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Igor Rodnianski · Jared Speck

On the nature of Hawking's incompleteness for the Einstein-vacuum equations: The regime of moderately spatially anisotropic initial data

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Abstract. In the mathematical physics literature, there are heuristic arguments, going back three decades, suggesting that for an open set of initially smooth solutions to the Einstein-vacuum equations in high dimensions, stable, approximately monotonic curvature singularities can dynamically form along a spacelike hypersurface. In this article, we study the Cauchy problem and give a rigorous proof of this phenomenon in sufficiently high dimensions, thereby providing the first constructive proof of stable curvature-blowup (without symmetry assumptions) along a spacelike hypersurface as an effect of pure gravity. Our proof applies to an open subset of regular initial data satisfying the assumptions of Hawking's celebrated "singularity" theorem, which shows that the solution is geodesically incomplete but does not reveal the nature of the incompleteness. Specifically, our main result is a proof of the dynamic stability of the Kasner curvature singularity for a subset of Kasner solutions whose metrics exhibit only moderately (as opposed to severely) spatially anisotropic behavior. Of independent interest is our method of proof, which is more robust than earlier approaches in that (i) it does not rely on approximate monotonicity identities and (ii) it accommodates the possibility that the solution develops very singular high-order spatial derivatives, whose blowup-rates are allowed to be, within the scope of our bootstrap argument, much worse than those of the base-level quantities driving the fundamental blowup. For these reasons, our approach could be used to obtain similar blowup-results for various Einstein-matter systems in any number of spatial dimensions for solutions corresponding to an open set of moderately spatially anisotropic initial data, thus going beyond the nearly spatially isotropic regime treated in earlier works.

Keywords. Big Bang, constant mean curvature, curvature singularity, geodesically incomplete, Hawking's theorem, Kasner solutions, maximal globally hyperbolic development, singularity theorem, stable blowup, transported spatial coordinates

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Igor Rodnianski: Princeton University, Department of Mathematics, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA; irod@math.princeton.edu

Jared Speck: Vanderbilt University, Department of Mathematics, 1326 Stevenson Center Lane, Room SC1412, Nashville, TN 37212, USA; jared.speck@vanderbilt.edu

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Contents

1. Introduction

Hawking's celebrated "singularity" theorem (see, e.g., [59, Theorem 9.5.1]) shows that an interestingly large set of initial data for the Einstein-vacuum[1] equations leads to geodesically incomplete solutions. The chief drawback of this result is that the proof is by contradiction and therefore does not reveal the nature of the incompleteness; see Section 1.4 for further discussion. In this article, for an open[2] subset of regular initial data in high spatial dimensions that satisfy the assumptions of Hawking's theorem, we show that the solution's incompleteness is due to the formation of a Big Bang, that is, the formation of a curvature singularity along a spacelike hypersurface. For more detailed statements of our main results, readers can jump ahead to Theorem 1.8d for a rough summary or to Theorem 11.1 for the precise versions.

Before proceeding, we note that the Einstein-vacuum equations in D spatial dimensions are

$$\mathbf{Ric}_{\mu\nu} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\mu\nu} = 0 \quad (\mu, \nu = 0, 1, \ldots, D), \tag{1.1}$$

where $\mathbf{Ric}$ is the Ricci curvature of the Lorentzian spacetime metric $\mathbf{g}$ (which has signature $(-, +, \cdots, +)$), and $\mathbf{R} = \mathbf{Ric}_{\alpha}^{\alpha}$ is[3] its scalar curvature. Throughout, $\mathcal{M}$ denotes the $(1 + D)$-dimensional spacetime manifold, on which system (1.1) is posed. In this article, the spacetimes under study are *cosmological*, i.e., they have compact spacelike Cauchy hypersurfaces. In particular, the solutions that we study are such that $\mathcal{M} = I \times \mathbb{T}^D$, where I is an interval of time and D is the standard D-dimensional torus (i.e., $[0, 1]^D$ with the endpoints identified).

Our work provides the first constructive proof of stable curvature-blowup along a spacelike hypersurface *as an effect of pure gravity* (i.e., without the presence of matter) for Einstein's equations in more than one spatial dimension without symmetry assumptions. The present work can be viewed as complementary to our previous works [50, 51, 53], in which we proved similar results in the presence of scalar field or stiff fluid[4] matter. As we explain below, out of necessity, we had to devise a new and more robust analytic

[1]Hawking's theorem also applies to any Einstein-matter system whose energy-momentum tensor verifies the strong energy condition.

[2]By "open," we mean relative to a suitable Sobolev norm topology.

[3]We summarize our index and summation conventions in Section 1.8.1.

[4]A stiff fluid is such that the speed of sound is equal to the speed of light, i.e., it obeys the equation of state $p = \rho$, where p is the pressure and ρ is the density. The stiff fluid is a generalization of the scalar field in the sense that it reduces to the scalar field matter model when the fluid's vorticity vanishes.

framework to treat the Einstein-vacuum equations, since many of the special structures that we exploited in [50, 51, 53] are not available in the vacuum case.

For the singularities that we study here, the blowup is rather "controlled" in the sense that the solutions exhibit approximately monotonic behavior as the singularity is approached. For example, relative to an appropriately constructed time coordinate t, the Kretschmann scalar $\mathbf{Riem}_{\alpha\beta\gamma\delta}\,\mathbf{Riem}^{\alpha\beta\gamma\delta}$ blows up like Ct^{-4} as $t \downarrow 0$ (see Lemma 10.1 for the precise statement), where **Riem** is the Riemann curvature of $\mathbf{g}$. A main theme of this paper is that the monotonic behavior is not just a curiosity, but rather it lies at the heart of our analysis. Heuristic evidence for the existence of a large family of (non-spatially homogeneous) approximately monotonic spacelike-singularity-forming Einstein-vacuum solutions for large D goes back more than 30 years to the work [25], which was preceded by other works of a similar vein that we review in Section 1.4.1. In [25] and in the present work as well, the largeness of D is used only to ensure the existence of background solutions with certain quantitative properties (in our case Kasner solutions whose exponents verify the condition (1.8e)); for the Einstein-vacuum equations in low space dimensions, the only obstacle to the existence of such solutions is the Hamiltonian constraint equation (1.2a). Given such a background solution, the rest of our analysis is essentially dimensionally independent.[5] Although our main theorem applies only when $D \geq 38$, our approach here is of interest in itself since it is more robust compared to prior works on stable blowup for various Einstein-matter systems, and since it has further applications, for example in three spatial dimensions; see the end of Section 1.2 for further discussion of this point.

1.1. The evolution problem and the initial data

Before further describing our results, we first provide some background material on the evolution problem for Einstein's equations. The foundational works [14, 26] collectively imply that the Einstein-vacuum equations (1.1) have an evolution problem formulation in which all sufficiently regular initial data verifying the constraint equations (1.2a)–(1.2b) launch a unique[6] maximal classical solution $(\mathcal{M}_{(\mathrm{Max})}, \mathbf{g}_{(\mathrm{Max})})$, known as the *maximal globally hyperbolic development of the data* (MGHD for short). Below we will discuss the evolution equations. For now, we recall that the initial data are $(\Sigma_1, \mathring{g}, \mathring{k})$, where

- Σ_1 is a D-dimensional manifold,
- $\mathring{g}$ is a Riemannian metric on Σ_1, and
- $\mathring{k}$ is a symmetric type $\binom{0}{2}$ Σ_1-tangent tensorfield.

The subscript "1" on Σ_1 emphasizes that in our main theorem, Σ_1 will be identified with a hypersurface of constant time 1. $\mathring{g}$ and $\mathring{k}$ represent, respectively, the first and second fundamental form of Σ_1, viewed as a Riemannian submanifold of the spacetime to be

[5] Aside from the number of derivatives that we need to close the estimates.

[6] More precisely, the maximal globally hyperbolic development (MGHD) is unique up to isometry in the class of globally hyperbolic spacetimes.

constructed. The constraints are the well-known *Gauss* and *Codazzi* equations (which are often referred to, respectively, as the Hamiltonian and momentum constraint equations):

$$\mathring{\mathrm{R}} - \mathring{k}^a{}_b \mathring{k}^b{}_a + (\mathring{k}^a{}_a)^2 = 0, \tag{1.2a}$$

$$\nabla_a \mathring{k}^a{}_j - \nabla_j \mathring{k}^a{}_a = 0. \tag{1.2b}$$

In (1.2a)–(1.2b), ∇ denotes the Levi-Civita connection of $\mathring{g}$.

In analyzing solutions to (1.1), we will use the well-known constant-mean-curvature-transported spatial coordinates gauge. The corresponding PDE system involves hyperbolic evolution equations for the first and second fundamental forms of the constant-time hypersurface Σ_t coupled to an elliptic PDE for the lapse $n > 0$, which verifies $\mathbf{g}(\partial_t, \partial_t) = -n^2$. See Section 2.1.1 for a review of this gauge. Here we only note that *the elliptic PDE for n is essential for synchronizing the singularity across space*. To employ this gauge in the context of our main theorem, we assume that the initial data verify the constant-mean-curvature (CMC from now on) condition

$$\mathring{k}^a{}_a = -1. \tag{1.3}$$

Remark 1.4 (The CMC assumption is not a further restriction). In the context of our main theorem, equation (1.3) should not be viewed as a further restriction on the initial data since for the near-Kasner solutions that we study, we can always construct a CMC hypersurface lying near the initial data hypersurface. This can be achieved by making minor modifications to the arguments given in [51], where we constructed a CMC hypersurface in a closely related context.

1.2. Some additional context and preliminary comments on the new ideas in the proof

The aforementioned results [14, 26], while philosophically of great importance, reveal very little information about the nature of the MGHD. In our main theorem, we derive sharp information about the MGHDs of an open set of nearly spatially homogeneous initial data (i.e., initial data along a spacelike hypersurface that "vary only slightly" from point to point) given along the D-dimensional torus $\Sigma_1 = \mathbb{T}^D$, where $D \geq 38$. The largeness of D allows us to consider initial data that are only moderately (as opposed to severely) spatially anisotropic, in the sense that the data are close to a Kasner solution (1.6) whose Kasner exponents verify (1.8e) (see below for further discussion). Broadly speaking, the main analytic themes of our work can be summarized as follows (see Section 1.5 for a more detailed outline of our proof):

> The amount of spatial anisotropy exhibited by the solutions under study is tied to the strength of various nonlinear error terms that depend on spatial derivatives. Below a certain threshold, the error terms become sub-critical (in the sense of the strength of their singularity) with respect to the main terms. This allows us to give a perturbative proof of stable blowup through a bootstrap argument, where the blowup is driven by "ODE-type" *singular terms that are linear in time derivatives*, and, at least at the low derivative levels, **the nonlinear spatial-**

derivative-involving error terms are strictly less singular than the singular linear time derivative terms. Put differently, at the low derivative levels, the spatial derivative terms remain negligible (in a relative sense) throughout the evolution, a phenomenon that, in the general relativity literature, is sometimes referred to as *asymptotically velocity term dominated* (AVTD) behavior. In contrast, at the high derivative levels, the spatial derivative terms can be very singular; bounding the maximum strength of the high-order spatial derivative singularities and showing that highly singular behavior does not propagate down into the low derivative levels together constitute the main technical challenges of the proof.

We stress that in the absence of matter, spatially homogeneous and isotropic Big Bang solutions *do not exist*; they are precluded by the Hamiltonian constraint equation (1.2a). Hence, it is out of necessity that our main theorem concerns spatially anisotropic solutions. In our previous works [50, 51, 53], we proved related stable blowup-results for the Einstein-scalar field and Einstein-stiff fluid systems in three spatial dimensions. In those works, the presence of matter allowed for the existence of spatially homogeneous, spatially isotropic Big-Bang-containing solutions, specifically, the famous Friedmann–Lemaître–Robertson–Walker (FLRW) solutions, whose perturbations we studied. The approximate spatial isotropy of the perturbed solutions lied at the heart of the analysis of [50, 51, 53], and we therefore had to develop a new approach to handle the moderately spatially anisotropic solutions under study here. We now quickly highlight the main new contributions of the present work; in Section 1.5, we provide a more in depth overview of how they fit into our analytical framework.

- In [50, 51, 53], the *presence* and the precise structure of the matter model was used in several key places, leaving open the possibility that the blowup was essentially "caused" by the matter. In contrast, our present work shows that for suitable initial data (which exist for $D \geq 38$), stable curvature-blowup can occur in general relativity as an effect of pure gravity.
- Our previous works [50, 51, 53] relied on *approximate monotonicity identities*, which were L^2-type integral identities in which the special structure of the matter models and the spatially isotropic nature of the FLRW background solutions were exploited to exhibit miraculous cancellations. The cancellations were such that dangerous singular error integrals with large coefficients were replaced, up to small error terms, with *coercive ones*. That is, the identities led to the availability of friction-type spacetime integrals that were used to control various error terms up to the singularity. The net effect was that we were able to prove that the very high spatial derivatives of the solution are not much more singular than the low-order derivatives; this was a fundamental ingredient in the analytic framework that we used to control error terms. In contrast, in the present article, **we completely avoid relying on approximate monotonicity identities**, which might not even exist for the kinds of solutions that we study here. This leads, at the high spatial derivative levels, to very singular energy estimates near the Big Bang, which our proof is able to accommodate; see Section 1.5 for an outline of the main ideas behind the estimates. For these reasons, our approach here is significantly more robust compared to [50, 51, 53] and could be used, for example,

to substantially enlarge the set of initial data for the Einstein-scalar field and Einstein-stiff fluid systems in three spatial dimensions that are guaranteed to lead to curvature-blowup. More precisely, it could be used to prove stable blowup for perturbations of generalized Kasner solutions (i.e., non-vacuum Kasner solutions) to these systems that exhibit moderate spatial anisotropy. We stress that it might not be possible to extend these stable blowup-results to Einstein-matter systems that feature $\mathbf{g}$-timelike characteristics, such as the Euler-Einstein system under a general equation of state; in [50, 51, 53], the analysis relied on the fact that for the matter models studied, the characteristics of the system are exactly the null hypersurfaces of $\mathbf{g}$, a feature that is tied to the following crucial structural property: the absence of time-derivative-involving error-terms in various equations. Readers can consult [50] for further discussion on this point.

1.3. Kasner solutions and a rough summary of the main theorem

As we have mentioned, in our main theorem, we consider initial data given on

$$\Sigma_1 := \mathbb{T}^D, \tag{1.5}$$

where D is the standard D-dimensional torus (i.e., $[0,1]^D$ with the endpoints identified) and, relative to the coordinates that we use in proving our main results, Σ_1 will be identified with a spacelike hypersurface of constant time 1, i.e., $\Sigma_1 = \{1\} \times \mathbb{T}^D$. Our assumption on the topology of Σ_1 is for convenience and is not of fundamental importance; we expect that similar results hold for other spatial topologies, much in the same way that the blowup-results of [50, 51] in the case of $\mathbb{T}^3$ spatial topology were extended to the case of $\mathbb{S}^3$ spatial topology in [53] (see Section 1.4.4 for further discussion of these works). Our main theorem addresses the evolution of perturbations of the initial data (at time 1) of the *Kasner solutions* [32]

$$\widetilde{\mathbf{g}} := -dt \otimes dt + \widetilde{g}, \quad (t, x) \in (0, \infty) \times \mathbb{T}^D, \tag{1.6}$$

where

$$\widetilde{g} := \sum_{i=1}^{D} t^{2q_i} dx^i \otimes dx^i, \tag{1.7}$$

and the constants $q_i \in (-1, 1]$, known as the *Kasner exponents*, are constrained by

$$\sum_{i=1}^{D} q_i = 1, \tag{1.8a}$$

$$\sum_{i=1}^{D} q_i^2 = 1. \tag{1.8b}$$

In the rest of the paper, we will also use the notation

$$\widetilde{k} := -t^{-1} \sum_{i=1}^{D} q_i \partial_i \otimes dx^i \tag{1.8c}$$

to denote the Kasner mixed second fundamental form (see definition (2.3)), i.e.,

$$\widetilde{k}^i{}_j = -\frac{1}{2}(\widetilde{g}^{-1})^{ia}\partial_t \widetilde{g}_{aj}$$

relative to the coordinates featured in (1.6)–(1.7). Aside from exceptional cases in which one of the q_i are equal to 1 and the rest are equal to 0, Kasner solutions have Big Bang singularities at $\{t = 0\}$, where their Kretschmann scalar $\mathbf{Riem}_{\alpha\beta\gamma\delta}\,\mathbf{Riem}^{\alpha\beta\gamma\delta}$ blows up like t^{-4} (see Lemma 10.1 for a proof of this fact). We now briefly summarize our main results. See Theorem 11.1 for the precise statements.

Theorem 1.8d (Rough summary of main results). *Kasner solutions whose exponents satisfy the inequality*

$$\max_{i=1,\dots,D} |q_i| < \frac{1}{6}, \tag{1.8e}$$

which is possible when $D \geq 38$ (see Section 2.3), are nonlinearly stable solutions to the Einstein-vacuum equations (1.1) near their Big Bang singularities. More precisely, perturbations (belonging to a suitable high-order Sobolev space) of the Kasner initial data along $\Sigma_1 = \{t = 1\}$ launch a perturbed solution that also has a Big Bang singularity in the past of Σ_1. In particular, relative to a set of CMC-transported spatial coordinates normalized by $k^a{}_a = -t^{-1}$, where $k^i{}_j$ is the (mixed) second fundamental form of Σ_t, the perturbed solution's Kretschmann scalar blows up like Ct^{-4}. Hence, the past of Σ_1 in the maximal (classical) globally hyperbolic development of the data is foliated by a family of spacelike CMC hypersurfaces Σ_t. Furthermore, every past-directed causal geodesic that emanates from Σ_1 crashes into the singular hypersurface Σ_0 in finite affine parameter time. That is, the perturbed solutions are geodesically incomplete to the past, and the incompleteness coincides with curvature-blowup.

Remark 1.9 (Additional information about the solution). Theorems 1.8d and 11.1 reveal only the most fundamental aspects of the singularity formation. It is possible to derive substantial additional information about the solution using the estimates that we prove in this paper. For example, one could show that various time-rescaled variables, such as the time-rescaled second fundamental form components $tk^i{}_j(t,x)$, converge to regular functions of x as $t \downarrow 0$, which is a manifestation of the AVTD behavior mentioned above. For brevity, we have chosen to neither state nor prove such additional results in this article since, thanks to the a priori estimates yielded by Proposition 9.3, their statements and proofs are similar to the ones given in [51, Theorem 2].

Remark 1.10 (On the bound of $1/6$). The value of $1/6$ on the right-hand side of (1.8e) is possibly not optimal. This bound for q emerges from considering the strength of various error terms; see Section 1.5 for further discussion on this point.

1.4. Previous breakdown results for Einstein's equations

The study of the breakdown of solutions in general relativity was ignited by the classic Hawking–Penrose "singularity" theorems (see, e.g., [28, 40], and the discussion in [29]), which show that for appropriate matter models and spatial topologies, a compellingly large set of initial conditions (with non-empty interior relative to a suitable function space topology) leads to geodesically incomplete solutions. In particular, Hawking's theorem (see [59, Theorem 9.5.1]) guarantees that under assumptions satisfied by the initial data featured in our main theorem, all past-directed timelike geodesics are incomplete. Although these theorems are robust with respect to the kinds of initial data and matter models to which they apply, they are "soft" in that they do not reveal the nature of the incompleteness, leaving open the possibilities that (i) it is tied the blowup of some invariant quantity, such as a spacetime curvature scalar, or (ii) it is due to some other more sinister phenomenon, such as the formation of a Cauchy horizon (beyond which the solution cannot be classically extended in a unique sense due to lack of sufficient information). In the wake of the Hawking–Penrose theorems, many authors studied the nature of the breakdown, though, with only a few key exceptions, the works produced were heuristic, numerical, concerned initial data with symmetry, or yielded information only about "special" solutions (as opposed to an open set of solutions corresponding to regular initial data given along a spacelike Cauchy hypersurface). In the next several subsubsections, we review some of these works.

1.4.1. Heuristic and numerical work. The famous-but-controversial work of Belinskiĭ, Khalatnikov and Lifshitz [11] gave heuristic arguments suggesting that for the Einstein-vacuum equations in three spatial dimensions, the "generic" (in an unspecified sense) solution that breaks down should (i) be local near the breakdown, i.e., be well-modeled by a family of Bianchi IX[7] ODE solutions[8] that are parameterized by space; (ii) become highly oscillatory in time near the breakdown-points,[9] like the Bianchi IX solutions do; and (iii) the breakdown-points should collectively form a spacelike singularity. We refer readers to the recent monograph [9] for a detailed, modern account of [11] and related works. Although the work [11] has stimulated a lot of research activity, as of the present, it is not clear to what extent the heuristic picture it painted holds true. The picture is almost certainly not true in a generic sense, for the recent breakthrough work [22] shows, assuming only a widely believed (but not yet proven) quantitative version of the stability of the Kerr black hole family of solutions to the Einstein-vacuum equations, that the Cauchy horizon inside the black hole is dynamically stable. Specifically, in [22], the

[7]Readers can consult [18] for an overview of the Bianchi IX symmetry class and other symmetry classes that we mention later.

[8]More precisely, the authors argued that they should be well-modeled by the so-called "Mixmaster" [36] solutions, which are of the form

$$\mathbf{g} = -dt^2 + \sum_{i=1}^{3} \ell_i^2(t)\omega^{(i)} \otimes \omega^{(i)},$$

where $\{\omega^{(i)}\}_{i=1,2,3}$ are one-forms on $\mathbb{S}^3$ verifying $d\omega^{(i)} = \frac{1}{2}[ijk]\omega^{(j)} \wedge \omega^{(k)}$, and $[ijk]$ is the fully antisymmetric symbol normalized by $[123] = 1$.

[9]Here, by "breakdown-points," we roughly mean points on the boundary of the MGHD.

authors showed that the metric is C^0-extendible past the Cauchy horizon, which is a null hypersurface that is contained in the boundary of the MGHD. This result in particular contradicts the vision of spacelike singularities posited in [11]. In the opposite direction (i.e., in accordance with [11]), in three spatial dimensions, Ringström [45] confirmed the oscillatory picture[10] of solutions near singularities for solutions with Bianchi IX symmetry to the Einstein-vacuum equations and to the Euler–Einstein equations under the equations of state $p = c_s^2 \rho$, where p is the pressure, ρ is the density, and the constant $0 < c_s < 1$ is the speed of sound. However, outside of the class of spatially homogeneous solutions, there are currently no known examples of Einstein-vacuum solutions that exhibit the oscillatory behavior suggested by [11].

There are also heuristic results concerning the existence of *non-oscillatory* solutions to various Einstein-matter systems that form a spacelike singularity. Specifically, in [10], Belinskiĭ and Khalatnikov noted that if one considers the Einstein-scalar field system in three spatial dimensions, then the heuristic arguments given in [11] concerning oscillations no longer seem plausible. They argued that instead, the generic incomplete solution should exhibit monotonic behavior near the breakdown-points, which should still collectively form a spacelike singularity. In [8], Barrow gave a similar heuristic argument for the Einstein-stiff fluid system. Most relevant for our work here is the work [25] that we mentioned at the beginning, in which, for $D \geq 10$ (i.e., at least eleven spacetime dimensions), the authors gave heuristic arguments, similar to the type given in [8, 10], suggesting that there is a *nontrivial regime* for the Einstein-vacuum equations in which solutions exhibit non-oscillatory spacelike singularity formation. This suggests that indeed, in high spatial dimensions, stable blowup-results of the type that we prove in our main theorem should hold. Note that there is a significant gap between the assumption $D \geq 10$ used in the heuristic arguments of [25] and the assumption $D \geq 38$ that we make in our main theorem. In Section 1.6, we will further discuss this gap.

We close this subsubsection by noting that the above works and others like them have stimulated a large number of numerical studies that have been designed to probe the validity of the heuristic predictions. We do not attempt to survey the vast literature here, but instead refer to the works [3, 12, 27, 43], which serve as useful starting points for exploring the subject of numerical analysis in the context of singularities in general relativity.

1.4.2. Construction of (but not the stability of) singular solutions. There are a variety of works showing the existence of, but not the *stability* of, singularity-containing solutions to various Einstein-matter systems that exhibit the same kind of AVTD behavior[11] near the singularity exhibited by the solutions in our main theorem. Typically, the constructions rely on deriving/solving a Fuchsian PDE system. Roughly speaking, a Fuchsian PDE is

[10] Ringström also studied the stiff fluid case $c_s = 1$ in Bianchi IX symmetry and proved monotonic-type singularity formation results similar to the ones we obtained in [50, 51, 53] and in the present work.

[11] That is, time-derivative dominated behavior, as we described in Section 1.2.

a system of the following form, the key point being that the equation degenerates as $t \downarrow 0$ (where for convenience we restrict out attention to one spatial dimension):

$$tA^0(t,x,u)\partial_t u + tA^1(t,x,u)\partial_x u + B(t,x,u)u = f(t,x,u). \tag{1.11}$$

In (1.11), u is the array of unknowns, A^α and B are matrices (which are often assumed to be symmetric so that the symmetric hyperbolic framework for energy estimates can be invoked), and f is an array, all of which must satisfy a collection of technical assumptions. A typical Fuchsian analysis is based on splitting the solution as $u = u_0 + w$, where u_0 is the leading order part and w is an error term that one would like to show is small compared to u_0 as $t \downarrow 0$. The leading order part u_0, which typically is singular as $t \downarrow 0$, must be guessed/solved for using an appropriate ansatz. Here we do not describe how this is typically accomplished; readers can consult [50] for a further description of Fuchsian analysis in the context of singular solutions in general relativity. Although the Fuchsian approach can sometimes be used to show the existence of singular solutions, it is inadequate for treating the true stability problem, i.e., the problem of posing initial data on a regular Cauchy hypersurface $\{t = \text{const}\}$ with const > 0 and then solving the equations all the way down to the singular hypersurface $\{t = 0\}$.

We now describe some of the Fuchsian-type existence results in more detail. Notable among these is the work of Andersson and Rendall [5], in which they constructed a family of spatially analytic Big-Bang-containing solutions to the Einstein-scalar field and Einstein-stiff fluid systems in three spatial dimensions that exhibit approximately monotonic behavior near the singularities. The family of solutions that the authors constructed was large in the sense that its number of degrees of freedom coincides with the number of free functions in Einstein initial data for the standard Cauchy problem. However, the results of [5] do not show the *stability* of the blowup under Sobolev-class perturbations of initial data given along a spacelike hypersurface near the expected singularity. Another notable work in the spirit of [5] is [24], in which the authors proved similar results for various Einstein-matter systems in various spatial dimensions, including for the Einstein-vacuum equations in ten or more spatial dimensions (i.e., $D \geq 10$ in the notation of the present article). Note that the Einstein-vacuum result with $D \geq 10$ supports the heuristic work [25] mentioned in the previous subsubsection.

There are many additional works that yield the construction of (but not the stability of) singularity-containing solutions to select Einstein-matter systems. We do not attempt to exhaustively survey the literature here, but we mention the following ones, which concern various symmetry-reduced equations: [1, 6, 13, 15, 30, 33, 41, 54].

There are also results in which the authors constructed singular solutions by essentially prescribing "singular data" on the singular hypersurface itself and then solving to the future; see, for example, [7, 20, 37, 38, 55–57]. One drawback of this approach is that there are fewer degrees of freedom in the singular data compared to the initial data corresponding to a standard Cauchy problem on a regular spacelike hypersurface. Thus, generic solutions cannot be constructed in this fashion; this is explained in more detail in [44, Section 6.1].

We close this subsubsection by mentioning that the Fuchsian techniques behind the above results have applications outside of general relativity. There are techniques for constructing solutions to large classes of hyperbolic Fuchsian PDEs; see, for example, [2]. Readers can consult Rendall's work [42] for a more detailed comparison of many of the results described above as well as application of the Fuchsian framework to prove the existence of singular solutions to the Einstein-vacuum equations with Gowdy symmetry.

1.4.3. Constructive stable blowup-results under symmetry assumptions. There are a variety of works on symmetry-reduced Einstein-matter systems in which the authors gave a constructive proof of *stable* singularity formation. In the spatially homogeneous case, in which the equations reduce to a system of ODEs, there are many results, including the constructive work of Ringström [45] mentioned earlier. We do not attempt to survey the literature here; instead we direct readers to [44, 58] for an overview of ODE-blowup for solutions to Einstein's equations.

There are also constructive proofs of *stable* singularity formation for various symmetry-reduced Einstein-matter systems in which the equations reduce to a $1+1$-dimensional system of PDEs; see, e.g., [19, 31, 46, 48]. Chief among these are Christodoulou's remarkable works [16, 17] on the Einstein-scalar field system in three spatial dimensions in spherical symmetry for 1- or 2-ended asymptotically flat data. In those works, he showed that the maximal globally hyperbolic future developments of generic data are future-inextendible as time-oriented Lorentzian manifolds with a C^0 metric; i.e., the breakdown is severe, at the level of the metric itself. See also the recent works [34, 35] on the spherically symmetric Einstein–Maxwell-(real) scalar field system with asymptotically flat 2-ended initial data, in which the authors proved that the maximal globally hyperbolic future developments of generic data are future-inextendible as time-oriented Lorentzian manifolds with a C^2 metric. This is an especially intriguing result in view of the fact that Dafermos and Rodnianski [21, 23] showed that the statement is *false* for this system if one replaces C^2 with C^0, i.e., the maximal globally hyperbolic future development can sometimes be extended as a time-oriented Lorentzian manifold with a C^0 metric.

1.4.4. Constructive stable blowup-results without symmetry assumptions. The only prior works exhibiting stable spacelike singularity formation for solutions to Einstein's equations without symmetry assumptions are [51, 53], which are closely related to the present work. In [51, 53], we showed that a curvature singularity develops along a spacelike hypersurface for open sets of solutions to two Einstein-matter systems: the Einstein-scalar field system and the Einstein-stiff fluid system; see also our related work [50] concerning the linear analysis and Ringström's recent monograph [49], which concerns estimates for solutions to a large family of linear wave equations whose corresponding metrics model the behavior that can occur in solutions to Einstein's equations near cosmological singularities.

Specifically, in [51], we showed that in three spatial dimensions with spatial topology $\mathbb{T}^3$, the FLRW solution is nonlinearly stable in a neighborhood of its Big Bang singularity. In [53], we proved the same result in the case of $\mathbb{S}^3$ spatial topology, a key new feature being that the solutions are not approximately spatially flat in the $\mathbb{S}^3$ case (although they are nearly spatially homogeneous and isotropic). As we stressed earlier, the analytic framework of [50, 51, 53] was quite different from that of the present work, due to the availability of L^2-based approximate monotonicity identities tied to the special structure of the matter models and the spatially isotropic nature of the FLRW background solutions.

1.5. An overview of the proof of the main theorem

In this subsection, we outline the ideas behind the proof of our main results, i.e., Theorem 11.1. As in our prior works [50, 51, 53], we analyze solutions relative CMC-transported spatial coordinates, in which the spacetime metric is decomposed into the lapse n and a Riemannian metric g on the constant time hypersurfaces Σ_t as follows:

$$\mathbf{g} = -n^2 dt \otimes dt + g_{ab} dx^a \otimes dx^b. \tag{1.12}$$

In such coordinates, the Einstein-vacuum evolution problem consists of the Hamiltonian and momentum constraint equations, hyperbolic evolution equations for the first fundamental form g of Σ_t and the second fundamental form k of Σ_t, and an elliptic PDE for n along Σ_t, supplemented by initial data for g and k given along Σ_1 that satisfy the constraints; see Proposition 2.10 for the details. Here we only note that the mixed second fundamental form $k^i{}_j$ verifies $\partial_t g_{ij} = -2n g_{ia} k^a{}_j$ and that we normalize t so that $k^a{}_a = -t^{-1}$, i.e., Σ_t is a hypersurface of constant mean curvature $-D^{-1}t^{-1}$.

The main part of the proof is showing that the solution (g, k, n) exists classically for $(t, x) \in (0, 1] \times \mathbb{T}^D$, i.e., long enough to form a curvature singularity. The proofs that the Kretschmann scalar blows up as $t \downarrow 0$ and that the spacetime is geodesically incomplete follow as straightforward consequences of estimates that we use in proving existence on $(0, 1] \times \mathbb{T}^D$; we refer readers to Section 10 for details on the nature of the breakdown, which we will not discuss here.

The main step in proving that the solution exists classically for $(t, x) \in (0, 1] \times \mathbb{T}^D$ is to obtain suitable a priori estimates showing that various solution norms along Σ_t do not blow up until $t = 0$. For this reason, in our discussion here, we describe only the a priori estimates. At the heart of the proof lies the following task, whose importance we describe below:

> Showing that for perturbed solutions, the t-rescaled type $\binom{1}{1}$ spatial Ricci components $t\mathrm{Ric}^i{}_j$ are, at each fixed spatial point $x \in \mathbb{T}^D$, *integrable in time* over $t \in (0, 1]$. Here and throughout, Ric denotes the Ricci curvature of g.

Remark 1.13 (On the significance of working with the type $\binom{1}{1}$ spatial Ricci tensor). One might wonder why we work with the spatial Ricci tensor in type $\binom{1}{1}$ form, i.e., as $\mathrm{Ric}^i{}_j$, instead of working with it in type $\binom{0}{2}$ form. The reason is that our work crucially relies on working with the evolution equation for the type $\binom{1}{1}$ tensorfield $tk^i{}_j$, which features a source term proportional to $t\mathrm{Ric}^i{}_j$; see equation (6.5). One might now wonder what the advantage is of working with $tk^i{}_j$ instead of tk_{ij}. The answer is that the evolution equation (6.5) for $tk^i{}_j$ does not feature any Riccati-type term that is proportional to $k \cdot k$, while an evolution equation of type $\partial_t(tk_{ij}) = \cdots$ *would* feature such a Riccati term. The absence of Riccati-type terms in the evolution equation for $tk^i{}_j$ is quite helpful for proving the boundedness of $tk^i{}_j$ as $t \downarrow 0$, which as we describe below, is central for our results. See also Remark 2.6.

The task of proving that $t\mathrm{Ric}^i{}_j(t,x)$ is integrable in time over $(0,1]$ is essentially a quantified version of the following idea, which has its origins in the heuristic works discussed in Section 1.4.1:

For near-Kasner initial data, we can prove stable blowup in regimes where we can prove that time-derivative terms dominate spatial derivative terms in the equations, i.e., when we can prove that the AVTD behavior described in Section 1.2 occurs.

In practice, to prove the time-integrability of $t\mathrm{Ric}^i{}_j$ (and the many other estimates that we need to close the proof), we rely on a bootstrap argument involving various norms that capture the following behavior: the solution's high-level derivatives are allowed to be substantially more singular than its low-level derivatives. Here we will not describe the logical flow of our bootstrap argument in detail, but rather only describe how the various estimates fit together consistently. Moreover, we will not focus on the "smallness assumptions" (i.e., near-Kasner assumptions) on the initial data that we need in our detailed proof, but rather only on the main feature of the analysis: the various powers of t that arise; in this subsection, we will simply denote all quantities that can be estimated in terms of the near-Kasner initial data by "Data". We stress already that our proof relies on commuting the evolution equations with up to N spatial derivatives, where N has to be chosen sufficiently large in a manner that we explain below.

We now recall that we are studying perturbations of a Kasner solution (1.6) whose exponents satisfy (1.8e) (see Section 2.3 for a proof that such Kasner solutions exist when $\mathcal{D} \geq 38$). We also recall that for the Kasner solution $(\widetilde{g}, \widetilde{k}, \widetilde{n})$, we have $\widetilde{n} \equiv 1$, while $\widetilde{g}$ and $\widetilde{k}$ are respectively given by (1.7) and (1.8c). Note also that the Kasner solution is spatially flat and thus the Ricci tensor of $\widetilde{g}$ vanishes. For the perturbed solution, to obtain the desired time integrability of the components $t\mathrm{Ric}^i{}_j$ described in the previous paragraph, it clearly suffices to prove that

$$|\mathrm{Ric}^i{}_j| \lesssim t^{-p} \quad \text{for some constant } p < 2. \tag{1.14}$$

We stress that (1.14) is essentially the same as the heuristic estimates featured in the works [8,10,25], and that the estimate was verified by the solutions that we studied in [50,51,53]. Before describing how we prove (1.14), we first outline the two main consequences that it affords:

(1) $n - 1 \to 0$ as $t \downarrow 0$.

(2) The components $tk^i{}_j$ remain bounded as $t \downarrow 0$.

The proof of (1) follows from a simple application of the maximum principle to the elliptic PDE

$$t^2 g^{ab} \nabla_a \nabla_b n = (n-1) + t^2 n \mathrm{R},$$

where $\mathrm{R} = \mathrm{Ric}^a{}_a$ is the scalar curvature of g; see the proof of (5.2) for the details. The proof of (2) essentially follows from freezing the spatial point and integrating the following evolution equation (see equation (6.5)) from time t to time 1: $\partial_t(tk^i{}_j) = t\text{Ric}^i{}_j + \cdots$, and from using a few additional estimates that allow one to show that, like $t\text{Ric}^i{}_j$, the error terms denoted by $\cdots$ are integrable over $t \in (0, 1]$; see the proof of Proposition 6.1 for the details.

We now return to the crucial issue of proving that $|\text{Ric}^i{}_j| \lesssim t^{-p}$ for some constant $p < 2$, a bound that is tied to all aspects of the proof. To achieve this, we first, in view of (1.8e), fix a constant q such that

$$\max_{i=1,\ldots,D} |q_i| < q < \frac{1}{6}.$$

To control $\text{Ric}^i{}_j$, we rely on the following estimates, whose proof we will describe below:

$$\max_{i,j=1,\ldots,D} |g_{ij}| \lesssim t^{-2q}, \qquad \max_{i,j=1,\ldots,D} |g^{ij}| \lesssim t^{-2q}. \tag{1.15}$$

Note that *the bounds* (1.15) *represent an absolute worst-case scenario*, in which all components of g and g^{-1} are allowed to be slightly more singular than the most singular component of the background Kasner spatial metric and its inverse. As will become clear, the bounds (1.15) are the most fundamental ones in our analysis. One might think of (1.15) as allowing for the "complete mixing" of the Kasner exponents in the perturbed solutions; this is the most glaring spot in the proof that has potential for improvements in future studies. To control $\text{Ric}^i{}_j$, we first express it in terms of the Christoffel symbols of the transported spatial coordinates (see (2.15b) for the precise expression). For the solutions under study, the most singular term in the component $\text{Ric}^i{}_j$ is not a top-order term, but rather lower-order terms (i.e., the last two products on the right-hand side of (2.15b)) that have the following schematic form: $g^{-3}(\partial g)^2$, where ∂ denotes the spatial gradient with respect to the transported spatial coordinates. An interpolation argument, which heavily relies on (1.15) and which we explain below, yields that *for large* N (where we again stress that N is the maximum number of times that we commute the equations with spatial derivatives), the low-level spatial derivatives of the components of g, including ∂g, are *only slightly more singular than the components of* g *itself*. Thus, in view of (1.15), we see that $g^{-3}(\partial g)^2$ is only slightly more singular than $(t^{-2q})^5 = t^{-10q}$. Since $q < 1/6$, we conclude that the product $g^{-3}(\partial g)^2$ is *less singular than* t^{-2}, consistent with the desired bound (1.14).

Remark 1.16. Since $g^{-3}(\partial g)^2$ is fifth-order in $(g, \partial g)$, the discussion above suggests that the proof should close assuming only $q < 1/5$. However, in the top-order energy estimates, we encounter some below-top-order error terms that seem to prevent the proof from closing unless we assume $q < 1/6$; as we further describe below, we encounter such error terms, for example, in the proof of (7.21b).

Having outlined how to obtain the desired bound for $\text{Ric}^i{}_j$, we can, as we described above, show that as $t \downarrow 0$, $n - 1 \to 0$ and $tk^i{}_j$ remains bounded. Then, given these bounds for n and k, we can integrate the evolution equations

$$\begin{aligned}\partial_t g_{ij} &= -2ng_{ia}k^a{}_j,\\ \partial_t g^{ij} &= 2ng^{ia}k^j{}_a\end{aligned}$$

and use a near-Kasner assumption on the initial data to obtain, through standard arguments, the desired estimates (1.15) (see Proposition 6.1 and its proof for the precise statements).

We now return to the issue of the interpolation argument mentioned above, which we used to show, for example, that ∂g is only slightly more singular than g. To appreciate the role of interpolation, it is essential to already know that the best energy estimates we are able to obtain allow for the following rather singular behavior:

> The top-order derivatives of g can be as large (in L^2) as Data $\times\, t^{-(A+1)}$, where $A > 0$ is a large universal constant, independent of the maximum number of times (i.e., N) that we commute the equations with spatial derivatives.

That is, under appropriate bootstrap assumptions, we can prove only that[12]

$$t^{A+1}\|\partial g\|_{\dot{H}^N(\Sigma_t)} \lesssim \text{Data}. \tag{1.17}$$

Then standard Sobolev interpolation (see Lemma 4.5) yields the following bound for the components of ∂g:

$$\|\partial g\|_{L^\infty(\Sigma_t)} \lesssim \|g\|_{L^\infty(\Sigma_t)}^{1-(1+\lfloor D/2\rfloor)/(N+1)} \|\partial g\|_{\dot{H}^N(\Sigma_t)}^{(1+\lfloor D/2\rfloor)/(N+1)} + \|g\|_{L^\infty(\Sigma_t)}. \tag{1.18}$$

From (1.18), we see that even if the top-order homogeneous norm $\|\partial g\|_{\dot{H}^N(\Sigma_t)}$ is as singular as Data $\times\, t^{-(A+1)}$, as long as we take N to be sufficiently large (relative to A and D), the singular nature of $\|\partial g\|_{L^\infty(\Sigma_t)}$ will not be much worse than that of $\|g\|_{L^\infty(\Sigma_t)}$, i.e., no worse than $t^{-(2q+\alpha)}$, where α is small; see the beginning of Section 4.4 for further discussion of this issue.

It remains for us to discuss the top-order energy estimates for g and k. In reality, these must be complemented with top-order elliptic estimates for n, but we will ignore this (relatively standard) issue in this subsection in order to condense our summary of the proof. Let $\vec{I}$ be a top-order spatial derivative multi-index, i.e., $|\vec{I}| = N$. Using the evolution equations and integration by parts, and using appropriate bootstrap assumptions to control error terms, we are able to derive an estimate of the following form, valid for $t \in (0, 1]$ (see Proposition 7.1 for the details and Section 3.3 for the definition of the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$):

$$\begin{aligned} &\left\|t^{A+1}\partial_{\vec{I}}k\right\|_{L^2_g(\Sigma_t)}^2 + \frac{1}{4}\left\|t^{A+1}\partial\partial_{\vec{I}}g\right\|_{L^2_g(\Sigma_t)}^2 \\ &\quad \le \text{Data} - \{2A - C_*\}\int_{s=t}^1 s^{-1}\left\{\left\|s^{A+1}\partial_{\vec{I}}k\right\|_{L^2_g(\Sigma_s)}^2 + \frac{1}{4}\left\|s^{A+1}\partial_{\vec{I}}\partial g\right\|_{L^2_g(\Sigma_s)}^2\right\} ds \\ &\qquad + \cdots, \end{aligned} \tag{1.19}$$

where $\cdots$ denotes error terms that, while they require some care to handle, are less singular. We stress the following points.

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[12]The bound (1.17) is slightly inaccurate in that it does not feature the precise norm that we use in proving the main theorem; see Section 3.3 for the precise definitions of the norms.

- The constant C_* on the right-hand side of (1.19) is *universal*, i.e., independent of N and A. Roughly, C_* is generated by the coefficients of the most singular *linear* terms in the evolution equations and the elliptic PDE for n; the most singular linear terms involve the top-order derivatives of g, k, and n, and because the terms are linear, their coefficient does not change when the term is differentiated (i.e., when N increases), nor does it depend on A. In our prior works [50, 51, 53], we were able to show that C_* is small or vanishing, thanks to the approximate monotonicity[13] identities that we mentioned earlier in the paper. For the solutions under study here, C_* can be large but fixed (and we do not bother to carefully track the precise value of C_*).
- We inserted the time weights t^{A+1} "by hand" into the energies in equation (1.19). We note that when proving (1.19) (roughly, by taking the time derivative of the left-hand side and integrating by parts), one encounters terms in which ∂_t falls on the weights. Roughly, this leads to the integrals preceded by the factor of $-2A$ on the right-hand side of (1.19) (note that $t \in (0, 1]$, which explains why the factors $-2A$ are on the right-hand side).

The key point is that if we choose A to be sufficiently large, then the factor $-\{2A - C_*\}$ on the right-hand side of (1.19) is negative, and the corresponding integral has an overall "friction" sign. In particular, it can be discarded, leaving only the less singular error terms "$\cdots$" on the right-hand side of (1.19). A careful analysis of the error integrals "$\cdots$" allows one to conclude, via Gronwall's inequality, the top-order a priori energy estimate

$$\|t^{A+1}\partial_{\vec{I}}k\|^2_{L^2_g(\Sigma_t)} + \frac{1}{4}\|t^{A+1}\partial\partial_{\vec{I}}g\|^2_{L^2_g(\Sigma_t)} \lesssim \text{Data}$$

as desired.

Although the above discussion summarizes the main ideas, in our detailed proof, we encounter several hurdles that we overcome using additional ideas. While conceptually simple, these ideas are somewhat technically involved. We close this subsection by highlighting some of the features of this analysis.

(1). To control error terms, we rely on a variety of norms. In the next point below, we will shed some light on how we use the different kinds of norms; see Sections 3.2 and 3.3 for the precise definitions. For example, when bounding Σ_t-tangent tensors, we rely on pointwise norms $|\cdot|_{\text{Frame}}$ that measure the size of components relative to the transported spatial coordinate frame as well as the more geometric norm $|\cdot|_g$, which measures the size of tensors using the dynamic spatial metric g. Similarly, our analysis relies on Sobolev norms for the frame components, such as $\|\cdot\|_{H^N_{\text{Frame}}(\Sigma_t)}$, as well as more geometric Sobolev norms $\|\cdot\|_{H^N_g(\Sigma_t)}$.

[13] One might say that inequality (1.19) also signifies a form of "approximate monotonicity." The difference between (1.19) and the "approximate monotonicity identities" from [50, 51, 53] is that these works relied on the "natural" time weights in which $A = 0$, whereas in the present paper, we must choose A to be sufficiently large in order to "force the approximate monotonicity to reveal itself."

(2). Although the interpolation estimates described above are sufficient for controlling various error terms at the low derivative levels, interpolation is not quite sufficient, in itself, for controlling some of the error terms in the high-order energy estimates. For example, our proof of the error term estimate (7.21b) relies not only on interpolation, but also on controlling the term in the norm on the left-hand side of (7.21b) (which we decompose in (7.5d)) by exploiting the structure of Einstein's equations in our gauge to derive *improved estimates* for the below-top-order derivatives of the solution via derivative-losing transport equation estimates. In Remark 1.20, we explain these issues in detail in the context of proving estimate (7.21b). Here we will provide a schematic overview of these issues. First, the top-order energy estimates can be directly obtained only in terms of geometric norms such as $\|\cdot\|_{\dot{H}_g^N(\Sigma_t)}$, since the basic energy identities involve these kinds of norms. In contrast, for $M \leq N$, Sobolev interpolation estimates involving the "background differential operators" ∂_i are most naturally stated and derived in terms of the less geometric norms $\|\cdot\|_{\dot{H}^M_{\mathrm{Frame}}(\Sigma_t)}$. Therefore, to close our estimates, we must compare the geometric norms with the frame-based norms. For tensorfields, the discrepancy between the norms $\|\cdot\|_{\dot{H}^M_{\mathrm{Frame}}(\Sigma_t)}$ and $\|\cdot\|_{\dot{H}_g^M(\Sigma_t)}$ is factors of g and g^{-1} (where the number of factors depends on the order of the tensorfield inside the norms), and by (1.15), the two norms can differ in strength by (singular) powers of t^{-q}; in some cases, these powers of t^{-q} are strong enough so that in the energy estimates, certain below-top-order error terms seem, at first glance, to be more singular than the main top-order terms. If this were the case, then our bootstrap argument would not close. Fortunately, this is not the case, but to prove this, we use an argument that involves deriving energy estimates not only at the very highest level, but also at down-to-two derivatives below top. In deriving these below-top-order energy estimates, we use arguments that lose one derivative, by treating the evolution equations like transport equations along the integral curves of ∂_t with derivative-losing source terms. These transport-type estimates lead to better below-top-order estimates compared to the ones that pure interpolation would afford, and we stress that this approach is viable only because in the solution regime under study, in our gauge, the source terms in the transport equations exhibit a favorable size (in various norms) with respect to powers of t. To implement this procedure in a consistent fashion, we rely on a hierarchy of energies that features different t weights at different orders and that involves both geometric norms $\|\cdot\|_{\dot{H}_g^M(\Sigma_t)}$ and coordinate frame norms $\|\cdot\|_{\dot{H}^M_{\mathrm{Frame}}(\Sigma_t)}$. See Definitions 3.14 and 3.16 for the precise definitions of the t-weighted norms that use to control the solution; in our main theorem, we prove that the norms from Definitions 3.14 and 3.16 are uniformly bounded up to the singularity.

(3). In carrying out our bootstrap argument, we find it convenient to derive, as a preliminary step, estimates showing that the lapse n can be controlled in terms of g and k. This is the content of Section 5. We have downplayed these estimates in this subsection since they are based on deriving standard estimates for elliptic equations. One feature worth noting is that for most lapse estimates, in particular the estimates at the lower-order derivative levels, we rely on the elliptic PDE

$$\left\{g^{ab}\nabla_a\nabla_b - (t^{-2} + \mathrm{R})\right\}(n-1) = \mathrm{R}.$$

This is a "good" equation for $n-1$ in the sense that it involves source terms that depend only on spatial derivatives of g, which are less singular than time derivatives of g (as represented by k). However, to obtain the top-order lapse estimates, we do not use the elliptic lapse PDE in the form stated above. Instead, we first use the Hamiltonian constraint (2.11a) to algebraically replace R with terms that depend on k; see equation (5.11) and its proof. This algebraic replacement leads to error terms that can be controlled within the scope of our bootstrap approach, both from the point of view of regularity and from the point of view of the structure of the singular error terms that our framework can accommodate.

Remark 1.20 (We need below-top-order estimates that are better than interpolation). For concreteness, here we explain why our proof of (7.21b) cannot be carried out using only interpolation. That is, we will explain why we need more than just interpolation, the highest-order energies, and the lowest-order L^∞-type bootstrap assumptions to bound the terms in the sum (7.5d) by $\lesssim$ RHS (7.21b). This remark would perhaps more naturally fit in with the analysis in Section 7, but we have placed it here so that the interested reader can obtain further preliminary insights on why we need to supplement our top-order energy norms with norms that control the solution down to two derivatives below top. We refer readers to Section 3 for the definitions of the norms relevant for the discussion here. Using only the aforementioned ingredients, which are sufficient for allowing us to use the comparison estimate (4.4) and the product estimate (4.10), and the ideas used in the proof of the interpolation results provided by Lemma 4.11, we can deduce that the terms in the sum (7.5d) verify the following bound:

$$\begin{aligned}
&t^{A+1}\|(\partial_{\vec{I}_1} g^{-1})(\partial_{\vec{I}_2} g^{-1})(\partial\partial_{\vec{I}_3} g)\partial_{\vec{I}_4} k\|_{L^2_g(\Sigma_t)} \\
&\quad \lesssim t^{A+1}\|g^{-1}\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)} + \cdots \\
&\quad \lesssim t^{A+1-8q-\alpha}\|k\|_{\dot{H}^N_g(\Sigma_t)} + \cdots ,
\end{aligned}$$

where $\alpha > 0$ goes to 0 as $N \to \infty$. The exponent portion $-8q$ comes from the fact that each factor of g or g^{-1} contributes a factor of t^{-2q} (coming from our bootstrap assumptions), which yields three factors of t^{-2q}, while bounding $\|k\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)}$ in terms of $\|k\|_{\dot{H}^N_g(\Sigma_t)}$ produces yet another factor of t^{-2q}, coming from equation (4.4) and the fact that k is type $\binom{1}{1}$. The factor $t^{-\alpha}$ comes from interpolation and is not important for the present discussion. In view of definition (3.17a), we see that the terms under consideration are $\lesssim t^{-8q-\alpha}\mathbb{H}_{(g,k)}(t) + \cdots$. Since q can be as large as $1/6$, $t^{-8q-\alpha}$ can be more singular than t^{-1}, which is singular enough to destroy the viability of our proof and in any case is not good enough to yield our estimate (7.21b). Thus, our proof of (7.21b) relies on the additional below-top-order norms mentioned above.

1.6. Further comparisons with two related works and open questions

In this subsection, we compare and contrast the reasons behind the assumed minimum values of D in the present work and in the aforementioned works [24, 25]. We start by recalling that, as we described in Section 1.4.1, the work [25] provided heuristic evidence for the existence of a large family of monotonic spacelike singularity-forming Einstein-vacuum solutions whenever $D \geq 10$, and that such solution families were constructed in [24] (though stability in the sense of the present article was not proved there). There is a substantial gap between the assumption $D \geq 10$ and the assumption $D \geq 38$ of the present article. In the remainder of this subsection, we provide an overview of how the assumption $D \geq 10$ is used in [24, 25], and shed some light on how one might approach the problem of extending the results of the present article to apply to a larger range of D values. Actually, we will focus on the heuristic work [25], which allows for a simplified presentation of the main ideas.

In [24, 25] and the present work, a crucial step is justifying that, in some sense, the spatial derivative terms in the Einstein-vacuum equations are negligible compared to the time derivative terms near the singularity. For example, in the present work (see in particular Section 1.5), this step is embodied by estimate (1.14), that is, our proof that the components of the Ricci tensor of g obey the following bound: $|\mathrm{Ric}^{i}{}_{j}| \lesssim t^{-p}$ for some constant $p < 2$. Similarly, the heuristic arguments given in [25] were designed exactly to yield a bound of this type. From this perspective, a primary analytic difference between the present work and the works [24, 25] is that the latter works precisely accounted for, in a tensorial fashion, *partial cancellations of singular powers of* t that can occur in the products featured in the coordinate expression of the Ricci tensor $\mathrm{Ric}^{i}{}_{j}$ of g (i.e., the products on the right-hand side of (2.15b)), at least for metrics that can be interpreted as having "spatially dependent Kasner exponents" (see below for more on this point). In contrast, in the present article, we allow for the possibility that all components of g and g^{-1} are as singular as t^{-2q} (see, for example, the discussion surrounding (1.15)), and we have therefore ignored the possibility of exploiting such cancellations; this is apparent from the proof outline that we gave in Section 1.5. We now further explain the connection between the work [25] and the notion of a spatial metric having "spatially dependent Kasner exponents."

The starting point of [25] is the hope that there are singular solutions to the Einstein-vacuum equations that are somehow well-described by a spacetime metric having "spatially dependent Kasner exponents", that is, a metric of the following form, defined for[14] $(t, x) \in (0, 1] \times \mathbb{T}^D$:

$$\mathbf{g} = -dt^2 + \sum_{i=1}^{D} t^{2q_i(x)} \omega^{(i)}(x) \otimes \omega^{(i)}(x), \tag{1.21}$$

where $\{\omega^{(i)}(x)\}_{i=1,\ldots,D}$ are a linearly independent set of time-independent one-forms on $\mathbb{T}^D$ (in particular, $\omega^{(i)}(x) = \omega^{(i)}_a(x) dx^a$ relative to local coordinates), $\{q_i(x)\}_{i=1,\ldots,D}$ are "x-dependent" Kasner exponents, subject to the following "spatially dependent Kasner constraints" (i.e., x-dependent analogs of (1.8a)–(1.8b)):

$$\sum_{i=1}^{D} q_i(x) = 1 = \sum_{i=1}^{D} (q_i(x))^2. \tag{1.22}$$

[14] As in our main results, here we have assumed the spatial topology $\mathbb{T}^D$; in [25], the spatial topology was not specified.

Given the above assumptions, the authors of [25] then observe (through straightforward but tedious computations) that for metrics of the form (1.21), the Ricci tensor components of the spatial metric verify

$$\lim_{t \downarrow 0} |t^2 \mathrm{Ric}^i{}_j(t,x)| = 0 \quad (i,j = 1,\dots,D), \tag{1.23}$$

as long as the following system of inequalities holds:

$$2q_i(x) + \sum_{l \neq i,j,k} q_l(x) > 0 \quad \text{whenever } i \neq j, i \neq k, \text{and } j \neq k. \tag{1.24}$$

Note that (1.23) is essentially equivalent to estimate (1.14) that we rigorously obtain in proving our main results; as we described in Section 1.5, (1.14) is a quantified version of the idea that spatial derivative terms should be negligible. The main conclusions of [25] can be summarized as follows.

> In view of (1.22), it is not possible to simultaneously satisfy all inequalities (1.24) when $D \leq 9$. However, there *does* exist an open set of $\{q_i\}_{i=1,\dots,D}$ satisfying (1.22) and (1.24) whenever $D \geq 10$.

We now stress that the metric $\mathbf{g}$ featured in equation (1.21) does not generally solve the Einstein-vacuum equations. However, it does solve a truncated version of the equations in which all spatial derivative terms, including $\mathrm{Ric}^i{}_j$, are discarded; in the mathematical general relativity literature, the truncated system is often referred to as the Velocity Term Dominated (VTD) system. For this reason, condition (1.23), which is supposed to capture the negligibility of the spatial derivative terms, suggests that one can think of the VTD solution $\mathbf{g}$ as "asymptotically solving" the Einstein-vacuum equations of Proposition 2.10 as $t \downarrow 0$. That is, for metrics of the form (1.21), the k-involving product in equation (2.12c) (specifically $t^{-1}k^i{}_j$) is of size t^{-2} while, by (1.23), the term $\mathrm{Ric}^i{}_j$ in equation (2.12c) is less singular. Put differently, the metrics $\mathbf{g}$ featured in (1.21) solve a PDE system obtained from the Einstein equations by throwing away terms that can be shown to be small in the sense of (1.23). The work [25] can therefore be viewed as providing a kind of "consistency argument" for metrics (1.21) that satisfy (1.24), i.e., an argument based on ignoring terms in the evolution equations that, for the *non-solution* $\mathbf{g}$, are small compared to the main terms. For this perspective, it is reasonable to speculate that, for metrics $\mathbf{g}$ of the form (1.21) that satisfy (1.23), there might be true Einstein-vacuum solutions lying "close" to $\mathbf{g}$. We clarify that this speculation was rigorously shown to be true in [24] whenever $D \geq 10$, though the results of [24] did not yield the dynamic stability of the singularity and relied on the assumption that the tensorfields $q_i(x)$ and $\omega^{(i)}(x)$ appearing in (1.21) are analytic.

We now explain some further connections between the heuristic picture painted in [25] and the results of the present article. As we described in Section 1.5, estimate (1.14) (which is almost the same as condition (1.23) from [25]) is a main ingredient needed to show that at each fixed x, $\partial_t(tk^i{}_j(t,x))$ is integrable over the time interval $(0,1]$ (we refer the readers to Remark 1.13 for a discussion on why we work with k in type $\binom{1}{1}$ form). From this integrability condition, one can easily show not only that $tk^i{}_j(t,x)$ remains bounded as $t \downarrow 0$ (which is what we prove in our main theorem), but also that the following stronger result holds: $tk^i{}_j(t,x)$ converges uniformly to a function $K^i{}_j(x)$ as $t \downarrow 0$; see [50, 51] for proofs of these kinds of convergent results in a related context, and see also Remark 1.9. Combining these kinds of convergence estimates with related ones (in particular for the lapse n), *one could rigorously prove that for the Einstein-vacuum solutions* $\mathbf{g}$ *under study in this article,* $\mathbf{g}$ *is asymptotic to a metric that behaves like the family of metrics that satisfy* (1.21)–(1.24); again, readers can consult [50,51] for proofs of these kinds of results in a related context.

The situation can be summarized as follows: the singular solutions that can be shown to be dynamically stable under our framework are asymptotic (as the singularity is approached) to a metric that is well-described by the family of metrics satisfying conditions (1.21)–(1.24). The following question is glaring: whether or not, under assumptions (1.22) and (1.23), the "VTD solutions" $\mathbf{g}$ *defined* by (1.21) are always the asymptotic end-state of a true singularity-forming solution of the Einstein-vacuum equations. As of present, not much is rigorously known about this issue. In fact, there are no results outside of symmetry for any Einstein-matter or Einstein-vacuum system that definitively show that the set of "asymptotic end-states near the Big Bang" is open with respect to a function space topology that is natural from the point of view of well-posedness theory (such as a topology induced by a Sobolev norm). A second question also stands out: whether or not it is possible to extend the stable blowup-results of the present article to apply to perturbations of all Kasner solutions (1.6) whose Kasner exponents $\{q_i\}_{i=1,\dots,D}$ satisfy (1.8a)–(1.8b) and (1.24), which in particular would extend our results to the cases $D \geq 10$. If such a result is in fact true, then its proof would almost certainly involve detecting the kinds of tensorial cancellations that the authors of [25] exploited to derive the bound (1.23), a feat that we did not attempt in the present work. It is likely that capturing these kinds of tensorial cancellations would involve substantial new ideas and techniques, going beyond our work here. This is a worthy avenue for future investigation, especially since it is intimately tied to the fundamental question of which terms in Einstein's equations are the ones driving the breakdown of solutions.

1.7. Paper outline

The remainder of the paper is organized as follows.

- In Section 1.8, we summarize the notation and conventions that we use in the rest of the article.
- In Section 2, we set up the ensuing analysis by providing the Einstein-vacuum equations in CMC-transported spatial coordinates and showing that Kasner solutions verifying our assumptions exist when $D \geq 38$.
- In Section 3, we define the norms that we use to control solutions and formulate the bootstrap assumptions that we use in our analysis.
- In Section 4, we provide some preliminary technical estimates, which are standard.
- In Section 5, we derive elliptic estimates showing that the lapse n can be controlled, in various norms, in terms of g and k.

- In Section 6, we derive preliminary estimates for g and k at the low derivative levels.
- In Section 7, we derive preliminary estimates for g and k at the top derivative levels, i.e., our main top-order energy estimates.
- In Section 8, we derive preliminary energy estimates for g and k at the near-top-order derivative levels; as we explained near the end of Section 1.5, for technical reasons, we need these estimates to close our bootstrap argument.
- In Section 9, we combine the results of Sections 5–8 to obtain the main technical result of the article: a priori estimates showing that appropriately defined solution norms can be uniformly controlled by the initial data, all the way up to the singularity.
- In Section 10, we derive some results that describe the breakdown as $t \downarrow 0$, e.g., the curvature blows up and past-directed timelike geodesics terminate with a finite length.
- In Section 11, we synthesize the results of the previous sections and give a relatively short proof of the main stable blowup theorem.

1.8. Notation and conventions

For the reader's convenience, in this subsection, we provide some notation and conventions that we use throughout the article. Some of the concepts referred to here are not defined until later.

1.8.1. Indices and their lowering and raising. Greek "spacetime" indices $\alpha, \beta, \ldots$ take on the values $0, 1, \ldots, D$, while Latin "spatial" indices $a, b, \ldots$ take on the values $1, 2, \ldots, D$. Repeated indices are summed over (from 0 to D if they are Greek, and from 1 to D if they are Latin). We use the same conventions for primed indices such as a' as we do for their non-primed counterparts. Spacetime indices are lowered and raised with the Lorentzian metric $\mathbf{g}_{\alpha\beta}$ and its inverse $(\mathbf{g}^{-1})^{\alpha\beta}$. Spatial indices are lowered and raised with the Riemannian metric g_{ij} and its inverse g^{ij}.

1.8.2. Spacetime tensorfields and Σ_t-tangent tensorfields. We denote spacetime tensorfields $\mathbf{T}_{\nu_1 \cdots \nu_m}{}^{\mu_1 \cdots \mu_l}$ in bold font. We denote Σ_t-tangent tensorfields $T_{b_1 \cdots b_m}{}^{a_1 \cdots a_l}$ in non-bold font.

1.8.3. Coordinate systems and differential operators. We often work in a fixed standard local coordinate system $(x^1, \ldots, x^D)$ on $\mathbb{T}^D$. The vectorfields $\partial_j := \frac{\partial}{\partial x^j}$ are globally well-defined even though the coordinates themselves are not. Hence, in a slight abuse of notation, we use $\{\partial_1, \ldots, \partial_D\}$ to denote the globally defined vectorfield frame. We denote the corresponding basis-dual co-frame by $\{dx^1, \ldots, dx^D\}$. In CMC-transported spatial coordinates, the spatial coordinate functions are transported along the unit normal to Σ_t, thus producing a local coordinate system $(x^0, x^1, \ldots, x^D)$ on manifolds-with-boundary of the form $(T, 1] \times \mathbb{T}^D$, and we often write t instead of x^0. The corresponding vectorfield frame on $(T, 1] \times \mathbb{T}^D$ is $\{\partial_0, \partial_1, \ldots, \partial_D\}$, and the corresponding basis-dual co-frame is $\{dx^0, dx^1, \ldots, dx^D\}$. Relative to this frame, Kasner solutions are of the form (1.6)–(1.7). The symbol ∂_μ denotes the frame derivative $\frac{\partial}{\partial x^\mu}$, and we often write ∂_t instead of ∂_0 and

dt instead of dx^0. Many of our equations and estimates are stated relative to the frame $\{\partial_\mu\}_{\mu=0,1,\ldots,D}$ and basis-dual co-frame $\{dx^\mu\}_{\mu=0,1,\ldots,D}$.

We use the notation ∂f to denote the *spatial coordinate* gradient of the scalar-valued function f relative to the coordinates described above, which we view to be a Σ_t-tangent one-form. That is, $(\partial f)_i := \partial_i f$. More generally, if ω is a type $\binom{l}{m}$ Σ_t-tangent tensorfield with components $\omega_{b_1\cdots b_m}{}^{a_1\cdots a_l}$ relative to the frame described above, then $\partial\omega$ denotes the type $\binom{l}{m+1}$ Σ_t-tangent tensorfield with components

$$(\partial\omega)_{b_1\cdots b_{m+1}}{}^{a_1\cdots a_l} := \partial_{b_1}\omega_{b_2\cdots b_{m+1}}{}^{a_1\cdots a_l}$$

relative to the same frame. Similarly, $\partial^2\omega$ denotes the type $\binom{l}{m+2}$ Σ_t-tangent tensorfield with components

$$(\partial^2\omega)_{b_1\cdots b_{m+2}}{}^{a_1\cdots a_l} := \partial_{b_1}\partial_{b_2}\omega_{b_3\cdots b_{m+2}}{}^{a_1\cdots a_l}.$$

If $\vec{I} = (n_1, n_2, \ldots, n_D)$ is an array comprising D non-negative integers, then we define the *spatial* multi-indexed differential operator $\partial_{\vec{I}}$ by

$$\partial_{\vec{I}} := \partial_1^{n_1}\partial_2^{n_2}\cdots\partial_D^{n_D}.$$

The notation $|\vec{I}| := n_1 + n_2 + \cdots + n_D$ denotes the order of $\vec{I}$.

If ω is a type $\binom{l}{m}$ Σ_t-tangent tensorfield with components $\omega_{b_1\cdots b_m}{}^{a_1\cdots a_l}$ relative to the frame described above, then $\partial_{\vec{I}}\omega$ denotes the type $\binom{l}{m}$ Σ_t-tangent tensorfield with components

$$(\partial_{\vec{I}}\omega)_{b_1\cdots b_m}{}^{a_1\cdots a_l} := \partial_{\vec{I}}(\omega_{b_1\cdots b_m}{}^{a_1\cdots a_l}).$$

In particular, under this definition, the operators $\partial_{\vec{I}}$ do not change the order of tensorfields; this is in contrast to the operators ∂ and ∂^2 introduced two paragraphs above.

Throughout, $\mathbf{D}$ denotes the Levi-Civita connection of $\mathbf{g}$. We write

$$\begin{aligned}\mathbf{D}_\nu\mathbf{T}_{\nu_1\cdots\nu_m}{}^{\mu_1\cdots\mu_l} &= \partial_\nu\mathbf{T}_{\nu_1\cdots\nu_m}{}^{\mu_1\cdots\mu_l} + \sum_{r=1}^{l}\mathbf{\Gamma}_{\nu\ \alpha}^{\ \mu_r}\mathbf{T}_{\nu_1\cdots\nu_m}{}^{\mu_1\cdots\mu_{r-1}\alpha\mu_{r+1}\cdots\mu_l} \\ &\quad - \sum_{s=1}^{m}\mathbf{\Gamma}_{\nu\ \nu_s}^{\ \alpha}\mathbf{T}_{\nu_1\cdots\nu_{s-1}\alpha\nu_{s+1}\cdots\nu_m}{}^{\mu_1\cdots\mu_l}\end{aligned} \tag{1.25}$$

to denote a component of the covariant derivative of a tensorfield $\mathbf{T}$ (with components $\mathbf{T}_{\nu_1\cdots\nu_m}{}^{\mu_1\cdots\mu_l}$) defined on the set $(T, 1] \times \mathbb{T}^D$. The Christoffel symbols of $\mathbf{g}$, which we denote by $\mathbf{\Gamma}_{\mu\ \nu}^{\ \lambda}$, are defined by

$$\mathbf{\Gamma}_{\mu\ \nu}^{\ \lambda} := \frac{1}{2}(\mathbf{g}^{-1})^{\lambda\sigma}\{\partial_\mu\mathbf{g}_{\sigma\nu} + \partial_\nu\mathbf{g}_{\mu\sigma} - \partial_\sigma\mathbf{g}_{\mu\nu}\}. \tag{1.26}$$

We use similar notation to denote the covariant derivative of a Σ_t-tangent tensorfield T (with components $T_{b_1\cdots b_n}{}^{a_1\cdots a_m}$) with respect to the Levi-Civita connection ∇ of the Riemannian metric g, i.e., the first fundamental form of Σ_t. The Christoffel symbols of g, which we denote by $\Gamma_{j\ k}^{\ i}$, are defined by

$$\Gamma_{j\ k}^{\ i} := \frac{1}{2}g^{ia}\{\partial_j g_{ak} + \partial_k g_{ja} - \partial_a g_{jk}\}. \tag{1.27}$$

1.8.4. Integrals and basic norms. Throughout this subsection, f denotes a scalar function defined on the hypersurface $\Sigma_t = \{(s,x) \in \mathbb{R} \times \mathbb{T}^D \mid s = t\}$. We define

$$\int_{\Sigma_t} f \, dx := \int_{\mathbb{T}^D} f(t, x^1, \ldots, x^D) \, dx. \tag{1.28}$$

Above, the notation "$\int_{\mathbb{T}^D} f \, dx$" denotes the integral of f over $\mathbb{T}^D$ with respect to the measure corresponding to the volume form of the *standard Euclidean metric* E on $\mathbb{T}^D$, which has the components $\mathrm{diag}(1, 1, \ldots, 1)$ relative to the coordinate frame described in Section 1.8.3. Note that dx is *not the canonical integration measure associated to the Riemannian metric* g.

All of our Sobolev norms are built out of the (spatial) L^2 norms of scalar quantities (which may be the components of a tensorfield). We define the standard L^2 norm $\| \cdot \|_{L^2(\Sigma_t)}$ as follows:

$$\|f\|_{L^2(\Sigma_t)} := \left(\int_{\Sigma_t} f^2 \, dx \right)^{1/2}. \tag{1.29}$$

For integers $M \geq 0$, we define the standard H^M norm $\| \cdot \|_{H^M(\Sigma_t)}$ as follows:

$$\|f\|_{H^M(\Sigma_t)} := \left(\sum_{|\vec{I}| \leq M} \|\partial_{\vec{I}} f\|_{L^2(\Sigma_t)}^2 \right)^{1/2}. \tag{1.30}$$

We also define the following standard homogeneous analog of (1.30):

$$\|f\|_{\dot{H}^M(\Sigma_t)} := \left(\sum_{|\vec{I}| = M} \|\partial_{\vec{I}} f\|_{L^2(\Sigma_t)}^2 \right)^{1/2}. \tag{1.31}$$

Finally, we define the Lebesgue norm $\| \cdot \|_{L^\infty(\Sigma_t)}$ of scalar functions f in the usual way:

$$\|f\|_{L^\infty(\Sigma_t)} := \operatorname*{ess\,sup}_{x \in \mathbb{T}^D} |f(t,x)|. \tag{1.32}$$

In Sections 3.2 and 3.3, we will introduce additional norms for tensorfields, many of which are built out of the basic ones from this subsubsection.

1.8.5. Parameters.

- $A \geq 1$ denotes a "time-weight exponent parameter" that is featured in the high-order solution norms from Definition 3.16. To close our estimates, we will choose A to be large enough to overwhelm various universal constants C_* (see Section 1.8.6). This corresponds to our use of high-order energies featuring large powers of t, which leads to weak high-order energies near $t = 0$.
- $0 < q < 1/6$ is a constant, fixed throughout the proof, that bounds the magnitude of the background Kasner exponents.
- $\sigma > 0$ is a small constant, fixed throughout the proof, that we use to simplify the proofs of various estimates that "have room in them."
- q and σ are constrained by (3.2).

- N denotes the maximum number of times that we commute the equations with spatial derivatives (e.g., $k \in H^N(\Sigma_t)$ and $g \in H^{N+1}(\Sigma_t)$). To close our estimates, we will choose N to be sufficiently large in a (non-explicit) manner that depends on A, D, q, σ.
- $\delta > 0$ is a small (N, D)-dependent parameter that is allowed to vary from line to line and that is generated by the estimates of Lemma 4.11. We use the convention that a sum of two constants δ is another δ. The only important feature of δ that we exploit in the proof is the following: at fixed D, we have $\lim_{N\to\infty} \delta = 0$. In particular, if A is also fixed, then $\lim_{N\to\infty} A\delta = 0$.
- ε is a small "bootstrap parameter" that, in our bootstrap argument, bounds the size of the solution norms; see (3.18). The smallness of ε needed to close the estimates is allowed to depend on the parameters N, A, D, q, and σ.

1.8.6. Constants.

- C denotes a positive constant that is free to vary from line to line. C can depend on N, A, D, q, and σ, but it can be chosen to be independent of all $\varepsilon > 0$ that are sufficiently small in the manner described in Section 1.8.5.
- C_* denotes a positive constant that is free to vary from line to line and that can depend on D. Like C, C_* can be chosen to be independent of all $\varepsilon > 0$ that are sufficiently small in the manner described in Section 1.8.5. However, unlike C, C_* can be chosen to be **independent** of N and A. The constant C_* can also be chosen to be independent of q and σ, but that is less important in the sense that we view q and σ to be fixed throughout the article; see Remark 3.3. For example, $1 + CN!\varepsilon \leq C_*$ while $N! = C$ and $N!/\sigma = C$, where C and C_* are as above.
- We write $v \lesssim w$ to indicate that $v \leq Cw$, with C as above.
- We write $v = \mathcal{O}(w)$ to indicate that $|v| \leq C|w|$, with C as above.

1.8.7. Schematic notation.

- We write $v = \prod_{r=1}^{R} v_r$ to indicate that the Σ_t-tangent tensorfield v is a tensor product, possibly involving contractions, of the Σ_t-tangent tensorfields v_r. We use this notation only when the precise details of the tensor product are not important for our analysis. We sometimes display the indices of v to indicate its order; for example, the expression $v^i{}_j = \prod_{r=1}^{R} v_r$ emphasizes that v is type $\binom{1}{1}$.
- We write $v \simeq \sum_{r=1}^{R} v_r$ to indicate that the Σ_t-tangent tensorfield v is a linear combination of the Σ_t-tangent tensorfields v_r, where **the coefficients in the linear combination are constants $\pm C$**, with C as in Section 1.8.6. As above, we sometimes display the indices of v to indicate its order. For example, $v_{ijk} \simeq v_1 + v_2$ means that v is type $\binom{0}{3}$ and that $v = \pm C_1 v_1 \pm C_2 v_2$, where the C_i can depend on N, A, D, q, and σ.
- We write $v \overset{*}{\simeq} \sum_{r=1}^{R} v_r$ to indicate that the Σ_t-tangent tensorfield v is a linear combination of the Σ_t-tangent tensorfields v_r, where **the coefficients in the linear combination are constants $\pm C_*$**, with C_* universal constants enjoying the properties described in Section 1.8.6. As above, we sometimes display the indices of v to indi-

cate its order. For example, $v^i{}_j \overset{*}{\simeq} v_1 + v_2$ means that v is type $\binom{1}{1}$ and that $v = \pm C_{*,1} v_1 \pm C_{*,1} v_2$, where the $C_{*,i}$ are **independent of** N and A.

2. Setting up the analysis

In this section, we start by providing the Einstein-vacuum equations in the gauge that we use to prove our main theorem. Next, we provide some standard expressions for the curvature tensors of the first fundamental form of Σ_t. Finally, we show that when $D \geq 38$, there exist Kasner solutions that satisfy the Kasner exponent assumptions in our main theorem.

2.1. The Einstein-vacuum equations in CMC-transported spatial coordinates

In this subsection, we recall the Einstein-vacuum equations in CMC-transported spatial coordinates gauge; this is the gauge that we use throughout the article.

2.1.1. Basic ingredients in the setup. In CMC-transported spatial coordinates, the spacetime metric is decomposed into the lapse n and a Riemannian metric g on the constant time hypersurfaces Σ_t (known as the first fundamental form of Σ_t) as follows:

$$\mathbf{g} = -n^2 dt \otimes dt + g_{ab} dx^a \otimes dx^b. \tag{2.1}$$

We use g^{ab} to denote the components of the inverse Riemannian metric g^{-1}. Above and throughout, t is a time function on the spacetime manifold $\mathcal{M}$ that we will describe just below and $\{x^a\}_{a=1,\dots,D}$ are standard (locally defined) "spatial coordinates" on the constant-time hypersurfaces $\Sigma_t := \{(s,x) \in \mathcal{M} \mid s = t\}$, which are diffeomorphic to $\mathbb{T}^D$. We refer readers to Section 1.8.3 for further discussion of these coordinates and notation tied to them. In view of (2.1), we see that the future-directed unit normal $\hat{\mathbf{N}}$ to Σ_t can be expressed as

$$\hat{\mathbf{N}} = n^{-1} \partial_t. \tag{2.2}$$

Note that $\hat{\mathbf{N}} x^a = 0$ for $a = 1, \dots, D$. Thus, in the gauge under consideration, the spatial coordinates are transported along the flow lines of $\hat{\mathbf{N}}$.

The second fundamental form k of Σ_t is defined by requiring that the following relation holds for all vectorfields X, Y tangent to Σ_t:

$$\mathbf{g}(\mathbf{D}_X \hat{\mathbf{N}}, Y) = -k(X,Y), \tag{2.3}$$

where $\mathbf{D}$ is the Levi-Civita connection of $\mathbf{g}$. It is a standard fact that k is symmetric:

$$k(X,Y) = k(Y,X). \tag{2.4}$$

For such X, Y, the action of the spacetime connection $\mathbf{D}$ can be decomposed into the action of the Levi-Civita connection ∇ of g and k as follows:

$$\mathbf{D}_X Y = \nabla_X Y - k(X,Y)\hat{\mathbf{N}}. \tag{2.5}$$

Remark 2.6. As we described in Section 1.5, when analyzing the components of k (and in particular when differentiating the components of k), **we will always assume that it is written in mixed form (i.e., type $\binom{1}{1}$ form) as $k^i{}_j$ with the first index upstairs and the second one downstairs.** This convention is absolutely essential for some of our analysis; in the problem of interest to us, the evolution and constraint equations verified by the components $k^i{}_j$ have a more favorable structure than the corresponding equations verified by k_{ij}. We refer to Section 1.8.1 regarding our conventions for lowering and raising indices.

Throughout the vast majority of our analysis, we normalize the CMC hypersurfaces Σ_t as follows:

$$k^a{}_a = -\frac{1}{t}, \quad t \in (0,1]. \tag{2.7}$$

In order for (2.7) to hold, the lapse has to verify the elliptic equation (2.13b).

We adopt the following sign convention for the Riemann curvature **Riem** of $\mathbf{g}$:

$$\mathbf{D}_\alpha \mathbf{D}_\beta \mathbf{X}_\mu - \mathbf{D}_\beta \mathbf{D}_\alpha \mathbf{X}_\mu = \mathbf{Riem}_{\alpha\beta\mu\nu}\, \mathbf{X}^\nu. \tag{2.8}$$

Similarly, we adopt the following sign convention for the Riemann curvature Riem of g:

$$\nabla_a \nabla_b X_c - \nabla_b \nabla_a X_c = \mathrm{Riem}_{abcd}\, X^d. \tag{2.9}$$

2.1.2. Statement of the equations. In this subsubsection, we state a proposition that provides the Einstein-vacuum equations relative to CMC-transported spatial coordinates. The proposition can be proved using standard calculations; we refer readers to [51, Appendix B] and [52, Section 6.2] for details.

Proposition 2.10 (The Einstein-vacuum equations in CMC-transported spatial coordinates). *In CMC-transported spatial coordinates normalized by $k^a{}_a = -t^{-1}$, the Einstein-vacuum equations* (1.1) *take the following form.*

- *The **Hamiltonian and momentum constraint equations** verified by g and k are respectively*

$$\mathrm{R} - k^a{}_b k^b{}_a + t^{-2} = 0, \tag{2.11a}$$
$$\nabla_a k^a{}_i = 0, \tag{2.11b}$$

where ∇ denotes the Levi-Civita connection of g, $\mathrm{R} = \mathrm{Ric}^a{}_a$ denotes the scalar curvature of g, and Ric *denotes the Ricci curvature of g (a precise expression is given in* (2.15b)*).*

- *The **evolution equations** verified by g, g^{-1}, and k are*

$$\partial_t g_{ij} = -2n g_{ia} k^a{}_j, \tag{2.12a}$$
$$\partial_t g^{ij} = 2n g^{ia} k^j{}_a, \tag{2.12b}$$
$$\partial_t (k^i{}_j) = -g^{ia} \nabla_a \nabla_j n + n(\mathrm{Ric}^i{}_j - t^{-1} k^i{}_j). \tag{2.12c}$$

- *The* ***elliptic lapse equation*** *can be written in either of the following two forms (by virtue of the constraint* (2.11a)*):*

$$\{g^{ab}\nabla_a\nabla_b - t^{-2}\}(n-1) = (n-1)\{k^a{}_b k^b{}_a - t^{-2}\} + \{k^a{}_b k^b{}_a - t^{-2}\}, \quad (2.13a)$$

$$\{g^{ab}\nabla_a\nabla_b - (t^{-2} + \mathrm{R})\}(n-1) = \mathrm{R}. \quad (2.13b)$$

Remark 2.14 (The form of equations (2.13a)–(2.13b)). We have written (2.13a)–(2.13b) in a form that is useful for studying perturbations of the background Kasner solution $(\widetilde{g}, \widetilde{k}, \widetilde{n})$, which is such that

$$\widetilde{k}^a{}_b \widetilde{k}^b{}_a \equiv t^{-2} \quad \text{and} \quad \widetilde{n} \equiv 1$$

and such that $\widetilde{g}$ has vanishing curvature. In the relevant spots in Sections 5–8, we will rewrite some of the other equations in Proposition 2.10 to make it easier to study perturbations of the background Kasner solution, and we will also derive equations satisfied by the derivatives of perturbed solutions.

2.2. *Standard expressions for curvature tensors of g*

For future use, we note the following standard facts: relative to an arbitrary coordinate system on Σ_t (and in particular relative to the transported spatial coordinates that we use in our analysis), the components of the type $\binom{0}{4}$ Riemann curvature Riem of g and the type $\binom{1}{1}$ Ricci curvature Ric of g can be expressed, respectively, as

$$\begin{aligned}\mathrm{Riem}_{ijkl} &= \frac{1}{2}\{\partial_i\partial_l g_{jk} + \partial_j\partial_k g_{il} - \partial_i\partial_k g_{jl} - \partial_j\partial_l g_{ik}\} \\ &\quad + g^{ab}\Gamma_{ial}\Gamma_{jbk} - g^{ab}\Gamma_{iak}\Gamma_{jbl}, \end{aligned} \quad (2.15a)$$

$$\begin{aligned}\mathrm{Ric}^i{}_j &= \frac{1}{2} g^{cd} g^{ie}\{\partial_e\partial_c g_{dj} + \partial_c\partial_j g_{ed} - \partial_e\partial_j g_{cd} - \partial_c\partial_d g_{ej}\} \\ &\quad + g^{ab} g^{cd} g^{ie}\Gamma_{eac}\Gamma_{jbd} - g^{ab} g^{cd} g^{ie}\Gamma_{eaj}\Gamma_{cbd}, \end{aligned} \quad (2.15b)$$

where $\Gamma_{ijk} := g_{ja}\Gamma_{i\ k}^{\ a}$ and $\Gamma_{j\ k}^{\ i}$ are the Christoffel symbols of g (see (1.27)).

2.3. *The existence of Kasner solutions verifying our exponent assumptions when $D \geq 38$*

Note that in view of (1.8b), if $D \leq 36$, then there do not exist any Kasner solutions that satisfy the exponent assumption (1.8e). In this subsection, we show that for $D \geq 38$, such Kasner solutions *do* exist; recall that this is equivalent to finding real numbers $\{q_i\}_{i=1,\ldots,D}$ that satisfy (1.8a)–(1.8b) and (1.8e).

To start, we note that in the case $D = 36$, the following Kasner exponents satisfy (1.8a)–(1.8b) but just barely fail to satisfy (1.8e):

$$\begin{aligned} q_1 = q_2 = \cdots = q_{15} &:= -\frac{1}{6}, \\ q_{16} = q_{17} = \cdots q_{36} &= \frac{1}{6}. \end{aligned} \quad (2.16)$$

Considering now the case $D = 38$, we let $\epsilon > 0$ be a small parameter, and we perturb the 36 Kasner exponents from (2.16) by ϵ so that they are bounded in magnitude by $< 1/6$:

$$\begin{aligned} q_1 = q_2 = \cdots = q_{15} &:= -\frac{1}{6} + \epsilon, \\ q_{16} = q_{17} = \cdots q_{36} &= \frac{1}{6} - \epsilon. \end{aligned} \tag{2.17}$$

Notice that by (1.8a)–(1.8b), assuming (2.17), any solution (q_{37}, q_{38}) to the following system yields, when complemented with the exponents (2.17), a complete set of Kasner exponents:

$$q_{37} + q_{38} = 6\epsilon, \tag{2.18a}$$

$$q_{37}^2 + q_{38}^2 = 12\epsilon - 36\epsilon^2. \tag{2.18b}$$

Using (2.18a) to solve for q_{38} in terms of q_{37} and then substituting into (2.18b), we obtain the equation $q_{37}^2 - 6\epsilon q_{37} - 6\epsilon + 36\epsilon^2 = 0$, which has the solutions

$$q_{37} = 3\epsilon \pm \sqrt{6\epsilon - 27\epsilon^2}. \tag{2.19}$$

We now observe that for any $\epsilon > 0$ sufficiently small, the corresponding solutions q_{37} to (2.19) are real and bounded in magnitude by $\lesssim \sqrt{\epsilon}$. From (2.18a), we deduce that the same statement holds for the corresponding exponent q_{38}. In view of (2.17), we see that for $\epsilon > 0$ sufficiently small, the exponents $\{q_i\}_{i=1,\ldots,38}$ constructed in this fashion satisfy (1.8a)–(1.8b) and (1.8e). Moreover, in the cases $D \geq 39$, we can complement these 38 Kasner exponents with others as follows: $q_i = 0$ for $39 \leq i \leq D$. In total, we have constructed, for any $D \geq 38$, sets of Kasner exponents $\{q_i\}_{i=1,\ldots,D}$ that satisfy (1.8a)–(1.8b) and (1.8e). That is, the Kasner solutions whose perturbations we study in our main theorem exist when $D \geq 38$.

3. Norms and bootstrap assumptions

In this section, we define the norms that we use to control the solution. We also state bootstrap assumptions for the solution norms; the bootstrap assumptions are convenient for our analysis in subsequent sections.

3.1. Fixed constants appearing in the norms

The norms that we define in Section 3.4 involve the positive numbers q and σ featured in the following definition.

Definition 3.1 (The constants q and σ). Assuming that the Kasner exponents satisfy the condition (1.8e), we fix positive numbers q and σ verifying the following inequalities:

$$0 < \sigma < \sigma + \max_{i=1,\ldots,D} |q_i| < q < q + 2\sigma < \frac{1}{6}. \tag{3.2}$$

Remark 3.3. We consider q and σ to be fixed for the remainder of the article. In particular, q and σ do not vary when we choose N and A to be large.

3.2. *Pointwise norms*

To control Σ_t-tangent tensorfields, we will rely on two kinds of pointwise norms: one that refers to the transported spatial coordinate frame, and the standard geometric norm that is based on the Riemannian metric g.

Definition 3.4 (Pointwise norms). For Σ_t-tangent type $\binom{l}{m}$ tensorfields T, we define

$$|T|_{\text{Frame}} := \left\{ \sum_{a_1,\ldots,a_l,b_1,\ldots,b_m=1}^{D} |T_{b_1\cdots b_m}{}^{a_1\cdots a_l}|^2 \right\}^{1/2}, \tag{3.5a}$$

$$|T|_g := \left\{ g_{a_1 a_1'} \cdots g_{a_l a_l'} g^{b_1 b_1'} \cdots g^{b_m b_m'} T_{b_1\cdots b_m}{}^{a_1\cdots a_l} T_{b_1'\cdots b_m'}{}^{a_1'\cdots a_l'} \right\}^{1/2}. \tag{3.5b}$$

3.3. *Lebesgue and Sobolev norms*

In this subsection, we define the Lebesgue and Sobolev norms that we will use to control the solution. We start by defining the $\partial_{\vec{I}}$-derivative of a tensorfield.

Definition 3.6 (Derivative of a tensorfield). If $T_{b_1\cdots b_m}{}^{a_1\cdots a_l}$ is a type $\binom{l}{m}$ Σ_t-tangent tensorfield and $\vec{I}$ is a spatial multi-index, then we define $\partial_{\vec{I}} T$ to be the type $\binom{l}{m}$ tensorfield whose components $(\partial_{\vec{I}} T)_{b_1\cdots b_m}{}^{a_1\cdots a_l}$ relative to the CMC-transported spatial coordinate frame are the following:

$$(\partial_{\vec{I}} T)_{b_1\cdots b_m}{}^{a_1\cdots a_l} := \partial_{\vec{I}} (T_{b_1\cdots b_m}{}^{a_1\cdots a_l}). \tag{3.7}$$

Remark 3.8. The operator $\partial_{\vec{I}}$, when acting on Σ_t-tangent tensorfields, can be given the following invariant interpretation: it can be viewed as repeated Lie differentiation with respect to the (globally defined) spatial coordinate partial derivative vectorfields $\{\partial_i\}_{i=1,\ldots,D}$.

Remark 3.9. We remind the reader that, as we described in Section 1.8.3, if T is a type $\binom{l}{m}$ Σ_t-tangent tensorfield, then ∂T is by definition type $\binom{l}{m+1}$ and $\partial^2 T$ is by definition type $\binom{l}{m+2}$. Thus, for example, in the ensuing equations and estimates, readers should carefully distinguish between the meaning of $\partial^2 T$ and $\partial \partial_{\vec{I}} T$ when $|\vec{I}| = 1$ (the former tensorfield is type $\binom{l}{m+2}$ while the latter is type $\binom{l}{m+1}$).

In what follows, $\|\cdot\|_{L^2(\Sigma_t)}$ and $\|\cdot\|_{L^\infty(\Sigma_t)}$ denote the standard Lebesgue norms for scalar functions on Σ_t; see Section 1.8.4. We now define additional Sobolev and Lebesgue norms that we will use in our analysis.

Definition 3.10 (Sobolev and Lebesgue norms). If T is a type $\binom{l}{m}$ Σ_t-tangent tensorfield, $p \in \{2, \infty\}$, and $M \geq 0$ is an integer, then we define

$$\|T\|_{L^p_{\text{Frame}}(\Sigma_t)} := \| |T|_{\text{Frame}} \|_{L^p(\Sigma_t)}, \quad \|T\|_{L^p_g(\Sigma_t)} := \| |T|_g \|_{L^p(\Sigma_t)}, \tag{3.11a}$$

$$\begin{aligned} \|T\|_{W^{M,\infty}_{\text{Frame}}(\Sigma_t)} &:= \sum_{|\vec{I}| \leq M} \| |\partial_{\vec{I}} T|_{\text{Frame}} \|_{L^\infty(\Sigma_t)}, \\ \|T\|_{W^{M,\infty}_g(\Sigma_t)} &:= \sum_{|\vec{I}| \leq M} \| |\partial_{\vec{I}} T|_g \|_{L^\infty(\Sigma_t)}, \end{aligned} \tag{3.11b}$$

$$\|T\|_{\dot{W}^{M,\infty}_{\text{Frame}}(\Sigma_t)} := \sum_{|\vec{I}|=M} \||\partial_{\vec{I}} T|_{\text{Frame}}\|_{L^\infty(\Sigma_t)},$$
$$\|T\|_{\dot{W}^{M,\infty}_{g}(\Sigma_t)} := \sum_{|\vec{I}|=M} \||\partial_{\vec{I}} T|_{g}\|_{L^\infty(\Sigma_t)}, \tag{3.11c}$$

$$\|T\|_{H^M_{\text{Frame}}(\Sigma_t)} := \left\{ \sum_{|\vec{I}|\leq M} \||\partial_{\vec{I}} T|_{\text{Frame}}\|^2_{L^2(\Sigma_t)} \right\}^{1/2},$$
$$\|T\|_{H^M_{g}(\Sigma_t)} := \left\{ \sum_{|\vec{I}|\leq M} \||\partial_{\vec{I}} T|_{g}\|^2_{L^2(\Sigma_t)} \right\}^{1/2}, \tag{3.11d}$$

$$\|T\|_{\dot{H}^M_{\text{Frame}}(\Sigma_t)} := \left\{ \sum_{|\vec{I}|= M} \||\partial_{\vec{I}} T|_{\text{Frame}}\|^2_{L^2(\Sigma_t)} \right\}^{1/2},$$
$$\|T\|_{\dot{H}^M_{g}(\Sigma_t)} := \left\{ \sum_{|\vec{I}|= M} \||\partial_{\vec{I}} T|_{g}\|^2_{L^2(\Sigma_t)} \right\}^{1/2}. \tag{3.11e}$$

Remark 3.12 (The omission of norm subscripts when appropriate). If T is a scalar function, then we typically omit the subscripts "Frame" and "g" in the norms since there is no danger of confusion over how to measure the norm of T. For example if f is scalar function, then we write $\|f\|_{L^\infty(\Sigma_t)}$ instead of $\|f\|_{L^\infty_{\text{Frame}}(\Sigma_t)}$ or $\|f\|_{L^\infty_g(\Sigma_t)}$.

Remark 3.13 (Simple comparison estimates that we use silently use in our analysis). Here, by way of example, we note some simple but important comparison estimates that we often silently use in our analysis. First, by using the g-Cauchy–Schwarz inequality, it is straightforward to see that if T is any type $\binom{l}{m}$ Σ_t-tangent tensorfield and $\partial^2 T$ is the type $\binom{l}{m+2}$ tensorfield defined as in Section 1.8.3, then

$$|\partial^2 T|_g^2 \leq g^{ab} |\partial_a \partial T|_g |\partial_b \partial T|_g,$$

where we are viewing $\partial_a \partial T$ and $\partial_b \partial T$ as type $\binom{l}{m+1}$ Σ_t-tangent tensorfields (i.e., we are not imposing any tensorial structure on ∂_a or ∂_b). It follows that

$$|\partial^2 T(t,x)|_g \lesssim \|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)} \sum_{|\vec{I}|=1} |\partial \partial_{\vec{I}} T(t,x)|_g$$

and that

$$\|\partial^2 T\|_{\dot{H}^M_g(\Sigma_t)} \lesssim \|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)} \|\partial T\|_{\dot{H}^{M+1}_g(\Sigma_t)}.$$

The same reasoning yields, for example, that

$$|\partial T(t,x)|_g \lesssim \|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)} \sum_{|\vec{I}|=1} |\partial_{\vec{I}} T(t,x)|_g$$

and that

$$\|\partial T\|_{\dot{H}^M_g(\Sigma_t)} \lesssim \|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)} \|T\|_{\dot{H}^{M+1}_g(\Sigma_t)}.$$

Estimates of this type, together with the singular behavior of $\|g^{-1}\|_{L^\infty_{\text{Frame}}(\Sigma_t)}$ as $t \downarrow 0$, are the primary reason that our analysis relies on the complete hierarchy of norms defined in Section 3.4.

3.4. *The specific norms that we use to control the deviation of the solution from the Kasner background*

Let $A \gg 1$ be a large parameter, to be chosen later. We recall that $q > 0$ and $\sigma > 0$ are the real numbers fixed in Section 3.1.

To control the solution variables (g,k,n), we will rely on a combination of norms for the low-order derivatives of the solution and norms for its high-order derivatives. Our norms are designed to measure the deviation of the perturbed solution from the background Kasner solution $(\widetilde{g},\widetilde{k},\widetilde{n})$, which is spatially homogeneous (in particular, $\widetilde{g}_{ij}$ does not depend on x) with $\widetilde{k}^a_{\ b}\widetilde{k}^b_{\ a} \equiv t^{-2}$ and $\widetilde{n} \equiv 1$.

We now define the low-order norms.

Definition 3.14 (Low norms). Let $\widetilde{g}$ and $\widetilde{k}$ denote the background Kasner solution variables and let q and σ be the fixed constants (which depend on the background Kasner solution $(\widetilde{g},\widetilde{k})$) satisfying (3.2). We define

$$\mathbb{L}_{(g,k)}(t) := \max\Big\{t^{2q}\|g-\widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_t)},\ t^{2q}\|g^{-1}-\widetilde{g}^{-1}\|_{L^\infty_{\text{Frame}}(\Sigma_t)},\ t\|k-\widetilde{k}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)},\ \||tk|_g-1\|_{L^\infty(\Sigma_t)}\Big\}, \tag{3.15a}$$

$$\mathbb{L}_{(n)}(t) := t^{-(2-10q-\sigma)}\|n-1\|_{L^\infty(\Sigma_t)}. \tag{3.15b}$$

We clarify that $t|\widetilde{k}|_{\widetilde{g}} = 1$ by (1.8b)–(1.8c) and thus the term $\||tk|_g-1\|_{L^\infty(\Sigma_t)}$ on the right-hand side of (3.15a) is a measure of the deviation of k from $\widetilde{k}$.

We will use the following norms to control the high-order derivatives of the solution.

Definition 3.16 (High norms). Let q and σ be the fixed constants (which depend on the background Kasner solution $(\widetilde{g},\widetilde{k})$) that satisfy (3.2). Let $A \gg 1$ be a real parameter (to be chosen later) and let $N \gg 1$ be an integer-valued parameter (also to be chosen later). We define

$$\begin{aligned}\mathbb{H}_{(g,k)}(t) := \max\Big\{&t^{A+1}\|k\|_{\dot{H}^N_g(\Sigma_t)}, t^{A+1}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)},\\ &t^{A+3q+\sigma}\|k\|_{\dot{H}^{N-1}_g(\Sigma_t)}, t^{A+3q+\sigma}\|k\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)},\\ &t^{A+q+\sigma}\|g\|_{\dot{H}^N_g(\Sigma_t)}, t^{A+q+\sigma}\|g^{-1}\|_{\dot{H}^N_g(\Sigma_t)},\\ &t^{A+2q+\sigma}\|\partial g\|_{\dot{H}^{N-1}_g(\Sigma_t)}, t^{A+2q+\sigma}\|g\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)},\\ &t^{A+2q+\sigma}\|g^{-1}\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)}, t^{A+5q+3\sigma-1}\|g\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)},\\ &t^{A+5q+3\sigma-1}\|g^{-1}\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)}\Big\},\end{aligned} \tag{3.17a}$$

$$\mathbb{H}_{(n)}(t) := \max\Big\{t^{A+1}\|\partial n\|_{\dot{H}^N_g(\Sigma_t)}, t^A\|n\|_{\dot{H}^N(\Sigma_t)}, t^{A+q-1}\|n\|_{\dot{H}^{N-1}(\Sigma_t)}\Big\}. \tag{3.17b}$$

3.5. Bootstrap assumptions

To facilitate our analysis, we find it convenient to rely on bootstrap assumptions. Let $T_{(\text{Boot})} \in (0, 1)$ be a "bootstrap time". Until the proof of Theorem 11.1, we assume that the perturbed solution exists classically for $(t, x) \in (T_{(\text{Boot})}, 1] \times \mathbb{T}^D$ and that the following bootstrap assumptions hold for the norms from Definitions 3.14 and 3.16:

$$\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t) + \mathbb{L}_{(n)}(t) + \mathbb{H}_{(n)}(t) \le \varepsilon, \quad t \in (T_{(\text{Boot})}, 1], \tag{3.18}$$

where $\varepsilon > 0$ is a small bootstrap parameter.

Remark 3.19 (The required smallness of ε depends on various parameters). We will continually adjust the required smallness of ε throughout our analysis. The required smallness of ε is allowed to depend on N, A, D, q, and σ, but we will often avoid pointing this out.

4. Estimates for the Kasner solution and preliminary technical estimates

In this section, we derive some simple estimates for the background Kasner solution and provide other basic estimates that we will use throughout the paper.

4.1. Basic estimates for the Kasner solution

In controlling various error terms, we will rely on the following simple estimates for the background Kasner solution.

Lemma 4.1 (Basic estimates for the Kasner solution). *The following estimates hold for* $t \in (0, 1]$:

$$\|\widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_t)} \le t^{\sigma - 2q}, \quad \|\widetilde{g}^{-1}\|_{L^\infty_{\text{Frame}}(\Sigma_t)} \le t^{\sigma - 2q}, \tag{4.2a}$$

$$\|\widetilde{k}\|_{L^\infty_{\text{Frame}}(\Sigma_t)} \le t^{-1}. \tag{4.2b}$$

Proof. Recall that the Kasner solution variables $\widetilde{g}$ and $\widetilde{k}$ are given, relative to the transported spatial coordinates, respectively, by expressions (1.7) and (1.8c). All estimates stated in the lemma follow as straightforward consequences of these expressions and the inequalities in (3.2). ∎

4.2. Norm comparisons

In controlling various error terms, we will compare the pointwise norms $|\cdot|_{\text{Frame}}$ and $|\cdot|_g$. Our comparisons will often rely on the following lemma.

Lemma 4.3 (Pointwise norm comparisons). *Let T be a type $\binom{l}{m}$ Σ_t-tangent tensorfield. Under the bootstrap assumptions (3.18), there exists a universal constant $C_* > 1$ independent of N and A (but depending on m and l) such that if $\varepsilon \le 1$, then the following estimates hold for the pointwise norms of Definition 3.4 for $t \in (T_{(\text{Boot})}, 1]$:*

$$C_*^{-1} t^{(l+m)q} |T|_g \le |T|_{\text{Frame}} \le C_* t^{-(l+m)q} |T|_g. \tag{4.4}$$

Proof. Let δ denote the standard Euclidean metric on Σ_t, i.e., relative to the transported spatial coordinates, δ has components δ_{ij} equal to $\mathrm{diag}(1, 1, \ldots, 1)$, and likewise for the inverse Euclidean metric δ^{-1}. Then in view of the definition of the norm $|\cdot|_{\mathrm{Frame}}$, we have, schematically,

$$|T|_{\mathrm{Frame}} = |(\delta)^l (\delta^{-1})^m T^2|^{1/2},$$

and by g-Cauchy–Schwarz, the right-hand side of the previous expression is

$$\leq |\delta|_g^{l/2} |\delta^{-1}|_g^{m/2} |T|_g = \left\{g^{ac} g^{bd} \delta_{ab} \delta_{cd}\right\}^{l/4} \left\{g_{a'c'} g_{b'd'} (\delta^{-1})^{a'b'} (\delta^{-1})^{c'd'}\right\}^{m/4} |T|_g.$$

From Definitions 3.4 and 3.14, estimate (4.2a), and the bootstrap assumptions, we deduce that the right-hand side of the above expression (which we consider from the perspective of the transported coordinate frame) is

$$\leq |g^{-1}|_{\mathrm{Frame}}^{l/2} |g|_{\mathrm{Frame}}^{m/2} |T|_g \leq C_* t^{-(m+l)q} |T|_g,$$

which yields the second inequality in (4.4). To obtain the first inequality in (4.4), we note that we have, schematically,

$$|T|_g = |(g)^l (g^{-1})^m T^2|^{1/2}.$$

The same Cauchy–Schwarz argument as before, but with the role of g and δ interchanged, yields (relative to the transported coordinate frame) that the right-hand side of the previous expression is $\leq |g|_{\mathrm{Frame}}^{l/2} |g^{-1}|_{\mathrm{Frame}}^{m/2} |T|_{\mathrm{Frame}}$. From Definition 3.14, estimate (4.2a), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is $\leq C_* t^{-(m+l)q} |T|_{\mathrm{Frame}}$, which yields the first inequality in (4.4). ∎

4.3. *Sobolev interpolation and product inequalities*

In this subsection, we provide some Sobolev interpolation and product inequalities that we will use to control various error terms. We start with the following lemma, which provides basic interpolation estimates.

Lemma 4.5 (Basic interpolation estimates). *If M_1 and M_2 are non-negative integers with $M_1 \leq M_2$ and v is a scalar function, then the following inequalities hold:*

$$\|v\|_{\dot{H}^{M_1}(\Sigma_t)} \lesssim \|v\|_{L^\infty(\Sigma_t)}^{1 - M_1/M_2} \|v\|_{\dot{H}^{M_2}(\Sigma_t)}^{M_1/M_2} \lesssim \|v\|_{L^\infty(\Sigma_t)} + \|v\|_{\dot{H}^{M_2}(\Sigma_t)}. \tag{4.6}$$

If M_1 and M_2 are non-negative integers with $M_1 + 1 + \lfloor D/2 \rfloor \leq M_2$ and v is a scalar function, then the following inequalities hold:

$$\|v\|_{\dot{W}^{M_1,\infty}(\Sigma_t)} \lesssim \|v\|_{H^{M_1+1+\lfloor D/2 \rfloor}(\Sigma_t)} \lesssim \|v\|_{L^\infty(\Sigma_t)} + \|v\|_{\dot{H}^{M_2}(\Sigma_t)}. \tag{4.7}$$

Proof. The first inequality in (4.6) follows as a special case of Nirenberg's famous interpolation results [39], except that on the right-hand side, we have replaced the norm $\|\cdot\|_{L^2(\Sigma_t)}$ with $\|\cdot\|_{L^\infty(\Sigma_t)}$; the replacement is possible because of the estimate

$$\|v\|_{L^2(\Sigma_t)} \lesssim \|v\|_{L^\infty(\Sigma_t)}$$

for scalar functions v (which holds because $\mathbb{T}^D$ is compact). Strictly speaking, Nirenberg stated his results for functions defined on $\mathbb{R}^D$, but the arguments given in his paper can be used to derive the same estimates for functions defined on $\mathbb{T}^D$. The second inequality in (4.6) follows from the first and Young's inequality.

The first inequality in (4.7) is a standard Sobolev embedding result on $\mathbb{T}^D$. The second inequality in (4.7) follows from the first inequality in (4.7) and the second inequality in (4.6). ∎

Lemma 4.8 (Sobolev product estimates for tensorfields). *Let M be a non-negative integer, let $\{v_r\}_{r=1}^R$ be a finite collection of Σ_t-tangent tensorfields, and let $v = \prod_{r=1}^R \partial_{\vec{I}_r} v_r$ be a (schematically denoted) tensor product, possibly involving contractions. Assume that the bootstrap assumptions (3.18) hold for some ε with $\varepsilon \leq 1$. Then the following estimate holds for $t \in (T_{(\mathrm{Boot})}, 1]$:*

$$\max_{\sum_{r=1}^R |\vec{I}_r| = M} \left\| \prod_{r=1}^R \partial_{\vec{I}_r} v_r \right\|_{L^2_{\mathrm{Frame}}(\Sigma_t)} \lesssim \sum_{r=1}^R \|v_r\|_{\dot{H}^M_{\mathrm{Frame}}(\Sigma_t)} \prod_{s \neq r} \|v_s\|_{L^\infty_{\mathrm{Frame}}(\Sigma_t)}. \tag{4.9}$$

Moreover, under the same assumptions as above, and assuming that v is type $\binom{l}{m}$, the following estimate holds for $t \in (T_{(\mathrm{Boot})}, 1]$:

$$\max_{\sum_{r=1}^R |\vec{I}_r| = M} \left\| \prod_{r=1}^R \partial_{\vec{I}_r} v_r \right\|_{L^2_g(\Sigma_t)} \lesssim t^{-(l+m)q} \sum_{r=1}^R \|v_r\|_{\dot{H}^M_{\mathrm{Frame}}(\Sigma_t)} \prod_{s \neq r} \|v_s\|_{L^\infty_{\mathrm{Frame}}(\Sigma_t)}. \tag{4.10}$$

Proof. If the $\{v_r\}_{r=1}^R$ are all scalar functions (and hence $l = m = 0$), then inequality (4.9) is standard; it is proved, for example, as [47, Lemma 6.16]. If one or more of the v_r are not scalar functions, then estimate (4.9) follows a straightforward consequence of the estimate (4.9) for scalar functions, essentially by writing out the definition of the left-hand side of (4.9) relative to the transported spatial coordinate frame and estimating the components of all tensorfields.

To prove (4.10), we first use (4.4) to deduce that

$$\max_{\sum_{r=1}^R |\vec{I}_r| = M} \left\| \prod_{r=1}^R \partial_{\vec{I}_r} v_r \right\|_{L^2_g(\Sigma_t)} \lesssim t^{-(l+m)q} \max_{\sum_{r=1}^R |\vec{I}_r| = M} \left\| \prod_{r=1}^R \partial_{\vec{I}_r} v_r \right\|_{L^2_{\mathrm{Frame}}(\Sigma_t)}.$$

We then use (4.9) to conclude that the right-hand side of the previous expression is $\lesssim$ RHS (4.10) as desired. ∎

4.4. Sobolev embedding

We will use the following lemma to obtain $L^\infty(\Sigma_t)$-control over some additional low-order derivatives that are not directly controlled by the low-order norms from Definition 3.14. To achieve the desired control, we borrow, via interpolation with sufficiently large N, a small amount of the high norm. Because the high norms are quite weak near $t = 0$ (at least when A is large), the interpolation introduces singular behavior into the estimates of the lemma, represented by the factors of $t^{-A\delta}$ on the right-hand sides of the estimates. However, $\delta \to 0$ as $N \to \infty$. Thus, at fixed A, if N is large, then the following basic principle, central to our approach, applies:

> The singular contribution to the behavior of the low-order norms coming from the high-order norms is very small.

Lemma 4.11 (Sobolev embedding, borrowing only a small amount of high norm). *There exists a parameter*[15] $\delta > 0$*, depending on* N *and* D*, such that* $\lim_{N\to\infty} \delta = 0$ *(at fixed* D*) and such that if* $N \geq 5 + \lfloor D/2 \rfloor$*, then the following estimates hold for* $t \in (T_{(\mathrm{Boot})}, 1]$*:*

$$\|g - \widetilde{g}\|_{W^{4,\infty}_{\mathrm{Frame}}(\Sigma_t)} \lesssim t^{-2q-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{4.12a}$$

$$\|g^{-1} - \widetilde{g}^{-1}\|_{W^{4,\infty}_{\mathrm{Frame}}(\Sigma_t)} \lesssim t^{-2q-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{4.12b}$$

$$\|n - 1\|_{W^{4,\infty}(\Sigma_t)} \lesssim t^{2-10q-\sigma-A\delta}\{\mathbb{L}_{(n)}(t) + \mathbb{H}_{(n)}(t)\}. \tag{4.12c}$$

Proof. To prove (4.12c), we first use the second inequality in (4.7) to deduce

$$\|n - 1\|_{W^{4,\infty}(\Sigma_t)} \lesssim \|n - 1\|_{L^\infty(\Sigma_t)} + \|n - 1\|_{\dot{H}^{5+\lfloor D/2 \rfloor}(\Sigma_t)}.$$

Then using the first inequality in (4.6), we deduce that for $N \geq 5 + \lfloor D/2 \rfloor$, the right-hand side of the previous expression is

$$\lesssim \|n - 1\|_{L^\infty(\Sigma_t)} + \|n - 1\|_{L^\infty(\Sigma_t)}^{1-\delta} \|n\|_{\dot{H}^N(\Sigma_t)}^{\delta},$$

where $\delta := (5 + \lfloor D/2 \rfloor)/N$. Combining these estimates and appealing to definitions (3.15b) and (3.17b), we find that

$$\|n - 1\|_{W^{4,\infty}(\Sigma_t)} \lesssim t^{2-10q-\sigma}\mathbb{L}_{(n)}(t) + \{t^{2-10q-\sigma}\mathbb{L}_{(n)}(t)\}^{1-\delta}\{t^{-A}\mathbb{H}_{(n)}(t)\}^{\delta}.$$

Finally, we use Young's inequality to deduce

$$\begin{aligned} &t^{2-10q-\sigma}\mathbb{L}_{(n)}(t) + \{t^{2-10q-\sigma}\mathbb{L}_{(n)}(t)\}^{1-\delta}\{t^{-A}\mathbb{H}_{(n)}(t)\}^{\delta} \\ &\quad = t^{2-10q-\sigma}\mathbb{L}_{(n)}(t) + t^{(2-10q-\sigma)(1-\delta)-A\delta}\{\mathbb{L}_{(n)}(t)\}^{1-\delta}\{\mathbb{H}_{(n)}(t)\}^{\delta} \\ &\quad \lesssim t^{2-10q-\sigma-A\delta}\{\mathbb{L}_{(n)}(t) + \mathbb{H}_{(n)}(t)\}, \end{aligned} \tag{4.13}$$

where in obtaining the last inequality in (4.13), we have used (3.2), the assumption $A \geq 1$, and our running convention that δ is free to vary from line to line, subject only to the restriction $\lim_{N\to\infty} \delta = 0$. In total, we have derived the desired bound (4.12c).

The remaining estimates in the lemma can be proved using similar arguments that take into account Definitions 3.14 and 3.16, and we omit the details. ∎

[15] Recall that, as we described in Section 1.8.5, we allow δ to vary from line to line and we use the convention that a sum of two parameters δ is another δ.

5. Control of n in terms of g and k

Our primary goal in this section is to prove the next proposition, in which we derive the main estimates for n. The proof of the proposition is located in Section 5.3. In Sections 5.1–5.2, we derive the identities and estimates that we will use when proving the proposition. The proposition shows that n is controlled, in various norms, by complementary norms of g and k. Achieving such control is possible since n solves the elliptic PDEs (2.13a)–(2.13b), which feature source terms that depend, respectively, on k and R.

Proposition 5.1 (Control of n in terms of g and k). *We recall that $\mathbb{L}_{(g,k)}(t)$, $\mathbb{L}_{(n)}(t)$, $\mathbb{H}_{(g,k)}(t)$, and $\mathbb{H}_{(n)}(t)$ are the norms from Definitions 3.14 and 3.16, and assume that the bootstrap assumptions (3.18) hold. There exists a constant $C_* > 0$* independent of N and *A and a constant $C = C_{N,A,D,q,\sigma} > 0$ such that if N is sufficiently large in a manner that depends on A, D, q, and σ, and if ε is sufficiently small (in a manner that depends on N, A, D, q, and σ), then the following estimates hold for $t \in (T_{(\mathrm{Boot})}, 1]$.*

- **Estimates at the lowest order.** *The following estimate holds:*

$$\mathbb{L}_{(n)}(t) \leq C\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \tag{5.2}$$

- **Estimate for $\partial_t n$.** *The following estimate holds:*

$$\|\partial_t n\|_{L^\infty(\Sigma_t)} \leq C t^{1-10q-\sigma}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \tag{5.3}$$

- **Top-order estimates.** *If $|\vec{I}| = N$, then the following estimates hold (see Remark 3.9):*

$$\begin{aligned} &t^{A+1}\|\partial\partial_{\vec{I}} n\|_{L^2_g(\Sigma_t)} + t^A\|\partial_{\vec{I}} n\|_{L^2(\Sigma_t)} \\ &\leq C_* t^{A+1}\|\partial_{\vec{I}} k\|_{L^2_g(\Sigma_t)} + C t^\sigma\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} \\ &\leq C_*\mathbb{H}_{(g,k)}(t) + C t^\sigma\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \end{aligned} \tag{5.4}$$

- **Near-top-order estimates.** *The following estimate holds:*

$$t^{A+q-1}\|n\|_{\dot{H}^{N-1}(\Sigma_t)} \leq C\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \tag{5.5}$$

5.1. *The equations*

In this subsection, we derive some equations that we use to control n in terms of g and k.

5.1.1. The equations verified by the time derivative of the lapse. The next lemma provides an analog of equation (2.13b) for $\partial_t n$, and we will use it to derive estimates for $\|\partial_t n\|_{L^\infty(\Sigma_t)}$. We remark that we need estimates for $\|\partial_t n\|_{L^\infty(\Sigma_t)}$ only in Section 10.2, when we bound the length of past-directed causal geodesic segments.

Lemma 5.6 (The ∂_t-commuted lapse equation). *The quantity $\partial_t n$ verifies the following elliptic PDE on Σ_t:*

$$g^{ab}\nabla_a\nabla_b\partial_t n - (\partial_t n)(t^{-2} + \mathrm{R}) = \mathfrak{N}', \tag{5.7}$$

where (see Remarks 2.6 and 3.9)

$$\begin{aligned}
\mathfrak{N}' \simeq & \sum_{i_1+i_2+i_3=2} \sum_{p=0}^{1} (n-1)^p (g^{-1})^2 (\partial^{i_1} n)(\partial^{i_2} g)\partial^{i_3} k \\
& + \sum_{i_1+i_2+i_3=1} \sum_{p=0}^{1} (n-1)^p (g^{-1})^3 (\partial g)(\partial^{i_1} n)(\partial^{i_2} g)\partial^{i_3} k + t^{-3}(n-1) \\
& + n g^{-1} k \partial^2 n + n (g^{-1})^2 k (\partial g)\partial n \\
& + \sum_{i_1+i_2+i_3=1} (g^{-1})^2 (\partial n)(\partial^{i_1} n)(\partial^{i_2} g)\partial^{i_3} k. \qquad (5.8)
\end{aligned}$$

Proof. Relative to CMC-transported spatial coordinates, the elliptic PDE (2.13b) can be expressed as

$$g^{ab}\partial_a\partial_b n - g^{ab}\Gamma_{a\ b}^{\ c}\partial_c n - (n-1)(t^{-2} + \mathrm{R}) = \mathrm{R},$$

where, schematically,

$$\mathrm{R} \simeq (g^{-1})^2 \partial^2 g + (g^{-1})^3 (\partial g)^2$$

(see (2.15b) and recall that $\mathrm{R} = \mathrm{Ric}^{a}_{\ a}$) and

$$g^{ab}\Gamma_{a\ b}^{\ c} \simeq g^{-2}\partial g.$$

Commuting the elliptic PDE with ∂_t, using equations (2.12a) and (2.12b) to algebraically substitute for $\partial_t g$ and $\partial_t g^{-1}$, and carrying out straightforward computations, we conclude the desired equation (5.7). ∎

5.1.2. The equations verified by the high-order derivatives. We will use the equations in the next lemma to control the $L^2(\Sigma_t)$-norms of high-order derivatives of n.

Remark 5.9 (Borderline error terms vs. junk error terms). In the rest of the paper, we will denote difficult "borderline" error terms by decorating them with "(Border)", e.g., ${}^{(\mathrm{Border};\vec{I})}\mathfrak{N}$. These error terms must treated with care since at the top derivative level, there is no room (in the sense of powers of t) in our estimates for such terms. In contrast, the error terms that we decorate with "(Junk)", such as ${}^{(\mathrm{Junk};\vec{I})}\mathfrak{N}$, are such that there is some room in our estimates, though we sometimes rely on subtle arguments to show that there is room.

Lemma 5.10 (The $\vec{I}$-commuted lapse equation). *For each spatial multi-index $\vec{I}$ with $|\vec{I}| = N$, $\partial_{\vec{I}} n$ verifies the following equation:*

$$t^{A+2} g^{ab}\partial_a\partial_b\partial_{\vec{I}} n - t^{A}\partial_{\vec{I}} n = {}^{(\mathrm{Border};\vec{I})}\mathfrak{N} + {}^{(\mathrm{Junk};\vec{I})}\mathfrak{N}, \qquad (5.11)$$

where[16] *(see Section 1.8.7 regarding our use of notation $\overset{*}{\simeq}$ and $\simeq$, and see Remarks 2.6*

[16] We clarify that the factor $(tk)\cdot(tk) - 1$ in braces on the right-hand side of (5.12a) can be written more precisely as $(tk^a_{\ b})(tk^b_{\ a}) - 1$, which, in view of the symmetry property $k_{ab} = k_{ba}$, can alternatively be expressed as the following term (that is controlled by the norm $\mathbb{L}_{(g,k)}(t)$ from Definition 3.14): $(|tk|_g - 1)(|tk|_g + 1)$.

and 3.9)

$$^{(Border;\vec{I})}\mathfrak{N} \overset{*}{\simeq} (t^A \partial_{\vec{I}} n)\{(tk) \cdot (tk) - 1\} + (tk)(t^{A+1}\partial_{\vec{I}} k), \tag{5.12a}$$

$$\begin{aligned} ^{(Junk;\vec{I})}\mathfrak{N} \simeq & \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I} \\ |\vec{I}_1|\leq|\vec{I}|-1}} t^{A+2}\{\partial_{\vec{I}_1}(n-1)\}(\partial_{\vec{I}_2} k)\partial_{\vec{I}_3} k \\ & + \sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I} \\ |\vec{I}_1|,|\vec{I}_2|\leq|\vec{I}|-1}} t^{A+2}(\partial_{\vec{I}_1} k)\partial_{\vec{I}_2} k \\ & + \sum_{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I}} t^{A+2}(\partial_{\vec{I}_1} g^{-1})(\partial_{\vec{I}_2} g^{-1})(\partial\partial_{\vec{I}_3} g)\partial\partial_{\vec{I}_4} n \\ & + \sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I} \\ |\vec{I}_2|\leq|\vec{I}|-1}} t^{A+2}(\partial_{\vec{I}_1} g^{-1})\partial^2\partial_{\vec{I}_2} n. \end{aligned} \tag{5.12b}$$

Furthermore, for each spatial multi-index $\vec{I}$ with $|\vec{I}| = N-1$, $\partial_{\vec{I}} n$ verifies the following equation:

$$t^{A+1+q} g^{ab}\partial_a\partial_b\partial_{\vec{I}} n - t^{A+q-1}(1+t^2\mathrm{R})\partial_{\vec{I}} n = {}^{(\vec{I})}\widetilde{\mathfrak{N}}, \tag{5.13}$$

where

$$\begin{aligned} ^{(\vec{I})}\widetilde{\mathfrak{N}} \simeq & \sum_{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}} t^{A+1+q}(\partial_{\vec{I}_1} g^{-1})(\partial_{\vec{I}_2} g^{-1})\partial^2\partial_{\vec{I}_3} g \\ & + \sum_{\vec{I}_1+\vec{I}_2+\cdots+\vec{I}_5=\vec{I}} t^{A+1+q}(\partial_{\vec{I}_1} g^{-1})(\partial_{\vec{I}_2} g^{-1})(\partial_{\vec{I}_3} g^{-1})(\partial\partial_{\vec{I}_4} g)\partial\partial_{\vec{I}_5} g \\ & + \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I} \\ |\vec{I}_1|\leq|\vec{I}|-1}} t^{A+1+q}\{\partial_{\vec{I}_1}(n-1)\}(\partial_{\vec{I}_2} g^{-1})(\partial_{\vec{I}_3} g^{-1})\partial^2\partial_{\vec{I}_4} g \\ & + \sum_{\substack{\vec{I}_1+\vec{I}_2+\cdots+\vec{I}_6=\vec{I} \\ |\vec{I}_1|\leq|\vec{I}|-1}} t^{A+1+q}\{\partial_{\vec{I}_1}(n-1)\}(\partial_{\vec{I}_2} g^{-1})(\partial_{\vec{I}_3} g^{-1})(\partial_{\vec{I}_4} g^{-1}) \\ & \qquad \times (\partial\partial_{\vec{I}_5} g)\partial\partial_{\vec{I}_6} g \\ & + \sum_{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I}} t^{A+1+q}(\partial_{\vec{I}_1} g^{-1})(\partial_{\vec{I}_2} g^{-1})(\partial\partial_{\vec{I}_3} g)\partial\partial_{\vec{I}_4} n \\ & + \sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I} \\ |\vec{I}_2|\leq|\vec{I}|-1}} t^{A+1+q}(\partial_{\vec{I}_1} g^{-1})\partial^2\partial_{\vec{I}_2} n. \end{aligned} \tag{5.14}$$

Remark 5.15 (The decay of $\|n-1\|_{L^\infty(\Sigma_t)}$ and its role in yielding "Junk" terms). From definition (3.15b) and the bootstrap assumption (3.18), we see that

$$\|n-1\|_{L^\infty(\Sigma_t)} \leq \varepsilon t^{(2-10q-\sigma)}.$$

Thus, in view of the parameter inequalities (3.2), we see that $\|n-1\|_{L^\infty(\Sigma_t)}$ decays to 0 at least as fast as t to a positive power as $t \downarrow 0$. This explains, for example, why we characterized the top-order product $t^{A+2}(n-1)k\partial_{\vec{I}}k$ featured in the first sum on the right-hand side of (5.12b) as a "Junk" term, even though we characterized the similar-looking top-order product $(tk)(t^{A+1}\partial_{\vec{I}}k)$ on the right-hand side of (5.12a) as a "Borderline" term; the extra decay yielded by the factor $n-1$ makes it easy for us to control the top-order product $t^{A+2}(n-1)k\partial_{\vec{I}}k$ compared to the difficult product $(tk)(t^{A+1}\partial_{\vec{I}}k)$. Similar remarks apply throughout the rest of the paper to top-order products featuring the factor $n-1$.

Proof of Lemma 5.10. First, we multiply equation (2.13a) by t^{A+2} and use the relation $g^{ab}\nabla_a\nabla_b n = g^{ab}\partial_a\partial_b n - g^{ab}\Gamma_{a\ b}^{\ c}\partial_c n$ to obtain the equation

$$\begin{aligned} t^{A+2}g^{ab}\partial_a\partial_b n - t^A(n-1) = t^A(n-1)\{(tk^a_{\ b})(tk^b_{\ a}) - 1\} - t^A \\ + t^{A+2}k^a_{\ b}k^b_{\ a} + t^{A+2}g^{ab}\Gamma_{a\ b}^{\ c}\partial_c n. \end{aligned} \tag{5.16}$$

Commuting equation (5.16) with $\partial_{\vec{I}}$ and using the schematic identity

$$g^{ab}\Gamma_{a\ b}^{\ c}\partial_c n \simeq g^{-2}(\partial g)\partial n,$$

we easily deduce (5.11). We clarify that the terms on the right-hand side of (5.12a) arise, respectively, when all $|\vec{I}|$ derivatives fall on the factor $n-1$ in the first product on the right-hand side of (5.16) or on one of the factors of k in the next-to-last product $t^{A+2}k^a_{\ b}k^b_{\ a}$ on the right-hand side of (5.16). It is for this reason that the coefficients in the corresponding products do not depend on N or A, as is indicated by the symbol $\overset{*}{\simeq}$ in equation (5.12a).

Equation (5.13) follows similarly from multiplying equation (2.13b) by t^{A+1+q} and using the schematic identity

$$\mathrm{R} \simeq (g^{-1})^2\partial^2 g + (g^{-1})^3(\partial g)^2,$$

which follows from (2.15b) and the fact that $\mathrm{R} = \mathrm{Ric}^a_{\ a}$. ∎

5.2. *Control of the error terms in the lapse estimates*

In this subsection, we derive estimates for the error terms in the equations of Lemmas 5.6 and 5.10.

Lemma 5.17 (Control of the error terms in the elliptic estimates). *Assume that the bootstrap assumptions* (3.18) *hold. There exists a constant* $C_* > 0$ independent of N and A *and a constant* $C = C_{N,A,D,q,\sigma} > 0$ *such that if* N *is sufficiently large in a manner that depends on* A, D, q, *and* σ, *and if* ε *is sufficiently small (in a manner that depends on* N, A, D, q, *and* σ*), then the following estimates hold for* $t \in (T_{(\mathrm{Boot})}, 1]$.

- **Control of the error term in the equation verified by $\partial_t n$.** *The following estimate holds for the error term $\mathfrak{N}'$ defined in* (5.8):

$$\|\mathfrak{N}'\|_{L^\infty(\Sigma_t)} \leq C t^{-1-10q-\sigma}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t) + \mathbb{L}_{(n)}(t) + \mathbb{H}_{(n)}(t)\}. \quad (5.18)$$

- **Borderline top-order error term estimates.** *If $|\vec{I}| = N$, then the following estimate holds for the error term from* (5.12a):

$$\|{}^{(\mathrm{Border};\vec{I})}\mathfrak{N}\|_{L^2(\Sigma_t)} \leq C_* \varepsilon t^A \|\partial_{\vec{I}} n\|_{L^2(\Sigma_t)} + C_* t^{A+1}\|\partial_{\vec{I}} k\|_{L^2_g(\Sigma_t)}. \quad (5.19)$$

- **Non-borderline top-order error term estimates.** *The following estimate holds for the error term from* (5.12b):

$$\max_{|\vec{I}|=N} \|{}^{(\mathrm{Junk};\vec{I})}\mathfrak{N}\|_{L^2(\Sigma_t)} \leq C t^\sigma\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \quad (5.20)$$

- **Just-below-top-order error term estimates.** *The following estimate holds for the error term from* (5.14):

$$\max_{|\vec{I}|=N-1} \|{}^{(\vec{I})}\widetilde{\mathfrak{N}}\|_{L^2(\Sigma_t)} \leq C\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \quad (5.21)$$

Proof. Throughout this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta < \sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large. We also freely use the observations of Remark 3.13.

Proof of (5.18). This estimate follows in a straightforward fashion from using (3.2), Definition 3.14, Lemma 4.1, and Lemma 4.11 to control the products on the right-hand side of (5.8) in the norm $\|\cdot\|_{L^\infty(\Sigma_t)}$, where we bound each factor in the products in the norm $\|\cdot\|_{L^\infty_{\mathrm{Frame}}(\Sigma_t)}$. We remark that the most singular (in the sense of powers of t) products on the right-hand side of (5.8) are $t^{-3}(n-1)$ and the terms in the second sum with $p = i_1 = 0$.

Proof of (5.19). From Definition 3.14 and the g-Cauchy-Schwarz inequality, we deduce that the magnitude of the right-hand side of (5.12a) is

$$\leq C_*\{1 + \mathbb{L}_{(g,k)}(t)\}\big\{\mathbb{L}_{(g,k)}(t)|t^A \partial_{\vec{I}} n| + |t^{A+1}\partial_{\vec{I}} k|_g\big\}.$$

From this bound and the bootstrap assumption bound $\mathbb{L}_{(g,k)}(t) \leq \varepsilon$, we conclude (5.19).

Proof of (5.20). : Let $\vec{I}$ be a spatial multi-index with $|\vec{I}| = N$. To bound the first sum on the right-hand side of (5.12b), we first consider the top-order case in which $|\vec{I}_2| = N$ or $|\vec{I}_3| = N$. Then using g-Cauchy–Schwarz, we see that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ by

$$\lesssim t^{A+2}\|n-1\|_{L^\infty(\Sigma_t)}\|k\|_{L^\infty_g(\Sigma_t)}\|k\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definitions 3.14 and 3.16 and the bootstrap assumptions, we see that the right-hand side of the previous expression is $\lesssim t^{2-10q-\sigma}\mathbb{H}_{(g,k)}(t)$. In view of (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (5.20) as desired. We now consider the remaining cases, in which $|\vec{I}_1|, |\vec{I}_2|, |\vec{I}_3| \leq N-1$. Using (4.6) and (4.9), we bound (using that $|\vec{I}| = N$) the terms under consideration as follows:

$$\begin{aligned}
\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I} \\ |\vec{I}_1|,|\vec{I}_2|,|\vec{I}_3|\leq N-1}} & t^{A+2} \|\{\partial_{\vec{I}_1}(n-1)\}(\partial_{\vec{I}_2} k)\partial_{\vec{I}_3} k\|_{L^2(\Sigma_t)} \\
& \lesssim t^{A+2}\|n-1\|_{W^{1,\infty}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\mathrm{Frame}}(\Sigma_t)} \\
& \quad + t^{A+2}\|k\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|k\|_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)} \\
& \quad + t^{A+2}\|n-1\|_{W^{1,\infty}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|k\|_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}. \qquad (5.22)
\end{aligned}$$

From Definitions 3.14 and 3.16, estimates (4.2b) and (4.12c), and the bootstrap assumptions, we deduce that

$$\text{RHS (5.22)} \lesssim t^{3-13q-2\sigma-A\delta}\mathbb{H}_{(g,k)}(t) + t^{1-q}\mathbb{L}_{(g,k)}(t).$$

In view of (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (5.20) as desired.

To bound the second sum on the right-hand side of (5.12b), we first use (4.9) to deduce that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ by

$$\lesssim t^{A+2}\|k\|_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\mathrm{Frame}}(\Sigma_t)}.$$

From Definitions 3.14 and 3.16 and the bootstrap assumptions, we see that the right-hand side of the previous expression is $\lesssim t^{1-3q-\sigma}\mathbb{H}_{(g,k)}(t)$, which, in view of (3.2), is $\lesssim$ RHS (5.20) as desired.

To bound the third sum on the right-hand side of (5.12b), we first consider the top-order case in which $|\vec{I}_3| = N$. Using g-Cauchy–Schwarz and the fact that $|g^{-1}|_g \lesssim 1$, we see that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ by

$$\lesssim t^{A+2}\|\partial n\|_{L^\infty_g(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.$$

Applying (4.4) (with $l=0$ and $m=1$) to $\|\partial n\|_{L^\infty_g(\Sigma_t)}$, we see that the right-hand side of the previous expression is

$$\lesssim t^{A+2-q}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definition 3.16, estimate (4.12c), and the bootstrap assumptions, we see that the right-hand side of the previous expression is

$$\lesssim t^{3-11q-\sigma-A\delta}\mathbb{H}_{(g,k)}(t),$$

which, in view of (3.2), is $\lesssim$ RHS (5.20) as desired. We now consider the top-order case in which $|\vec{I}_4| = N$. Arguing as above, we deduce that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ by

$$\lesssim t^{A+2}\|\partial g\|_{L^\infty_g(\Sigma_t)}\|\partial n\|_{\dot{H}^N_g(\Sigma_t)} \lesssim t^{A+2-3q}\|g\|_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|\partial n\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definition 3.16, estimate (4.12a), and the bootstrap assumptions, we see that the right-hand side of the previous expression is

$$\lesssim t^{1-5q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}\mathbb{H}_{(n)}(t)\lesssim t^{1-5q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (5.20) as desired. It remains for us to handle the cases in which $|\vec{I}_3|,|\vec{I}_4|\leq N-1$. Using (4.6) and (4.9), we bound (using that $|\vec{I}|=N$) the terms under consideration as follows:

$$\begin{aligned}\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I}\\ |\vec{I}_3|,|\vec{I}_4|\leq N-1}} & t^{A+2}\|(\partial_{\vec{I}_1}g^{-1})(\partial_{\vec{I}_2}g^{-1})(\partial\partial_{\vec{I}_3}g)\partial\partial_{\vec{I}_4}n\|_{L^2(\Sigma_t)}\\ &\lesssim t^{A+2}\|n-1\|_{W^{2,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_t)}\\ &\quad+t^{A+2}\|g-\widetilde{g}\|_{W^{2,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}\\ &\quad+t^{A+2}\|n-1\|_{W^{2,\infty}(\Sigma_t)}\|g^{-1}\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\\ &\qquad\times\|g-\widetilde{g}\|_{W^{2,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_t)}\\ &\quad+t^{A+2}\|n-1\|_{W^{2,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\mathrm{Frame}}(\Sigma_t)}.\end{aligned}\tag{5.23}$$

From Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), and the bootstrap assumptions, we deduce that

$$\text{RHS (5.23)}\lesssim t^{4-16q-2\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}+t^{2-6q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.$$

In view of (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (5.20) as desired.

To bound the last sum on the right-hand side of (5.12b), we first consider the case in which $|\vec{I}_2|=N-1$. Using (4.4), we bound the terms under consideration in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ by

$$\lesssim t^{A+2}\|g^{-1}\|_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|\partial n\|_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_t)}\lesssim t^{A+2-q}\|g^{-1}\|_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|\partial n\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definition 3.16, estimate (4.12b), and the bootstrap assumptions, we see that the right-hand side of the previous expression is

$$\lesssim t^{1-3q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}\mathbb{H}_{(n)}(t)\lesssim t^{1-3q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (5.20) as desired. We now consider the remaining cases, in which $|\vec{I}_2|\leq N-2$. Using (4.9), we bound (using that $|\vec{I}|=N$) the terms under consideration as follows:

$$\begin{aligned}\sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I}\\ |\vec{I}_2|\leq N-2}} & t^{A+2}\|(\partial_{\vec{I}_1}g^{-1})\partial^2\partial_{\vec{I}_2}n\|_{L^2(\Sigma_t)}\\ &\lesssim t^{A+2}\|g^{-1}\|_{\dot{W}^{2,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}+t^{A+2}\|n\|_{\dot{W}^{2,\infty}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_t)}.\end{aligned}\tag{5.24}$$

From Definition 3.16, estimates (4.12b) and (4.12c), and the bootstrap assumptions, we deduce that

$$\text{RHS (5.24)} \lesssim t^{2-2q-A\delta}\mathbb{L}_{(g,k)}(t) + t^{4-12q-2\sigma-A\delta}\mathbb{H}_{(g,k)}(t).$$

In view of (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (5.20) as desired.

Proof of (5.21). Let $\vec{I}$ be a spatial multi-index with $|\vec{I}| = N-1$. To bound the first sum on the right-hand side of (5.14), we first consider the case in which $|\vec{I}_3| = N-1$. Using g-Cauchy–Schwarz and the fact that $|g^{-1}|_g \lesssim 1$, we see that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ by

$$\lesssim t^{A+1+q}\|\partial^2 g\|_{\dot{H}_g^{N-1}(\Sigma_t)}.$$

From the definitions of $\|\cdot\|_{\dot{H}_g^M(\Sigma_t)}$ and $\|\cdot\|_{L^\infty_{\text{Frame}}(\Sigma_t)}$, we deduce (see Remark 3.13) that the right-hand side of the previous estimate is

$$\lesssim t^{A+1+q}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}_g^N(\Sigma_t)}.$$

From Definitions 3.14 and 3.16, estimate (4.2a), and the bootstrap assumptions, we see that the right-hand side of the previous expression is $\lesssim \mathbb{H}_{(g,k)}(t)$, which is $\lesssim$ RHS (5.21) as desired. It remains for us to consider the remaining cases, in which $|\vec{I}_3| \le N-2$. Using (4.6) and (4.9), we bound (using that $|\vec{I}| = N-1$) the terms under consideration as follows:

$$\begin{aligned}
\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_3|\le N-2}} & t^{A+1+q}\|(\partial_{\vec{I}_1}g^{-1})(\partial_{\vec{I}_2}g^{-1})\partial^2\partial_{\vec{I}_3}g\|_{L^2(\Sigma_t)} \\
&\lesssim t^{A+1+q}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)} \\
&\quad + t^{A+1+q}\|g^{-1}\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{W}^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)} \\
&\quad + t^{A+1+q}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{W}^{2,\infty}_{\text{Frame}}(\Sigma_t)}.
\end{aligned} \tag{5.25}$$

From Definition 3.16, estimates (4.2a), (4.12a), and (4.12b), and the bootstrap assumptions, we see that

$$\text{RHS (5.25)} \lesssim t^{1-5q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (5.21) as desired.

To bound the second sum on the right-hand side of (5.14), we first consider the cases in which $|\vec{I}_4| = N-1$ or $|\vec{I}_5| = N-1$. Using g-Cauchy–Schwarz and the fact that $|g^{-1}|_g \lesssim 1$, we see that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ by

$$\lesssim t^{A+1+q}\|\partial g\|_{L^\infty_g(\Sigma_t)}\|\partial g\|_{\dot{H}_g^{N-1}(\Sigma_t)}.$$

Applying (4.4) to $\|\partial g\|_{L^\infty_g(\Sigma_t)}$ (with $l = 0$ and $m = 3$), we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+1-2q}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}_g^{N-1}(\Sigma_t)}.$$

From Definition 3.16, estimate (4.12a), and the bootstrap assumptions, we see that the right-hand side of the previous expression is $\lesssim t^{1-6q-\sigma-A\delta}\mathbb{H}_{(g,k)}(t)$, which, in view of (3.2), is $\lesssim$ RHS (5.21) as desired. It remains for us to consider the remaining cases, in which $|\vec{I}_4|, |\vec{I}_5| \leq N-2$. Using (4.6) and (4.9), we bound (using that $|\vec{I}| = N-1$) the terms under consideration as follows:

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\cdots+\vec{I}_5=\vec{I} \\ |\vec{I}_4|,|\vec{I}_5|\leq N-2}} t^{A+1+q}\left\|(\partial_{\vec{I}_1}g^{-1})(\partial_{\vec{I}_2}g^{-1})(\partial_{\vec{I}_3}g^{-1})(\partial\partial_{\vec{I}_4}g)\partial\partial_{\vec{I}_5}g\right\|_{L^2(\Sigma_t)} \\
&\quad\lesssim t^{A+1+q}\|g^{-1}\|^3_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} \\
&\qquad + t^{A+1+q}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} \\
&\qquad + t^{A+1+q}\|g^{-1}\|^3_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}.
\end{aligned} \tag{5.26}$$

From Definition 3.16, estimates (4.2a), (4.12a), and (4.12b), and the bootstrap assumptions, we see that

$$\text{RHS (5.26)} \lesssim t^{2-12q-3\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (5.21) as desired.

To bound the third sum on the right-hand side of (5.14), we first consider the case $|\vec{I}_4| = N-1$. Using g-Cauchy–Schwarz and the fact that $|g^{-1}|_g \lesssim 1$, we see that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ by

$$\lesssim t^{A+1+q}\|n-1\|_{L^\infty(\Sigma_t)}\|\partial^2 g\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

Considering the definitions of $\|\cdot\|_{\dot{H}^M_g(\Sigma_t)}$ and $\|\cdot\|_{L^\infty_{\text{Frame}}(\Sigma_t)}$, we deduce (see Remark 3.13) that the right-hand side of the previous expression is

$$\lesssim t^{A+1+q}\|n-1\|_{L^\infty(\Sigma_t)}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definitions 3.14 and 3.16, estimate (4.2a), and the bootstrap assumptions, we see that the right-hand side of the previous expression is $\lesssim t^{2-10q-\sigma}\mathbb{H}_{(g,k)}(t)$, which, in view of (3.2), is $\lesssim$ RHS (5.21) as desired. It remains for us to consider the case $|\vec{I}_4| \leq N-2$. Using (4.6) and (4.9), we bound (using that $|\vec{I}| = N-1$) the terms under consideration as

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I} \\ |\vec{I}_1|,|\vec{I}_4|\leq N-2}} t^{A+1+q}\|\{\partial_{\vec{I}_1}(n-1)\}(\partial_{\vec{I}_2}g^{-1})(\partial_{\vec{I}_3}g^{-1})\partial^2\partial_{\vec{I}_4}g\|_{L^2(\Sigma_t)} \\
&\quad\lesssim t^{A+1+q}\|n-1\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)} \\
&\qquad + t^{A+1+q}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{3,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)} \\
&\qquad + t^{A+1+q}\|n-1\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)} \\
&\qquad\qquad \times\|g-\widetilde{g}\|_{W^{3,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)} \\
&\qquad + t^{A+1+q}\|n-1\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{3,\infty}_{\text{Frame}}(\Sigma_t)}.
\end{aligned} \tag{5.27}$$

From Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), and the bootstrap assumptions, we see that

$$\text{RHS (5.27)} \lesssim t^{3-15q-2\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}+t^{2-6q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (5.21) as desired.

To bound the fourth sum on the right-hand side of (5.14), we first consider the cases in which $|\vec{I}_5| = N-1$ or $|\vec{I}_6| = N-1$. Using g-Cauchy–Schwarz and the fact that $|g^{-1}|_g \lesssim 1$, we see that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ by

$$\lesssim t^{A+1+q}\|n-1\|_{L^\infty(\Sigma_t)}\|\partial g\|_{L^\infty_g(\Sigma_t)}\|\partial g\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

Applying (4.4) to $\|\partial g\|_{L^\infty_g(\Sigma_t)}$ (with $l=0$ and $m=3$), we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+1-2q}\|n-1\|_{L^\infty(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

From Definitions 3.14 and 3.16, estimates (4.2a) and (4.12a), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is

$$\lesssim t^{3-16q-2\sigma-A\delta}\mathbb{H}_{(g,k)}(t),$$

which, in view of (3.2), is $\lesssim$ RHS (5.21) as desired. It remains for us to consider the remaining cases, in which $|\vec{I}_1|, |\vec{I}_5|, |\vec{I}_6| \leq N-2$. Using (4.6) and (4.9), we bound (using that $|\vec{I}| = N-1$) the terms under consideration as follows:

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\cdots+\vec{I}_6=\vec{I}\\ |\vec{I}_1|,|\vec{I}_5|,|\vec{I}_6|\leq N-2}} t^{A+1+q}\left\|\{\partial_{\vec{I}_1}(n-1)\}(\partial_{\vec{I}_2}g^{-1})(\partial_{\vec{I}_3}g^{-1})(\partial_{\vec{I}_4}g^{-1})(\partial\partial_{\vec{I}_5}g)\partial\partial_{\vec{I}_6}g\right\|_{L^2(\Sigma_t)}\\
&\quad\lesssim t^{A+1+q}\|n-1\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^3_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)}\\
&\qquad+t^{A+1+q}\|g^{-1}\|^3_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)}\\
&\qquad+t^{A+1+q}\|n-1\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\\
&\qquad\qquad\times\|g-\widetilde{g}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)}\\
&\qquad+t^{A+1+q}\|n-1\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^3_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}.
\end{aligned}\tag{5.28}$$

From Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), and the bootstrap assumptions, we see that

$$\begin{aligned}\text{RHS (5.28)} \lesssim\ & t^{4-22q-4\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}\\ &+t^{2-10q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},\end{aligned}$$

which, in view of (3.2), is $\lesssim$ RHS (5.21) as desired.

To bound the fifth sum on the right-hand side of (5.14), we first use (4.6), (4.9), and the fact that $|\vec{I}| = N-1$ to deduce that the terms under consideration are bounded as follows:

$$\begin{aligned}
\sum_{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I}} & t^{A+1+q}\left\|(\partial_{\vec{I}_1} g^{-1})(\partial_{\vec{I}_2} g^{-1})(\partial\partial_{\vec{I}_3} g)(\partial\partial_{\vec{I}_4} n)\right\|_{L^2(\Sigma_t)} \\
& \lesssim t^{A+1+q}\|g^{-1}\|^2_{L^\infty_{Frame}(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{Frame}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)} \\
& \quad + t^{A+1+q}\|g^{-1}\|^2_{L^\infty_{Frame}(\Sigma_t)}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|g\|_{\dot{H}^N_{Frame}(\Sigma_t)} \\
& \quad + t^{A+1+q}\|g^{-1}\|_{L^\infty_{Frame}(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{Frame}(\Sigma_t)}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{Frame}(\Sigma_t)} \\
& \quad + t^{A+1+q}\|g^{-1}\|^2_{L^\infty_{Frame}(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{Frame}(\Sigma_t)}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}.
\end{aligned} \tag{5.29}$$

From Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), and the bootstrap assumptions, we see that

$$\text{RHS } (5.29) \lesssim t^{1-5q-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} + t^{3-15q-2\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (5.21) as desired.

To bound the last sum on the right-hand side of (5.14), we first use (4.9) and the fact that $|\vec{I}| = N-1$ to deduce that the terms under consideration are bounded as follows:

$$\begin{aligned}
\sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I} \\ |\vec{I}_2|\leq N-2}} & \|t^{A+1+q}(\partial_{\vec{I}_1} g^{-1})\partial^2\partial_{\vec{I}_2} n\|_{L^2(\Sigma_t)} \\
& \lesssim t^{A+1+q}\|g^{-1}\|_{\dot{W}^{1,\infty}_{Frame}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)} \\
& \quad + t^{A+1+q}\|n\|_{\dot{W}^{2,\infty}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^{N-1}_{Frame}(\Sigma_t)}.
\end{aligned} \tag{5.30}$$

From Definition 3.16, estimates (4.12b) and (4.12c), and the bootstrap assumptions, we see that

$$\text{RHS } (5.30) \lesssim t^{1-q-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} + t^{4-14q-4\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (5.21) as desired. This completes the proof of (5.21) and finishes the proof of the lemma. ∎

5.3. *Proof of Proposition* 5.1

In this subsection, we prove Proposition 5.1. Throughout this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta < \sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large.

Proof of (5.2). From (2.15b), we deduce that $\mathrm{R} \simeq (g^{-1})^2\partial^2 g + (g^{-1})^3(\partial g)^2$. Hence, we can bound these products by bounding each factor in the norm $\|\cdot\|_{L^\infty_{Frame}(\Sigma_t)}$ with the help of estimates (4.2a) and (4.12a)–(4.12b) and the bootstrap assumptions, thereby deducing that

$$|\mathrm{R}| \lesssim t^{-10q-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} \lesssim \varepsilon t^{-10q-A\delta}. \tag{5.31}$$

From (3.2), both inequalities in (5.31), and equation (2.13b) (multiplied by t^2), we deduce that if $A\delta < \sigma$, then

$$\begin{aligned}|t^2 g^{ab}\nabla_a\nabla_b n - (n-1)\{1+\mathcal{O}(\varepsilon)\}| &\lesssim t^{2-10q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\} \\ &\lesssim t^{2-10q-\sigma}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}. \end{aligned} \tag{5.32}$$

At any point $p_{(\mathrm{Max})}\in\Sigma_t$ at which $n-1$ achieves its maximum value, we have that $g^{ab}\nabla_a\nabla_b n \le 0$. From this fact and estimate (5.32), it follows that

$$(n-1)|_{p_{(\mathrm{Max})}} \lesssim t^{2-10q-\sigma}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.$$

Using similar reasoning, we deduce that at any point $p_{(Min)}\in\Sigma_t$ at which $n-1$ achieves its minimum value, we have the estimate

$$(n-1)|_{p_{(Min)}} \gtrsim -t^{2-10q-\sigma}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.$$

Combining these two estimates, we find that

$$\|n-1\|_{L^\infty(\Sigma_t)} \lesssim t^{2-10q-\sigma}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},$$

which, in view of definition (3.15b), yields (5.2).

Proof of (5.4) *and* (5.5). In view of Definition 3.16, we see that to obtain (5.4) and (5.5), it suffices to prove the following estimates:

$$\begin{aligned}&\left[\int_{\Sigma_t}(|t^{A+1}\partial\partial_{\vec{I}}n|_g^2+|t^A\partial_{\vec{I}}n|^2)\,dx\right]^{1/2} \\ &\quad\le C_*\|t^{A+1}\partial_{\vec{I}}k\|_{L^2_g(\Sigma_t)}+Ct^{\sigma}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},\quad (|\vec{I}|=N),\end{aligned} \tag{5.33}$$

$$\begin{aligned}&\left[\int_{\Sigma_t}(|t^{A+q}\partial\partial_{\vec{I}}n|_g^2+|t^{A+q-1}\partial_{\vec{I}}n|^2)\,dx\right]^{1/2} \\ &\quad\le C\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}\quad (|\vec{I}|=N-1).\end{aligned} \tag{5.34}$$

To prove (5.33), we let $\vec{I}$ be any spatial derivative multi-index with $|\vec{I}|=N$. Multiplying equation (5.11) by $t^A\partial_{\vec{I}}n$ and integrating by parts over Σ_t, we deduce

$$\begin{aligned}\int_{\Sigma_t}(|t^{A+1}\partial\partial_{\vec{I}}n|_g^2+(t^A\partial_{\vec{I}}n)^2)\,dx &\le \int_{\Sigma_t}|(t\partial_b g^{ab})(t^{A+1}\partial_a\partial\partial_{\vec{I}}n)(t^A\partial_{\vec{I}}n)|\,dx \\ &\quad+\int_{\Sigma_t}|^{(\mathrm{Border};\vec{I})}\mathfrak{N}||t^A\partial_{\vec{I}}n|\,dx \\ &\quad+\int_{\Sigma_t}|^{(\mathrm{Junk};\vec{I})}\mathfrak{N}||t^A\partial_{\vec{I}}n|\,dx.\end{aligned} \tag{5.35}$$

Next, we use (3.2), (4.2a), (4.12b), the bootstrap assumptions, and g-Cauchy–Schwarz to deduce that

$$\begin{aligned}|t\partial_b g^{ab}||t^{A+1}\partial_a\partial_{\vec{I}}n| &\lesssim |g_{ac}(t\partial_b g^{ab})(t\partial_d g^{cd})|^{1/2}|t^{A+1}\partial\partial_{\vec{I}}n|_g \\ &\lesssim t\|g\|^{1/2}_{L^\infty_{\mathrm{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}|t^{A+1}\partial\partial_{\vec{I}}n|_g \\ &\lesssim \varepsilon|t^{A+1}\partial\partial_{\vec{I}}n|_g.\end{aligned} \tag{5.36}$$

From (5.35), (5.36), and Young's inequality, we deduce that if ε is sufficiently small, then

$$\begin{aligned}
&\int_{\Sigma_t} (|t^{A+1}\partial\partial_{\vec{I}} n|_g^2 + |t^A \partial_{\vec{I}} n|^2)\, dx \\
&\quad \leq C\varepsilon \int_{\Sigma_t} |t^{A+1}\partial\partial_{\vec{I}} n|_g^2\, dx + \frac{1}{2}\int_{\Sigma_t} |t^A \partial\partial_{\vec{I}} n|^2\, dx \\
&\qquad + 4\int_{\Sigma_t} |^{(\mathrm{Border};\vec{I})}\mathfrak{N}|^2\, dx + 4\int_{\Sigma_t} |^{(\mathrm{Junk};\vec{I})}\mathfrak{N}|^2\, dx.
\end{aligned} \tag{5.37}$$

Using (5.19) and (5.20) to bound the last two integrals on the right-hand side of (5.37), and soaking (assuming ε is sufficiently small) the first two terms on the right-hand side of (5.37) and the first term on the right-hand side of (5.19) back into the left-hand side of (5.37), we arrive at (5.33).

Similarly, to prove inequality (5.34), we let $\vec{I}$ be any spatial derivative multi-index with $|\vec{I}| = N - 1$. We multiply equation (5.13) by $t^{A+q-1}\partial_{\vec{I}} n$, use (5.31), use the bound

$$|(t\partial_b g^{ab})(t^{A+q}\partial_a \partial\partial_{\vec{I}} n)(t^{A+q-1}\partial_{\vec{I}} n)| \lesssim \varepsilon |t^{A+q}\partial\partial_{\vec{I}} n|_g |t^{A+q-1}\partial_{\vec{I}} n|$$

(which follows from essentially the same reasoning we used to prove (5.36)), and argue as in the previous paragraph to deduce the following analog of (5.37):

$$\begin{aligned}
&\int_{\Sigma_t} (|t^{A+q}\partial\partial_{\vec{I}} n|_g^2 + \{1 + \mathcal{O}(\varepsilon)\}|t^{A+q-1}\partial_{\vec{I}} n|^2)\, dx \\
&\leq C\varepsilon \int_{\Sigma_t} |t^{A+q}\partial\partial_{\vec{I}} n|_g^2\, dx + \frac{1}{2}\int_{\Sigma_t} |t^{A+q-1}\partial\partial_{\vec{I}} n|^2\, dx + 4\int_{\Sigma_t} |^{(\vec{I})}\widetilde{\mathfrak{N}}|^2\, dx.
\end{aligned} \tag{5.38}$$

Soaking the first two terms on the right-hand side of (5.38) back into the left-hand side and using (5.21) to bound the last integral on the right-hand side of (5.38), we arrive at (5.34).

Proof of (5.3). We argue as in the proof of (5.2), but using estimate (5.18) to control the right-hand side of (5.7). This leads to the bound

$$\|\partial_t n\|_{L^\infty(\Sigma_t)} \leq C t^{1-10q-\sigma}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t) + \mathbb{L}_{(n)}(t) + \mathbb{H}_{(n)}(t)\}.$$

Using Definition 3.16 and estimates (5.2), (5.4), and (5.5), we have that

$$\mathbb{L}_{(n)}(t) + \mathbb{H}_{(n)}(t) \leq C\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, when combined with the previous estimate, yields the desired bound (5.3).

6. Estimates for the low-order derivatives of g and k

Our main goal in this section is to prove the following proposition, which provides the integral inequality that we use to control the low norm $\mathbb{L}_{(g,k)}(t)$. The proof of the proposition is located in Section 6.2. In Section 6.1, we derive the equations that we will use in proving it.

Proposition 6.1 (Integral inequality for $\mathbb{L}_{(g,k)}(t)$). *Recall that $\mathbb{L}_{(g,k)}(t)$ is the low-order norm from Definition 3.14, and assume that the bootstrap assumptions (3.18) hold. If N is sufficiently large in a manner that depends on A, D, q, and σ, then there exists a constant $C = C_{N,A,D,q,\sigma} > 0$ such that if ε is sufficiently small (in a manner that depends on N, A, D, q, and σ), then the following estimate holds for $t \in (T_{(\mathrm{Boot})}, 1]$:*

$$\mathbb{L}_{(g,k)}(t) \leq C\mathbb{L}_{(g,k)}(1) + C \int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s) + \mathbb{H}_{(g,k)}(s)\}\, ds. \tag{6.2}$$

6.1. The equations

In this subsection, we derive the equations that we use to control g and k at the lowest derivative levels.

Lemma 6.3 (A rewriting of the equations verified by g and k). *Let $\widetilde{g}$ and $\widetilde{k}$ denote the background Kasner solution variables with corresponding Kasner exponents $\{q_i\}_{i=1,\ldots,D}$. Then the following evolution equations hold, where $i \leq j$ in (6.4a)–(6.4b) and there is* no summation over j in (6.4a)–(6.4b) *(note that $t^{-2q_j}\widetilde{g}_{ij} = t^{2q_j}(\widetilde{g}^{-1})^{ij} = \mathrm{diag}(1, 1, \ldots, 1)$):*

$$\begin{aligned}
\partial_t\{t^{-2q_j} g_{ij} - t^{-2q_j}\widetilde{g}_{ij}\} = &-2t^{-2q_j}\{g_{ia} - \widetilde{g}_{ia}\}\{k^a{}_j - \widetilde{k}^a{}_j\} \\
&- 2t^{2q_i-2q_j}\{k^i{}_j - \widetilde{k}^i{}_j\} \\
&- 2t^{-2q_j}(n-1)g_{ia}k^a{}_j,
\end{aligned} \tag{6.4a}$$

$$\begin{aligned}
\partial_t\{t^{2q_j} g^{ij} - t^{2q_j}(\widetilde{g}^{-1})^{ij}\} = &\, 2t^{2q_j}\{g^{ia} - (\widetilde{g}^{-1})^{ia}\}\{k^j{}_a - \widetilde{k}^j{}_a\} \\
&+ 2t^{-2q_i+2q_j}\{k^j{}_i - \widetilde{k}^j{}_i\} \\
&+ 2t^{2q_j}(n-1)g^{ia}k^j{}_a.
\end{aligned} \tag{6.4b}$$

Moreover, the following evolution equation holds (note that $t\widetilde{k}^i{}_j = -\mathrm{diag}(q_1, \ldots, q_D)$):

$$\partial_t\{tk^i{}_j - t\widetilde{k}^i{}_j\} = (1-n)k^i{}_j - tg^{ia}\partial_a\partial_j n + tg^{ia}\Gamma_a{}^b{}_j\partial_b n + tn\mathrm{Ric}^i{}_j. \tag{6.5}$$

Proof. Throughout this proof, we do not use Einstein summation over j. To derive equation (6.4a), we first use equation (2.12a) and the fact that $t^{-2q_j}\widetilde{g}_{ij} = \delta_{ij}$ (where δ_{ij} is the standard Kronecker delta) to deduce

$$\begin{aligned}
\partial_t\{t^{-2q_j} g_{ij} - t^{-2q_j}\widetilde{g}_{ij}\} = &-2t^{-2q_j} g_{ia}k^a{}_j - 2q_j t^{-2q_j-1}g_{ij} \\
&- 2t^{-2q_j}(n-1)g_{ia}k^a{}_j.
\end{aligned} \tag{6.6}$$

Since $\widetilde{k} = -t^{-1}\mathrm{diag}(q_1, \ldots, q_D)$, we can express the first two products on the right-hand side of (6.6) as follows:

$$-2t^{-2q_j} g_{ia}k^a{}_j - 2q_j t^{-2q_j-1}g_{ij} = -2t^{-2q_j} g_{ia}\{k^a{}_j - \widetilde{k}^a{}_j\}.$$

Next, using that $\widetilde{g} = \mathrm{diag}(t^{2q_1}, \ldots, t^{2q_D})$ and $\widetilde{k} = -t^{-1}\mathrm{diag}(q_1, \ldots, q_D)$, we express the right-hand side of the previous expression as follows:

$$-2t^{-2q_j} g_{ia}\{k^a{}_j - \widetilde{k}^a{}_j\} = -2t^{-2q_j}\{g_{ia} - \widetilde{g}_{ia}\}\{k^a{}_j - \widetilde{k}^a{}_j\} - 2t^{2q_i-2q_j}\{k^i{}_j - \widetilde{k}^i{}_j\}.$$

Combining these calculations, we arrive at (6.4a).

Equation (6.4b) can be derived by applying similar arguments to equation (2.12b), and we omit the details.

Equation (6.5) follows from multiplying both sides of equation (2.12c) by t and using $t\widetilde{k} = -\mathrm{diag}(q_1, \ldots, q_D)$. ■

6.2. *Proof of Proposition* 6.1

In this subsection, we prove Proposition 6.1. In this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta < \sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large.

First, we note that to obtain (6.2), it suffices to show that the following bounds hold:

$$\|t^{2q} g - t^{2q}\widetilde{g}\|_{L^\infty_{\mathrm{Frame}}(\Sigma_t)} \leq C\mathbb{L}_{(g,k)}(1) + C\int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s) + \mathbb{H}_{(g,k)}(s)\}\,ds, \tag{6.7}$$

$$\|t^{2q} g^{-1} - t^{2q}\widetilde{g}^{-1}\|_{L^\infty_{\mathrm{Frame}}(\Sigma_t)} \leq C\mathbb{L}_{(g,k)}(1) + C\int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s) + \mathbb{H}_{(g,k)}(s)\}\,ds, \tag{6.8}$$

$$\|tk - t\widetilde{k}\|_{W^{2,\infty}_{\mathrm{Frame}}(\Sigma_t)} \leq C\mathbb{L}_{(g,k)}(1) + C\int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s) + \mathbb{H}_{(g,k)}(s)\}\,ds. \tag{6.9}$$

For we can then use the symmetry property $k_{ab} = k_{ba}$ and the fact that $(t\widetilde{k}^a_{\ b})(t\widetilde{k}^b_{\ a}) = 1$ to derive the identity

$$|tk|_g - 1 = \sqrt{1 + 2(t\widetilde{k}^a_{\ b})\{tk^b_{\ a} - t\widetilde{k}^b_{\ a}\} + \{tk^a_{\ b} - t\widetilde{k}^a_{\ b}\}\{tk^b_{\ a} - t\widetilde{k}^b_{\ a}\}} - 1,$$

which, in conjunction with the bootstrap assumptions (3.18) and estimates (4.2b) and (6.9), also yields (upon Taylor expanding the square root) the estimate

$$\||tk|_g - 1\|_{L^\infty(\Sigma_t)} \leq C\mathbb{L}_{(g,k)}(1) + \int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s) + \mathbb{H}_{(g,k)}(s)\}\,ds. \tag{6.10}$$

In view of definition (3.15a), from (6.7)–(6.10), we conclude the desired estimate (6.2).

It remains for us to prove (6.7)–(6.9). We first prove (6.7) by analyzing equation (6.4a). From Definition 3.14, estimates (4.2a), (4.2b), and (5.2), and the bootstrap assumptions, we deduce, by bounding each factor on the right-hand side of (6.4a) in the norm $\|\cdot\|_{L^\infty_{\mathrm{Frame}}(\Sigma_t)}$, that the following estimate holds:

$$\begin{aligned} |\mathrm{RHS}\ (6.4a)| &\leq C\varepsilon t^{-1-2q-2q_j}\|t^{2q} g - t^{2q}\widetilde{g}\|_{L^\infty_{\mathrm{Frame}}(\Sigma_t)} \\ &\quad + Ct^{-1+2q_i-2q_j}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} \\ &\quad + Ct^{1-12q-\sigma-2q_j}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \end{aligned} \tag{6.11}$$

From (3.2), we deduce that the last two products on the right-hand side of (6.11) are $\leq C t^{\sigma-1-2q-2q_j}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}$ for $t \in (T_{(\text{Boot})}, 1]$. Hence, with the help of these bounds, we can integrate equation (6.4a) from time t to time 1 and use the initial data bound $\|g-\widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_1)} \leq C\mathbb{L}_{(g,k)}(1)$ to deduce the following estimate for components, where we do not sum over j on the left-hand side of (6.12):

$$\begin{aligned} t^{-2q_j}|g_{ij}-\widetilde{g}_{ij}| \leq C\mathbb{L}_{(g,k)}(1) & \\ + C\varepsilon \int_{s=t}^{1} & s^{-1-2q-2q_j}\|s^{2q}g - s^{2q}\widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_s)}\, ds \\ + C \int_{s=t}^{1} & s^{\sigma-1-2q-2q_j}\{\mathbb{L}_{(g,k)}(s)+\mathbb{H}_{(g,k)}(s)\}\, ds. \end{aligned} \tag{6.12}$$

Multiplying both sides of (6.12) by t^{2q+2q_j} and using (3.2), we deduce

$$\begin{aligned} |t^{2q}g_{ij}-t^{2q}\widetilde{g}_{ij}| \leq C\mathbb{L}_{(g,k)}(1) & \\ + C\varepsilon t^{2q+2q_j}\int_{s=t}^{1} & s^{-1-2q-2q_j}\|s^{2q}g - s^{2q}\widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_s)}\, ds \\ + C \int_{s=t}^{1} & s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s)+\mathbb{H}_{(g,k)}(s)\}\, ds. \end{aligned} \tag{6.13}$$

We now define $G(t) := \sup_{s\in[t,1]}\|s^{2q}g - s^{2q}\widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_s)}$. Next, with the help of (3.2), we deduce the following estimate for the first integral on the right-hand side of (6.13):

$$\begin{aligned} & C\varepsilon t^{2q+2q_j}\int_{s=t}^{1} s^{-1-2q-2q_j}\|s^{2q}g - s^{2q}\widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_s)}\, ds \\ & \quad \leq C\varepsilon G(t) t^{2q+2q_j}\int_{s=t}^{1} s^{-1-2q-2q_j}\, ds \leq C\varepsilon G(t). \end{aligned} \tag{6.14}$$

From (6.13) and (6.14), we deduce that for $(t,x) \in (T_{(\text{Boot})}, 1] \times \mathbb{T}^D$, we have

$$\begin{aligned} |t^{2q}g_{ij}-t^{2q}\widetilde{g}_{ij}| & \leq C\mathbb{L}_{(g,k)}(1) + C\varepsilon G(t) \\ & \quad + C\int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s)+\mathbb{H}_{(g,k)}(s)\}\, ds. \end{aligned} \tag{6.15}$$

From (6.15), it follows that

$$G(t) \leq C\mathbb{L}_{(g,k)}(1) + C\varepsilon G(t) + C\int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s)+\mathbb{H}_{(g,k)}(s)\}\, ds. \tag{6.16}$$

For ε sufficiently small, we can absorb the product $C\varepsilon G(t)$ on the right-hand side of (6.16) back into the left-hand side (at the minor expense of increasing the constants C on the right-hand side). From these arguments, we conclude the bound

$$G(t) \leq C\mathbb{L}_{(g,k)}(1) + C\int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s)+\mathbb{H}_{(g,k)}(s)\}\, ds,$$

which in particular implies the desired bound (6.7).

Estimate (6.8) can be proved using a similar argument based on the evolution equation (6.4b), and we omit these details.

To prove (6.9), we note that the right-hand side of (6.5) can be expressed (with the help of (2.15b)) in the schematic form

$$(1-n)k + tg^{-1}\partial^2 n + tg^{-2}(\partial g)\partial n + tng^{-2}\partial^2 g + tng^{-3}(\partial g)\partial g.$$

Hence, commuting (6.5) with up to two spatial derivatives and bounding all of the resulting products by bounding each factor in the norm $\|\cdot\|_{L^\infty_{\text{Frame}}(\Sigma_t)}$ with the help of Definition 3.14, Lemma 4.1, Lemma 4.11, (5.2), (5.4), (5.5), and the bootstrap assumptions, and also using Young's inequality, we find, in view of (3.2), that if $A\delta$ is sufficiently small, then

$$\begin{aligned}\|\partial_t\{tk - t\widetilde{k}\}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)} &\leq Ct^{1-10q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} \\ &\quad + Ct^{3-16q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} \\ &\leq Ct^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \end{aligned} \tag{6.17}$$

Integrating (6.17) in time and using the initial data bound $\|k - \widetilde{k}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_1)} \leq \mathbb{L}_{(g,k)}(1)$, we conclude (6.9). We have therefore proved the proposition.

7. Estimates for the top-order derivatives of g and k

Our main goal in this section is to prove the following proposition, which provides the main integral inequality for the top-order derivatives of g and k. The proof is located in Section 7.4. In Sections 7.1–7.3, we derive the identities and estimates that we will use when proving the proposition.

Proposition 7.1 (Integral inequality for the top-order derivatives of g and k). *Let $\vec{I}$ be a top-order spatial multi-index, that is, a multi-index with $|\vec{I}| = N$. Assume that the bootstrap assumptions (3.18) hold. There exists a constant $C_* > 0$* independent of N and A *and a constant $C = C_{N,A,D,q,\sigma} > 0$ such that if N is sufficiently large in a manner that depends on A, D, q, and σ, and if ε is sufficiently small (in a manner that depends on N, A, D, q, and σ), then the following estimates hold for $t \in (T_{(\text{Boot})}, 1]$:*

$$\begin{aligned}&\|t^{A+1}\partial_{\vec{I}}k\|^2_{L^2_g(\Sigma_t)} + \frac{1}{4}\|t^{A+1}\partial\partial_{\vec{I}}g\|^2_{L^2_g(\Sigma_t)} \\ &\leq C\mathbb{H}^2_{(g,k)}(1) - \{2A - C_*\}\int_{s=t}^1 s^{-1}\left\{\|s^{A+1}\partial_{\vec{I}}k\|^2_{L^2_g(\Sigma_s)} + \frac{1}{4}\|s^{A+1}\partial\partial_{\vec{I}}g\|^2_{L^2_g(\Sigma_s)}\right\} ds \\ &\quad + C\int_{s=t}^1 s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds. \end{aligned} \tag{7.2}$$

7.1. The equations

In this subsection, we derive the equations that we will use when deriving estimates for g and k at the high derivative levels.

Lemma 7.3 (The equations verified by the high-order derivatives of the metric and second fundamental form). *Let $\vec{I}$ be a top-order spatial multi-index, that is, a multi-index with $|\vec{I}| = N$. Then the following* ***commuted momentum constraint equations*** *hold (see Remark 3.9):*

$$t^{A+1}\partial_a\partial_{\vec{I}}k^a_{\ i} = {}^{(Border;\vec{I})}\mathfrak{M}_i + {}^{(Junk;\vec{I})}\mathfrak{M}_i, \tag{7.4a}$$

$$t^{A+1}g^{ab}\partial_a\partial_{\vec{I}}k^i_{\ b} = {}^{(Border;\vec{I})}\widetilde{\mathfrak{M}}^i + {}^{(Junk;\vec{I})}\widetilde{\mathfrak{M}}^i, \tag{7.4b}$$

where (see Section 1.8.7 regarding our use of notation $\overset{*}{\simeq}$ and $\simeq$, and see Remarks 2.6 and 3.9)

$${}^{(Border;\vec{I})}\mathfrak{M}_i \overset{*}{\simeq} t^{A+1}g^{-1}(\partial\partial_{\vec{I}}g)k, \tag{7.5a}$$

$${}^{(Junk;\vec{I})}\mathfrak{M}_i \simeq \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_2|\leq N-1}} t^{A+1}(\partial_{\vec{I}_1}g^{-1})(\partial\partial_{\vec{I}_2}g)\partial_{\vec{I}_3}k, \tag{7.5b}$$

$${}^{(Border;\vec{I})}\widetilde{\mathfrak{M}}^i \overset{*}{\simeq} t^{A+1}(g^{-1})^2(\partial\partial_{\vec{I}}g)k, \tag{7.5c}$$

$${}^{(Junk;\vec{I})}\widetilde{\mathfrak{M}}^i \simeq \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I}\\ |\vec{I}_3|\leq N-1}} t^{A+1}(\partial_{\vec{I}_1}g^{-1})(\partial_{\vec{I}_2}g^{-1})(\partial\partial_{\vec{I}_3}g)\partial_{\vec{I}_4}k. \tag{7.5d}$$

Moreover, for any constant $P \geq 0$ and for any spatial multi-index $\vec{I}$, the following ***commuted evolution equations*** *hold:*

$$\begin{aligned}\partial_t(t^{A+P}\partial_e\partial_{\vec{I}}g_{ij}) &= \frac{1}{t}\{(A+P)\delta^a_{\ j} - 2tk^a_{\ j}\}(t^{A+P}\partial_e\partial_{\vec{I}}g_{ia}) - 2t^{A+P}ng_{ia}\partial_e\partial_{\vec{I}}k^a_{\ j}\\ &\quad + {}^{(Border;P;\vec{I})}\mathfrak{H}_{eij} + {}^{(Junk;P;\vec{I})}\mathfrak{H}_{eij},\end{aligned} \tag{7.6a}$$

$$\begin{aligned}\partial_t(t^{A+P}\partial_{\vec{I}}k^i_{\ j}) &= (A+P-1)t^{A+P-1}\partial_{\vec{I}}k^i_{\ j} - t^{A+P}g^{ia}\partial_a\partial_j\partial_{\vec{I}}n\\ &\quad + \frac{1}{2}t^{A+P}ng^{ic}g^{ab}\{\partial_a\partial_c\partial_{\vec{I}}g_{bj} + \partial_a\partial_j\partial_{\vec{I}}g_{bc} - \partial_a\partial_b\partial_{\vec{I}}g_{cj}\\ &\qquad - \partial_c\partial_j\partial_{\vec{I}}g_{ab}\}\\ &\quad + {}^{(Border;P;\vec{I})}\mathfrak{K}^i_{\ j} + {}^{(Junk;P;\vec{I})}\mathfrak{K}^i_{\ j},\end{aligned} \tag{7.6b}$$

where $\delta^i_{\ j}$ is the standard Kronecker delta, and

$${}^{(Border;P;\vec{I})}\mathfrak{H}_{eij} \overset{*}{\simeq} t^{A+P}(\partial\partial_{\vec{I}}n)gk, \tag{7.7a}$$

$$\begin{aligned}{}^{(Junk;P;\vec{I})}\mathfrak{H}_{eij} &\simeq t^{A+P}(n-1)(\partial\partial_{\vec{I}}g)k + \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_1|\leq|\vec{I}|-1}} t^{A+P}(\partial\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g)\partial_{\vec{I}_3}k\\ &\quad + \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_2|\leq|\vec{I}|-1}} t^{A+P}(\partial_{\vec{I}_1}n)(\partial\partial_{\vec{I}_2}g)\partial_{\vec{I}_3}k\\ &\quad + \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_3|\leq|\vec{I}|-1}} t^{A+P}(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g)\partial\partial_{\vec{I}_3}k,\end{aligned} \tag{7.7b}$$

and

$$^{(Border;P;\vec{I})}\mathfrak{K}^i{}_j \overset{*}{\simeq} t^{A+P-1}(\partial_{\vec{I}} n)k, \tag{7.8a}$$

$$\begin{aligned}
^{(Junk;P;\vec{I})}\mathfrak{K}^i{}_j \simeq & \sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I} \\ |\vec{I}_1|\leq|\vec{I}|-1}} t^{A+P-1}\{\partial_{\vec{I}_1}(n-1)\}\partial_{\vec{I}_2}k \\
& + \sum_{\vec{I}_1+\vec{I}_2+\cdots+\vec{I}_6=\vec{I}} t^{A+P}(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g^{-1})(\partial_{\vec{I}_3}g^{-1})(\partial_{\vec{I}_4}g^{-1}) \\
& \qquad \times (\partial\partial_{\vec{I}_5}g)\partial\partial_{\vec{I}_6}g \\
& + \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I} \\ |\vec{I}_4|\leq|\vec{I}|-1}} t^{A+P}(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g^{-1})(\partial_{\vec{I}_3}g^{-1})\partial^2\partial_{\vec{I}_4}g \\
& + \sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I} \\ |\vec{I}_2|\leq|\vec{I}|-1}} t^{A+P}(\partial_{\vec{I}_1}g^{-1})\partial^2\partial_{\vec{I}_2}n \\
& + \sum_{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I}} t^{A+P}(\partial_{\vec{I}_1}g^{-1})(\partial_{\vec{I}_2}g^{-1})(\partial\partial_{\vec{I}_3}g)\partial\partial_{\vec{I}_4}n.
\end{aligned} \tag{7.8b}$$

Proof. To prove (7.4a), we first write equation (2.11b) relative to the transported spatial coordinates in the schematic form $\partial_a k^a_{\ i} = g^{-1}(\partial g)k$. Commuting this equation with $t^{A+1}\partial_{\vec{I}}$, we arrive at (7.4a). The proof of (7.4b) is similar, but we start by raising the index i in equation (2.11b) to deduce $g^{ab}\nabla_a k^i{}_b = 0$ and then (schematically) writing this equation in coordinates as $g^{ab}\partial_a k^i{}_b = g^{-2}(\partial g)k$.

To prove (7.6a), we commute equation (2.12a) with $\partial_e\partial_{\vec{I}}$ and then with t^{A+P} and carry out straightforward computations.

To prove (7.6b), we first use (2.15b) to decompose

$$\mathrm{Ric}^i{}_j = \frac{1}{2}g^{ic}g^{ab}\{\partial_a\partial_c g_{bj} + \partial_a\partial_j g_{bc} - \partial_a\partial_b g_{cj} - \partial_c\partial_j g_{ab}\} + \mathrm{Error}^i{}_j, \tag{7.9}$$

where (schematically) $\mathrm{Error} = g^{-3}(\partial g)^2$. We now use (7.9) to decompose the term $\mathrm{Ric}^i{}_j$ on the right-hand side of (2.12c), commute the evolution equation (2.12c) with $\partial_{\vec{I}}$ and then with t^{A+P}, and carry out straightforward computations, thereby arriving at (7.6b). ■

7.2. *Energy currents*

When deriving top-order energy estimates (see the proof of Proposition 7.1), we will integrate by parts by applying the divergence theorem on spacetime regions of the form $[t,1]\times\mathbb{T}^D$ with the help of the vectorfields $^{(\vec{I})}\mathbf{J}$ featured in the following definition. Put differently, the vectorfields $^{(\vec{I})}\mathbf{J}$ are convenient for bookkeeping during integration by parts.

Remark 7.10 (Another way to think about the results of this subsection). Roughly speaking, Definition 7.11 and the ensuing Lemma 7.14 are equivalent to setting $P = 1$ in the equations of Lemma 7.3, then multiplying equation (7.6a) by the g-dual of $\partial\partial_{\vec{I}} g$, multiplying (7.6b) by the g-dual of $\partial_{\vec{I}} k$, differentiating by parts, and using the constraints (7.4a)–(7.4b) to eliminate all terms involving $|\vec{I}|+2$ derivatives of g, $|\vec{I}|+2$ derivatives of n, and $|\vec{I}|+1$ derivatives of k, up to a perfect divergence term (which appears on the left-hand side of (7.15) as $\partial_\alpha {}^{(\vec{I})}\mathbf{J}^\alpha$). That is, the results of this subsection are just an intricate differential version of integration by parts that relies on the evolution equations (7.6a)–(7.6b) and the constraint equations (7.4a)–(7.4b).

Definition 7.11 (Energy current vectorfields). To each top-order spatial multi-index $\vec{I}$ (i.e., $|\vec{I}| = N$), we associate the energy current ${}^{(\vec{I})}\mathbf{J}$, which we define to be the vectorfield with the following components relative to the CMC-transported spatial coordinates:

$$
\begin{aligned}
{}^{(\vec{I})}\mathbf{J}^0 &:= |t^{A+1}\partial_{\vec{I}} k|_g^2 + \frac{1}{4}|t^{A+1}\partial\partial_{\vec{I}} g|_g^2, && (7.12a)\\
{}^{(\vec{I})}\mathbf{J}^j &:= 2g^{ab}(t^{A+1}\partial_{\vec{I}} k^j{}_a)(t^{A+1}\partial_b\partial_{\vec{I}} n) + n g^{ij} g^{bc}(t^{A+1}\partial_{\vec{I}} k^a{}_b)(t^{A+1}\partial_i\partial_{\vec{I}} g_{ac})\\
&\quad + n g^{ab} g^{cd}(t^{A+1}\partial_{\vec{I}} k^j{}_c)(t^{A+1}\partial_d\partial_{\vec{I}} g_{ab})\\
&\quad - n g^{ab} g^{cd}(t^{A+1}\partial_{\vec{I}} k^j{}_c)(t^{A+1}\partial_a\partial_{\vec{I}} g_{bd})\\
&\quad - n g^{ab} g^{jc}(t^{A+1}\partial_{\vec{I}} k^d{}_c)(t^{A+1}\partial_a\partial_{\vec{I}} g_{bd}). && (7.12b)
\end{aligned}
$$

Remark 7.13. The components $\{{}^{(\vec{I})}\mathbf{J}^\alpha\}_{\alpha=0,\ldots,D}$ are quadratic forms in $(\partial_{\vec{I}} k, \partial\partial_{\vec{I}} g, \partial\partial_{\vec{I}} n)$ with coefficients that depend on g, g^{-1}, and n.

Lemma 7.14 (Divergence identity verified by ${}^{(\vec{I})}\mathbf{J}$). *Let $\vec{I}$ be a top-order spatial multi-index, that is, a multi-index with $|\vec{I}| = N$. Then for solutions to the commuted equations of Lemma 7.3 with $P = 1$ in equations (7.6a)–(7.6b), the spacetime vectorfield ${}^{(\vec{I})}\mathbf{J}$ from Definition 7.11 verifies the following divergence identity relative to the CMC-transported spatial coordinates:*

$$
\partial_\alpha {}^{(\vec{I})}\mathbf{J}^\alpha = \frac{2A}{t}|t^{A+1}\partial_{\vec{I}} k|_g^2 + \frac{A+1}{2t}|t^{A+1}\partial\partial_{\vec{I}} g|_g^2 + {}^{(\mathrm{Border};\vec{I})}\mathfrak{J} + {}^{(\mathrm{Junk};\vec{I})}\mathfrak{J}, \qquad (7.15)
$$

where

$$
\begin{aligned}
{}^{(\mathrm{Border};\vec{I})}\mathfrak{J} &:= 2t^{A+1} g_{ac} g^{bd} (\partial_{\vec{I}} k^a{}_b)^{(\mathrm{Border};1;\vec{I})}\mathfrak{K}^c{}_d\\
&\quad + \frac{1}{2} t^{A+1} g^{ad} g^{be} g^{cf} (\partial_a\partial_{\vec{I}} g_{bc})^{(\mathrm{Border};1;\vec{I})}\mathfrak{H}_{def}\\
&\quad + 2t^{A+1} g^{ab} (\partial_a\partial_{\vec{I}} n)^{(\mathrm{Border};\vec{I})}\mathfrak{M}_b\\
&\quad + t^{A+1} n g^{ab} g^{cd} (\partial_c\partial_{\vec{I}} g_{ab})^{(\mathrm{Border};\vec{I})}\mathfrak{M}_d\\
&\quad - t^{A+1} n g^{ab} g^{cd} (\partial_a\partial_{\vec{I}} g_{bc})^{(\mathrm{Border};\vec{I})}\mathfrak{M}_d\\
&\quad - t^{A+1} n g^{ab} (\partial_a\partial_{\vec{I}} g_{bc})^{(\mathrm{Border};\vec{I})}\widetilde{\mathfrak{M}}^c\\
&\quad + 2t^{2A+2} n g_{ac} g^{id} k^b{}_i (\partial_{\vec{I}} k^a{}_b)(\partial_{\vec{I}} k^c{}_d)\\
&\quad - 2t^{2A+2} n g_{ai} g^{bd} k^i{}_c (\partial_{\vec{I}} k^a{}_b)(\partial_{\vec{I}} k^c{}_d)
\end{aligned}
$$

$$+ \frac{1}{2} t^{2A+2} n g^{be} g^{cf} g^{id} k^{a}_{\ i} (\partial_a \partial_{\vec{I}} g_{bc})(\partial_d \partial_{\vec{I}} g_{ef})$$
$$+ t^{2A+2} n g^{ad} g^{ie} g^{cf} k^{b}_{\ e} (\partial_a \partial_{\vec{I}} g_{bc})(\partial_d \partial_{\vec{I}} g_{if}), \tag{7.16a}$$

$$\begin{aligned}
{}^{(Junk;\vec{I})}\mathfrak{J} := {} & 2t^{A+1} g_{ac} g^{bd} (\partial_{\vec{I}} k^{a}_{\ b})^{(Junk;1;\vec{I})} \mathfrak{K}^{c}_{\ d} \\
& + \frac{1}{2} t^{A+1} g^{ad} g^{be} g^{cf} (\partial_a \partial_{\vec{I}} g_{bc})^{(Junk;1;\vec{I})} \mathfrak{H}_{def} \\
& + 2t^{A+1} g^{ab} (\partial_a \partial_{\vec{I}} n)^{(Junk;\vec{I})} \mathfrak{M}_b \\
& + t^{A+1} n g^{ab} g^{cd} (\partial_c \partial_{\vec{I}} g_{ab})^{(Junk;\vec{I})} \mathfrak{M}_d \\
& - t^{A+1} n g^{ab} g^{cd} (\partial_a \partial_{\vec{I}} g_{bc})^{(Junk;\vec{I})} \mathfrak{M}_d \\
& - t^{A+1} n g^{ab} (\partial_a \partial_{\vec{I}} g_{bc})^{(Junk;\vec{I})} \widetilde{\mathfrak{M}}^{c} \\
& + 2t^{2A+2} (\partial_j g^{ab})(\partial_{\vec{I}} k^{j}_{\ a})(\partial_b \partial_{\vec{I}} n) \\
& + t^{2A+2} \{\partial_j (n g^{ij} g^{bc})\}(\partial_{\vec{I}} k^{a}_{\ b})(\partial_{\vec{I}} \partial_i g_{ac}) \\
& + t^{2A+2} \{\partial_j (n g^{ab} g^{cd})\}(\partial_{\vec{I}} k^{j}_{\ c})(\partial_d \partial_{\vec{I}} g_{ab}) \\
& - t^{2A+2} \{\partial_j (n g^{ab} g^{cd})\}(\partial_{\vec{I}} k^{j}_{\ c})(\partial_a \partial_{\vec{I}} g_{bd}) \\
& - t^{2A+2} \{\partial_j (n g^{ab} g^{jc})\}(\partial_{\vec{I}} k^{d}_{\ c})(\partial_a \partial_{\vec{I}} g_{bd}).
\end{aligned} \tag{7.16b}$$

Proof. We view the right-hand sides of equations (7.12a)–(7.12b) as quadratic forms in $(\partial_{\vec{I}} k, \partial\partial_{\vec{I}} g, \partial\partial_{\vec{I}} n)$ with coefficients that depend on g, g^{-1}, and n. We now consider the expression $\partial_\alpha {}^{(\vec{I})}\mathbf{J}^\alpha$. On the right-hand side of (7.16b), we place all terms in which spatial derivatives ∂_j fall on the coefficients. In contrast, when ∂_t falls on g or g^{-1}, we use (2.12a)–(2.12b) to substitute for $\partial_t g$ and $\partial_t g^{-1}$ and then place the resulting terms on the right-hand side of (7.16a) (as the last four products). Next, we consider all of the terms in the expression $\partial_\alpha {}^{(\vec{I})}\mathbf{J}^\alpha$ in which a derivative falls on one of $\partial_{\vec{I}} k$, $\partial\partial_{\vec{I}} g$, or $\partial\partial_{\vec{I}} n$. For these terms, we use equations (7.4a)–(7.4b) and (7.6a)–(7.6b) for algebraic substitution, where $P = 1$ (by assumption) in (7.6a)–(7.6b). More precisely, in the expression $\partial_\alpha {}^{(\vec{I})}\mathbf{J}^\alpha$, we use equation (7.6b) to substitute for the factor $\partial_t (t^{A+1} \partial_{\vec{I}} k^{c}_{\ d})$ in

$$2 g_{ac} g^{bd} (t^{A+1} \partial_{\vec{I}} k^{a}_{\ b}) \partial_t (t^{A+1} \partial_{\vec{I}} k^{c}_{\ d}),$$

equation (7.6a) to substitute for the factor $\partial_t (t^{A+1} \partial_{\vec{I}} \partial_d \partial g_{ef})$ in

$$\frac{1}{2} g^{ad} g^{be} g^{cf} (t^{A+1} \partial_{\vec{I}} \partial_a \partial g_{bc}) \partial_t (t^{A+1} \partial_{\vec{I}} \partial_d \partial g_{ef}),$$

equation (7.4a) to substitute for the factor $(t^{A+1} \partial_j \partial_{\vec{I}} k^{j}_{\ a})$ in

$$2 g^{ab} (t^{A+1} \partial_j \partial_{\vec{I}} k^{j}_{\ a})(t^{A+1} \partial_b \partial_{\vec{I}} n),$$

the factor $(t^{A+1} \partial_j \partial_{\vec{I}} k^{j}_{\ c})$ in

$$n g^{ab} g^{cd} (t^{A+1} \partial_j \partial_{\vec{I}} k^{j}_{\ c})(t^{A+1} \partial_d \partial_{\vec{I}} g_{ab}),$$

and the factor $(t^{A+1}\partial_j\partial_{\vec{I}}k^j{}_c)$ in

$$-ng^{ab}g^{cd}(t^{A+1}\partial_j\partial_{\vec{I}}k^j{}_c)(t^{A+1}\partial_a\partial_{\vec{I}}g_{bd}),$$

and equation (7.4b) to substitute for the factor $g^{jc}(t^{A+1}\partial_j\partial_{\vec{I}}k^d{}_c)$ in

$$-ng^{ab}g^{jc}(t^{A+1}\partial_j\partial_{\vec{I}}k^d{}_c)(t^{A+1}\partial_a\partial_{\vec{I}}g_{bd}).$$

These algebraic substitutions lead to the elimination of all products that depend on a derivative of $(\partial_{\vec{I}}k, \partial\partial_{\vec{I}}g, \partial\partial_{\vec{I}}n)$, at the expense of introducing products that depend on the inhomogeneous terms, for example

$$2g_{ac}g^{bd}(t^{A+1}\partial_{\vec{I}}k^a{}_b)^{(\text{Border};1;\vec{I})}\mathfrak{R}^c{}_d \quad \text{and} \quad 2g_{ac}g^{bd}(t^{A+1}\partial_{\vec{I}}k^a{}_b)^{(\text{Junk};1;\vec{I})}\mathfrak{R}^c{}_d.$$

We place the borderline error term products such as

$$2g_{ac}g^{bd}(t^{A+1}\partial_{\vec{I}}k^a{}_b)^{(\text{Border};1;\vec{I})}\mathfrak{R}^c{}_d$$

on the right-hand side of (7.16a) and the junk error term products such as

$$2g_{ac}g^{bd}(t^{A+1}\partial_{\vec{I}}k^a{}_b)^{(\text{Junk};1;\vec{I})}\mathfrak{R}^c{}_d$$

on the right-hand side of (7.16b). Moreover, as the first product on the right-hand side of (7.15), we place

$$\frac{2A}{t}|t^{A+1}\partial_{\vec{I}}k|_g^2,$$

which is generated by the first term on the right-hand side of (7.6b) when we use equation (7.6b) (with $P = 1$) to substitute for the factor $\partial_t(t^{A+1}\partial_{\vec{I}}k^c{}_d)$ in the expression

$$2g_{ac}g^{bd}(t^{A+1}\partial_{\vec{I}}k^a{}_b)\partial_t(t^{A+1}\partial_{\vec{I}}k^c{}_d).$$

Finally, as the second product on the right-hand side of (7.15), we place

$$\frac{A+1}{2t}|t^{A+1}\partial\partial_{\vec{I}}g|_g^2,$$

which is generated by the first term on the right-hand side of (7.6a) when we use equation (7.6a) to substitute for the factor $\partial_t(t^{A+1}\partial_{\vec{I}}\partial_d\partial g_{ef})$ in the expression

$$\frac{1}{2}g^{ad}g^{be}g^{cf}(t^{A+1}\partial_{\vec{I}}\partial_a\partial g_{bc})\partial_t(t^{A+1}\partial_{\vec{I}}\partial_d\partial g_{ef}).$$

In total, these steps yield the lemma. ∎

7.3. Control of the error terms

In this subsection, we derive estimates for the error terms that will arise when we derive energy estimates for solutions to the equations of Lemma 7.1.

7.3.1. Pointwise estimates for the error terms in the divergence of the energy current. In this subsubsection, we derive pointwise estimates for the error terms ${}^{(Border;\vec{I})}\mathfrak{J}$ and ${}^{(Junk;\vec{I})}\mathfrak{J}$ that appear in the expression (7.15) for $\partial_\alpha {}^{(\vec{I})}\mathbf{J}^\alpha$.

Lemma 7.17 (Pointwise estimates for the error terms in the divergence of the energy current). *Let $\vec{I}$ be a top-order spatial multi-index, that is, a multi-index with $|\vec{I}| = N$. Assume that the bootstrap assumptions* (3.18) *hold. There exist a constant $C_* > 0$ independent of N and A and a constant $C = C_{N,A,D,q,\sigma} > 0$ such that if N is sufficiently large in a manner that depends on A, D, q, and σ, and if ε is sufficiently small (in a manner that depends on N, A, D, q, and σ), then the error terms ${}^{(Border;\vec{I})}\mathfrak{J}$ and ${}^{(Junk;\vec{I})}\mathfrak{J}$ from* (7.16a) *and* (7.16b) *verify the following pointwise estimates for $t \in (T_{(Boot)}, 1]$:*

$$\begin{aligned} |{}^{(Border;\vec{I})}\mathfrak{J}| \leq{}& C_* t^{2A+1}|\partial_{\vec{I}} k|_g^2 + C_* t^{2A+1}|\partial\partial_{\vec{I}} g|_g^2 + C_* t^{2A+1}|\partial\partial_{\vec{I}} n|_g^2 \\ &+ C_* t|{}^{(Border;1;\vec{I})}\mathfrak{K}|_g^2 + C_* t|{}^{(Border;1;\vec{I})}\mathfrak{H}|_g^2 \\ &+ C_* t|{}^{(Border;\vec{I})}\mathfrak{M}|_g^2 + C_* t|{}^{(Border;\vec{I})}\widetilde{\mathfrak{M}}|_g^2, \end{aligned} \tag{7.18a}$$

$$\begin{aligned} |{}^{(Junk;\vec{I})}\mathfrak{J}| \leq{}& C t^{2A+1+\sigma}|\partial_{\vec{I}} k|_g^2 + C t^{2A+1+\sigma}|\partial\partial_{\vec{I}} g|_g^2 + C t^{2A+1+\sigma}|\partial\partial_{\vec{I}} n|_g^2 \\ &+ C t^{1-\sigma}|{}^{(Junk;1;\vec{I})}\mathfrak{K}|_g^2 + C t^{1-\sigma}|{}^{(Junk;1;\vec{I})}\mathfrak{H}|_g^2 \\ &+ C t^{1-\sigma}|{}^{(Junk;\vec{I})}\mathfrak{M}|_g^2 + C t^{1-\sigma}|{}^{(Junk;\vec{I})}\mathfrak{M}|_g^2 \\ &+ C t^{1-\sigma}|{}^{(Junk;\vec{I})}\widetilde{\mathfrak{M}}|_g^2. \end{aligned} \tag{7.18b}$$

Proof. Throughout this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta < \sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large.

Estimate (7.18a) follows in a straightforward fashion from applying the g-Cauchy–Schwarz inequality and Young's inequality (in the form $|ab| \leq \frac{1}{2}(a^2 + b^2)$) to the products on the right-hand side of (7.16a) and using the estimate $|g|_g = |g^{-1}|_g \leq C_*$ as well as the pointwise estimates $|n| \leq C_*$ and $|k|_g \leq C_* t^{-1}$, which follow from (3.2), Definition 3.14, and the bootstrap assumptions.

To prove (7.18b), we first note the estimate $\|n - 1\|_{L^\infty(\Sigma_t)} \leq C t^\sigma$, which follows from Definition 3.14, (3.2), and the bootstrap assumptions. Next, we use (4.4) with $l = 2$ and $m = 1$ to deduce

$$\|\partial g^{-1}\|_{L_g^\infty(\Sigma_t)} \leq t^{-3q}\|g^{-1}\|_{\dot{W}^{1,\infty}_{Frame}(\Sigma_t)}.$$

Also using (3.2) and (4.12b), we deduce that

$$\|\partial g^{-1}\|_{L_g^\infty(\Sigma_t)} \leq C t^{-5q-A\delta} \leq C t^{\sigma-1}.$$

Using similar reasoning and the bound (4.12c), we deduce that

$$\|\partial n\|_{L_g^\infty(\Sigma_t)} \leq C t^{2-11q-\sigma-A\delta} \leq C t^{\sigma-1}.$$

Using these three bounds to help control the $|\cdot|_g$ norms of the products on the right-hand side of (7.16b), we can derive inequality (7.18b) using arguments similar to the ones we used to deduce (7.18a). ∎

7.3.2. L^2 estimates for the error terms in the divergence of the energy current. In this subsubsection, we bound the error terms from Lemma 7.3 in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$.

Lemma 7.19 (L^2 control of the error terms in the top-order energy estimates for g and k). *Assume that the bootstrap assumptions (3.18) hold. There exists a constant $C_* > 0$ independent of N and A and a constant $C = C_{N,A,D,q,\sigma} > 0$ such that if N is sufficiently large in a manner that depends on A, D, q, and σ, and if ε is sufficiently small (in a manner that depends on N, A, D, q, and σ), then the following estimates hold for $t \in (T_{(\mathrm{Boot})}, 1]$.*

- **Borderline top-order error term estimates**. *For each top-order spatial multi-index $\vec{I}$ (i.e., $|\vec{I}| = N$), the following estimates hold for the error terms from (7.5a), (7.5c), (7.7a), and (7.8a), where $P = 1$ on the right-hand sides of (7.7a) and (7.8a):*

$$\|^{(\mathrm{Border};\vec{I})}\mathfrak{M}\|_{L^2_g(\Sigma_t)} \leq C_* \|t^A \partial\partial_{\vec{I}} g\|_{L^2_g(\Sigma_t)}, \tag{7.20a}$$

$$\|^{(\mathrm{Border};\vec{I})}\widetilde{\mathfrak{M}}\|_{L^2_g(\Sigma_t)} \leq C_* \|t^A \partial\partial_{\vec{I}} g\|_{L^2_g(\Sigma_t)}, \tag{7.20b}$$

$$\|^{(\mathrm{Border};1,\vec{I})}\mathfrak{H}\|_{L^2_g(\Sigma_t)} \leq C_* \|t^A \partial_{\vec{I}} k\|_{L^2_g(\Sigma_t)} + C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{7.20c}$$

$$\|^{(\mathrm{Border};1;\vec{I})}\mathfrak{K}\|_{L^2_g(\Sigma_t)} \leq C_* \|t^A \partial_{\vec{I}} k\|_{L^2_g(\Sigma_t)} + C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \tag{7.20d}$$

- **Non-borderline top-order error term estimates**. *The following estimates hold for the error terms from (7.5b), (7.5d), (7.7b), and (7.8b), where $P = 1$ on the right-hand sides of (7.7b) and (7.8b):*

$$\max_{|\vec{I}|=N} \|^{(\mathrm{Junk};\vec{I})}\mathfrak{M}\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{7.21a}$$

$$\max_{|\vec{I}|=N} \|^{(\mathrm{Junk};\vec{I})}\widetilde{\mathfrak{M}}\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{7.21b}$$

$$\max_{|\vec{I}|=N} \|^{(\mathrm{Junk};1;\vec{I})}\mathfrak{H}\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{7.21c}$$

$$\max_{|\vec{I}|=N} \|^{(\mathrm{Junk};1;\vec{I})}\mathfrak{K}\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \tag{7.21d}$$

Proof. Throughout this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta < \sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large. We also freely use the observations of Remark 3.13.

The proof of the lemma is lengthy but straightforward. The main task is to control derivatives of products in $\|\cdot\|_{L^2_g(\Sigma_t)}$. To help the reader navigate the estimates, we make the following remarks.

- To control products involving n, we use Proposition 5.1; this allows us to state all of our estimates in term of norms of g and k.
- The "Borderline" error terms that we have identified throughout the article are the easiest to estimate. The reason is that they are all of the schematic form Lowest $*$ Highest,

where "Lowest" denotes products of g, g^{-1}, and k, and "Highest" denotes a top-order term. We bound all of these products in $\|\cdot\|_{L^2_g(\Sigma_t)}$ by first bounding their $|\cdot|_g$ norms by $\lesssim |\text{Lowest}|_g \cdot |\text{Highest}|_g$, using that $|g|_g \lesssim 1$, $|g^{-1}|_g \lesssim 1$, $|k|_g \lesssim t^{-1}$ (where the last estimate is an easy consequence of the bootstrap assumptions), and then bounding

$$\sqrt{\int_{\Sigma_t} |\text{Highest}|_g^2 \, dx}$$

in terms of $\mathbb{H}_{(g,k)}(t)$ by directly appealing to Definition 3.16 if "Highest" is a top-order derivative of g of k, or by using Proposition 5.1 if "Highest" is a top-order derivative of n.

- We handle the "Junk" products on a case-by-case basis. To bound them in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$, we rely on all of the norms from Definitions 3.14 and 3.16, the interpolation results provided by Lemma 4.5, the product estimates provided by Lemma 4.8, and the Sobolev embedding results provided by Lemma 4.11. We highlight three crucial things that we have to carefully track:
  - (i) the number of factors of g and g^{-1} in the products (these factors can be hit with derivatives), since our framework allows for the possibility that each of these factors contributes at least a factor of t^{-2q} to the estimates;
  - (ii) the norm comparison results of Lemma 4.3, which show that additional factors of t^{-2q} can be generated in translating from the frame norms $|\cdot|_{\text{Frame}}$ to the geometric norms $|\cdot|_g$; and
  - (iii) the precise details of the way that the t-weights appear in Definitions 3.14 and 3.16.

Proof of (7.20a)–(7.20d). Let $\vec{I}$ be a spatial multi-index with $|\vec{I}| = N$. We first use Definition 3.14, the fact that $|g^{-1}|_g \leq C_*$, the bootstrap assumptions, and g-Cauchy–Schwarz to bound the product on the right-hand side of (7.5a) as follows:

$$|^{(\text{Border};\vec{I})}\mathfrak{M}|_g \leq C_* t^{A+1} \|k\|_{L^\infty_g(\Sigma_t)} |\partial\partial_{\vec{I}} g|_g \leq C_* t^A |\partial\partial_{\vec{I}} g|_g.$$

Taking the norm $\|\cdot\|_{L^2(\Sigma_t)}$ of this inequality, we conclude that

$$\|^{(\text{Border};\vec{I})}\mathfrak{M}\|_{L^2_g(\Sigma_t)} \leq C_* \|t^A \partial\partial_{\vec{I}} g\|_{L^2_g(\Sigma_t)}$$

as desired. Estimate (7.20b) follows similarly based on equation (7.5c), and we omit the details. The estimates (7.20c) and (7.20d) follow similarly based on equations (7.7a) and (7.8a) (with $P = 1$ by assumption) and the elliptic estimate (5.4), and we omit the details.

Proof of (7.21a)–(7.21b). We prove only (7.21b); estimate (7.21a) can be proved by applying a similar argument to the products on the right-hand side of (7.5b) and we omit those details. Let $\vec{I}$ be a spatial multi-index with $|\vec{I}| = N$. To obtain the desired bound for the sum on the right-hand side of (7.5d), we first consider the cases in which $|\vec{I}_1| = N$ or $|\vec{I}_2| = N$. Using that $|g^{-1}|_g \lesssim 1$ and g-Cauchy–Schwarz, we first bound the products under consideration in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+1} \|\partial g\|_{L^\infty_g(\Sigma_t)} \|k\|_{L^\infty_g(\Sigma_t)} \|g^{-1}\|_{\dot{H}^N_g(\Sigma_t)}.$$

Applying (4.4) to $\|\partial g\|_{L_g^\infty(\Sigma_t)}$ (with $l=0$ and $m=3$), we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+1-3q}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}(\Sigma_t)}}\|k\|_{L_g^\infty(\Sigma_t)}\|g^{-1}\|_{\dot{H}_g^N(\Sigma_t)}.$$

From Definitions 3.14 and 3.16, estimate (4.12a), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is $\lesssim t^{-6q-\sigma-A\delta}\mathbb{H}_{(g,k)}(t)$, which, in view of (3.2), is $\lesssim$ RHS (7.21b) as desired. We now consider the case in which $|\vec{I}_3| = N-1$ on the right-hand side of (7.5d). Using that $|g^{-1}|_g \lesssim 1$ and g-Cauchy–Schwarz, we first bound the products under consideration in the norm $\|\cdot\|_{L_g^2(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|g^{-1}\|_{\dot{W}_g^{1,\infty}(\Sigma_t)}\|k\|_{L_g^\infty(\Sigma_t)}\|\partial g\|_{\dot{H}_g^{N-1}(\Sigma_t)} + t^{A+1}\|k\|_{\dot{W}_g^{1,\infty}(\Sigma_t)}\|\partial g\|_{\dot{H}_g^{N-1}(\Sigma_t)}.$$

Applying (4.4) to $\|g^{-1}\|_{\dot{W}_g^{1,\infty}(\Sigma_t)}$ (with $l=2$ and $m=0$) and to $\|k\|_{\dot{W}_g^{1,\infty}(\Sigma_t)}$ (with $l=m=1$), we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+1-2q}\|g^{-1}\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{L_g^\infty(\Sigma_t)}\|\partial g\|_{\dot{H}_g^{N-1}(\Sigma_t)} + t^{A+1-2q}\|k\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}_g^{N-1}(\Sigma_t)}.$$

From Definitions 3.14 and 3.16, estimate (4.12b), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is $\lesssim t^{-6q-\sigma-A\delta}\mathbb{H}_{(g,k)}(t)$, which, in view of (3.2), is $\lesssim$ RHS (7.21b) as desired. We now consider the case in which $|\vec{I}_4| = N$ on the right-hand side of (7.5d). Using that $|g^{-1}|_g \lesssim 1$ and g-Cauchy–Schwarz, we first bound the products under consideration in the norm $\|\cdot\|_{L_g^2(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|\partial g\|_{L_g^\infty(\Sigma_t)}\|k\|_{\dot{H}_g^N(\Sigma_t)}.$$

Applying (4.4) to $\|\partial g\|_{L_g^\infty(\Sigma_t)}$ (with $l=0$ and $m=3$), we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+1-3q}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}_g^N(\Sigma_t)}.$$

From Definition 3.16, estimate (4.12a), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is $\lesssim t^{-5q-A\delta}\mathbb{H}_{(g,k)}(t)$, which, in view of (3.2), is $\lesssim$ RHS (7.21b) as desired. It remains for us to consider the remaining terms on the right-hand side of (7.5d), in which $|\vec{I}_1|, |\vec{I}_2|, |\vec{I}_4| \leq N-1$ and $|\vec{I}_3| \leq N-2$. We first use inequality (4.10) with $l=1$ and $m=0$ (since ${}^{(\text{Junk};\vec{I})}\widetilde{\mathfrak{M}}^i$ is a type $\binom{1}{0}$ tensorfield) and (4.6) to bound (using that $|\vec{I}| = N$) the terms under consideration as follows:

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I}\\ |\vec{I}_1|,|\vec{I}_2|,|\vec{I}_4|\leq N-1\\ |\vec{I}_3|\leq N-2}} t^{A+1}\|(\partial_{\vec{I}_1}g^{-1})(\partial_{\vec{I}_2}g^{-1})(\partial\partial_{\vec{I}_3}g)\partial_{\vec{I}_4}k\|_{L_g^2(\Sigma_t)} \\
&\lesssim t^{A+1-q}\|g^{-1}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} \\
&\quad + t^{A+1-q}\|g^{-1}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} \\
&\quad + t^{A+1-q}\|g^{-1}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} \\
&\quad + t^{A+1-q}\|g^{-1}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}.
\end{aligned} \tag{7.22}$$

From Definitions 3.14 and 3.16, estimates (4.2a), (4.2b), (4.12a), and (4.12b), and the bootstrap assumptions, we deduce that

$$\text{RHS (7.22)} \lesssim t^{1-10q-3\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21b) as desired.

Proof of (7.21c). We stress that for this estimate, on the right-hand side of (7.7b), we have $P = 1$ and $|\vec{I}| = N$.

To bound the first product on the right-hand side of (7.7b), we first use g-Cauchy–Schwarz to deduce that it is bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\leq t^{A+1}\|n-1\|_{L^\infty(\Sigma_t)}\|k\|_{L^\infty_g(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definitions 3.14 and 3.16, (3.2), and the bootstrap assumptions, we see that the right-hand side of the previous expression is

$$\lesssim t^{1-10q-\sigma}\mathbb{H}_{(g,k)}(t) \lesssim t^{\sigma-1}\mathbb{H}_{(g,k)}(t)$$

as desired.

To complete the proof of (7.21c), we must bound the three sums on the right-hand side of (7.7b) in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$. To bound the first sum on the right-hand side of (7.7b), we first consider the products with $|\vec{I}_3| = N$. Using that $|g|_g \lesssim 1$ and g-Cauchy–Schwarz, we first bound the products under consideration in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|\partial n\|_{L^\infty_g(\Sigma_t)}\|k\|_{\dot{H}^N_g(\Sigma_t)}.$$

Applying (4.4) to $\|\partial n\|_{L^\infty_g(\Sigma_t)}$ (with $l = 0$ and $m = 1$), we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+1-q}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|k\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definition 3.16, estimate (4.12c), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is

$$\lesssim t^{2-11q-\sigma-A\delta}\mathbb{H}_{(g,k)}(t),$$

which, in view of (3.2), is $\lesssim$ RHS (7.21c) as desired. It remains for us to bound the products in the first sum on the right-hand side of (7.7b) with $|\vec{I}_1|, |\vec{I}_3| \leq N-1$. We first use inequality (4.10) with $l = 0$ and $m = 3$ (since ${}^{(\text{Junk};P;\vec{I})}\mathfrak{H}$ is a type $\binom{0}{3}$ tensorfield) and (4.6) to deduce (using that $|\vec{I}| = N$) that the products under consideration are bounded as follows:

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_1|,|\vec{I}_3|\leq N-1}} t^{A+1}\|(\partial\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g)\partial_{\vec{I}_3}k\|_{L^2_g(\Sigma_t)} \qquad (7.23)\\
&\quad\lesssim t^{A+1-3q}\|n-1\|_{W^{2,\infty}(\Sigma_t)}\|g\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)}\\
&\qquad + t^{A+1-3q}\|n-1\|_{W^{2,\infty}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)}\\
&\qquad + t^{A+1-3q}\|g\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}\\
&\qquad + t^{A+1-3q}\|n-1\|_{W^{2,\infty}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}.
\end{aligned}$$

From Definitions 3.14 and 3.16, estimates (4.2a), (4.2b), (4.12a), and (4.12c), the elliptic estimates (5.2) and (5.4), and the bootstrap assumptions, we deduce that

$$\begin{aligned}\text{RHS (7.23)} \lesssim t^{3-18q-2\sigma-A\delta}&\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}\\&+t^{2-15q-2\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}\\&+t^{-5q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.\end{aligned} \tag{7.24}$$

In view of (3.2), we see that RHS (7.24) $\lesssim$ RHS (7.21c) as desired.

To bound the second sum on the right-hand side of (7.7b), we first consider the products with $|\vec{I}_3| = N$. Using g-Cauchy–Schwarz, we first bound the products under consideration in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|n\|_{L^\infty(\Sigma_t)}\|\partial g\|_{L^\infty_g(\Sigma_t)}\|k\|_{\dot{H}^N_g(\Sigma_t)}.$$

Applying (4.4) to $\|\partial g\|_{L^\infty_g(\Sigma_t)}$ (with $l = 0$ and $m = 3$), we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+1-3q}\|n\|_{L^\infty(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^N_g(\Sigma_t)}.$$

From (3.2), Definitions 3.14 and 3.16, estimate (4.12a), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is $\lesssim t^{-5q-A\delta}\mathbb{H}_{(g,k)}(t)$, which, in view of (3.2), is $\lesssim$ RHS (7.21c) as desired. It remains for us to consider the remaining terms in the second sum on the right-hand side of (7.7b), in which $|\vec{I}_2|, |\vec{I}_3| \le N-1$. We first use (4.10) with $l = 0$ and $m = 3$ (since ${}^{(\text{Junk};\,P;\,\vec{I})}\mathfrak{H}$ is a type $\binom{0}{3}$ tensorfield) and (4.6) to deduce (using that $|\vec{I}| = N$) that the terms under consideration are bounded as follows:

$$\begin{aligned}\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_2|,|\vec{I}_3|\le N-1}} & t^{A+1}\|(\partial_{\vec{I}_1}n)(\partial\partial_{\vec{I}_2}g)\partial_{\vec{I}_3}k\|_{L^2_g(\Sigma_t)}\\ &\lesssim t^{A+1-3q}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)}\\ &\quad+t^{A+1-3q}\|n\|_{W^{1,\infty}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)}\\ &\quad+t^{A+1-3q}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}\\ &\quad+t^{A+1-3q}\|n\|_{W^{1,\infty}(\Sigma_t)}\|k\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}.\end{aligned} \tag{7.25}$$

From (3.2), Definitions 3.14 and 3.16, estimates (4.2b), (4.12a), and (4.12c), and the bootstrap assumptions, we deduce that

$$\text{RHS (7.25)} \lesssim t^{1-8q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}+t^{-5q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21c) as desired.

To bound the last sum on the right-hand side of (7.7b), we first consider the products with $|\vec{I}_3| = N-1$. Using that $|g|_g \lesssim 1$ and g-Cauchy–Schwarz, we first bound the products under consideration in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|\partial k\|_{\dot{H}^{N-1}_g(\Sigma_t)}+t^{A+1}\|n\|_{L^\infty(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_g(\Sigma_t)}\|\partial k\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

Using (4.4) (with $l = 0$ and $m = 2$) to estimate $\|g\|_{\dot{W}_g^{1,\infty}(\Sigma_t)}$, and using the definition of the norms $\|\cdot\|_{\dot{H}_g^M(\Sigma_t)}$ and $\|\cdot\|_{L^2_{\text{Frame}}(\Sigma_t)}$, we deduce (see Remark 3.13) that the right-hand side of the previous expression is

$$\lesssim t^{A+1}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}_g^N(\Sigma_t)} + t^{A+1-2q}\|n\|_{L^\infty(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}_g^N(\Sigma_t)}.$$

From (3.2), Definitions 3.14 and 3.16, estimates (4.2a), (4.12a) and (4.12c), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is

$$\lesssim t^{2-11q-\sigma-A\delta}\mathbb{H}_{(g,k)}(t) + t^{-5q-A\delta}\mathbb{H}_{(g,k)}(t),$$

which, in view of (3.2), is $\lesssim$ RHS (7.21c) as desired. It remains for us to consider the remaining terms in the last sum on the right-hand side of (7.7b), in which $|\vec{I}_3| \leq N-2$. We first use (4.10) with $l = 0$ and $m = 3$ (since ${}^{(\text{Junk};P;\vec{I})}\mathfrak{H}$ is a type $\binom{0}{3}$ tensorfield) and (4.6) to deduce (using that $|\vec{I}| = N$) that the terms under consideration are bounded as follows:

$$\begin{aligned}\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_3|\leq N-2}} & t^{A+1}\|(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g)\partial\partial_{\vec{I}_3}k\|_{L^2_g(\Sigma_t)}\\ &\lesssim t^{A+1-3q}\|n\|_{W^{2,\infty}(\Sigma_t)}\|g\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)}\\ &\quad + t^{A+1-3q}\|n\|_{W^{2,\infty}(\Sigma_t)}\|k\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)}\\ &\quad + t^{A+1-3q}\|g\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}\\ &\quad + t^{A+1-3q}\|n\|_{W^{2,\infty}(\Sigma_t)}\|k\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}.\end{aligned} \tag{7.26}$$

From (3.2), Definitions 3.14 and 3.16, estimates (4.2a), (4.12a), and (4.12c), and the bootstrap assumptions, we deduce that

$$\text{RHS (7.26)} \lesssim t^{1-8q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} + t^{-5q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21c) as desired.

Proof of (7.21d). We stress that for this estimate, on the right-hand side of (7.8b), we have $P = 1$ and $|\vec{I}| = N$.

To bound the first sum on the right-hand side of (7.8b), we first consider the case in which $|\vec{I}_2| = N$. Then the terms under consideration are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^A\|n-1\|_{L^\infty(\Sigma_t)}\|k\|_{\dot{H}_g^N(\Sigma_t)}.$$

From Definitions 3.14 and 3.16 and (3.2), we see that the right-hand side of the previous expression is

$$\lesssim t^{1-10q-\sigma}\mathbb{H}_{(g,k)}(t) \lesssim t^{\sigma-1}\mathbb{H}_{(g,k)}(t)$$

as desired. We now consider the remaining cases, in which $|\vec{I}_1| \leq N-1$ and $|\vec{I}_2| \leq N-1$. First, using (4.10) with $l = m = 1$ (since ${}^{(Junk;\,1;\,\vec{I})}\mathfrak{K}$ is a type $\binom{1}{1}$ tensorfield) and (4.6), we bound (using that $|\vec{I}| = N$) the products under consideration as follows:

$$\begin{aligned}
\sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I}\\ |\vec{I}_1|,|\vec{I}_2|\leq N-1}} & \|t^A\{\partial_{\vec{I}_1}(n-1)\}\partial_{\vec{I}_2}k\|_{L^2_g(\Sigma_t)} \\
& \lesssim t^{A-2q}\|n-1\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\mathrm{Frame}}(\Sigma_t)} \\
& \quad + t^{A-2q}\|k\|_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)}.
\end{aligned} \tag{7.27}$$

From Definitions 3.14 and 3.16, estimate (4.12c), and the bootstrap assumptions, we deduce that

$$\text{RHS (7.27)} \lesssim t^{2-15q-2\sigma-A\delta}\mathbb{H}_{(g,k)}(t) + t^{-3q}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21d) as desired.

To bound the second sum on the right-hand side of (7.8b), we first consider the cases in which $|\vec{I}_5| = N$ or $|\vec{I}_6| = N$. Using that $|g^{-1}|_g \lesssim 1$, we deduce that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|n\|_{L^\infty(\Sigma_t)}\|\partial g\|_{L^\infty_g(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.$$

Using (4.4) (with $l = 0$ and $m = 3$) to estimate $\|\partial g\|_{L^\infty_g(\Sigma_t)}$, we see that the right-hand side of the previous expression is

$$\lesssim t^{A+1-3q}\|n\|_{L^\infty(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.$$

From (3.2), Definitions 3.14 and 3.16, estimate (4.12a), and the bootstrap assumptions, we see that the right-hand side of the previous expression is $\lesssim t^{-5q-A\delta}\mathbb{H}_{(g,k)}(t)$. Using (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21d) as desired. We now consider the remaining cases, in which $|\vec{I}_5| \leq N-1$ and $|\vec{I}_6| \leq N-1$. Using (4.10) with $l = m = 1$ (since ${}^{(Junk;\,1;\,\vec{I})}\mathfrak{K}$ is a type $\binom{1}{1}$ tensorfield) and (4.6), we deduce (using that $|\vec{I}| = N$) that

$$\begin{aligned}
\sum_{\substack{\vec{I}_1+\vec{I}_2+\cdots+\vec{I}_6=\vec{I}\\ |\vec{I}_5|,|\vec{I}_6|\leq N-1}} & t^{A+1}\|(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g^{-1})(\partial_{\vec{I}_3}g^{-1})(\partial_{\vec{I}_4}g^{-1})(\partial\partial_{\vec{I}_5}g)\partial\partial_{\vec{I}_6}g\|_{L^2_g(\Sigma_t)} \\
& \lesssim t^{A+1-2q}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^3_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_t)} \\
& \quad + t^{A+1-2q}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_t)} \\
& \quad + t^{A+1-2q}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g^{-1}\|^3_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)} \\
& \quad + t^{A+1-2q}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^3_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{\mathrm{Frame}}(\Sigma_t)}.
\end{aligned} \tag{7.28}$$

From (3.2), Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), the elliptic estimate (5.4), and the bootstrap assumptions, we deduce that

$$\text{RHS (7.28)} \lesssim t^{1-12q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21d) as desired.

To bound the third sum on the right-hand side of (7.8b), we first consider the case in which $|\vec{I}_4| = N-1$. Using that $|g^{-1}|_g \lesssim 1$ and g-Cauchy–Schwarz, we deduce that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|\partial^2 g\|_{\dot{H}^{N-1}_g(\Sigma_t)} + t^{A+1}\|n\|_{L^\infty(\Sigma_t)}\|\partial g\|_{L^\infty_g(\Sigma_t)}\|\partial^2 g\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

Using (4.4) (with $l=0$ and $m=3$) to estimate $\|\partial g\|_{L^\infty_g(\Sigma_t)}$, and using the definition of the norms $\|\cdot\|_{L^\infty_{\text{Frame}}(\Sigma_t)}$, $\|\cdot\|_{L^\infty_g(\Sigma_t)}$, and $\|\cdot\|_{\dot{H}^M_g(\Sigma_t)}$, we deduce (see Remark 3.13) that the right-hand side of the previous expression is

$$\begin{aligned}
&\lesssim t^{A+1}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)} \\
&\quad + t^{A+1-3q}\|n\|_{L^\infty(\Sigma_t)}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.
\end{aligned}$$

From (3.2), Definitions 3.14 and 3.16, estimates (4.2a), (4.12a) and (4.12c), and the bootstrap assumptions, we see that the right-hand side of the previous expression is

$$\lesssim t^{2-11q-\sigma-A\delta}\mathbb{H}_{(g,k)}(t) + t^{-6q-A\delta}\mathbb{H}_{(g,k)}(t).$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21d) as desired. We now consider the remaining cases, in which $|\vec{I}_4| \leq N-2$. Using (4.10) with $l=m=1$ (since ${}^{(Junk;1;\vec{I})}\mathfrak{K}$ is a type $\binom{1}{1}$ tensorfield) and (4.6), we deduce (using that $|\vec{I}| = N$) that

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I} \\ |\vec{I}_4|\leq N-2}} t^{A+1}\|(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g^{-1})(\partial_{\vec{I}_3}g^{-1})\partial^2\partial_{\vec{I}_4}g\|_{L^2_g(\Sigma_t)} \\
&\quad \lesssim t^{A+1-2q}\|n\|_{W^{2,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)} \\
&\qquad + t^{A+1-2q}\|n\|_{W^{2,\infty}(\Sigma_t)}\|g^{-1}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)} \\
&\qquad + t^{A+1-2q}\|g^{-1}\|^2_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}.
\end{aligned} \tag{7.29}$$

From (3.2), Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), the elliptic estimate (5.4), and the bootstrap assumptions, we deduce that

$$\text{RHS (7.29)} \lesssim t^{1-8q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21d) as desired.

To bound the fourth sum on the right-hand side of (7.8b), we first consider the case in which $|\vec{I}_2| = N-1$. Using g-Cauchy–Schwarz, we deduce that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|g^{-1}\|_{\dot{W}^{1,\infty}_g(\Sigma_t)}\|\partial^2 n\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

Using (4.4) (with $l=2$ and $m=0$) to estimate $\|g^{-1}\|_{\dot{W}^{1,\infty}_g(\Sigma_t)}$, and using the definition of the norms $\|\cdot\|_{L^\infty_{\text{Frame}}(\Sigma_t)}$, $\|\cdot\|_{L^\infty_g(\Sigma_t)}$, and $\|\cdot\|_{\dot{H}^M_g(\Sigma_t)}$, we deduce (see Remark 3.13) that the right-hand side of the previous expression is

$$\lesssim t^{A+1-2q}\|g^{-1}\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|\partial n\|_{\dot{H}^N_g(\Sigma_t)}.$$

From (3.2), Definition 3.16, estimates (4.2a) and (4.12b), and the bootstrap assumptions, we see that the right-hand side of the previous expression is

$$\lesssim t^{-5q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21d) as desired. We now consider the remaining cases, in which $|\vec{I}_2|\leq N-2$. Using (4.10) with $l=m=1$ (since ${}^{(\text{Junk};1;\vec{I})}\mathfrak{K}$ is a type $\binom{1}{1}$ tensorfield) and (4.6), we deduce (using that $|\vec{I}|=N$) that

$$\begin{aligned}\sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I}\\|\vec{I}_2|\leq N-2}} & t^{A+1}\|(\partial_{\vec{I}_1}g^{-1})\partial^2\partial_{\vec{I}_2}n\|_{L^2_g(\Sigma_t)}\\ &\lesssim t^{A+1-2q}\|n\|_{\dot{W}^{2,\infty}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)}\\ &\quad+t^{A+1-2q}\|g^{-1}\|_{\dot{W}^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}.\end{aligned}\tag{7.30}$$

From (3.2), Definition 3.16, estimates (4.2a), (4.12b), and (4.12c), and the bootstrap assumptions, we deduce that

$$\text{RHS (7.30)}\lesssim t^{3-14q-2\sigma-A\delta}\mathbb{H}_{(g,k)}(t)+t^{1-4q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21d) as desired.

To bound the last sum on the right-hand side of (7.8b), we first consider the case in which $|\vec{I}_3|=N$. Using that $|g^{-1}|_g\lesssim 1$ and g-Cauchy–Schwarz, we deduce that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|\partial n\|_{L^\infty_g(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.$$

Next, using (4.4) (with $l=0$ and $m=1$) to estimate $\|\partial n\|_{L^\infty_g(\Sigma_t)}$, we see that the right-hand side of the previous expression is

$$\lesssim t^{A+1-q}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definition 3.16, estimate (4.12c), and the bootstrap assumptions, we see that the right-hand side of the previous expression is $\lesssim t^{2-11q-\sigma-A\delta}\mathbb{H}_{(g,k)}(t)$, which, in view of (3.2), is $\lesssim$ RHS (7.21d) as desired. We now consider the case in which $|\vec{I}_4|=N$. Using that $|g^{-1}|_g \lesssim 1$ and g-Cauchy–Schwarz, we deduce that the terms under consideration are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+1}\|\partial g\|_{L^\infty_g(\Sigma_t)}\|\partial n\|_{\dot{H}^N_g(\Sigma_t)}.$$

Next, using (4.4) (with $l=0$ and $m=3$) to estimate $\|\partial g\|_{L^\infty_g(\Sigma_t)}$, we see that the right-hand side of the previous expression is

$$\lesssim t^{A+1-3q}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|\partial n\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definition 3.16 and estimate (4.12a), we see that the right-hand side of the previous expression is

$$\lesssim t^{-5q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}\mathbb{H}_{(n)}(t)\lesssim t^{-5q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21d) as desired. We now consider the remaining cases, in which $|\vec{I}_3|\leq N-1$ and $|\vec{I}_4|\leq N-1$. Using (4.10) with $l=m=1$ (since ${}^{(\text{Junk};1;\vec{I})}\mathfrak{K}$ is a type $\binom{1}{1}$ tensor-field) and (4.6), we deduce (using that $|\vec{I}|=N$) that

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I}\\ |\vec{I}_3|,|\vec{I}_4|\leq N-1}} t^{A+1}\|(\partial_{\vec{I}_1}g^{-1})(\partial_{\vec{I}_2}g^{-1})(\partial\partial_{\vec{I}_3}g)\partial\partial_{\vec{I}_4}n\|_{L^2_g(\Sigma_t)}\\
&\quad\lesssim t^{A+1-2q}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}\\
&\qquad+t^{A+1-2q}\|n\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)}\\
&\qquad+t^{A+1-2q}\|n\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)}\\
&\qquad+t^{A+1-2q}\|n\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{\text{Frame}}(\Sigma_t)}.
\end{aligned}\tag{7.31}$$

From (3.2), Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), and the bootstrap assumptions, we deduce that

$$\text{RHS (7.31)}\lesssim t^{1-8q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.$$

In view of inequalities (3.2), we see that the right-hand side of the previous expression is $\lesssim$ RHS (7.21d) as desired. We have therefore proved (7.21d), which completes the proof of the lemma. ∎

7.4. Proof of Proposition 7.1

In this subsection, we prove Proposition 7.1. Throughout this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta<\sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large.

Let $\vec{I}$ be a spatial multi-index with $|\vec{I}|=N$ and let ${}^{(\vec{I})}\mathbf{J}$ be the energy current from Definition 7.11. Applying the divergence theorem on the region $[t,1]\times\mathbb{T}^D$, and considering equation (7.12a), we deduce that for $t \in (T_{(\mathrm{Boot})}, 1]$, we have

$$\int_{\Sigma_t} \left\{ |t^{A+1}\partial_{\vec{I}} k|_g^2 + \frac{1}{4}|t^{A+1}\partial\partial_{\vec{I}} g|_g^2 \right\} dx \\ = \int_{\Sigma_1} \left\{ |\partial_{\vec{I}} k|_g^2 + \frac{1}{4}|\partial\partial_{\vec{I}} g|_g^2 \right\} dx - \int_{s=t}^{1} \int_{\Sigma_s} \partial_\alpha {}^{(\vec{I})}\mathbf{J}^\alpha \, dx \, ds. \tag{7.32}$$

We now use equation (7.15) to substitute for the last integral on the right-hand side of (7.32), thereby obtaining

$$\begin{aligned} &\int_{\Sigma_t} \left\{ |t^{A+1}\partial_{\vec{I}} k|_g^2 + \frac{1}{4}|t^{A+1}\partial\partial_{\vec{I}} g|_g^2 \right\} dx \\ &= \int_{\Sigma_1} \left\{ |\partial_{\vec{I}} k|_g^2 + \frac{1}{4}|\partial\partial_{\vec{I}} g|_g^2 \right\} dx \\ &\quad - \int_{s=t}^{1} s^{-1} \int_{\Sigma_s} \left\{ 2A|s^{A+1}\partial_{\vec{I}} k|_g^2 + \frac{A+1}{2}|s^{A+1}\partial\partial_{\vec{I}} g|_g^2 \right\} dx \, ds \\ &\quad - \int_{s=t}^{1} \int_{\Sigma_s} {}^{(\mathrm{Border};\vec{I})}\mathfrak{J} \, dx \, ds - \int_{s=t}^{1} \int_{\Sigma_s} {}^{(\mathrm{Junk};\vec{I})}\mathfrak{J} \, dx \, ds. \end{aligned} \tag{7.33}$$

Next, we use the pointwise estimates (7.18a) and (7.18b) and the elliptic estimate (5.4) to deduce that the last two integrals on the right-hand side of (7.33) are bounded in magnitude by

$$\begin{aligned} &\leq C_* \int_{s=t}^{1} s^{-1} \left\{ \|s^{A+1}\partial_{\vec{I}} k\|_{L_g^2(\Sigma_s)}^2 + \frac{1}{4}\|s^{A+1}\partial\partial_{\vec{I}} g\|_{L_g^2(\Sigma_s)}^2 \right\} ds \\ &\quad + C_* \int_{s=t}^{1} s\|{}^{(\mathrm{Border};\vec{I})}\mathfrak{K}\|_{L_g^2(\Sigma_s)}^2 \, ds + C_* \int_{s=t}^{1} s\|{}^{(\mathrm{Border};\vec{I})}\mathfrak{H}\|_{L_g^2(\Sigma_s)}^2 \, ds \\ &\quad + C_* \int_{s=t}^{1} s\|{}^{(\mathrm{Border};\vec{I})}\mathfrak{M}\|_{L_g^2(\Sigma_s)}^2 \, ds + C_* \int_{s=t}^{1} s\|{}^{(\mathrm{Border};\vec{I})}\widetilde{\mathfrak{M}}\|_{L_g^2(\Sigma_s)}^2 \, ds \\ &\quad + C \int_{s=t}^{1} s^{1-\sigma}\|{}^{(\mathrm{Junk};\vec{I})}\mathfrak{K}\|_{L_g^2(\Sigma_s)}^2 \, ds + C \int_{s=t}^{1} s^{1-\sigma}\|{}^{(\mathrm{Junk};\vec{I})}\mathfrak{H}\|_{L_g^2(\Sigma_s)}^2 \, ds \\ &\quad + C \int_{s=t}^{1} s^{1-\sigma}\|{}^{(\mathrm{Junk};\vec{I})}\mathfrak{M}\|_{L_g^2(\Sigma_s)}^2 \, ds + C \int_{s=t}^{1} s^{1-\sigma}\|{}^{(\mathrm{Junk};\vec{I})}\widetilde{\mathfrak{M}}\|_{L_g^2(\Sigma_s)}^2 \, ds \\ &\quad + C \int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}^2(s) + \mathbb{H}_{(g,k)}^2(s)\} \, ds. \end{aligned} \tag{7.34}$$

Next, we use the estimates (7.20a)–(7.21d) to bound the terms on lines two to five of the right-hand side of (7.34). Also noting that the term

$$\int_{\Sigma_1} \left\{ |\partial_{\vec{I}} k|_g^2 + \frac{1}{4}|\partial\partial_{\vec{I}} g|_g^2 \right\} dx$$

on the right-hand side of (7.33) is $\leq C\mathbb{H}_{(g,k)}^2(1)$, we arrive at the desired bound (7.2).

8. Estimates for the near-top-order derivatives of g and k

Our primary goal in this section is to prove the following proposition, which provides the main integral inequalities that we will use to control the near-top-order derivatives of g and k; recall that, as we explained near the end of Section 1.5, for technical reasons, we need these estimates to close our bootstrap argument. The proof of the proposition is located in Section 8.3. In Sections 8.1–8.2, we provide the identities and estimates that we will use when proving the proposition.

Proposition 8.1 (Integral inequalities for the near-top-order derivatives of g and k). *We assume that the bootstrap assumptions* (3.18) *hold. There exist a constant* $C_* > 0$ independent of N and A *and a constant* $C = C_{N,A,D,q,\sigma} > 0$ *such that if* N *is sufficiently large in a manner that depends on* A, D, q, *and* σ, *and if* ε *is sufficiently small (in a manner that depends on* N, A, D, q, *and* σ*), then the following integral inequalities hold for* $t \in (T_{(\mathrm{Boot})}, 1]$:

$$\|t^{A+2q+\sigma} g\|^2_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_t)} \leq C\mathbb{H}^2_{(g,k)}(1) - 2A \int_{s=t}^1 s^{-1} \|s^{A+2q+\sigma} g\|^2_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_s)}\, ds + C \int_{s=t}^1 s^{\sigma-1} \{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds, \tag{8.2a}$$

$$\|t^{A+2q+\sigma} g^{-1}\|^2_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_t)} \leq C\mathbb{H}^2_{(g,k)}(1) - 2A \int_{s=t}^1 s^{-1} \|s^{A+2q+\sigma} g^{-1}\|^2_{\dot{H}^N_{\mathrm{Frame}}(\Sigma_s)}\, ds + C \int_{s=t}^1 s^{\sigma-1} \{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds, \tag{8.2b}$$

and

$$\|t^{A+q+\sigma} g\|^2_{\dot{H}^N_g(\Sigma_t)} \leq C\mathbb{H}^2_{(g,k)}(1) - 2A \int_{s=t}^1 s^{-1} \|s^{A+q+\sigma} g\|^2_{\dot{H}^N_g(\Sigma_s)}\, ds + C \int_{s=t}^1 s^{\sigma-1} \{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds, \tag{8.3a}$$

$$\|t^{A+q+\sigma} g^{-1}\|^2_{\dot{H}^N_g(\Sigma_t)} \leq C\mathbb{H}^2_{(g,k)}(1) - 2A \int_{s=t}^1 s^{-1} \|s^{A+q+\sigma} g^{-1}\|^2_{\dot{H}^N_g(\Sigma_s)}\, ds + C \int_{s=t}^1 s^{\sigma-1} \{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds, \tag{8.3b}$$

and

$$\|t^{A+2q+\sigma} \partial g\|^2_{\dot{H}^{N-1}_g(\Sigma_t)} \leq C\mathbb{H}^2_{(g,k)}(1) - \{2A - C_*\} \int_{s=t}^1 s^{-1} \|s^{A+2q+\sigma} \partial g\|^2_{\dot{H}^{N-1}_g(\Sigma_s)}\, ds + C \int_{s=t}^1 s^{\sigma-1} \{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds, \tag{8.4}$$

and

$$\begin{aligned}
&\|t^{A+5q+3\sigma-1}g\|^2_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} \\
&\quad \leq C\mathbb{H}^2_{(g,k)}(1) - \{2A - C_*\}\int_{s=t}^{1} s^{-1}\|s^{A+5q+3\sigma-1}g\|^2_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_s)}\,ds \\
&\qquad + C\int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\,ds, \qquad (8.5a)
\end{aligned}$$

$$\begin{aligned}
&\|t^{A+5q+3\sigma-1}g^{-1}\|^2_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} \\
&\quad \leq C\mathbb{H}^2_{(g,k)}(1) - \{2A - C_*\}\int_{s=t}^{1} s^{-1}\|s^{A+5q+3\sigma-1}g^{-1}\|^2_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_s)}\,ds \\
&\qquad + C\int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\,ds, \qquad (8.5b)
\end{aligned}$$

and

$$\begin{aligned}
\|t^{A+3q+\sigma}k\|^2_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} &\leq C\mathbb{H}^2_{(g,k)}(1) - \{2A - C_*\}\int_{s=t}^{1} s^{-1}\|s^{A+3q+\sigma}k\|^2_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_s)}\,ds \\
&\quad + \int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\,ds, \qquad (8.6a)
\end{aligned}$$

and

$$\begin{aligned}
\|t^{A+3q+\sigma}k\|^2_{\dot{H}^{N-1}_{g}(\Sigma_t)} &\leq C\mathbb{H}^2_{(g,k)}(1) - \{2A - C_*\}\int_{s=t}^{1} s^{-1}\|s^{A+3q+\sigma}k\|^2_{\dot{H}^{N-1}_{g}(\Sigma_s)}\,ds \\
&\quad + \int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\,ds. \qquad (8.6b)
\end{aligned}$$

8.1. *The equations*

In this subsection, we derive the evolution equations verified by the near-top-order derivatives of g.

Remark 8.7. Different from the case of g, to prove the desired estimates in Section 8, we do not need to derive a new evolution equation for the derivatives of k; it will suffice for us to use equation (7.6b). We note, however, that in Section 8, we choose the constant P from equation (7.6b) to have a smaller value than it did in Section 7. This corresponds to the fact that our estimates for the below-top-order derivatives for k are less singular as $t \downarrow 0$ compared to our top-order estimates.

Lemma 8.8 (The equations). *Let $\vec{I}$ be a spatial multi-index and let $P \geq 0$ be a constant. Then the following* ***commuted metric evolution equations*** *hold:*

$$\partial_t(t^{A+P}\partial_{\vec{I}}g_{ij}) = \frac{1}{t}\{(A+P)\delta^a_{\ j} - 2ntk^a_{\ j}\}(t^{A+P}\partial_{\vec{I}}g_{ia}) + {}^{(P;\vec{I})}\mathfrak{G}_{ij}, \qquad (8.9a)$$

$$\partial_t(t^{A+P}\partial_{\vec{I}}g^{ij}) = \frac{1}{t}\{(A+P)\delta^j_{\ a} + 2ntk^j_{\ a}\}(t^{A+P}\partial_{\vec{I}}g^{ia}) + {}^{(P;\vec{I})}\widetilde{\mathfrak{G}}^{ij}, \qquad (8.9b)$$

where

$$ {}^{(P;\vec{I})}\mathfrak{G}_{ij} \simeq \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_2|\leq|\vec{I}|-1}} t^{A+P}(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g)\partial_{\vec{I}_3}k, \tag{8.10a} $$

$$ {}^{(P;\vec{I})}\widetilde{\mathfrak{G}}^{ij} \simeq \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_2|\leq|\vec{I}|-1}} t^{A+P}(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g^{-1})\partial_{\vec{I}_3}k. \tag{8.10b} $$

Proof. Equation (8.9a) follows in a straightforward fashion from commuting equation (2.12a) first with $\partial_{\vec{I}}$, and then with t^{A+P}. Equation (8.9b) follows from applying the same procedure to equation (2.12b). ■

8.2. *Control of the error terms in the near-top-order energy estimates*

We now bound various L^2 norms of the error terms from Lemma 8.8 in terms of the solution norms.

Lemma 8.11 (L^2 control of the error terms in the near-below-top-order energy estimates for the g and k). *Assume that the bootstrap assumptions* (3.18) *hold. If N is sufficiently large in a manner that depends on A, D, q, and σ, then there exists a constant $C = C_{N,A,D,q,\sigma} > 0$ such that if ε is sufficiently small (in a manner that depends on N, A, D, q, and σ), then the following estimates hold for $t \in (T_{(Boot)}, 1]$.*

- **Error term estimates for the near-top-order derivatives of g.** *The following estimates hold for the error terms from* (8.10a)–(8.10b):

$$ \max_{|\vec{I}|=N} \|{}^{(2q+\sigma;\vec{I})}\mathfrak{G}\|_{L^2_{Frame}(\Sigma_t)} \leq Ct^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{8.12a} $$

$$ \max_{|\vec{I}|=N} \|{}^{(2q+\sigma;\vec{I})}\widetilde{\mathfrak{G}}\|_{L^2_{Frame}(\Sigma_t)} \leq Ct^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{8.12b} $$

$$ \max_{|\vec{I}|=N} \|{}^{(q+\sigma;\vec{I})}\mathfrak{G}\|_{L^2_g(\Sigma_t)} \leq Ct^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{8.13a} $$

$$ \max_{|\vec{I}|=N} \|{}^{(q+\sigma;\vec{I})}\widetilde{\mathfrak{G}}\|_{L^2_g(\Sigma_t)} \leq Ct^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \tag{8.13b} $$

Furthermore, the following estimates hold for the error terms from (8.10a)–(8.10b):

$$ \max_{|\vec{I}|=N-1} \|{}^{(A+5q+3\sigma-1;\vec{I})}\mathfrak{G}\|_{L^2_{Frame}(\Sigma_t)} \leq C\varepsilon t^{A+5q+3\sigma-2}\|g\|_{\dot{H}^{N-1}_{Frame}(\Sigma_t)} + Ct^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{8.14a} $$

$$ \max_{|\vec{I}|=N-1} \|{}^{(A+5q+3\sigma-1;\vec{I})}\widetilde{\mathfrak{G}}\|_{L^2_{Frame}(\Sigma_t)} \leq C\varepsilon t^{A+5q+3\sigma-2}\|g^{-1}\|_{\dot{H}^{N-1}_{Frame}(\Sigma_t)} + Ct^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \tag{8.14b} $$

In addition, the following estimates hold for the error terms from (7.7a)–(7.7b):

$$\max_{|\vec{I}|=N-1} \|{}^{(\text{Border};2q+\sigma;\vec{I})}\mathfrak{H}\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{8.15a}$$

$$\max_{|\vec{I}|=N-1} \|{}^{(\text{Junk};2q+\sigma;\vec{I})}\mathfrak{H}\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \tag{8.15b}$$

Let $\vec{I}$ be a spatial multi-index with $|\vec{I}| = N-1$ and let T be the type $\binom{0}{3}$ Σ_t-tangent tensorfield with the following components relative to the transported spatial coordinates:

$$T_{eij} =: -2n g_{ia}\partial_e\partial_{\vec{I}} k^a{}_j.$$

Then the following estimate holds:

$$t^{A+2q+\sigma}\|T\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\mathbb{H}_{(g,k)}(t). \tag{8.16}$$

- **Error term estimates for the just-below-top-order derivatives of** k. *The following estimates hold for the error terms from* (7.8a)–(7.8b):

$$\max_{|\vec{I}|=N-1} \|{}^{(\text{Border};3q+\sigma;\vec{I})}\mathfrak{K}\|_{L^2_{\text{Frame}}(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{8.17a}$$

$$\max_{|\vec{I}|=N-1} \|{}^{(\text{Junk};3q+\sigma;\vec{I})}\mathfrak{K}\|_{L^2_{\text{Frame}}(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{8.17b}$$

$$\max_{|\vec{I}|=N-1} \|{}^{(\text{Border};3q+\sigma;\vec{I})}\mathfrak{K}\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}, \tag{8.17c}$$

$$\max_{|\vec{I}|=N-1} \|{}^{(\text{Junk};3q+\sigma;\vec{I})}\mathfrak{K}\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}. \tag{8.17d}$$

Let $\vec{I}$ be any spatial multi-index with $|\vec{I}| = N-1$ and let T be the type $\binom{1}{1}$ Σ_t-tangent tensorfield with the following components relative to the transported spatial coordinates:

$$\begin{aligned} T^i{}_j =: &-g^{ia}\partial_a\partial_j\partial_{\vec{I}} n \\ &+ \frac{1}{2} n g^{ic} g^{ab}\{\partial_a\partial_c\partial_{\vec{I}} g_{bj} + \partial_a\partial_j\partial_{\vec{I}} g_{bc} - \partial_a\partial_b\partial_{\vec{I}} g_{cj} - \partial_c\partial_j\partial_{\vec{I}} g_{ab}\}. \end{aligned}$$

Then the following estimates hold:

$$t^{A+3q+\sigma}\|T\|_{L^2_{\text{Frame}}(\Sigma_t)} \leq C t^{\sigma-1}\mathbb{H}_{(g,k)}(t), \tag{8.18a}$$

$$t^{A+3q+\sigma}\|T\|_{L^2_g(\Sigma_t)} \leq C t^{\sigma-1}\mathbb{H}_{(g,k)}(t). \tag{8.18b}$$

Proof. Throughout this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta < \sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large. We also freely use the observations of Remark 3.13.

Proof of (8.12a)–(8.12b). We first prove (8.12a). We stress that for this estimate, on the right-hand side of (8.10a), we have $P = 2q + \sigma$ and $|\vec{I}| = N$.

We first consider the case in which $|\vec{I}_1| = N$ on the right-hand side of (8.10a). Using now inequalities (4.4) with $l = 0$ and $m = 2$ (since ${}^{(2q+\sigma;\vec{I})}\mathfrak{G}$ is type $\binom{0}{2}$), the fact that $|g|_g \lesssim 1$, and g-Cauchy–Schwarz, we deduce that the products under consideration are bounded in the norm $\|\cdot\|_{L^2_{Frame}(\Sigma_t)}$ by

$$\begin{aligned}
&\lesssim t^{A+2q+\sigma}\|g\cdot k\|_{L^\infty_{Frame}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}\\
&\lesssim t^{A+\sigma}\|g\cdot k\|_{L^\infty_g(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}\\
&\lesssim t^{A+\sigma}\|k\|_{L^\infty_g(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}.
\end{aligned}$$

From Definitions 3.14 and 3.16, the elliptic estimate (5.4), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is

$$\lesssim t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}$$

as desired. We next consider the case in which $|\vec{I}_3| = N$ on the right-hand side of (8.10a). Using now inequalities (4.4) with $l = 0$ and $m = 2$ (since ${}^{(2q+\sigma;\vec{I})}\mathfrak{G}$ is type $\binom{0}{2}$), and g-Cauchy–Schwarz, we deduce that the products under consideration are bounded in the norm $\|\cdot\|_{L^2_{Frame}(\Sigma_t)}$ by

$$\lesssim t^{A+\sigma}\|n\|_{L^\infty(\Sigma_t)}\|k\|_{\dot{H}^N_g(\Sigma_t)}.$$

From inequalities (3.2), Definitions 3.14 and 3.16, and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is $\lesssim t^{\sigma-1}\mathbb{H}_{(g,k)}(t)$ as desired. It remains for us to consider the cases in which $|\vec{I}_1|, |\vec{I}_2|, |\vec{I}_3| \leq N-1$ on the right-hand side of (8.10a). We first use (4.6) and (4.9) to bound (using that $|\vec{I}| = N$) the products under consideration as follows:

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_1|,|\vec{I}_2|,|\vec{I}_3|\leq N-1}} t^{A+2q+\sigma}\|(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g)\partial_{\vec{I}_3}k\|_{L^2_{Frame}(\Sigma_t)}\\
&\quad\lesssim t^{A+2q+\sigma}\|n\|_{W^{1,\infty}_{Frame}(\Sigma_t)}\|k\|_{W^{1,\infty}_{Frame}(\Sigma_t)}\|g\|_{\dot{H}^{N-1}_{Frame}(\Sigma_t)}\\
&\qquad+ t^{A+2q+\sigma}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g\|_{W^{1,\infty}_{Frame}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{Frame}(\Sigma_t)}\\
&\qquad+ t^{A+2q+\sigma}\|g\|_{W^{1,\infty}_{Frame}(\Sigma_t)}\|k\|_{W^{1,\infty}_{Frame}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)}\\
&\qquad+ t^{A+2q+\sigma}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g\|_{W^{1,\infty}_{Frame}(\Sigma_t)}\|k\|_{W^{1,\infty}_{Frame}(\Sigma_t)}.
\end{aligned} \tag{8.19}$$

From (3.2), Definitions 3.14 and 3.16, estimates (4.2a), (4.2b), (4.12a), and (4.12c), the elliptic estimate (5.5), and the bootstrap assumptions, we deduce that

$$\text{RHS (8.19)} \lesssim t^{-3q-2\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, in view of inequalities (3.2), is $\lesssim$ RHS (8.12a) as desired. We have therefore proved (8.12a).

Estimate (8.12b) can be proved by applying nearly identical arguments to the right-hand side of (8.10b) (with $P = 2q + \sigma$ and $|\vec{I}| = N$), and we omit the details.

Proof of (8.13a)–(8.13b). We first prove (8.13a). We stress that for this estimate, on the right-hand side of (8.10a), we have $P = q + \sigma$ and $|\vec{I}| = N$.

To bound the right-hand side of (8.10a), we first consider the case in which $|\vec{I}_3| = N$. Using that $|g|_g \lesssim 1$ and g-Cauchy–Schwarz, we deduce that the products under consideration are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+q+\sigma}\|n\|_{L^\infty(\Sigma_t)}\|k\|_{\dot{H}^N_g(\Sigma_t)}.$$

Next, from inequalities (3.2), Definitions 3.14 and 3.16, and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is $\lesssim t^{q+\sigma-1}\mathbb{H}_{(g,k)}(t)$, which is $\lesssim$ RHS (8.13a) as desired. We now consider the case in which $|\vec{I}_1| = N$ on the right-hand side of (8.10a). Using that $|g|_g \lesssim 1$ and g-Cauchy–Schwarz, we deduce that the products under consideration are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+q+\sigma}\|k\|_{L^\infty_g(\Sigma_t)}\|n\|_{\dot{H}^N_g(\Sigma_t)}.$$

Using Definition 3.14, the bootstrap assumptions and the elliptic estimate (5.4), we deduce that the right-hand side of the previous expression is

$$\lesssim t^{q+\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which is $\lesssim$ RHS (8.13a) as desired. It remains for us to consider the remaining cases, in which $|\vec{I}_1|, |\vec{I}_2|, |\vec{I}_3| \le N-1$ on the right-hand side of (8.10a). We first use (4.4) with $l = 0$ and $m = 2$ (since ${}^{(q+\sigma;\vec{I})}\mathfrak{G}$ is type $\binom{0}{2}$) to deduce (using that $|\vec{I}| = N$) that the products under consideration are bounded as follows:

$$\begin{aligned}\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_1|,|\vec{I}_2|,|\vec{I}_3|\le N-1}} & t^{A+q+\sigma}\|(\partial_{\vec{I}_1} n)(\partial_{\vec{I}_2} g)\partial_{\vec{I}_3} k\|_{L^2_g(\Sigma_t)}\\ \lesssim & \sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_1|,|\vec{I}_2|,|\vec{I}_3|\le N-1}} t^{A-q+\sigma}\|(\partial_{\vec{I}_1} n)(\partial_{\vec{I}_2} g)\partial_{\vec{I}_3} k\|_{L^2_{\mathrm{Frame}}(\Sigma_t)}. \end{aligned} \tag{8.20}$$

Since the right-hand side of (8.20) is equal to t^{-3q} times the left-hand side of (8.19), the arguments surrounding (8.19) imply that

$$\text{RHS (8.20)} \lesssim t^{-6q-2\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, in view of inequalities (3.2), is $\lesssim$ RHS (8.13a) as desired. We have therefore proved (8.13a).

Estimate (8.13b) can be proved by applying nearly identical arguments to the products on the right-hand side of (8.10b) (with $P = q + \sigma$ and $|\vec{I}| = N$), and we omit the details.

Proof of (8.14a)–(8.14b). We first prove (8.14a). We stress that for this estimate, on the right-hand side of (8.10a), we have

$$P = A + 5q + 3\sigma - 1$$

and $|\vec{I}| = N - 1$.

We first use (4.6) and (4.9) to bound the terms on the right-hand side of (8.10a) as follows:

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_2|\leq N-2}} t^{A+5q+3\sigma-1}\|(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g)\partial_{\vec{I}_3}k\|_{L^2_{\text{Frame}}(\Sigma_t)}\\
&\lesssim t^{A+5q+3\sigma-1}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)}\\
&\quad + t^{A+5q+3\sigma-1}\|g\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)}\\
&\quad + t^{A+5q+3\sigma-1}\sum_{|\vec{I}_1|+|\vec{I}_2|=1}\|\partial_{\vec{I}_1}n\|_{L^\infty(\Sigma_t)}\|\partial_{\vec{I}_2}k\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^{N-2}_{\text{Frame}}(\Sigma_t)}\\
&\quad + t^{A+5q+3\sigma-1}\sum_{|\vec{I}_1|+|\vec{I}_2|=1}\|\partial_{\vec{I}_1}n\|_{L^\infty(\Sigma_t)}\|\partial_{\vec{I}_2}k\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|g\|_{L^\infty_{\text{Frame}}(\Sigma_t)}. \qquad (8.21)
\end{aligned}$$

From (3.2), Definitions 3.14 and 3.16, estimates (4.2a), (4.2b), and (4.12c), the elliptic estimates (5.2) and (5.5), and the bootstrap assumptions, we deduce that the products on the right-hand side of (8.21), except for the sum on the next-to-last line, are bounded by

$$\lesssim t^{2\sigma-1-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},$$

which is $\lesssim$ RHS (8.14a) as desired. To handle the remaining sum on the next-to-last line of the right-hand side of (8.21), we first note the bound

$$\sum_{|\vec{I}_1|+|\vec{I}_2|=1}\|\partial_{\vec{I}_1}n\|_{L^\infty(\Sigma_t)}\|\partial_{\vec{I}_1}k\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\lesssim \varepsilon t^{-1},$$

which follows from Definitions 3.14 and 3.16, (3.2), estimates (4.2b) and (4.12c), and the bootstrap assumptions. From this bound and the interpolation estimate (4.6), it follows that the sum on the next-to-last line of the right-hand side of (8.21) is

$$\begin{aligned}
&\lesssim \varepsilon t^{A+5q+3\sigma-2}\|g\|_{\dot{H}^{N-2}_{\text{Frame}}(\Sigma_t)}\\
&\lesssim \varepsilon t^{A+5q+3\sigma-2}\|g-\widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_t)}+\varepsilon t^{A+5q+3\sigma-2}\|g\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)}.
\end{aligned}$$

From Definition 3.14, it follows that since (by assumption) $A\geq 1$, the right-hand side of the previous estimate is

$$\lesssim \varepsilon t^{\sigma-1}\mathbb{L}_{(g,k)}(t)+\varepsilon t^{A+5q+3\sigma-2}\|g\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)},$$

which is $\lesssim$ RHS (8.14a) as desired. We have therefore proved (8.14a).

Estimate (8.14b) can be proved by applying nearly identical arguments to the right-hand side of (8.10b) (with $P=A+5q+3\sigma-1$ and $|\vec{I}|=N-1$), and we omit the details.

Proof of (8.15a). We stress that for this estimate, on the right-hand side of (7.7a), we have $P=2q+\sigma$ and $|\vec{I}|=N-1$.

Using that $|g|_g \lesssim 1$ and g-Cauchy–Schwarz, we deduce that the product on the right-hand side of (7.7a) is bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+2q+\sigma}\|k\|_{L^\infty_g(\Sigma_t)}\|\partial n\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

Using (4.4) (with $l = 0$ and $m = 1$) to estimate $\|\partial n\|_{\dot{H}^{N-1}_g(\Sigma_t)}$, we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+q+\sigma}\|k\|_{L^\infty_g(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}.$$

From Definition 3.14, the elliptic estimate (5.4) and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+q+\sigma-1}\|n\|_{\dot{H}^N(\Sigma_t)} \lesssim t^{q+\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which is $\lesssim$ RHS (8.15a) as desired.

Proof of (8.15b). We stress that for this estimate, on the right-hand side of (7.7b), we have $P = 2q + \sigma$ and $|\vec{I}| = N - 1$.

To bound the first product on the right-hand side of (7.7b), we first use g-Cauchy–Schwarz to deduce that it is bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\leq t^{A+2q+\sigma}\|n-1\|_{L^\infty(\Sigma_t)}\|k\|_{L^\infty_g(\Sigma_t)}\|\partial g\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

From Definitions 3.14 and 3.16 and the bootstrap assumptions, we see that the right-hand side of the previous expression is

$$\lesssim t^{1-10q-\sigma}\mathbb{H}_{(g,k)}(t),$$

which, in view of (3.2), is $\lesssim$ RHS (8.15b) as desired.

To bound the first sum on the right-hand side of (7.7b), we first use (4.10) with $l = 0$ and $m = 3$ (since ${}^{(\mathrm{Junk};\,2q+\sigma;\,\vec{I})}\mathfrak{H}$ is type $\binom{0}{3}$) and (4.6) to deduce (using that $|\vec{I}| = N - 1$) that the products under consideration are bounded as follows:

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3=\vec{I}\\ |\vec{I}_1|\leq N-2}} t^{A+2q+\sigma}\|(\partial\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g)\partial_{\vec{I}_3}k\|_{L^2_g(\Sigma_t)}\\
&\quad\lesssim t^{A-q+\sigma}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|g\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\mathrm{Frame}}(\Sigma_t)}\\
&\qquad + t^{A-q+\sigma}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^{N-1}_{\mathrm{Frame}}(\Sigma_t)}\\
&\qquad + t^{A-q+\sigma}\|g\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)}\\
&\qquad + t^{A-q+\sigma}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|g\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}\|k\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)}.
\end{aligned} \tag{8.22}$$

From Definitions 3.14 and 3.16, estimates (4.2a), (4.2b), (4.12a), and (4.12c), the elliptic estimates (5.2) and (5.5), and the bootstrap assumptions, we see that

$$\begin{aligned}
\text{RHS (8.22)} \lesssim{}& t^{2-16q-3\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}\\
&+ t^{-4q+\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},
\end{aligned}$$

which, in view of (3.2), is $\lesssim$ RHS (8.15b) as desired.

Using essentially the same reasoning, we find that the second and third sums on the right-hand side of (7.7b) are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by $\lesssim$ (8.22) and thus are $\lesssim$ RHS (8.15b) as well. We have therefore proved (8.15b).

Proof of (8.16). First, using that $|g|_g \lesssim 1$ and g-Cauchy–Schwarz, we deduce that

$$\|T\|_{L^2_g(\Sigma_t)} \lesssim t^{A+2q+\sigma}\|n\|_{L^\infty(\Sigma_t)}\|\partial k\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

Using the definition of the norms $\|\cdot\|_{\dot{H}^M_g(\Sigma_t)}$ and $\|\cdot\|_{L^2_{\text{Frame}}(\Sigma_t)}$, we deduce (see Remark 3.13) that the right-hand side of the previous expression is

$$\lesssim t^{A+2q+\sigma}\|n\|_{L^\infty(\Sigma_t)}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|k\|_{\dot{H}^N_g(\Sigma_t)}.$$

From (3.2), Definitions 3.14 and 3.16, estimate (4.2a), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is $\lesssim t^{q+\sigma-1}\mathbb{H}_{(g,k)}(t)$, which is $\lesssim$ RHS (8.16) as desired.

Proof of (8.17a) *and* (8.17c). We stress that for these estimates, on the right-hand side of (7.8a), we have $P = 3q + \sigma$ and $|\vec{I}| = N - 1$.

We first prove (8.17a). We start by noting that the right-hand side of (7.8a) is bounded in the norm $\|\cdot\|_{L^2_{\text{Frame}}(\Sigma_t)}$ by

$$\leq Ct^{A+3q+\sigma-1}\|k\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)}.$$

From Definitions 3.14 and 3.16, estimate (4.2b), the elliptic estimate (5.5), and the bootstrap assumptions, we find that the right-hand side of the previous expression is

$$\lesssim t^{2q+\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which is $\lesssim$ RHS (8.17a) as desired.

Estimate (8.17c) can be proved using a nearly identical argument, the key point being that like $\|k\|_{L^\infty_{\text{Frame}}(\Sigma_t)}$, the term $\|k\|_{L^\infty_g(\Sigma_t)}$ is bounded by $\lesssim t^{-1}$.

Proof of (8.17b). We stress that for this estimate, on the right-hand side of (7.8b), we have $P = 3q + \sigma$ and $|\vec{I}| = N - 1$.

To bound the first sum on the right-hand side of (7.8b), we first use (4.6) and (4.9) to bound (using that $|\vec{I}| = N - 1$) the terms under consideration as follows:

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I}\\ |\vec{I}_1|\leq N-2}} t^{A+3q+\sigma-1}\|\{\partial_{\vec{I}_1}(n-1)\}\partial_{\vec{I}_2}k\|_{L^2_{\text{Frame}}(\Sigma_t)} \\
&\quad \lesssim t^{A+3q+\sigma-1}\|n-1\|_{L^\infty(\Sigma_t)}\|k\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} \\
&\qquad + t^{A+3q+\sigma-1}\|k\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)} \\
&\qquad + t^{A+3q+\sigma-1}\|n-1\|_{L^\infty(\Sigma_t)}\|k\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}.
\end{aligned} \tag{8.23}$$

From Definitions 3.14 and 3.16 and the bootstrap assumptions, we see that

$$\text{RHS (8.23)} \lesssim t^{1-10q-\sigma}\mathbb{H}_{(g,k)}(t) + t^{2q+\sigma-1}\mathbb{L}_{(g,k)}(t),$$

which, in view of (3.2), is $\lesssim$ RHS (8.17b) as desired.

To bound the second sum on the right-hand side of (7.8b), we first consider the cases in which $|\vec{I}_5| = N-1$ or $|\vec{I}_6| = N-1$. Using that $|g^{-1}|_g \lesssim 1$, using (4.4) with $l = m = 1$ (since ${}^{(Junk;\, 3q+\sigma;\, \vec{I})}\mathfrak{K}$ is type $\binom{1}{1}$), and then again using (4.4) (this time with $l = 0$ and $m = 3$) to estimate the term $\|\partial g\|_{L^\infty_g(\Sigma_t)}$, we deduce that the products under consideration are bounded in the norm $\|\cdot\|_{L^2_{Frame}(\Sigma_t)}$ as follows:

$$\begin{aligned} &\lesssim t^{A+q+\sigma}\|n\|_{L^\infty(\Sigma_t)}\|\partial g\|_{L^\infty_g(\Sigma_t)}\|\partial g\|_{\dot{H}^{N-1}_g(\Sigma_t)} \\ &\lesssim t^{A-2q+\sigma}\|n\|_{L^\infty(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{Frame}(\Sigma_t)}\|\partial g\|_{\dot{H}^{N-1}_g(\Sigma_t)}. \end{aligned} \tag{8.24}$$

From (3.2), Definitions 3.14 and 3.16, estimate (4.12a), and the bootstrap assumptions, we see that

$$\text{RHS (8.24)} \lesssim t^{-6q-A\delta}\mathbb{H}_{(g,k)}(t),$$

which, in view of (3.2), is $\lesssim$ RHS (8.17b) as desired. It remains for us to consider the cases in which $|\vec{I}_5| \le N-2$ and $|\vec{I}_6| \le N-2$. Using (4.6) and (4.9), we bound (using that $|\vec{I}| = N-1$) the terms under consideration as follows:

$$\begin{aligned} &\sum_{\substack{\vec{I}_1+\vec{I}_2+\cdots+\vec{I}_6=\vec{I} \\ |\vec{I}_5|,|\vec{I}_6|\le N-2}} t^{A+3q+\sigma}\|(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g^{-1})(\partial_{\vec{I}_3}g^{-1})(\partial_{\vec{I}_4}g^{-1})(\partial\partial_{\vec{I}_5}g)\partial\partial_{\vec{I}_6}g\|_{L^2_{Frame}(\Sigma_t)} \\ &\lesssim t^{A+3q+\sigma}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^3_{W^{1,\infty}_{Frame}(\Sigma_t)}\|g-\widetilde{g}\|_{W^{2,\infty}_{Frame}(\Sigma_t)}\|g\|_{\dot{H}^{N-1}_{Frame}(\Sigma_t)} \\ &\quad + t^{A+3q+\sigma}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{Frame}(\Sigma_t)}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{Frame}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^{N-1}_{Frame}(\Sigma_t)} \\ &\quad + t^{A+3q+\sigma}\|g^{-1}\|^2_{W^{1,\infty}_{Frame}(\Sigma_t)}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{Frame}(\Sigma_t)}\|g^{-1}\|_{W^{1,\infty}_{Frame}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)} \\ &\quad + t^{A+3q+\sigma}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^3_{W^{1,\infty}_{Frame}(\Sigma_t)}\|g-\widetilde{g}\|^2_{W^{2,\infty}_{Frame}(\Sigma_t)}. \end{aligned} \tag{8.25}$$

From (3.2), Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), and the bootstrap assumptions, we see that

$$\text{RHS (8.25)} \lesssim t^{1-10q-2\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (8.17b) as desired.

To bound the third sum on the right-hand side of (7.8b), we first use (4.6) and (4.9) to bound (using that $|\vec{I}| = N-1$) the terms under consideration as follows:

$$\begin{aligned} &\sum_{\substack{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I} \\ |\vec{I}_4|\le N-2}} t^{A+3q+\sigma}\|(\partial_{\vec{I}_1}n)(\partial_{\vec{I}_2}g^{-1})(\partial_{\vec{I}_3}g^{-1})\partial^2\partial_{\vec{I}_4}g\|_{L^2_{Frame}(\Sigma_t)} \\ &\lesssim t^{A+3q+\sigma}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{Frame}(\Sigma_t)}\|g\|_{\dot{H}^N_{Frame}(\Sigma_t)} \\ &\quad + t^{A+3q+\sigma}\|g^{-1}\|^2_{W^{1,\infty}_{Frame}(\Sigma_t)}\|g\|_{\dot{W}^{2,\infty}_{Frame}(\Sigma_t)}\|n\|_{\dot{H}^{N-1}(\Sigma_t)} \\ &\quad + t^{A+3q+\sigma}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|_{W^{1,\infty}_{Frame}(\Sigma_t)}\|g\|_{\dot{W}^{2,\infty}_{Frame}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^N_{Frame}(\Sigma_t)} \\ &\quad + t^{A+3q+\sigma}\|n\|_{W^{1,\infty}(\Sigma_t)}\|g^{-1}\|^2_{W^{1,\infty}_{Frame}(\Sigma_t)}\|g\|_{\dot{W}^{2,\infty}_{Frame}(\Sigma_t)}. \end{aligned} \tag{8.26}$$

From (3.2), Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), and the bootstrap assumptions, we see that

$$\text{RHS (8.26)} \lesssim t^{-3q-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} + t^{1+\sigma-4q-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (8.17b) as desired.

To bound the fourth sum on the right-hand side of (7.8b), we first use (4.6) and (4.9) to bound (using that $|\vec{I}| = N-1$) the terms under consideration as follows:

$$\begin{aligned}
&\sum_{\substack{\vec{I}_1+\vec{I}_2=\vec{I} \\ |\vec{I}_2|\leq|\vec{I}|-1}} t^{A+3q+\sigma}\|(\partial_{\vec{I}_1} g^{-1})\partial^2\partial_{\vec{I}_2} n\|_{L^2_{\text{Frame}}(\Sigma_t)} \\
&\quad \lesssim t^{A+3q+\sigma}\|n\|_{\dot{W}^{2,\infty}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)} \\
&\qquad + t^{A+3q+\sigma}\|g^{-1}\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)}.
\end{aligned} \tag{8.27}$$

From (3.2), Definition 3.16, estimates (4.2a), (4.12b), and (4.12c), and the bootstrap assumptions, we see that

$$\text{RHS (8.27)} \lesssim t^{3-12q-3\sigma-A\delta}\mathbb{H}_{(g,k)}(t) + t^{q+\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (8.17b) as desired.

To bound the last sum on the right-hand side of (7.8b), we first use (4.9) to bound (using that $|\vec{I}| = N-1$) the terms under consideration as follows:

$$\begin{aligned}
&\sum_{\vec{I}_1+\vec{I}_2+\vec{I}_3+\vec{I}_4=\vec{I}} t^{A+3q+\sigma}\|(\partial_{\vec{I}_1} g^{-1})(\partial_{\vec{I}_2} g^{-1})(\partial\partial_{\vec{I}_3} g)\partial\partial_{\vec{I}_4} n\|_{L^2_{\text{Frame}}(\Sigma_t)} \\
&\lesssim t^{A+3q+\sigma}\|g^{-1}\|^2_{L^\infty_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|n\|_{\dot{H}^N(\Sigma_t)} \\
&\quad + t^{A+3q+\sigma}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|g^{-1}\|^2_{L^\infty_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{H}^N_{\text{Frame}}(\Sigma_t)} \\
&\quad + t^{A+3q+\sigma}\|n\|_{\dot{W}^{1,\infty}(\Sigma_t)}\|g^{-1}\|_{L^\infty_{\text{Frame}}(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|g^{-1}\|_{\dot{H}^{N-1}_{\text{Frame}}(\Sigma_t)}.
\end{aligned} \tag{8.28}$$

From Definition 3.16, estimates (4.2a) and (4.12a)–(4.12c), and the bootstrap assumptions, we see that

$$\begin{aligned}
\text{RHS (8.28)} \lesssim\ & t^{\sigma-3q-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} \\
& + t^{2-13q-\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\} \\
& + t^{3-16q-3\sigma-A\delta}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\}
\end{aligned} \tag{8.29}$$

which, in view of (3.2), is $\lesssim$ RHS (8.17b) as desired.

Proof of (8.17d). We stress that for this estimate, on the right-hand side of (7.8b), we have $P = 3q + \sigma$ and $|\vec{I}| = N - 1$.

We claim that we only have to bound (in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$) the second sum on the right-hand side of (7.8b) in the cases in which $|\vec{I}_5| = N-1$ or $|\vec{I}_6| = N-1$. For by inspecting the proof of (8.17b) given above, and using (3.2), we see that all remaining products on the right-hand side of (7.8b) are bounded in the norm $\|\cdot\|_{L^2_{\text{Frame}}(\Sigma_t)}$ by

$$\lesssim t^{2q+\sigma-1}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\}.$$

Hence, using (4.4) with $l=m=1$ (since ${}^{(\text{Junk};3q+\sigma;\vec{I})}\mathfrak{K}$ is type $\binom{1}{1}$), we find that these same products are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},$$

which is $\lesssim$ RHS (8.17d) as desired.

To handle the remaining cases in which $|\vec{I}_5|=N-1$ or $|\vec{I}_6|=N-1$ in the second sum on the right-hand side of (7.8b), we first use that $|g^{-1}|_g\lesssim 1$ and g-Cauchy–Schwarz to deduce that the products under consideration are bounded in the norm $\|\cdot\|_{L^2_g(\Sigma_t)}$ by

$$\lesssim t^{A+3q+\sigma}\|n\|_{L^\infty(\Sigma_t)}\|\partial g\|_{L^\infty_g(\Sigma_t)}\|\partial g\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

Using (4.4) (with $l=0$ and $m=3$) to estimate $\|\partial g\|_{L^\infty_g(\Sigma_t)}$, we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+\sigma}\|n\|_{L^\infty(\Sigma_t)}\|g\|_{\dot{W}^{1,\infty}_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

From (3.2), Definitions 3.14 and 3.16, estimate (4.12a), and the bootstrap assumptions, we deduce that the right-hand side of the previous expression is

$$\lesssim t^{-4q-A\delta}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},$$

which, in view of (3.2), is $\lesssim$ RHS (8.17d) as desired.

Proof of (8.18a). First, using (4.4) with $l=m=1$ (since T is a type $\binom{1}{1}$ tensorfield), we deduce that

$$t^{A+3q+\sigma}\|T\|_{L^2_{\text{Frame}}(\Sigma_t)}\lesssim t^{A+q+\sigma}\|T\|_{L^2_g(\Sigma_t)}.$$

Next, using that $|g^{-1}|_g\lesssim 1$, g-Cauchy–Schwarz, and the estimate $\|n\|_{L^\infty(\Sigma_t)}\lesssim 1$ (which is a simple consequence of (3.2), Definition 3.14, and the bootstrap assumptions), we find that

$$t^{A+q+\sigma}\|T\|_{L^2_g(\Sigma_t)}\lesssim t^{A+q+\sigma}\|\partial^2 n\|_{\dot{H}^{N-1}_g(\Sigma_t)}+t^{A+q+\sigma}\|\partial^2 g\|_{\dot{H}^{N-1}_g(\Sigma_t)}.$$

Using the definition of the norms $\|\cdot\|_{\dot{H}^M_g(\Sigma_t)}$ and $\|\cdot\|_{L^\infty_{\text{Frame}}(\Sigma_t)}$ (see Remark 3.13), we deduce that the right-hand side of the previous expression is

$$\lesssim t^{A+q+\sigma}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|\partial n\|_{\dot{H}^N_g(\Sigma_t)}+t^{A+q+\sigma}\|g^{-1}\|^{1/2}_{L^\infty_{\text{Frame}}(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)}.$$

From Definitions 3.14 and 3.16, (4.2a), the elliptic estimate (5.4), and the bootstrap assumptions, we find that the right-hand side of the previous expression is

$$\lesssim t^{\sigma-1}\{\mathbb{L}_{(g,k)}(t)+\mathbb{H}_{(g,k)}(t)\},$$

which is $\lesssim$ RHS (8.18a) as desired.

Proof of (8.18b). The arguments given in the proof of (8.18a) yield that

$$
\begin{aligned}
t^{A+3q+\sigma}\|T\|_{L^2_g(\Sigma_t)} &\lesssim t^{A+3q+\sigma}\|g^{-1}\|^{1/2}_{L^\infty_{Frame}(\Sigma_t)}\|\partial n\|_{\dot{H}^N_g(\Sigma_t)} \\
&\quad + t^{A+3q+\sigma}\|g^{-1}\|^{1/2}_{L^\infty_{Frame}(\Sigma_t)}\|\partial g\|_{\dot{H}^N_g(\Sigma_t)} \\
&\lesssim t^{2q+\sigma-1}\{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t)\},
\end{aligned}
$$

which is a better bound than we need. ■

8.3. Proof of Proposition 8.1

In this subsection, we prove Proposition 8.1. Throughout this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta < \sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large.

To prove (8.2a), we first use the fundamental theorem of calculus and the evolution equation (8.9a) with $P = 2q + \sigma$ to deduce that for any multi-index $\vec{I}$ with $|\vec{I}| = N$, we have (where we do not use Einstein summation over i, j and we stress that $0 < t \leq 1$)

$$
\begin{aligned}
\|t^{A+2q+\sigma}\partial_{\vec{I}} g_{ij}\|^2_{L^2(\Sigma_t)} &= \|\partial_{\vec{I}} g_{ij}\|^2_{L^2(\Sigma_1)} \\
&\quad - 2\int_{s=t}^1 \int_{\Sigma_s} s^{-1}\{(A+2q+\sigma)\delta^a{}_j - 2nsk^a{}_j\} \\
&\qquad \times (s^{A+2q+\sigma}\partial_{\vec{I}} g_{ia})(s^{A+2q+\sigma}\partial_{\vec{I}} g_{ij})dx\,ds \\
&\quad - 2\int_{s=t}^1 \int_{\Sigma_s} (s^{A+2q+\sigma}\partial_{\vec{I}} g_{ij})^{(2q+\sigma;\vec{I})}\mathfrak{G}_{ij}\,dx\,ds. \qquad (8.30)
\end{aligned}
$$

From Definition 3.16, Cauchy–Schwarz, Young's inequality, and estimate (8.12a), we deduce that the last integral on the right-hand side of (8.30) can be bounded as follows:

$$
2\int_{s=t}^1 \int_{\Sigma_s} (s^{A+2q}\partial_{\vec{I}} g_{ij})^{(P;\vec{I})}\mathfrak{G}_{ij}\,dx\,ds \leq C\int_{s=t}^1 s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\,ds. \qquad (8.31)
$$

Use Definition 3.14, the fact that $s\widetilde{k}(s,x)$ equals the diagonal tensor $-\mathrm{diag}(q_1, \ldots, q_D)$, (3.2), and the bootstrap assumptions to bound the (scalar) component $2nsk^a{}_j$ on the right-hand side of (8.30) as

$$
|2nsk^a{}_j| \leq 2q\delta^a{}_j + C\varepsilon,
$$

where $\delta^a{}_j$ is the standard Kronecker delta. It follows that the integral on the second line of the right-hand side of (8.30) is bounded from above by

$$
\begin{aligned}
&\leq -\{2A+\sigma\}\int_{s=t}^1 \int_{\Sigma_s} s^{-1}(s^{A+2q+\sigma}\partial_{\vec{I}} g_{ij})^2 dx\,ds \\
&\quad + C\varepsilon\int_{s=t}^1 s^{-1}\|s^{A+2q+\sigma} g\|^2_{\dot{H}^N_{Frame}(\Sigma_t)}\,ds.
\end{aligned}
$$

Combining these estimates, noting that

$$\|\partial_{\vec{I}} g_{ij}\|^2_{L^2(\Sigma_1)} \leq \mathbb{H}^2_{(g,k)}(1),$$

summing the resulting estimates over $1 \leq i, j \leq D$ and over $\vec{I}$ with $|\vec{I}| = N$, and taking ε to be sufficiently small, we arrive at (8.2a).

Estimate (8.2b) can be proved using a similar argument based on the evolution equation (8.9b) with $P = 2q + \sigma$ and $|\vec{I}| = N$ and estimate (8.12b), and we omit the details.

Estimate (8.5a) can be proved using a similar argument based on the evolution equation (8.9a) with $P = 5q + 3\sigma - 1$ and $|\vec{I}| = N - 1$ and estimate (8.14a), and we omit the details.

Estimate (8.5b) can be proved using a similar argument based on the evolution equation (8.9b) with $P = 5q + 3\sigma - 1$ and $|\vec{I}| = N - 1$ and estimate (8.14b), and we omit the details.

Estimate (8.6a) can be proved using a similar argument based on the evolution equation (7.6b) with $P = 3q + \sigma$ and $|\vec{I}| = N - 1$ and estimates (8.17a), (8.17b), and (8.18a), and we omit the details.

To prove (8.3a), we let $\vec{I}$ be any multi-index with $|\vec{I}| = N$. Using the definition of the norm $|\cdot|_g$ and equation (2.12b) (to substitute for the factors of $\partial_t g^{-1}$ that appear when ∂_t falls on the factors of g^{-1} that are inherent in the definition of $|\cdot|_g$), we deduce that

$$\begin{aligned}\partial_t\{|t^{A+q+\sigma}\partial_{\vec{I}} g|_g^2\} = {} & 4n g^{ac} k^{bd} (t^{A+q+\sigma}\partial_{\vec{I}} g_{ab})(t^{A+q+\sigma}\partial_{\vec{I}} g_{cd}) \\ & + 2 g^{ac} g^{bd} (t^{A+q+\sigma}\partial_{\vec{I}} g_{ab})\partial_t(t^{A+q+\sigma}\partial_{\vec{I}} g_{cd}). \end{aligned} \tag{8.32}$$

Next, using equation (8.9a) with $P = q + \sigma$ to substitute for the factor $\partial_t(t^{A+2q+\sigma}\partial_{\vec{I}} g_{cd})$ on the right-hand side of (8.32), we obtain

$$\begin{aligned}\partial_t\{|t^{A+q+\sigma}\partial_{\vec{I}} g|_g^2\} = {} & \frac{2(A+q+\sigma)}{t}|t^{A+q+\sigma}\partial_{\vec{I}} g|_g^2 \\ & + 2 g^{ac} g^{bd} (t^{A+q+\sigma}\partial_{\vec{I}} g_{ab})^{(q+\sigma;\vec{I})}\mathfrak{G}_{cd}. \end{aligned} \tag{8.33}$$

Integrating (8.33) over the spacetime slab $(t, 1] \times \mathbb{T}^D$, using g-Cauchy–Schwarz, and appealing to Definition 3.16, we obtain the following estimate (where we stress that $t < 1$):

$$\begin{aligned}\|t^{A+q+\sigma}\partial_{\vec{I}} g\|^2_{L^2_g(\Sigma_t)} \leq {} & \|\partial_{\vec{I}} g\|^2_{L^2_g(\Sigma_1)} \\ & - 2(A+q+\sigma)\int_{s=t}^1 s^{-1}\|s^{A+q+\sigma}\partial_{\vec{I}} g\|^2_{L^2_g(\Sigma_s)}\, ds \\ & + 2\int_{s=t}^1 \int_{\Sigma_s} \mathbb{H}_{(g,k)}(s)\|^{(q+\sigma;\vec{I})}\mathfrak{G}\|_{L^2_g(\Sigma_s)}\, ds. \end{aligned} \tag{8.34}$$

Using (8.13a) to bound the integrand factor $\|^{(q+\sigma;\vec{I})}\mathfrak{G}\|_{L^2_g(\Sigma_s)}$ on the right-hand side of (8.34), using Young's inequality, noting that

$$\|\partial_{\vec{I}} g\|^2_{L^2_g(\Sigma_1)} \leq \mathbb{H}^2_{(g,k)}(1),$$

and summing the resulting estimates over $\vec{I}$ with $|\vec{I}| = N$, we arrive at the desired bound (8.3a).

Estimate (8.3b) can be proved using a similar argument based on equation (8.9b) with $P = q + \sigma$ and $|\vec{I}| = N$, equation (2.12a) (which one uses to substitute for the factors of $\partial_t g$ that appear when ∂_t falls on the factors of g that are inherent in the definition of $|\cdot|_g$), and estimate (8.13b); we omit the details.

Estimate (8.4) can be proved using a similar argument based on the evolution equation (7.6a) with $P = 2q + \sigma$, and $|\vec{I}| = N - 1$, equation (2.12b) (to substitute for the factors of $\partial_t g^{-1}$ that appear when ∂_t falls on the factors of g^{-1} that are inherent in the definition of $|\cdot|_g$), and estimates (8.15a)–(8.15b) and (8.16). We omit the details, noting only that the factors of $\partial_t g^{-1}$ and the factor $-2tk^a{}_j$ in the first braces on the right-hand side of (7.6a) lead to the terms

$$2nk^{ad} g^{be} g^{cf} (t^{A+q+\sigma}\partial_a\partial_{\vec{I}} g_{bc})(t^{A+q+\sigma}\partial_d\partial_{\vec{I}} g_{ef}) \\ + 4(n-1) g^{ad} k^{be} g^{cf} (t^{A+q+\sigma}\partial_a\partial_{\vec{I}} g_{bc})(t^{A+q+\sigma}\partial_d\partial_{\vec{I}} g_{ef}),$$

which we pointwise bound in magnitude as follows by using g-Cauchy–Schwarz, the fact that $|g^{-1}|_g \leq C_*$, (3.2), Definition 3.14, and the bootstrap assumptions:

$$\begin{aligned} &\big|2nk^{ad} g^{be} g^{cf} (t^{A+q+\sigma}\partial_a\partial_{\vec{I}} g_{bc})(t^{A+q+\sigma}\partial_d\partial_{\vec{I}} g_{ef}) \\ &\qquad + 4(n-1) g^{ad} k^{be} g^{cf} (t^{A+q+\sigma}\partial_a\partial_{\vec{I}} g_{bc})(t^{A+q+\sigma}\partial_d\partial_{\vec{I}} g_{ef})\big| \\ &\leq C_*\{\|n\|_{L^\infty(\Sigma_t)} + \|n-1\|_{L^\infty(\Sigma_t)}\}\|k\|_{L^\infty_g(\Sigma_t)} |t^{A+q+\sigma}\partial\partial_{\vec{I}} g|_g^2 \\ &\leq C_* t^{-1} |t^{A+q+\sigma}\partial\partial_{\vec{I}} g|_g^2. \end{aligned} \tag{8.35}$$

We further remark that these factors of C_* lead to the C_*-dependent products on the right-hand side of (8.4).

Estimate (8.6b) can be proved using a similar argument based on equation (7.6b) with $P = 3q + \sigma$ and $|\vec{I}| = N - 1$ (where we use equations (2.12a)–(2.12b) to substitute for the factors of $\partial_t g$ and $\partial_t g^{-1}$ that arise when ∂_t falls on the factors of g and g^{-1} inherent in the definition of $|\cdot|_g$), and estimates (8.17c), (8.17d), and (8.18b); we omit the details.

9. The main a priori estimates

In this section, we use the estimates derived in Sections 5–8 to prove the main technical result of the article: Proposition 9.3, which provides a priori estimates for the solution norms from Definitions 3.14 and 3.16. The proposition in particular yields a strict improvement of the bootstrap assumptions.

9.1. Integral inequality for the high norm

We start with the following lemma, in which we derive an integral inequality for the high norm $\mathbb{H}_{(g,k)}(t)$. The lemma is an analog of Proposition 6.1, in which we derived a similar but simpler inequality for the low norm $\mathbb{L}_{(g,k)}(t)$.

Lemma 9.1 (Integral inequality for the high norm). *Assume that the bootstrap assumptions* (3.18) *hold. There exist a constant* $C_* > 0$ independent of N and A *and a constant*

$C = C_{N,A,D,q,\sigma} > 0$ such that if N is sufficiently large in a manner that depends on A, D, q, and σ, and if ε is sufficiently small (in a manner that depends on N, A, D, q, and σ), then the following integral inequality holds for $t \in (T_{(\text{Boot})}, 1]$:

$$\mathbb{H}^2_{(g,k)}(t) \leq C\mathbb{H}^2_{(g,k)}(1) - \{2A - C_*\} \int_{s=t}^{1} s^{-1}\mathbb{H}^2_{(g,k)}(s)\, ds + C \int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds. \tag{9.2}$$

Proof. Sum estimates (7.2) over all $\vec{I}$ with $|\vec{I}| = N$ together with estimates (8.2a)–(8.6b). In view of definition (3.17a), we conclude the estimate (9.2). ■

9.2. A priori estimates for the solution norms

In the next proposition, we provide the main result of Section 9.

Proposition 9.3 (A priori estimates for the solution norms). *Recall that $\mathbb{L}_{(g,k)}(t)$, $\mathbb{L}_{(n)}(t)$, $\mathbb{H}_{(g,k)}(t)$, and $\mathbb{H}_{(n)}(t)$ are the norms from Definitions 3.14 and 3.16, and assume that the bootstrap assumptions (3.18), which involve the smallness parameter ε, hold. Let $\mathring{\epsilon}$ be the following norm of the difference between the Kasner initial data and the perturbed initial data:*

$$\mathring{\epsilon} := \|g - \widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_1)} + \|k - \widetilde{k}\|_{L^\infty_{\text{Frame}}(\Sigma_1)} + \|g\|_{\dot{H}^{N+1}_{\text{Frame}}(\Sigma_1)} + \|k\|_{\dot{H}^{N}_{\text{Frame}}(\Sigma_1)}. \tag{9.4}$$

If A is sufficiently large and if N is sufficiently large in a manner that depends on A, D, q, and σ, there exists a constant $C_{N,A,D,q,\sigma} > 1$ such that if ε is sufficiently small in a manner that depends on N, A, D, q, and σ, then the following estimates hold for $t \in (T_{(\text{Boot})}, 1]$:

$$\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t) + \mathbb{L}_{(n)}(t) + \mathbb{H}_{(n)}(t) \leq C_{N,A,D,q,\sigma}\mathring{\epsilon}. \tag{9.5}$$

In particular, if $C_{N,A,D,q,\sigma}\mathring{\epsilon} < \varepsilon$, then (9.5) yields a strict improvement of the bootstrap assumptions.

Proof. We first square inequality (6.2) and use the Cauchy–Schwarz estimate

$$\begin{aligned}
&\left(\int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}_{(g,k)}(s) + \mathbb{H}_{(g,k)}(s)\}\, ds\right)^2 \\
&\leq C \int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds \times \underbrace{\int_{s=t}^{1} s^{\sigma-1}\, ds}_{\leq C} \\
&\leq C \int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds
\end{aligned} \tag{9.6}$$

to deduce

$$\mathbb{L}^2_{(g,k)}(t) \leq C\mathbb{L}^2_{(g,k)}(1) + C \int_{s=t}^{1} s^{\sigma-1}\{\mathbb{L}^2_{(g,k)}(s) + \mathbb{H}^2_{(g,k)}(s)\}\, ds. \tag{9.7}$$

We now fix $A \geq \max\{C_*/2, 1\}$, where $C_* > 0$ is the universal constant (independent of N and A) on the right-hand side of (9.2), while the assumption $A \geq 1$ was used in other parts of the paper (for example, the proof of (8.14a)). Given this choice of A, we fix N large enough such that whenever ε is sufficiently small, the estimates of Proposition 6.1 and Lemma 9.1 hold. Note that our choice of A ensures that the factor $\{2A - C_*\}$ on the right-hand side of (9.2) is non-positive; hence the first time integral on the right-hand side of (9.2) is non-positive and can be discarded. In particular, using (9.7) and Lemma 9.1, we see that for $t \in (T_{(\mathrm{Boot})}, 1]$, the quantity

$$Q(t) := \mathbb{L}^2_{(g,k)}(t) + \mathbb{H}^2_{(g,k)}(t)$$

verifies

$$Q(t) \leq CQ(1) + C \int_{s=t}^{1} s^{\sigma-1} Q(s)\, ds.$$

Because the function $s^{\sigma-1}$ is integrable over the interval $s \in (0, 1]$, we conclude from Gronwall's inequality that

$$Q(t) \leq CQ(1)$$

for $t \in (T_{(\mathrm{Boot})}, 1]$. Moreover, from Definitions 3.14 and 3.16, definition (9.4), standard Sobolev interpolation (i.e., (4.6)), and Sobolev embedding, we deduce that if N is sufficiently large, then $Q(1) \leq C\mathring{\epsilon}^2$. From this bound and the bound $Q(t) \leq CQ(1)$, we deduce that $Q(t) \leq C\mathring{\epsilon}^2$ for $t \in (T_{(\mathrm{Boot})}, 1]$. From this bound, (5.2), (5.4), and (5.5), we conclude, in view of Definitions 3.14 and 3.16, the desired bound (9.5). ∎

10. Estimates tied to curvature-blowup and the length of past-directed causal geodesics

In this section, we derive the main estimates needed to show curvature-blowup and geodesic incompleteness for the solutions under study.

10.1. Curvature estimates

In the following lemma, we derive a pointwise estimate that shows in particular that the Kretschmann scalar blows up as $t \downarrow 0$.

Lemma 10.1 (Pointwise estimate for the Kretschmann scalar). *Under the hypotheses and conclusions of Proposition 9.3, perhaps enlarging N if necessary, the following pointwise estimate holds for $t \in (T_{(\mathrm{Boot})}, 1]$:*

$$\mathbf{Riem}_{\alpha\beta\gamma\delta}\, \mathbf{Riem}^{\alpha\beta\gamma\delta} = 4t^{-4} \left\{ \sum_{i=1}^{D} (q_i^2 - q_i)^2 + \sum_{1 \leq i < j = D} q_i^2 q_j^2 \right\} + t^{-4} \mathcal{O}(\mathring{\epsilon}). \quad (10.2)$$

Proof. Throughout this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta < \sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large.

First, we observe the following identity, which holds relative to CMC-transported spatial coordinates in view of (2.1) and the curvature properties $\mathbf{Riem}_{\alpha\beta\gamma\delta} = -\mathbf{Riem}_{\beta\alpha\gamma\delta} = -\mathbf{Riem}_{\alpha\beta\delta\gamma} = \mathbf{Riem}_{\gamma\delta\alpha\beta}$:

$$t^4\,\mathbf{Riem}_{\alpha\beta\gamma\delta}\,\mathbf{Riem}^{\alpha\beta\gamma\delta} = t^4\,\mathbf{Riem}_{ab}{}^{cd}\,\mathbf{Riem}_{cd}{}^{ab} + 4t^4\,\mathbf{Riem}_{a0}{}^{c0}\,\mathbf{Riem}_{c0}{}^{a0} - 4n^{-2}t^4|\mathbf{Riem}_{0\cdot}|_g^2, \tag{10.3}$$

where $\mathbf{Riem}_{0\cdot}$ is defined to be the type $\binom{0}{3}$ Σ_t-tangent tensorfield with components $\mathbf{Riem}_{0bcd}$ relative to the transported spatial coordinates. Next, from standard calculations based in part on the Gauss and Codazzi equations (see, for example, [52, equation (4.14)] and [52, equation (4.18)]), we find that relative to the CMC-transported spatial coordinates, the components $\mathbf{Riem}_{\alpha\beta}{}^{\mu\nu}$ of the (type $\binom{2}{2}$) Riemann curvature tensor of $\mathbf{g}$ can be decomposed into principal terms and error terms as follows:

$$\mathbf{Riem}_{ab}{}^{cd} = k^c_{\ a}k^d_{\ b} - k^d_{\ a}k^c_{\ b} + \Delta_{ab}{}^{cd}, \tag{10.4a}$$
$$\mathbf{Riem}_{a0}{}^{c0} = t^{-1}k^c_{\ a} + k^c_{\ e}k^e_{\ a} + \Delta_{a0}{}^{c0}, \tag{10.4b}$$
$$n^{-1}\,\mathbf{Riem}_{0b}{}^{cd} = \Delta_{0b}{}^{cd}, \tag{10.4c}$$

where the error terms are defined by

$$\Delta_{ab}{}^{cd} := \mathrm{Riem}_{ab}{}^{cd}, \tag{10.5a}$$
$$\Delta_{a0}{}^{c0} := -t^{-1}n^{-1}\partial_t(tk^c_{\ a}) + t^{-1}(n^{-1}-1)k^c_{\ a} - n^{-1}g^{ec}\partial_a\partial_e n + n^{-1}g^{ec}\Gamma_{a\ e}^{\ f}\partial_f n, \tag{10.5b}$$
$$\Delta_{0b}{}^{cd} := g^{ce}\partial_e(k^d_{\ b}) - g^{de}\partial_e(k^c_{\ b}) + g^{ce}\Gamma_{e\ f}^{\ d}k^f_{\ b} - g^{ce}\Gamma_{e\ b}^{\ f}k^d_{\ f} - g^{de}\Gamma_{e\ f}^{\ c}k^f_{\ b} + g^{de}\Gamma_{e\ b}^{\ f}k^c_{\ f}. \tag{10.5c}$$

In (10.5a), $\mathrm{Riem}_{ab}{}^{cd}$ denotes a component of the (type $\binom{2}{2}$) Riemann curvature tensor of g.

We now claim that the following estimates hold, where the $\widetilde{k}^i_{\ j}$ are the components of the Kasner mixed second fundamental form (see (1.8c)):

$$\|t^2\,\mathbf{Riem}_{ab}{}^{cd} - (t^2\widetilde{k}^c_{\ a}\widetilde{k}^d_{\ b} - t^2\widetilde{k}^d_{\ a}\widetilde{k}^c_{\ b})\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}, \tag{10.6}$$
$$\|t^2\,\mathbf{Riem}_{a0}{}^{c0} - (t\widetilde{k}^c_{\ a} + t^2\widetilde{k}^c_{\ e}\widetilde{k}^e_{\ a})\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}, \tag{10.7}$$
$$\|n^{-1}t^2\,\mathbf{Riem}_{0\cdot}\|_{L^\infty_g(\Sigma_t)} \lesssim \mathring{\epsilon}, \tag{10.8}$$

where we stress that (10.6)–(10.7) are estimates for *components* of tensorfields and (10.8) is an estimate for the *norm* $|\cdot|_g$ of the tensorfield $\mathbf{Riem}_{0\cdot}$. Let us momentarily accept (10.6)–(10.8). Then from (10.3), (10.6)–(10.8), Definition 3.14, and estimate (9.5) (which in particular implies the component bound $tk^a_{\ b} = t\widetilde{k}^a_{\ b} + \mathcal{O}(\mathring{\epsilon})$), we deduce that

$$t^4\,\mathbf{Riem}_{\alpha\beta\gamma\delta}\,\mathbf{Riem}^{\alpha\beta\gamma\delta} = 2t^4(\widetilde{k}^c_{\ a}\widetilde{k}^a_{\ c})^2 + 4t^2\widetilde{k}^c_{\ a}\widetilde{k}^a_{\ c} + 8t^3\widetilde{k}^c_{\ a}\widetilde{k}^a_{\ d}\widetilde{k}^d_{\ c} + 2t^4\widetilde{k}^c_{\ e}\widetilde{k}^e_{\ a}\widetilde{k}^a_{\ d}\widetilde{k}^d_{\ c} + \mathcal{O}(\mathring{\epsilon}). \tag{10.9}$$

Next, using the fact that in CMC-transported spatial coordinates, $t\widetilde{k}$ is equal to the diagonal tensor $-\mathrm{diag}(q_1, \ldots, q_D)$, we compute that

$$t^2 \widetilde{k}^c{}_a \widetilde{k}^a{}_c = \sum_{i=1}^{D} q_i^2, \tag{10.10}$$

$$t^3 \widetilde{k}^c{}_a \widetilde{k}^a{}_d \widetilde{k}^d{}_c = -\sum_{i=1}^{D} q_i^3, \tag{10.11}$$

$$t^4 \widetilde{k}^c{}_e \widetilde{k}^e{}_a \widetilde{k}^a{}_d \widetilde{k}^d{}_c = \sum_{i=1}^{D} q_i^4. \tag{10.12}$$

From (10.9) and (10.10)–(10.12), we arrive at the desired bound (10.2).

It remains for us to prove (10.6)–(10.8). To prove (10.6), we first note the schematic identity $\mathrm{Riem}_{ab}{}^{cd} \simeq g^{-2}\partial^2 g + g^{-3}(\partial g)^2$, which follows from (2.15a). Hence, bounding each factor on the right-hand side of the schematic identity in the norm $\| \cdot \|_{L^\infty_{\mathrm{Frame}}(\Sigma_t)}$ with the help of estimates (4.2a), (4.12a)–(4.12b), and (9.5), we deduce the estimate $\|\mathrm{Riem}_{ab}{}^{cd}\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon} t^{-10q-A\delta}$. From this estimate, (10.4a), (10.5a), the aforementioned component estimate $t k^a{}_b = t\widetilde{k}^a{}_b + \mathcal{O}(\mathring{\epsilon})$, and (3.2), we conclude (10.6). Estimate (10.7) can be proved by combining similar arguments based on equation (10.4b) with the additional bounds (4.12c) and (6.17), which are needed to help control the terms on the right-hand side of (10.5b); we omit the straightforward details. To prove (10.8), we first use the fact that $|g^{-1}|_g \lesssim 1$ and the g-Cauchy–Schwarz inequality to deduce that the norm $|\cdot|_g$ of the right-hand side of (10.5c) is bounded by $\lesssim |\partial k|_g + |\partial g|_g |k|_g$. Next, using (4.4), we bound the right-hand side of the previous expression as follows:

$$|\partial k|_g + |\partial g|_g |k|_g \lesssim t^{-3q} \|k\|_{W^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)} + t^{-3q} \|g\|^{1/2}_{\dot{W}^{1,\infty}_{\mathrm{Frame}}(\Sigma_t)} \|k\|_{L^\infty_g(\Sigma_t)}. \tag{10.13}$$

From Definition 3.14 and estimates (4.2b), (4.12a), and (9.5), we deduce, in view of (3.2), that RHS (10.13) $\lesssim \mathring{\epsilon} t^{-1-4q-A\delta} \lesssim \mathring{\epsilon} t^{-2}$. Considering also equation (10.4c), we see that we have proved the desired bound (10.8). ∎

10.2. Estimates for the length of past-directed causal geodesic segments

In this subsection, we show that for the solutions under consideration, the length of any past-directed causal geodesic segment is uniformly bounded from above by a constant.

Lemma 10.14 (Estimates for the length of past-directed causal geodesic segments). *Under the hypotheses and conclusions of Proposition 9.3, perhaps enlarging N and shrinking $\mathring{\epsilon}$ if necessary, the following holds: any past-directed causal geodesic $\boldsymbol{\zeta}$ that emanates from Σ_1 and is contained in the region $(T_{(\mathrm{Boot})}, 1] \times \mathbb{T}^D$ has an affine length that is bounded from above by*

$$\leq \mathscr{A}(T_{(\mathrm{Boot})}) \leq \mathscr{A}(0) \leq \frac{|\mathscr{A}'(1)|}{1-q}, \tag{10.15}$$

where $\mathscr{A}(t)$ is the affine parameter along $\boldsymbol{\zeta}$ viewed as a function of t along $\boldsymbol{\zeta}$ (normalized by $\mathscr{A}(1) = 0$).

Proof. Throughout this proof, we will assume that $A\delta$ is sufficiently small (and in particular that $A\delta < \sigma$); in view of the discussion in Section 4.4, we see that at fixed A, this can be achieved by choosing N to be sufficiently large.

Let $\boldsymbol{\zeta}(\mathscr{A})$ be a past-directed affinely parametrized causal geodesic verifying the hypotheses of the lemma. Note that the component $\boldsymbol{\zeta}^0$ can be identified with the CMC time coordinate. In the rest of the proof, we view the affine parameter $\mathscr{A}$ as a function of $t = \boldsymbol{\zeta}^0$ along $\boldsymbol{\zeta}$. We normalize $\mathscr{A}(t)$ by setting $\mathscr{A}(1) = 0$. We also define

$$\dot{\boldsymbol{\zeta}}^\mu := \frac{d}{d\mathscr{A}}\boldsymbol{\zeta}^\mu, \quad \ddot{\boldsymbol{\zeta}}^\mu := \frac{d^2}{d\mathscr{A}^2}\boldsymbol{\zeta}^\mu,$$

and $\mathscr{A}' := \frac{d}{dt}\mathscr{A}$. By the chain rule, we have

$$\mathscr{A}' = \frac{1}{\dot{\boldsymbol{\zeta}}^0}, \quad \ddot{\boldsymbol{\zeta}}^0 = \dot{\boldsymbol{\zeta}}^0 \frac{d}{dt}\dot{\boldsymbol{\zeta}}^0 = -(\mathscr{A}')^{-3}\mathscr{A}''. \tag{10.16}$$

For use below, we note that since $\boldsymbol{\zeta}$ is a causal curve, we have (by the definition of a causal curve) that $\mathbf{g}(\dot{\boldsymbol{\zeta}}, \dot{\boldsymbol{\zeta}}) \leq 0$. Considering also the expression (2.1) for $\mathbf{g}$, we deduce that relative to the CMC-transported coordinates, causal curves satisfy

$$g_{ab}\dot{\zeta}^a\dot{\zeta}^b \leq n^2(\dot{\boldsymbol{\zeta}}^0)^2. \tag{10.17}$$

Next, we note that (relative to CMC-transported spatial coordinates), the 0 component of the geodesic equation is

$$\ddot{\boldsymbol{\zeta}}^0 + \boldsymbol{\Gamma}_{\alpha\ \beta}^{\ 0}|_{\boldsymbol{\zeta}}\dot{\boldsymbol{\zeta}}^\alpha\dot{\boldsymbol{\zeta}}^\beta = 0,$$

where $\boldsymbol{\Gamma}_{\alpha\ \beta}^{\ 0}$ are Christoffel symbols of $\mathbf{g}$ (see (1.26)). Using (2.1) and (2.12a), we compute that this geodesic equation component can be written in the following more explicit form:

$$\ddot{\boldsymbol{\zeta}}^0 + (\partial_t \ln n)|_{\boldsymbol{\zeta}}(\dot{\boldsymbol{\zeta}}^0)^2 + 2(\partial_a \ln n)|_{\boldsymbol{\zeta}}\dot{\boldsymbol{\zeta}}^a\dot{\boldsymbol{\zeta}}^0 - (n^{-1}g_{ac}k^c_{\ b})|_{\boldsymbol{\zeta}}\dot{\boldsymbol{\zeta}}^a\dot{\boldsymbol{\zeta}}^b = 0. \tag{10.18}$$

Multiplying (10.18) by $-(\mathscr{A}')^3$ and using the g-Cauchy–Schwarz inequality, (10.16), and (10.17), we deduce

$$|\mathscr{A}''| \leq |n^{-1}k^a_{\ b}\dot{\boldsymbol{\zeta}}_a\dot{\boldsymbol{\zeta}}^b(\mathscr{A}')^3| + \big(n^{-1}|\partial_t n| + 2|\partial n|_g\big)|\mathscr{A}'|. \tag{10.19}$$

From Definition 3.14 and estimate (9.5), we see that in CMC-transported spatial coordinates, tk is equal to the diagonal tensor diag $-(q_1, \ldots, q_D)$ plus an $\mathcal{O}(\mathring{\epsilon})$ correction. Hence, the eigenvalues of tk are all bounded in magnitude by $q_{(\mathrm{Max})} + \mathcal{O}(\mathring{\epsilon})$, where $q_{(\mathrm{Max})} := \max_{i=1,\ldots,D}|q_i|$. Also using (10.16)–(10.17), we see that

$$\begin{aligned}
|n^{-1}k^a_{\ b}\dot{\boldsymbol{\zeta}}_a\dot{\boldsymbol{\zeta}}^b(\mathscr{A}')^3| &\leq \{q_{(\mathrm{Max})} + \mathcal{O}(\mathring{\epsilon})\}t^{-1}n^{-1}(\mathscr{A}')^3|\dot{\boldsymbol{\zeta}}|_g^2 \\
&\leq \{q_{(\mathrm{Max})} + \mathcal{O}(\mathring{\epsilon})\}t^{-1}|\mathscr{A}'| \\
&\quad + \{q_{(\mathrm{Max})} + \mathcal{O}(\mathring{\epsilon})\}t^{-1}|n-1||\mathscr{A}'|.
\end{aligned} \tag{10.20}$$

Moreover, from Definition 3.14, (3.2), and estimate (9.5), we see that the last product on the right-hand side of (10.20) obeys the bound

$$\{q_{(\mathrm{Max})} + \mathcal{O}(\mathring{\epsilon})\}t^{-1}|n-1||\mathscr{A}'| \leq C\mathring{\epsilon}t^{-1}|\mathscr{A}'|. \tag{10.21}$$

Furthermore, also using (4.2a), (4.4), (4.12c), and (5.3), we see that the last term on the right-hand side of (10.19) is bounded as follows:

$$\big(n^{-1}|\partial_t n| + 2|\partial n|_g\big)|\mathscr{A}'| \leq C\mathring{\epsilon}t^{-1}|\mathscr{A}'|. \tag{10.22}$$

Combining (10.19)–(10.22) and taking into account (3.2), we deduce that if $\mathring{\epsilon}$ is sufficiently small, then the following bound holds:

$$|\mathscr{A}''| \leq t^{-1}q|\mathscr{A}'|, \quad t \in (T_{(\text{Boot})}, 1]. \tag{10.23}$$

Applying Gronwall's inequality to (10.23), we deduce that

$$|\mathscr{A}'(t)| \leq |\mathscr{A}'(1)|t^{-q}, \quad t \in (T_{(\text{Boot})}, 1]. \tag{10.24}$$

Integrating (10.24) from time t to time 1 and using the assumption $\mathscr{A}(1) = 0$, we find that

$$\mathscr{A}(t) \leq \frac{|\mathscr{A}'(1)|}{1-q}(1 - t^{1-q}), \quad t \in (T_{(\text{Boot})}, 1], \tag{10.25}$$

from which the desired estimate (10.15) follows. ∎

11. The main stable blowup theorem

We now state and prove our main stable blowup theorem. As we noted in Remark 1.9, it is possible to derive substantial additional information about the solution, going beyond that provided by the theorem.

Theorem 11.1 (The main stable blowup theorem). *Let* $\widetilde{\mathbf{g}} = -dt \otimes dt + \widetilde{g}_{ab}dx^a \otimes dx^b$ *be an Einstein-vacuum Kasner solution on* $(0, \infty) \times \mathbb{T}^D$, *i.e.,*

$$\widetilde{g} = \operatorname{diag}(t^{2q_1}, t^{2q_2}, \ldots, t^{2q_D}),$$

where $\sum_{i=1}^D q_i = \sum_{i=1}^D q_i^2 = 1$, *and assume that*

$$\max_{i=1,\ldots,D} |q_i| < \frac{1}{6}. \tag{11.2}$$

Recall that in Section 2.3, we showed that such Kasner solutions exist when $D \geq 38$. *Let* $\widetilde{k} = -t^{-1}\operatorname{diag}(q_1, q_2, \ldots, q_D)$ *denote the corresponding Kasner (mixed) second fundamental form. Let* $(\Sigma_1 = \mathbb{T}^D, \mathring{g}, \mathring{k})$ *be initial data for the Einstein-vacuum equations verifying the constraints (2.11a)–(2.11b) and the CMC condition* $\mathring{k}^a{}_a = -1$ *(see, however, Remark 1.4), and let*

$$\mathring{\epsilon} := \|\mathring{g} - \widetilde{g}\|_{L^\infty_{\text{Frame}}(\Sigma_1)} + \|\mathring{k} - \widetilde{k}\|_{L^\infty_{\text{Frame}}(\Sigma_1)} + \|\mathring{g}\|_{\dot{H}^{N+1}_{\text{Frame}}(\Sigma_1)} + \|\mathring{k}\|_{\dot{H}^N_{\text{Frame}}(\Sigma_1)}. \tag{11.3}$$

Assume that

- $A > 0$ *is sufficiently large.*
- $N > 0$ *is sufficiently large, where the required largeness depends on* A, D, q, *and* σ. *Here we recall that* $q > 0$ *and* $\sigma > 0$ *are the constants fixed in Section 3.1.*
- $\mathring{\epsilon}$ *is sufficiently small, where the required smallness depends on* N, A, D, q, *and* σ.

Then the following conclusions hold.

- **Existence and norm estimates on $(0,1] \times \mathbb{T}^D$.** *The initial data launch a solution (g, k, n) to the Einstein-vacuum equations in CMC-transported spatial coordinates (that is, the equations of Proposition 2.10, where $\mathbf{g} = -n^2 dt \otimes dt + g_{ab} dx^a \otimes dx^b$ is the spacetime metric) that exists classically for $(t, x) \in (0, 1] \times \mathbb{T}^D$. Moreover, there exists a constant $C = C_{N,A,D,q,\sigma} > 1$ such that the (N, A, q, σ)-dependent norms from Definitions 3.14 and 3.16 verify the following estimate for $t \in (0, 1]$:*

$$\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t) + \mathbb{L}_{(n)}(t) + \mathbb{H}_{(n)}(t) \leq C\mathring{\epsilon}. \tag{11.4}$$

- **Description of the "past-half" of the MGHD and curvature-blowup.** *The spacetime Kretschmann scalar verifies the following estimate for $t \in (0, 1]$:*

$$t^4\, \mathbf{Riem}_{\alpha\beta\gamma\delta}\, \mathbf{Riem}^{\alpha\beta\gamma\delta} = 4\left\{ \sum_{i=1}^{D} (q_i^2 - q_i)^2 + \sum_{1 \leq i < j = D} q_i^2 q_j^2 \right\} + \mathcal{O}(\mathring{\epsilon}). \tag{11.5}$$

In particular, for $\mathring{\epsilon}$ sufficiently small, $\mathbf{Riem}_{\alpha\beta\gamma\delta}\, \mathbf{Riem}^{\alpha\beta\gamma\delta}$ blows up like

$$\{C + \mathcal{O}(\mathring{\epsilon})\} t^{-4}$$

as $t \downarrow 0$, where

$$C = 4\left\{ \sum_{i=1}^{D} (q_i^2 - q_i)^2 + \sum_{1 \leq i < j = D} q_i^2 q_j^2 \right\}.$$

Consequently, the "past-half" of the maximal (classical) globally hyperbolic development of the data is $((0, 1] \times \mathbb{T}^D, \mathbf{g})$, and $\mathbf{g}$ cannot be continued as a C^2 Lorentzian metric to the past of the singular hypersurface Σ_0. That is, the past of Σ_1 in the maximal (classical) globally hyperbolic development of the data is foliated by the family of spacelike hypersurfaces Σ_t, along which the CMC condition $k^a{}_a = -t^{-1}$ holds.

- **Bounded length of past-directed causal geodesics.** *Let $\boldsymbol{\zeta}$ be any past-directed causal geodesic that emanates from Σ_1, and let $\mathscr{A} = \mathscr{A}(t)$ be an affine parameter along $\boldsymbol{\zeta}$, where $\mathscr{A}$ is viewed as a function of t along $\boldsymbol{\zeta}$ that is normalized by $\mathscr{A}(1) = 0$. Then $\boldsymbol{\zeta}$ crashes into the singular hypersurface Σ_0 in finite affine parameter time*

$$\mathscr{A}(0) \leq \frac{|\mathscr{A}'(1)|}{1 - q}, \tag{11.6}$$

where $\mathscr{A}'(t) := \frac{d}{dt}\mathscr{A}(t)$.

Proof. We first fix A and N to be large enough so that all of the estimates proved (under the bootstrap assumptions) in the previous sections hold true. By standard local well-posedness, if $\mathring{\epsilon}$ is sufficiently small and the constant C' is sufficiently large, then there exists a maximal time $T_{(\mathrm{Max})} \in [0, 1)$ such that the solution (g, k, n) exists classically for $(t, x) \in (T_{(\mathrm{Max})}, 1] \times \mathbb{T}^D$ and such that the bootstrap assumptions (3.18) hold with $T_{(\mathrm{Boot})} := T_{(\mathrm{Max})}$ and $\varepsilon := C'\mathring{\epsilon}$. By enlarging the constant C' if necessary, we can assume that $C' \geq 2C_{N,A,D,q,\sigma}$, where $C_{N,A,D,q,\sigma} > 1$ is the constant from inequality (9.5). Readers can consult [4] for the main ideas behind the proof of local well-posedness in a similar but distinct gauge for Einstein's equations, or [51, Theorem 14.1] for a sketch of a proof of local well-posedness in CMC-transported spatial coordinates. Moreover, in view of Definitions 3.14 and 3.16, it is a standard result (again, see [4] for the main ideas) that if ε is sufficiently small, then either $T_{(\mathrm{Max})} = 0$ or the bootstrap assumptions are saturated on the time interval $(T_{(\mathrm{Max})}, 1]$, that is,

$$\sup_{t\in(T_{(\mathrm{Max})},1]} \{\mathbb{L}_{(g,k)}(t) + \mathbb{H}_{(g,k)}(t) + \mathbb{L}_{(n)}(t) + \mathbb{H}_{(n)}(t)\} = C'\mathring{\epsilon}.$$

The latter possibility is ruled out by inequality (9.5). Thus, $T_{(\mathrm{Max})} = 0$. In particular, the solution exists classically for $(t,x) \in (0,1] \times \mathbb{T}^D$, and estimate (11.4) holds for $t \in (0,1]$.

The remaining aspects of the theorem follow from Lemmas 10.1 and 10.14. ∎

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