^{t} \left(\frac{1}{(t-s)} + \frac{\mathbf{r}(x,t)^2}{8(t-s)^2} \right) e^{-\frac{\mathbf{r}(x,t)^2}{8(t-s)}}\, ds \\
&\leq C_3 \sup_{(t-\lambda,t)} \frac{|G(s)|}{t-s} \cdot |\log \mathbf{r}(x,t)|,
\end{aligned}$$

where C_3 depends on σ and λ. Thus, we have

$$
\begin{aligned}
&\int_{t_1}^{t_2} dt \int_{A_t(\delta,R)} \frac{|I_3(x,t)|}{\mathbf{r}(x,t)|\log \mathbf{r}(x,t)|} \, d\mathrm{vol} \\
&\le C(\sigma,\lambda) \sup_{(t-\lambda,t)} \frac{|G(s)|}{t-s} \int_{t_1}^{t_2} dt \int_{A_t(\delta,R)} \frac{1}{\mathbf{r}(x,t)} \, d\mathrm{vol} \\
&\le C(\sigma,\lambda) \sup_{(t-\lambda,t)} \frac{|G(s)|}{t-s} \cdot (R-\delta)(t_2-t_1)
\end{aligned} \tag{4.35}
$$

and

$$
\frac{1}{\delta} \int_{t_1}^{t_2} dt \int_{\frac{\delta}{2} < \mathbf{r}(x,t) < \delta} |I_3(x,t)| \, d\mathrm{vol} \le C(\sigma,\lambda) \sup_{(t-\lambda,t)} \frac{|G(s)|}{t-s} \cdot (t_2-t_1)|\log \delta|\delta. \tag{4.36}
$$

Combining (4.30)–(4.36), we have (4.28)–(4.29). ■

4.3. Estimates of the solution with disjoint singularities

In this subsection, we follow [42, Section 4.1] to construct some cutoff functions and show the integrability of the solution across the singular set when the singular curves are disjoint. First, we construct some cutoff functions.

Definition 4.7 (cf. [42, Section 4.1]). (1) Let $t_3 < t_1 < t_2 < t_4$ and $0 < \delta < r_1$. Define $\zeta = \zeta(t; t_1, t_2, t_3, t_4, \delta, r_1) \in C^\infty(\mathbb{R})$ such that

$$
\zeta(t) = \begin{cases} \delta, & t \in [t_1, t_2], \\ r_1, & t \in (-\infty, t_3] \cup [t_4, \infty), \end{cases}
$$

$$
0 < \left| \frac{\partial \zeta}{\partial t} \right| \le 2r_1 \left(\frac{1}{t_1 - t_3} + \frac{1}{t_4 - t_2} \right). \tag{4.37}
$$

(2) Let η be a smooth function on $\mathbb{R}$ satisfying

$$
\eta(z) = \begin{cases} 0, & z \le 0, \\ 1, & z \ge 1, \end{cases} \qquad 0 < \eta'(z) \le 2 \quad (0 < z < 1), \tag{4.38}
$$

and define $H(z) = \int_0^z \eta(\tau)\, d\tau$. Then $H(z)$ satisfies the inequality

$$
0 \le zH'(z) - H(z) \le H'(z). \tag{4.39}
$$

We keep the same notation $H(z)$ as in [42]. Throughout this section, H always denotes the function as above and it should not be confused with the mean curvature.

(3) Let $0 < \underline{r} < \bar{r} < 1$, $T_1 < \underline{T} < \underline{t} < \bar{t} < \bar{T} < T_2$ and let $\xi : [T_1, T_2] \to \Sigma$ be a continuous curve. We define $\phi_\xi = \phi_\xi(x, t; \underline{r}, \bar{r}, \underline{t}, \bar{t}, \underline{T}, \bar{T}, T_1, T_2) \in C^\infty(Q_{1,T_1,T_2})$ satisfying

$$
0 \le \phi_\xi \le 1, \quad \phi_\xi = \begin{cases} 1 & \text{on } \overline{Q_{\underline{r},\underline{t},\bar{t}}}, \\ 0 & \text{on } Q_{1,T_1,T_2} \backslash Q_{\bar{r},\underline{T},\bar{T}}, \end{cases} \qquad \nabla_x \phi_\xi = 0 \text{ in } Q_{\underline{r},T_1,T_2}. \tag{4.40}
$$

A direct corollary of Lemma 4.5 is the following result.

Lemma 4.8. *Under the assumption of Lemma 4.5, we define*

$$V(x,t) = e^{-2v(x,t)} \in C^\infty(\overline{Q_{1,\underline{t},\bar{t}}}\backslash\overline{\Gamma_{\underline{t},\bar{t}}}).$$

Then $V(x,t)$ satisfies

$$\frac{\partial V}{\partial t} + \Delta V = 4e^{-2v}|\nabla v|^2 \quad \text{in } Q_{1,\underline{t},\bar{t}}\backslash\Gamma_{\underline{t},\bar{t}}$$

By using inequalities (4.25)–(4.26), for all $(x,t) \in Q_{r_0,\underline{t},\bar{t}}\backslash\Gamma_{\underline{t},\bar{t}}$ the following inequalities hold:

$$\mathbf{r}(x,t)^2 \le V(x,t) \le \mathbf{r}(x,t)^{2\gamma}, \tag{4.41}$$

$$1 \le V(x,t)^{-1}|\nabla V(x,t)|^2 \le 4\mathbf{r}(x,t)^{2\gamma-2}, \tag{4.42}$$

$$1 \le \frac{\partial V}{\partial t} + \Delta V \le 4\mathbf{r}(x,t)^{2\gamma-2}, \tag{4.43}$$

where $\gamma \in (\frac{1}{2}, 1)$.

Consider the case that there is only one singular curve. We show that the solution of (4.1) is in L^1 across the singular set. The argument is the same as that of [42] and we give all the details for the readers' convenience.

Lemma 4.9 (cf. [42, Lemma 4.2]). *Fix $\gamma \in (\frac{1}{2}, 1)$. Under the same assumption as in Theorem 4.2, if there is only one singular curve $\xi : [T_1, T_2] \to \Sigma$, then for any interval $(t_1, t_2) \subset (T_1, T_2)$ there exists $r_1 = r_1(\rho_0, \Xi_0, \sigma, t_1, t_2, T_1, T_2, \gamma) > 0$ such that*

$$\|u\|_{L^1(Q_{r_1,t_1,t_2})} \le C\|u\|_{L^1(K)}, \tag{4.44}$$

where C is a constant depending on $\|c\|_{L^\infty(Q_{1,T_1,T_2})}, \gamma, \rho_0, \Xi_0, \sigma, t_1, t_2, T_1, T_2$ and K is defined by $K = \overline{Q_{2r_1,T_1,T_2}\backslash Q_{r_1,T_1,T_2}}$.

Proof. Let

$$T_1 < t_5 < t_3 < t_1 < t_2 < t_4 < t_6 < T_2,$$

$\gamma \in (\frac{1}{2}, 1)$ and $r_0 = r_0(\rho_0, \Xi_0, \sigma, t_5, t_6, \gamma) > 0$ as in Lemma 4.5. Let $0 < \delta < r_1 < \frac{r_0}{2}$. We construct the function

$$\phi(x,t) = \phi(x,t; r_1, 2r_1, t_3, t_4, t_5, t_6, T_1, T_2)$$

satisfying (4.40), and the function $v \in C^\infty(\overline{Q_{\rho_0,t_5,t_6}}\backslash\overline{\Gamma_{t_5,t_6}})$ satisfying (4.24) with properties (4.25)–(4.26). Moreover, we define

$$V(x,t) = e^{-2v(x,t)}, \quad w(x,t) = \zeta(t)^{-1}V(x,t) - 1 \tag{4.45}$$

and

$$\varphi(x,t) = \phi(x,t)\zeta(t)(H \circ w)(x,t),$$

where $\zeta = \zeta(t; t_1, t_2, t_3, t_4, \delta, r_1)$ and H are given in Definition 4.7. Note that $H \circ w = 0$ near Γ in Q_{r_0,t_5,t_6}. This implies that $\varphi \in C_0^\infty(Q_{r_0,t_5,t_6}\backslash\Gamma_{t_5,t_6})$. By (4.1) we have

$$-\int_{Q_{r_0,t_5,t_6}} u\left(\frac{\partial \varphi}{\partial t} + \Delta\varphi\right) = \int_{Q_{r_0,t_5,t_6}} cu\varphi. \tag{4.46}$$

Note that (4.39) and (4.45) imply that

$$\zeta H \circ w \le \zeta w H' \circ w \le V H' \circ w, \tag{4.47}$$

we have $\varphi \le \phi V H' \circ w$. Thus, the right-hand side of (4.46) can be estimated by

$$\int_{Q_{r_0,t_5,t_6}} c u \varphi \ge -\|cV\|_{L^\infty(Q_{r_0,t_5,t_6})} \int_{Q_{r_0,t_5,t_6}} u\,\phi H' \circ w. \tag{4.48}$$

On the other hand, direct calculation shows that

$$\frac{\partial \varphi}{\partial t} + \Delta \varphi = \phi A + B,$$

where

$$\begin{aligned} A &= (\partial_t V + \Delta V) H' \circ w - \partial_t \zeta \big((w+1) H' \circ w - H \circ w \big) + \zeta^{-1} |\nabla V|^2 H'' \circ w, \\ B &= (\partial_t \phi + \Delta \phi) \zeta H \circ w + 2 \langle \nabla \phi, \nabla V \rangle H' \circ w. \end{aligned}$$

By (4.39) and (4.43) we have

$$A \ge (\partial_t V + \Delta V) H' \circ w - 2|\partial_t \zeta| H' \circ w \ge (1 - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}) H' \circ w.$$

Note that

$$\begin{aligned} \mathrm{Supp}(B) &\subset \mathrm{Supp}(|\nabla \phi| + |\partial_t \phi|) \cap \{(x,t) \in Q_{2r_1,t_5,t_6} \mid w \ge 0\} \\ &\subset \{(x,t) \in \overline{Q_{2r_1,t_5,t_6}} \setminus Q_{r_1,t_3,t_4} \mid w \ge 0\} \\ &\subset \{(x,t) \in \overline{Q_{2r_1,t_5,t_6}} \setminus Q_{r_1,t_3,t_4} \mid \mathbf{r}(x,t) \ge \zeta(t)^{\frac{1}{2\gamma}}\} \\ &\subset \{(x,t) \in \overline{Q_{2r_1,t_5,t_6}} \mid \mathbf{r}(x,t) \ge r_1\} =: K, \end{aligned}$$

where we used the construction of $\zeta(t)$ in Definition 4.7. Thus, we have

$$\begin{aligned} |B| &\le \big(\|\partial_t \phi + \Delta \phi\|_{L^\infty(K)} \|V\|_{L^\infty(K)} + 2\|\nabla \phi\|_{L^\infty(K)} \|\nabla V\|_{L^\infty(K)} \big) \chi_K \\ &\le C(c_{\phi,K}, \gamma, r_1) \chi_K, \end{aligned}$$

where $c_{\phi,K} = \sup_K (|\partial_t \phi| + |\Delta \phi| + |\nabla \phi|)$. Combining the above estimates, we have

$$\frac{\partial \varphi}{\partial t} + \Delta \varphi \ge (1 - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}) \phi H' \circ w - C(c_{\phi,K}, \gamma, r_1) \chi_K. \tag{4.49}$$

Combining (4.48)–(4.49) with (4.46), we have

$$\big(1 - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})} - \|cV\|_{L^\infty(Q_{r_0,t_5,t_6})}\big) \int_{Q_{r_0,t_5,t_6}} u\,\phi H' \circ w \le C(c_{\phi,K}, \gamma, r_1) \int_K u.$$

Taking r_0 sufficiently small and using the assumption that $c(x,t)$ is locally bounded, we have

$$1 - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})} - \|cV\|_{L^\infty(Q_{r_0,t_5,t_6})} \ge \frac{1}{2}.$$

Therefore, by the definition of ϕ we have

$$\int_{Q_{r_1,t_1,t_2}} u\,H' \circ w \le \int_{Q_{r_0,t_5,t_6}} u\,\phi H' \circ w \le C(c_{\phi,K}, \gamma, r_1) \int_K u. \tag{4.50}$$

Note that the function $H' \circ w$ converges to 1 on $Q_{r_1,t_1,t_2} \backslash \Gamma_{t_1,t_2}$ as $\delta \to 0$. Thus, taking $\delta \to 0$ in (4.50), we have that u is integrable on Q_{r_1,t_1,t_2}. The lemma is proved. ■

4.4. Estimates of the solution with multiple singularities

In this subsection, we show that the solution of (4.1) is L^1 across the singularities when multiple singular curves exist. If any two singular curves do not coincide at any time, we can use Lemma 4.9 for each singular curve and get the L^1 estimates. Otherwise, the proof will be much more difficult. The idea comes from a combination of the arguments in [42, Lemma 4.2 and Lemma 4.3], but we need to use some new cutoff functions in Definition 4.12. We sketch the proof as follows. First, we control the L^1 norm of u near the intersection point (x_0, t_0) by an integral which characterizes the growth of u near the singular curves away from (x_0, t_0) (cf. (4.68)). Next, the integral of u is bounded by the L^1 norm of u on some compact set K away from the singular curves (cf. (4.80)). Combining the above two steps, we can bound the L^1 norm of u near the intersection point.

First, we introduce the following definition.

Definition 4.10. Let $\{\xi_1, \xi_2, \dots, \xi_l\}$ $(t \in [T_1, T_2])$ be continuous curves in Σ, and let $I \subset [T_1, T_2]$. We say that $\{\xi_1(t), \dots, \xi_l(t)\}$ are disjoint on I if for any time $t_0 \in I$, we have

$$\xi_i(t_0) \neq \xi_j(t_0) \quad \text{for all } i \neq j.$$

Let (x_0, t_0) be a point in the singular set. By Lemma 2.11, there exists finitely many singular curves passing through (x_0, t_0). There are two cases for the singular curves:

(A) There exists an interval (t_1, t_2) with $t_0 \in (t_1, t_2)$ and singular curves $\{\xi_1(t), \dots, \xi_l(t)\}$ $(t \in (t_1, t_2))$ such that $\{\xi_1(t), \dots, \xi_l(t)\}$ are disjoint on $(t_1, t_2)\backslash\{t_0\}$ and

$$\xi_1(t_0) = \xi_2(t_0) = \cdots = \xi_l(t_0).$$

(B) There exists an interval (t_1, t_2) with $t_0 \in (t_1, t_2)$, singular curves $\{c_1(t), \dots, c_k(t)\}$ $(t \in (t_1, t_0])$ and $\{\tilde{c}_1(t), \dots, \tilde{c}_l(t)\}$ $(t \in [t_0, t_2))$ such that

(a) $\{c_1(t), \dots, c_k(t)\}$ are disjoint on (t_1, t_0),

(b) $\{\tilde{c}_1(t), \dots, \tilde{c}_l(t)\}$ are disjoint on (t_0, t_2),

(c) The singular curves coincide at t_0:

$$c_1(t_0) = \cdots = c_k(t_0) = \tilde{c}_1(t_0) = \cdots = \tilde{c}_l(t_0) = x_0.$$

If $k = l$, then this is just the case (A). Note that the union of two Lipschitz curves is still a Lipschitz curve. Thus, for $k < l$ we can construct the curves

$$\xi_i(t) = \begin{cases} c_i(t), & t \in (t_1, t_0], \\ \tilde{c}_i(t), & t \in (t_0, t_2), \end{cases} \quad \text{for } 1 \leq i \leq k,$$

$$\xi_i(t) = \begin{cases} c_k(t), & t \in (t_1, t_0], \\ \tilde{c}_i(t), & t \in (t_0, t_2), \end{cases} \quad \text{for } k < i \leq l,$$

Then $\{\xi_1(t), \dots, \xi_l(t)\}(t \in (t_1, t_2))$ are Lipschitz curves. For $k > l$ we can also construct similar curves $\{\xi_1(t), \dots, \xi_k(t)\}(t \in (t_1, t_2))$.

Summarizing the above discussion, we define:

Definition 4.11. Let $I = [t_1, t_2]$ or (t_1, t_2) where $t_1 < t_0 < t_2$. We call that the singular curves $\{\xi_1, \xi_2, \dots, \xi_l\}(t \in I)$ are around (x_0, t_0) on I, if the curves satisfy the conditions in Case (A) or are constructed as in Case (B) on I.

We construct some cutoff functions when the singular curves are not disjoint.

Definition 4.12. Let $0 < \underline{r} < \bar{r} < 1$, $T_1 < \underline{T} < \underline{t} < \bar{t} < \bar{T} < T_2$ and let $\{\xi_1, \xi_2, \dots, \xi_l\}$ $(t \in [T_1, T_2])$ be σ-Lipschitz curves. We assume that $\{\xi_1(t), \dots, \xi_l(t)\}$ are around (x_0, t_0) on (T_1, T_2) for some $t_0 \in (\underline{t}, \bar{t})$.

(1) For each ξ_k and $(t_1, t_2) \subset [T_1, T_2]$, we define the notations $Q^{(k)}_{r,t_1,t_2}$ and $\Gamma^{(k)}_{t_1,t_2}$ as in (4.17)–(4.18), and we define

$$Q_{r,t_1,t_2} = \bigcup_{k=1}^{l} Q^{(k)}_{r,t_1,t_2}, \quad \Gamma_{t_1,t_2} = \bigcup_{k=1}^{l} \Gamma^{(k)}_{t_1,t_2}, \quad \hat{Q}_{r,t_1,t_2} = \bigcap_{k=1}^{l} Q^{(k)}_{r,t_1,t_2}. \tag{4.51}$$

(2) For each ξ_k we define the function $\phi_{\xi_k}(x, t; \underline{r}, \bar{r}, \underline{t}, \bar{t}, \underline{T}, \bar{T}, T_1, T_2) \in C^\infty(Q^{(k)}_{1,T_1,T_2})$ as in (4.40). Then the function

$$\phi(x,t;\underline{r},\bar{r},\underline{t},\bar{t},\underline{T},\bar{T}) = 1-(1-\phi_{\xi_1})(1-\phi_{\xi_2})\cdots(1-\phi_{\xi_l}) \in C^\infty(\hat{Q}_{1,T_1,T_2}) \tag{4.52}$$

satisfies the properties

$$0 \le \phi \le 1, \quad \phi = \begin{cases} 1 & \text{on} \, \hat{Q}_{1,T_1,T_2} \cap \overline{Q_{\underline{r},\underline{t},\bar{t}}}, \\ 0 & \text{on } \hat{Q}_{1,T_1,T_2} \backslash \overline{Q_{\bar{r},\underline{T},\bar{T}}}, \end{cases}$$

Moreover, ϕ satisfies the properties

$$\operatorname{Supp}(\phi) \cap \hat{Q}_{1,\underline{T},\bar{T}} \subset Q_{\bar{r},\underline{T},\bar{T}} = \bigcup_{k=1}^{l} \{(x,t) \in \hat{Q}_{1,\underline{T},\bar{T}} \mid \mathbf{r}_k(x,t) \le \bar{r}\}$$

and

$$\begin{aligned} &\operatorname{Supp}(|\nabla\phi| + |\partial_t\phi|) \cap \hat{Q}_{1,\underline{t},\bar{t}} \subset \overline{Q_{\bar{r},\underline{t},\bar{t}}} \backslash Q_{\underline{r},\underline{t},\bar{t}} \\ &\subset \bigcup_{k=1}^{l} \{(x,t) \in \hat{Q}_{1,\underline{t},\bar{t}} \mid \underline{r} \le \mathbf{r}_k(x,t) \le \bar{r},\ \mathbf{r}_i(x,t) \ge \underline{r} \text{ for all } i \ne k\}. \end{aligned} \tag{4.53}$$

Here we assumed that $Q_{\bar{r},\underline{T},\bar{T}} \subset \hat{Q}_{1,\underline{T},\bar{T}}$ by shrinking the interval $[T_1, T_2]$ if necessary.

(3) Fix $\gamma \in (\frac{1}{2}, 1)$. For each ξ_k, we define

$$v_k \in C^\infty(\overline{Q^{(k)}_{1,\underline{T},\bar{T}}} \backslash \overline{\Gamma^{(k)}_{\underline{T},\bar{T}}})$$

as in (2) of Lemma 4.5, and let $r_0^{(k)}$ the constant in (2) of Lemma 4.5 such that inequalities (4.25)–(4.26) hold for

$$(x,t) \in \overline{Q^{(k)}_{r_0^{(k)},\underline{T},\bar{T}}} \backslash \overline{\Gamma^{(k)}_{\underline{T},\bar{T}}}.$$

Set

$$r_0 := \min\{r_0^{(1)}, r_0^{(2)}, \ldots, r_0^{(l)}\}. \tag{4.54}$$

After shrinking the interval $[T_1, T_2]$ if necessary, we can assume that

$$\Gamma_{T_1,T_2} \subset \hat{Q}_{r_0,T_1,T_2}. \tag{4.55}$$

By (4.54)–(4.55), we know that inequalities (4.25)–(4.26) hold for all functions v_k and all $(x,t) \in \hat{Q}_{r_0,\underline{T},\bar{T}} \backslash \Gamma_{\underline{T},\bar{T}}$.

(4) For any $\epsilon > 0$ and $(x,t) \in \hat{Q}_{r_0,\underline{T},\bar{T}} \backslash \Gamma_{\underline{T},\bar{T}}$, we define

$$\begin{aligned}\tilde{v}(x,t) &= \sum_{k=1}^{l} v_k(x,t),\\ \tilde{w}_\epsilon(x,t) &= 2 - \frac{\tilde{v}(x,t)}{\log \frac{1}{\epsilon}},\\ \tilde{\varphi}_\epsilon(x,t) &= (H \circ \tilde{w}_\epsilon)(x,t),\end{aligned} \tag{4.56}$$

where H is defined in (2) of Definition 4.7. Note that $H \circ \tilde{w}_\epsilon = 0$ near each ξ_k and this implies that $\tilde{\varphi}_\epsilon$ vanishes near $\Gamma_{\underline{T},\bar{T}}$. Moreover, for any $(x,t) \in \hat{Q}_{r_0,\underline{T},\bar{T}} \backslash \Gamma_{\underline{T},\bar{T}}$ we have

$$\lim_{\epsilon\to 0} \tilde{\varphi}_\epsilon(x,t) = H(2), \quad \lim_{\epsilon\to 0} |\nabla \tilde{\varphi}_\epsilon|(x,t) = 0, \tag{4.57}$$

$$\frac{\partial \tilde{\varphi}_\epsilon}{\partial t} + \Delta \tilde{\varphi}_\epsilon = H'' \circ \tilde{w}_\epsilon |\nabla \tilde{w}_\epsilon|^2. \tag{4.58}$$

Let

$$\tilde{\mathbf{r}}(x,t) = e^{-\tilde{v}(x,t)}.$$

Then inequalities (4.25) imply that for any $(x,t) \in \hat{Q}_{r_0\underline{T},\bar{T}} \backslash \Gamma_{\underline{T},\bar{T}}$

$$\mathbf{r}_1\mathbf{r}_2\cdots\mathbf{r}_l \le \tilde{\mathbf{r}}(x,t) \le (\mathbf{r}_1\mathbf{r}_2\cdots\mathbf{r}_l)^\gamma. \tag{4.59}$$

(5) Under the above assumptions, for $\rho > 0$ and $h \in L^1(Q_{r_0,\underline{t},\bar{t}})$ we define

$$I(\rho; \underline{t}, \bar{t}, h, r_0) = \int_{Q_{r_0,\underline{t},\bar{t}} \cap \{\rho \le \tilde{\mathbf{r}}(x,t) \le 1\}} \frac{h|\nabla \tilde{v}|^2}{|\log \rho|^2},$$

where $\tilde{\mathbf{r}}(x,t)$ and $\tilde{v}(x,t)$ are the function defined in (4) above.

(6) Assume that $\{\xi_1(t), \ldots, \xi_l(t)\}$ are disjoint on $[T_1, T_2]$. We choose $\bar{\rho} > 0$ such that

$$Q^{(i)}_{\bar{\rho},T_1,T_2} \cap Q^{(j)}_{\bar{\rho},T_1,T_2} = \emptyset$$

for any $1 \le i \ne j \le l$. For any $\rho \in (0, \bar{\rho})$, $T_1 \le \underline{t} < \bar{t} \le T_2$ and $h \in L^1(Q^{(k)}_{1,\underline{t},\bar{t}})$, we define

$$I_{\xi_k}(\rho; \underline{t}, \bar{t}, h, \bar{\rho}) = \frac{1}{|\log \rho|^2} \int_{\underline{t}}^{\bar{t}} \int_{\rho \le \mathbf{r}_k(x,t) \le \bar{\rho}} \frac{h}{\mathbf{r}_k(x,t)^2}.$$

The next result gives the L^1 estimate of the solution near the singularities when the singular curves are not disjoint.

Lemma 4.13. *Under the same assumption as in Theorem 4.2, for any $(t_1, t_2) \subset (T_1, T_2)$ there exists $r_1 = r_1(\rho_0, \Xi_0, l, t_1, t_2, T_1, T_2, \gamma) > 0$ such that*

$$\|u\|_{L^1(Q_{r_1,t_1,t_2})} \le C\|u\|_{L^1(K)}, \tag{4.60}$$

where C is a constant depending on $\|c\|_{L^\infty(Q_{1,T_1,T_2})}, \gamma, \rho_0, \Xi_0, \sigma, t_1, t_2, T_1, T_2$ and K is defined by $K = \overline{Q_{2r_1,T_1,T_2}} \backslash Q_{r_1,T_1,T_2}$. Moreover, we have

$$\sup_{\rho\in(0,\frac{1}{2})} I(\rho; t_1, t_2, u, r_0) < +\infty. \tag{4.61}$$

Proof. We divide the proof into several steps:

Step 1. Without loss of generality, we can assume that $c(x,t) \ge 0$ on Q_{1,T_1,T_2}. In fact, let $u(x,t)$ be a solution of (4.1). Then for any $k \in \mathbb{R}$ the function $\tilde{u}(x,t) = u(x,t)e^{kt}$ satisfies the equation

$$\frac{\partial \tilde{u}}{\partial t} = \Delta\tilde{u} + (c+k)\tilde{u} \quad \text{for all } (x,t) \in Q_{1,T_1,T_2} \backslash \Gamma_{T_1,T_2}.$$

Since c is locally bounded by the assumption, we can choose k large such that $c + k \ge 0$ on Q_{1,T_1,T_2}. Thus, it suffices to show Lemma 4.13 for $c(x,t) \ge 0$.

Step 2. Assume that $\{\xi_1(t), \dots, \xi_l(t)\}$ are around (x_0, t_0) on $[T_1, T_2]$. Let $T_1 < t_5 < t_3 < t_1 < t_0 < t_2 < t_4 < t_6 < T_2$. We construct $v_k, r_0, \tilde{w}_\epsilon$ and $\tilde{\varphi}_\epsilon$ as in Definition 4.12 by setting

$$\underline{T} = t_5, \quad \bar{T} = t_6, \quad \underline{t} = t_3, \quad \bar{t} = t_4.$$

Assume that (4.55) holds. Let $0 < \delta < r_1 < \frac{r_0}{2}$ and set $\underline{r} = r_1, \bar{r} = 2r_1$. After shrinking r_1 and the interval $[T_1, T_2]$ if necessary, we assume that $Q_{2r_1,T_1,T_2} \subset \hat{Q}_{r_0,T_1,T_2}$. We choose t_7, t_8 such that $T_1 < t_7 < t_5 < t_6 < t_8 < T_2$ and define the function

$$\phi = \phi(x, t; r_1, 2r_1, t_5, t_6, t_7, t_8, T_1, T_2)$$

as in Definition 4.12. Then by (4.53) the function ϕ satisfies the properties

$$\begin{aligned} &\mathrm{Supp}(|\nabla\phi| + |\partial_t\phi|) \cap Q_{2r_1,t_5,t_6} \\ &\quad \subset Q_{2r_1,t_5,t_6} \backslash Q_{r_1,t_5,t_6} \\ &\quad \subset \bigcup_{k=1}^{l} \{(x,t) \in Q_{2r_1,t_5,t_6} \mid r_1 \le \mathbf{r}_k(x,t) \le 2r_1,\ \mathbf{r}_i(x,t) \ge r_1 \text{ for all } i \ne k\}. \end{aligned}$$

Moreover, we define the following functions on $\hat{Q}_{r_0,t_5,t_6} \backslash \Gamma_{t_5,t_6}$:

$$V_k(x,t) = e^{-2v_k(x,t)}, \qquad V(x,t) = \sum_{k=1}^{l} V_k(x,t),$$

$$w(x,t) = \zeta(t)^{-1}V(x,t) - 1, \quad \varphi_0(x,t) = \phi(x,t)\zeta(t)(H \circ w)(x,t),$$

where $\zeta = \zeta(t; t_1, t_2, t_3, t_4, \delta, r_1)$ is the function defined in (1) of Definition 4.7. By using properties (4.41)–(4.43), for any $(x,t) \in \hat{Q}_{r_0,t_5,t_6} \backslash \Gamma_{t_5,t_6}$ we have

$$\sum_{k=1}^{l} \mathbf{r}_k(x,t)^2 \le V(x,t) \le \sum_{k=1}^{l} \mathbf{r}_k(x,t)^{2\gamma}, \tag{4.62}$$

$$l \le \frac{\partial V}{\partial t} + \Delta V \le 4 \sum_{k=1}^{l} \mathbf{r}_k(x,t)^{2\gamma-2}, \tag{4.63}$$

$$|\nabla V| \le 2 \sum_{k=1}^{l} \mathbf{r}_k(x,t)^{2\gamma-1}. \tag{4.64}$$

Note that the function $\varphi_0(x,t)$ vanishes near the point $(\xi_1(t_0), t_0)$, but $\varphi_0(x,t)$ may not be zero on Γ_{t_5,t_6}. The function $\tilde{\varphi}_\epsilon$ defined in Definition 4.12 vanishes near Γ_{t_5,t_6}, but it does not satisfy inequality (4.63) and inequality (4.49). Therefore, the argument of Lemma 4.9 does not work any more.

Step 3. Direct calculation as in the proof of Lemma 4.9 yields, for any $(x,t) \in Q_{2r_1,t_5,t_6}$,

$$\varphi_0(x,t) \le \phi(x,t) V(x,t) H' \circ w, \tag{4.65}$$

$$\frac{\partial \varphi_0}{\partial t} + \Delta \varphi_0 \ge \big(l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}\big)\phi H' \circ w - C(c_{\phi,K}, \gamma, r_1)\chi_K, \tag{4.66}$$

where K and $c_{\phi,K}$ are defined by

$$K = \operatorname{Supp}(|\nabla\phi| + |\partial_t \phi|) \cap Q_{2r_1,t_5,t_6}, \quad c_{\phi,K} = \sup_K (|\partial_t \phi| + |\Delta\phi| + |\nabla\phi|). \tag{4.67}$$

Let $\varphi = \varphi_0 \tilde{\varphi}_\epsilon \in C_0^\infty(Q_{2r_1,t_5,t_6} \backslash \Gamma_{t_5,t_6})$. Then we have

$$\begin{aligned}
\frac{\partial \varphi}{\partial t} + \Delta\varphi &= \left(\frac{\partial \varphi_0}{\partial t} + \Delta\varphi_0\right)\tilde{\varphi}_\epsilon + \left(\frac{\partial \tilde{\varphi}_\epsilon}{\partial t} + \Delta\tilde{\varphi}_\epsilon\right)\varphi_0 + 2\langle \nabla\tilde{\varphi}_\epsilon, \nabla\varphi_0\rangle \\
&= \left(\frac{\partial \varphi_0}{\partial t} + \Delta\varphi_0\right)\tilde{\varphi}_\epsilon + H'' \circ \tilde{\omega}_\epsilon |\nabla\tilde{\omega}_\epsilon|^2 \varphi_0 + 2\phi H' \circ \tilde{\omega}_\epsilon H' \circ w \langle \nabla\tilde{\omega}_\epsilon, \nabla V\rangle \\
&\quad + 2\zeta H' \circ \tilde{\omega}_\epsilon H \circ w \langle \nabla\tilde{\omega}_\epsilon, \nabla\phi\rangle \\
&\ge \big((l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})})\phi H' \circ w - C(c_{\phi,K}, \gamma, r_1)\chi_K\big)\tilde{\varphi}_\epsilon \\
&\quad - 2\phi H' \circ w |\nabla V| \cdot |\nabla\tilde{\omega}_\epsilon| \chi_{\{\tilde{\omega}_\epsilon > 0\}} - 2V H' \circ w H' \circ \tilde{\omega}_\epsilon |\nabla\tilde{\omega}_\epsilon| \cdot |\nabla\phi|,
\end{aligned}$$

where we used (4.47), (4.58), (4.66) and the definition of w. Combining this with (4.46) and using the assumption $c(x,t) \ge 0$, we have

$$\begin{aligned}
0 &\le \int_{Q_{2r_1,t_5,t_6}} cu\varphi = -\int_{Q_{2r_1,t_5,t_6}} u\left(\frac{\partial\varphi}{\partial t} + \Delta\varphi\right) \\
&\le -\big(l - 2\|\partial_t\zeta\|_{L^\infty(\mathbb{R})}\big)\int_{Q_{2r_1,t_5,t_6}} u\,\phi\tilde{\varphi}_\epsilon H' \circ w + C(c_{\phi,K}, \gamma, r_1)\int_K u\,\tilde{\varphi}_\epsilon \\
&\quad + 2\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} \int_{Q_{2r_1,t_5,t_6}} u\,\phi H' \circ w |\nabla\tilde{\omega}_\epsilon| \chi_{\{\tilde{\omega}_\epsilon > 0\}} \\
&\quad + 2\|V\|_{L^\infty(Q_{2r_1,t_5,t_6})} \int_{Q_{2r_1,t_5,t_6}} uH' \circ w H' \circ \tilde{\omega}_\epsilon |\nabla\tilde{\omega}_\epsilon| \cdot |\nabla\phi|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\big(l-2\|\partial_t\zeta\|_{L^\infty(\mathbb{R})}\big)\int_{Q_{2r_1,t_5,t_6}} u\,\phi H'\circ w\,\tilde{\varphi}_\epsilon \\
&\le 2\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})}\int_{Q_{2r_1,t_5,t_6}} u\,\phi H'\circ w|\nabla\tilde{\omega}_\epsilon|\chi_{\{\tilde{\omega}_\epsilon>0\}} \\
&\quad + C(c_{\phi,K},\gamma,r_1)\int_K u\,\tilde{\varphi}_\epsilon \\
&\quad + 2\|V\|_{L^\infty(Q_{2r_1,t_5,t_6})}\int_{Q_{2r_1,t_5,t_6}} u|\nabla\phi|\cdot|\nabla\tilde{\omega}_\epsilon| \\
&\le 2\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})}\int_{Q_{2r_1,t_5,t_6}} u\,\phi H'\circ w\chi_{\{\tilde{\omega}_\epsilon>0\}} \\
&\quad + 2\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})}\int_{Q_{2r_1,t_5,t_6}} u\,\phi H'\circ w|\nabla\tilde{\omega}_\epsilon|^2\chi_{\{\tilde{\omega}_\epsilon>0\}} \\
&\quad + C(c_{\phi,K},\gamma,r_1)\int_K u\,\tilde{\varphi}_\epsilon \\
&\quad + 2\|V\|_{L^\infty(Q_{2r_1,t_5,t_6})}\int_{Q_{2r_1,t_5,t_6}} u|\nabla\phi|\cdot|\nabla\tilde{\omega}_\epsilon|. \qquad (4.68)
\end{aligned}$$

The main difficulty is to estimate the integral

$$\int_{Q_{2r_1,t_5,t_6}} u\,\phi H'\circ w|\nabla\tilde{\omega}_\epsilon|^2\chi_{\{\tilde{\omega}_\epsilon>0\}} \qquad (4.69)$$

on the right-hand side of (4.68).

Step 4. We estimate the integral (4.69). For any $\rho\in(0,\frac{1}{2})$, we define the functions

$$\bar{w}_\rho(x,t)=\frac{1}{3}\left(2-\frac{\tilde{v}(x,t)}{\log(\frac{1}{\rho})}\right),\qquad \psi(x,t)=\phi(x,t)H\circ\bar{w}_\rho\in C_0^\infty(Q_{2r_1,t_5,t_6}\backslash\Gamma_{t_5,t_6}),$$

where $\tilde{v}$ is the function defined in (4.56). Note that $\bar{w}_\rho(x,t)$ satisfies $\partial_t\bar{w}_\rho+\Delta\bar{w}_\rho=0$. Direct calculation shows that

$$\frac{\partial\psi}{\partial t}+\Delta\psi=\phi|\nabla\bar{w}_\rho|^2H''\circ\bar{w}_\rho+\left(\frac{\partial\phi}{\partial t}+\Delta\phi\right)H\circ\bar{w}_\rho+2\langle\nabla\phi,\nabla\bar{w}_\rho\rangle H'\circ\bar{w}_\rho.$$

Since u satisfies

$$-\int_{Q_{2r_1,t_5,t_6}} u\left(\frac{\partial\psi}{\partial t}+\Delta\psi\right)=\int_{Q_{2r_1,t_5,t_6}} cu\psi\ge 0,$$

we have

$$\begin{aligned}
&\int_{Q_{2r_1,t_5,t_6}} u\,\phi|\nabla\bar{w}_\rho|^2H''\circ\bar{w}_\rho \\
&\quad\le-\int_{Q_{2r_1,t_5,t_6}} u\left(\left(\frac{\partial\phi}{\partial t}+\Delta\phi\right)H\circ\bar{w}_\rho+2\langle\nabla\phi,\nabla\bar{w}_\rho\rangle H'\circ\bar{w}_\rho\right). \qquad (4.70)
\end{aligned}$$

We estimate each term of (4.70). Note that

$$\begin{aligned}
&\left\{(x,t)\in Q_{2r_1,t_5,t_6}\;\middle|\; H''\circ\bar{w}_\rho \ge \min_{\frac13\le z\le\frac23} H''(z)\right\}\\
&\supset \left\{(x,t)\in Q_{2r_1,t_5,t_6}\;\middle|\;\frac13\le\bar{\omega}_\rho\le\frac23\right\}\\
&=\{(x,t)\in Q_{2r_1,t_5,t_6}\mid 1\le\tilde{\omega}_\rho\le 2\}\\
&=\{(x,t)\in Q_{2r_1,t_5,t_6}\mid \rho\le\tilde{\mathbf{r}}(x,t)\le 1\},
\end{aligned}\tag{4.71}$$

where $\tilde{\omega}_\rho$ is defined in (4.56) and $\tilde{\mathbf{r}}(x,t)=e^{-\tilde{v}(x,t)}$. Thus, the left-hand side of (4.70) satisfies the inequality

$$\begin{aligned}
\int_{Q_{2r_1,t_5,t_6}} u\,\phi|\nabla\bar{w}_\rho|^2 H''\circ\bar{w}_\rho &\ge C\int_{Q_{2r_1,t_5,t_6}}\frac{u\phi|\nabla\tilde{v}|^2\chi_{\{1\le\tilde{\omega}_\rho\le2\}}}{|\log\rho|^2}\\
&=C\int_{Q_{2r_1,t_5,t_6}} u\phi|\nabla\tilde{\omega}_\rho|^2\chi_{\{1\le\tilde{\omega}_\rho\le2\}},
\end{aligned}\tag{4.72}$$

where C is a universal constant. We choose $2r_1<1$ and by (4.59) we have $\tilde{\mathbf{r}}(x,t)<1$ on $\mathrm{Supp}(\phi)\cap Q_{2r_1,t_5,t_6}$. Thus, on $\mathrm{Supp}(\phi)\cap Q_{2r_1,t_5,t_6}$ we have

$$\bar{\omega}_\rho=\frac13\left(2-\frac{\tilde{v}(x,t)}{\log(\frac1\rho)}\right)\le\frac23$$

and

$$H\circ\bar{\omega}_\rho\le\bar{\omega}_\rho\le\frac23.$$

Combining this with (4.70), the first term of the right-hand side of (4.70) satisfies the inequality

$$-\int_{Q_{2r_1,t_5,t_6}} u\left(\frac{\partial\phi}{\partial t}+\Delta\phi\right)H\circ\bar{w}_\rho\le C(c_{\phi,K})\int_K u,\tag{4.73}$$

where K and $c_{\phi,K}$ are given by (4.67). Note that by (4.53) for any i we have $\mathbf{r}_i(x,t)\ge r_1$ on $\mathrm{Supp}(|\nabla\phi|)\cap Q_{2r_1,t_5,t_6}$. Combining this with (4.26), we have

$$|\nabla\tilde{v}|^2\le l\sum_{k=1}^{l}|\nabla v_k|^2\le l\sum_{k=1}^{l}\frac{1}{\mathbf{r}_k^2}\le\frac{l^2}{r_1^2}\quad\text{for all }(x,t)\in\mathrm{Supp}(|\nabla\phi|)\cap Q_{2r_1,t_5,t_6}.$$

Thus, when $\rho\in(0,\frac12)$, we have

$$|\nabla\bar{\omega}_\rho|=\frac23\frac{|\nabla\tilde{v}|}{\log\frac1\rho}\le\frac{2l^2}{(3\log 2)\,r_1^2}\quad\text{for all }(x,t)\in\mathrm{Supp}(|\nabla\phi|)\cap Q_{2r_1,t_5,t_6}.\tag{4.74}$$

This implies that the second term of the right-hand side of (4.70) satisfies

$$-\int_{Q_{2r_1,t_5,t_6}} 2u\langle\nabla\phi,\nabla\bar{w}_\rho\rangle H'\circ\bar{w}_\rho\le C(c_{\phi,K},r_1,l)\int_K u.\tag{4.75}$$

Let $\rho = \epsilon^2$. Note that $\tilde{\mathbf{r}}(x,t) \le 1$ on $\mathrm{Supp}(\phi) \cap Q_{2r_1,t_5,t_6}$. By (4.71) we have

$$\begin{aligned}
&\{(x,t) \in Q_{2r_1,t_5,t_6} \mid \tilde{\omega}_\epsilon > 0\} \cap \mathrm{Supp}(\phi) \\
&\quad = \{(x,t) \in Q_{2r_1,t_5,t_6} \mid \tilde{\mathbf{r}}(x,t) > \epsilon^2\} \cap \mathrm{Supp}(\phi) \\
&\quad = \{(x,t) \in Q_{2r_1,t_5,t_6} \mid \rho \le \tilde{\mathbf{r}}(x,t) \le 1\} \cap \mathrm{Supp}(\phi), \\
&\quad = \{(x,t) \in Q_{2r_1,t_5,t_6} \mid 1 \le \tilde{\omega}_\rho \le 2\} \cap \mathrm{Supp}(\phi).
\end{aligned} \tag{4.76}$$

Combining (4.73)–(4.75) with (4.70), we have

$$\int_{Q_{2r_1,t_5,t_6}} u\,\phi |\nabla \bar{w}_\rho|^2 H'' \circ \bar{w}_\rho \le C(c_{\phi,K}, r_1, l) \int_K u. \tag{4.77}$$

This together with (4.72) implies that

$$\begin{aligned}
\int_{Q_{2r_1,t_5,t_6}} u\phi |\nabla \tilde{\omega}_\rho|^2 \chi_{\{1\le\tilde{\omega}_\rho\le 2\}} &\le C \int_{Q_{2r_1,t_5,t_6}} u\,\phi |\nabla \bar{w}_\rho|^2 H'' \circ \bar{w}_\rho \\
&\le C(c_{\phi,K}, r_1, l) \int_K u.
\end{aligned} \tag{4.78}$$

Thus, by (4.78) and (4.76) we have

$$\begin{aligned}
\int_{Q_{2r_1,t_5,t_6} \cap \{\rho < \tilde{\mathbf{r}} < 1\}} u\,\phi |\nabla \tilde{\omega}_\rho|^2 &= \int_{Q_{2r_1,t_5,t_6}} u\phi |\nabla \tilde{\omega}_\rho|^2 \chi_{\{1\le\tilde{\omega}_\rho\le 2\}} \\
&\le C(c_\phi, r_1, l) \int_K u.
\end{aligned} \tag{4.79}$$

Moreover, we have the estimate for the integral (4.69)

$$\begin{aligned}
\int_{Q_{2r_1,t_5,t_6}} u\,\phi H' \circ w |\nabla \tilde{\omega}_\epsilon|^2 \chi_{\{\tilde{\omega}_\epsilon>0\}} &\le \int_{Q_{2r_1,t_5,t_6} \cap \{\tilde{\omega}_\epsilon > 0\}} \frac{u\,\phi |\nabla \tilde{v}|^2}{|\log \epsilon|^2} \\
&= 4 \int_{Q_{2r_1,t_5,t_6} \cap \{1 \le \tilde{\omega}_\rho \le 2\}} u\,\phi |\nabla \tilde{\omega}_\rho|^2 \\
&\le C(c_{\phi,K}, r_1, l) \int_K u.
\end{aligned} \tag{4.80}$$

Step 5. Now we turn back to inequality (4.68). Moreover, by (4.74) we have

$$\int_{Q_{2r_1,t_5,t_6}} u |\nabla \phi| \cdot |\nabla \tilde{\omega}_\epsilon| = 2 \int_{Q_{2r_1,t_5,t_6}} u |\nabla \phi| \cdot \frac{|\nabla \tilde{v}|}{\log \frac{1}{\rho}} \le C(c_{\phi,K}, l, r_1) \int_K u. \tag{4.81}$$

Combining (4.68), (4.81) with (4.80), we have

$$\begin{aligned}
&\big(l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}\big) \int_{Q_{2r_1,t_5,t_6}} u\,\phi H' \circ w\, \tilde{\varphi}_\epsilon \\
&\quad \le 2\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} \int_{Q_{2r_1,t_5,t_6}} u\,\phi H' \circ w\, \chi_{\{\tilde{\omega}_\epsilon > 0\}} \\
&\qquad + C(c_{\phi,K}, r_1, l)\big(\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} + \|V\|_{L^\infty(Q_{2r_1,t_5,t_6})}\big) \int_K u \\
&\qquad + C(c_{\phi,K}, \gamma, r_1) \int_K u\, \tilde{\varphi}_\epsilon.
\end{aligned} \tag{4.82}$$

Since all singular curves are disjoint on $Q_{2r_1,t_5,t_6} \cap \{w \geq 0\}$ by our assumption, by Lemma 4.9 u is integrable on $Q_{2r_1,t_5,t_6} \cap \{w \geq 0\}$. Taking $\epsilon \to 0$ in (4.82) and using the dominated convergence theorem, we have

$$\begin{aligned}
&\big(l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}\big) \int_{Q_{2r_1,t_5,t_6}} u\,\phi H' \circ w \\
&\quad \leq 2\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} \int_{Q_{2r_1,t_5,t_6}} u\,\phi H' \circ w + C(c_{\phi,K}, \gamma, r_1) \int_K u \\
&\qquad + C(c_{\phi,K}, r_1, l)\big(\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} + \|V\|_{L^\infty(Q_{2r_1,t_5,t_6})}\big) \int_K u.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\big(l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})} - 2\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})}\big) \int_{Q_{2r_1,t_5,t_6}} u\,\phi H' \circ w \\
&\quad \leq C(c_{\phi,K}, \gamma, r_1) \int_K u. \qquad (4.83)
\end{aligned}$$

By (4.37) and (4.64), we choose r_1 small such that

$$l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})} - 2\|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} > \frac{1}{2} l.$$

Combining this with (4.83), we have

$$\int_{Q_{2r_1,t_5,t_6}} u\,\phi H' \circ w \leq C(c_{\phi,K}, \gamma, r_1, l) \int_K u. \qquad (4.84)$$

Note that the function $H' \circ w$ converges to 1 on $Q_{r_1,t_1,t_2} \backslash \Gamma_{t_1,t_2}$ as $\delta \to 0$. Thus, taking $\delta \to 0$ in (4.84), we have

$$\int_{Q_{r_1,t_1,t_2}} u \leq C(c_{\phi,K}, \gamma, r_1, l) \int_K u,$$

which implies (4.60). Note that (4.79) implies (4.61) since

$$\begin{aligned}
I(\rho; t_1, t_2, u, r_0) &= \int_{Q_{r_0,t_1,t_2} \cap \{\rho \leq \tilde{\mathbf{r}} \leq 1\}} \frac{u|\nabla \tilde{v}|}{|\log \rho|^2} \\
&= \int_{Q_{r_1,t_5,t_6} \cap \{\rho < \tilde{\mathbf{r}} < 1\}} u|\nabla \tilde{\omega}_\rho|^2 + \int_{Q_{r_0,t_5,t_6} \backslash Q_{r_1,t_5,t_6}} u|\nabla \tilde{\omega}_\rho|^2 \\
&\leq \int_{Q_{2r_1,t_5,t_6} \cap \{\rho < \tilde{\mathbf{r}} < 1\}} u\,\phi|\nabla \tilde{\omega}_\rho|^2 + C(l, r_1) \int_{Q_{r_0,t_5,t_6} \backslash Q_{r_1,t_5,t_6}} u \\
&\leq C(c_\phi, r_1, l) \int_K u + C(l, r_1) \int_{Q_{r_0,t_5,t_6} \backslash Q_{r_1,t_5,t_6}} u < +\infty. \qquad (4.85)
\end{aligned}$$

The lemma is proved. ∎

As a byproduct of the above proof, we have the following result.

Lemma 4.14. *Under the same assumption as in Lemma 4.9, we have, for the singular curve* $\xi : [T_1, T_2] \to \Sigma$,

$$\sup_{\rho\in(0,\frac{1}{2})} I_\xi(\rho; t_1, t_2, u, 2r_1) < +\infty. \tag{4.86}$$

Proof. Inequality (4.86) follows directly from inequality (4.85) and Step 4 of the proof of Lemma 4.13 by choosing $l = 1$. ∎

By using Lemma 4.9, Lemma 4.14 and following the same arguments as in [42], we have the following results when the singular curves are disjoint.

Lemma 4.15 (cf. [42]). *Under the same assumption as in Theorem 4.2, if we assume that* $\{\xi_1(t), \dots, \xi_l(t)\}$ *are disjoint on* $[T_1, T_2]$ *and* $\bar\rho$ *is the constant in (6) of Definition 4.12, then we have:*

(1) *For each* ξ_k *and* $(t_1, t_2) \subset [T_1, T_2]$*, the mapping* $\mathcal{J}_k : C_0^\infty(Q^{(k)}_{\bar\rho,t_1,t_2}) \to \mathbb{R}$

$$\mathcal{J}_k(f) = \int_{Q^{(k)}_{\bar\rho,t_1,t_2}} \left(u\left(-\frac{\partial f}{\partial t} - \Delta f \right) - cuf \right) dx\, dt$$

defines a distribution whose support is contained in $\Gamma^{(k)}_{t_1,t_2}$*, and satisfies*

$$|\mathcal{J}_k(f)| \le C \Big(\sup_{\Gamma^{(k)}_{t_1,t_2}} |f| \Big) \liminf_{\rho\to 0} I_{\xi_k}(\rho; t_1, t_2, u, \bar\rho),$$

where C *is a universal constant. Here* $I_{\xi_k}(\rho; t_1, t_2, u, \bar\rho)$ *is defined in (6) of Definition 4.12 and it is finite by Lemma 4.14.*

(2) *There exists linear functionals* $\{\mu_1, \dots, \mu_l\}$ *with each* $\mu_k \in (C_0((T_1, T_2)))'$ *such that for all* $\varphi \in C_0^\infty(Q_{1,T_1,T_2})$,

$$\int_{Q_{1,T_1,T_2}} u\left(-\frac{\partial\varphi}{\partial t} - \Delta\varphi \right) = \int_{Q_{1,T_1,T_2}} cu\varphi + \sum_{k=1}^{l} \int_{(T_1,T_2)} \varphi(\xi_k(t), t)\, d\mu_k(t). \tag{4.87}$$

Identity (4.87) *can be rewritten as*

$$\frac{\partial u}{\partial t} - \Delta u = cu + \sum_{k=1}^{l} \delta_{\xi_k} \otimes \mu_k \quad \textit{in } \mathcal{D}'(Q_{1,T_1,T_2}).$$

(3) *Let* μ_k *be one of the measures in part* (2)*. For any function* $\psi \in C_0^\infty((T_1, T_2))$ *with* Supp$(\psi) \subset (t_1, t_2)$*, we have*

$$\int_{T_1}^{T_2} \psi\, d\mu_k = 2 \lim_{\rho\to 0} \frac{1}{|\log\rho|^2} \int_{Q^{(k)}_{\bar\rho,T_1,T_2}} |\nabla v_k|^2 \chi_{\{v_k \le |\log\rho|\}} \psi u. \tag{4.88}$$

(4) *Each measure* μ_k *obtained in* (3) *is positive.*

Proof. Since $\{\xi_1(t),\dots,\xi_l(t)\}$ are disjoint on $[t_1,t_2]$, we can consider each ξ_k as in [42]. After replacing the function g in [42, (4.21)] by the function $c(x,t)$, we know that part (1) follows directly from [42, Lemma 4.4]. Part (2) follows from the proof of [42, Theorem 2.1] (see [42, p. 7303]), (3) follows from [42, Lemma 5.2] and (4) follows from the non-negativity of the right-hand side of (4.88). Since the proof is exactly the same as in [42], we omit the details here. ∎

When the singular curves are around (x_0,t_0), the measures μ_k constructed in Lemma 4.15 may blow up as $t\to t_0$. The next result shows that μ_k is actually bounded when t is close to t_0.

Lemma 4.16. *The same assumption as in Theorem 4.2. Suppose that the singular curves $\{\xi_1(t),\dots,\xi_l(t)\}$ are around (x_0,t_0) on $[t_1,t_2]$ as in Definition 4.11. Define the measure μ on (t_1,t_2) by*

$$\int_{t_1}^{t_2}\psi\,d\mu=\lim_{\rho\to 0}\frac{2}{|\log\rho|^2}\int_{Q_{1,t_1,t_2}}|\nabla\tilde{v}|^2\chi_{\{\tilde{v}\le|\log\rho|\}}\psi u\,d\mathrm{vol}\,dt, \tag{4.89}$$

where the right-hand side is finite by (4.61). Here $\tilde{v}$ is the function defined by (4.56). Then $\mu\in(C_0((t_1,t_2)))'$ and for each ξ_k the measure μ_k obtained by Lemma 4.15 satisfies

$$0\le\gamma^4\,\mu_k(t)\le\mu(t)\quad\textit{for all } t\in(t_1,t_0)\cup(t_0,t_2),$$

where $\gamma\in(\frac{1}{2},1)$ is the constant chosen in Lemma 4.5.

Proof. Since $\{\xi_1(t),\dots,\xi_l(t)\}$ are around (x_0,t_0) on (t_1,t_2), by Definition 4.11 we can assume that $\{\xi_1(t),\dots,\xi_{l'}(t)\}$ are disjoint for some $l'\le l$ on $[t_1,t_0)$ and

$$\xi_{l'}(t)=\xi_{l'+1}(t)=\cdots=\xi_l(t)\quad\text{for all } t\in(t_1,t_0).$$

Let r_0 be the constant defined by (4.54). After shrinking (t_1,t_2) if necessary, we can assume that $Q_{r_0',t_1,t_2}\subset\hat{Q}_{r_0,t_1,t_2}$ for some $r_0'>0$. Let $\rho_1>0$ be the constant such that for any $(x,t)\in Q_{r_0',t_1,t_2}$ and $1\le i\le l'$ we have $\mathbf{r}_i(x,t)\le\rho_1$. Since for any $\delta>0$ the curves $\{\xi_1(t),\dots,\xi_{l'}(t)\}$ are disjoint on $[t_1,t_0-\delta]$, we define

$$d_\delta:=\min\{d_g(\xi_i(t),\xi_j(t))\mid 1\le i\ne j\le l',\ t\in[t_1,t_0-\delta]\}>0.$$

Let $\alpha_0=\frac{d_\delta}{2}$ and $(x,t)\in Q_{r_0',t_1,t_0-\delta}$. By the choice of α_0, if $\mathbf{r}_k(x,t)<\alpha_0$, then we have

$$\mathbf{r}_i(x,t)\ge\alpha_0\quad\text{for all } i\ne k. \tag{4.90}$$

For any $\rho_2>0$, we can find some integer $k\in[1,l']$ such that if $t\in[t_1,t_0-\delta]$ and $\mathbf{r}_1\mathbf{r}_2\cdots\mathbf{r}_{l'}\le\rho_2$, we have

$$\mathbf{r}_i\ge\alpha_0\quad\text{for all } i\ne k,\qquad \mathbf{r}_k\le\frac{\rho_2}{\alpha_0^{l'-1}}. \tag{4.91}$$

We choose ρ_2 such that

$$\frac{\rho_2}{\rho_1^{l'-1}}=\alpha_0. \tag{4.92}$$

By (4.91) for any $k \in \{1, 2, \ldots, l'\}$ and $\rho \in (0, \rho_2)$,

$$\begin{aligned}\left\{(x,t) \in Q_{r_0', t_1, t_0-\delta} \;\middle|\; \tilde{v}(x,t) \le \log \frac{1}{\rho}\right\} &\supseteq \{(x,t) \in Q_{r_0', t_1, t_0-\delta} \mid \mathbf{r}_1 \mathbf{r}_2 \cdots \mathbf{r}_{l'} \ge \rho\} \\ &\supseteq \{(x,t) \in Q_{r_0', t_1, t_0-\delta} \mid \rho_2 \ge \mathbf{r}_1 \mathbf{r}_2 \cdots \mathbf{r}_{l'} \ge \rho\} \\ &= \bigcup_{k=1}^{l'} \Omega_{k,\rho},\end{aligned}$$

where $\Omega_{k,\rho}$ is defined by

$$\begin{aligned}\Omega_{k,\rho} &:= \left\{(x,t) \in Q_{r_0', t_1, t_0-\delta} \;\middle|\; \alpha_0 \le \mathbf{r}_i \le \rho_1,\ i \ne k,\ \frac{\rho}{\alpha_0^{l'-1}} \le \mathbf{r}_k \le \frac{\rho_2}{\rho_1^{l'-1}}\right\} \\ &= \left\{(x,t) \in Q_{r_0', t_1, t_0-\delta} \;\middle|\; \alpha_0 \le \mathbf{r}_i \le \rho_1,\ i \ne k,\ \frac{\rho}{\alpha_0^{l'-1}} \le \mathbf{r}_k \le \alpha_0\right\}. \qquad (4.93)\end{aligned}$$

Note that we used (4.92) in the equality of (4.93). By the definition of r_0' and Lemma 4.5, for any $(x,t) \in \Omega_{k,\rho}$ with $1 \le k \le l'-1$ we have

$$|\nabla \tilde{v}|^2 \ge |\nabla v_k|^2 - \sum_{i \ne k} |\nabla v_i|^2 \ge \frac{\gamma^2}{\mathbf{r}_k^2} - \sum_{i \ne k} \frac{1}{\mathbf{r}_i^2} \ge \frac{\gamma^2}{\mathbf{r}_k^2} - \frac{l-1}{\alpha_0^2},$$

and for any $(x,t) \in \Omega_{k,\rho}$ with $l' \le k \le l$ we have

$$\begin{aligned}|\nabla \tilde{v}|^2 &\ge (l - l' + 1)|\nabla v_k|^2 - \sum_{i=1}^{l'-1} |\nabla v_i|^2 \\ &\ge (l - l' + 1)\frac{\gamma^2}{\mathbf{r}_k^2} - \sum_{i=1}^{l'-1} \frac{1}{\mathbf{r}_i^2} \ge (l - l' + 1)\frac{\gamma^2}{\mathbf{r}_k^2} - \frac{l'-1}{\alpha_0^2}.\end{aligned}$$

Consequently, by (4.89) for any $k \in \{1, 2, \ldots, l'-1\}$ we have

$$\begin{aligned}\int_{t_1}^{t_0-\delta} \psi\, d\mu &\ge \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k,\rho}} |\nabla \tilde{v}|^2 \psi u \\ &\ge \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k,\rho}} \left(\frac{\gamma^2}{\mathbf{r}_k^2} - \frac{l-1}{\alpha_0^2}\right)\psi u \\ &= \gamma^2 \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k,\rho}} \frac{\psi u}{\mathbf{r}_k^2} \qquad (4.94)\end{aligned}$$

and for $k \in \{l', \ldots, l\}$ we have

$$\begin{aligned}\int_{t_1}^{t_0-\delta} \psi\, d\mu &\ge (l - l' + 1)\gamma^2 \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k,\rho}} \frac{\psi u}{\mathbf{r}_k^2} \\ &\ge \gamma^2 \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k,\rho}} \frac{\psi u}{\mathbf{r}_k^2}. \qquad (4.95)\end{aligned}$$

On the other hand, taking

$$\tilde{\rho}^{\frac{1}{\gamma}} = \frac{\rho}{\alpha_0^{l'-1}}$$

and using (4.88), we have

$$\begin{aligned}
\int_{t_1}^{t_0-\delta} \psi \, d\mu_k &= 2 \lim_{\tilde{\rho}\to 0} \frac{1}{|\log \tilde{\rho}|^2} \int_{Q^{(k)}_{\tilde{\rho},t_1,t_0-\delta}} |\nabla v_k|^2 \chi_{\{v_k \le |\log \tilde{\rho}|\}} \psi u \\
&= 2 \lim_{\tilde{\rho}\to 0} \frac{1}{\gamma^2 |\log \rho|^2} \int_{Q^{(k)}_{\alpha_0,t_1,t_0-\delta}} |\nabla v_k|^2 \chi_{\{v_k \le |\log \tilde{\rho}|\}} \psi u \\
&\le \frac{2}{\gamma^2} \lim_{\rho\to 0} \frac{1}{|\log \rho|^2} \int_{Q^{(k)}_{\alpha_0,t_1,t_0-\delta} \cap \{\frac{\rho}{\alpha_0^{l'-1}} \le \mathbf{r}_k\}} \frac{\psi u}{\mathbf{r}_k^2},
\end{aligned} \tag{4.96}$$

where we used the fact that $\{v_k \le \log \frac{1}{\tilde{\rho}}\} \subseteq \{\tilde{\rho}^{\frac{1}{\gamma}} \le \mathbf{r}_k\}$. Note that

$$\begin{aligned}
&Q^{(k)}_{\alpha_0,t_1,t_0-\delta} \cap \left\{\frac{\rho}{\alpha_0^{l'-1}} \le \mathbf{r}_k\right\} \\
&= \left\{(x,t) \in Q_{\alpha_0,t_1,t_0-\delta} \;\middle|\; \frac{\rho}{\alpha_0^{l'-1}} \le \mathbf{r}_k \le \alpha_0\right\} \\
&= \left\{(x,t) \in Q_{r'_0,t_1,t_0-\delta} \;\middle|\; \frac{\rho}{\alpha_0^{l'-1}} \le \mathbf{r}_k \le \alpha_0,\ \alpha_0 \le \mathbf{r}_i \le \rho_1,\ i \ne k\right\} = \Omega_{k,\rho},
\end{aligned} \tag{4.97}$$

where we used (4.90) and (4.93). Combining (4.97) with (4.96), we have

$$\int_{t_1}^{t_0-\delta} \psi \, d\mu_k \le \frac{2}{\gamma^2} \lim_{\rho\to 0} \frac{1}{|\log \rho|^2} \int_{\Omega_{k,\rho}} \frac{\psi u}{\mathbf{r}_k^2}. \tag{4.98}$$

Inequalities (4.94)–(4.95) and (4.98) implies that

$$\int_{t_1}^{t_0-\delta} \psi \, d\mu \ge \gamma^4 \int_{t_1}^{t_0-\delta} \psi \, d\mu_k.$$

Thus, we have

$$0 \le \gamma^4 \mu_k \le \mu \quad \text{for all } t \in (t_1, t_0).$$

Similarly, we can consider the case when $\{\xi_1(t), \dots, \xi_{l'}(t)\}$ are disjoint for some $l' \le l$ on $(t_0, t_2]$. The lemma is proved. ∎

4.5. *Proof of Theorem* 4.2

In this subsection we show Theorem 4.2. Part (1) of Theorem 4.2 follows from (4.44) and (4.60). For part (2), the proof divides into the following steps.

Step 1. Without loss of generality, we can assume that $c(x,t) \le 0$. In fact, let $u(x,t)$ be a solution of (4.1). Then for any $k \in \mathbb{R}$ the function $\tilde{u}(x,t) = u(x,t)e^{kt}$ satisfies

$$\frac{\partial \tilde{u}}{\partial t} = \Delta \tilde{u} + (c+k)\tilde{u}.$$

Therefore, for any compact set K in $\Sigma \times [T_1, T_2]$ we can choose k such that the function $\tilde{c} := c + k$ is nonpositive on K. Thus, it suffices to show Theorem 4.2 for $c(x,t) \le 0$.

Step 2. Suppose that the curves $\{\xi_1(t), \dots, \xi_l(t)\}$ are disjoint on the interval $[T_1, T_2]$. Let $T_1 < t_1 < t_2 < T_2$. Lemma 4.9 implies that u is in L^1. For any $(x,t) \in \Sigma \times (t_1, t_2)$, we define

$$\begin{aligned} w_k(x,t) &= \int_{t_1}^{t} ds \int_{\Sigma} p(x,y,t-s)\tilde{g}_k(y,s)\, d\,\mathrm{vol}_y, \\ \tilde{g}_k(x,t) &= c(x,t)u(x,t)\chi^{(k)}_{Q_{\frac{1}{2},t_1-\delta_0,t_2}}, \\ U_k(x,t) &= \int_{(t_1,t)} p(x,\xi_k(s),t-s)\, d\mu_k(s), \end{aligned} \tag{4.99}$$

where μ_k is the measure obtained in Lemma 4.15, and $\delta_0 > 0$ is a constant chosen such that $t_1 - \delta_0 > T_1$. Then $u - \sum_{k=1}^{l}(U_k + w_k)$ satisfies the heat equation in $\mathcal{D}'(Q_{\frac{1}{2},t_1,t_2})$, which implies that $u - \sum_{k=1}^{l}(U_k + w_k)$ is bounded in $Q_{\frac{1}{2},t_1,t_2}$. Since $c(x,t) \le 0$, we have $w_k(x,t) \le 0$ and

$$u(x,t) \le \sum_{k=1}^{l} U_k(x,t) + f(x,t),$$

where $f(x,t)$ is a bounded function on $Q_{\frac{1}{2},t_1,t_2}$. Therefore, by Lemma 4.6 u satisfies inequalities (4.3)–(4.4).

Step 3. In general, the singular curves may not be disjoint. In this case, we assume that the curves $\{\xi_1(t), \dots, \xi_l(t)\}$ are around (x_0, t_0) on (t_1, t_2). Consider the interval (t_1, t_0). By Definition 4.11, we can find an integer $l' \in [1, l]$ such that $\{\xi_1(t), \dots, \xi_{l'}(t)\}$ are disjoint on (t_1, t_0). By Lemma 4.15 we get positive measures $\mu_k \in (C_0((t_1, t_0)))'$ for each ξ_k with $k \in [1, l']$, and by Lemma 4.16 we have

$$0 \le \gamma^4 \mu_k(t) \le \mu(t) \quad \text{for all } t \in (t_1, t_0). \tag{4.100}$$

For each k, we define U_k as in (4.99). Using the same argument as in (2), for any $t \in (t_1, t_0)$ we have

$$u(x,t) \le \sum_{k=1}^{l} U_k(x,t) + f(x,t) \quad \text{for all } t \in (t_1, t_0), \tag{4.101}$$

where $f(x,t)$ is a bounded function. By (4.100), we have

$$u(x,t) \le \frac{1}{\gamma^4} \sum_{k=1}^{l'} \int_{t_1}^{t} p(x,\xi_k(s),t-s)\, d\mu + f(x,t) \quad \text{for all } t \in (t_1, t_0). \tag{4.102}$$

Similarly, we can prove that (4.102) also holds for $t \in (t_0, t_2)$. Therefore, by Lemma 4.6, u satisfies inequalities (4.3)–(4.4). The theorem is proved.

5. Proof of main theorems

In this section, we prove Theorem 1.2 and Corollary 1.4.

Proof of Theorem 1.2. Suppose that the mean curvature flow (1.1) reaches a singularity at (x_0, T) with $T < +\infty$. Then [25, Corollary 3.6] implies that for all $t < T$ we have

$$d(\Sigma_t, x_0) \le 2\sqrt{T-t}.$$

We rescale the flow by

$$s = -\log(T-t), \quad \tilde{\Sigma}_s = e^{\frac{s}{2}}(\Sigma_{T-e^{-s}} - x_0) \tag{5.1}$$

such that the flow $\{(\tilde{\Sigma}_s, \tilde{\mathbf{x}}(p,s)), -\log T \le s < +\infty\}$ satisfies the following properties:

(1) $\tilde{\mathbf{x}}(p,s)$ satisfies the equation

$$\left(\frac{\partial \tilde{\mathbf{x}}}{\partial s}\right)^{\perp} = -\left(\tilde{H} - \frac{1}{2}\langle \tilde{\mathbf{x}}, \mathbf{n}\rangle\right)\mathbf{n},$$

(2) the mean curvature of $\tilde{\Sigma}_s$ satisfies $|\tilde{H}(p,s)| \le \Lambda_0$ for some $\Lambda_0 > 0$,

(3) $d(\tilde{\Sigma}_s, 0) \le 2$.

Fix $\tau > 0$. By Theorem 3.1, for any sequence $s_i \to +\infty$ there exists a subsequence, still denoted by $\{s_i\}$, such that the flow $\{\tilde{\Sigma}_{s_i+s}, -\tau < s < \tau\}$ converges smoothly to a self-shrinker with multiplicity one. In other words, taking

$$c_j = e^{\frac{s_j}{2}},$$

the flow $\{\tilde{\Sigma}^j_s, -\tau < s < \tau\}$, where

$$\tilde{\Sigma}^j_s := c_j e^{\frac{s}{2}}(\Sigma_{T-c_j^{-2}e^{-s}} - x_0),$$

converges smoothly to a self-shrinker with multiplicity one as $j \to +\infty$. Consider the corresponding flow

$$\tilde{t} = -e^{-s}, \quad \Sigma^j_{\tilde{t}} := \sqrt{-\tilde{t}}\,\tilde{\Sigma}^j_{-\log(-\tilde{t})} = c_j(\Sigma_{T+c_j^{-2}\tilde{t}} - x_0).$$

Thus, for fixed $\tau > 0$ the flow $\{\Sigma^j_{\tilde{t}}, -e^{\tau} < \tilde{t} < -e^{-\tau}\}$ converges smoothly to a smooth self-shrinker flow with multiplicity one as $j \to +\infty$. Theorem 1.2 is proved. ∎

Proof of Corollary 1.4. We follow the argument in the proof of Theorem 1.2. Suppose that

$$\delta_0 := \sup_{\Sigma\times[0,T)} \left(\sqrt{T-t}\cdot|H|(p,t)\right) < +\infty.$$

Then the rescaled mean curvature flow (5.1) satisfies $|\tilde{H}| \le \delta_0$. There exists a sequence of times $s_i \to +\infty$ such that for any fixed $\tau > 0$ the flow $\{\tilde{\Sigma}_{s_i+s}, -\tau < s < \tau\}$ converges smoothly to a self-shrinker $\Sigma_\infty \in \mathcal{C}(N,\rho)$ with multiplicity one. Moreover, the mean curvature of the limit self-shrinker satisfies $\sup_{\Sigma_\infty}|H| \le \delta_0$. On the other hand, we have:

Lemma 5.1. *For any $N > 0$ and any increasing function ρ, there exists a positive constant $\delta(N,\rho)$ such that any self-shrinker $\Sigma \in \mathcal{C}(N,\rho)$ with $|H| \le \delta$ must be a plane passing through the origin.*

Proof. For otherwise, there exists a sequence of non-flat self-shrinkers $\Sigma_i \in \mathcal{C}(N,\rho)$ with $\sup_{\Sigma_i} |H| \le \delta_i \to 0$. By the smooth compactness result of self-shrinkers in [20], we can assume that Σ_i converges smoothly to a self-shrinker $\Sigma_\infty \in \mathcal{C}(N,\rho)$ with multiplicity one. Since the convergence is smooth, the limit self-shrinker Σ_∞ has zero mean curvature and by [19, Corollary 2.8] it must be a plane passing through the origin.

Let $\Sigma_{i,t} = \sqrt{1-t}\,\Sigma_i$. Then $\{\Sigma_{i,t}, 0 \le t < 1\}$ is a solution of mean curvature flow (1.1) which reaches $x_0 = 0$ at $T = 1$. Consider the Heat kernel function

$$\Phi_{(x_0,T)}(x,t) = \frac{1}{4\pi(T-t)} e^{-\frac{|x-x_0|^2}{4(T-t)}} \quad \text{for all } (x,t) \in \Sigma_{i,t} \times [0,T).$$

Thus, Huisken's monotonicity formula (cf. [36, Theorem 3.1]) implies that

$$\begin{aligned}\Theta(\Sigma_{i,t},0,1) &:= \lim_{t\to 1} \int_{\Sigma_{i,t}} \Phi_{(0,1)}(x,t)\, d\mu_{i,t} \\ &= \frac{1}{4\pi}\int_{\Sigma_i} e^{-\frac{|x|^2}{4}}\, d\mu_i \to \frac{1}{4\pi}\int_{\Sigma_\infty} e^{-\frac{|x|^2}{4}}\, d\mu_\infty = 1,\end{aligned}$$

where we used the fact Σ_i converges smoothly to the plane Σ_∞ with multiplicity one. Therefore, by [60, Theorem 3.5] or [25, Theorem 5.6] we have

$$|A_{\Sigma_{i,t}}|(x,t) \le \frac{C}{r_0} \tag{5.2}$$

for some $C, r_0 > 0$ and for all $(x,t) \in (\Sigma_{i,t} \cap B_{r_0}(0)) \times (1-r_0^2, 1)$. For any $p \in \Sigma_i$, there exists $t_p \in (1-r_0^2, 1)$ such that for all $t \in (t_p, 1)$ we have

$$\sqrt{1-t}\,p \in \Sigma_{i,t} \cap B_{r_0}(0).$$

Thus, (5.2) implies that for any $t \in (t_p, 1)$,

$$|A_{\Sigma_i}|(p) = \sqrt{1-t}\,|A_{\Sigma_{i,t}}|(\sqrt{1-t}\,p, t) \le \frac{C}{r_0}\sqrt{1-t}.$$

Letting $t \to 1$, we have $|A_{\Sigma_i}|(p) = 0$ which contradicts our assumption that Σ_i is non-flat.

Alternatively, one can also quote the results of C. Bao (cf. [4, Theorem 1.2]) or Guang and Zhu (cf. [32]) to obtain that each Σ_i is a plane and derive the same contradiction. The lemma is proved. ■

Therefore, by Lemma 5.1 the limit self-shrinker Σ_∞ must be a plane passing through the origin. Thus, Huisken's monotonicity formula implies that

$$\begin{aligned}\Theta(\Sigma_t, x_0, T) &:= \lim_{t\to T}\int_{\Sigma_t} \Phi_{(x_0,T)}(x,t)\, d\mu_t \\ &= \lim_{s_i\to+\infty} \frac{1}{4\pi}\int_{\tilde{\Sigma}_{s_i}} e^{-\frac{|x|^2}{4}}\, d\tilde{\mu}_{s_i} = 1,\end{aligned}$$

which implies that (x_0, T) is a regular point by [60, Theorem 3.1]. It follows that the flow $\{\Sigma_t, 0 \le t < T\}$ cannot blow up at (x_0, T). The corollary is proved. ■

Appendix A. Krylov–Safonov's parabolic Harnack inequality

In this appendix, we include the parabolic Harnack inequality from Krylov and Safonov [43]. First, we introduce some notations. Let $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$. Denote

$$|x| = \left(\sum_{i=1}^n (x^i)^2\right)^{\frac{1}{2}}, \quad B_R(x) = \{y \in \mathbb{R}^n \mid |x-y| < R\},$$
$$Q(\theta, R) = B_R(0) \times (0, \theta R^2).$$

Consider the parabolic operator

$$Lu = -\frac{\partial u}{\partial t} + a^{ij}(x,t)u_{ij} + b^i(x,t)u_i - c(x,t)u, \tag{A.1}$$

where the coefficients are measurable and satisfy the conditions

$$\mu|\xi|^2 \le a^{ij}(x,t)\xi_i\xi_j \le \frac{1}{\mu}|\xi|^2, \tag{A.2}$$
$$|b(x,t)| \le \frac{1}{\mu}, \tag{A.3}$$
$$0 \le c(x,t) \le \frac{1}{\mu}. \tag{A.4}$$

Here $b(x,t) = (b^1(x,t), \dots, b^n(x,t))$. Then we have

Theorem A.1 ([43, Theorem 1.1]). *Suppose the operator L in (A.1) satisfies conditions (A.2)–(A.4). Let $\theta > 1$, $R \le 2$, $u \in W^{1,2}_{n+1}(Q(\theta, R))$, $u \ge 0$ in $Q(\theta, R)$, and $Lu = 0$ on $Q(\theta, R)$. Then there exists a constant C, depending only on θ, μ and n, such that*

$$u(0, R^2) \le C\, u(x, \theta R^2) \quad \textit{for all } x \in B_{\frac{R}{2}}(0). \tag{A.5}$$

Moreover, when $\frac{1}{\theta-1}$ and $\frac{1}{\mu}$ vary within finite bounds, C also varies within finite bounds.

Note that in our case equation (3.28) does not satisfy the assumption that $c(x,t) \ge 0$ in (A.4). Therefore, we cannot use Theorem A.1 directly. The following result shows that the Harnack inequality still works when $c(x,t)$ is bounded.

Theorem A.2. *Let $\theta > 1$, $R \le 2$. Suppose that $u(x,t) \in W^{1,2}_{n+1}(Q(\theta, R))$ is a nonnegative solution to the equation*

$$Lu = -\frac{\partial u}{\partial t} + a^{ij}(x,t)u_{ij} + b^i(x,t)u_i + c(x,t)u = 0, \tag{A.6}$$

where the coefficients $a^{ij}(x,t)$ and $b^i(x,t)$ satisfy (A.2)–(A.3), and $c(x,t)$ satisfies

$$|c(x,t)| \le \frac{1}{\mu} \quad \textit{for all } (x,t) \in Q(\theta, R). \tag{A.7}$$

Then there exists a constant C, depending only on θ, μ and n, such that

$$u(0, R^2) \le C\, u(x, \theta R^2) \quad \textit{for all } |x| < \frac{1}{2}R.$$

Proof. Since $u(x,t)$ is a solution of (A.6) and $c(x,t)$ satisfies (A.7), the function

$$v(x,t) = e^{-\frac{1}{\mu}t}u$$

satisfies

$$-\frac{\partial v}{\partial t} + a^{ij}(x,t)v_{ij} + b^i(x,t)v_i + \tilde{c}(x,t) = 0. \tag{A.8}$$

where

$$-\frac{2}{\mu} \le \tilde{c}(x,t) = c(x,t) - \frac{1}{\mu} \le 0.$$

Applying Theorem A.1 to equation (A.8), we have

$$v(0,R^2) \le C\, v(x,\theta R^2) \quad \text{for all } |x| < \frac{1}{2}R,$$

where C depends only on θ, μ and n. Thus, for any $x \in B_{\frac{R}{2}}(0)$ we have

$$u(0,R^2) \le Ce^{-k(\theta-1)R^2}u(x,\theta R^2) \le C'u(x,\theta R^2),$$

where C' depends only on θ, μ and n. Here we used $R \le 2$ by the assumption. The theorem is proved. ∎

We generalize Theorem A.2 to a general bounded domain in $\mathbb{R}^n$.

Theorem A.3. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $u(x,t) \in W^{1,2}_{n+1}(\Omega \times (0,T))$ is a nonnegative solution to the equation*

$$Lu = -\frac{\partial u}{\partial t} + a^{ij}(x,t)u_{ij} + b^i(x,t)u_i + c(x,t)u = 0,$$

where the coefficients $a^{ij}(x,t)$ and $b^i(x,t)$ satisfy (A.2)–(A.3), and $c(x,t)$ satisfies (A.7) for a constant $\mu > 0$. For any s,t satisfying $0 < s < t < T$ and any $x,y \in \Omega$ with the following properties:

(1) *x and y can be connected by a line segment γ with the length $|x-y| \le l$,*

(2) *Each point in γ has a positive distance at least $\delta > 0$ from the boundary of Ω,*

(3) *s and t satisfy $T_1 \le t-s \le T_2$ for some $T_1, T_2 > 0$,*

we have

$$u(y,s) \le C\, u(x,t),$$

where C depends only on $n, \mu, \min\{s,\delta^2\}, l, T_1$ and T_2.

Proof. Let γ be the line segment with properties (1) and (2) connecting x and y. We set

$$p_0 = y, \quad p_N = x, \quad p_i = p_0 + \frac{x-y}{N}i \in \gamma$$

for any $0 \le i \le N$. Here we choose N to be the smallest integer satisfying

$$N > \max\left\{\frac{2(t-s)}{s}, \frac{l}{\min\{\frac{\sqrt{s}}{4}, \frac{\delta}{4}\}}\right\}. \tag{A.9}$$

We define

$$R = \frac{2l}{N}, \quad \theta = 1 + \frac{t-s}{R^2 N}. \tag{A.10}$$

We can check that $R \le \frac{\delta}{2}$. For any $s, t \in (0, T)$, choose $\{t_i\}_{i=0}^N$ such that $t_0 = s, t_N = t$ and

$$t_i - t_{i-1} = \frac{t-s}{N} \tag{A.11}$$

for all integers $1 \le i \le N$. Note that (A.9)–(A.11) imply that for any $0 \le i \le N-1$,

$$t_{i+1} - \theta R^2 \ge s - \theta R^2 = s - R^2 - \frac{t-s}{N} \ge \frac{s}{4} > 0$$

and

$$|p_{i+1} - p_i| = \frac{|x-y|}{N} \le \frac{l}{N} = \frac{R}{2}.$$

Therefore, for any $0 \le i \le N-1$ we have

$$(t_{i+1} - \theta R^2, t_{i+1}) \subset (0, T) \quad \text{and} \quad p_{i+1} \in B_{\frac{R}{2}}(p_i).$$

Applying Theorem A.2 on $B_R(p_i) \times (t_{i+1} - \theta R^2, t_{i+1}) \subset \Omega \times (0, T)$, we have

$$u(p_i, t_i) \le C\, u(p_{i+1}, t_{i+1}),$$

where C depends only on c, n, μ and $\frac{1}{\theta - 1} = \frac{R^2 N}{t-s}$. Here we used the fact that

$$t_i = (t_{i+1} - \theta R^2) + R^2.$$

Therefore,

$$u(y, s) = u(p_0, t_0) \le C^N u(p_N, t_N) = C' u(x, t), \tag{A.12}$$

where the constant C' in (A.12) depends only on $c, n, \mu, \min\{s, \delta^2\}, l, T_1$ and T_2. The theorem is proved. ∎

A direct corollary of Theorem A.3 is the following result.

Theorem A.4. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $u(x, t) \in W^{1,2}_{n+1}(\Omega \times (0, T))$ is a nonnegative solution to the equation*

$$Lu = -\frac{\partial u}{\partial t} + a^{ij}(x, t) u_{ij} + b^i(x, t) u_i + c(x, t) u = 0,$$

where the coefficients $a^{ij}(x, t)$ and $b^i(x, t)$ satisfy (A.2)–(A.3), and $c(x, t)$ satisfies (A.7) for a constant $\mu > 0$. Suppose that Ω', Ω'' are subdomains in Ω satisfying the following properties:

(1) *$\Omega' \subset \Omega'' \subset \Omega$, and Ω'' has a positive distance $\delta > 0$ from the boundary of Ω,*

(2) *Ω' can be covered by k balls with radius r, and all balls are contained in Ω''.*

Then for any s, t satisfying $0 < s < t < T$ and any $x, y \in \Omega'$, we have

$$u(y, s) \le C\, u(x, t), \tag{A.13}$$

where C depends only on $n, \mu, \min\{s, \delta^2\}, t - s, r$ and k.

Proof. By the assumption, we can find finite many points $\mathcal{A} = \{q_1, q_2, \dots, q_k\}$ such that

$$\Omega' \subset \bigcup_{q \in \mathcal{A}} B_r(q) \subset \Omega''. \tag{A.14}$$

For any $x, y \in \Omega'$, there exists two points in $\mathcal{A}$, which we denote by q_1 and q_2, such that $x \in B_r(q_1)$ and $y \in B_r(q_2)$. Then x and y can be connected by a polygonal chain γ, which consists of two line segments $\overline{xq_1}, \overline{yq_2}$ and a polygonal chain with vertices in $\mathcal{A}$ connecting q_1 and q_2. Clearly, the number of the vertices of γ is bounded by $k+2$ and the total length of γ is bounded by $(k+2)r$. Moreover, by assumption we have $\gamma \subset \Omega''$ and each point in γ has a positive distance at least $\delta > 0$ from the boundary of Ω.

Assume that the polygonal chain γ has consecutive vertices $\{p_0, p_1, \dots, p_N\}$ with

$$p_0 = y, \quad p_N = x \quad \text{and} \quad 1 \le N \le k+2.$$

We apply Theorem A.3 for each line segment $\overline{p_i p_{i+1}}$ and the interval $[t_i, t_{i+1}]$, where $\{t_i\}$ is chosen as in (A.11). Note that

$$\frac{t-s}{k+2} \le t_{i+1} - t_i = \frac{t-s}{N} \le t - s.$$

Thus, for any $0 \le i \le N-1$ we have

$$u(p_i, t_i) \le C u(p_{i+1}, t_{i+1}), \tag{A.15}$$

where C depends only on $c, n, \mu, \min\{s, \delta^2\}, r, k$ and $t-s$, and (A.15) implies (A.13). This finishes the proof of Theorem A.4. ■

Theorem A.4 can be generalized to Riemannian manifolds by using the partition of unity. Here we omit the proof since the argument is standard. Note that the constant in (A.13) depends on the geometry of (M, g).

Theorem A.5. *Let (M, g) be a Riemannian manifold with boundary ∂M and $\Omega \subset M$ a bounded domain which does not intersect with ∂M. Suppose $u(x,t) \in W^{1,2}_{n+1}(\Omega \times (0,T))$ is a nonnegative solution to the equation*

$$Lu = -\frac{\partial u}{\partial t} + a^{ij}(x,t)\nabla_i\nabla_j u + b^i(x,t)\nabla_i u + c(x,t)u = 0,$$

where the coefficients $a^{ij}(x,t)$ and $b^i(x,t)$ satisfy (A.2)–(A.3), and $c(x,t)$ satisfies (A.7) for a constant $\mu > 0$. Suppose that Ω', Ω'' are subdomains in Ω satisfying the following properties:

(1) *$\Omega' \subset \Omega'' \subset \Omega$, and Ω'' has a positive distance $\delta > 0$ from the boundary of Ω,*

(2) *Ω' can be covered by k balls with radius r, and all balls are contained in Ω''.*

Then for any s, t satisfying $0 < s < t < T$ and any $x, y \in \Omega'$, we have

$$u(y, s) \le C\, u(x, t),$$

where C depends only on $c, n, \mu, \min\{s, \delta^2\}, t-s, r, k$ and (M, g).

Appendix B. Li–Yau's parabolic Harnack inequality

In this appendix, we include Li–Yau's parabolic Harnack inequality in [49]. Compared with the Harnack inequality in Appendix A, Li–Yau's result gives explicit dependence of the constants on the geometric quantities of the metric. Thus, we can apply Li–Yau's result to a class of Riemannian manifolds and we obtain uniform bounds of the constants in the Harnack inequality.

Theorem B.1 (cf. [49, Theorem 2.1]). *Let M be a Riemannian manifold with boundary ∂M. Assume $p \in M$ and let $\mathcal{B}_{2R}(p)$ be a geodesic ball of radius $2R$ centered at p which does not intersect ∂M. We denote $-K(2R)$, with $K(2R) \geq 0$, to be a lower bound of the Ricci curvature on $\mathcal{B}_{2R}(p)$. Let $q(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in the x-variable and C^1 in the t-variable. Assume that*

$$\Delta q \leq \theta(2R), \quad |\nabla q| \leq \gamma(2R)$$

on $\mathcal{B}_{2R}(p) \times [0,T]$ for some constants $\theta(2R)$ and $\gamma(2R)$. If $u(x,t)$ is a positive solution of the equation

$$\Big(\Delta - q - \frac{\partial}{\partial t}\Big)u(x,t) = 0$$

on $M \times (0,T]$, then for any $\alpha > 1$, $0 < t_1 < t_2 \leq T$, and $x, y \in \mathcal{B}_R(p)$, we have the inequality

$$u(x,t_1) \leq u(y,t_2)\Big(\frac{t_2}{t_1}\Big)^{\frac{n\alpha}{2}} e^{A(t_2-t_1)+\rho_{\alpha,R}(x,y,t_2-t_1)},$$

where

$$A = C\big(\alpha R^{-1}\sqrt{K} + \alpha^3(\alpha-1)^{-1}R^{-2} + \gamma^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}\alpha^{-\frac{1}{3}} + (\alpha\theta)^{\frac{1}{2}} + \alpha(\alpha-1)^{-1}K\big)$$

and

$$\rho_{\alpha,R}(x,y,t_2-t_1) = \inf_{\gamma\in\Gamma(R)}\bigg(\frac{\alpha}{4(t_2-t_1)}\int_0^1 |\dot{\gamma}|^2 + (t_2-t_1)\int_0^1 q(\gamma(s),(1-s)t_2+st_1)\,ds\bigg),$$

with inf *taken over all paths in $\mathcal{B}_R(p)$ parametrized by $[0,1]$ joining y to x.*

A direct corollary of Theorem B.1 is the following result.

Theorem B.2. *The same assumptions as in Theorem B.1 on M, $\mathcal{B}_{2R}(p)$ and the function $q(x,t)$. If $u(x,t)$ is a positive solution of the equation*

$$\Big(\Delta - q - \frac{\partial}{\partial t}\Big)u(x,t) = 0$$

on $\Omega \times (0,T]$, where Ω is a connected open subset of $\mathcal{B}_R(p)$. Let Ω', Ω'' be connected open subsets of Ω satisfying the following properties, which we called (δ,k,r) property:

(1) *$\Omega' \subset \Omega'' \subset \Omega$, and Ω'' has a positive distance $\delta > 0$ from the boundary of Ω,*

(2) *Ω' can be covered by k geodesic balls with radius r, and all balls are contained in Ω''.*

Then for any $0 < t_1 < t_2 \leq T$ *and* $x, y \in \Omega'$, *we have the inequality*

$$u(x, t_1) \leq C u(y, t_2), \tag{B.1}$$

where C *depends only on* $n, K(2R), \theta(2R), \gamma(2R), t_1, t_2 - t_1, k, \delta$ *and* r.

Proof. By the assumption on Ω', Ω'' and Ω, x and y can be connected by a path γ in Ω'' with bounded length and every point in γ has a distance at least δ from the boundary of Ω. Thus, the theorem follows directly from Theorem B.1 by choosing $R = \delta$ and $\alpha = 2$. ∎

In the proof of Lemma 3.21, we need to use Theorem B.2 to a class of surfaces with bounded geometry. In order to show that the constants in the Harnack inequality is uniformly bounded, we have the following result.

Theorem B.3. *Fix* $R > 0$. *We assume that:*

(1) $\Sigma_i^2 \subset \mathbb{R}^3$ *is a sequence of complete surfaces which converges smoothly to a complete surface* Σ *in* $\mathbb{R}^3$.
(2) *The Ricci curvature of* $\Sigma \cap B_R(0)$ *is bounded by a constant* $-K$ *with* $K \geq 0$. *Here* $B_R(0) \subset \mathbb{R}^3$ *denotes the extrinsic ball centered at* 0 *with radius* R.
(3) Ω_i, Ω_i', Ω_i'' *are bounded domains in* $\Sigma_i \cap B_{R/2}(0)$ *with* $\Omega_i' \subset \Omega_i'' \subset \Omega_i$, *and* Ω_i, Ω_i', Ω_i'' *converges smoothly to* Ω_i, Ω_i', Ω_i'' *with* $\Omega' \subset \Omega'' \subset \Omega \subset \Sigma \cap B_{R/2}(0)$, *respectively. Here the smooth convergence of* Ω_i *to* Ω *means that for any* $\epsilon > 0$ *and sufficiently large* i, *there exists a smooth function* u_i *on* Ω *with* $|u_i|_{C^2(\Omega)} \leq \epsilon$ *such that* Ω_i *can be written as a normal exponential graph of* u_i *over* Ω.
(4) Ω'' *has a positive geodesic distance* $\delta > 0$ *from the boundary of* Ω.
(5) Ω' *can be covered by* k *geodesic balls with radius* $r \in (0, \frac{\delta}{2})$, *and all balls are contained in* Ω''.
(6) $q_i(x, t)$ *is a function defined on* $\Sigma_i \times [0, T]$ *which is* C^2 *in the* x*-variable and* C^1 *in the* t*-variable. Assume that*

$$\Delta_{g_i} q_i \leq \theta, \quad |\nabla q_i|_{g_i} \leq \theta$$

on $\Omega_i \times [0, T]$ *for some constant* θ.

If $f_i(x, t)$ *are positive functions satisfying*

$$\left(\Delta_{g_i} - q_i(x, t) - \frac{\partial}{\partial t} \right) f_i(x, t) = 0 \tag{B.2}$$

on $\Omega_i \times (0, T]$, *where* $q_i(x, t) \in C^2(\Sigma_i \times [0, T])$, *then for any* $0 < t_1 < t_2 \leq T$ *and any points* $x, y \in \Omega_i'$, *we have the inequality*

$$f_i(x, t_1) \leq C f_i(y, t_2),$$

where C *depends only on* $n, K, \theta, t_1, t_2 - t_1, k, \delta$ *and* r.

Proof. It suffices to show that Ω_i', Ω_i'' and Ω_i satisfy the (δ', k', r') property of Theorem B.2 with uniform constants δ', k' and r'. By the smooth convergence of Ω_i to Ω, we define the map $\varphi_i : \Omega \to \Omega_i$ by

$$\varphi_i(x) = x + u_i(x)\mathbf{n}(x) \quad \text{for all } x \in \Omega, \tag{B.3}$$

where $u_i(x)$ is the graph function of Ω_i over Ω and $\mathbf{n}(x)$ denotes the normal vector of Σ at x. Note that $\varphi_i(\Omega) = \Omega_i$ and φ_i converges in C^2 to the identity map on Ω as $i \to +\infty$. By the assumption (5), there exists k points $\{p_\alpha\}_{\alpha=1}^k \subset \Omega'$ and $\epsilon > 0$ such that

$$\Omega' \subset \bigcup_{\alpha=1}^{k} \mathcal{B}_r(p_\alpha), \quad \mathcal{B}_r(p_\alpha) \subset \Omega''_{4\epsilon},$$

where $\Omega''_{4\epsilon} = \{x \in \Omega'' \mid d_\Sigma(x, \partial\Omega'') \geq 4\epsilon\}$. Therefore, we have

$$\Omega'_i = \varphi_i(\Omega') \subset \varphi_i\left(\bigcup_{\alpha=1}^{k} \mathcal{B}_r(p_\alpha)\right) = \bigcup_{\alpha=1}^{k} \varphi_i(\mathcal{B}_r(p_\alpha)) \tag{B.4}$$

Since the C^l norms of u_i in (B.3) are small, for large i we have

$$\varphi_i(\mathcal{B}_r(p_\alpha)) \subset \mathcal{B}_{i,r+\epsilon}(\varphi_i(p_\alpha)) \subset \Omega''_{i,2\epsilon}, \tag{B.5}$$

where $\Omega''_{i,2\epsilon} = \{x \in \Omega''_i \mid d_{\Sigma_i}(x, \partial\Omega''_i) \geq 2\epsilon\}$ and $\mathcal{B}_{i,r}(p)$ denotes the geodesic ball of Σ_i centered at p with radius r. Combining (B.4) with (B.5), we have

$$\Omega'_i \subset \bigcup_{\alpha=1}^{k} \mathcal{B}_{r+\epsilon}(\varphi_i(p_\alpha)) \subset \Omega''_{i,2\epsilon} \subset \Omega''_i.$$

Therefore, Ω'_i can be covered by k geodesic balls with radius $r + \epsilon$, and all balls are contained in Ω''_i. It is clear that Ω''_i has a positive geodesic distance $\frac{\delta}{2} > 0$ from the boundary of Ω_i for large i. Thus, Ω'_i, Ω''_i and Ω_i satisfy the $(\frac{\delta}{2}, k, r + \epsilon)$ property and the theorem follows directly from Theorem B.2. ■

Appendix C. The linearized equation of rescaled mean curvature flow

In this appendix, we follow the calculation in [21, Appendix A] to show (3.26). See also [23, Appendix A]. Let Σ be a hypersurface in $\mathbb{R}^{n+1}$ and Σ_u the graph of a function u over Σ. Then Σ_u is given by

$$\Sigma_u = \{x + u(x)\mathbf{n}(x) \mid x \in \Sigma\},$$

where $\mathbf{n}(x)$ denotes the normal vector of Σ at x. We assume that $|u|$ is small. Let e_{n+1} be the gradient of the signed distance function to Σ and e_{n+1} equals $\mathbf{n}$ on Σ. We define

$$\nu_u(p) = \sqrt{\frac{\det g^u_{ij}(p)}{\det g_{ij}(p)}}, \quad w_i(p) = \langle e_{n+1}, \mathbf{n}_u\rangle, \quad \eta_u(p) = \langle p + u(p)\mathbf{n}(p), \mathbf{n}_u\rangle,$$

where g_{ij} denotes the metric on Σ at p, g^u_{ij} is the induced metric on Σ_u and $\mathbf{n}_u$ is the normal to Σ_u.

Lemma C.1 ([21, Lemma A.3]). *There exist three functions w, ν and η depending on $(p,s,y)\in\Sigma\times\mathbb{R}\times T_p\Sigma$ that are smooth for $|s|$ less than the normal injectivity radius of Σ so that*

$$w_u(p)=w(p,s,y)=\sqrt{1+|B^{-1}(p,s)(y)|^2}, \tag{C.1}$$

$$\nu_u(p)=\nu(p,s,y)=w(p,s,y)\det(B(p,s)), \tag{C.2}$$

$$\eta_u(p)=\eta(p,s,y)=\frac{\langle p,\mathbf{n}(p)\rangle+s-\langle p,B^{-1}(p,s)(y)\rangle}{w(p,s,y)}, \tag{C.3}$$

where the linear operator $B(p,s)=\mathrm{Id}-sA(p)$. Finally, we have:

(1) *w satisfies*

$$w(p,s,0)=1,\qquad \partial_s w(p,s,0)=0, \tag{C.4}$$

$$\partial_{y_\alpha}w(p,s,0)=0,\quad \partial_{y_\alpha}\partial_{y_\beta}w(p,0,0)=\delta_{\alpha\beta}. \tag{C.5}$$

(2) *ν satisfies*

$$\nu(p,0,0)=1,\qquad \partial_s\nu(p,0,0)=H(p),$$
$$\partial_{p_j}\partial_s\nu(p,0,0)=H_j(p),\qquad \partial_{y_\alpha}\partial_{y_\beta}\nu(p,0,0)=\delta_{\alpha\beta},$$
$$\partial_s^2\nu(p,0,0)=H^2(p)-|A|^2(p).$$

(3) *η satisfies*

$$\eta(p,0,0)=\langle p,\mathbf{n}\rangle,\quad \partial_s\eta(p,0,0)=1,$$
$$\partial_{y_\alpha}\eta(p,0,0)=-p_\alpha.$$

(4) *Furthermore, we have*

$$\partial_{y_i}\nu(p,0,0)=0,\qquad \partial_{p_j}\partial_{y_i}\nu(p,0,0)=0, \tag{C.6}$$

$$\partial_s\partial_{p_j}\partial_{y_i}\nu(p,0,0)=0,\quad \partial_{y_k}\partial_{p_j}\partial_{y_i}\nu(p,0,0)=0. \tag{C.7}$$

Proof. Parts (1)–(3) and (C.1)–(C.3) follow directly from [21, Lemma A.3]. It suffices to show part (4). Following the notations in the proof of [21, Lemma A.3], we assume that (p,s) is the Fermi coordinates on the normal tubular neighborhood of Σ so that s measures the signed distance to Σ. We define

$$B(p,s)\equiv(\mathrm{Id}-sA(p)):T_p\Sigma\to T_p\Sigma.$$

Let $\mathcal{B}(p,s)=\det(B(p,s))$ and $J(p,s)=B^{-1}(p,s)$. Then we have

$$\mathcal{B}(p,0)=1,\qquad \partial_s\mathcal{B}(p,0)=H(p), \tag{C.8}$$

$$\partial_{y_i}\mathcal{B}(p,s)\equiv 0,\qquad \partial_{p_j}B(p,0)=-s\partial_{p_j}A|_{s=0}=0, \tag{C.9}$$

$$\partial_{p_j}\mathcal{B}(p,0)=\mathcal{B}(p,0)\cdot\mathrm{tr}(\partial_{p_j}B(p,0))=0. \tag{C.10}$$

Since $J=B^{-1}$, we have

$$\partial_{p_j}JB+J\partial_{p_j}B=0.$$

This implies that

$$\partial_{p_j}J(p,0)=-J(p,0)\cdot\partial_{p_j}B(p,0)\cdot J(p,0)=0. \tag{C.11}$$

Note that by (C.1), w can be rewritten as

$$w(p,s,y) = \sqrt{1 + J_{\alpha\beta} J_{\alpha\gamma} y_\beta y_\gamma}.$$

It follows immediately that

$$\begin{aligned}
\partial_{y_i} w &= \frac{1}{2w} J * J * y, \\
\partial_{p_i} w &= \frac{1}{w} \partial_{p_i} J * J * y * y, \\
\partial_s \partial_{y_i} w &= -\frac{1}{2w^2} \partial_s w \cdot J * J * y + \frac{1}{w} \partial_s J * J * y, \\
\partial_{p_j} \partial_{y_i} w &= -\frac{1}{2w^2} \partial_{p_j} w \cdot J * J * y + \frac{1}{w} \partial_{p_j} J * J * y,
\end{aligned}$$

where the notation "*" denotes the multiplication of two matrices. Furthermore, we calculate

$$\begin{aligned}
\partial_s \partial_{p_j} \partial_{y_i} w = {} & w^{-3} \partial_s w \partial_{p_j} w \cdot J * J * y - \frac{1}{2w^2} \partial_s \partial_{p_j} w \cdot J * J * y \\
& - \frac{1}{w^2} \partial_{p_j} w \cdot \partial_s J * J * y - \frac{1}{w^2} \partial_s w \partial_{p_j} J * J * y \\
& + \frac{1}{w} \partial_s \partial_{p_j} J * J * y + \frac{1}{w} \partial_{p_j} J * \partial_s J * y, \\
\partial_{y_k} \partial_{p_j} \partial_{y_i} w = {} & w^{-3} \partial_{y_k} w \partial_{p_j} w \cdot J * J * y - \frac{1}{2w^2} \partial_{y_k} \partial_{p_j} w \cdot J * J * y \\
& - \frac{1}{w^2} \partial_{p_j} w \cdot \partial_{y_k} J * J * y - \frac{1}{2w^2} \partial_{p_j} w \cdot J * J \\
& - \frac{1}{w^2} \partial_{y_k} w \partial_{p_j} J * J * y + \frac{1}{w} \partial_{y_k} \partial_{p_j} J * J * y \\
& + \frac{1}{w} \partial_{p_j} J * \partial_{y_k} J * y + \frac{1}{w} \partial_{p_j} J * J.
\end{aligned}$$

Combining the above identities with (C.4), (C.5) and (C.11), we have

$$\partial_{y_i} w(p,0,0) = 0, \qquad \partial_{p_i} w(p,0,0) = 0, \tag{C.12}$$

$$\partial_s \partial_{y_i} w(p,0,0) = 0, \qquad \partial_{p_j} \partial_{y_i} w(p,0,0) = 0, \tag{C.13}$$

$$\partial_s \partial_{p_j} \partial_{y_i} w(p,0,0) = 0, \quad \partial_{y_k} \partial_{p_j} \partial_{y_i} w(p,0,0) = 0. \tag{C.14}$$

Moreover, we calculate the derivatives of the function $\nu(p,s,y) = w(p,s,y)\mathcal{B}(p,s)$:

$$\begin{aligned}
\partial_{y_i} \nu = {} & \partial_{y_i} w \mathcal{B} + w \partial_{y_i} \mathcal{B}, \\
\partial_{p_j} \partial_{y_i} \nu = {} & \partial_{p_j} \partial_{y_i} w \mathcal{B} + \partial_{y_i} w \partial_{p_j} \mathcal{B} + \partial_{p_j} w \partial_{y_i} \mathcal{B} + w \partial_{p_j} \partial_{y_i} \mathcal{B}, \\
\partial_s \partial_{p_j} \partial_{y_i} \nu = {} & \partial_s \partial_{p_j} \partial_{y_i} w \mathcal{B} + \partial_{p_j} \partial_{y_i} w \partial_s \mathcal{B} + \partial_s \partial_{y_i} w \partial_{p_j} \mathcal{B} + \partial_{y_i} w \partial_s \partial_{p_j} \mathcal{B} \\
& + \partial_s \partial_{p_j} w \partial_{y_i} \mathcal{B} + \partial_{p_j} w \partial_{y_i} \partial_s \mathcal{B} + \partial_s w \partial_{p_j} \partial_{y_i} \mathcal{B} + w \partial_s \partial_{p_j} \partial_{y_i} \mathcal{B}, \\
\partial_{y_k} \partial_{p_j} \partial_{y_i} \nu = {} & \partial_{y_k} \partial_{p_j} \partial_{y_i} w \mathcal{B} + \partial_{p_j} \partial_{y_i} w \partial_{y_k} \mathcal{B} + \partial_{y_k} \partial_{y_i} w \partial_{p_j} \mathcal{B} + \partial_{y_i} w \partial_{y_k} \partial_{p_j} \mathcal{B} \\
& + \partial_{y_k} \partial_{p_j} w \partial_{y_i} \mathcal{B} + \partial_{p_j} w \partial_{y_i} \partial_{y_k} \mathcal{B} + \partial_{y_k} w \partial_{p_j} \partial_{y_i} \mathcal{B} + w \partial_{y_k} \partial_{p_j} \partial_{y_i} \mathcal{B}.
\end{aligned}$$

Combining this with (C.9)–(C.10), (C.12)–(C.14), we have (C.6)–(C.7). The lemma is proved. ■

We have the following expression for the mean curvature of Σ_u.

Lemma C.2 ([21, Corollary A.30]). *The mean curvature H_u of Σ_u is given by*

$$H_u(p) = \frac{w}{v}\big(\partial_s \nu - \partial_{p_\alpha}\partial_{y_\alpha}\nu - (\partial_s\partial_{y_\alpha}\nu)u_\alpha(p) - (\partial_{y_\beta}\partial_{y_\alpha}\nu)u_{\alpha\beta}(p)\big), \tag{C.15}$$

where w, ν and their derivatives are all evaluated at $(p, u(p), \nabla u(p))$.

Combining Lemma C.1 with Lemma C.2, we can show (3.26).

Lemma C.3. *The function $u_i = u_i^+ - u_i^-$ satisfies the following parabolic equations on $\Omega_{\epsilon,R}(I)\times I$*

$$\frac{\partial u_i}{\partial t} = \Delta_0 u_i - \frac{1}{2}\langle x, \nabla u_i\rangle + |A|^2 u_i + \frac{u_i}{2} + a_i^{pq}u_{i,pq} + b_i^p u_{i,p} + c_i\, u_i, \tag{C.16}$$

where Δ_0 denotes the Laplacian operator on Σ_∞ with respect to the induced metric, and the coefficients a_i^{pq}, b_i^p and c_i are small and tend to zero as u_i^+ and u_i^- tend to zero.

Proof. We divide the proof into several steps.

Step 1. We calculate the difference of the mean curvature of $\Sigma_{u_i^+}$ and $\Sigma_{u_i^-}$. Let

$$u = u_i^+ - u_i^- \quad \text{and} \quad \tilde u_\tau = u_i^- + \tau u$$

for $\tau \in [0,1]$. Thus, we have $\tilde u_0 = u_i^-$ and $\tilde u_1 = u_i^+$. Note that

$$H_{u_i^+}(p) - H_{u_i^-}(p) = \int_0^1 \partial_\tau(H_{\tilde u_\tau}(p))\, d\tau. \tag{C.17}$$

For any function $f(p,s,y)$, we calculate the derivative with respect to τ

$$\partial_\tau(f(p,\tilde u_\tau, \nabla\tilde u_\tau)) = (\partial_s f)(p,\tilde u_\tau,\nabla\tilde u_\tau)\cdot u + (\partial_{y_\alpha} f)(p,\tilde u_\tau,\nabla\tilde u_\tau)\cdot u_\alpha, \tag{C.18}$$

where $u_\alpha = \partial_{x_\alpha} u$. Therefore, we have

$$\begin{aligned}
\partial_\tau(\partial_s\nu) &= \partial_s^2\nu\cdot u + \partial_{y_i}\partial_s\nu\cdot u_i,\\
\partial_\tau(\partial_{p_\alpha}\partial_{y_\alpha}\nu) &= \partial_s\partial_{p_\alpha}\partial_{y_\alpha}\nu\cdot u + \partial_{y_i}\partial_{p_\alpha}\partial_{y_\alpha}\nu\cdot u_i,\\
\partial_\tau\big((\partial_s\partial_{y_\alpha}\nu)\tilde u_{\tau,\alpha}\big) &= (\partial_s\partial_{y_\alpha}\nu)u_\alpha + (\partial_s^2\partial_{y_\alpha}\nu)u\tilde u_{\tau,\alpha} + (\partial_s\partial_{y_i}\partial_{y_\alpha}\nu)u_i\tilde u_{\tau,\alpha}\\
\partial_\tau\big((\partial_{y_\beta}\partial_{y_\alpha}\nu)\tilde u_{\tau,\alpha\beta}\big) &= (\partial_{y_\beta}\partial_{y_\alpha}\nu)u_{\alpha\beta} + \partial_s\partial_{y_\beta}\partial_{y_\alpha}\nu\cdot u\tilde u_{\tau,\alpha\beta} + \partial_{y_i}\partial_{y_\beta}\partial_{y_\alpha}\nu\cdot u_i\tilde u_{\tau,\alpha\beta},
\end{aligned}$$

where $\tilde u_{\tau,\alpha} = \partial_{x_\alpha}\tilde u_\tau$ and $\tilde u_{\tau,\alpha\beta} = \partial_{x_\alpha}\partial_{x_\beta}\tilde u_\tau$. By Lemma C.2 we have

$$\begin{aligned}
\partial_\tau(H_{\tilde u_\tau}(p)) &= \left(\partial_s\left(\frac{w}{\nu}\right)u + \partial_{y_i}\left(\frac{w}{\nu}\right)u_i\right)\\
&\qquad\cdot\big(\partial_s\nu - \partial_{p_\alpha}\partial_{y_\alpha}\nu - (\partial_s\partial_{y_\alpha}\nu)\tilde u_{\tau,\alpha} - (\partial_{y_\beta}\partial_{y_\alpha}\nu)\tilde u_{\tau,\alpha\beta}\big)\\
&\quad + \left(\frac{w}{\nu}\right)\cdot\partial_\tau\big(\partial_s\nu - \partial_{p_\alpha}\partial_{y_\alpha}\nu - (\partial_s\partial_{y_\alpha}\nu)\tilde u_{\tau,\alpha} - (\partial_{y_\beta}\partial_{y_\alpha}\nu)\tilde u_{\tau,\alpha\beta}\big)\\
&= Eu + F_\alpha u_\alpha + G_{\alpha\beta}u_{\alpha\beta},
\end{aligned} \tag{C.19}$$

where E, F and G are given by

$$
\begin{aligned}
E(p,\tilde{u}_\tau) = {} & \partial_s\Big(\frac{w}{\nu}\Big)\big(\partial_s\nu - \partial_{p_\alpha}\partial_{y_\alpha}\nu - (\partial_s\partial_{y_\alpha}\nu)\tilde{u}_{\tau,\alpha} - (\partial_{y_\beta}\partial_{y_\alpha}\nu)\tilde{u}_{\tau,\alpha\beta}\big) \\
& + \Big(\frac{w}{\nu}\Big)\cdot\big(\partial_s^2\nu - \partial_s\partial_{p_\alpha}\partial_{y_\alpha}\nu - (\partial_s^2\partial_{y_\alpha}\nu)\tilde{u}_{\tau,\alpha} \\
& \qquad - \partial_s\partial_{y_\beta}\partial_{y_\alpha}\nu\cdot\tilde{u}_{\tau,\alpha\beta}\big),
\end{aligned}
\tag{C.20}
$$

$$
\begin{aligned}
F_\gamma(p,\tilde{u}_\tau) = {} & \partial_{y_\gamma}\Big(\frac{w}{\nu}\Big)\big(\partial_s\nu - \partial_{p_\alpha}\partial_{y_\alpha}\nu - (\partial_s\partial_{y_\alpha}\nu)\tilde{u}_{\tau,\alpha} - (\partial_{y_\beta}\partial_{y_\alpha}\nu)\tilde{u}_{\tau,\alpha\beta}\big) \\
& + \Big(\frac{w}{\nu}\Big)\big(\partial_{y_\gamma}\partial_s\nu - \partial_{y_\gamma}\partial_{p_\alpha}\partial_{y_\alpha}\nu - \partial_s\partial_{y_\gamma}\nu - (\partial_s\partial_{y_i}\partial_{y_\gamma}\nu)\tilde{u}_{\tau,i} \\
& \qquad - \partial_{y_\gamma}\partial_{y_\beta}\partial_{y_\alpha}\nu\cdot\tilde{u}_{\tau,\alpha\beta}\big),
\end{aligned}
\tag{C.21}
$$

$$
G_{\alpha\beta}(p,\tilde{u}_\tau) = -\Big(\frac{w}{\nu}\Big)\cdot\partial_{y_\beta}\partial_{y_\alpha}\nu. \tag{C.22}
$$

In view of (C.20)–(C.22), we define the functions depending on

$$
(p,s,y,Q)\in\Sigma\times\mathbb{R}\times T_p(\Sigma)\times \mathrm{GL}(2,\mathbb{R})
$$

such that

$$
\begin{aligned}
E(p,s,y,Q) = {} & \partial_s\Big(\frac{w}{\nu}\Big)(p,s,y)\big(\partial_s\nu - \partial_{p_\alpha}\partial_{y_\alpha}\nu - (\partial_s\partial_{y_\alpha}\nu)y_\alpha - (\partial_{y_\beta}\partial_{y_\alpha}\nu)Q_{\alpha\beta}\big) \\
& + \Big(\frac{w}{\nu}\Big)(p,s,y)\cdot\big(\partial_s^2\nu - \partial_s\partial_{p_\alpha}\partial_{y_\alpha}\nu - (\partial_s^2\partial_{y_\alpha}\nu)y_\alpha \\
& \qquad - \partial_s\partial_{y_\beta}\partial_{y_\alpha}\nu\cdot Q_{\alpha\beta}\big), \\
F_\gamma(p,s,y,Q) = {} & \partial_{y_\gamma}\Big(\frac{w}{\nu}\Big)(p,s,y)\cdot\big(\partial_s\nu - \partial_{p_\alpha}\partial_{y_\alpha}\nu - (\partial_s\partial_{y_\alpha}\nu)y_\alpha - (\partial_{y_\beta}\partial_{y_\alpha}\nu)Q_{\alpha\beta}\big) \\
& + \Big(\frac{w}{\nu}\Big)\big(\partial_{y_\gamma}\partial_s\nu - \partial_{y_\gamma}\partial_{p_\alpha}\partial_{y_\alpha}\nu - \partial_s\partial_{y_\gamma}\nu - (\partial_s\partial_{y_i}\partial_{y_\gamma}\nu)y_i \\
& \qquad - \partial_{y_\gamma}\partial_{y_\beta}\partial_{y_\alpha}\nu\cdot Q_{\alpha\beta}\big), \\
G_{\alpha\beta}(p,s,y) = {} & -\Big(\frac{w}{\nu}\Big)(p,s,y)\cdot\partial_{y_\beta}\partial_{y_\alpha}\nu.
\end{aligned}
$$

Let $\hat{u}_\lambda = \lambda\tilde{u}_\tau$ for $\lambda\in[0,1]$. Then we have

$$
E(p,u_\tau) = E(p,0) + \int_0^1 \partial_\lambda(E(p,\hat{u}_\lambda))\,d\lambda, \tag{C.23}
$$

$$
F_\gamma(p,u_\tau) = F_\gamma(p,0) + \int_0^1 \partial_\lambda(F_\gamma(p,\hat{u}_\lambda))\,d\lambda, \tag{C.24}
$$

$$
G_{\alpha\beta}(p,u_\tau) = G_{\alpha\beta}(p,0) + \int_0^1 \partial_\lambda(G_{\alpha\beta}(p,\hat{u}_\lambda))\,d\lambda. \tag{C.25}
$$

Note that

$$\partial_\lambda(E(p,\hat{u}_\lambda)) = (\partial_s E)\cdot\tilde{u}_\tau + (\partial_{y_i}E)\cdot\tilde{u}_{\tau,i} + (\partial_{Q_{\alpha\beta}}E)\cdot\tilde{u}_{\tau,\alpha\beta}, \tag{C.26}$$

$$\partial_\lambda(F_\gamma(p,\hat{u}_\lambda)) = (\partial_s F_\gamma)\cdot\tilde{u}_\tau + (\partial_{y_i}F_\gamma)\cdot\tilde{u}_{\tau,i} + (\partial_{Q_{\alpha\beta}}F_\gamma)\cdot\tilde{u}_{\tau,\alpha\beta}, \tag{C.27}$$

$$\partial_\lambda(G_{\alpha\beta}(p,\hat{u}_\lambda)) = (\partial_s G_{\alpha\beta})\cdot\tilde{u}_\tau + (\partial_{y_i}G_{\alpha\beta})\cdot\tilde{u}_{\tau,i}, \tag{C.28}$$

where the right-hand sides of (C.26)–(C.28) are evaluated at

$$(p,s,y,Q) = (p,\hat{u}_\lambda,\nabla\hat{u}_\lambda,\nabla^2\hat{u}_\lambda).$$

By Lemma C.1, we have

$$E(p,0) = -|A|^2, \tag{C.29}$$

$$F_\gamma(p,0) = 0, \tag{C.30}$$

$$G_{\alpha\beta}(p,0) = -\delta_{\alpha\beta}. \tag{C.31}$$

Combining (C.23) and (C.26) with (C.29), we have

$$\begin{aligned} E(p,u_\tau) = -|A|^2 &+ \tilde{u}_\tau\int_0^1 (\partial_s E)(p,\hat{u}_\tau,\nabla\hat{u}_\tau,\nabla^2\hat{u}_\tau)\,d\lambda \\ &+ \tilde{u}_{\tau,i}\int_0^1 (\partial_{y_i}E)(p,\hat{u}_\tau,\nabla\hat{u}_\tau,\nabla^2\hat{u}_\tau)\,d\lambda \\ &+ \tilde{u}_{\tau,\alpha\beta}\int_0^1 (\partial_{Q_{\alpha\beta}}E)(p,\hat{u}_\tau,\nabla\hat{u}_\tau,\nabla^2\hat{u}_\tau)\,d\lambda. \end{aligned} \tag{C.32}$$

Similar, we have

$$\begin{aligned} F_\gamma(p,u_\tau) = \tilde{u}_\tau &\int_0^1 (\partial_s F_\gamma)(p,\hat{u}_\tau,\nabla\hat{u}_\tau,\nabla^2\hat{u}_\tau)\,d\lambda \\ &+ \tilde{u}_{\tau,i}\int_0^1 (\partial_{y_i}F_\gamma)(p,\hat{u}_\tau,\nabla\hat{u}_\tau,\nabla^2\hat{u}_\tau)\,d\lambda \\ &+ \tilde{u}_{\tau,\alpha\beta}\int_0^1 (\partial_{Q_{\alpha\beta}}F_\gamma)(p,\hat{u}_\tau,\nabla\hat{u}_\tau,\nabla^2\hat{u}_\tau)\,d\lambda \end{aligned} \tag{C.33}$$

and

$$\begin{aligned} G_{\alpha\beta}(p,u_\tau) = -\delta_{\alpha\beta} &+ \tilde{u}_\tau\int_0^1 (\partial_s G_{\alpha\beta})(p,\hat{u}_\tau,\nabla\hat{u}_\tau)\,d\lambda \\ &+ \tilde{u}_{\tau,i}\int_0^1 (\partial_{y_i}G_{\alpha\beta})(p,\hat{u}_\tau,\nabla\hat{u}_\tau)\,d\lambda. \end{aligned} \tag{C.34}$$

Combining (C.32)–(C.34), (C.17) with (C.19), we have

$$\begin{aligned} H_{u_i^+}(p) - H_{u_i^-}(p) &= \int_0^1 \partial_\tau(H_{u_\tau}(p))\,d\tau \\ &= -|A|^2u - \Delta u + a_1^{\alpha\beta}u_{\alpha\beta} + b_1^\gamma u_\gamma + c_1 u, \end{aligned} \tag{C.35}$$

where the coefficients $a_1^{\alpha\beta}$, b_1^i and c_1 are small, and tend to zero as u_i^+ and u_i^- tend to zero.

Step 2. We calculate the difference of $\eta_{u_i^+}$ and $\eta_{u_i^-}$. Note that

$$\eta_{u_i^+}(p) = \eta_{u_i^-}(p) + \int_0^1 \partial_\tau(\eta_{\tilde{u}_\tau}(p))\,d\tau. \tag{C.36}$$

By (C.18), we have

$$\partial_\tau(\eta_{\tilde{u}_\tau}(p)) = (\partial_s\eta)(p,\tilde{u}_\tau,\nabla\tilde{u}_\tau)\cdot u + (\partial_{y_\alpha}\eta)(p,\tilde{u}_\tau,\nabla\tilde{u}_\tau)\cdot u_\alpha, \tag{C.37}$$

where the function η of the right-hand side is defined by (C.3). Let $\hat{u}_\lambda = \lambda\tilde{u}_\tau$ as in Step 1. Then we have

$$\begin{aligned}(\partial_s\eta)(p,\tilde{u}_\tau,\nabla\tilde{u}_\tau) = (\partial_s\eta)(p,0,0) + \tilde{u}_\tau \int_0^1 (\partial_s^2\eta)(p,\hat{u}_\lambda,\nabla\hat{u}_\lambda)\,d\lambda \\ + \tilde{u}_{\tau,\alpha}\int_0^1 (\partial_{y_\alpha}\partial_s\eta)(p,\hat{u}_\lambda,\nabla\hat{u}_\lambda)\,d\lambda.\end{aligned} \tag{C.38}$$

Similarly, we have

$$\begin{aligned}(\partial_{y_\alpha}\eta)(p,\tilde{u}_\tau,\nabla\tilde{u}_\tau) = (\partial_{y_\alpha}\eta)(p,0,0) + \tilde{u}_\tau \int_0^1 (\partial_s\partial_{y_\alpha}\eta)(p,\hat{u}_\lambda,\nabla\hat{u}_\lambda)\,d\lambda \\ + \tilde{u}_{\tau,\beta}\int_0^1 (\partial_{y_\alpha}\partial_{y_\beta}\eta)(p,\hat{u}_\lambda,\nabla\hat{u}_\lambda)\,d\lambda.\end{aligned} \tag{C.39}$$

Combining (C.36)–(C.39) with part (3) of Lemma C.1, we have

$$\eta_{u_i^+}(p) = \eta_{u_i^-}(p) + u - \langle p,\nabla u\rangle + b_2^i u_i + c_2 u, \tag{C.40}$$

where c_2 and b_2^α are small and tend to zero as u_i^+ and u_i^- tend to zero.

Step 3. We calculate the difference of $\phi_{u_i^+}(p)$ and $\phi_{u_i^-}(p)$, where $\phi_u = H_u - \frac{1}{2}\langle \mathbf{x}_u,\mathbf{n}_u\rangle$. Combining (C.40) with (C.35), we have

$$\phi_{u_i^+}(p) - \phi_{u_i^-}(p) = -Lu + a_3^{\alpha\beta}u_{\alpha\beta} + b_3^\gamma u_\gamma + c_3 u, \tag{C.41}$$

where $a_3^{\alpha\beta}, b_3^\gamma$ and c_3 are small and tend to zero as u_i^+ and u_i^- tend to zero. Note that

$$(\partial_t\mathbf{x}_{u_i^+})^\perp = \langle \partial_t\mathbf{x}_{u_i^+},\mathbf{n}_{u_i^+}\rangle = \partial_t u_i^+\langle \mathbf{n},\mathbf{n}_{u_i^+}\rangle = \partial_t u_i^+ w_{u_i^+}, \tag{C.42}$$

where $w_{u_i^+}$ is defined by (C.1), and $\mathbf{x}_{u_i^-}$ satisfies a similar equation as (C.42). Moreover, we have

$$\partial_t u_i^+ w_{u_i^+} - \partial_t u_i^- w_{u_i^-} = \int_0^1 \partial_\tau(\partial_t\tilde{u}_\tau w_{\tilde{u}_\tau})\,d\tau.$$

As in (C.18), we have

$$\begin{aligned}\partial_\tau(\partial_t\tilde{u}_\tau w_{\tilde{u}_\tau}) &= \partial_t u\, w_{\tilde{u}_\tau} + \partial_t\tilde{u}_\tau\,\partial_\tau w_{\tilde{u}_\tau} \\ &= \partial_t u\, w_{\tilde{u}_\tau} + \partial_t\tilde{u}_\tau\big((\partial_s w)(p,\tilde{u}_\tau,\nabla\tilde{u}_\tau)\cdot u + (\partial_{y_i}w)(p,\tilde{u}_\tau,\nabla\tilde{u}_\tau)\cdot u_i\big).\end{aligned}$$

Since $w(p,0,0) = 1$ by (C.4), we have

$$w_{\tilde{u}_\tau}(p) = 1 + \int_0^1 (\partial_\lambda w_{\hat{u}_\lambda})(p)\, d\lambda$$
$$= 1 + \tilde{u}_\tau \int_0^1 (\partial_s w)(p, \hat{u}_\lambda, \nabla \hat{u}_\lambda)\, d\lambda + \tilde{u}_{\tau,i} \int_0^1 (\partial_{y_i} w)(p, \hat{u}_\lambda, \nabla \hat{u}_\lambda)\, d\lambda.$$

Combining the above identities, we have

$$\partial_t u_i^+ w_{u_i^+} - \partial_t u_i^- w_{u_i^-} = \partial_t u(1 + b_4^i \tilde{u}_{\tau,i} + c_4 \tilde{u}_\tau) + b_5^i u_i + c_5 u, \tag{C.43}$$

where c_4, c_5, b_4^i and b_5^i are small and tend to zero as u_i^+ and u_i^- tend to zero. Combining (C.43) and (C.41) with the equation of rescaled mean curvature flow, we have

$$\frac{\partial u}{\partial t} = \frac{1}{1 + b_4^i \tilde{u}_{\tau,i} + c_4 \tilde{u}_\tau} \big(\partial_t u_i^+ w_{u_i^+} - \partial_t u_i^- w_{u_i^-} - b_5^i u_i - c_5 u\big)$$
$$= \frac{1}{1 + b_4^i \tilde{u}_{\tau,i} + c_4 \tilde{u}_\tau} \big(Lu + a_6^{\alpha\beta} u_{\alpha\beta} + b_6^\gamma u_\gamma + c_6 u\big)$$
$$= Lu + a_7^{\alpha\beta} u_{\alpha\beta} + b_7^\gamma u_\gamma + c_7 u,$$

where $a_6^{\alpha\beta}, b_6^\gamma, c_6, a_7^{\alpha\beta}, b_7^\gamma$ and c_7 are small and tend to zero as u_i^+ and u_i^- tend to zero. The lemma is proved. ■

Acknowledgments. Bing Wang would like to thank Lu Wang for many helpful conversations. Both authors are grateful to the anonymous referees for many useful suggestions to improve the exposition of this paper.

Funding. The first author was supported by NSFC grant No. 12071449 and by the Fundamental Research Funds for the Central Universities. The second author was supported by NSFC grant No. 11971452 and No. 12026251.

References

[1] Andrews, B.: Noncollapsing in mean-convex mean curvature flow. Geom. Topol. **16**, 1413–1418 (2012) Zbl 1250.53063 MR 2967056

[2] Andrews, B., Bryan, P.: Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson's theorem. J. Reine Angew. Math. **653**, 179–187 (2011) Zbl 1218.53070 MR 2794630

[3] Bamler, R.: Convergence of Ricci flows with bounded scalar curvature. Ann. of Math. (2) **188**, 753–831 (2018) Zbl 1410.53063 MR 3866886

[4] Bao, C.: A note on the entropy of mean curvature flow. Sci. China Math. **58**, 2611–2620 (2015) Zbl 1342.53088 MR 3429272

[5] Berline, N., Getzler, E., Vergne, M.: Heat Kernels and Dirac Operators. Grundlehren Text Ed., Springer, Berlin (2004) Zbl 1037.58015 MR 2273508

[6] Brakke, K. A.: The Motion of a Surface by Its Mean Curvature. Math. Notes 20, Princeton University, Princeton (1978) Zbl 0386.53047 MR 485012

[7] Brendle, S.: Embedded self-similar shrinkers of genus 0. Ann. of Math. (2) **183**, 715–728 (2016) Zbl 1343.53062 MR 3450486

[8] Brendle, S., Huisken, G.: Mean curvature flow with surgery of mean convex surfaces in $\mathbb{R}^3$. Invent. Math. **203**, 615–654 (2016) Zbl 1381.53113 MR 3455158

[9] Calabi, E.: Extremal Kähler metrics. In: Seminar on Differential Geometry, Ann. of Math. Stud. 102, Princeton University, Princeton, 259–290 (1982) Zbl 0487.53057 MR 645743

[10] Chen, X., Cheng, J. R.: On the constant scalar curvature Kähler metrics (II)-existence results. arXiv:1801.00656 (2018)

[11] Chen, X., Wang, B.: Remarks of weak-compactness along Kähler Ricci flow. In: Proceedings of the Seventh International Congress of Chinese Mathematicians. Vol. II, Adv. Lect. Math. (ALM) 44, International Press, Somerville, 203–233 (2019) Zbl 1442.53070 MR 3971900

[12] Chen, X., Wang, B.: Space of Ricci flows I. Comm. Pure Appl. Math. **65**, 1399–1457 (2012) Zbl 1252.53076 MR 2957704

[13] Chen, X., Wang, B.: Space of Ricci flow (II). arXiv:1405.6797 (2014)

[14] Chen, X., Wang, B.: Space of Ricci flows (II)—Part A: Moduli of singular Calabi–Yau spaces. Forum Math. Sigma **5**, art. e32 (2017) Zbl 1385.53033 MR 3739253

[15] Chen, X., Wang, B.: Space of Ricci flows (II)—Part B: Weak compactness of the flows. J. Differential Geom. **116**, 1–123 (2020) Zbl 07246681 MR 4146357

[16] Chen, B.-L., Yin, L.: Uniqueness and pseudolocality theorems of the mean curvature flow. Comm. Anal. Geom. **15**, 435–490 (2007) Zbl 1138.53055 MR 2379801

[17] Choi, H. I., Schoen, R.: The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature. Invent. Math. **81**, 387–394 (1985) Zbl 0577.53044 MR 807063

[18] Colding, T. H., Minicozzi, W. P., II: A Course in Minimal Surfaces. Grad. Stud. Math. 121, American Mathematical Society, Providence (2011) Zbl 1242.53007 MR 2780140

[19] Colding, T. H., Minicozzi, W. P., II: Generic mean curvature flow I: Generic singularities. Ann. of Math. (2) **175**, 755–833 (2012) Zbl 1239.53084 MR 2993752

[20] Colding, T. H., Minicozzi, W. P., II: Smooth compactness of self-shrinkers. Comment. Math. Helv. **87**, 463–475 (2012) Zbl 1258.53069 MR 2914856

[21] Colding, T. H., Minicozzi, W. P., II: Uniqueness of blowups and łojasiewicz inequalities. Ann. of Math. (2) **182**, 221–285 (2015) Zbl 1337.53082 MR 3374960

[22] Colding, T. H., Minicozzi, W. P., II: The singular set of mean curvature flow with generic singularities. Invent. Math. **204**, 443–471 (2016) Zbl 1341.53098 MR 3489702

[23] T. H. Colding, W. P. Minicozzi II, Dynamics of closed singularities. Ann. Inst. Fourier **69**, 2973–3016 (2019) Zbl 1453.58007

[24] Cooper, A. A.: A characterization of the singular time of the mean curvature flow. Proc. Amer. Math. Soc. **139**, 2933–2942 (2011) Zbl 1220.53080 MR 2801634

[25] Ecker, K.: Regularity theory for mean curvature flow. Progr. Nonlinear Differential Equations Appl. 57, Birkhäuser, Boston (2004) Zbl 1058.53054 MR 2024995

[26] Ecker, K., Huisken, G.: Mean curvature evolution of entire graphs. Ann. of Math. (2) **130**, 453–471 (1989) Zbl 0696.53036 MR 1025164

[27] Ecker, K., Huisken, G.: Interior estimates for hypersurfaces moving by mean curvature. Invent. Math. **105**, 547–569 (1991) Zbl 0707.53008 MR 1117150

[28] Gage, M. E.: An isoperimetric inequality with applications to curve shortening. Duke Math. J. **50**, 1225–1229 (1983) Zbl 0534.52008 MR 726325

[29] Gage, M. E.: Curve shortening makes convex curves circular. Invent. Math. **76**, 357–364 (1984) Zbl 0542.53004 MR 742856

[30] Gage, M., Hamilton, R. S.: The heat equation shrinking convex plane curves. J. Differential Geom. **23**, 69–96 (1986) Zbl 0621.53001 MR 840401

[31] Grayson, M. A.: The heat equation shrinks embedded plane curves to round points. J. Differential Geom. **26**, 285–314 (1987) Zbl 0667.53001 MR 906392

[32] Guang, Q., Zhu, J. J.: Rigidity and curvature estimates for graphical self-shrinkers. Calc. Var. Partial Differential Equations **56**, art. 176, 18 pp. (2017) Zbl 1380.53072 MR 3721649

[33] Haslhofer, R., Kleiner, B.: Mean curvature flow of mean convex hypersurfaces. Comm. Pure Appl. Math. **70**, 511–546 (2017) Zbl 1360.53069 MR 3602529

[34] Haslhofer, R., Kleiner, B.: Mean curvature flow with surgery. Duke Math. J. **166**, 1591–1626 (2017) Zbl 1370.53046 MR 3662439

[35] Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Differential Geom. **20**, 237–266 (1984) Zbl 0556.53001 MR 772132

[36] Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Differential Geom. **31**, 285–299 (1990) Zbl 0694.53005 MR 1030675

[37] Huisken, G., Sinestrari, C.: Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math. **183**, 45–70 (1999) Zbl 0992.53051 MR 1719551

[38] Huisken, G., Sinestrari, C.: Mean curvature flow singularities for mean convex surfaces. Calc. Var. Partial Differential Equations **8**, 1–14 (1999) Zbl 0992.53052 MR 1666878

[39] Huisken, G., Sinestrari, C.: Mean curvature flow with surgeries of two-convex hypersurfaces. Invent. Math. **175**, 137–221 (2009) Zbl 1170.53042 MR 2461428

[40] Ilmanen, T.: Singularities of mean curvature flow of surfaces. Preprint

[41] Ilmanen, T.: Lectures on mean curvature flow and related equations. Preprint

[42] Kan, T., Takahashi, J.: Time-dependent singularities in semilinear parabolic equations: Behavior at the singularities. J. Differential Equations **260**, 7278–7319 (2016) Zbl 1439.35285 MR 3473442

[43] Krylov, N. V., Safonov, M. V.: A property of the solutions of parabolic equations with measurable coefficients. Izv. Akad. Nauk SSSR Ser. Mat. **44**, 161–175, 239 (1980) MR 563790

[44] Le, N. Q., Sesum, N.: The mean curvature at the first singular time of the mean curvature flow. Ann. Inst. H. Poincaré Anal. Non Linéaire **27**, 1441–1459 (2010) Zbl 1237.53067 MR 2738327

[45] Le, N. Q., Sesum, N.: Blow-up rate of the mean curvature during the mean curvature flow and a gap theorem for self-shrinkers. Comm. Anal. Geom. **19**, 633–659 (2011) Zbl 1244.53078 MR 2880211

[46] Li, H., Wang, B.: The extension problem of the mean curvature flow (I). Invent. Math. **218**, 721–777 (2019) Zbl 07127766 MR 4022071

[47] Li, H., Wang, B., Zheng, K.: Regularity scales and convergence of the Calabi flow. J. Geom. Anal. **28**, 2050–2101 (2018) Zbl 1406.53073 MR 3833786

[48] Li, P.: Geometric Analysis. Cambridge Stud. Adv. Math. 134, Cambridge University, Cambridge (2012) Zbl 1246.53002 MR 2962229

[49] Li, P., Yau, S.-T.: On the parabolic kernel of the Schrödinger operator. Acta Math. **156**, 153–201 (1986) Zbl 0611.58045 MR 834612

[50] Lieberman, G. M.: Second Order Parabolic Differential Equations. World Scientific, River Edge, (1996) Zbl 0884.35001 MR 1465184

[51] Šešum, N.: Curvature tensor under the Ricci flow. Amer. J. Math. **127**, 1315–1324 (2005) Zbl 1093.53070 MR 2183526

[52] Sun, A.: Local entropy and generic multiplicity one singularities of mean curvature flow of surfaces. arXiv:1810.08114 (2018)

[53] Tian, G., Zhang, Z.: Regularity of Kähler–Ricci flows on Fano manifolds. Acta Math. **216**, 127–176 (2016) Zbl 1356.53067 MR 3508220

[54] Wang, B.: On the conditions to extend Ricci flow(II). Int. Math. Res. Not. IMRN **2012**, 3192–3223 (2012) Zbl 1251.53040 MR 2946223

[55] Wang, L.: Asymptotic structure of self-shrinkers. arXiv:1610.04904 (2016)

[56] Wang, M.-T.: The mean curvature flow smoothes Lipschitz submanifolds. Comm. Anal. Geom. **12**, 581–599 (2004) Zbl 1059.53053 MR 2128604

[57] White, B.: Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals. Invent. Math. **88**, 243–256 (1987) Zbl 0615.53044 MR 880951

[58] White, B.: The size of the singular set in mean curvature flow of mean-convex sets. J. Amer. Math. Soc. **13**, 665–695 (2000) Zbl 0961.53039 MR 1758759

[59] White, B.: The nature of singularities in mean curvature flow of mean-convex sets. J. Amer. Math. Soc. **16**, 123–138 (2003) Zbl 1027.53078 MR 1937202

[60] White, B.: A local regularity theorem for mean curvature flow. Ann. of Math. (2) **161**, 1487–1519 (2005) Zbl 1091.53045 MR 2180405