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Topological regularity of spaces with an upper curvature bound

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Abstract. We prove that a locally compact space with an upper curvature bound is a topological manifold if and only if all of its spaces of directions are homotopy equivalent and not contractible. We discuss applications to homology manifolds, limits of Riemannian manifolds and deduce a sphere theorem.

Keywords. Curvature bounds, regularity, homology manifold

1. Introduction

1.1. Main results

We prove the following:

Theorem 1.1. *Let X be a connected, locally compact metric space with an upper curvature bound. Then the following are equivalent:*

(1) *X is a topological manifold.*

(2) *All tangent spaces T_pX of X are homeomorphic to the same space T, and T is of finite topological dimension.*

(3) *All spaces of directions Σ_pX are homotopy equivalent to the same space Σ, and Σ is non-contractible.*

Theorem 1.1 answers a folklore question about the infinitesimal characterization of topological manifolds among spaces with upper curvature bounds, cf. [4]. It implies the following affirmative answer to a question of F. Quinn, [54, Problem 7.2].

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Theorem 1.2. *Let X be a metric space with an upper curvature bound. If X is a homology manifold, then there exists a locally finite subset E of X such that $X \setminus E$ is a topological manifold.*

We refer the reader to [44] and to Section 2.4 below for basics on homology manifolds and to Section 6 for a stronger result.

If X in Theorem 1.1 is a topological manifold of dimension n, then all tangent spaces T_pX turn out to be homeomorphic to $\mathbb{R}^n$ and all spaces of directions turn out to be homotopy equivalent to $\mathbb{S}^{n-1}$.

For $n \geq 5$, the spaces of directions may not all be homeomorphic to $\mathbb{S}^{n-1}$, [7], as a consequence of the double suspension theorem of R. Edwards [23], [17]. However, for $n \leq 4$, all spaces of directions $\Sigma_p X$ are homeomorphic to $\mathbb{S}^{n-1}$, see Theorem 6.4. This answers a question of V. Berestovskii [9, Problem 1].

We deduce the following topological stability theorem:

Theorem 1.3. *For $\kappa \in \mathbb{R}$ and $r > 0$, let a sequence of complete n-dimensional Riemannian manifolds M_i with sectional curvature $\leq \kappa$ and injectivity radius $\geq r$ converge in the pointed Gromov–Hausdorff topology to a locally compact space X. Then X is a topological manifold and any space of directions $\Sigma_x X$ of X is homeomorphic to $\mathbb{S}^{n-1}$.*

Moreover, if X is compact, then M_i is homeomorphic to X, for all i large enough.

In particular, the double suspension of a non-simply connected homology sphere, Example 2.5, is not a limit of CAT(κ) Riemannian manifolds, proving the conjecture formulated in [7].

1.2. Analogies and differences

For spaces with lower curvature bounds the analogs of Theorem 1.1 and the stability part of Theorem 1.3 are special cases of the fundamental topological stability theorem of G. Perelman, [49], [38]. Moreover, Theorem 1.2 for Alexandrov spaces is a direct consequence of Perelman's stability theorem, as observed in [62]. The analog of the additional statement in Theorem 1.3 about the homeomorphism type of the spaces of directions (see also Theorem 7.1 below, for a more general statement) has been proved for Alexandrov spaces by V. Kapovitch in [37].

However, for spaces with an upper curvature bound there is no analog of the stability theorem, even for finite graphs. Moreover, already in dimension 2, locally compact, geodesically complete spaces with an upper curvature bound do not need to admit a topological triangulation, as has been observed by B. Kleiner, [39]. Thus, unlike their analogs for Alexandrov spaces, our results are not special cases of much more general statements.

On the other hand, our approach requires less geometric control and should be applicable beyond our setting. For instance, it might simplify Perelman's stability theorem for Alexandrov spaces.

As in Perelman's topological theory of Alexandrov spaces, a major role in our topological results play the so-called *strainer maps* investigated in [41]. Perelman has proved in [49] that in the realm of Alexandrov spaces strainer maps are local fiber bundles. Similarly to the failure of topological stability, the example in [39] demonstrates that in spaces with upper curvature bounds the local fiber bundle structure can not be expected. Nevertheless, from the homotopy point of view, strainer maps behave well and turn out to be (local) *Hurewicz fibrations*. This result, Theorem 5.1, is deduced from general topological statements and the local contractibility of fibers of strainer maps obtained in [41]. Theorem 5.1 might be useful in further investigations of spaces with upper curvature bounds and beyond.

We further mention, that the main theorems of [30] ([31]), [26] imply (in a more general situation) the finiteness of topological types of manifolds in the sequence appearing in the final statement of Theorem 1.3. However, no conclusion about the limit space itself can be deduced in the generality of [30], [26] besides the fact that the limit space is a homology manifold.

Finally, Theorems 1.1, 1.2 in dimensions ≤ 3 and some related insights in dimension 4 are due to P. Thurston, [58].

1.3. Two applications

In order to state yet another manifold characterization we recall, [41], that a space X with an upper curvature bound is *locally geodesically complete* if any local geodesic $\gamma : [a, b] \to X$ can be extended as a local geodesic to some larger interval $[a - \epsilon, a + \epsilon]$. All homology manifolds, thus all spaces appearing in the previous theorems, are always locally geodesically complete, Lemma 2.2.

Theorem 1.4. *Let X be a connected, locally compact space which has an upper curvature bound and is locally geodesically complete. If X is not a topological manifold, then it contains an isometrically embedded compact metric tree different from an interval.*

Theorem 1.4 states that a non-manifold must have geodesics which branch at an angle at least π. It can be seen as a soft version of the following much more special and rigid result. If a connected, locally compact space X with an upper curvature bound is locally geodesically complete and has no branching geodesics, then X is a smooth manifold whose distance is defined by a continuous Riemannian metric g (with some additional properties), [8], [41, Theorem 1.3].

Theorem 1.4 is a consequence of Theorem 1.1 and the following sphere theorem.

Theorem 1.5. *Let Σ be a compact, locally geodesically complete space with curvature bounded from above by* 1. *If the injectivity radius of Σ is at least π and Σ does not contain a triple of points with pairwise distances at least π, then Σ is homeomorphic to a sphere.*

Our Theorem 1.5 has a well-known analog for spaces with lower curvature bounds, due to K. Grove and P. Petersen, later reproved by A. Petrunin: An Alexandrov space of curvature at least 1 and *radius* larger than $\frac{\pi}{2}$ is homeomorphic to a sphere, [29], [52]. In terms of the *packing radii* investigated in [28], [32], the assumption about the triple of points in X reads as $\mathrm{pack}_3(X) < \frac{\pi}{2}$.

From Theorem 1.5 it is easy to deduce a volume sphere theorem, see Theorem 8.3 below, generalizing [20], [46].

1.4. *Structure of the paper*

After Preliminaries in Section 2, we study in Section 3 homology manifolds with upper curvature bounds and prove those parts of our main theorems, which do not rely on properties of strainer maps. In Section 4 we recall several topological results relating fibrations, fiber bundles and local uniform contractibility. In Section 5 we recall from [41] basic properties of strainer maps and apply results from Section 4 to deduce Theorem 5.1 discussed above. In Section 6, we apply the general topological statements inductively to strainer maps and prove Theorems 1.1–1.2. In Section 7 we discuss iterated spaces of directions and prove Theorem 1.3 and its generalization. In the final Section 8 we discuss basic properties of *pure-dimensional spaces* and prove Theorems 1.4, 1.5 and 8.3.

2. Preliminaries

2.1. *Notations*

We refer to [2], [13], [16], [3], [41] for the basics on upper curvature bounds in the sense of Alexandrov.

We will stick to the following notations. By d we denote the distance functions on metric spaces. For a point p in a metric space X, we denote by $d_x : X \to \mathbb{R}$ the distance function to the point x. By $B_r(p)$ (respectively, by $\bar{B}_r(p)$) we denote the open (respectively, closed) metric ball of radius r around the point p. By $B_r^*(p)$ we will denote the punctured ball $B_r(p) \setminus \{p\}$. By $S_r(p)$ we will denote the metric sphere of radius r around the point p. Geodesics will always be globally minimizing and parametrized by arclength.

For two maps φ, ψ from a set G into a metric space Y, we set

$$d(\varphi, \psi) := \sup_{x \in G} d\big(\varphi(x), \psi(x)\big).$$

The dimension of a metric space X will always be the covering dimension and will be denoted by $\dim(X)$.

A metric space X has curvature bounded from above by κ if every point of X has a CAT(κ) neighborhood. If X is a space with an upper curvature bound and $x \in X$ a point, we denote by Σ_x or by $\Sigma_x X$ the *space of directions* at x and by T_x or by $T_x X$ the *tangent cone* at x of X, which is canonically identified with the Euclidean cone over Σ_x.

We denote by $H_k(X, Y)$ the k-th singular homology with integer coefficients of the pair $Y \subset X$ of topological spaces.

2.2. *Basic topological properties of spaces with upper curvature bounds*

Any space X with an upper curvature bound is an *absolute neighborhood retract*, abbreviated as ANR, [48], [40]. In particular, X is homotopy equivalent to a simplicial complex. We have, [40]:

Lemma 2.1. *For any point x in a space X with an upper curvature bound, there exists some $r > 0$ such that for each $0 < s \leq r$, the ball $B_s(x)$ is contractible and the punctured ball $B_s^*(x)$ is homotopy equivalent to the space of directions Σ_x.*

Due to [39], for any separable space X with an upper curvature bound

$$\dim(X) = 1 + \sup_{x \in X}(\dim(\Sigma_x X)) = \sup_{x \in X}(\dim(T_x X)).$$

Moreover, if $\dim(X)$ is a finite number n, there exists some $x \in X$ such that $H_{n-1}(\Sigma_x X)$ is not 0.

2.3. *Tiny balls, GCBA spaces*

Let X be a locally compact space with an upper curvature bound κ. We will say that a ball $B_r(x)$ is *tiny* if the closed ball with radius $10 \cdot r$ around x is a compact CAT(κ) space and if $100 \cdot r$ is smaller than the diameter of the simply connected, complete surface of constant curvature κ.

In a tiny ball all geodesics are determined by their endpoints.

A space X with an upper curvature bound is *locally geodesically complete* if every local geodesic defined on any compact interval can be extended as a local geodesic beyond its endpoints. The following observation, [42, Theorem 1.5], goes back to H. Busemann:

Lemma 2.2. *Let X be a space with an upper curvature bound. If for any $x \in X$ there exist arbitrary small r such that the punctured ball $B_r^*(x)$ is non-contractible, then X is locally geodesically complete.*

Due to the long exact sequence and the contractibility of small balls, the *local homology* $H_m(X, X \setminus \{x\})$ at x coincides with $H_{m-1}(B_r^*(x))$, for any x in a space X with an upper curvature bound, any small $r > 0$ and any natural m. Thus, the non-vanishing of $H_*(X, X \setminus \{x\})$, for all $x \in X$ implies, that X is locally geodesically complete.

A *GCBA space* is a locally compact, separable metric space with an upper curvature bound, which is locally geodesically complete. If X is GCBA, then so is any tangent space $T_x X$ and space of directions $\Sigma_x X$. Moreover, any space of directions $\Sigma_x X$ is compact and any tangent space $T_x X$ is the limit in the pointed Gromov–Hausdorff topology of rescaled balls around x, [2], [41, Section 5].

Every GCBA space X has locally finite dimension, [41, Theorem 1.1], and contains an open and dense topological manifold (possibly of non-constant dimension), [41, Theorem 1.2]. Moreover, X contains a dense set of points with tangent spaces isometric to Euclidean spaces, possibly of different dimension, [41, Theorem 1.3].

The following result has been shown in [41, Theorems 1.12 and 13.1]:

Proposition 2.3. *Let x be a point in a GCBA space X. Then there exists some $r_x > 0$ such that for all $r < r_x$ the following hold true:*

(1) *The metric sphere $S_r(x)$ is homotopy equivalent to Σ_x.*

(2) *Let $B_{10 \cdot r}(x_i)$ be a sequence of tiny metric balls in GCBA spaces X_i with the same upper curvature bound κ. If $\bar{B}_{10 \cdot r}(x_i)$ converge to $\bar{B}_{10 \cdot r}(x)$ in the Gromov–Hausdorff topology, then, for all i large enough, $S_r(x_i)$ is homotopy equivalent to $S_r(x)$.*

2.4. Homology manifolds

We denote by D^n the closed unit ball in $\mathbb{R}^n$. We call D^2 the *unit disk* and denote it by D.

Let M be a locally compact, separable metric space of finite topological dimension. We say that M is a *homology n-manifold with boundary* if for any $p \in M$ we find a point $x \in D^n$ such that the local homology $H_*(M, M \setminus \{p\})$ at p is isomorphic to $H_*(D^n, D^n \setminus \{x\})$. The *boundary* ∂M of M is defined as the set of all points at which the n-th local homologies are trivial. In the case where the boundary of M is empty, we simply say that M is a *homology n-manifold*.

If M is a homology n-manifold with boundary, then ∂M is a closed subset of M and it is a homology $(n-1)$-manifold by [45].

Any homology n-manifold (with boundary) has dimension n. For $n \leq 2$, we have the following theorem of R. Moore, see [61, Chapter IX].

Theorem 2.4. *Any homology n-manifold with $n \leq 2$ is a topological manifold.*

2.5. Examples

The following example is well known, [6], [29].

Example 2.5. Consider a closed Riemannian $(n-2)$-manifold Y which has the homology of $\mathbb{S}^{n-2}$ but is not simply connected (such manifolds exist for $n \geq 5$). Rescaling the metric, we may assume that Y is CAT(1). The spherical suspension $X_1 = \mathbb{S}^0 * Y$ of Y is a CAT(1) space which is a homology manifold and has exactly two non-manifold points. The double suspension $X = \mathbb{S}^1 * Y$ of Y is a CAT(1) space homeomorphic to $\mathbb{S}^n$ by the double suspension theorem, [17], [23]. But for any point x on the $\mathbb{S}^1$-factor, the space of directions $\Sigma_x X$ is isometric to X_1, hence not homeomorphic to $\mathbb{S}^{n-1}$.

Some additional assumption on the tangent spaces and spaces of directions are needed in Theorem 1.1:

Example 2.6. Let X be the Hilbert cube, hence a compact CAT(0) space. At any $x \in X$ the space of directions Σ_x is contractible. Moreover, at any $x \in X$, the tangent space T_x is homeomorphic to the Hilbert space, as can be deduced from [10], [59].

3. Homology manifolds with upper curvature bounds

3.1. General observations

We start with the following:

Lemma 3.1. *Let X be a locally compact space with an upper curvature bound. The space X is a homology n-manifold if and only if $H_*(\Sigma_x) = H_*(\mathbb{S}^{n-1})$, for all $x \in X$. In this case, X is locally geodesically complete.*

Proof. For any $x \in X$ and $r > 0$ as in Lemma 2.1, Σ_x is homotopy equivalent to $B_r^*(x)$ and $\tilde{H}_k(B_r^*(x)) = H_{k+1}(X, X \setminus \{x\})$, for any k.

Thus, if X is a homology n-manifold, then any space of directions $\Sigma_x X$ has the homology of $\mathbb{S}^{n-1}$.

If all spaces of directions have the homology of $\mathbb{S}^{n-1}$, then they are non-contractible and, therefore, X is locally geodesically complete. For any $x \in X$, we have

$$H_m(X, X \setminus \{x\}) = H_m(\mathbb{S}^{n-1}).$$

Thus, X has the same local homology as $\mathbb{R}^n$. Since X, as any GCBA space, has locally finite dimension, X is locally a homology n-manifold. Thus X has topological dimension n and it is a homology n-manifold. ■

Using the contraction along geodesics to the center of a ball, [45] and the Poincaré duality, [12], we obtain [58, Proposition 2.7]:

Lemma 3.2. *Let $B_r(x)$ be a tiny ball in a homology n-manifold X with an upper curvature bound. Then $\bar{B}_r(x)$ is a compact, contractible homology manifold with boundary $S_r(x)$. In particular, $S_r(x)$ is a homology $(n-1)$-manifold with the same homology as $\mathbb{S}^{n-1}$.*

3.2. Stability under convergence

The following observation is a special case of result of E. Begle, [5], see also [30, Theorem 2.1]. In our situation the proof can be simplified using the homotopy properties of distance spheres.

Lemma 3.3. *Let X_i be compact* CAT(κ) *spaces converging in the Gromov–Hausdorff topology to a compact space X. Let $x_i \in X_i$ converge to $x \in X$ and let $r > 0$ be such that for all i, the ball $B_r(x_i) \subset X_i$ is a homology n-manifold. Then $B_r(x) \subset X$ is a homology n-manifold.*

Proof. Due to Lemma 3.1, the open balls $B_r(x_i)$ are GCBA spaces, hence so is $B_r(x)$, compare [41, Example 4.3]. In particular, the dimension of $B_r(x)$ is locally finite.

It remains to prove that $H_*(\Sigma_z X) = H_*(\mathbb{S}^{n-1})$ for all $z \in X$. Write z as a limit of points $z_i \in B_r(x_i)$. By Proposition 2.3 we find some $r > t > 0$, such that $S_t(z)$ is homotopy equivalent to $\Sigma_z X$ and to $S_t(z_i)$, for all i large enough. Thus, the homology of $\Sigma_z X$ coincides with the homology of $\mathbb{S}^{n-1}$ by Lemma 3.2. ■

As a consequence we deduce:

Corollary 3.4. *Let X be a homology n-manifold with an upper curvature bound. Then, for any $x \in X$, the tangent space $T_x X$ is a homology n-manifold and Σ_x is a homology $(n-1)$-manifold.*

Proof. The space X is GCBA by Lemma 3.1. Thus, the tangent space $(T_x X, 0)$ is a limit of rescaled metric balls around x in X. Due to Lemma 3.3, T_x is a homology n-manifold. The homology n-manifold $T_x \setminus \{0\}$ is homeomorphic to $\Sigma_x \times \mathbb{R}$. Therefore, Σ_x is a homology $(n-1)$-manifold. ■

3.3. *Simple implications in the main theorem*

We can already discuss the simple implications in our main theorem. We start with the following (folklore) result, compare [15, Proposition 3.12]:

Lemma 3.5. *Let X be a topological n-manifold with an upper curvature bound. Then any space of directions Σ_x is homotopy equivalent to $\mathbb{S}^{n-1}$.*

Proof. Any space of directions Σ_x is an ANR and compact homology $(n-1)$-manifold with the homology of $\mathbb{S}^{n-1}$, Lemma 3.1, Corollary 3.4.

If $n \leq 3$, then Σ_x is homeomorphic to $\mathbb{S}^{n-1}$ by Theorem 2.4.

If $n \geq 3$, then, by Whitehead's theorem, it suffices to prove that Σ_x is simply connected. To this end, consider a small neighborhood U of x homeomorphic to a Euclidean ball and small numbers $r_1, r_2 > 0$ such that $B_{r_1}(x) \subset U \subset B_{r_2}(x)$. Note that the inclusion $B^*_{r_1}(x) \to B^*_{r_2}(x)$ is a homotopy equivalence factoring through the simply connected space $U \setminus \{x\}$. This implies that all small punctured balls $B^*_r(x)$ are simply connected. Due to Lemma 2.1, Σ_x is simply connected, finishing the proof. ∎

Lemma 3.6. *Let X be a locally compact space with an upper curvature bound. Assume that all spaces of directions $\Sigma_x X$ are homotopy equivalent to the same non-contractible space Σ. Then Σ is homotopy equivalent to $\mathbb{S}^{n-1}$, for some n, and X is a homology manifold.*

Proof. By assumption, all spaces of directions are non-contractible. By Lemma 2.2, X is a GCBA space. Any GCBA space has a point with space of directions isometric to a sphere $\mathbb{S}^{n-1}$, [41, Theorem 1.3]. Then, by assumption, all spaces of directions are homotopy equivalent to $\mathbb{S}^{n-1}$. By Lemma 3.1, X must be a homology n-manifold. ∎

Lemma 3.7. *Let X be a locally compact space with an upper curvature bound. Assume that all tangent spaces $T_x X$ are homeomorphic to the same finite-dimensional space T. Then T is homeomorphic to $\mathbb{R}^n$, for some n, and all spaces of directions are homotopy equivalent to $\mathbb{S}^{n-1}$.*

Proof. For points $x, y \in X$ there is, by assumption, a homeomorphism

$$I : T_x = C(\Sigma_x) \to T_y = C(\Sigma_y).$$

Restricting I to a large distance sphere in T_x around the origin, we obtain an embedding $I : \Sigma_x \to C(\Sigma_y) \setminus \{y\} = (0, \infty) \times \Sigma_y$. Composing with the projection to the second factor we obtain a map $\hat{I} : \Sigma_x \to \Sigma_y$, and it is easy to see (using the cone structures of T_x and T_y) that $\hat{I}$ is a weak homotopy equivalence. Since the spaces of directions are ANRs, $\hat{I}$ is a homotopy equivalence. Thus, all spaces of directions are homotopy equivalent.

Due to [39], X has finite dimension n, equal to the dimension of T. Then, by [39] there exists some x such that Σ_x is not contractible. By Lemma 3.6, there exists some n such that all spaces of directions are homotopy equivalent to $\mathbb{S}^{n-1}$.

Moreover, X is a homology n-manifold and a GCBA space by Lemma 3.1. By [41] there exists a point $x \in X$ with tangent space isometric to $\mathbb{R}^n$. Therefore, T is homeomorphic to $\mathbb{R}^n$. ∎

4. Homotopy stability and Hurewicz fibrations

4.1. Uniform local contractibility

Following [51], we say that a function $\rho : [0, r_0) \to [0, \infty)$ is a *contractibility function* if $\rho(0) = 0$, $\rho(t) \geq t$, for all $t \in [0, r_0)$, and ρ is continuous at 0.

Definition 4.1. We say that a family $\mathcal{F}$ of metric spaces is *locally uniformly contractible* if there exists a contractibility function $\rho : [0, r_0) \to [0, \infty)$ such that for any space X in the family $\mathcal{F}$, any point $x \in X$ and any $0 < r < r_0$, the ball $B_r(x)$ is contractible within the ball $B_{\rho(r)}(x)$.

For example, the family of all CAT(κ) spaces is locally uniformly contractible with $\rho : [0, \frac{\pi}{\sqrt{\kappa}}) \to [0, \infty)$ being the identity map.

A compact, finite-dimensional space is locally uniformly contractible if and only if it is an ANR.

We will use the notion of ϵ*-equivalence*, [19], a controlled version of homotopy equivalence. A continuous map $f : X \to Y$ between metric spaces is called an ϵ-equivalence if there exists a continuous map $g : Y \to X$ with the following property. There exist homotopies F and G of $f \circ g$ and $g \circ f$ to the respective identity map of Y and X such that the F-flow line of any point in Y and the f-image of the G-flow line of any point in X has diameter less than ϵ in Y.

The following result is due to P. Petersen, [51, Theorem A]:

Theorem 4.2. *For any $n, \epsilon > 0$ and any family $\mathcal{F}$ of locally uniformly contractible metric spaces of dimension at most n, there exists some $\delta > 0$ such that the following holds true. Any pair of spaces $X, Y \in \mathcal{F}$, with Gromov–Hausdorff distance at most δ are ϵ-equivalent.*

When dealing with the family of fibers of a map the following variant of Definition 4.1 seems more suitable, compare [60].

Definition 4.3. Let $F : X \to Y$ be a map between metric spaces. We say that F *has locally uniformly contractible fibers* if the following condition holds true for any point $x \in X$ and every neighborhood U of x in X. There exists a neighborhood $V \subset U$ of x in X such that for any fiber $F^{-1}(y)$ with non-empty intersection $F^{-1}(y) \cap V$, this intersection is contractible in $F^{-1}(y) \cap U$.

For X compact, a map $F : X \to Y$ has locally uniformly contractible fibers in the sense of Definition 4.3 if and only if the family of the fibers is locally uniformly contractible in the sense of Definition 4.1.

4.2. Relation to Hurewicz fibrations

A map $F : X \to Y$ between metric spaces is called a *Hurewicz fibration* if it satisfies the homotopy lifting property with respect to all spaces, [33, Section 4.2], [60].

The map F is called open if the images of open sets are open. It is called proper if the preimage of any compact set is compact.

Any locally compact metric space carries a complete metric. This allows us to formulate Theorems 4.4–4.6 below for locally compact metric spaces, while the original formulations in [43], [60] are done for complete metric spaces.

We formulate a special case of the continuous selection theorems of E. Michael, [43, Theorem 1.2], as in [22, Theorem M]:

Theorem 4.4. *Let $F : X \to Y$ be an open map with locally uniformly contractible fibers between finite dimensional, locally compact metric spaces. Then, for any $x \in X$, there exist a neighborhood U of $F(x)$ in Y and a continuous map $s : U \to X$ such that $F \circ s$ is the identity.*

The following result is proved in [60, Theorem 1], see also [1] and [24] for related statements.

Theorem 4.5. *Let X, Y be finite-dimensional, compact metric spaces and let Y be an* ANR. *Let $F : X \to Y$ be an open, surjective map with locally uniformly contractible fibers. Then F is a Hurewicz fibration.*

In the locally compact case one can not expect that an open, surjective map with locally uniformly contractible fibers is a Hurewicz fibration, as we see by restricting a Hurewicz fibration to a complicated open subset. However, the following result is deduced in [60, Theorem 2] from Michael's selection theorem mentioned above.

Theorem 4.6. *Let X and Y be finite-dimensional locally compact metric spaces. Assume that an open, surjective map $F : X \to Y$ has locally uniformly contractible fibers. If all fibers $F^{-1}(y)$ of F are contractible then F is a Hurewicz fibration.*

4.3. Fibrations and fiber bundles

In some situations, Hurewicz fibrations turn out to be fiber bundles. Most results in this direction are based on the famous α-approximation theorem, proved by T. Chapman and S. Ferry in dimensions $n \geq 5$, [19], and extended by S. Ferry and S. Weinberger to dimension $n = 4$, [27, Theorem 4], and by W. Jakobsche to dimensions $n = 2, 3$, [35], [36]. Note that the 3-dimensional statement in [36] relies on the resolution of the Poincaré conjecture. For $n = 1$, the α-approximation theorem is rather clear.

Theorem 4.7. *Let the metric space M be a closed topological n-manifold. For any $\alpha > 0$ there is some $\epsilon = \epsilon(M, \alpha) > 0$ such that for any closed topological n-manifold M' and any ϵ-equivalence $f : M' \to M$ there exists a homeomorphism $f' : M' \to M$ with $d(f, f') < \alpha$.*

This theorem combined with the fiber-bundle recognition developed in [22] implies [25, Theorems 1.1–1.4], [55, Theorem 2]:

Theorem 4.8. *Let X, Y be finite-dimensional locally compact* ANR*s. Let $F : X \to Y$ be a Hurewicz fibration. If all fibers of F are closed n-manifolds, then F is a locally trivial fiber bundle.*

We also apply the following local variant of this global result proved in [25, Proposition 4.2]. The case $n = 3$, excluded in [25, Proposition 4.2], need not be excluded due to the solution of the Poincaré conjecture (and [36]):

Theorem 4.9. *Let $F : X \to I$ be a Hurewicz fibration from a locally compact metric space X to an open interval I. Assume that all fibers are topological n-manifolds. Then X is a topological $(n+1)$-manifold.*

4.4. Fibrations and homology manifolds

Finally, we will use the following result proved by F. Raymond in [55, Theorem 1]. Relying on the local orientability of homology n-manifolds [12], Raymond's Theorem 1 can be slightly strengthened as explained in [55, pp. 52–53]. (The result will be used only for Euclidean balls Y.)

Theorem 4.10. *Let X be a homology n-manifold and let $F : X \to Y$ be a Hurewicz fibration. If Y is connected and locally contractible, then there exists some $k \le n$ such that any fiber of F is a homology $(n-k)$-manifold and Y is a homology k-manifold.*

5. Strainer maps

5.1. Basic properties

We recall the basic properties of strainer maps in GCBA spaces, a tool invented in [14] for Alexandrov space, and applied to GCBA and investigated in this context in [41]. We are not going to recall the exact definition but state instead the properties of strainer maps which will be used below.

Let O be a tiny ball of a GCBA space X. For any natural $k \ge 0$, and any $\delta > 0$ there is the family

$$\mathcal{F}_{k,\delta} = \mathcal{F}_{k,\delta}(O)$$

of the so-called (k,δ)-*strainer maps* $F : U \to \mathbb{R}^k$ defined on open subsets U of O with the following properties, [41, Sections 7–8].

(0) By convention, for $k = 0$, any $\delta > 0$ and any open $U \subset O$, we let the constant map $F : U \to \{0\} = \mathbb{R}^0$ be a $(0,\delta)$-strainer map.

(1) For any $F \in \mathcal{F}_{k,\delta}(O)$, the coordinates f_i of F are distance functions to some points $p_1, \dots, p_k \in O$.

(2) For $\delta_1 > \delta_2$, we have the inclusion

$$\mathcal{F}_{k,\delta_2}(O) \subset \mathcal{F}_{k,\delta_1}(O).$$

(3) For any $F \in \mathcal{F}_{k,\delta}(O)$ and $l < k$, the first l coordinate functions of $F : U \to \mathbb{R}^k$ define a map $\tilde{F} : U \to \mathbb{R}^l$ contained in $\mathcal{F}_{l,\delta}(O)$.

(4) The restriction of any (k,δ)-strainer map to any open subset is a (k,δ)-strainer map.

5.2. *Extension properties*

All extendability properties of strainer maps and the "largeness" of the sets $\mathcal{F}_{k,\delta}(O)$ depend on the following:

(5) For any $x \in X$ and any $\delta > 0$, there exists some $r > 0$ such that $d_x : B_r^*(x) \to \mathbb{R}$ is contained in $\mathcal{F}_{1,\delta}$, [41, Proposition 7.3].

This result has the following generalization, [41, Proposition 9.4]:

(6) Let $F : U \to \mathbb{R}^k$ be a map in $\mathcal{F}_{k,\delta}$. Let $x \in U$ be a point and let Π be the fiber $F^{-1}(F(x))$. Then there is $r > 0$ and an open set $V \subset U$ containing $B_r^*(x) \cap \Pi$ such that the map $\hat{F} = (F, f) : V \to \mathbb{R}^{k+1}$ with last coordinate $f = d_x$ is contained in $\mathcal{F}_{k+1,12\cdot\delta}$.

This property (6) is the "fiber-wise" statement of the following closely related result, contained in [41, Theorem 10.5] in a stronger form:

(7) Let $F : U \to \mathbb{R}^k$ be in $\mathcal{F}_{k,\delta}$. Consider the set K of points $x \in U$ at which F can not be locally extended to a $(k+1, 12\cdot\delta)$-map $\hat{F} = (F, f) : U_x \to \mathbb{R}^{k+1}$. Then the closed set K intersects any fiber of F in U in a finite set of points.

5.3. *Topological properties*

The following property is contained in [41, Theorem 1.10]:

(8) Let $F : U \to \mathbb{R}^k$ be a map in $\mathcal{F}_{k,\delta}$ with $\delta < \frac{1}{20\cdot k}$. Then the map F is open. Moreover, for any compact subset K of U, there exists some $\epsilon > 0$ such that for all $r < \epsilon$ and all $x \in K$ the intersection $B_r(x) \cap F^{-1}(F(x))$ is contractible.

Now we easily derive:

Theorem 5.1. *Let U be an open subset of a GCBA space X, and let $F : U \to \mathbb{R}^k$ be a (k, δ)-strainer map, for some k and any $\delta < \frac{1}{20\cdot k}$. Then any $x \in U$ has arbitrary small open contractible neighborhoods V such that the restriction $F : V \to F(V)$ is a Hurewicz fibration with contractible fibers. If a fiber $F^{-1}(b)$ is compact, there exists an open neighborhood V of $F^{-1}(b)$ in U such that $F : V \to F(V)$ is a Hurewicz fibration.*

Proof. By property (8) of strainer maps, the map F is open and it has locally uniformly contractible fibers.

Let $x \in U$ be arbitrary. Using Theorem 4.4, we find a neighborhood W of $F(x)$ in $\mathbb{R}^k$ and a continuous section $s : W \to U$ such that $F \circ s$ is the identity. Making W smaller, if needed, we may assume that W is an open ball and $s(W)$ is contained in a compact subset $K \subset U$.

Take a positive number ϵ provided by property (8). Making ϵ smaller, we may assume that the distance from K to the boundary of U in X is larger than ϵ. Consider the set $V \subset U$ (the union of balls-in-the-fiber of radius ϵ) of all $z \in F^{-1}(W)$ such that $d(z, s(F(z))) < \epsilon$.

Then V is open in U and contains x. We have $F(V) = W$ and every fiber $F^{-1}(\mathrm{t}) \cap V$ of F in V is a contractible. By Theorem 4.6, $F : V \to F(V)$ is a Hurewicz fibration.

Since $W = F(V)$ and the fibers of the Hurewicz fibration $F : V \to W$ are contractible, V is contractible as well.

Let now $F^{-1}(b)$ be a compact fiber of F in U. Take a compact neighborhood V_0 of $F^{-1}(b)$ in U. Let C be its boundary ∂V_0. Consider a closed ball B around b which is contained in the neighborhood $F(V_0)$ of b but does not intersect the compact image $F(C)$. Let V_1 be the intersection $V_0 \cap F^{-1}(B)$.

Since $F^{-1}(B)$ does not intersect C, the set V_1 is compact. The restriction $F : V_1 \to B$ has locally uniformly contractible fibers. Applying Theorem 4.5, we deduce that the map $F : V_1 \to B$ is a Hurewicz fibration. If we take W to be any open ball around b contained in B and let V be the preimage $F^{-1}(B) \cap V_1$, then $F : V \to B$ is a Hurewicz fibration as well. This finishes the proof. ∎

Since being a homology k-manifold is a local property, we directly deduce from Theorem 5.1 and Theorem 4.10:

Corollary 5.2. *Let $F : U \to \mathbb{R}^k$ be a (k, δ)-strainer map with $\delta < \frac{1}{20 \cdot k}$ defined on an open subset of a GCBA space X. If U is a homology n-manifold, then any non-empty fiber Π of F is a homology $(n-k)$-manifold.*

6. Topological regularity

6.1. Disjoint disk property

A metric space M has the *disjoint disk property* if for any two continuous maps

$$\varphi_i : D \to M, \quad i = 1, 2,$$

on the unit disk D and for any $\epsilon > 0$, there are two continuous maps

$$\widetilde{\varphi}_i : D \to M$$

such that

$$d(\varphi_i, \widetilde{\varphi}_i) \leq \epsilon \quad \text{and} \quad \widetilde{\varphi}_1(D) \cap \widetilde{\varphi}_2(D) = \emptyset.$$

For a homology n-manifold Y we denote by $\mathcal{M}(Y)$ the set of manifold points in Y, thus of all points in Y with a neighborhood homeomorphic to $\mathbb{R}^n$. We recall the following special case of the celebrated manifold recognition theorem of Edwards and Quinn, [44, Theorem 2.7]:

Theorem 6.1. *Let the connected metric space Y be an* ANR *and a homology n-manifold with $n \geq 5$. Then Y is a topological manifold if and only if the set of manifold points $\mathcal{M}(Y)$ is not empty and Y has the disjoint disk property.*

For $n \geq 5$ the next result easily follows from Theorem 6.1 and is a very special case of the main theorem of [18]. For $n = 4$, the next result is a very special case of the main theorem of [11].

Theorem 6.2. *Let Y be an* ANR *and a homology n-manifold with $n \geq 4$. Let $K \subset Y$ be a discrete set of points such that $Y \setminus K$ is a topological n-manifold. If every point $x \in K$ has arbitrary small neighborhoods U in Y such that $U \setminus \{x\}$ is simply connected, then Y is a topological n-manifold.*

6.2. Structure of GCBA homology manifolds

We are going to formulate and prove the main technical result.

Theorem 6.3. *For natural numbers k and n with $0 \leq k \leq n$, let $U \subset X$ be an open subset of a GCBA space X. Assume that U is a homology n-manifold. Let $F : U \to \mathbb{R}^{n-k}$ be an $(n-k, \delta)$-strainer map and let Π be a fiber of the map F. Let $E \subset \Pi$ be the set of points at which F does not have a local extension to an $(n-k+1, 12 \cdot \delta)$-strainer map $\hat{F} = (F, f)$.*

Assume finally that $\delta < 20^{-n+k-1}$. Then the set E is finite and the complement $\Pi \setminus E$ is a topological k-manifold. Moreover, if $k \leq 3$, then Π is a topological k-manifold.

Proof. By our assumption,

$$\delta < \frac{1}{20 \cdot (n-k)} \quad \text{and} \quad 12 \cdot \delta < \frac{1}{20 \cdot (n-k+1)}.$$

Thus, Corollary 5.2 and Theorem 5.1 apply to F and the extensions of F provided by Section 5.2. Hence, Π is a homology k-manifold by Corollary 5.2. Due to Section 5.2, the set $E \subset \Pi$ is finite.

We fix n and proceed by induction on k. For $k \leq 2$, we deduce from Theorem 2.4 that Π is a topological k-manifold.

Assume $k = 3$. Let $x \in \Pi$ be arbitrary. By Section 5.2, we find some $r > 0$ such that the ball $\bar{B}_r(x) \subset U$ is compact and has the following property. There exists an open set $V \subset X$ containing $B_r^*(x) \cap \Pi$ such that the map $\hat{F} = (F, f) : V \to \mathbb{R}^{n-k+1}$ is an $(n-k+1, 12 \cdot \delta)$-strainer map, where f is the distance function $f = d_x$.

The fibers of the map $\hat{F}$ through points $z \in B_r^*(x) \cap \Pi$ are compact distance spheres $\Pi_t := S_t(x) \cap \Pi$ around x in Π. By Theorem 5.1 the restriction of $\hat{F}$ to a neighborhood of any such fiber Π_t is a Hurewicz fibration. Hence, the restriction of f to a neighborhood of Π_t in Π is a Hurewicz fibration, for any $0 < t < r$. Therefore, $f : B_r^*(x) \cap \Pi \to (0, r)$ is a Hurewicz fibration.

By the already verified case $k = 2$, the fibers of $\hat{F}$ (hence of f) are topological 2-manifolds. Due to Theorem 4.8, the Hurewicz fibration f must be a fiber bundle. Since the base of the bundle is a contractible interval, the bundle must be trivial. Thus, $B_r^*(x) \cap \Pi$ is homeomorphic to $(0, r) \times M$ for a topological 2-manifold M.

From the uniqueness of one-point compactifications, we see that $B_r(x) \cap \Pi$ is homeomorphic to the cone CM over M. Since Π is a homology 3-manifold, M must have the homology of $\mathbb{S}^2$. Therefore, M is homeomorphic to $\mathbb{S}^2$ and $B_r(x) \cap \Pi$ is homeomorphic to $\mathbb{R}^3$. Since the point x was arbitrary, Π is a topological 3-manifold.

Assume $k = 4$. For any $x \in \Pi \setminus E$, there exists a neighborhood U_x of x in X and an extension of F to an $(n-k+1, 12 \cdot \delta)$-strainer map $\hat{F} = (F, f) : U_x \to \mathbb{R}^{n-k+1}$. Applying the case $k = 3$, we know that the fibers of $\hat{F}$ are topological 3-manifolds. Due to Theorem 5.1, we may restrict to a smaller neighborhood of x and assume that $\hat{F} : U_x \to \hat{F}(U_x)$ is a Hurewicz fibration. Then so is the restriction $\hat{F} : \Pi \cap U_x \to \hat{F}(\Pi \cap U_x)$, which is nothing else but the last coordinate f. Applying Theorem 4.9, we see that $\Pi \cap U_x$ is a 4-manifold. Since $x \in \Pi \setminus E$ was arbitrary, this finishes the proof for $k = 4$.

Assume now $k \geq 5$ and that the claim is true for $k - 1$. We consider an arbitrary point $x \in \Pi \setminus E$, a neighborhood U_x of x and an extension of F to an $(n-k+1, 12 \cdot \delta)$-strainer map $\hat{F} = (F, f)$ as before. Making U_x smaller, we may assume by Theorem 5.1, that the restriction $\hat{F} : U_x \to \hat{F}(U_x)$ is a Hurewicz fibration with contractible fibers.

Consider the intersection $W := \Pi \cap U_x$ and the restriction $f : W \to f(W) \subset \mathbb{R}$ which is a Hurewicz fibration with contractible fibers. By making U_x smaller (if needed), we may and will assume that $f(W)$ is an open interval $J \subset \mathbb{R}$. In this setting we will prove that W is a topological k-manifold.

For $t \in J$ we let W_t the preimage $f^{-1}(t) \subset W$, which is a contractible fiber of the strainer map $\hat{F} : U_x \to \mathbb{R}^{n-k+1}$.

Let K_1 be the closed subset of points of U_x at which $\hat{F}$ does not (locally) extend to an $(n-k+2, (12)^2 \cdot \delta)$-strainer map. By the inductive assumption, the intersection of any fiber of $\hat{F}$ with $U_x \setminus K_1$ is a topological $(k-1)$-manifold. Applying Theorem 4.9 to the Hurewicz fibration $f : W \to J$, we deduce that $W \setminus K_1$ is a topological k-manifold.

Consider the set of manifold points $\mathcal{M}(W)$ and its complement $K_0 := W \setminus \mathcal{M}(W)$, the set of non-manifold points in W. We have just shown that K_0 is contained in K_1. We assume that K_0 is not empty, and we are going to derive a contradiction.

By the inductive assumption, the set K_1 intersects every fiber of $\hat{F}$ only in finitely many points. Hence, for any $t \in J$ the intersection $W_t \cap K_0$ is finite.

The Hurewicz fibration $f : W \to J$ has contractible base and fibers, hence W is contractible, in particular, it is connected. The set $W \setminus K_0$ is not empty, as we have seen. Due to Theorem 6.1, it suffices to prove that W satisfies the disjoint disk property, in order to conclude that W is a topological k-manifold and to achieve a contradiction.

The verification of the disjoint disk property occupies the rest of the proof and happens in several steps.

Step 1. *For any map $\gamma : \mathbb{S}^1 \to W$ and any $\epsilon > 0$ there exists a map $\hat{\gamma} : \mathbb{S}^1 \to W$ with $d(\gamma, \hat{\gamma}) < \epsilon$ such that $f \circ \hat{\gamma}$ is piecewise monotone.*

Indeed, we easily find a homotopy of the map $\eta_0 := f \circ \gamma : \mathbb{S}^1 \to J$ through maps η_t such that each η_t for $t > 0$ is piecewise linear. Using that f is a Hurewicz fibration we can lift η_t to a homotopy of $\gamma = \gamma_0$. Then we find the required map $\hat{\gamma}$ as γ_t for a small t.

Step 2. *The set $\mathcal{M}(W) = W \setminus K_0$ is connected.*

Indeed, for any $t \in J$, the fiber W_t is a connected homology $(k-1)$-manifold. Since $W_t \cap K_0$ is discrete, it follows that the complement $W_t \setminus K_0$ is not empty and connected, see [21, Lemma 2.1]. For any connected component W' of $\mathcal{M}(W)$ we deduce

$$W' = f^{-1}(f(W')) \cap \mathcal{M}(W).$$

Since J is connected, this implies $f(W') = J$ and $W' = \mathcal{M}(W)$.

Step 3. *For any $y \in W$, the complement $W \setminus \{y\}$ is simply connected.*

Indeed, consider an arbitrary curve $\gamma : \mathbb{S}^1 \to W \setminus \{y\}$. In order to fill γ by a disk, we use the local contractibility of W and Step 1, and may assume that $\eta = f \circ \gamma$ is piecewise monotone. If the image of η does not contain $t_0 := f(y)$, then γ lies in the contractible set $f^{-1}(\eta(\mathbb{S}^1))$ (which does not contain the point y) and the statement is clear.

If the image of η contains t_0, we can write η as a concatenation of finitely many curves η_i based in t_0, each of them completely contained either in $[t_0, \infty)$ or in $(-\infty, t_0]$. The corresponding decomposition of $\mathbb{S}^1$ decomposes γ in a finite concatenation of possibly non-closed curves γ_i each of them ending and starting on W_{t_0}.

The homology $(k-1)$-manifold W_{t_0} is connected, hence so is $W_{t_0} \setminus \{y\}$. Therefore, we can connect the endpoints of each γ_i in W_{t_0}.

Concatenating these "connection curves" with γ, we obtain a closed curve $\hat{\gamma}$, homotopy equivalent to γ in $W \setminus \{y\}$. Moreover, $\hat{\gamma}$ is a concatenation of finitely many closed curves $\tilde{\gamma}$, such that $f \circ \tilde{\gamma}$ is contained either in $(-\infty, t_0]$ or $[t_0, \infty)$.

For any such curve $\tilde{\gamma}$ we can now fill $f \circ \tilde{\gamma}$ in J by a disk none of whose interior point is sent to t_0. Using the homotopy lifting property, we can lift this disk to a filling of $\tilde{\gamma}$ in $W \setminus \{y\}$. Thus, any of the curves $\tilde{\gamma}$ and hence γ are contractible in $W \setminus \{y\}$.

Step 4. *For any curve $\gamma : \mathbb{S}^1 \to W \setminus K_0$, there exists an extension of γ to a disk*

$$\phi : D \to W$$

intersecting K_0 only in finitely many points.

Indeed, arguing as in Step 3, we can assume that $f \circ \gamma$ is piecewise monotone. Subdividing $f \circ \gamma$ and using connection curves in single fibers of f, as in the previous step, we reduce the question to the case that $f \circ \gamma$ is the concatenation of two monotone curves. Reparametrizing γ we can assume that γ is parametrized on an interval $[-a, a]$ such that $f \circ \gamma(q) = f \circ \gamma(-q)$ for all $q \in [0, a]$.

For any $q \in [0, a]$ we choose any curve γ_q in $W_{f(\gamma(q))} \setminus K_0$ connecting the points $\gamma(-q)$ and $\gamma(q)$. Let Q denote the set of numbers $q \in [0, a]$ such that the concatenation of γ_q and $\gamma|_{[-q,q]}$ can be filled by a disk in W intersecting only finitely many points in K_0.

Clearly Q contains 0. We are done if Q contains a. Using a connectedness argument it suffices to prove that for any q_0 there exists some $\epsilon > 0$ such that for any q with $|q - q_0| < \epsilon$ the concatenation γ_{q,q_0} of γ_q, γ_{q_0} and the parts of γ between $\pm q$ and $\pm q_0$ can be filled in W by a disk intersecting K_0 only in a finite number of points.

We fix $q_0 \in J$.

Since the Hurewicz fibration $f : W \to J$ has contractible fibers, we can find a continuous family $P_s, s \in J$, of homotopy retractions $P_s : W \times [0,1]$ from W to W_s. Indeed, the map f satisfies the homotopy extension property for every pair of finite-dimensional spaces, see [43, Theorem 1.2]. Thus, we can extend a continuous map

$$P : W \times J \times [0,1] \to W$$

such that $P(w, f(w), t) = P(w, s, 0) = w$ for all $w \in W, t \in [0,1]$ and $s \in J$ and such that $f \circ P(w, s, t) = (1-t) \cdot f(w) + t \cdot s$ for all $(w, s, t) \in W \times J \times [0,1]$.

By continuity we find some $\epsilon > 0$ such that for all $q \in [0, a]$ with $|q - q_0| < \epsilon$ the homotopy retraction $P_{f(\gamma(q))}$ from W onto the fiber $W_{f(\gamma(q))}$ has the following property: The trace under this homotopy retraction of γ_{q_0} and both parts of γ between $\pm q_0$ and $\pm q$ do not intersect K_0.

Therefore, the homotopy retraction $P_{f(\gamma(q))}$ defines a homotopy (not intersecting K_0) of the curve γ_{q,q_0} to a closed curve c completely contained in the fiber $W_{f(\gamma(q))}$. Filling the curve c inside the contractible fiber $W_{f(\gamma(q))}$ by any disk, we obtain the required filling of the curve γ_{q,q_0}. This finishes the proof of Step 4.

Step 5. *For all $z \in W$ and all $\epsilon > 0$ there exists an open contractible neighborhood V^z of z in W with diameter smaller than ϵ such that the restriction $f : V^z \to f(V^z)$ is a Hurewicz fibration with contractible fibers.*

Indeed, this follows from Theorem 5.1 in the same way as in the construction of W.

Step 6. *The conclusions of Steps 3 and 4 are valid for all neighborhoods V^z constructed in Step 5.*

Indeed, the proofs of the respective steps apply literally.

Step 7. *For every disk $\phi : D \to W$ and every $\epsilon > 0$, there exists a disk $\phi_\epsilon : D \to W$ with pointwise distance to ϕ at most ϵ and such that the image of ϕ_ϵ meets K_0 only in a finite set.*

Indeed, we consider a covering of the $\phi(D)$ by the sets V^z described above each of them of diameter at most $\frac{\epsilon}{3}$. Using the Lemma of Lebesgue, we find a triangulation of the disk D by a finite graph Γ such that for any 2-simplex Δ of the triangulation, the image $\phi(\Delta)$ is contained in one of the sets V^z.

We slightly move the images of the vertices of Γ and use Step 2 and Step 6 in order to find a map $\phi_\epsilon : \Gamma \to U$ which does not meet K_0 and such that for any 2-simplex Δ of the triangulation Γ the images $\phi_\epsilon(\partial\Delta)$ and $\phi_2(\Delta)$ are contained in one set V^z. Applying Step 4 and Step 6, we can extend ϕ_ϵ from the boundary $\partial\Delta$ of any 2-simplex Δ such that this extension lies inside the same open set V^z and intersects K_0 only in a finite set of points. Taking all these extensions together, we obtain the required disk ϕ_ϵ.

Step 8. *The disjoint disk property holds in W.*

Thus, let $\phi_1, \phi_2 : D \to W$ and $\epsilon > 0$ be given. Apply the previous Step 4 and obtain a map $\tilde{\phi}_1 : D \to W$ with distance at most $\frac{\epsilon}{2}$ to ϕ_1, whose image intersects K_0 only in a finite set of points $Q = \{x_1, \ldots, x_l\}$.

We find a covering of the compact image $\phi_2(D)$ by finitely many open neighborhoods V^z as above of diameter smaller than $\frac{\epsilon}{2}$, such that any subset V^z contains at most one of the points x_i.

We find a triangulation of the disk D by a finite graph Γ, such that for any 2-simplex Δ of the triangulation, the image $\phi_2(\Delta)$ is contained in one of these sets V^z. Arguing as in the previous Step 7 (applying Step 2), we find a map $\tilde{\phi}_2 : \Gamma \to W$ which does not meet K_0 and such that for any 2-simplex Δ of the triangulation the image $\phi_2(\partial\Delta)$ is contained in one of the sets V^z.

By Step 3 and Step 6, for any of our sets V^z, the complement $V^z \setminus Q$ is simply connected. Therefore, we can extend $\tilde{\phi}_2 : \Gamma \to W$ to a map $\tilde{\phi}_2 : D \to W \setminus Q$ such that $\tilde{\phi}_2(\Delta)$ and $\phi_2(\Delta)$ are in the same set V^z of our covering.

By construction, the intersection $\tilde{\phi}_2(D) \cap \tilde{\phi}_1(D)$ is contained in the set of manifold points $U \setminus K_0$. Since in the n-manifold $U \setminus K$ the disjoint disk property holds true, we can slightly perturb $\tilde{\phi}_2$ and $\tilde{\phi}_1$ (outside of K), so that the arising disks do not intersect.

This finishes the proof of Step 8 and therefore of the theorem. ∎

6.3. *Main theorems*

We now finish the proof of the main theorems.

Proof of Theorem 1.2. Let X be a metric space with an upper curvature bound, which is a homology n-manifold. By Lemma 3.1, X is a GCBA space. We cover X by tiny balls O, and apply Theorem 6.3 in the case $k = n$ and the constant map $F : O \to \mathbb{R}^0 = \{0\}$. We deduce that X is a topological manifold outside a discrete set of points. ∎

Proof of Theorem 1.1. We have seen in Lemma 3.5 and Lemma 3.7 that (1) implies (3) and that (2) implies (3).

Assume now that (3) holds, thus all spaces of directions are homotopy equivalent to a non-contractible space. We have seen in Lemma 3.6 that X must be a homology n-manifold and all spaces of directions are homotopy equivalent to $\mathbb{S}^{n-1}$.

By Theorem 6.3, X is a topological manifold if $\dim(X) \leq 3$.

Let the dimension of X be at least 4. Then all Σ_x are simply connected, hence so are all small punctured balls $B_r^*(x)$. The result that X is a topological manifold follows now directly as a combination of Theorem 1.2 and Theorem 6.2.

Thus (3) implies (1).

It remains to prove that (1) implies (2). Assuming that X is a topological n-manifold let $x \in X$ be arbitrary. We deduce from Corollary 3.4 that any space of directions Σ_x is a homology $(n-1)$-manifold and any tangent space $T_x = C(\Sigma_x)$ is a homology n-manifold. Moreover, any space of direction is homotopy equivalent to $\mathbb{S}^{n-1}$ by Lemma 3.5.

If $n \leq 3$, then Σ_x is a topological manifold, Theorem 6.3, homeomorphic to $\mathbb{S}^{n-1}$ by Lemma 3.5. Thus, T_x is homeomorphic to $\mathbb{R}^n$.

Assume $n \geq 4$. By Theorem 1.2, the set of non-manifold points of T_x is discrete. Due to the conical structure, this directly implies that $T_x \setminus \{0\}$ is a topological manifold. But Σ_x and, therefore, all punctured balls around 0 in T_x are simply connected. From Theorem 6.2 we deduce that T_x is a topological n-manifold.

Thus, T_x is a contractible n-manifold, simply connected at infinity, since Σ_x is simply connected. Therefore, T_x is homeomorphic to $\mathbb{R}^n$. ∎

6.4. *Some improvements*

Theorem 1.1 can be slightly strengthened in dimensions ≤ 4. The first of these results is contained in [58].

Theorem 6.4. *Let X be a locally compact space with an upper curvature bound. If X is a homology n-manifold with $n \leq 3$, then X is a topological manifold. If X is a topological n-manifold with $n \leq 4$, then any space of directions Σ_x in X is homeomorphic to the sphere $\mathbb{S}^{n-1}$.*

Proof. The first statement is local and is therefore contained (as the case $k = n = 3$) in Theorem 6.3.

To prove the second statement, we use Lemma 3.4 and Lemma 3.5 to deduce that Σ_x is a homology $(n-1)$-manifold, homotopy equivalent to $\mathbb{S}^{n-1}$. By the first statement and the resolution of the Poincaré conjecture, Σ_x is homeomorphic to $\mathbb{S}^{n-1}$. ∎

Theorem 6.3 and, therefore, Theorem 1.1 can be strengthened as follows. Since the result is not used in the sequel, the proof will be somewhat sketchy. For definitions and fundamental results about ends of manifolds we refer to [56] and [34].

Theorem 6.5. *Under the assumptions of Theorem 6.3, for any point $x \in \Pi$ there exists a neighborhood of x in Π homeomorphic to the open cone $C(M)$ over a topological $(k-1)$-manifold M, with the homology of $\mathbb{S}^{k-1}$.*

Proof. We proceed by induction on k.

Cases $k \leq 3$. In this cases the statement is clear, since Π is a topological manifold.

Case $k = 4$. One argues in the same way as in the case $k = 3$ in the proof of Theorem 6.3. Using that the fibers of the Hurewicz fibration $f : B_r^*(x) \cap \Pi \to (0, r)$ are topological 3-manifolds, one concludes that $B_r^*(x) \cap \Pi$ is homeomorphic to $(0, r) \times M$ for a topological 3-manifold M. Thus $B_r(x) \cap \Pi$ is homeomorphic to the cone $C(M)$. Since Π is a homology 4-manifold, M must be a homology 3-sphere.

Cases $k \geq 5$. Find a small number $r > 0$ such that the $(n-k, \delta)$-strainer map F extends to an $(n-k+1, 12 \cdot \delta)$-strainer map $\hat{F} = (F, f)$ on a neighborhood V of $N := B_r^*(x) \cap \Pi$ in X. Here f denotes as before, the distance function $f = d_x$. By Theorem 6.3, N is a topological k-manifold and, as we have seen in the proof of Theorem 6.3, the map $f : N \to (0, r)$ is a Hurewicz fibration.

We claim that the end of the manifold N corresponding to the point x is *collared*. Thus, N contains a subset homeomorphic to $M \times [0, \infty)$, whose closure in Π contains a neighborhood of x, for some manifold M. Since Π is a homology manifold, this would imply that M must have the homology of $\mathbb{S}^{k-1}$ and finish the proof of the theorem.

For $k \geq 6$ the statement is a direct consequence of Siebenmann's theorem on collared ends, [56], [34, Theorem 10.2], and the observation that N homotopically retracts onto any compact fiber of f.

The following elegant argument due to Steven Ferry covers the case $k = 5$ as well as the case $k \geq 6$.

Fix a fiber $\Pi_t = f^{-1}(t)$ for some t. By induction, Π_t is a homology $(k-1)$-manifold with a finite set K of singularities (each of whom has a neighborhood in Π_t homeomorphic to a cone). By the main result of [53], there exists a *resolution* $g : M \to \Pi_t$ which is a homeomorphism outside the preimages $g^{-1}(K)$.

Consider the space N^+ obtained by gluing the cylinder $M \times (-1, 1]$ to $f^{-1}(0, t) \subset N$ by identifying $M \times \{1\}$ with Π_t along the map g. The space N^+ is by construction a topological manifold outside the finitely many singularities in $K \subset \Pi_t$. Computing the local homology at points in K, we see that N^+ is a homology k-manifold. Finally, arguing as in the proof of Theorem 6.3, we see that all points in K have arbitrary small simply connected punctured neighborhoods in N^+. Applying Theorem 6.2, we conclude that N^+ is a topological k-manifold.

We apply the main theorem from [57] and see that the topological k-manifold with boundary $N^+ \setminus M \times (-1, 0)$ (the boundary is $M \times \{0\}$) is homeomorphic to $M \times [0, \infty)$. Thus the end of the manifold N is collared. ■

7. Limits of manifolds

7.1. Topological stability

We start with a part of Theorem 1.3:

Theorem 7.1. *Let a sequence of complete* CAT(κ) *Riemannian manifolds M_i of dimension n converge in the pointed Gromov–Hausdorff topology to a space X. Then X is a topological n-manifold.*

Proof. Note that X is a GCBA space, [41, Example 4.3]. Due to Theorem 1.1, it suffices to prove that for all $x \in X$ the space of directions Σ_x is homotopy equivalent to $\mathbb{S}^{n-1}$. Due to Proposition 2.3, it suffices to prove that for all r small enough, the distance sphere $S_r(x)$ is homotopy equivalent to $\mathbb{S}^{n-1}$.

Fix a sequence $x_i \in X_i$ converging to x. For all r small enough, the injectivity radius of the Riemannian manifold X_i is larger than r, hence the distance sphere $S_r(x_i)$ is homeomorphic to $\mathbb{S}^{n-1}$. According to Proposition 2.3, the spheres $S_r(x_i)$ are homotopy equivalent to $S_r(x)$, for all i large enough. This proves the claim. ■

The α-approximation theorem (Theorem 4.7) and Petersen's stability theorem (Theorems 4.2) give us:

Corollary 7.2. *Under the assumptions of Theorem 7.1, assume in addition that X is compact. Then M_i is homeomorphic to X, for all i large enough.*

7.2. Iterated spaces of directions

In order to prove the remaining statement in Theorem 1.3, we need to understand spaces of directions of spaces of directions. For a GCBA space X, we call any space of directions $\Sigma_x X$ of X a *first order space of directions* of X. Inductively we define a *k-th iterated space of directions of* X to be a space of directions $\Sigma_z Y$ of a $(k-1)$-st iterated space of directions Y of X.

Using Theorem 1.1, we can easily derive the following lemma, clarifying the second statement in Theorem 1.3.

Lemma 7.3. *Let X be a locally compact space with an upper curvature bound. Then the following are equivalent:*

(1) *X is a topological manifold and, for any $1 \le k < n$, all k-th iterated spaces of directions of X are homeomorphic to $\mathbb{S}^{n-k}$.*

(2) *For any $1 \le k < n$, all k-th iterated spaces of directions of X are homotopy equivalent to $\mathbb{S}^{n-k}$.*

Proof. Clearly (1) implies (2).

Assuming (2), we deduce from Theorem 1.1 that X is a topological manifold. Thus, X is a GCBA space and all of its iterated spaces of directions are compact CAT(1) spaces. Let Σ be a k-th iterated space of directions of X. By assumption, all of its spaces of directions are homotopy equivalent to $\mathbb{S}^{n-k-1}$. By Theorem 1.1, the space Σ is a topological manifold. Due to the resolution of the (generalized) Poincaré conjecture, Σ is homeomorphic to $\mathbb{S}^{n-k}$. ∎

Iterated spaces of directions can be seen in factors of blow-ups of the original space. More generally, we have:

Lemma 7.4. *Let X_i be complete GCBA spaces which are* CAT(κ) *for a fixed κ, and assume that (X_i, x_i) converge in the pointed Gromov–Hausdorff topology to a GCBA space (X, x). Then, for any non-empty k-th iterated space of directions Σ^k of X, there exist a sequence of points $z_i \in X_i$ and a sequence $t_i \ge 1$ such that, possibly after choosing a subsequence, we have the following convergence in the pointed Gromov–Hausdorff topology:*

$$(\mathbb{R}^{k-1} \times C\,\Sigma^k, 0) = \lim_{i\to\infty} (t_i \cdot X_i, z_i).$$

Proof. Consider the set $\mathcal{L}$ of (isometry classes of) all pointed locally compact spaces (Y, y) which can be obtained as a pointed Gromov–Hausdorff limit of a subsequence of a sequence $(t_i \cdot X_i, y_i)$, for some $y_i \in X_i$ and some sequence $t_i \ge 1$.

The set $\mathcal{L}$ consists of complete GCBA spaces, it contains the space (X, x). With any space (Y, y), the family $\mathcal{L}$ contains the space (Y, y'), for any $y' \in Y$. Thus, we may ignore the base points. Moreover, $\mathcal{L}$ is closed under rescaling with numbers $t \ge 1$ and under pointed Gromov–Hausdorff convergence. Thus, with every space Z the family $\mathcal{L}$ contains any of the tangent spaces $T_z Z$.

For any $k \ge 1$ and any non-empty k-th iterated space of directions Σ^k of X, we need to prove that $Z = \mathbb{R}^{k-1} \times C\,\Sigma^k$ is contained in $\mathcal{L}$.

We proceed by induction on k. The case $k = 1$, thus $C\,\Sigma^1$ being a tangent cone of X at some point is already verified.

Assume that we have already verified the claim for k. Let Σ^k be any k-th iterated space of directions of X and let $v \in \Sigma^k$ be an arbitrary point such that $\Sigma := \Sigma_v \Sigma^k$ is not empty.

By the inductive assumption, the space $Z = \mathbb{R}^{k-1} \times C\,\Sigma^k$ is contained in $\mathcal{L}$. Then also $T_v Z = \mathbb{R}^k \times C\,\Sigma$ is contained in $\mathcal{L}$. This verifies the claim for $k+1$ and finishes the proof of the lemma. ∎

7.3. Limits of Riemannian manifolds

We are in a position to formulate and to prove the following generalization of the remaining part of Theorem 1.3. Its proof relies on some stability properties of strainer maps.

Theorem 7.5. *Under the assumptions of Theorem 1.3, for any $k \leq n$, any k-th iterated space of directions of X is homeomorphic $\mathbb{S}^{n-k}$.*

Proof. It suffices to prove that, for all $k < n$, any k-th iterated space of directions Σ^k of X is homotopy equivalent to $\mathbb{S}^{n-k}$, Lemma 7.3.

Let us fix such $\Sigma = \Sigma^k$. Due to Corollary 7.4 we get (by rescaling the manifolds M_i) a sequence of pointed Riemannian manifolds (N_i, p_i) with the following properties. The manifold N_i is CAT(κ_i), with κ_i converging to 0, and the sequence (N_i, p_i) converges in the pointed Gromov–Hausdorff topology to $Y = (\mathbb{R}^{k-1} \times C\Sigma, 0)$.

We fix the standard coordinate vectors $e_1, \ldots, e_{k-1} \in \mathbb{R}^{k-1} \times \{0\} \subset Y$ and consider the map $F : Y \to \mathbb{R}^{k-1}$ whose coordinates $f_1, \ldots, f_{k-1}$ are the distance functions to the points e_j. The geodesics from e_j to 0 do not branch at 0. By the definition of strainer maps, [41, Sections 7, 8], we know that for every $\delta > 0$ there exists some $\epsilon > 0$ such that F is a $(k-1, \delta)$-strainer map in the ball B of radius ϵ around 0 in Y. We fix $\delta < \frac{1}{20 \cdot k^2}$ and ϵ as above.

We take $(k-1)$-tuples of points in N_i converging to $(e_1, \ldots, e_{k-1})$ and consider the correspondingly defined maps $F_i : N_i \to \mathbb{R}^{k-1}$ which converge to F. By the openness property of strainers the following holds true, [41, Lemma 7.8]: For all i large enough, the map F_i is a $(k-1, \delta)$-strainer in the ball B^i of radius ϵ around p_i.

Denote by Π the fiber of F in B through 0 and by Π_i the fiber of F_i in B^i through p_i. The fibers Π_i are locally uniformly contractible. More precisely, by [41, Lemma 7.11 and Theorem 9.1], there exists some $\epsilon_1 > 0$ with the following property: For all i large enough, any $q \in \Pi_i$ and any $r < \epsilon_1$ such that $\bar{B}_r(q) \cap \Pi_i$ is compact, this compact set is contractible.

Denote by $g : \Pi \to \mathbb{R}$ and by $g_i : \Pi_i \to \mathbb{R}$ the distance functions to the points 0 and p_i, respectively.

By the extension property of strainers, we may assume, after making ϵ smaller, that the map $\hat{F} = (F, g) : V \to \mathbb{R}^k$ is a $(k, 12 \cdot \delta)$-strainer map on an open neighborhood V of $B^*_\epsilon(0) \cap \Pi$ in Y. The fibers of the map $\hat{F}$ through points on Π are (compact) distance spheres in Π around 0.

From the homotopy stability of fibers of strainer maps [41, Theorem 13.1], for any $0 < r < \epsilon$ there exists some i_0 such that for all $i > i_0$ the following holds true: The distance sphere $S_r(p_i) \cap \Pi_i$ is compact and homotopy equivalent to the distance sphere $S_r(0) \cap \Pi$.

We subdivide the rest of the proof into six steps.

Step 1. *The manifold N_i has injectivity radius larger than 2, for all large i. The strainer map $F_i : B^i \to \mathbb{R}^{k-1}$ is smooth. The distance function $g_i : B^i \to \mathbb{R}$ is smooth outside p_i.*

Indeed, the first statement is a consequence of our assumption that N_i is CAT(κ_i) with κ_i converging to 0. The remaining statements follow from the first one.

Step 2. *The strainer map $F_i : B^i \to \mathbb{R}^{k-1}$ is a submersion. Thus Π_i is a smooth submanifold of N_i.*

Indeed, the strainer map F_i is a $2\sqrt{k}$-open map, see [41, Lemma 8.2]. In particular, the differential of F_i at any point of B^i is surjective. This implies the first and, therefore, the second claim.

Step 3. *There exists ϵ_0 with $0 < \epsilon_0 < \frac{\epsilon_1}{2}$ such that, for all i large enough, the map*

$$g_i : \Pi_i \cap B^*_{\epsilon_0}(p_i) \to \mathbb{R}$$

has at all points a gradient of norm between $\frac{1}{2}$ and 1, with respect to the intrinsic metric of Π_i.

Indeed, for any $x \in B^*_\epsilon(p_i)$ the gradient $\nabla_x g_i$ of the map $g_i : B^i \to \mathbb{R}$ is the unit velocity vector of the geodesic connecting p_i with x. Thus, for every $x \in \Pi_i \setminus \{p_i\}$ the gradient of g_i at x with respect to the induced metric of Π_i is the projection of $\nabla_x g_i$ to the tangent space $T_x \Pi_i$.

There exists some $\epsilon_0 > 0$ such that for all i large enough and all $x \in \Pi_i \cap B^*_{\epsilon_0}(p_i)$, the inequality

$$|DF_i(\nabla_x g_i)| \leq k \cdot 2 \cdot \delta < \frac{1}{10 \cdot k}$$

holds, due to [41, Lemmas 7.6, 7.10 and 7.11].

Since the differential of F_i at x is $2\sqrt{k}$-open, this implies that the projection of $\nabla_x g_i$ to the tangent space $T_x \Pi_i$ has norm at least $\frac{1}{2}$.

Step 4. *In the notations above, for all i large enough and all $0 < r < \epsilon_0$, the distance sphere $S_r(p_i) = g_i^{-1}(r) \subset \Pi_i$ is diffeomorphic to $\mathbb{S}^{n-k}$. Moreover, $S_r(p_i)$ is locally uniformly contractible with respect to the contractibility function $\rho : [0, r) \to \mathbb{R}$ given by $\rho(s) = 2s$.*

Indeed, for all sufficiently small r (depending on i), the fact that $S_r(p_i)$ is diffeomorphic to a sphere is true for every smooth submanifold of a smooth Riemannian manifold, as easily seen in local coordinates.

The fact that for all $r < \epsilon_0$ the level sets of g_i are diffeomorphic among each other is a consequence of the (easy part) of Morse theory, since g_i has no critical points in Π_i.

Finally, the gradient flow of the function g_i on Π_i retracts $\Pi_i \setminus \{p_i\}$ onto $S_r(p_i)$. Moreover, along this retraction, any point moves with velocity less than 1 and the distance to $S_r(p_i)$ decreases with velocity at least $\frac{1}{2}$. Thus, for any point $q \in S_r(p_i)$ and any $s < r$ the retraction sends the ball $\bar{B}_s(q) \cap \Pi_i$ into the ball $\bar{B}_{2s}(q) \cap S_r(p_i)$.

Since the ball $\bar{B}_s(q) \cap \Pi_i$ is contractible, we deduce that the ball $\bar{B}_s(q) \cap S_r(p_i)$ is contractible inside the ball $\bar{B}_{2s}(q) \cap S_r(p_i)$. Finishing the proof of Step 4.

Step 5. *For every $r < \epsilon_0$ the distance sphere $S_r(0) \cap \Pi$ is homotopy equivalent to $\mathbb{S}^{n-k}$. Moreover, $S_r(0) \cap \Pi$ is uniformly locally contractible with respect to the contractibility function $\rho : [0, r) \to \mathbb{R}$ given by $\rho(s) = 2s$.*

Indeed, the sets $S_r(p_i)$ converge to $S_r(p)$ in the Gromov–Hausdorff topology. Hence the result follows from [50, Section 5].

Step 6. *The space Y is a cone, hence invariant under rescalings. However, the rescaled sequence $(m \cdot \Pi, 0) \subset Y$ converges to the factor $(C\Sigma, 0)$, for $m \to \infty$.*

Under this convergence the rescaled spheres $m \cdot (S_{\frac{1}{m}}(0) \cap \Pi)$ converge to Σ. All these spheres are homotopy equivalent to $\mathbb{S}^{n-k}$ and are uniformly locally contractible. Applying Theorem 4.2 once more, we deduce that Σ is homotopy equivalent to $\mathbb{S}^{n-k}$. ∎

8. A sphere theorem

8.1. Pure-dimensional spaces

In order to deduce Theorem 1.4 from Theorem 1.5, we need to show that a GCBA space all of whose spaces of directions are spheres (of a priori different dimensions) must be a manifold. We address this question in a slightly more general setting.

A GCBA space X is *purely n-dimensional* if all of its non-empty open subsets have dimension n. We say that X is *pure-dimensional* if X is purely n-dimensional for some n.

Due to [41, Corollary 11.6], a GCBA space X is purely n-dimensional if and only if all of its tangent spaces T_xX have dimension n. This happens if and only if all spaces of directions Σ_xX have dimension $n-1$. Using the stability of dimension under convergence proved in [41], we can show:

Proposition 8.1. *A connected GCBA space X is pure-dimensional if and only if all tangent spaces of X are pure-dimensional.*

Proof. Let X be purely n-dimensional and $x \in X$ arbitrary. Applying [41, Lemma 11.5] to the convergence of the rescaled balls in X around x to T_xX we deduce, that for any $v \in T_xX$ the dimension of $T_v(T_xX)$ is n. Thus, T_xX is purely n-dimensional.

Assume that all spaces of directions of X are pure-dimensional. By the connectedness of X it suffices to prove that every point $x \in X$ has a pure-dimensional open neighborhood. Therefore, we may replace X by a small ball around some of its points and assume that X is a geodesic space and has finite dimension n. Consider the set X^n of all points in X for which T_xX has dimension n. By [41, Corollary 11.6] the set X^n is closed in X.

Assume that X^n is not X. Then we find a point $y \in X \setminus X^n$ such that there exists a point $x \in X^n$ closest to y among all points of X^n. Consider the geodesic γ from x to y and set $x_i = \gamma(\frac{1}{i})$. Then the ball $B_{\frac{1}{i}}(x_i)$ does not intersect X^n, hence has dimension less than n. Under the convergence of $(i \cdot X, x)$ to T_xX, the closed balls $i \cdot \bar{B}_{\frac{1}{i}}(x_i)$ in X converge to the closed ball of radius 1 around the starting direction $v = \gamma'(0) \in \Sigma_x \subset T_x$.

Applying [41, Lemma 11.5] again, we see that the open ball $B_1(v)$ in T_x has dimension less than n. Since T_xX is n-dimensional, this contradicts the assumption that T_xX is pure-dimensional. This contradiction shows $X = X^n$ and finishes the proof. ∎

8.2. Conclusions

The proof of Theorem 1.5 relies on the following observation, well known to experts. We could not find a reference and include a short proof.

Lemma 8.2. *If a closed topological n-manifold M is covered by two contractible open subsets U, V, then M is homeomorphic to $\mathbb{S}^n$.*

Proof. By assumption, for any commutative ring R, the cup product of any two elements in the reduced cohomology $\tilde{H}^*(M, R)$ is 0, [33, Section 3.2, Exercise 2].

By Poincaré duality, M has the same cohomology with R-coefficients as $\mathbb{S}^n$ if M is R-orientable. Applying this for $R = \mathbb{Z}_2$, we deduce that $H_{n-1}(M, \mathbb{Z}_2) = 0$. Therefore, M is orientable with respect to integer coefficients, [33, Chapter 3, Corollary 3.28], and has the same integer homology and cohomology as $\mathbb{S}^n$.

By the theorem of Mayer–Vietoris, $0 = H_1(M) = H_0(U \cap V)$. Thus, $U \cap V$ is connected. Applying van Kampen's theorem, we deduce that M is simply connected. By the theorem of Whitehead, M is homotopy equivalent to $\mathbb{S}^n$. By the resolution of the (generalized) Poincaré conjecture, M is homeomorphic to $\mathbb{S}^n$. ■

Now we can finish:

Proof of Theorem 1.5. We proceed by induction on the (always finite) dimension of the compact GCBA space Σ. By assumption, Σ is CAT(1).

If the dimension of Σ is 0, then Σ is discrete and not a singleton. All points in Σ have distance at least π from each other. The assumption on the triples of points implies that Σ has exactly two points. Hence Σ is homeomorphic to $\mathbb{S}^0$.

Assume now that the statement is proven for all spaces of dimension less than n and let Σ be n-dimensional. Let $x \in \Sigma$ be a point. Then the space of directions Σ_x is a GCBA space of dimension less than n. If there exists a triple of points v_1, v_2, v_3 in Σ_x with pairwise distances at least π, then we consider geodesics γ_i in Σ starting in x in the directions of v_i (which exist by the geodesical completeness, see [41, Section 5.5]). Then the points $x_i = \gamma_i(\frac{\pi}{2})$ have in Σ pairwise distances π, in contradiction to our assumption.

Thus, by the inductive assumption, each space of directions Σ_x is homeomorphic to some sphere. Therefore, all spaces of directions Σ_x in Σ and hence all tangent spaces T_x are pure-dimensional. Due to Proposition 8.1, the space Σ must be purely n-dimensional. Then all spaces of directions Σ_x are $(n-1)$-dimensional, hence homeomorphic to $\mathbb{S}^{n-1}$ by the inductive hypothesis.

By Theorem 1.1, the space Σ is a topological n-manifold.

Consider a pair of points $x, y \in \Sigma$ at distance π. By the assumption on triple of points, there are no points $z \in \Sigma$ with distance at least π to x and y. Therefore, all of Σ is contained in $B_\pi(x) \cup B_\pi(y)$. By the CAT(1) assumption, both balls are contractible. Thus, Σ is homeomorphic to $\mathbb{S}^n$, by Lemma 8.2. ■

Proof of Theorem 1.4. Let X be a connected GCBA space that does not contain an isometrically embedded tree different from an interval.

Let $x \in X$ be a point. If there is a triple of points v_1, v_2, v_3 in Σ_x with pairwise distances at least π, then we obtain an isometrically embedded tree by taking the union of three short geodesics γ_i starting in the direction of v_i, in contradiction to our assumption. By Theorem 1.5, Σ_x must be homeomorphic to some sphere.

As in the first part of the proof of Theorem 1.5, we now deduce from Proposition 8.1 and Theorem 1.1 that X is a topological manifold. ∎

Finally, from Theorem 1.5 and the optimal lower bound on the volume of balls, [47, Proposition 6.1], we deduce:

Theorem 8.3. *Let Σ be a purely n-dimensional, compact, locally geodesically complete* CAT(1) *space. If $\mathcal{H}^n(\Sigma) < \frac{3}{2} \cdot \mathcal{H}^n(\mathbb{S}^n)$, then Σ is homeomorphic to $\mathbb{S}^n$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure.*

Proof. Otherwise, X contains a triple of points at pairwise distances at least π. The open balls of radius $\frac{\pi}{2}$ around these points are disjoint. Each of these balls has $\mathcal{H}^n$-measure not less than $\frac{1}{2} \cdot \mathcal{H}^n(\mathbb{S}^n)$. This contradicts the prescribed upper volume bound of X. ∎

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