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On restriction of unitarizable representations of general linear groups and the non-generic local Gan–Gross–Prasad conjecture

Abstract. We prove the first direction of a recently posed conjecture by Gan, Gross and Prasad, which predicts branching laws that govern restriction from p-adic GL_n to GL_{n-1} of irreducible smooth representations within the Arthur-type class. We extend this prediction to the full class of unitarizable representations, by exhibiting a combinatorial relation that must be satisfied for any pair of irreducible representations, in which one appears as a quotient of the restriction of the other. We settle the full conjecture for the cases in which either one of the representations in the pair is generic. The method of proof involves a transfer of the problem, using the Bernstein decomposition and the quantum affine Schur–Weyl duality, into the realm of quantum affine algebras. This restatement of the problem allows for an application of the combined power of a result of Hernandez on cyclic modules together with the Lapid–Mínguez criterion from the p-adic setting.

Keywords. Representation theory, branching laws, p-adic groups, quantum affine algebras

1. Introduction

Let π be a smooth irreducible representation of the group $\mathrm{GL}_n(F)$, where F is a p-adic field. Let us consider $\mathrm{GL}_{n-1}(F)$ as a subgroup of $\mathrm{GL}_n(F)$, embedded in a natural way, which in matrix form is described as

$$\begin{pmatrix} \mathrm{GL}_{n-1}(F) & 0_{n-1,1} \\ 0_{1,n-1} & 1 \end{pmatrix} < \mathrm{GL}_n(F).$$

We study the branching laws that govern the decomposition of the restricted representation $\pi|_{\mathrm{GL}_{n-1}(F)}$ into irreducible representations of $\mathrm{GL}_{n-1}(F)$. As an approachable goal, we focus on the description of the possible isomorphism classes of irreducible representations which appear as *quotients* of $\pi|_{\mathrm{GL}_{n-1}(F)}$.

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One fundamental result in that direction was achieved in [1], where it was shown that, for all $\pi \in \mathrm{Irr}\,\mathrm{GL}_n(F)$ and $\sigma \in \mathrm{Irr}\,\mathrm{GL}_{n-1}(F)$, $\mathrm{Hom}_{\mathrm{GL}_{n-1}(F)}(\pi, \sigma)$ is at most one-dimensional.

As a step forward, we are seeking for a meaningful description of the collection of pairs (π, σ), for which the latter morphism space is non-zero.

An irreducible representation is said to be generic, if it can be produced on a Whittaker model of functions. It is a classical fact proved in [29] through the study of L-functions, that for every pair of generic representations (π, σ) as above, $\mathrm{Hom}_{\mathrm{GL}_{n-1}(F)}(\pi, \sigma) \neq 0$.

In later years, great efforts were focused on posing and proving analogous rules for (quasi-split) classical groups, in place of the general linear group [20, 21, 37, 44]. The resulting branching laws became known as the *local Gan–Gross–Prasad* conjectures. In similarity with GL_n, these laws for classical groups were always set to pertain the generic case, in the sense of generic Langlands parameters which parameterize irreducible representations.

Back in the GL_n case, an effective answer for the general restriction problem is still largely considered unpractical. Yet, recently Gan, Gross and Prasad [18] have revisited this problem in an attempt to formulate branching laws that would extend beyond the generic case.

They stipulated a principle that clear combinatorial rules should describe the pairs (π, σ) with non-zero $\mathrm{Hom}_{\mathrm{GL}_{n-1}(F)}(\pi, \sigma)$, when π and σ belong to a well-behaved class of representations. More precisely, they formulated a conjecture which concerns a sub-class of unitarizable representations which is described by *Arthur parameters*.

An Arthur parameter for $\mathrm{GL}_n(F)$, in the definition of [18], stands for an admissible homomorphism

$$\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}),$$

such that the image of W_F is bounded, and the restriction of ϕ to each $\mathrm{SL}_2(\mathbb{C})$ component is algebraic. Here W_F stands for the Weil group of the field F.

In particular, an Arthur parameter ϕ is a completely reducible representation, which decomposes as

$$\phi = \bigoplus_{i=1}^{k} \psi_i \otimes V_{a_i} \otimes V_{b_i},$$

where $\{\psi_i\}$ are irreducible representations of W_F with bounded image, and V_d, $d \in \mathbb{Z}_{>0}$, denotes the unique isomorphism class of a d-dimensional irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$.

Given an Arthur parameter ϕ, one can attach an *L-parameter* to it. Consequently, by the established local Langlands reciprocity this L-parameter gives rise to an irreducible representation $\pi(\phi)$ of $\mathrm{GL}_n(F)$. We will say that a representation which is constructed in this manner is of *Arthur type*.

Conjecture 1.1 ([18]). *Suppose that*

$$\phi_1 = \bigoplus_{i=1}^{k} \psi_i \otimes V_{a_i} \otimes V_{b_i}, \quad \phi_2 = \bigoplus_{i=1}^{l} \psi_i' \otimes V_{a_i'} \otimes V_{b_i'},$$

are two Arthur parameters for $\mathrm{GL}_n(F), \mathrm{GL}_{n-1}(F)$, *respectively. Then*

$$\mathrm{Hom}_{\mathrm{GL}_{n-1}(F)}(\pi(\phi_1)|_{\mathrm{GL}_{n-1}(F)}, \pi(\phi_2)) \neq 0$$

holds, if and only if, there are disjoint partitions

$$\{1, \ldots, k\} = I_1 \cup I_2 \cup I_3, \quad \{1, \ldots, l\} = J_1 \cup J_2 \cup J_3,$$

and bijections $u : I_1 \to J_2$, $d : I_2 \to J_1$, *which satisfy*

$$(a'_{u(i)}, b'_{u(i)}) = (a_i, b_i + 1), \quad \psi'_{u(i)} \cong \psi_i \quad \text{for all } i \in I_1,$$
$$(a'_{d(i)}, b'_{d(i)}) = (a_i, b_i - 1), \quad \psi'_{d(i)} \cong \psi_i \quad \text{for all } i \in I_2,$$
$$b_i = 1 \quad \text{for all } i \in I_3, \qquad b'_j = 1 \quad \text{for all } j \in J_3.$$

The results of this article are as follows:

- Theorem 5.6, which provides a full proof of one direction of Conjecture 1.1 through the use of tools from the representation theory of quantum affine algebras.

- Theorem 5.7, which extends the conjecture's statement to the class of all unitarizable representations.

- Theorem 5.10, which settles some families of cases of the converse direction of the conjecture.

Combining these results, we settle both directions for the case when at least one of $\pi(\phi_1)$ and $\pi(\phi_2)$ is generic.

One motivation for the formulation of Conjecture 1.1 were the works of Clozel [15], Venkatesh [43] and Lapid and Rogawski [35] in the setting of *unitary* representations. They studied a restriction problem in the sense of direct integral decomposition, and showed that the Burger-Sarnak principle for automorphic representations implies some necessary combinatorial conditions to occur in a restriction.

Even though there is no immediate relation between the smooth and unitary problems, our Theorem 5.6 is consistent with [43, Proposition 2 (1)] in a certain sense. Namely, the cases where $I_1 = J_2 = \emptyset$ hold in the statement of Conjecture 1.1 produce a refinement of the SL(2)-type condition required by the mentioned proposition from [43]. Further discussion of this relation can be found in [18].

Following the publication of results stated here, a new encompassing treatment of Conjecture 1.1, using rather different techniques, was made public in [8].

1.1. Outline of proof

When attempting to tackle Conjecture 1.1 with standard techniques in hand, it appears unavoidable to encounter the intricacies of the structure of the Bernstein–Zelevinsky product. As explain below, we are able to isolate the required information on such products in the form of Theorem 1.2.

We next prove Theorem 1.2 by transferring it into other Lie-theoretic Abelian categories. Namely, we move into the representation theories of affine Hecke algebras and quantum affine algebras of type A.

Let us discuss the steps of the proof in more detail.

We first make use of the classical Bernstein–Zelevinsky filtration for the space of $\pi|_{\mathrm{GL}_{n-1}}$. This special feature of general linear groups allows for a translation (Proposition 5.4) of the restriction problem into questions on spaces of the form

$$\mathrm{Hom}_{\mathrm{GL}_{n-i}(F)}(\nu^{1/2} \otimes \pi^{(i)}, {}^{(i-1)}\sigma), \tag{1.1}$$

where $\nu^{1/2}$ is a certain character, and $\pi \mapsto \pi^{(i)}$, $\pi \mapsto {}^{(i)}\pi$ are the Bernstein–Zelevinsky derivative functors, which attach finite-length representations of a smaller rank group to a given irreducible representation.

Thus, we are left with questions on morphism spaces inside a category of finite-length representations. Moreover, derivatives of Arthur-type representations are built out of (Bernstein–Zelevinsky) products of derivatives of *Speh* representations. We call these derived Speh representations (which happen to be irreducible) *quasi-Speh* representations.

The reasoning portrayed thus far was employed already in [18] to tackle the problem and to prove some basic cases of Conjecture 1.1 (such as when $\pi(\phi_1), \pi(\phi_2)$ are Speh representations themselves).

Yet, the spaces (1.1) for the general case of Arthur-type representations were discovered to be substantially more intricate: Products of quasi-Speh representations are often reducible. The subrepresentation structure of such Bernstein–Zelevinsky products is often a highly non-trivial issue, as evidenced by a batch of recent works ([23, 33, 34, 42]).

Such difficulties are manifested in our problem through the following key statement, on which the resolution of the first direction of Conjecture 1.1 (Theorem 5.6) is highly dependent.

Theorem 1.2 (restatement of Proposition 4.3). *For any choice of quasi-Speh representations $\pi_1, \ldots, \pi_k$, there is an ordering ω (permutation of $\{1, \ldots, k\}$), for which the Bernstein–Zelevinsky product representation*

$$\pi_{\omega(1)} \times \cdots \times \pi_{\omega(k)}$$

has a unique irreducible quotient, whose Langlands parameter is given as the sum of Langlands parameters of $\pi_1, \ldots, \pi_k$.

We show (Proposition 4.2) that the case of $k = 2$ in Theorem 1.2 follows from the recent work of Lapid and Mínguez [33] on behavior of products of representations in the *ladder* class.

As a consequence, we are now left with the problem of whether Theorem 1.2 can be settled by looking on products of pairs of representations. Namely, we prove the following general phenomenon.

Theorem 1.3. *Suppose that $\sigma_1, \ldots, \sigma_k$ are irreducible smooth representations of general linear groups, such that for all $1 \leq i < j \leq k$, the product representation $\sigma_i \times \sigma_j$ has a unique irreducible quotient whose Langlands parameter is given as the sum of Langlands parameters of σ_i and σ_j. Then the product representation*

$$\sigma_1 \times \cdots \times \sigma_k$$

has a unique irreducible quotient, whose Langlands parameter is given as the sum of Langlands parameters of $\sigma_1, \ldots, \sigma_k$.

The mechanism of *Bernstein decomposition* (Section 3.1) presents the category of smooth representations of $\mathrm{GL}_n(F)$ as a product of smaller Abelian categories called Bernstein blocks. It is sufficient to prove Theorem 1.3 for each such block.

It is well known that each block can be described as modules over a complex algebra. The identification of these algebras with *affine Hecke algebras* (for GL_n) was done in [6, 7] through type theory, and independently in [25] through a more explicit approach. Thus, it is enough to solve the problem of Theorem 1.3 for modules over affine Hecke algebras (stated as Theorem 3.3).

Next, we are able to pass from modules over affine Hecke algebras to modules over quantum affine algebras by using the *quantum affine Schur–Weyl duality* functors (Section 6.3), developed by Chari and Pressley [14]. Since all functors involved turn out to be monoidal, in a suitable sense, we are finally left with a statement regarding the simple quotients of tensor products of quantum affine modules.

The advantage of posing our problem in a language of quantum groups lies in a recent result of Hernandez [27] dealing with *cyclic* modules. These are not necessarily simple modules, that are generated by their highest weight vector. It states that a product

$$V_1 \otimes \cdots \otimes V_k$$

of simple modules $V_1, \ldots, V_k$ is cyclic if all the products $V_i \otimes V_j$, $1 \le i < j \le k$, are cyclic.

When transferring the notion of a cyclic products back to the p-adic setting through our sequence of functors, we see that the theorem of Hernandez gives the precise statement of Theorem 1.3.

We comment that in a later work with Alberto Mínguez [24], we were able to extract to the key arguments of [27] and use them to construct an alternative "purely p-adic" proof of Theorem 1.3 for certain (so-called square irreducible) cases. Those cases are sufficient for the applications of this theorem in the current article.

1.2. Paper structure

Section 2 surveys the basic tools needed to study the smooth representation theory of p-adic GL_n, together with some necessary lemmas. In particular, we recall the Zelevinsky multisegment parametrization, which is an essential tool in our work. We make note of the basic Proposition 2.3, whose statement has not appeared previously in the literature to the best of our knowledge. Section 2.5 recovers the necessary results from [33].

Section 3 portrays the categorical passage from representations of $\mathrm{GL}_n(F)$ into those of affine Hecke algebras. Proposition 3.2 delves into the resulting correspondence between irreducible representations. The proof of the key Theorem 3.3 is delayed to Section 6.

Section 4 contains the gist of the combinatorial work in the class of quasi-Speh representations and their products, which is necessary for the proof of Conjecture 1.1.

In Section 5 we prove the main theorems discussed above.

Finally, Section 6 surveys the necessary ingredients from the representations theory of $U_q(\widehat{\mathfrak{sl}}_N)$ with the aim of proving Theorem 3.3, which is shown to be essentially a translation of the main result of [27].

2. Background on representation theory of p-adic GL_n

Let F be a p-adic field. Unless explicitly stated, F will be fixed and omitted from our notation. We are interested in the representation theory of the groups $G_n := \mathrm{GL}_n(F)$, for all $n \geq 1$.

For a given n, let $\alpha = (n_1, \ldots, n_r)$ be a composition of n. We denote by M_α the subgroup of G_n isomorphic to $G_{n_1} \times \cdots \times G_{n_r}$ consisting of matrices which are diagonal by blocks of size $n_1, \ldots, n_r$ and by P_α the subgroup of G_n generated by M_α and the upper unitriangular matrices. A standard parabolic subgroup of G_n is a subgroup of the form P_α and its standard Levi factor is M_α.

For a p-adic group G, let $\mathfrak{S}(G)$ be the category of smooth complex representations of G, and let $\mathfrak{R}(G)$ be the subcategory of objects of finite length. Denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible objects in $\mathfrak{R}(G)$. Denote by $\mathcal{C}(G) \subseteq \mathrm{Irr}(G)$ the subset of irreducible supercuspidal representations.

We write $\mathbf{i}_\alpha : \mathfrak{R}(M_\alpha) \to \mathfrak{R}(G_n)$ for the parabolic induction functor associated to P_α. For $\pi_i \in \mathfrak{R}(G_{n_i})$, $i = 1, \ldots, r$, we write

$$\pi_1 \times \cdots \times \pi_r := \mathbf{i}_{(n_1, \ldots, n_r)}(\pi_1 \otimes \cdots \otimes \pi_r) \in \mathfrak{R}(G_{n_1 + \cdots + n_r}).$$

We also write

$$\mathrm{Irr} = \bigcup_{m \geq 0} \mathrm{Irr}(G_m) \quad \text{and} \quad \mathcal{C} = \bigcup_{m \geq 1} \mathcal{C}(G_m).$$

For any n, let

$$\nu^s = |\det|_F^s, \quad s \in \mathbb{C},$$

denote the family of one-dimensional representations of G_n, where $|\cdot|_F$ is the absolute value of F. For $\pi \in \mathfrak{R}(G_n)$, we write

$$\nu^s \pi = \pi \nu^s := \pi \otimes \nu^s \in \mathfrak{R}(G_n).$$

The map $s \mapsto \nu^s$ is a group homomorphism from $\mathbb{C}$ to the group of characters of G_n, whose kernel is $(2\pi i / \log q_F)\mathbb{Z}$, where q_F is the residue characteristic of F.

The group of (unramified) characters $\{\nu^s : s \in \mathbb{C}\}$ of G_n acts on $\mathcal{C}(G_n)$ by $\rho \mapsto \rho\nu^s$. For $\rho \in \mathcal{C}(G_n)$, we write $o(\rho)$ for the (finite) order of the stabilizer of ρ for that action.

For $\pi \in \mathrm{Irr}$, the central character χ_π of π is a representation of G_1. Hence, $|\chi_\pi| = \nu^{s_\pi}$ for a number $s_\pi \in \mathbb{R}$. We will write $\mathrm{Re}(\pi) = s_\pi \in \mathbb{R}$.

Given a set X, we write $\mathbb{N}(X)$ for the commutative monoid of maps from X to $\mathbb{N} = \mathbb{Z}_{\geq 0}$ with finite support. We will sometimes treat an element $A \in \mathbb{N}(X)$ as a finite subset of X, by writing $x \in A$ for a given $x \in X$, in case $A(x) > 0$.

There is a natural embedding $X \to \mathbb{N}(X)$ which sends an element to its indicator function. Thus, we will often treat elements of X as elements in $\mathbb{N}(X)$.

2.1. Multisegments

The elements of Irr are classified by *multisegments* in the following manner.

A *segment* $\Delta = [a, b]_\rho$ is a formal object defined by a triple $([\rho], a, b)$, where $\rho \in \mathcal{C}$ and $a \leq b$ are two integers, up to the equivalence $[a, b]_\rho = [a', b']_{\rho'}$, when $\rho v^a \cong \rho' v^{a'}$ and $\rho v^b \cong \rho' v^{b'}$.

It is also useful to refer to the empty segment given in the form $[a, a-1]_\rho$, for any $\rho \in \mathcal{C}$ and integer a.

For a segment $\Delta = [a, b]_\rho$, we will write $b_\rho(\Delta) = a$ and $e_\rho(\Delta) = b$.

A segment Δ_1 is said to precede a segment Δ_2 if $\Delta_1 = [a_1, b_1]_\rho$, $\Delta_2 = [a_2, b_2]_\rho$ and $a_1 \leq a_2 - 1 \leq b_1 < b_2$. We will write $\Delta_1 \prec \Delta_2$ in this case.

Let Seg denote the collection of all segments. Elements of Mult $:= \mathbb{N}(\text{Seg})$ are called multisegments.

The *Zelevinsky classification* [45] defines a bijection

$$Z : \text{Mult} \to \text{Irr}.$$

Let us briefly recall the process that constructs Z. First, $Z(\Delta)$ is defined directly for every segment Δ. It is easily verified that each $\mathfrak{m} \in \text{Mult}$ can be written additively as $\mathfrak{m} = \Delta_1 + \cdots + \Delta_k$, where Δ_i are segments whose order is chosen so that $\Delta_i \not\prec \Delta_j$, for any $i < j$. Next, the *standard* representation

$$\zeta(\mathfrak{m}) = Z(\Delta_1) \times \cdots \times Z(\Delta_k)$$

is constructed. While $\zeta(\mathfrak{m})$ may be reducible, its unique irreducible subrepresentation is defined to be $Z(\mathfrak{m})$.

We say that $\mathfrak{n} \leq \mathfrak{m}$ if the isomorphism class of $Z(\mathfrak{n})$ appears as a subquotient in $\zeta(\mathfrak{m})$. It is shown in [45] that this is a partial order on Mult (the Zelevinsky order).

Similarly, the *Langlands (quotient) classification*, which describes irreducible representations of any reductive p-adic group, can be stated for the case of the groups $\{G_n\}_{n=1}^\infty$, as another[1] bijection

$$L : \text{Mult} \to \text{Irr}.$$

The relation between both bijections Z and L is further discussed in Section 3.2.

For a segment $\Delta = [a, b]_\rho$ and $s \in \mathbb{C}$, we write $\Delta v^s := [a, b]_{\rho v^s}$. We naturally extend this to an operation $\mathfrak{m} \mapsto \mathfrak{m} v^s$ on Mult. It is easy to check that, $Z(\mathfrak{m} v^s) = Z(\mathfrak{m}) v^s$ and $L(\mathfrak{m} v^s) = L(\mathfrak{m}) v^s$ hold.

Let $\pi_1 = Z(\mathfrak{m}_1), \pi_2 = Z(\mathfrak{m}_2), \ldots, \pi_t = Z(\mathfrak{m}_t) \in \text{Irr}$ be given. Then $\pi_1 \times \cdots \times \pi_t$ contains $Z(\mathfrak{m}_1 + \cdots + \mathfrak{m}_t)$ with multiplicity one in its composition series (see, e.g., [33, Proposition 2.5 (5)]). In particular, when $\pi_1 \times \cdots \times \pi_t$ is irreducible, we must have

$$\pi_1 \times \cdots \times \pi_t \cong Z(\mathfrak{m}_1 + \cdots + \mathfrak{m}_t).$$

The analogous statements remain true when replacing the Z bijection with L.

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[1]In fact, the Zelevinsky classification can also be obtained as a variation of the Langlands classification, when considering the anti-tempered spectrum of the group, rather than the tempered spectrum.

Lemma 2.1. *Let* $\pi_1 = L(\mathfrak{m}_1), \ldots, \pi_k = L(\mathfrak{m}_k)$ *and* $\sigma_1 = L(\mathfrak{n}_1), \ldots, \sigma_l = L(\mathfrak{n}_l)$ *be irreducible representations, for which* $L(\mathfrak{m}_1 + \cdots + \mathfrak{m}_k)$ *is the unique irreducible quotient of* $\pi_1 \times \cdots \times \pi_k$ *and* $L(\mathfrak{n}_1 + \cdots + \mathfrak{n}_l)$ *is the unique irreducible subrepresentation of* $\sigma_1 \times \cdots \times \sigma_l$. *Suppose that*

$$\mathrm{Hom}(\pi_1 \times \cdots \times \pi_k, \sigma_1 \times \cdots \times \sigma_l) \neq \{0\}.$$

Then $\mathfrak{m}_1 + \cdots + \mathfrak{m}_k = \mathfrak{n}_1 + \cdots + \mathfrak{n}_l$.

Proof. Let $0 \neq f$ be a homomorphism in $\mathrm{Hom}(\pi_1 \times \cdots \times \pi_k, \sigma_1 \times \cdots \times \sigma_l)$, and let κ denote its image. By assumption κ contains $L(\mathfrak{m}_1 + \cdots + \mathfrak{m}_k)$ as a quotient, and $L(\mathfrak{n}_1 + \cdots + \mathfrak{n}_l)$ as a subrepresentation. In particular, we see that $L(\mathfrak{m}_1 + \cdots + \mathfrak{m}_k)$ is a subquotient of $\sigma_1 \times \cdots \times \sigma_l$. It is known that for every subquotient $L(\mathfrak{t})$ of $\sigma_1 \times \cdots \times \sigma_l$, we have $\mathfrak{t} \leq \mathfrak{n}_1 + \cdots + \mathfrak{n}_l$. Hence, $\mathfrak{m}_1 + \cdots + \mathfrak{m}_k \leq \mathfrak{n}_1 + \cdots + \mathfrak{n}_l$. We claim analogously that $\mathfrak{n}_1 + \cdots + \mathfrak{n}_l \leq \mathfrak{m}_1 + \cdots + \mathfrak{m}_k$ and the statement follows. ∎

Given a supercuspidal representation $\rho \in \mathcal{C}$, let Seg_ρ denote the collection of segments of the form $[a, b]_{\rho \nu^s}$, for some integers a, b and $s \in \mathbb{C}$.

We also define the submonoid

$$\mathrm{Mult}_\rho := \mathbb{N}(\mathrm{Seg}_\rho) \subseteq \mathrm{Mult}.$$

When $\nu^0 \in \mathrm{Irr}(G_1)$ is the trivial representation (thought of as a supercuspidal representation), we write

$$\mathrm{Mult}_0 = \mathrm{Mult}_{\nu^0}.$$

We will also let the field F vary for this part of the discussion, and write Seg_ρ^F, Mult_ρ^F, Mult_0^F for the corresponding objects, defined for an arbitrary p-adic field F.

Given $\rho_1 \in \mathcal{C}(\mathrm{GL}_{n_1}(F_1))$ and $\rho_2 \in \mathcal{C}(\mathrm{GL}_{n_2}(F_2))$ with

$$q_{F_1}^{o(\rho_1)} = q_{F_2}^{o(\rho_2)},$$

we have a well-defined bijection

$$\phi_{\rho_1, \rho_2} : \mathrm{Seg}_{\rho_1}^{F_1} \to \mathrm{Seg}_{\rho_2}^{F_2}, \quad \phi([a, b]_{\rho_1 \nu^s}) = [a, b]_{\rho_2 \nu^s},$$

which naturally extends to an isomorphism of monoids $\phi_{\rho_1, \rho_2} : \mathrm{Mult}_{\rho_1}^{F_1} \to \mathrm{Mult}_{\rho_2}^{F_2}$.

2.2. Gelfand–Kazhdan involution

The outer automorphism $g \mapsto (g^t)^{-1}$ on G_n, gives an involutive auto-equivalence η of the category $\Re(G_n)$. Let us write $\overline{\eta}$ for the composition of η with the operation of taking the contragredient (smooth dual) representation.

The resulting involution $\overline{\eta}$ is a contragredient functor with satisfies the special property $\overline{\eta}(\pi) \cong \pi$, for all $\pi \in \mathrm{Irr}$, as shown by a classical result of Gelfand and Kazhdan [19].

It is easy to check that $\overline{\eta}$ is compatible with the parabolic induction product, in the sense of

$$\overline{\eta}(\pi_1 \times \cdots \times \pi_t) \cong \overline{\eta}(\pi_t) \times \cdots \times \overline{\eta}(\pi_1)$$

for any $\pi_i \in \Re(G_{n_i})$, $i = 1, \ldots, t$.

The existence of $\overline{\eta}$ also gives the following well-known corollary.

Proposition 2.2. *For all $\pi_1, \ldots, \pi_t \in \text{Irr}$, the socle (maximal semisimple subrepresentation) of $\pi_1 \times \cdots \times \pi_t$ is isomorphic to the co-socle (maximal semisimple quotient) of $\pi_t \times \cdots \times \pi_1$. In particular, when $\pi_1 \times \cdots \times \pi_t$ is irreducible, it is isomorphic to $\pi_{w(1)} \times \cdots \times \pi_{w(t)}$, for any permutation w on $\{1, \ldots, t\}$.*

2.3. Supercuspidal support

For every class $\pi \in \text{Irr}$ there exist $\rho_1, \ldots, \rho_r \in \mathcal{C}$ for which π is a subrepresentation of $\rho_1 \times \cdots \times \rho_r$. The notion of *supercuspidal support* can then be defined as the multiset

$$\text{supp}(\pi) \in \mathbb{N}(\mathcal{C}),$$

given by the tuple $(\rho_1, \ldots, \rho_r)$. It is known that for any $\pi_1, \pi_2 \in \text{Irr}$, the product $\pi_1 \times \pi_2$ is irreducible, unless there is a supercuspidal representation $\rho \in \text{supp}(\pi_1)$, so that either $\rho v^1 \in \text{supp}(\pi_2)$ or $\rho v^{-1} \in \text{supp}(\pi_2)$.

For $\sigma \in \mathfrak{R}(G_n)$, we will write $\text{supp}(\sigma) = I$, in case $\text{supp}(\pi) = I$ holds, for all irreducible subquotients π of σ.

We recall from the general theory of the Bernstein center, that any representation $\sigma \in \mathfrak{R}(G_n)$ splits uniquely to the form $\sigma = \bigoplus_I \sigma_I$, where the sum goes over distinct multisets $I \in \mathbb{N}(\mathcal{C})$, so that $\text{supp}(\sigma_I) = I$.

In particular, for $\pi, \sigma \in \mathfrak{R}(G_n)$, we have $\text{Hom}(\pi, \sigma) = \bigoplus_I \text{Hom}(\pi_I, \sigma_I)$.

Proposition 2.3. *Suppose that $\pi_i, \sigma_i \in \mathfrak{R}(G_{n_i})$ are given for $i = 1, \ldots, t$ such that*

$$\text{supp}(\pi_i) = \text{supp}(\sigma_i) = I_i$$

for pairwise disjoint multisets $I_1, \ldots, I_t \in \mathbb{N}(\mathcal{C})$. Then we have a natural identification

$$\text{Hom}(\pi_1 \times \cdots \times \pi_t, \sigma_1 \times \cdots \times \sigma_t) \cong \text{Hom}(\pi_1, \sigma_1) \otimes \cdots \otimes \text{Hom}(\pi_t, \sigma_t).$$

Proof. Consider the parabolic induction functor

$$\mathbf{i}_\alpha : \mathfrak{R}(G_{n_1} \times \cdots \times G_{n_t}) \to \mathfrak{R}(G_N),$$

where $N = n_1 + \cdots + n_t$. It clearly gives an embedding of the space $\bigotimes_{i=1}^t \text{Hom}(\pi_i, \sigma_i)$ into $\text{Hom}(\pi_1 \times \cdots \times \pi_t, \sigma_1 \times \cdots \times \sigma_t)$. We are left to show that any morphism in the latter space is contained in the image of $\mathbf{i}_\alpha$.

Recall that parabolic induction has an exact left-adjoint functor $\mathbf{r}_\alpha$, that is, the Jacquet functor. This adjunction gives a canonical morphism

$$\epsilon_{\underline{\pi}} : \mathbf{r}_\alpha(\pi_1 \times \cdots \times \pi_t) \to \pi_1 \otimes \cdots \otimes \pi_t.$$

Similarly, we have

$$\epsilon_{\underline{\sigma}} : \mathbf{r}_\alpha(\sigma_1 \times \cdots \times \sigma_t) \to \sigma_1 \otimes \cdots \otimes \sigma_t.$$

The suitable Mackey theory in the form of the geometric lemma of Bernstein and Zelevinsky ([4], or see [33, Section 2.2]) predicts a more precise information on $\epsilon_{\underline{\pi}}$.

First, $\epsilon_{\underline{\pi}}$ is surjective. Second, for any irreducible subquotient $\tau = \tau_1 \otimes \cdots \otimes \tau_t$ of $\mathbf{r}_\alpha(\pi_1 \times \cdots \times \pi_t)$, there is a matrix of multisets $(J_i^j)_{i,j=1}^t$ in $\mathbb{N}(\mathcal{C})$ such that

$$I_i = J_i^1 + \cdots + J_i^t \quad \text{and} \quad \text{supp}(\tau_j) = J_1^j + \cdots + J_t^j.$$

Moreover, when τ is a subquotient of $\ker \epsilon_{\underline{\pi}}$, we know that $J_i^j \neq 0$, some $i \neq j$. In particular, by the disjointness assumption $(\mathrm{supp}(\tau_1), \ldots, \mathrm{supp}(\tau_t)) \neq (I_1, \ldots, I_t)$ in that case. Hence, the projection $\epsilon_{\underline{\pi}}$ splits, and we can view $\pi_1 \otimes \cdots \otimes \pi_t$ as a subrepresentation of $\mathbf{r}_\alpha(\pi_1 \times \cdots \times \pi_t)$.

Now, let ϕ be a morphism in $\mathrm{Hom}(\pi_1 \times \cdots \times \pi_t, \sigma_1 \times \cdots \times \sigma_t)$. By considerations of supercuspidal support, the image of $\psi := \mathbf{r}_\alpha(\phi)|_{\pi_1 \otimes \cdots \otimes \pi_t}$ is contained in $\sigma_1 \otimes \cdots \otimes \sigma_t$ (after running similar arguments on $\epsilon_{\underline{\sigma}}$). Thus, we obtain the commutative diagram

$$
\begin{array}{ccc}
\mathbf{r}_\alpha(\sigma_1 \times \cdots \times \sigma_t) & \xrightarrow{\ \epsilon_{\underline{\pi}}\ } & \sigma_1 \otimes \cdots \otimes \sigma_t \\[2pt]
{\scriptstyle \mathbf{r}_\alpha(\phi)} \big\uparrow & & \big\uparrow {\scriptstyle \psi} \\[2pt]
\mathbf{r}_\alpha(\pi_1 \times \cdots \times \pi_t) & \xrightarrow{\ \epsilon_{\underline{\sigma}}\ } & \pi_1 \otimes \cdots \otimes \pi_t,
\end{array}
$$

which implies by adjunction of functors the commutativity of the diagram

$$
\begin{array}{ccc}
\sigma_1 \times \cdots \times \sigma_t & \xrightarrow{\ \mathrm{id}\ } & \sigma_1 \times \cdots \times \sigma_t \\[2pt]
{\scriptstyle \phi} \big\uparrow & & \big\uparrow {\scriptstyle \mathbf{i}_\alpha(\psi)} \\[2pt]
\pi_1 \times \cdots \times \pi_t & \xrightarrow{\ \mathrm{id}\ } & \pi_1 \times \cdots \times \pi_t.
\end{array}
$$

The proposition is proved.   ∎

Corollary 2.4. *Suppose that $\pi_i, \sigma_i \in \Re(G_{n_i})$ are given for $i = 1, \ldots, t$, together with pairwise disjoint multisets $I_1, \ldots, I_t \in \mathbb{N}(\mathcal{C})$, such that for every irreducible subquotient τ of either π_i or σ_i, $\mathrm{supp}(\tau) \subseteq I_i$ holds. Then we have a natural identification*

$$\mathrm{Hom}(\pi_1 \times \cdots \times \pi_t, \sigma_1 \times \cdots \times \sigma_t) \cong \mathrm{Hom}(\pi_1, \sigma_1) \otimes \cdots \otimes \mathrm{Hom}(\pi_t, \sigma_t).$$

Proof. It follows directly by decomposing each π_i, σ_i into direct sums according to supercuspidal support, and applying Proposition 2.3 on each of the summands.   ∎

2.4. Tadic classification of the unitary spectrum

Let $\mathrm{Irr}^u \subseteq \mathrm{Irr}$ be the subset of irreducible representations that are unitarizable, that is, those whose space can be equipped with a positive definite Hermitian form invariant under the group action.

A classification of Irr^u in terms of multisegments was achieved by Tadic in [41]. Let us briefly recall this classification. It is easy to check that $\mathrm{Irr}^u \cap \mathcal{C} = \{\rho \in \mathcal{C} : \mathrm{Re}(\rho) = 0\}$.

For each pair of integers $a, b \in \mathbb{Z}_{>0}$ and a supercuspidal $\rho \in \mathrm{Irr}^u \cap \mathcal{C}$, we define the *Speh* multisegment

$$\mathfrak{m}_\rho^{a,b} = \sum_{i=1}^{b} \left[\frac{b-a}{2} + 1 - i, \frac{b+a}{2} - i \right]_\rho \in \mathrm{Mult}.$$

We then set $\pi_\rho^{a,b} := L(\mathfrak{m}_\rho^{a,b})$ to be a (unitary[2]) *Speh representation*.

---

[2]It is also common to define a general Speh representation without the requirement that ρ is unitarizable.

Let $B \subseteq \text{Mult}$ denote the collection of Speh multisegments. Let us also define

$$B^{\text{comp}} = \{\mathfrak{m}\nu^{\alpha} + \mathfrak{m}\nu^{-\alpha} : \mathfrak{m} \in B, \, 0 < \alpha < \tfrac{1}{2}\}.$$

Theorem 2.5. *Let $U \subseteq \text{Mult}$ be the submonoid generated by $B \sqcup B^{\text{comp}}$ in* Mult. *Then*

$$L(U) = Z(U) = \text{Irr}^{u}.$$

It is known ([3]) that $\pi_1 \times \pi_2$ is irreducible, for any $\pi_1, \pi_2 \in \text{Irr}^{u}$. Hence, the theorem above states that any $\pi \in \text{Irr}^{u}$ can be written in the form $\pi = \pi_1 \times \cdots \times \pi_t$, where $\pi_i = L(\mathfrak{m}_i)$ with $\mathfrak{m}_i \in B \sqcup B^{\text{comp}}$ for all $i = 1, \ldots, t$.

Definition 2.6. We say that $\pi \in \text{Irr}^{u}$ is of *Arthur type* if $\mathfrak{m}_i \in B$ for all $i = 1, \ldots, t$ in the decomposition above.[3]

In other words, Arthur-type representations are products of Speh representations.

Definition 2.7. We say that $\pi \in \text{Irr}^{u}$ is of *proper Arthur type* if $\pi \cong \pi_{\rho}^{a_1, b_1} \times \cdots \times \pi_{\rho}^{a_k, b_k}$ for a fixed $\rho \in \text{Irr}^{u} \cap \mathcal{C}$ and integers $a_i, b_i \in \mathbb{Z}_{>0}$.

Note that proper Arthur-type representations built out of non-isomorphic elements of $\text{Irr}^{u} \cap \mathcal{C}$ will always have disjoint supercuspidal supports.

Given an Arthur-type representation $\pi = L(\mathfrak{m})$ and a real number $0 < \alpha < \tfrac{1}{2}$, let us write

$$\pi(\alpha) := \pi\nu^{\alpha} \times \pi\nu^{-\alpha} = L(\mathfrak{m}\nu^{\alpha} + \mathfrak{m}\nu^{-\alpha}) \in \text{Irr}^{u}.$$

It is easy to deduce from Theorem 2.5 that for every $\pi \in \text{Irr}^{u}$ there is a unique, up to order, factorization of the form

$$\pi \cong \pi_0 \times \pi_1(\alpha_1) \times \cdots \times \pi_k(\alpha_k),$$

where $\pi_0, \pi_1, \ldots, \pi_k \in \text{Irr}^{u}$ are of Arthur type[4] and $0 < \alpha_1, \ldots, \alpha_k < \tfrac{1}{2}$ are distinct real numbers.

2.5. *Ladder representations and the Lapid-Mínguez criterion*

Speh representations are a special case of a class of irreducible representations known as *ladder* representations. A representation $\pi \in \text{Irr}$ is called a *proper*[5] *ladder* representation if it is given as $\pi = L(\Delta_1 + \cdots + \Delta_k)$ for segments $\Delta_i, i = 1, \ldots, k$, satisfying

$$\Delta_k \prec \cdots \prec \Delta_2 \prec \Delta_1.$$

---

[3]There may be an ambiguity regarding this definition, when compared to other sources. Arthur-type representations were defined with the goal of parameterizing the local components of the discrete automorphic spectrum of a group. Assuming the Ramanujan conjecture for GL_n, our definition should coincide with this global notion.

[4]At least for the purposes of this decomposition, we need to consider $L(0) \in \text{Irr}(G_0)$ (treated as neutral to multiplication) as an Arthur-type representation.

[5]For ease of exposition, we do not define a general ladder here. The ladder representations that occur in this manuscript are all proper ladder.

Ladder representations were shown (e.g., [22,32,33]) to possess certain remarkable properties, which often make the ladder class more approachable for treatment of questions on general irreducible representations. Essentially the same class of representations was also studied in the literature under various names in various type A settings, such as calibrated affine Hecke algebra modules, snake modules for quantum affine algebras and homogeneous modules for KLR algebras.

Given a ladder representation $\pi \in \mathrm{Irr}$ and any representation $\sigma \in \mathrm{Irr}$, it was shown [33, Proposition 6.15] that both $\pi \times \sigma$ and $\sigma \times \pi$ have a unique irreducible subrepresentation. Lapid and Mínguez have also devised an algorithm in [33] for computing the multisegment classifying that subrepresentation. We recall in what follows one corollary of that algorithm.

Suppose, for that purpose, that

$$\mathfrak{m}_1 = \Delta_1 + \cdots + \Delta_{k_1}, \quad \mathfrak{m}_2 = \Delta'_1 + \cdots + \Delta'_{k_2},$$

are two multisegments with

$$\Delta_{k_1} \prec \cdots \prec \Delta_1 \quad \text{and} \quad \Delta'_{k_2} \prec \cdots \prec \Delta'_1.$$

Recall that $Z(\mathfrak{m}_1 + \mathfrak{m}_2)$ always appears as a subquotient in $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$. Among the results of [33] is a combinatorial criterion for determining when does $Z(\mathfrak{m}_1 + \mathfrak{m}_2)$ actually appear as a subrepresentation of $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$.

Consider the set of indices $I = \{1, \ldots, k_1\} \times \{1, \ldots, k_2\}$, and the following bipartite graph on the set of vertices $I_1 \sqcup I_2$, where $I_1 = I_2 = I$ (two copies of I). We say that elements $(i_1, j_1) \in I_1$ and $(i_2, j_2) \in I_2$ are in relation $(i_1, j_1) \leftrightarrow (i_2, j_2)$ if either $i_1 = i_2$ and $j_2 = j_1 + 1$, or $j_1 = j_2$ and $i_2 = i_1 - 1$.

Consider the sets

$$X_{\mathfrak{m}_1;\mathfrak{m}_2} = \{(i, j) \in I_1 : \Delta_i \prec \Delta'_j\}, \quad Y_{\mathfrak{m}_1;\mathfrak{m}_2} = \{(i, j) \in I_2 : \Delta_i \prec \overrightarrow{\Delta'_j}\}.$$

Here $\overrightarrow{\Delta}$ means the segment $[s + 1, t + 1]_\rho$, for a segment $\Delta = [s, t]_\rho$.

We can consider the bipartite graph $(X_{\mathfrak{m}_1;\mathfrak{m}_2}, Y_{\mathfrak{m}_1;\mathfrak{m}_2}, \leftrightarrow)$ created by restricting the relation $\leftrightarrow$ onto edges between $X_{\mathfrak{m}_1;\mathfrak{m}_2}$ (as a subset of I_1) and $Y_{\mathfrak{m}_1;\mathfrak{m}_2}$ (as a subset of I_2).

Recall that a *matching function* on a bipartite graph whose two parts of vertices are described by sets X, Y, is a function $f : X \to Y$, such that there is an edge between x and $f(x)$, for all $x \in X$.

Theorem 2.8 ([33, Lemma 6.21]). *The unique irreducible subrepresentation of $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ is isomorphic to $Z(\mathfrak{m}_1 + \mathfrak{m}_2)$ if and only if there is a matching function for the bipartite graph $(X_{\mathfrak{m}_1;\mathfrak{m}_2}, Y_{\mathfrak{m}_1;\mathfrak{m}_2}, \leftrightarrow)$, i.e., there exists an injective function*

$$f : X_{\mathfrak{m}_1;\mathfrak{m}_2} \to Y_{\mathfrak{m}_1;\mathfrak{m}_2},$$

which satisfies $x \leftrightarrow f(x)$ for all $x \in X_{\mathfrak{m}_1;\mathfrak{m}_2}$.

2.6. Bernstein–Zelevinsky derivatives

For given n_1, n_2, consider the subgroup $U < G_{n_1} \times G_{n_2}$ of upper unitriangular matrices in G_{n_2}. Let ψ be a non-degenerate character of U.

We then have an obvious functor of taking co-invariants:

$$W_{n_1,n_2} : \Re(G_{n_1} \times G_{n_2}) \to \Re(G_{n_1}), \quad V \mapsto V/\operatorname{span}\{g \cdot v - \psi(g)v : g \in G_{n_2}, \, v \in V\}.$$

The Bernstein–Zelevinsky derivatives of [4] can be defined as functors

$$\Re(G_n) \to \Re(G_{n-i})$$

constructed by composing W with the Jacquet functor. More precisely, given $\pi \in \Re(G_n)$, we set its i-th derivative to be

$$\pi^{(i)} := W_{n-i,i}(\mathbf{r}_{(n-i,i)}(\pi)) \in \Re(G_{n-i})$$

for all $1 \le i \le n$. Here $\mathbf{r}_{(n-i,i)}$ is the Jacquet functor left-adjoint to $\mathbf{i}_{(n-i,i)}$.

We set $\pi^{(0)} = \pi$.

The derivatives comply with a Leibniz rule, in the following sense.

Proposition 2.9. *Suppose that $\pi_i \in \Re(G_{n_i})$, $i = 1, \ldots, k$, are given.*

(1) *The representation $(\pi_1 \times \cdots \times \pi_k)^{(j)}$ has a filtration whose constituents are given by all representations of the form*

$$\pi_1^{(s_1)} \times \cdots \times \pi_k^{(s_k)},$$

with $0 \le s_i \le n_i$ for all $i = 1, \ldots, k$ and $j = s_1 + \cdots + s_k$.

(2) *Suppose that $\pi_i^{(m_i)}$ is the highest non-zero derivative of π_i, for all $i = 1, \ldots, k$. Suppose that $j = m_t + m_{t+1} + \cdots + m_k$, for some $1 \le t \le k$. Then the representation*

$$\pi_1 \times \cdots \times \pi_{t-1} \times \pi_t^{(m_t)} \times \cdots \times \pi_k^{(m_k)}$$

is a quotient of $(\pi_1 \times \cdots \times \pi_k)^{(j)}$, that is, the uppermost piece of the above filtration.

Proof. See [4, Corollary 4.14 (c)] together with its proof. ∎

3. Affine Hecke algebras

Given $n \in \mathbb{Z}_{>0}$ and $q \in \mathbb{C}$ (which for our needs will be assumed to be non-root of unity), the root datum of GL_n gives rise to the (extended) *affine Hecke algebra* $H(n, q)$. In fact, we will not be using here the concrete algebraic structure of these algebras, but rather some abstract information on their categories of representations.

Yet, to avoid confusion let us recall a possible presentation of $H(n, q)$: This is the complex algebra generated by $T_1, \ldots, T_{n-1}$ and invertible $y_1, \ldots, y_n$, subject to the relations

$$
\begin{aligned}
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for all } 1 \le i \le n-2, \\
(T_i - q)(T_i + 1) &= 0 && \text{for all } 1 \le i \le n-1, \\
T_i T_j &= T_j T_i && \text{for all } |j - i| > 1, \\
y_i y_j &= y_j y_i && \text{for all } 1 \le i, j \le n, \\
T_i y_i T_i &= q y_{i+1} && \text{for all } 1 \le i \le n-1, \\
T_i y_j &= y_j T_i && \text{for all } j \ne i, i+1.
\end{aligned}
$$

We denote by $\mathcal{M}_n^q$ the category of finite-dimensional modules over the algebra $H(n, q)$.

For any $n_1, \ldots, n_t \in \mathbb{Z}_{>0}$, there is a natural embedding of algebras

$$H(n_1, q) \otimes \cdots \otimes H(n_t, q) \hookrightarrow H(n_1 + \cdots + n_t, q).$$

This embedding gives rise to an induction functor

$$\mathbf{i}_{(n_1, \ldots, n_t)} : \mathcal{M}_{n_1}^q \times \cdots \times \mathcal{M}_{n_t}^q \to \mathcal{M}_{n_1 + \cdots + n_t}^q,$$

which we will simply denote as a product operation, i.e., $\pi_1 \times \pi_2 := \mathbf{i}_{(n_1, n_2)}(\pi_1 \otimes \pi_2)$, for $\pi_i \in \mathcal{M}_{n_i}^q$.

3.1. Equivalence to Bernstein blocks

Let $\sim$ be an equivalence relation on the elements of $\mathbb{N}(\mathcal{C})$ defined as follows: For two elements $A, B \in \mathbb{N}(\mathcal{C})$, we say that $A \sim B$ if there are representations $\rho_1, \ldots, \rho_t \in \mathcal{C}$ and numbers $s_1, \ldots, s_t \in \mathbb{C}$ such that

$$A = \rho_1 + \cdots + \rho_t, \quad B = \rho_1 \nu^{s_1} + \cdots + \rho_t \nu^{s_t}.$$

Note that the number $N_A := n_1 + \cdots + n_t$, where $\rho_i \in \mathcal{C}(G_{n_i})$, for $i = 1, \ldots, t$, is an invariant of the $\sim$-equivalence class of A (also known as the inertia class). Hence, we will write N_Θ, where Θ denotes that equivalence class.

Each inertia class Θ defines the *Bernstein block*[6] $\Re(\Theta)$, which is the full subcategory of $\Re(G_{N_\Theta})$ consisting of representations π, such that $\mathrm{supp}(\sigma)$ belongs to Θ for all irreducible subquotients of π.

The Bernstein decomposition [2] (in the case of GL_n) states that we have a decomposition of Abelian categories

$$\Re(G_n) = \prod_\Theta \Re(\Theta),$$

where the product goes over all inertia classes Θ, for which $N_\Theta = n$.

When writing such a decomposition, we mean that every object M in $\Re(G_n)$ is decomposed uniquely as $M \cong \bigoplus_\Theta M_\Theta$, where each M_Θ is an object in $\Re(\Theta)$, all but finitely many M_Θ are zero (we deal with finite length objects), and

$$\mathrm{Hom}_{\Re(G_n)} = \bigoplus_\Theta \mathrm{Hom}_{\Re(\Theta)}(M_\Theta, N_\Theta)$$

holds, for all M, N in $\Re(G_n)$.

We will call $\Re(\Theta)$ a *simple block* if $\Theta = \Theta(\rho, d)$ has a representative of the form $d \cdot \rho \in \mathbb{N}(\mathcal{C})$, where $\rho \in \mathcal{C}$ and $d \geq 1$ is an integer.

It is evident that for $\pi_1 \in \Re(\Theta(\rho, n_1))$ and $\pi_2 \in \Re(\Theta(\rho, n_2))$, the representation $\pi_1 \times \pi_2$ belongs to the block $\Re(\Theta(\rho, n_1 + n_2))$.

The clear consequence of the above is that for any $\rho \in \mathcal{C}$, the irreducible representations appearing in the sequence of blocks $\{\Re(\Theta(\rho, n))\}_{n=0}^\infty$ are precisely those given by $Z(\mathrm{Mult}_\rho)$ or $L(\mathrm{Mult}_\rho)$.

---

[6]These subcategories may become blocks in the more axiomatic treatment of Abelian categories when dealing with larger categories of all smooth representations of a group.

For the trivial representation ν^0 of G_1, we set $\Theta_n = \Theta(\nu^0, n)$ for all $n \geq 1$. The simple block $\mathfrak{R}(\Theta_n)$ is called the *principal* (or *Iwahori-invariant*) block of $\mathfrak{R}(G_n)$.

It was shown in [7] and [25] that for every simple block $\Theta = \Theta(\rho, d)$, we have an equivalence of categories

$$U_\Theta : \mathfrak{R}(\Theta) \xrightarrow{\sim} \mathcal{M}_d^{q_F^{o(\rho)}}.$$

The equivalences $\{U_\Theta\}$ are not canonical, yet they can be chosen in a way that is compatible with parabolic induction [39]. Namely, we are allowed to assume that

$$U_{\Theta(\rho, n_1 + n_2)}(\pi_1 \times \pi_2) \cong U_{\Theta(\rho, n_1)}(\pi_1) \times U_{\Theta(\rho, n_2)}(\pi_2) \tag{3.1}$$

holds for all $\pi_1 \in \mathfrak{R}(\Theta(\rho, n_1))$ and $\pi_2 \in \mathfrak{R}(\Theta(\rho, n_2))$.

The particular case of equivalences for principal blocks

$$U_n = U_{\Theta_n} : \mathfrak{R}(\Theta_n) \xrightarrow{\sim} \mathcal{M}_n^{q_F} \quad \text{for all } n$$

is in fact a classical theorem[7] of Borel [5] and Casselman.

We will fix a canonical choice of $\{U_n\}$ which is supplied by said theorem.

Let us note that the irreducible representations of the algebra $H(1, q_F) = \mathbb{C}[y_1, y_1^{-1}]$ are naturally given by the variety $\mathbb{C}^\times$. Note further that the irreducible representations in $\mathfrak{R}(\Theta_1)$ are given by the group of unramified characters $\{\nu^s : s \in \mathbb{C}\}$. Under these identifications, the canonical equivalence U_1 takes ν^s to the character given by $q_F^s \in \mathbb{C}^\times$.

We can use the equivalence U_n to push a parametrization of the irreducible representations of $H(n, q_F)$ in terms of multisegments. Recall that the irreducible representations in $\mathfrak{R}(\Theta_n)$ are parameterized by Mult_0, either through Z or L. Hence, we write bijections

$$\hat{Z}, \hat{L} : \mathrm{Mult}_0 \to \bigcup_{n \geq 0} \mathrm{Irr}(\mathcal{M}_n^{q_F}),$$

defined by $\hat{Z}(\mathfrak{m}) = U_{n_\mathfrak{m}}(Z(\mathfrak{m}))$, where $Z(\mathfrak{m}) \in \mathrm{Irr}(G_{n_\mathfrak{m}})$. Similarly, $\hat{L}$ is defined by L.

Remark 3.1. Note that $\hat{Z}$ (or $\hat{L}$) can in fact be described intrinsically (that is, without use of p-adic groups) as was done in [40]. In particular, these classifications are independent of the field F.

The following proposition shows that all equivalences $\{U_\Theta\}$ preserve the Zelevinsky and Langlands parametrizations of irreducible representations, in a natural sense.

Proposition 3.2. *Let $\rho \in \mathcal{C}$ be given. Let E be a p-adic field with residue cardinality $q_E = q_F^{o(\rho)}$. Let*

$$\phi = \phi_{\nu_E^0, \rho} : \mathrm{Mult}_0^E \to \mathrm{Mult}_\rho$$

be the isomorphism defined in Section 2.1, where ν_E^0 stands for the trivial representation of $\mathrm{GL}_1(E)$. Let us write

$$\hat{Z} : \mathrm{Mult}_0^E \to \bigcup_{n \geq 0} \mathrm{Irr}(\mathcal{M}_n^{q_E})$$

---

[7] More precisely, the cited theorem relates the principal block to representations of the Iwahori-Hecke algebra of G_n, which is then known by a result of Bernstein to be isomorphic to our definition of $H(n, q)$ (see, for example, the lectures [28]).

for the map as defined above (but for the field E in place of F). Then we can choose the collection $\{U_{\Theta(\rho,n)}\}_n$, so that it satisfies

$$\hat{Z}(\mathfrak{m}) \cong U_{\Theta(\rho,n_{\mathfrak{m}})}(Z(\phi(\mathfrak{m})))$$

for all $\mathfrak{m} \in \mathrm{Mult}_0^E$, where $\hat{Z}(\mathfrak{m}) \in \mathcal{M}_{n_{\mathfrak{m}}}^{q_E}$.

Proof. Recall that for a multisegment $\mathfrak{m} = \sum_{i=1}^k \Delta_i \in \mathrm{Mult}_0^E$, $Z(\mathfrak{m})$ is defined to be the unique irreducible subrepresentation of $\zeta(\mathfrak{m}) = Z(\Delta_1) \times \cdots \times Z(\Delta_k)$ (for a prescribed ordering of the segments of $\mathfrak{m}$, see Section 2.1). Furthermore, each $Z(\Delta_i)$ is the unique irreducible subrepresentation of

$$\nu_E^{s_1^i} \times \cdots \times \nu_E^{s_{n_i}^i},$$

for some numbers $s_1^i, \ldots, s_{n_i}^i \in \mathbb{C}$.

Similarly, $Z(\phi(\mathfrak{m}))$ is constructed by the same process, replacing the role of each $\nu_E^{s_j^i}$ by $\rho\nu^{s_j^i}$.

Hence, using the compatibility property (3.1), the fact that $\{U_\Theta\}$ are all equivalences of abelian categories and Remark 3.1, it is enough to check the statement for $U_{\Theta(\rho,1)}$. In other words, we need to verify that $U_{\Theta(\rho,1)}$ is allowed to be chosen so that for all $s \in \mathbb{C}$, the representation $\rho\nu^s$ in $\mathfrak{R}(\Theta(\rho,1))$ is sent to the character of $H(1,q_E) = \mathbb{C}[y_1, y_1^{-1}]$ given by $y_1 \mapsto q_E^s$.

Let us recall the construction of $U_{\Theta(\rho,1)}$ in [25]. Let V be the space of the representation $\rho \in \mathcal{C}(G_m)$. Then G_m acts on the space $V[t,t^{-1}]$ by

$$\bar{\rho}(g) \cdot (vt^k) = \rho(g)vt^{k+\mathrm{val}(\det(g))} \quad \text{for all } g \in G_m, \, v \in V, \, k \in \mathbb{Z},$$

where $|a|_F = q_F^{\mathrm{val}(a)}$ for $a \in F$.

The natural action of the ring of Laurent polynomials $\mathbb{C}[t,t^{-1}]$ on $V[t,t^{-1}]$ intertwines the $\bar{\rho}$ action of G_m. The construction in [25] specifies a subrepresentation

$$W < V[t,t^{-1}]$$

of $\bar{\rho}$ which is stable under the action of the subalgebra $H = \mathbb{C}[t^{o(\rho)}, t^{-o(\rho)}] \subseteq \mathbb{C}[t,t^{-1}]$. Moreover, we have

$$\mathrm{End}(\bar{\rho}|_W) = H.$$

The functor $U_{\Theta(\rho,1)}$ is then defined by taking $\pi \in \mathfrak{R}(\Theta(\rho,1))$ to

$$U_{\Theta(\rho,1)}(\pi) := \mathrm{Hom}_{G_m}(W, \pi),$$

viewed as a $H \cong H(1, q_E)$-module.

For all $s \in \mathbb{C}$, there is a projection of representations $\psi_s : \bar{\rho} \to \rho\nu^s$ given by

$$\psi_s(vt^k) = q_F^{ks}v.$$

It is easy to verify that $U_{\Theta(\rho,1)}(\rho\nu^s)$ is a one-dimensional space spanned by $\psi_s|_W$.

Note that for the generator $t^{o(\rho)} \in H$ and $w \in W$, we have

$$\psi_s(t^{o(\rho)} \cdot w) = q_F^{o(\rho)s}\psi_s(w).$$

Hence, $H(1, q_E)$ acts on $U_{\Theta(\rho,1)}(\rho\nu^s)$ through the character $y_1 \mapsto q_E^s$. ∎

3.2. Consequences of a result of Hernandez

Section 6 deals with quantum affine algebras and the quantum affine Schur–Weyl duality functor. The following theorem on representations of affine Hecke algebras will be shown to be a manifestation of a theorem of Hernandez [27], when transferred through the duality functor.

Theorem 3.3. *Let* $\pi_1 = \hat{Z}(\mathfrak{m}_1), \pi_2 = \hat{Z}(\mathfrak{m}_2), \ldots, \pi_t = \hat{Z}(\mathfrak{m}_t)$ *be irreducible representations in* $\mathcal{M}_{k_1}^{q_F}, \mathcal{M}_{k_2}^{q_F}, \ldots, \mathcal{M}_{k_t}^{q_F}$, *respectively. If* $\pi_i \times \pi_j$ *has a unique irreducible quotient which is parameterized by* $\hat{Z}(\mathfrak{m}_i + \mathfrak{m}_j)$, *for all* $1 \leq i < j \leq t$, *then the representation* $\pi_1 \times \cdots \times \pi_t$ *has a unique irreducible quotient which is parameterized by* $\hat{Z}(\mathfrak{m}_1 + \cdots + \mathfrak{m}_t)$.

We would like to extend the statement of the theorem above slightly.

For that purpose let us recall that each algebra $H(n, q)$ is equipped with the *Iwahori–Matsumoto involutive automorphism* θ_n (see [36, Section I.6]). It gives rise to an involutive auto-equivalence of $\mathcal{M}_n^q$. In order to ease notation, we will simply write θ for all these involutions.

It is known [36, Lemme I.7.1][8] that the Iwahori–Matsumoto involution is compatible with the induction product, in the following sense. For all representations $\pi_1, \ldots, \pi_t$ in $\mathcal{M}_{k_1}^q, \ldots, \mathcal{M}_{k_t}^q$, respectively, we have

$$\theta(\pi_1 \times \cdots \times \pi_t) \cong \theta(\pi_t) \times \cdots \times \theta(\pi_1).$$

When restricting θ to irreducible representations, we obtain [36, Proposition I.7.3] what is known as the *Zelevinsky involution* in the context of p-adic groups, that is,

$$\theta(\hat{Z}(\mathfrak{m})) \cong \hat{L}(\mathfrak{m})$$

for all $\mathfrak{m} \in \mathrm{Mult}_0$.

Corollary 3.4. *The statement of Theorem 3.3 remains valid when* $\hat{Z}$ *is replaced with* $\hat{L}$. *In addition, "quotient" may be replaced with "subrepresentation".*

Proof. It follows from an application of the functor θ and Proposition 2.2, which remains valid for representations of affine Hecke algebras through the equivalences $\{U_n\}$. ∎

4. Classes of irreducible representations

We would like to study certain classes of representations in Irr which naturally occur in the derivatives of unitarizable representations.

4.1. Quasi-Speh representations

We will first deal with a subclass of ladder representations, which we will call *quasi-Speh* representations. These are parameterized by integers $a, b \in \mathbb{Z}_{>0}$, $c \in \mathbb{Z}$, so that

---

[8]There is an obvious typo in the statement of the lemma in that source.

$0 \leq c \leq a$, and a representation $\rho \in \mathrm{Irr}^u \cap \mathcal{C}$. For such data, we define the multisegment

$$\mathfrak{m}_\rho^{a,b,c} = \left[\frac{b+a}{2} - c, \frac{b+a}{2} - 1\right]_\rho + \sum_{i=2}^{b}\left[\frac{b-a}{2} + 1 - i, \frac{b+a}{2} - i\right]_\rho \in \mathrm{Mult}_\rho.$$

We then set $\pi_\rho^{a,b,c} := L(\mathfrak{m}_\rho^{a,b,c})$ to be a quasi-Speh representation.

Note that for $a = c$, these are the usual Speh representations $\pi_\rho^{a,b} = \pi_\rho^{a,b,a}$ defined in Section 2.4. We also note that for $b > 1$ and all a, we have $\pi_\rho^{a,b,0} = \pi_\rho^{a,b-1}\nu^{-1/2}$.

The following identities make the class of quasi-Speh representations relevant to our discussion. See [32, Section 5.4] for the proof, which is attributed to Tadic.

Proposition 4.1. *Let $\rho \in \mathrm{Irr}^u \cap \mathcal{C}$ be a given representations of G_d. Let $a, b \in \mathbb{Z}_{>0}$ be given. Then the formula*

$$(\pi_\rho^{a,b})^{(i)} \cong \begin{cases} \pi_\rho^{a,b,a-k}, & i = kd, \ 0 \leq k \leq a, \\ 0, & \text{otherwise}, \end{cases}$$

gives the derivatives of the Speh representation $\pi_\rho^{a,b}$.

Let us define a preorder on the class of quasi-Speh representations. We will write

$$\pi_\rho^{a_1,b_1,c_1} \preceq \pi_\rho^{a_2,b_2,c_2}$$

if $a_1 + b_1 < a_2 + b_2$, or if $a_1 + b_1 = a_2 + b_2$ and $a_1 \leq a_2$.

For any given $\rho \in \mathrm{Irr}^u \cap \mathcal{C}$, the restriction of $\preceq$ to the collection $\{\pi_\rho^{a,b,c}\}_{a,b,c}$ gives a total preorder.

Proposition 4.2. *If $\pi_1 = L(\mathfrak{m}_1)$ and $\pi_2 = L(\mathfrak{m}_2)$ are two quasi-Speh representations, which satisfy $\pi_2 \preceq \pi_1$, then $\pi_1 \times \pi_2$ has a unique irreducible quotient which is given by $L(\mathfrak{m}_1 + \mathfrak{m}_2)$.*

Proof. Suppose that $\pi_1 = \pi_\rho^{a_1,b_1,c_1}$ and $\pi_2 = \pi_\rho^{a_2,b_2,c_2}$. Let us denote

$$\mathfrak{m}_1 = \mathfrak{m}_\rho^{a_1,b_1,c_1} = \Delta_1 + \cdots + \Delta_{k_1}, \quad \mathfrak{m}_2 = \mathfrak{m}_\rho^{a_2,b_2,c_2} = \Delta_1' + \cdots + \Delta_{k_2}',$$

so that $\Delta_{k_1} \prec \cdots \prec \Delta_1$ and $\Delta_{k_2}' \prec \cdots \prec \Delta_1'$ are the defining segments. When c_1 (resp. c_2) equal to zero, the condition $\Delta_2 \prec \Delta_1$ (resp. $\Delta_2' \prec \Delta_1'$) is waived, but we will still write Δ_1 (resp. Δ_1'), while referring to the empty segment.

By the same considerations as in the proof of Corollary 3.4, it is enough to prove that $Z(\mathfrak{m}_1 + \mathfrak{m}_2)$ is the socle of $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$. To show that, we will apply the Lapid–Mínguez criterion of Theorem 2.8.

Recall the sets $X = X_{\mathfrak{m}_1;\mathfrak{m}_2}, Y = Y_{\mathfrak{m}_1;\mathfrak{m}_2}$ and the relation $\leftrightarrow$, as they were defined in Section 2.5.

We need to show there is an injective function $f : X \to Y$, which satisfies $x \leftrightarrow f(x)$ for all $x \in X$.

Let us write the set of indices

$$K = \{(i, i-1) \in [1, \ldots, k_1] \times [1, \ldots, k_2]\}.$$

Note that if $X \cap K \neq \emptyset$, we have $e_\rho(\Delta_{i_1}) < e_\rho(\Delta'_{i_1-1}) = e_\rho(\Delta'_{i_1}) + 1$ for some index i_1. It follows that $b_1 + a_1 \leq b_2 + a_2$. Yet, since $\pi_2 \preceq \pi_1$ holds, we must have

$$b_1 + a_1 = b_2 + a_2 \quad \text{and} \quad a_2 \leq a_1.$$

This implies $b_1 \leq b_2$ and that $(i, i) \in Y$ holds, for all $2 \leq i \leq k_1$.

For every $(i, i - 1) \in X \cap K$, let us set $f(i, i - 1) = (i, i)$.

Now, let us consider the Speh representations given as $\hat{\pi}_1 = \pi_\rho^{a_1, b_1}$ and $\hat{\pi}_2 = \pi_\rho^{a_2, b_2}$. Then

$$\hat{\pi}_1 = L(\hat{\mathfrak{m}}_1) = L\left(\sum_{i=1}^{k_1} \hat{\Delta}_i\right), \quad \hat{\pi}_2 = L(\hat{\mathfrak{m}}_2) = L\left(\sum_{i=1}^{k_2} \hat{\Delta}'_i\right),$$

where $\hat{\Delta}_i = \Delta_i$ and $\hat{\Delta}'_i = \Delta'_i$ for all $2 \leq i$, while $b_\rho(\Delta_1) \geq b_\rho(\hat{\Delta}_1)$, $b_\rho(\Delta'_1) \geq b_\rho(\hat{\Delta}'_1)$, $e(\Delta_1) = e(\hat{\Delta}_1)$ and $e(\Delta'_1) = e(\hat{\Delta}'_1)$.

It is known that $\hat{\pi}_1 \times \hat{\pi}_2$ is irreducible, as a product of unitarizable irreducible representations. Hence, we can apply Theorem 2.8 to obtain a matching function $g : \hat{X} \to \hat{Y}$ for the restricted bipartite graph $(\hat{X} = X_{\hat{\mathfrak{m}}_1; \hat{\mathfrak{m}}_2}, \hat{Y} = Y_{\hat{\mathfrak{m}}_1; \hat{\mathfrak{m}}_2}, \leftrightarrow)$.

Note that since $\pi_2 \preceq \pi_1$, we know that

$$e_\rho(\Delta_1) = \frac{1}{2}(b_1 + a_1) - 1$$

is no smaller than $e_\rho(\Delta'_j)$ for all $1 \leq j \leq k_2$. Thus, $X, \hat{X} \subseteq [2, \ldots, k_1] \times [1, \ldots, k_2]$.

Clearly, we also have

$$X \cap ([2, \ldots, k_1] \times [2, \ldots, k_2]) = \hat{X} \cap ([2, \ldots, k_1] \times [2, \ldots, k_2]),$$
$$Y \cap ([2, \ldots, k_1] \times [2, \ldots, k_2]) = \hat{Y} \cap ([2, \ldots, k_1] \times [2, \ldots, k_2]).$$

Suppose that $(i, j) \in (X \cap \hat{X}) \setminus K$, and denote

$$(i', j') = g(i, j) \in \hat{Y}.$$

In case $j' = j + 1$, we see that $i', j' \geq 2$, which implies that $(i', j') \in Y$. Otherwise, $(i', j') = (i - 1, j)$. In case that $i > 2$, we have

$$\Delta_{i-1} = \overrightarrow{\Delta_i} \prec \overrightarrow{\Delta'_j},$$

since $(i, j) \in X$. Thus, again we have $(i', j') \in Y$.

As for the case that $i = 2$, the inclusion $(2, j) \in X$ would have implied that

$$e_\rho(\Delta'_1) \geq e_\rho(\Delta'_j) \geq e_\rho(\Delta_2) + 1 = \frac{b_1 + a_1}{2} - 1.$$

On the other hand, it follows from $\pi_2 \preceq \pi_1$ that $e_\rho(\Delta'_1) \leq \frac{b_1 + a_1}{2} - 1$. Hence, $j = 1$ and $(i, j) \in K$ would have given a contradiction.

Having established that $g((X \cap \hat{X}) \setminus K)$ is always contained in Y, we can define $f = g$ on $(X \cap \hat{X}) \setminus K$. Injectivity is not interfered with the definition of $f|_K$, because when $x \in X$ satisfies $x \leftrightarrow (i, i)$ for an index i, we clearly must have $x \in K$.

We are left with the task of extending f injectively to

$$X \setminus (\hat{X} \cup K) \subseteq [3, \ldots, k_1] \times \{1\}.$$

Suppose that $(i, 1) \in X \setminus (\hat{X} \cup K)$. Since $i - 1 > 1$, we again have

$$\Delta_{i-1} = \overrightarrow{\Delta_i} \prec \overrightarrow{\Delta_1},$$

which gives $(i - 1, 1) \in Y$. Moreover, the same argument shows that $(i, 1) \notin \hat{X}$ implies $(i - 1, 1) \notin \hat{Y}$. Hence, we can extend f by setting $f(i, 1) = (i - 1, 1)$ without harming injectivity. ∎

4.2. Quasi-Arthur-type representations

We say that $\pi \in \mathrm{Irr}$ is of *quasi-Arthur type* if it has the form

$$\pi = L\big(\mathfrak{m}_\rho^{a_1,b_1,c_1} + \cdots + \mathfrak{m}_\rho^{a_k,b_k,c_k}\big)$$

for some integers $\{a_i, b_i, c_i\}_{i=1}^k$ and $\rho \in \mathrm{Irr}^u \cap \mathcal{C}$.

The significance of quasi-Arthur-type representations appears through the following corollary of previous discussions.

Proposition 4.3. *Let $\rho \in \mathrm{Irr}^u \cap \mathcal{C}$ be fixed. Suppose that*

$$\pi_\rho^{a_k,b_k,c_k} \preceq \cdots \preceq \pi_\rho^{a_1,b_1,c_1}$$

are given quasi-Speh representations, for some integers $\{a_i, b_i, c_i\}$. Then the product

$$\pi_\rho^{a_1,b_1,c_1} \times \cdots \times \pi_\rho^{a_k,b_k,c_k}$$

has a unique irreducible quotient, whose isomorphism class is given by the quasi-Arthur-type representation

$$L\big(\mathfrak{m}_\rho^{a_1,b_1,c_1} + \cdots + \mathfrak{m}_\rho^{a_k,b_k,c_k}\big).$$

Proof. By Proposition 3.2 and property (3.1), we can assume that we are dealing with finite-dimensional representations of the corresponding affine Hecke algebras. The statement then follows from Proposition 4.2 combined with Corollary 3.4. ∎

For $\pi \in \mathfrak{R}(G_n)$, we write $\pi^\vee \in \mathfrak{R}(G_n)$ for the contragredient (smooth dual) representation. Recall, that this involution of the category was composed with the Gelfand–Kazhdan involution of Section 2.2 to obtain Proposition 2.2.

For a segment $\Delta = [a, b]_\rho \in \mathrm{Seg}$, we write

$$\Delta^\vee = [-b, -a]_{\rho^\vee}.$$

This is an involution which naturally extends to Mult. When $\pi = L(\mathfrak{m}) = Z(\mathfrak{n}) \in \mathrm{Irr}$, we have

$$\pi^\vee = L(\mathfrak{m}^\vee) = Z(\mathfrak{n}^\vee)$$

(in particular, the operation of smooth dual commutes with the Zelevinsky involution, that is, sending $Z(\mathfrak{m})$ to $L(\mathfrak{m})$).

Let $\rho \in \mathrm{Irr}^u \cap \mathcal{C}$ be given. We would like to define a similar involution on Mult_ρ. Given $\Delta = [a, b]_{\rho\nu^s} \in \mathrm{Seg}_\rho$, we set

$$\Delta^! = [-b, -a]_{\rho\nu^{-s}},$$

and extend it to an involution $\mathfrak{m} \mapsto \mathfrak{m}^!$, for all $\mathfrak{m} \in \mathrm{Mult}_\rho$.

When $\rho \cong \rho^\vee$, we clearly have $\mathfrak{m}^! = \mathfrak{m}^\vee$.

Now, given $\mathfrak{m} \in \mathrm{Mult}_\rho$, we can write a unique decomposition $\mathfrak{m} = \mathfrak{m}_s + \mathfrak{m}_a$, with $\mathfrak{m}_s, \mathfrak{m}_a \in \mathrm{Mult}_\rho$, so that $\mathfrak{m}_s^! = \mathfrak{m}_s$ and $\mathfrak{m}_s$ is the maximal multisegment with that property.

Clearly, for a segment $\Delta \in \mathfrak{m}_a$, we have $\Delta^! \notin \mathfrak{m}_a$.

Lemma 4.4. *Suppose that π is a quasi-Arthur-type representation. If $\overline{\pi} := \nu^{-1/2}\pi^\vee$ is of quasi-Arthur type as well, given as*

$$\overline{\pi} = L\big(\mathfrak{m}_\rho^{a_1, b_1, c_1} + \cdots + \mathfrak{m}_\rho^{a_k, b_k, c_k}\big),$$

then we can write

$$\pi = L\left(\sum_{i \in I^-} \mathfrak{m}_{\rho^\vee}^{a_i, b_i - 1} + \sum_{i \in I^+} \mathfrak{m}_{\rho^\vee}^{a_i, b_i + 1, 0}\right)$$

for a certain disjoint partition $\{1, \ldots, k\} = I^- \cup I^+$, where $\mathfrak{m}_\rho^{a,o}$ is viewed as an empty multisegment. In particular,

$$\pi \cong L(\mathfrak{m} + \mathfrak{m}'\nu^{-1/2})$$

for some proper Arthur-type representations $L(\mathfrak{m}), L(\mathfrak{m}')$.

Proof. We can assume that

$$\pi = L\big(\mathfrak{m}_{\rho^\vee}^{a_1', b_1', c_1'} + \cdots + \mathfrak{m}_{\rho^\vee}^{a_m', b_m', c_m'}\big)$$

for some integers a_i', b_i', c_i'. Let us write

$$\mathfrak{n} = \big(\mathfrak{m}_{\rho^\vee}^{a_1', b_1', c_1'} + \cdots + \mathfrak{m}_{\rho^\vee}^{a_m', b_m', c_m'}\big)\nu^{1/2}.$$

Clearly, we have

$$\mathfrak{n}_s = \mathfrak{m}_{\rho^\vee}^{a_1', b_1' - 1} + \cdots + \mathfrak{m}_{\rho^\vee}^{a_m', b_m' - 1},$$

and for every segment $\Delta \in \mathfrak{n}_a$, we have

$$b_{\rho^\vee}(\Delta) + e_{\rho^\vee}(\Delta) > 0. \tag{4.1}$$

From $\nu^{1/2}\pi = \overline{\pi}^\vee$ we deduce that

$$\mathfrak{n} = (\mathfrak{m}_\rho^{a_1, b_1, c_1})^\vee + \cdots + (\mathfrak{m}_\rho^{a_k, b_k, c_k})^\vee$$

$$= \mathfrak{m}_{\rho^\vee}^{a_1, b_1 - 2} + \cdots + \mathfrak{m}_{\rho^\vee}^{a_k, b_k - 2} + \Delta_1^+ + \cdots + \Delta_k^+ + \Delta_1^- + \cdots + \Delta_k^-,$$

where

$$\Delta_i^+ = \begin{cases} \left[\frac{b_i - a_i}{2}, \frac{b_i + a_i}{2} - 1\right]_{\rho^\vee}, & b_i > 1, \\ 0, & b_i = 1, \end{cases}$$

and

$$\Delta_i^- = \left[-\frac{b_i + a_i}{2} + 1, -\frac{b_i + a_i}{2} + c_i \right]_{\rho^\vee},$$

and $\mathfrak{m}_{\rho^\vee}^{a,0}$, $\mathfrak{m}_{\rho^\vee}^{a,-1}$ are understood as empty multisegments. Clearly,

$$\mathfrak{t} := \mathfrak{m}_{\rho^\vee}^{a_1,b_1-2} + \cdots + \mathfrak{m}_{\rho^\vee}^{a_k,b_k-2} \subseteq \mathfrak{n}_s.$$

Note that $b_{\rho^\vee}(\Delta_i^-) + e_{\rho^\vee}(\Delta_i^-) \leq 0$ for all i with non-empty Δ_i^-. Suppose that Δ_i^- is such a non-empty segment. Then, by (4.1) $\Delta_i^- \notin \mathfrak{n}_a$, which means $\Delta_i^- \in \mathfrak{n}_s$. In case that $b_1 = 1$ and $c_i = a_i$, we have

$$\Delta_i^- = \mathfrak{m}_{\rho^\vee}^{a_i,1} = (\mathfrak{m}_{\rho^\vee}^{a_i,1})^!.$$

Otherwise, $\Delta_i^- \neq (\Delta_i^-)^! \in \mathfrak{n}$. Since $\mathfrak{t}^! = \mathfrak{t}$ and $b_{\rho^\vee}((\Delta_i^-)^!) + e_{\rho^\vee}((\Delta_i^-)^!) > 0$, we must have $(\Delta_i^-)^! = \Delta_j^+$, for some j. It follows that

$$\mathfrak{m}_{\rho^\vee}^{a_j,b_j} = \mathfrak{m}_{\rho^\vee}^{a_j,b_j-2} + \Delta_i^- + \Delta_j^+ \subseteq \mathfrak{n}_s.$$

We then can write

$$\mathfrak{n}_s = \sum_{j \in J} \mathfrak{m}_\rho^{a_j,b_j-2} + \sum_{j \in K} \mathfrak{m}_\rho^{a_j,b_j}, \quad \mathfrak{n}_a = \sum_{j \in J} \Delta_j^+$$

for disjoint subsets $J, K \subseteq \{1, \ldots, k\}$ such that $\{\Delta_j^+\}_{j \in J}$ are non-empty. Note that we have $b_i = 1$ for all $i \in \{1, \ldots, k\} \setminus (J \cup K)$.

The statement follows from the observations

$$\nu^{-1/2} \mathfrak{m}_{\rho^\vee}^{a_j,b_j} = \mathfrak{m}_{\rho^\vee}^{a_j,b_j+1,0}, \quad \nu^{-1/2}(\mathfrak{m}_{\rho^\vee}^{a_j,b_j-2} + \Delta_j^+) = \mathfrak{m}_{\rho^\vee}^{a_j,b_j-1}. \qquad \blacksquare$$

Corollary 4.5. *Suppose that*

$$\pi_1 = L\left(\mathfrak{m}_\rho^{a_1,b_1,c_1} + \cdots + \mathfrak{m}_\rho^{a_k,b_k,c_k} \right),$$

$$\pi_2 = L\left(\mathfrak{m}_{\rho^\vee}^{a_1',b_1',c_1'} + \cdots + \mathfrak{m}_{\rho^\vee}^{a_l',b_l',c_l'} \right),$$

are two quasi-Arthur-type representations satisfying $\pi_2 = \nu^{-1/2}\pi_1^\vee$. Then there are disjoint partitions $\{1, \ldots, k\} = I_1 \cup I_2 \cup I_3$ and $\{1, \ldots, l\} = J_1 \cup J_2 \cup J_3$ and bijections $u : I_1 \to J_2$ and $d : I_2 \to J_1$, which satisfy

$$
\begin{aligned}
(a_{u(i)}', b_{u(i)}') &= (a_i, b_i + 1) && \text{for all } i \in I_1, \\
(a_{d(i)}', b_{d(i)}') &= (a_i, b_i - 1) && \text{for all } i \in I_2, \\
b_i &= 1 && \text{for all } i \in I_3, \\
b_j' &= 1 && \text{for all } j \in J_3.
\end{aligned}
$$

Proof. By Lemma 4.4 we have an equality of multisegments of the form

$$\sum_{j \in J_1} \mathfrak{m}_\rho^{a_j',b_j'+1,0} + \sum_{j \in J_2} \mathfrak{m}_\rho^{a_j',b_j'-1} = \mathfrak{m}_\rho^{a_1,b_1,c_1} + \cdots + \mathfrak{m}_\rho^{a_k,b_k,c_k}$$

for a disjoint partition $\{1, \ldots, l\} = J_1 \cup J_2 \cup J_3$ such that $b_j' = 1$ for all $j \in J_3$, and $b_j' > 1$ for all $j \in J_2$.

Let us further write a disjoint partition $J_2 = J_2' \cup J_2''$, where $J_2' = \{j \in J_2 : b_j' = 2\}$. Similarly, we write $\{1, \ldots, k\} = I' \cup I''$, where $I' = \{i \in I : b_i = 1\}$.

For a multisegment $\mathfrak{m} = \sum_{s \in K} [\alpha_s, \beta_s]_\rho \in \mathrm{Seg}_\rho$, let us write a decomposition

$$\mathfrak{m} = \mathfrak{m}_- + \mathfrak{m}_0 + \mathfrak{m}_+,$$

where

$$\mathfrak{m}_- = \sum_{s \in K, \, \alpha_s + \beta_s < 0} [\alpha_s, \beta_s]_\rho,$$

$$\mathfrak{m}_0 = \sum_{s \in K, \, \alpha_s + \beta_s = 0} [\alpha_s, \beta_s]_\rho,$$

$$\mathfrak{m}_+ = \sum_{s \in K, \, \alpha_s + \beta_s > 0} [\alpha_s, \beta_s]_\rho.$$

We obtain the identity

$$
\begin{aligned}
\left(\mathfrak{m}_\rho^{a_1, b_1} + \cdots + \mathfrak{m}_\rho^{a_k, b_k} \right)_- &= \left(\mathfrak{m}_\rho^{a_1, b_1, c_1} + \cdots + \mathfrak{m}_\rho^{a_k, b_k, c_k} \right)_- \\
&= \left(\sum_{j \in J_1} \mathfrak{m}_\rho^{a_j', b_j' + 1, 0} + \sum_{j \in J_2} \mathfrak{m}_\rho^{a_j', b_j' - 1} \right)_- \\
&= \left(\sum_{j \in J_1} \mathfrak{m}_\rho^{a_j', b_j' + 1} + \sum_{j \in J_2} \mathfrak{m}_\rho^{a_j', b_j' - 1} \right)_-.
\end{aligned}
$$

A moment's reflection will show that such an identity forces a bijection $t : I'' \to J_1 \cup J_2''$, so that

$$(a_i, b_i) = (a_{t(i)}', b_{t(i)}' + 1),$$

in case that $t(i) \in J_1$, and

$$(a_i, b_i) = (a_{t(i)}', b_{t(i)}' - 1)$$

in case that $t(i) \in J_2''$.

In particular, we see that

$$
\begin{aligned}
\left(\sum_{j \in J_1} \mathfrak{m}_\rho^{a_j', b_j' + 1, 0} + \sum_{j \in J_2''} \mathfrak{m}_\rho^{a_j', b_j' - 1} \right)_0 &= \left(\sum_{j \in J_1} \mathfrak{m}_\rho^{a_j', b_j' + 1} + \sum_{j \in J_2''} \mathfrak{m}_\rho^{a_j', b_j' - 1} \right)_0 \\
&= \left(\sum_{i \in I''} \mathfrak{m}_\rho^{a_i, b_i} \right)_0 = \left(\sum_{i \in I''} \mathfrak{m}_\rho^{a_i, b_i, c_i} \right)_0,
\end{aligned}
$$

which also implies

$$\sum_{j \in J_2'} \mathfrak{m}_\rho^{a_j', 1} = \left(\sum_{j \in J_2'} \mathfrak{m}_\rho^{a_j', 1} \right)_0 = \left(\sum_{i \in I'} \mathfrak{m}_\rho^{a_i, 1, c_i} \right)_0.$$

Since

$$(\mathfrak{m}_\rho^{a, 1, c})_0 = \begin{cases} \mathfrak{m}_\rho^{a, 1}, & c = a, \\ 0, & c < a, \end{cases}$$

we see that

$$\sum_{j \in J_2'} \mathfrak{m}_\rho^{a_j',1} = \sum_{i \in \tilde{I}} \mathfrak{m}_\rho^{a_i,1}$$

for a subset $\tilde{I} \subseteq I'$. This implies a bijection $s : \tilde{I} \to J_2'$ such that

$$(a_i, b_i) = (a_{s(i)}', b_{s(i)}' - 1)$$

for all $i \in \tilde{I}$.

The desired bijections u and d are easily constructed out of t and s after denoting $I_1 = \tilde{I} \cup t^{-1}(J_2''), I_2 = t^{-1}(J_1)$ and $I_3 = I_1 \setminus \tilde{I}$. ∎

5. Main theorems

We would like to determine for which pairs of representations $\pi_1, \pi_2 \in \mathrm{Irr}^u$ such that $\pi_1 \in \mathrm{Irr}(G_n)$ and $\pi_2 \in \mathrm{Irr}(G_{n-1})$, the space

$$\mathrm{Hom}(\pi_1|_{G_{n-1}}, \pi_2)$$

is non-zero. Here we consider G_{n-1} as a subgroup of G_n embedded in the corner, as described in the introduction section.

Now, suppose that π_1, π_2 are Arthur-type representations. It is easy to verify that they can be written uniquely (up to ordering) in the form

$$\pi_1 = L\big(\mathfrak{m}_{\rho_1}^{a_1,b_1} + \cdots + \mathfrak{m}_{\rho_k}^{a_k,b_k}\big),$$
$$\pi_2 = L\big(\mathfrak{m}_{\rho_1'}^{a_1',b_1'} + \cdots + \mathfrak{m}_{\rho_l'}^{a_l',b_l'}\big).$$

Definition 5.1. We say that a pair (π_1, π_2) of Arthur-type representations is in *GGP position* if, in terms of the decomposition above, there are disjoint partitions

$$\{1, \ldots, k\} = I_1 \cup I_2 \cup I_3, \quad \{1, \ldots, l\} = J_1 \cup J_2 \cup J_3,$$

and bijections $u : I_1 \to J_2, d : I_2 \to J_1$, which satisfy

$$\begin{aligned}
(a_{u(i)}', b_{u(i)}') &= (a_i, b_i + 1), & \rho_{u(i)}' &\cong \rho_i & \text{for all } i \in I_1, \\
(a_{d(i)}', b_{d(i)}') &= (a_i, b_i - 1), & \rho_{d(i)}' &\cong \rho_i & \text{for all } i \in I_2, \\
b_i &= 1 \quad \text{for all } i \in I_3, & b_j' &= 1 & \text{for all } j \in J_3.
\end{aligned}$$

Our terminology *GGP position* stands for the authors' names Gan, Gross and Prasad, who have coined this condition in their Conjecture 1.1.

We are now going to prove two main intermediate steps that together will imply that a non-zero Hom space between two Arthur-type representations π_1, π_2 of G_n, G_{n-1} must imply that the pair (π_1, π_2) is in GGP position.

The first step (Section 5.1, Proposition 5.4) will transfer the non-vanishing of the restricted Hom space into the non-vanishing of another Hom space between certain der-

ivatives of π_1, π_2. This will have the immense advantage of translating the problem into one about the category of finite-length representations of a smaller group. It will be possible due the useful Bernstein–Zelevinsky filtration occurring when restricting to the mirabolic group situated amidst G_{n-1} and G_n.

The second step is proved in the following theorem, that states (using the tools developed in Section 4) that the non-zero derivative Hom space implies a combinatorial condition that can be read as the GGP position.

Put together, these steps will be summed up in Section 5.2 to state our main results.

Theorem 5.2. *Suppose that* $\pi_1, \pi_2 \in \mathrm{Irr}^u$ *are two Arthur-type representations, which satisfy*

$$\mathrm{Hom}(\nu^{1/2}\pi_1^{(i)}, {}^{(j)}\pi_2) \neq \{0\}$$

for some i, j. *Then* (π_1, π_2) *is in GGP position.*

Proof. Let us write

$$\pi_1 \cong \pi_{\rho_1'}^{a_1', b_1'} \times \cdots \times \pi_{\rho_k'}^{a_k', b_k'} \quad \text{and} \quad \pi_2 \cong \pi_{\rho_1}^{a_1, b_1} \times \cdots \times \pi_{\rho_l}^{a_l, b_l}.$$

Since the above products are independent of the order in which the factors are taken, we can assume that

$$a_i + b_i \geq a_j + b_j \quad \text{and} \quad a_i' + b_i' \geq a_j' + b_j'$$

for all $i < j$ and that $a_i \geq a_j$ (respectively, $a_i' \geq a_j'$), in case that $a_i + b_i = a_j + b_j$ (respectively, $a_i' + b_i' = a_j' + b_j'$).

Recall that

$$(\pi_\rho^{a,b})^\vee \cong \pi_{\rho^\vee}^{a,b}.$$

Then, by Propositions 2.9, 4.1 and the assumption on the non-vanishing homomorphism space, we deduce that

$$\mathrm{Hom}\left(\nu^{1/2}\left(\pi_{\rho_1'}^{a_1', b_1', c_1'} \times \cdots \times \pi_{\rho_l'}^{a_l', b_l', c_l'}\right), \left(\pi_{\rho_1^\vee}^{a_1, b_1, c_1} \times \cdots \times \pi_{\rho_k^\vee}^{a_k, b_k, c_k}\right)^\vee\right) \neq \{0\}$$

for some integers $c_1, \ldots, c_k, c_1', \ldots, c_l'$.

By rearranging the factors in the product if necessary and using Corollary 2.4, we can decompose the above homomorphism space as

$$\bigotimes_{i=1}^{t} \mathrm{Hom}\left(\nu^{1/2}\left(\pi_{\rho^i}^{a_{r_i'(1)}', b_{r_i'(1)}', c_{r_i'(1)}'} \times \cdots \times \pi_{\rho^i}^{a_{r_i'(s_i')}', b_{r_i'(s_i')}', c_{r_i'(s_i')}'}\right),\right.$$
$$\left.\left(\pi_{(\rho^i)^\vee}^{a_{r_i(1)}, b_{r_i(1)}, c_{r_i(1)}} \times \cdots \times \pi_{(\rho^i)^\vee}^{a_{r_i(s_i)}, b_{r_i(s_i)}, c_{r_i(s_i)}}\right)^\vee\right),$$

where $\{\rho^1, \ldots, \rho^t\} = \{\rho_i\}_{i=1}^k \cup \{\rho_i'\}_{i=1}^l$ such that the ρ^i are pairwise non-isomorphic unitary supercuspidal representations, $r_i : \{1, \ldots, s_i\} \to \{1, \ldots, k\}$ are ascending injections such that $\{1, \ldots, k\} = \dot{\bigcup}_i \mathrm{Im}\, r_i$ and $\rho_{r(i)} \cong \rho^i$, and $r_i' : \{1, \ldots, s_i'\} \to \{1, \ldots, l\}$ are ascending injections such that $\{1, \ldots, l\} = \dot{\bigcup}_i \mathrm{Im}\, r_i'$ and $\rho'_{r'(i)} \cong \rho^i$.[9]

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[9]Here $\dot{\bigcup}$ stands for a disjoint union.

In particular, the above decomposition shows that it is enough to consider the case where all ρ_i's and ρ_j''s are isomorphic. In other words, we are free to assume for the rest of the proof that π_1, π_2 are of proper Arthur type and that $\rho_i \cong \rho_j' \cong \rho$ for all i, j and one fixed $\rho \in \mathcal{C} \cap \mathrm{Irr}^u$.

Now, note that the given quasi-Speh representations satisfy

$$\pi_{\rho^\vee}^{a_k, b_k, c_k} \preceq \cdots \preceq \pi_{\rho^\vee}^{a_1, b_1, c_1} \quad \text{and} \quad \pi_\rho^{a_l', b_l', c_l'} \preceq \cdots \preceq \pi_\rho^{a_1', b_1', c_1'}.$$

Hence, by Proposition 4.3, we have the quasi-Arthur-type representation

$$\sigma_2 := L\big(\mathfrak{m}_{\rho^\vee}^{a_1, b_1, c_1} + \cdots + \mathfrak{m}_{\rho^\vee}^{a_k, b_k, c_k}\big)$$

as the unique irreducible quotient of $\pi_{\rho^\vee}^{a_1, b_1, c_1} \times \cdots \times \pi_{\rho^\vee}^{a_k, b_k, c_k}$, while

$$\sigma_1 := L\big(\mathfrak{m}_\rho^{a_1', b_1', c_1'} + \cdots + \mathfrak{m}_\rho^{a_l', b_l', c_l'}\big)$$

as the unique irreducible quotient of $\pi_\rho^{a_1', b_1', c_1'} \times \cdots \times \pi_\rho^{a_l', b_l', c_l'}$.

By applying Lemma 2.1, we can deduce from the non-vanishing Hom space that $\sigma_1 \nu^{1/2} \cong \sigma_2^\vee$. Finally, the statement follows from Corollary 4.5. ∎

5.1. Bernstein–Zelevinsky filtration

In order to study the morphism space above, we will make use of the analysis in [4] of restrictions of representations in Irr. Let us sketch the main ingredients used in that reference. The reader can also refer to [9, 38] for very similar discussions.

Recall the p-adic *mirabolic* groups $\{P_n\}$ situated within the inclusions

$$G_{n-1} < P_n < G_n.$$

In matrix form, P_n is defined to be the subgroup of G_n consisting of matrices whose bottom row is given by $(0 \ \ldots \ 0 \ 1)$.

Bernstein and Zelevinsky defined families of exact functors

$$\Phi^- : \mathfrak{S}(P_n) \to \mathfrak{S}(P_{n-1}), \quad \Phi^+, \hat{\Phi}^+ : \mathfrak{S}(P_n) \to \mathfrak{S}(P_{n+1}),$$
$$\Psi^- : \mathfrak{S}(P_n) \to \mathfrak{S}(G_{n-1}), \qquad \Psi^+ : \mathfrak{S}(G_n) \to \mathfrak{S}(P_{n+1}),$$

together with a set of identities between them. We will not use the definition of these functors, but let recall they are all defined through suitable induction (in case of $+$ functors) or restriction of coinvariants (in case of $-$ functors). The functor Φ^+ is the subfunctor of $\hat{\Phi}^+$ that takes the compactly supported sections of the induced representation.

In particular, Φ^- is both left adjoint to $\hat{\Phi}^+$ and right adjoint to Φ^+, while Ψ^- is left adjoint to Ψ^+.

In terms of these functors, for every $0 < i \leq n$, the i-th derivative of a representation $\pi \in \mathfrak{R}(G_n)$ is given as

$$\pi^{(i)} = \Psi^-(\Phi^-)^{i-1}(\pi|_{P_n}) \in \mathfrak{R}(G_{n-i}),$$

where $(\Phi^-)^{i-1}$ denotes the $(i-1)$-st consecutive composition of Φ^-.

Proposition 5.3. *For any representation $\tau \in \Re(P_n)$ on a vector space V, there is a filtration of P_n-representations*

$$\{0\} = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_0 = V$$

such that

$$V_i / V_{i+1} \cong (\Phi^+)^i \Psi^+ \Psi^- (\Phi^-)^i (\tau)$$

as a P_n-representation, for all $0 \leq i < n$.

Following the terminology of [9], let us define the *left i-th derivative* of $\pi \in \Re(G_n)$ by setting

$$^{(i)}\pi := \left((\pi^\vee)^{(i)}\right)^\vee.$$

Proposition 5.4. *Let $\pi_1 \in \Re(G_n)$ and $\pi_2 \in \Re(G_{n-1})$ be given. Let $\{V_i\}$ be the filtration of $\pi_1|_{P_n}$ as described in Proposition 5.3. Then we have a natural identification of homomorphism spaces*

$$\mathrm{Hom}_{G_{n-1}}(V_i / V_{i-1}, \pi_2) \cong \mathrm{Hom}_{G_{n-i-1}}(\nu^{1/2} \pi_1^{(i+1)}, {}^{(i)}\pi_2)$$

for all $0 \leq i < n$. In particular, a non-zero element in $\mathrm{Hom}_{G_{n-1}}(\pi_1|_{G_{n-1}}, \pi_2)$ gives rise to a non-zero element in $\mathrm{Hom}_{G_{n-i-1}}(\nu^{1/2} \pi_1^{(i+1)}, {}^{(i)}\pi_2)$ for some i.

Proof. Let I (respectively, i) denote the induction (respectively, compact induction) functor from $\Re(G_{n-1})$ to $\Re(P_n)$. Recall that by standard Mackey theory, the restriction functor $\Re(P_n) \to \Re(G_{n-1})$ is left adjoint to I (see, e.g., [4, Proposition 1.9]).

Thus, by Proposition 5.3,

$$\mathrm{Hom}_{G_{n-1}}(V_i / V_{i-1}, \pi_2) \cong \mathrm{Hom}_{G_{n-1}}\left(\left.\left((\Phi^+)^i \Psi^+ \left(\pi_1^{(i+1)}\right)\right)\right|_{G_{n-1}}, \pi_2\right)$$

$$\cong \mathrm{Hom}_{P_n}\left((\Phi^+)^i \Psi^+ \left(\pi_1^{(i+1)}\right), I(\pi_2)\right)$$

$$\cong \mathrm{Hom}_{P_n}\left(I(\pi_2)^\vee, \left((\Phi^+)^i \Psi^+ \left(\pi_1^{(i+1)}\right)\right)^\vee\right).$$

Now, by [4, Proposition 3.4], we know that

$$\left((\Phi^+)^i \Psi^+ \left(\pi_1^{(i+1)}\right)\right)^\vee \cong \nu^{-1}(\hat{\Phi}^+)^i \Psi^+ \left(\left(\pi_1^{(i+1)}\right)^\vee\right)$$

$$\cong \nu^{-1/2}(\hat{\Phi}^+)^i \Psi^+ \left(\nu^{-1/2}\left(\pi_1^{(i+1)}\right)^\vee\right)$$

holds, while similarly $I(\pi_2)^\vee \cong \nu^{-1} i(\pi_2^\vee) \cong \nu^{-1/2} i(\nu^{-1/2}\pi_2^\vee)$.

From the above mentioned adjunctions of functors, we now see that

$$\mathrm{Hom}_{G_{n-1}}(V_i / V_{i-1}, \pi_2) \cong \mathrm{Hom}_{P_n}\left(i(\nu^{-1/2}\pi_2^\vee), (\hat{\Phi}^+)^i \Psi^+ \left(\nu^{-1/2}\left(\pi_1^{(i+1)}\right)^\vee\right)\right)$$

$$\cong \mathrm{Hom}_{G_{n-i-1}}\left(\Psi^-(\Phi^-)^i \left(i(\nu^{-1/2}\pi_2^\vee)\right), \nu^{-1/2}\left(\pi_1^{(i+1)}\right)^\vee\right).$$

Yet, it follows from [4, Proposition 4.13 (c)] that $\Phi^-\left(i(\nu^{-1/2}\pi_2^\vee)\right) \cong \pi_2^\vee|_{P_{n-1}}$. Hence,

$$\mathrm{Hom}_{G_{n-1}}(V_i / V_{i-1}, \pi_2) \cong \mathrm{Hom}_{G_{n-i-1}}\left((\pi_2^\vee)^{(i)}, \nu^{-1/2}\left(\pi_1^{(i+1)}\right)^\vee\right)$$

$$\cong \mathrm{Hom}_{G_{n-i-1}}(\nu^{1/2} \pi_1^{(i+1)}, {}^{(i)}\pi_2). \qquad \blacksquare$$

Corollary 5.5. *Let $\pi_1 \in \Re(G_n)$ and $\pi_2 \in \Re(G_{n-1})$ be given. Let $\{V_i\}$ be the filtration of $\pi_1|_{P_n}$ as described in Proposition 5.3. Then we have a natural identification of homology spaces*

$$\mathrm{Ext}^1_{G_{n-1}}(V_i/V_{i-1}, \pi_2) \cong \mathrm{Ext}^1_{G_{n-i-1}}(\nu^{1/2}\pi_1^{(i+1)}, {}^{(i)}\pi_2)$$

for all $0 \le i < n$.

Proof. The proof of Proposition 5.4 works for this case as well, after recalling that adjunctions of exact functors give natural identifications of Ext-spaces, as well as Hom-spaces. See, e.g., [38, Proposition 2.2]. ∎

5.2. Branching laws

We will first treat the Arthur-type case which was considered in Conjecture 1.1.

Theorem 5.6. *Let $\pi_1, \pi_2 \in \mathrm{Irr}^u$ be Arthur-type representations such that $\pi_1 \in \mathrm{Irr}(G_n)$ and $\pi_2 \in \mathrm{Irr}(G_{n-1})$. If*

$$\mathrm{Hom}_{G_{n-1}}(\pi_1|_{G_{n-1}}, \pi_2) \neq \{0\}$$

holds, then (π_1, π_2) must be in GGP position.

Proof. The last part of Proposition 5.4 implies that under the stated condition,

$$\mathrm{Hom}_{G_{n-i-1}}(\nu^{1/2}\pi_1^{(i+1)}, {}^{(i)}\pi_2) \neq \{0\}$$

for a certain i. Hence, the result follows from Theorem 5.2. ∎

The above theorem can also be extended into a branching law governing the irreducible unitarizable quotients of a restriction of any unitarizable irreducible representation.

Theorem 5.7. *Let $\pi, \sigma \in \mathrm{Irr}^u$ be two representations with the property that $\pi \in \mathrm{Irr}(G_n)$ and $\sigma \in \mathrm{Irr}(G_{n-1})$. Let*

$$\pi \cong \pi_0 \times \pi_1(\alpha_1) \times \cdots \times \pi_k(\alpha_k),$$
$$\sigma \cong \sigma_0 \times \sigma_1(\beta_1) \times \cdots \times \sigma_l(\beta_l)$$

be their decomposition as described in Section 2.4, that is, $\pi_i, \sigma_i \in \mathrm{Irr}^u$ are Arthur-type representations and $\{\alpha_i\}, \{\beta_i\}$ are sets of real numbers. If

$$\mathrm{Hom}_{G_{n-1}}(\pi|_{G_{n-1}}, \sigma) \neq \{0\}$$

holds, then:

(1) *The pair (π_0, σ_0) is in GGP position.*

(2) *For every i, j for which $\alpha_i = \beta_j$ holds, the pair (π_i, σ_j) is in GGP position.*

(3) *If $\alpha_i \notin \{\beta_1, \ldots, \beta_l\}$ for some i, then π_i is generic (equivalently, π_i is tempered, or equivalently, $\pi_i(\alpha_i)$ is generic), i.e.*

$$\pi_i = L(\mathfrak{m}^{a_1,1}_{\rho_1} + \cdots + \mathfrak{m}^{a_t,1}_{\rho_t}).$$

(4) *Similarly, if $\beta_j \notin \{\alpha_1, \ldots, \alpha_k\}$ for some j, then σ_j is generic.*

Proof. By Proposition 5.4 we know that $\mathrm{Hom}(\nu^{1/2}\pi^{(i+1)}, {}^{(i)}\sigma)$ is non-zero, for some i.

By Proposition 2.9 that means

$$H := \mathrm{Hom}\Big(\nu^{1/2}\big(\pi_0^{(c_0)} \times \pi_1(\alpha_1)^{(c_1)} \times \cdots \times \pi_k(\alpha_k)^{(c_k)}\big),$$
$$^{(d_0)}\sigma_0 \times {}^{(d_1)}\big(\sigma_1(\beta_1)\big) \times \cdots \times {}^{(d_l)}\big(\sigma_l(\beta_l)\big)\Big) \neq \{0\}$$

holds, for some $c_0, \ldots, c_k, d_0, \ldots, d_l$.

It is easy to see that given Arthur-type representations τ, τ', τ'' and numbers γ, γ' with $0 < \gamma < \gamma' < \frac{1}{2}$, the representations $\tau, \tau'(\gamma'), \tau''(\gamma'')$ have pairwise disjoint supercuspidal supports. The same clearly holds for their derivatives.

Hence, for $\alpha_i \notin \{\beta_1, \ldots, \beta_l\}$, we must have $\pi_i(\alpha_i)^{(c_i)} \in \mathrm{Irr}(G_0)$. Claim (3) then follows, with claim (4) following similarly.

Also, arguing as in the proof of Theorem 5.2, using Corollary 2.4, we see that

$$H = \mathrm{Hom}\big(\nu^{1/2}\pi_0^{(c_0)}, {}^{(d_0)}\sigma_0\big) \otimes \Bigg(\bigotimes_{(i,j):\ \alpha_i = \beta_j} \mathrm{Hom}\Big(\nu^{1/2}\pi_i(\alpha_i)^{(c_i)}, {}^{(d_j)}\big(\sigma_j(\beta_j)\big)\Big)\Bigg).$$

Claim (1) now follows from Theorem 5.2.

In order to establish claim (2), we need to note that for $\alpha = \alpha_i = \beta_j$, by Proposition 2.9 and Corollary 2.4, we have

$$\mathrm{Hom}\big(\nu^{1/2}\pi_i(\alpha)^{(c_i)}, {}^{(d_j)}(\sigma_j(\alpha))\big) \neq \{0\} \implies \begin{cases} \mathrm{Hom}\big(\nu^{1/2+\alpha}\pi_i^{(c)}, \nu^{\alpha}({}^{(d)}\sigma_j)\big) \neq \{0\} \\ \mathrm{Hom}\big(\nu^{1/2-\alpha}\pi_i^{(c')}, \nu^{-\alpha}({}^{(d')}\sigma_j)\big) \neq \{0\} \end{cases}$$

for some $c + c' = c_i$, $d + d' = d_j$. The claim then follows again from Theorem 5.2. ∎

5.3. Converse direction

Let us first state a certain corollary from a discussion in [33].

Lemma 5.8. *Let $\sigma_1, \sigma_2 \in \mathrm{Irr}^u$ be two Speh representations. Then $\nu^{1/2}\sigma_1 \times \sigma_2$ is irreducible. In particular, $\nu^{1/2}\sigma_1 \times \sigma_2 \cong \sigma_2 \times \nu^{1/2}\sigma_1$.*

Proof. Let us write $\sigma_1 = L(\mathfrak{m}_{\rho_1}^{a_1,b_1})$ and $\sigma_2 = L(\mathfrak{m}_{\rho_2}^{a_2,b_2})$. If $\rho_1 \not\cong \rho_2$ holds, the statement is clear. Otherwise, let us apply the irreducibility criterion from [33, Corollary 5.10].

For the purposes of this proof, let us view $\mathrm{supp}(\nu^{1/2})\sigma_1$ and $\mathrm{supp}(\sigma_2)$ as elements of Seg in a natural way. Namely,

$$\mathrm{supp}(\nu^{1/2}\sigma_1) = \left[-\frac{a_1 + b_1 - 3}{2}, \frac{a_1 + b_1 - 1}{2}\right]_{\rho_1},$$
$$\mathrm{supp}(\sigma_2) = \left[-\frac{a_2 + b_2 - 2}{2}, \frac{a_2 + b_2 - 2}{2}\right]_{\rho_2}.$$

By [33, Corollary 5.10], it is enough to show that as segments, $\mathrm{supp}(\nu^{1/2}\sigma_1)$ and $\mathrm{supp}(\sigma_2)$ do not precede each other. If $a_1 + b_1$ and $a_2 + b_2$ have distinct parities, it follows immediately. Otherwise, it is easy to see that the inequalities implied by demanding either of the segments to precede the other do not have a solution. ∎

Proposition 5.9. *Let (π_1, π_2) be a pair of Arthur-type representations in GGP position such that $\pi_1 \in \mathrm{Irr}(G_n)$ and $\pi_2 \in \mathrm{Irr}(G_{n-1})$. Let $\{V_i\}$ be the filtration of $\pi_1|_{P_n}$ as described in Proposition* 5.3. *Then*

$$\mathrm{Hom}_{G_{n-1}}(V_i / V_{i-1}, \pi_2) \neq \{0\}$$

for some i.

Proof. From the assumption on the GGP position, we are able to write the Arthur-type representations as products of Speh representations of a certain form. Namely,

$$\pi_1 = \pi_{\rho_1}^{a_1, b_1} \times \cdots \times \pi_{\rho_t}^{a_t, b_t} \times \pi_{\varrho_1}^{d_1, 1} \times \cdots \times \pi_{\varrho_s}^{d_s, 1} \times \pi_{\rho_1'}^{a_1', b_1'+1} \times \cdots \times \pi_{\rho_{t'}'}^{a_{t'}', b_{t'}'+1},$$

$$\pi_2 = \pi_{\rho_1'}^{a_1', b_1'} \times \cdots \times \pi_{\rho_{t'}'}^{a_{t'}', b_{t'}'} \times \pi_{\varrho_1'}^{d_1', 1} \times \cdots \times \pi_{\varrho_{s'}'}^{d_{s'}', 1} \times \pi_{\rho_1}^{a_1, b_1+1} \times \cdots \times \pi_{\rho_t}^{a_t, b_t+1}$$

for some $\{\rho_i\}, \{\rho_i'\}, \{\varrho_i\}, \{\varrho_i'\} \subseteq \mathcal{C} \cap \mathrm{Irr}^u$ and positive integers $\{a_i\}$, $\{b_i\}$, $\{a_i'\}$, $\{b_i'\}$, $\{d_i\}, \{d_i'\}$.

Now, let us also define the representations

$$\Pi_1 = \pi_{\rho_1}^{a_1, b_1} \times \cdots \times \pi_{\rho_t}^{a_t, b_t} \times \nu^{-1/2} \pi_{\rho_1'}^{a_1', b_1'} \times \cdots \times \nu^{-1/2} \pi_{\rho_{t'}'}^{a_{t'}', b_{t'}'},$$

$$\Pi_2 = \pi_{(\rho_1')^\vee}^{a_1', b_1'} \times \cdots \times \pi_{(\rho_{t'}')^\vee}^{a_{t'}', b_{t'}'} \times \nu^{-1/2} \pi_{\rho_1^\vee}^{a_1, b_1} \times \cdots \times \nu^{-1/2} \pi_{\rho_t^\vee}^{a_t, b_t}.$$

By Propositions 2.9 and 4.1, Π_1 appears as a quotient of $\pi_1^{(i)}$, while Π_2 appears as a quotient of $(\pi_2^\vee)^{(j)}$, for a certain choice of i, j.

Note that by successive application of Lemma 5.8, we see that

$$\Pi_2 \cong \nu^{-1/2} \pi_{\rho_1^\vee}^{a_1, b_1} \times \cdots \times \nu^{-1/2} \pi_{\rho_t^\vee}^{a_t, b_t} \times \pi_{(\rho_1')^\vee}^{a_1', b_1'} \times \cdots \times \pi_{(\rho_{t'}')^\vee}^{a_{t'}', b_{t'}'},$$

which, in particular, implies the isomorphism

$$\nu^{1/2} \Pi_1 \cong \Pi_2^\vee.$$

Hence, $\mathrm{Hom}(\nu^{1/2} \pi_1^{(i)}, {}^{(j)}\pi_2) \neq \{0\}$.

Note also that since Π_1, Π_2 turn out to be representations of the same group, we can deduce $j = i - 1$ from the assumption that $\pi_1 \in \mathrm{Irr}(G_n)$ and $\pi_2 \in \mathrm{Irr}(G_{n-1})$. Finally, the statement follows from Proposition 5.4. ∎

Let $\pi_1 \in \mathrm{Irr}(G_n)$ and $\pi_2 \in \mathrm{Irr}(G_{n-1})$ be given. Let $\{V_i\}$ be the filtration of $\pi_1|_{P_n}$ as described in Proposition 5.3.

We would like to make the straightforward observation, that in order to show the non-vanishing of $\mathrm{Hom}_{G_{n-1}}(\pi_1|_{G_{n-1}}, \pi_2)$, it is enough to find a number $0 \leq i < n$, for which the conditions

$$\begin{cases} \mathrm{Hom}_{G_{n-1}}(V_i / V_{i-1}, \pi_2) \neq \{0\}, \\ \mathrm{Ext}^1_{G_{n-1}}(V_j / V_{j-1}, \pi_2) = \{0\} \quad \text{for all } 0 \leq j < i \end{cases} \tag{5.1}$$

hold.

By Proposition 5.4 and Corollary 5.5, the set of conditions (5.1) is equivalent to

$$\begin{cases} \mathrm{Hom}_{G_{n-1}}(\nu^{1/2}\pi_1^{(i+1)}, {}^{(i)}\pi_2) \neq \{0\}, \\ \mathrm{Ext}^1_{G_{n-1}}(\nu^{1/2}\pi_1^{(j+1)}, {}^{(j)}\pi_2) = \{0\} \quad \text{for all } 0 \leq j < i. \end{cases}$$

Recall that an Arthur-type representation

$$\pi = L\big(\mathfrak{m}_{\rho_1}^{a_1, b_1} + \cdots + \mathfrak{m}_{\rho_k}^{a_k, b_k}\big)$$

is generic, when $b_i = 1$, for all $i = 1, \ldots, k$.

Theorem 5.10. *Let (π_1, π_2) be a pair of Arthur-type representations in GGP position such that $\pi_1 \in \mathrm{Irr}(G_n)$ and $\pi_2 \in \mathrm{Irr}(G_{n-1})$ and at least one of π_1, π_2 is generic. Then*

$$\mathrm{Hom}_{G_{n-1}}(\pi_1|_{G_{n-1}}, \pi_2) \neq \{0\}.$$

Proof. Let us assume for simplicity that π_1 is generic. The case of generic π_2 is proved by same arguments, while exchanging the roles of π_1, π_2.

From the assumptions we are able to write (in similarity with the proof of Proposition 5.9)

$$\pi_1 = \pi_{\rho_1}^{a_1, 1} \times \cdots \times \pi_{\rho_t}^{a_t, 1} \times \pi_{\varrho_1}^{d_1, 1} \times \cdots \times \pi_{\varrho_s}^{d_s, 1},$$

$$\pi_2 = \pi_{\varrho_1'}^{d_1', 1} \times \cdots \times \pi_{\varrho_{s'}'}^{d_{s'}', 1} \times \pi_{\rho_1}^{a_1, 2} \times \cdots \times \pi_{\rho_t}^{a_t, 2}$$

for some $\{\rho_i\}, \{\varrho_i\}, \{\varrho_i'\} \subseteq \mathcal{C} \cap \mathrm{Irr}^u$ and positive integers $\{a_i\}, \{d_i\}, \{d_i'\}$.

From Proposition 5.9 we know that $\mathrm{Hom}_{G_{n-1}}(V_{i_0}/V_{i_0-1}, \pi_2) \neq \{0\}$ for some i_0. Hence, by the discussion above, it is enough to show that $\mathrm{Ext}^1_{G_{n-1}}(\nu^{1/2}\pi_1^{(i+1)}, {}^{(i)}\pi_2)$ vanishes, for all $0 \leq i < i_0$.

Let us assume the contrary, that is, $\mathrm{Ext}^1_{G_{n-1}}(\nu^{1/2}\pi_1^{(i_1+1)}, {}^{(i_1)}\pi_2) \neq \{0\}$ for a given $0 \leq i_1 < i_0$. By Propositions 2.9 and 4.1, that means we have

$$\mathrm{Ext}^1_{G_{n-1}}(\nu^{1/2}\sigma_1, \sigma_2^{\vee}) \neq \{0\}$$

for some representations

$$\sigma_1 = \pi_{\rho_1}^{a_1, 1, c_1} \times \cdots \times \pi_{\rho_t}^{a_t, 1, c_t} \times \pi_{\varrho_1}^{d_1, 1, e_1} \times \cdots \times \pi_{\varrho_s}^{d_s, 1, e_s},$$

$$\sigma_2 = \pi_{(\varrho_1')^{\vee}}^{d_1', 1, e_1'} \times \cdots \times \pi_{(\varrho_{s'}')^{\vee}}^{d_{s'}', 1, e_{s'}'} \times \pi_{\rho_1^{\vee}}^{a_1, 2, f_1} \times \cdots \times \pi_{\rho_t^{\vee}}^{a_t, 2, f_t}.$$

Since non-trivial extensions can occur only between representations with the same central character, we see that

$$\mathrm{Re}(\nu^{1/2}\sigma_1) = \mathrm{Re}(\sigma_2^{\vee}) = -\mathrm{Re}(\sigma_2).$$

As a consequence, the equality

$$\mathrm{Re}(\nu^{1/2}\sigma_2) = -\mathrm{Re}(\sigma_1) = -\sum_{i=1}^{t} \mathrm{Re}(\pi_{\rho_i}^{a_i, 1, c_i}) - \sum_{i=1}^{s} \mathrm{Re}(\pi_{\varrho_i}^{d_i, 1, e_i})$$

holds. Yet,

$$\mathrm{Re}(\pi_\rho^{a,1,c}) = \mathrm{Re}\big(\rho\nu^{a/2-c+1/2} \times \rho\nu^{a/2-c+3/2} \times \cdots \times \rho\nu^{a/2-1/2}\big)$$

$$= k_\rho \sum_{i=1}^{c} \left(\frac{a+1}{2} - i\right) \geq 0$$

for all $\rho \in \mathcal{C}(G_{k_\rho}) \cap \mathrm{Irr}^u$ and integers $0 \leq c \leq a$. Hence, $\mathrm{Re}(\nu^{1/2}\sigma_2) \leq 0$.

On the other hand, for all $1 \leq i \leq t$, we have

$$\mathrm{Re}\big(\nu^{1/2}\pi_{\rho_i^\vee}^{a_i,2,f_i}\big) = \mathrm{Re}\big(\rho_i^\vee\nu^{-\frac{a-1}{2}} \times \rho_i^\vee\nu^{-\frac{a-1}{2}+1} \times \cdots \times \rho_i^\vee\nu^{\frac{a-1}{2}}\big) + \mathrm{Re}\big(\nu^1\pi_{\rho_i^\vee}^{a_i,1,f_i}\big)$$

$$= \mathrm{Re}\big(\nu^1\pi_{\rho_i^\vee}^{a_i,1,f_i}\big) \geq \mathrm{Re}\big(\pi_{\rho_i^\vee}^{a_i,1,f_i}\big) \geq 0.$$

Moreover, it is clear that the above inequality is strict, unless $f_i = 0$. Thus,

$$\mathrm{Re}(\nu^{1/2}\sigma_2) = \sum_{i=1}^{s'} \mathrm{Re}\big(\nu^{1/2}\pi_{(\varrho_i')^\vee}^{d_i',1,e_i'}\big) + \sum_{i=1}^{t} \mathrm{Re}\big(\nu^{1/2}\pi_{\rho_i^\vee}^{a_i,2,f_i}\big) = 0$$

with $f_i = 0$ and $e_i' = 0$ for all i.

Now, this implies that σ_2 appears only in the highest non-zero derivative of $\pi_2^\vee$. This is a contradiction to the facts that $i_1 < i_0$ and that $^{(i_0)}\pi_2$ is non-zero. ∎

6. Quantum affine algebras

We would like to prove Theorem 3.3 by applying results from the representation theory of quantum affine algebras.

6.1. Setting

Let us recall parts of the theory of quantum affine algebras and their finite-dimensional representations. We refer the reader to [13], [12, Chapter 12], [17] for comprehensive study and discussions of these objects.

We are interested in the Hopf $\mathbb{C}$-algebra $\mathcal{A}_{N,q} = U_q(\hat{\mathfrak{sl}}_N)$, which is defined for a fixed parameter $q \in \mathbb{C}^\times$. Its definition involves generators and relations which depend on the Cartan matrix of the affine Lie algebra $\hat{\mathfrak{sl}}_N$. We will refrain from stating the full definition, which can be easily found in the sources mentioned above, since our applications will not require it. Our assumption for the rest of this section will be that q *is not a root of unity*.

Recall from [13, Proposition 2.3] that, as vector spaces, we have a triangular decomposition

$$\mathcal{A}_{N,q} = U^- \otimes U^0 \otimes U^+,$$

where U^-, U^0, U^+ are subalgebras of $\mathcal{A}_{N,q}$.

Also, recall that the definition of $\mathcal{A}_{N,q}$ involves the invertible elements

$$k_i \in U^0, \quad i = 1, \ldots, N-1,$$

which generate a commutative co-commutative Hopf-subalgebra $K < \mathcal{A}_{N,q}$. This subalgebra can be seen as the Cartan algebra (zero-part) in the triangular decomposition of the smaller quantum group $U_q(\mathfrak{sl}_N)$, which is naturally embedded in $\mathcal{A}_{N,q}$.

Let Ω be the set of complex characters λ of K, which satisfy $\lambda(k_i) \in q^{\mathbb{Z}}$ for all $i = 1, \ldots, N-1$. Since K is co-commutative, Ω is an abelian group. Given a complex character P of the algebra U^0, we will write $\omega(P)$ to be its restriction to K.

We are further interested in the category $\mathcal{C}_N^q$ of *type* 1 representations of $\mathcal{A}_{N,q}$. It consists of all finite-dimensional representations V in which a certain specified central element $c^{1/2} \in U^0$ acts trivially, and which comply with a weight space decomposition for the subalgebra K, of the form

$$V = \bigoplus_{\lambda \in \Omega} V_\lambda,$$

where V_λ is the λ-eigenspace of V.

6.2. Cyclic modules

For a representation V in $\mathcal{C}_N^q$, we say that a vector $v \in V$ is *highest weight* if v is an eigenvector for the algebra U^0 and $U^+ \cdot v = \mathbb{C} \cdot v$. The character of U^0 by which it acts on a highest weight vector v is called the ℓ-weight of v (not to be confused with the previous weight decomposition by characters of the smaller algebra K).

Given an irreducible representation V in $\mathcal{C}_N^q$, it is known there is a unique, up to scalar, highest weight vector $v \in V$. The ℓ-weight P of v characterizes the isomorphism class of V. We can write in this case $V = V(P)$.

Let us write $\mathcal{D}_N$ for the set of characters of U^0 which give rise to irreducible representations $V(P)$, $P \in \mathcal{D}_N$. The set $\mathcal{D}_N$ can be described using what is known as *Drinfeld polynomials*.

The set $\mathcal{D}_N$ also comes with a natural monoid structure, in the following sense. Given highest weight vectors $v_P \in V$, $v_Q \in W$ of respective ℓ-weights $P, Q \in \mathcal{D}_N$, the vector $v_P \otimes v_Q \in V \otimes W$ is also a highest weight vector of ℓ-weight $P \cdot Q \in \mathcal{D}_N$.

Let $V_1, \ldots, V_k$ be irreducible representations in $\mathcal{C}_N$. Let $v_i \in V_i$, $1 \leq i \leq k$, be the corresponding highest weight vectors. Let $W \subseteq V_1 \otimes \cdots \otimes V_k$ be the subrepresentation generated by the vector $v_1 \otimes \cdots \otimes v_k$.

Inspired by the terminology of [27], we say that the tuple $(V_1, \ldots, V_k)$ is *cyclic* if $W = V_1 \otimes \cdots \otimes V_k$.

We would like to reformulate the notion of cyclic tuples into a more categorical language.

Proposition 6.1. *Let*
$$\underline{V} = (V(P_1), \ldots, V(P_k))$$
be a tuple of irreducible representations in $\mathcal{C}_N^q$, with $P_1, \ldots, P_k \in \mathcal{D}_N$. The tuple $\underline{V}$ is cyclic if and only if the representation

$$V(P_1) \otimes \cdots \otimes V(P_k)$$

has a unique irreducible quotient whose isomorphism class is given by $V(P_1 \cdots P_k)$.

Proof. For $P \in \mathcal{D}_N$, note that $\omega(P) \in \Omega$ by the assumptions on $\mathcal{C}_N^q$.

Recall from [16, Theorem 1.3 (3)] that dim $V(P_i)_{w(P_i)} = 1$, and that for any $\lambda \in \Omega$ with $V(P_i)_\lambda \neq \{0\}$, we have $w(P_i) > \lambda$. This inequality is meant in a sense of the partial order, which comes from the natural identification of Ω with the wight lattice of the simple Lie algebra $\mathfrak{sl}_N$.

It easily follows that the representation $M := V(P_1) \otimes \cdots \otimes V(P_k)$ has

$$\dim M_{\omega(P_1)+\cdots+\omega(P_k)} = 1.$$

Define $v_i \in V(P_i)$, $i = 1, \ldots, k$, to be the corresponding highest weight vectors, with ℓ-weights P_i. Then

$$m := v_1 \otimes \cdots \otimes v_k \in M_{\omega(P_1)+\cdots+\omega(P_k)}.$$

In particular,

$$w(P_1 \cdots P_k) = w(P_1) + \cdots + w(P_k).$$

Let $W \subseteq M$ be the subrepresentation generated by the vector m. By standard arguments on weight decompositions, the one-dimensionality of $W_{\omega(P_1 \cdots P_k)} = M_{\omega(P_1 \cdots \cdots P_k)}$ implies that W has a unique irreducible quotient Z to which m has a non-zero projection. In particular, the projection of m in Z is a highest weight vector of ℓ-weight $P_1 \cdots P_k$, which means $Z \cong V(P_1 \cdots P_k)$. One direction of the proposition readily follows.

Conversely, suppose that M has an irreducible quotient $Y \cong V(P_1 \cdots P_k)$. Again, the one-dimensionality of $M_{\omega(P_1 \cdots P_k)}$ requires m, and hence W, to project non-trivially on Y. Since Y is irreducible, we must have $W = M$. ∎

Theorem 6.2 (Hernandez [27]). *Let $V_1, \ldots, V_k$ be irreducible representations in $\mathcal{C}_N^q$, such that for all $1 \leq i < j \leq k$, the pair (V_i, V_j) is cyclic. Then the tuple $(V_1, \ldots, V_k)$ is cyclic.*

Remark 6.3. The motivation for the above theorem slightly differs from its current application in our work. It was proved as part of a systematic study of the non-semisimple category $\mathcal{C}_N^q$. Since a cyclic representation whose dual is cyclic as well must be irreducible, the theorem implies an irreducibility criterion for tensor products of representations (which, in fact, was proved earlier in [26]). Thus, this result generalized the criterion and described a factorization of a family of cyclic representations into a product of irreducible ones.

Remark 6.4. In our considerations for the problem at hand we will apply Theorem 6.2 on a family of representations analogous (through a Schur–Weyl duality, see next section) to quasi-Speh representations as in Proposition 4.3. It would be interesting to compare this application to the special case of Theorem 6.2 that was presented in [10].

6.3. Quantum affine Schur–Weyl duality

Let $q \in \mathbb{C}^\times$ be fixed. For each $k \geq 1$ and $N \geq 2$, Chari and Pressley [14] have defined a quantum affine Schur–Weyl duality functor

$$\mathcal{F}_{k,N} : \mathcal{M}_k^{q^2} \to \mathcal{C}_N^q.$$

Recall from Section 3 that $\mathcal{M}_k^{q^2}$ stands for the category of finite-dimensional representations of the affine Hecke algebra $H(k, q^2)$.

These functors served as an affine generalization of the work of Jimbo [30] on a quantum Schur–Weyl duality between quantum groups and (finite) Hecke algebras.

Let us recall some of the properties of these duality functors, as were studied in [14].

When $k \leq N$, $\mathcal{F}_{k,N}$ is a fully faithful functor. In other words, it serves as an embedding of the category $\mathcal{M}_k^{q^2}$ into a certain full subcategory of $\mathcal{C}_N^q$. In fact, this subcategory can be easily characterized by an intrinsic property of representations in $\mathcal{C}_N^q$ (these are called *level k* representations). We will not need this characterization in our discussion.

Moreover, the duality functors are monoidal ([14, Proposition 4.7]), in the sense that

$$\mathcal{F}_{k_1,N}(\pi_1) \otimes \mathcal{F}_{k_2,N}(\pi_2) \cong \mathcal{F}_{k_1+k_2,N}(\pi_1 \times \pi_2) \tag{6.1}$$

for all representations π_1, π_2 in $\mathcal{M}_{k_1}^{q^2}, \mathcal{M}_{k_2}^{q^2}$, respectively. Recall that $\pi_1 \times \pi_2$ denotes the induction product which produces a representation in $\mathcal{M}_{k_1+k_2}^{q^2}$.

Let us now return to the setting of the first sections and assume that $q^2 = q_F$. Recall the bijection

$$\hat{Z} : \mathrm{Mult}_0 \to \bigcup_{k \geq 0} \mathrm{Irr}(\mathcal{M}_k^{q_F}),$$

that is given by the Zelevinsky(–Rogawski) classification.

The following proposition implies, in particular, that for all $k \leq N$, the duality functor $\mathcal{F}_{k,N}$ sends irreducible representations to irreducible representations.

This correspondence of irreducible objects from different categories is rather intriguing. For example, the family of irreducible Speh representations is sent by the composed functor $\mathcal{F}_{k,N} \circ U_\Theta$ (U_Θ as in Section 3.1) to the family of *Kirillov–Reshetikhin modules* in $\mathcal{C}_N^q$. The latter is an important family of modules for the study of quantum affine algebras, which was initially explored in [31] (see the survey [11], for example).

Proposition 6.5 ([14, Theorem 7.6]). *For every $N \geq 2$, there is a natural embedding of monoids $\iota_N : \mathcal{D}_N \to \mathrm{Mult}_0$, for which there holds*

$$\mathcal{F}_{k_P,N}(\hat{Z}(\iota_N(P))) \cong V(P) \in \mathrm{Irr}(\mathcal{C}_N^q)$$

for all $P \in \mathcal{D}_N$ with $k_P \leq N$, where k_P is the integer for which $\hat{Z}(\iota_N(P)) \in \mathrm{Irr}(\mathcal{M}_{k_P}^{q_F})$. Moreover, for every $1 \leq k \leq N$, the collection of multisegments $\hat{Z}^{-1}(\mathrm{Irr}(\mathcal{M}_k^{q_F}))$ is contained in the image of ι_N.

We finish by noting that the combination of Proposition 6.1, Theorem 6.2, Proposition 6.5 and (6.1) directly give the proof of Theorem 3.3.

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