



Yalong Cao · Davesh Maulik · Yukinobu Toda

Stable pairs and Gopakumar–Vafa type invariants for Calabi–Yau 4-folds

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Abstract. As an analogy to the Gopakumar–Vafa conjecture on CY 3-folds, Klemm–Pandharipande defined GV type invariants on CY 4-folds using GW theory and conjectured their integrality. In this paper, we define stable pair type invariants on CY 4-folds and use them to interpret these GV type invariants. Examples are computed for both compact and non-compact CY 4-folds to support our conjectures.

Keywords. Stable pairs, Gopakumar–Vafa type invariants, Calabi–Yau 4-folds

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Yalong Cao (corresponding author): Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo Institutes for Advanced Study, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan; yalong.cao@ipmu.jp

Davesh Maulik: Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA; maulik@mit.edu

Yukinobu Toda: Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo Institutes for Advanced Study, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan; yukinobu.toda@ipmu.jp

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1. Introduction

1.1. Background

Gromov–Witten invariants are rational numbers counting stable maps from complex curves to algebraic varieties (or symplectic manifolds). They are not necessarily integers because of multiple cover contributions. In [24], Klemm–Pandharipande gave a definition of Gopakumar–Vafa type invariants on Calabi–Yau 4-folds using GW theory and conjectured that they are integers. For dimensional reasons, GW invariants for genus $g \geq 2$ always vanish on Calabi–Yau 4-folds, so the integrality conjecture only applies in genus 0 and 1. In our previous paper [13], we gave a sheaf-theoretic interpretation of $g = 0$ GV type invariants using DT_4 invariants [4, 12] of one-dimensional stable sheaves, analogous to the work of Katz for 3-folds [22] (see [14] for an extension to the $g = 1$ case).

In this paper, we propose a sheaf-theoretic approach to both genus 0 and 1 GV type invariants using stable pairs on CY 4-folds. For CY 3-folds, a Pairs/GV conjecture was first developed in work of Pandharipande and Thomas [34, 36]. Our paper may be viewed as an analogue of their work in the setting of CY 4-folds.

1.2. GV type invariants on CY 4-folds

Let X be a smooth projective CY 4-fold. As mentioned above, Gromov–Witten invariants vanish for genus $g \geq 2$ for dimensional reasons, so we only consider the genus 0 and 1 cases.

The genus 0 GW invariants on X are defined using insertions: for integral classes $\gamma_i \in H^{m_i}(X, \mathbb{Z})$, $1 \leq i \leq n$, one defines

$$\text{GW}_{0,\beta}(\gamma_1, \dots, \gamma_n) = \int_{[\overline{M}_{0,n}(X,\beta)]^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^*(\gamma_i),$$

where $\text{ev}_i: \overline{M}_{0,n}(X, \beta) \rightarrow X$ is the i -th evaluation map.

The invariants

$$n_{0,\beta}(\gamma_1, \dots, \gamma_n) \in \mathbb{Q} \tag{1}$$

are defined in [24] by the identity

$$\sum_{\beta>0} \text{GW}_{0,\beta}(\gamma_1, \dots, \gamma_n) q^\beta = \sum_{\beta>0} n_{0,\beta}(\gamma_1, \dots, \gamma_n) \sum_{d=1}^\infty d^{n-3} q^{d\beta}.$$

For genus 1, virtual dimensions of GW moduli spaces without marked points are 0, so the GW invariants

$$\text{GW}_{1,\beta} = \int_{[\overline{M}_{1,0}(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Q}$$

can be defined without insertions. The invariants

$$n_{1,\beta} \in \mathbb{Q} \tag{2}$$

are defined in [24] by the identity

$$\begin{aligned} \sum_{\beta>0} \text{GW}_{1,\beta} q^\beta &= \sum_{\beta>0} n_{1,\beta} \sum_{d=1}^\infty \frac{\sigma(d)}{d} q^{d\beta} + \frac{1}{24} \sum_{\beta>0} n_{0,\beta}(c_2(X)) \log(1 - q^\beta) \\ &\quad - \frac{1}{24} \sum_{\beta_1, \beta_2} m_{\beta_1, \beta_2} \log(1 - q^{\beta_1 + \beta_2}), \end{aligned}$$

where $\sigma(d) = \sum_{i|d} i$ and where $m_{\beta_1, \beta_2} \in \mathbb{Z}$, called *meeting invariants*, can be inductively determined by genus 0 GW invariants. In [24], both of the invariants (1), (2) are conjectured to be integers, and GW invariants on X are computed to support the conjectures in many examples by either localization techniques or mirror symmetry.

1.3. Our proposal

The aim of this paper is to give a sheaf-theoretic interpretation for the above GV-type invariants (1), (2) via stable pairs, using Donaldson–Thomas theory for CY 4-folds introduced by Cao–Leung [12] and Borisov–Joyce [4].

We consider the moduli space $P_n(X, \beta)$ of stable pairs $(s : \mathcal{O}_X \rightarrow F)$ with $\text{ch}(F) = (0, 0, 0, \beta, n)$. By Theorem 2.4, one can construct a virtual class

$$[P_n(X, \beta)]^{\text{vir}} \in H_{2n}(P_n(X, \beta), \mathbb{Z}), \tag{3}$$

which depends on the choice of orientation of a certain (real) line bundle over $P_n(X, \beta)$. On each connected component of $P_n(X, \beta)$, there are two choices of orientation, which affect the corresponding contribution to the virtual class (3) by a sign (for each connected component).

When $n = 0$, the virtual dimension of the virtual class (3) is 0. By integrating, we define the stable pair invariant

$$P_{0,\beta} := \int_{[P_0(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Z}.$$

When $n = 1$, the (real) virtual dimension of the virtual class (3) is 2. We use insertions to define invariants as follows. For integral classes $\gamma_i \in H^{m_i}(X, \mathbb{Z})$, $1 \leq i \leq n$, let

$$\tau : H^m(X) \rightarrow H^{m-2}(P_1(X, \beta)), \quad \tau(\gamma) = \pi_{P*}(\pi_X^* \gamma \cup \text{ch}_3(\mathbb{F})), \tag{4}$$

where π_X, π_P are the projections from $X \times P_1(X, \beta)$ to the corresponding factors, $\mathbb{I} = (\pi_X^* \mathcal{O}_X \rightarrow \mathbb{F})$ is the universal pair, and $\text{ch}_3(\mathbb{F})$ is the Poincaré dual to the fundamental cycle of \mathbb{F} .

Then we define stable pair invariants

$$P_{1,\beta}(\gamma_1, \dots, \gamma_n) := \int_{[P_1(X,\beta)]^{\text{vir}}} \prod_{i=1}^n \tau(\gamma_i).$$

We propose the following interpretation of (1), (2) using stable pair invariants.

Conjecture 1.1 (Conjecture 2.5). *For a suitable choice of orientation, we have*

$$P_{1,\beta}(\gamma_1, \dots, \gamma_n) = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ \beta_1, \beta_2 \geq 0}} n_{0,\beta_1}(\gamma_1, \dots, \gamma_n) \cdot P_{0,\beta_2},$$

where the sum is over all possible effective classes, and we set $n_{0,0}(\gamma_1, \dots, \gamma_n) := 0$ and $P_{0,0} := 1$. In particular, when β is irreducible,

$$P_{1,\beta}(\gamma_1, \dots, \gamma_n) = n_{0,\beta}(\gamma_1, \dots, \gamma_n).$$

Conjecture 1.2 (Conjecture 2.6). *For a suitable choice of orientation, we have*

$$\sum_{\beta \geq 0} P_{0,\beta} q^\beta = \prod_{\beta > 0} M(q^\beta)^{n_{1,\beta}},$$

where $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function and $P_{0,0} := 1$.

For instance, when the Picard number of X is 1, for an irreducible curve class β , the above identity implies

$$\begin{aligned}
 P_{0,\beta} &= n_{1,\beta}, \\
 P_{0,2\beta} &= n_{1,2\beta} + 3n_{1,\beta} + \binom{n_{1,\beta}}{2}, \\
 P_{0,3\beta} &= n_{1,3\beta} + n_{1,\beta} \cdot n_{1,2\beta} + 6n_{1,\beta} + 6\binom{n_{1,\beta}}{2} + \binom{n_{1,\beta}}{3},
 \end{aligned}$$

by comparing the coefficients of q^β , $q^{2\beta}$ and $q^{3\beta}$.

One issue with our current proposal (as in our earlier conjecture [13]) is that we do not have a general mechanism for choosing the orientation in the above conjectures. Currently, in the cases we examine in this paper, we choose orientations on a case-by-case basis to show the correct matching. It would be very interesting to construct canonical choices of orientation for these moduli spaces and study our conjectures using them.

Our proposal is based on a heuristic argument given in Section 2.5, where we show Conjectures 1.1 and 1.2 assuming that the CY 4-fold X is ‘ideal’, i.e. curves in X deform in some family of expected dimensions. Apart from that, we verify our conjecture in examples as follows.

1.4. *Verification of the conjectures, I: compact examples*

We first prove our conjectures for some special compact Calabi–Yau 4-folds.

Sextic 4-folds. Let $X \subseteq \mathbb{P}^5$ be a degree 6 smooth hypersurface and $[l] \in H_2(X, \mathbb{Z}) \cong H_2(\mathbb{P}^5, \mathbb{Z})$ be the line class. We check our conjectures for $\beta = [l], 2[l]$.

Proposition 1.3 (Propositions 3.1, 3.2). *Let X be a smooth sextic 4-fold and $[l] \in H_2(X, \mathbb{Z})$ be the line class. Then Conjectures 1.1 and 1.2 are true for $\beta = [l], 2[l]$.*

Elliptic fibrations. We consider a projective CY 4-fold X which admits an elliptic fibration $\pi : X \rightarrow \mathbb{P}^3$, given by a Weierstrass model (17). Let f be a general fiber of π and h be a hyperplane in \mathbb{P}^3 , and set

$$B = \pi^*h, \quad E = \iota(\mathbb{P}^3) \in H_6(X, \mathbb{Z}),$$

where ι is a section of π . Then we have

Proposition 1.4 (Propositions 3.4, 3.6). (1) *Conjecture 1.1 is true for the fiber class $\beta = [f]$ and $\gamma = B^2, B \cdot E$.*

(2) *Conjecture 1.2 is true for the multiple fiber classes $\beta = r[f]$ ($r \geq 1$).*

In the above cases, we can directly compute the pair invariants and check the compatibility with the computation of GW invariants in [24].

Product of elliptic curve and Calabi–Yau 3-fold. Let $X = Y \times E$ be the product of a Calabi–Yau 3-fold and an elliptic curve E . We check our conjectures when the curve class comes from either Y or E .

Theorem 1.5 (Theorem 3.13, 3.15, Proposition 3.17). *Let $X = Y \times E$ be as above. Then:*

- (1) *Conjecture 1.1 is true for any irreducible curve class $\beta \in H_2(Y) \subseteq H_2(X)$ if Y is a complete intersection in a product of projective spaces.*
- (2) *Conjecture 1.2 is true for any irreducible curve class $\beta \in H_2(Y) \subseteq H_2(X)$.*
- (3) *Conjecture 1.2 is true for the classes $\beta = r[E]$ ($r \geq 1$).*

The proof of these results is briefly reviewed here.

For (1), when $\beta \in H_2(Y) \subseteq H_2(X)$ is an irreducible curve class, we have an isomorphism

$$P_n(X, \beta) \cong P_n(Y, \beta) \times E.$$

The corresponding virtual class satisfies (see Proposition 3.11)

$$[P_n(X, \beta)]^{\text{vir}} = [P_n(Y, \beta)]^{\text{vir}}_{\text{pair}} \otimes [E]$$

for a certain choice of orientation in defining the LHS, where the virtual class of $P_n(Y, \beta)$ is defined using the deformation-obstruction theory of pairs (Lemma 3.9) instead of the deformation-obstruction theory of complexes in the derived category used by [34].

In this case, we have a forgetful morphism

$$f : P_1(Y, \beta) \rightarrow M_{1,\beta}(Y), \quad (\mathcal{O}_Y \rightarrow F) \mapsto F, \tag{5}$$

to the moduli space $M_{1,\beta}(X)$ of 1-dimensional stable sheaves F with $[F] = \beta$ and $\chi(F) = 1$. We show that the map satisfies Manolache’s virtual push-forward formula (Proposition 3.10),

$$\int_{[P_1(Y,\beta)]^{\text{vir}}_{\text{pair}}} 1 = \int_{[M_{1,\beta}(Y)]^{\text{vir}}} 1.$$

Then Conjecture 1.1 can be reduced to Katz’s conjecture on the CY 3-fold Y [22] (Corollary 3.12). Combining with our previous proof of Katz’s conjecture for primitive classes [13, Cor. A.6], we can deduce (1) of Theorem 1.5.

As for (3), this is one of few cases where we can compute non-primitive curve classes and form generating series. The point is to identify pair moduli spaces on X with Hilbert schemes of points on Y and compute zero-dimensional DT invariants of Y .

Hyperkähler 4-folds. When the CY 4-fold X is hyperkähler, GW invariants vanish, and so do the GV type invariants. To verify our conjectures, it remains to prove the vanishing of pair invariants. A cosection map from the (trace-free) obstruction space is constructed and shown to be surjective and compatible with Serre duality (Proposition 3.18). We expect the following vanishing result then follows.

Claim 1.6 (Claim 3.19). *Let X be a projective hyperkähler 4-fold and $P_n(X, \beta)$ be the moduli space of stable pairs with $n \neq 0$ or $\beta \neq 0$. Then the virtual class satisfies*

$$[P_n(X, \beta)]^{\text{vir}} = 0.$$

At the moment, a Kiem–Li type theory of cosection localization for D-manifolds is not available in the literature. We believe that when such a theory is established, our claim should follow automatically. Nevertheless, we have the following evidence for the claim.

1. At least when $P_n(X, \beta)$ is smooth, Proposition 3.18 gives the vanishing of virtual class.
2. If there is a complex analytic version of (-2) -shifted symplectic geometry [37] and the corresponding construction of virtual classes [4], one could prove the vanishing result as in GW theory, i.e. taking a generic complex structure in the S^2 -twistor family of the hyperkähler 4-fold which does not support coherent sheaves, and then vanishing of virtual classes follows from their deformation invariance.

1.5. Verification of the conjectures, II: local 3-folds and surfaces

For a Fano 3-fold Y , we consider the non-compact CY 4-fold

$$X = K_Y.$$

In this case, the stable pair moduli space $P_n(X, \beta)$ is compact (Proposition 4.3), so we can formulate Conjectures 1.1 and 1.2 here (even though the target is not projective).

When the curve class $\beta \in H_2(X)$ is irreducible, we study this as follows. Similar to the case of the product of a CY 3-fold and an elliptic curve, for a certain choice of orientation, the virtual class of $P_n(X, \beta)$ satisfies (Proposition 4.3)

$$[P_n(X, \beta)]^{\text{vir}} = [P_n(Y, \beta)]_{\text{pair}}^{\text{vir}},$$

under the isomorphism

$$P_n(X, \beta) \cong P_n(Y, \beta).$$

And we have a virtual push-forward formula (Proposition 4.2)

$$f_*[P_1(Y, \beta)]_{\text{pair}}^{\text{vir}} = [M_{1,\beta}(Y)]^{\text{vir}},$$

where $f : P_1(Y, \beta) \rightarrow M_{1,\beta}(Y)$, $(\mathcal{O}_X \rightarrow F) \mapsto F$, is the morphism forgetting the section, $M_{1,\beta}(Y)$ is the moduli scheme of one-dimensional stable sheaves E on Y with $[E] = \beta$ and $\chi(E) = 1$. Then Conjecture 1.1 is easily reduced to our previous conjecture [13, Conjecture 0.2]. Combined with computations in [6], we have

Theorem 1.7 (Propositions 4.4, 4.5). *Let $X = K_Y$ be as above. Then:*

- (1) *Conjecture 1.1 is true for any irreducible curve class $\beta \in H_2(X) \cong H_2(Y)$ provided that (i) $Y \subseteq \mathbb{P}^4$ is a smooth hypersurface of degree $d \leq 4$, or (ii) $Y = S \times \mathbb{P}^1$ for a toric del Pezzo surface S .*
- (2) *Conjecture 1.2 is true for an irreducible curve class $\beta \in H_2(X) \cong H_2(Y)$ when $Y = \mathbb{P}^3$.*

Similarly for a smooth projective surface S , we consider the non-compact CY 4-fold

$$X = \text{Tot}_S(L_1 \oplus L_2),$$

where L_1, L_2 are line bundles on S satisfying $L_1 \otimes L_2 \cong K_S$. In particular, when β is irreducible and $L_i \cdot \beta < 0$ ($i = 1, 2$), the moduli space $P_n(X, \beta)$ of stable pairs on X is compact and smooth (Lemma 4.6 and Proposition 4.7). So pair invariants are well-defined and we can also study our conjectures in this case. In particular, we have

Proposition 1.8 (Proposition 4.8). *Let S be a del Pezzo surface and L_1^{-1}, L_2^{-1} be ample line bundles on S such that $L_1 \otimes L_2 \cong K_S$. Let $\beta \in H_2(X, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$ be an irreducible curve class on $X = \text{Tot}_S(L_1 \oplus L_2)$. Then Conjectures 1.1 and 1.2 are true for β .*

In fact such a del Pezzo surface must be \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$ (see the proof of Proposition 4.8), and the corresponding X is given by

$$\text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)), \quad \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1)).$$

By using computations due to Kool and Monavari [26], one can check Conjectures 1.1 and 1.2 for small degree curve classes on such X (see Section 4.3 and [10] for details).

1.6. Verification of the conjectures, III: local curves

Let C be a smooth projective curve. We consider a CY 4-fold X given by

$$X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3),$$

where L_1, L_2, L_3 are line bundles on C satisfying $L_1 \otimes L_2 \otimes L_3 \cong \omega_C$. The three-dimensional complex torus $T = (\mathbb{C}^*)^{\times 3}$ acts on X fiberwise over C . The T -equivariant GW invariants

$$\text{GW}_{g,d[C]}(X) \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)$$

can be defined via equivariant residue. Here λ_i are the equivariant parameters with respect to the T -action.

On the other hand, there is a two-dimensional subtorus $T_0 \subseteq (\mathbb{C}^*)^3$ which preserves the CY 4-form on X . We may define equivariant pair invariants

$$P_{n,d[C]}(X) \in \mathbb{Q}(\lambda_1, \lambda_2)$$

as rational functions in terms of equivariant parameters of T_0 following a localization principle for DT_4 invariants (see Section 5.2, [12], [13, Sect. 4.2]).

When $C = \mathbb{P}^1$ and $X = \mathcal{O}_{\mathbb{P}^1}(l_1, l_2, l_3)$, we explicitly determine $P_{1,d[C]}(X)$ for $d \leq 2$ (Proposition 5.5). Note in this case $P_{0, [\mathbb{P}^1]}(X) = 0$ and there are no insertions, so an equivariant analogue of Conjecture 1.1 is given by the following conjecture:

Conjecture 1.9 (Conjecture 5.6). *Let $X = \mathcal{O}_{\mathbb{P}^1}(l_1, l_2, l_3)$ for $l_1 + l_2 + l_3 = -2$. Then*

$$\text{GW}_{0,2}(X) = P_{1,2[\mathbb{P}^1]}(X) + \frac{1}{8} P_{1, [\mathbb{P}^1]}(X).$$

We can verify the above equivariant conjecture in a large number of examples.

Theorem 1.10 (Theorem 5.7). *Conjecture 1.9 is true if $|l_1|, |l_2| \leq 10$.*

When C is an elliptic curve and L_i 's are general degree 0 line bundles on C , one can define pair invariants and explicitly compute them.

Theorem 1.11 (Theorem 5.10). *Let C be an elliptic curve, $L_i \in \text{Pic}^0(C)$ ($i = 1, 2, 3$) general line bundles satisfying $L_1 \otimes L_2 \otimes L_3 \cong \omega_C$ and $X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3)$. Then stable pair invariants $P_{0,d[C]}(X)$ are well-defined and fit into the generating series*

$$\sum_{d \geq 0} P_{0,d[C]}(X) q^d = M(q),$$

where $M(q) := \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function.

Similarly, if we have $n_{1,\beta}$ ($\beta \in H_2(X, \mathbb{Z})$) such elliptic curves, then they contribute to pair invariants according to the formula

$$\sum_{\beta \geq 0} P_{0,\beta} q^\beta = \prod_{\beta > 0} M(q^\beta)^{n_{1,\beta}}.$$

This calculation arises in the heuristic argument for our genus 1 conjecture (Conjecture 1.2) in the ‘ideal’ situation as families of rational curves do not contribute to pair invariants $P_{0,\beta}$ (see Section 2.5 for more details).

1.7. Speculation on the generating series of stable pair invariants

As before, if we allow insertions, we can use the virtual class (3) and insertions to define stable pair invariants of $P_n(X, \beta)$ for any n .

For $\gamma \in H^4(X, \mathbb{Z})$, we have $\tau(\gamma) \in H^2(P_n(X, \beta), \mathbb{Z})$, so we may define

$$P_{n,\beta}(\gamma) := \int_{[P_n(X,\beta)]^{\text{vir}}} \tau(\gamma)^n.$$

Our computations and geometric arguments indicate that we may have the following formula, which generalizes the formula in Conjecture 1.1:

$$P_{n,\beta}(\gamma) = \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_n = \beta \\ \beta_0, \beta_1, \dots, \beta_n \geq 0}} P_{0,\beta_0} \cdot \prod_{i=1}^n n_{0,\beta_i}(\gamma). \tag{6}$$

To group these invariants into a generating series, we introduce notation

$$\text{PT}(X)(\exp(\gamma)) := \sum_{n,\beta} \frac{P_{n,\beta}(\gamma)}{n!} y^n q^\beta.$$

Assuming Conjecture 1.2, (6) is equivalent to the following Gopakumar–Vafa type formula:

$$\text{PT}(X)(\exp(\gamma)) = \prod_{\beta} (\exp(y q^\beta)^{n_{0,\beta}(\gamma)} \cdot M(q^\beta)^{n_{1,\beta}}),$$

where $n_{0,\beta}(\gamma)$ and $n_{1,\beta}$ are genus 0 and 1 GV type invariants of X (1), (2) respectively and $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function. As mentioned before, GW invariants on CY 4-folds vanish for $g > 1$, so they do not form a nice generating series as in the 3-fold case. Here the advantage of considering stable pair invariants is that we can use them to form a generating series which is conjecturally of GV form.

A heuristic explanation of the formula will be given in Section 2.5. Some more analysis will be pursued in a future work (see e.g. [15, 16]).

1.8. Notation and convention

In this paper, all varieties and schemes are defined over \mathbb{C} . For a morphism $\pi: X \rightarrow Y$ of schemes, and for $\mathcal{F}, \mathcal{G} \in D^b(\text{Coh}(X))$, we denote by $\mathbf{R}\mathcal{H}om_\pi(\mathcal{F}, \mathcal{G})$ the functor $\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$. We also denote by $\text{ext}^i(\mathcal{F}, \mathcal{G})$ the dimension of $\text{Ext}_X^i(\mathcal{F}, \mathcal{G})$.

A class $\beta \in H_2(X, \mathbb{Z})$ is called *irreducible* (resp. *primitive*) if it is not the sum of two non-zero effective classes (resp. if it is not a positive integer multiple of an effective class).

2. Definitions and conjectures

Throughout this paper, unless stated otherwise, X is always a smooth projective Calabi–Yau 4-fold, i.e. $K_X \cong \mathcal{O}_X$.

2.1. GW/GV conjecture on CY 4-folds

Let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of genus g , n -pointed stable maps to X with curve class β . Its virtual dimension is given by

$$-K_X \cdot \beta + (\dim X - 3)(1 - g) + n = 1 - g + n.$$

For integral classes

$$\gamma_i \in H^{m_i}(X, \mathbb{Z}), \quad 1 \leq i \leq n, \tag{7}$$

the GW invariant is defined by

$$\text{GW}_{g,\beta}(\gamma_1, \dots, \gamma_n) = \int_{[\overline{M}_{g,n}(X,\beta)]^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^*(\gamma_i), \tag{8}$$

where $\text{ev}_i: \overline{M}_{g,n}(X, \beta) \rightarrow X$ is the i -th evaluation map.

For $g = 0$, the virtual dimension of $\overline{M}_{0,n}(X, \beta)$ is $n + 1$, and (8) is 0 unless

$$\sum_{i=1}^n (m_i - 2) = 2. \tag{9}$$

In analogy with the Gopakumar–Vafa conjecture for CY 3-folds [18], Klemm–Pandharipande [24] defined invariants $n_{0,\beta}(\gamma_1, \dots, \gamma_n)$ on CY 4-folds by the identity

$$\sum_{\beta>0} \text{GW}_{0,\beta}(\gamma_1, \dots, \gamma_n)q^\beta = \sum_{\beta>0} n_{0,\beta}(\gamma_1, \dots, \gamma_n) \sum_{d=1}^\infty d^{n-3}q^{d\beta},$$

and conjecture the following

Conjecture 2.1 ([24, Conjecture 0]). *The invariants $n_{0,\beta}(\gamma_1, \dots, \gamma_n)$ are integers.*

For $g = 1$, the virtual dimension of $\overline{M}_{1,0}(X, \beta)$ is 0, so no insertions are needed. The genus 1 GW invariant

$$\text{GW}_{1,\beta} = \int_{[\overline{M}_{1,0}(X,\beta)]^{\text{vir}}} \in \mathbb{Q}$$

is also expected to be described in terms of certain integer valued invariants.

Let S_1, \dots, S_k be a basis of the free part of $H^4(X, \mathbb{Z})$ and

$$\sum_{i,j} g^{ij} [S_i \otimes S_j] \in H^8(X \times X, \mathbb{Z})$$

be the (4, 4)-component of the Künneth decomposition of the diagonal. For β_1, β_2 in $H_2(X, \mathbb{Z})$, the meeting number $m_{\beta_1,\beta_2} \in \mathbb{Z}$ is introduced in [24] as a virtual number of rational curves of class β_1 meeting rational curves of class β_2 . They are uniquely determined by the following rules:

- (i) The meeting invariants are symmetric: $m_{\beta_1,\beta_2} = m_{\beta_2,\beta_1}$.
- (ii) If either $\text{deg}(\beta_1) \leq 0$ or $\text{deg}(\beta_2) \leq 0$, then $m_{\beta_1,\beta_2} = 0$.
- (iii) If $\beta_1 \neq \beta_2$, then

$$m_{\beta_1,\beta_2} = \sum_{i,j} n_{0,\beta_1}(S_i)g^{ij}n_{0,\beta_2}(S_j) + m_{\beta_1,\beta_2-\beta_1} + m_{\beta_1-\beta_2,\beta_2}.$$

- (iv) If $\beta_1 = \beta_2 = \beta$, then

$$m_{\beta,\beta} = n_{0,\beta}(c_2(X)) + \sum_{i,j} n_{0,\beta}(S_i)g^{ij}n_{0,\beta}(S_j) - \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2}.$$

The invariants $n_{1,\beta}$ are uniquely defined by the identity

$$\begin{aligned} \sum_{\beta>0} \text{GW}_{1,\beta}q^\beta &= \sum_{\beta>0} n_{1,\beta} \sum_{d=1}^\infty \frac{\sigma(d)}{d} q^{d\beta} + \frac{1}{24} \sum_{\beta>0} n_{0,\beta}(c_2(X)) \log(1 - q^\beta) \\ &\quad - \frac{1}{24} \sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2} \log(1 - q^{\beta_1+\beta_2}), \end{aligned}$$

where $\sigma(d) = \sum_{i|d} i$.

Conjecture 2.2 ([24, Conjecture 1]). *The invariants $n_{1,\beta}$ are integers.*

For $g \geq 2$, GW invariants vanish for dimensional reasons, so the GW/GV type integrality conjecture on CY 4-folds only applies for genus 0 and 1. In [24], GW invariants are computed directly in many examples using localization or mirror symmetry to support the conjectures.

2.2. *Review of DT_4 invariants*

Let us first introduce the set-up of DT_4 invariants. We fix an ample divisor ω on X and take a cohomology class $v \in H^*(X, \mathbb{Q})$.

The coarse moduli space $M_\omega(v)$ of ω -Gieseker semistable sheaves E on X with $\text{ch}(E) = v$ exists as a projective scheme. We always assume that $M_\omega(v)$ is a fine moduli space, i.e. any point $[E] \in M_\omega(v)$ is stable and there is a universal family

$$\mathcal{E} \in \text{Coh}(X \times M_\omega(v)). \tag{10}$$

In [4, 12], under certain hypotheses, the authors construct a DT_4 virtual class

$$[M_\omega(v)]^{\text{vir}} \in H_{2-\chi(v,v)}(M_\omega(v), \mathbb{Z}), \tag{11}$$

where $\chi(-, -)$ is the Euler pairing. Notice that this class may not necessarily be algebraic.

Roughly speaking, in order to construct such a class, one chooses at every point $[E]$ in $M_\omega(v)$, a half-dimensional real subspace

$$\text{Ext}_+^2(E, E) \subset \text{Ext}^2(E, E)$$

of the usual obstruction space $\text{Ext}^2(E, E)$, on which the quadratic form Q defined by Serre duality is real and positive definite. Then one glues local Kuranishi-type models of the form

$$\kappa_+ = \pi_+ \circ \kappa : \text{Ext}^1(E, E) \rightarrow \text{Ext}_+^2(E, E),$$

where κ is a Kuranishi map of $M_\omega(v)$ at E and π_+ is the projection according to the decomposition

$$\text{Ext}^2(E, E) = \text{Ext}_+^2(E, E) \oplus \sqrt{-1} \cdot \text{Ext}_+^2(E, E).$$

In [12], local models are glued in three special cases:

- (1) when $M_\omega(v)$ consists of locally free sheaves only;
- (2) when $M_\omega(v)$ is smooth;
- (3) when $M_\omega(v)$ is a shifted cotangent bundle of a derived smooth scheme.

And the corresponding virtual classes are constructed using either gauge theory or algebro-geometric perfect obstruction theory.

The general gluing construction is due to Borisov–Joyce [4]¹, based on Pantev–Töen–Vaquié–Vezzosi’s theory of shifted symplectic geometry [37] and Joyce’s theory of derived C^∞ -geometry. The corresponding virtual class is constructed using Joyce’s D-manifold theory (a machinery similar to Fukaya–Oh–Ohta–Ono’s theory of Kuranishi space structures used in defining Lagrangian Floer theory).

In this paper, all computations and examples will only involve the virtual class constructions in situations (2), (3) mentioned above. We briefly review them as follows:

- When $M_\omega(v)$ is smooth, the obstruction sheaf $Ob \rightarrow M_\omega(v)$ is a vector bundle endowed with a quadratic form Q via Serre duality. Then the DT_4 virtual class is given by

$$[M_\omega(v)]^{\text{vir}} = \text{PD}(e(Ob, Q)).$$

Here $e(Ob, Q)$ is the half-Euler class of (Ob, Q) (i.e. the Euler class of its real form Ob_+), and $\text{PD}(-)$ is its Poincaré dual. Note that the half-Euler class satisfies

$$\begin{aligned} e(Ob, Q)^2 &= (-1)^{\text{rk}(Ob)/2} e(Ob) && \text{if } \text{rk}(Ob) \text{ is even,} \\ e(Ob, Q) &= 0 && \text{if } \text{rk}(Ob) \text{ is odd.} \end{aligned}$$

- When $M_\omega(v)$ is a shifted cotangent bundle of a derived smooth scheme, roughly speaking, this means that at any closed point $[F] \in M_\omega(v)$, we have a Kuranishi map of type

$$\kappa: \text{Ext}^1(F, F) \rightarrow \text{Ext}^2(F, F) = V_F \oplus V_F^*,$$

where κ factors through a maximal isotropic subspace V_F of $(\text{Ext}^2(F, F), Q)$. Then the DT_4 virtual class of $M_\omega(v)$ is, roughly speaking, the virtual class of the perfect obstruction theory formed by $\{V_F\}_{F \in M_\omega(v)}$. When $M_\omega(v)$ is furthermore smooth as a scheme, it is simply the Euler class of the vector bundle $\{V_F\}_{F \in M_\omega(v)}$ over $M_\omega(v)$.

On orientations. To construct the above virtual class (11) with coefficients in \mathbb{Z} (instead of \mathbb{Z}_2), we need an orientability result for $M_\omega(v)$, which is stated as follows. Let

$$\mathcal{L} := \det(\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{E}, \mathcal{E})) \in \text{Pic}(M_\omega(v)), \quad \pi_M: X \times M_\omega(v) \rightarrow M_\omega(v), \quad (12)$$

be the determinant line bundle of $M_\omega(v)$, equipped with a symmetric pairing Q induced by Serre duality. An *orientation* of (\mathcal{L}, Q) is a reduction of its structure group (from $O(1, \mathbb{C})$) to $SO(1, \mathbb{C}) = \{1\}$; in other words, we require a choice of square root of the isomorphism

$$Q : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_{M_\omega(v)} \quad (13)$$

to construct the virtual class (11). An orientability result was first obtained for $M_\omega(v)$ when the CY 4-fold X satisfies $\text{Hol}(X) = SU(4)$ and $H^{\text{odd}}(X, \mathbb{Z}) = 0$ [11, Thm. 2.2] and it has recently been generalized to arbitrary CY 4-folds by [7]. Notice that if an orientation exists, the set of orientations forms a torsor for $H^0(M_\omega(v), \mathbb{Z}_2)$.

¹One needs to assume that $M_\omega(v)$ can be given a (-2) -shifted symplectic structure as in [4, Claim 3.29] to apply their constructions. In the stable pair case, we show this can be done in Lemma 2.3.

2.3. *Stable pair invariants on CY 4-folds*

The notion of stable pairs on a CY 4-fold X can be defined similarly to the case of 3-folds [34]. It consists of data

$$(F, s), \quad F \in \text{Coh}(X), \quad s: \mathcal{O}_X \rightarrow F,$$

where F is a pure one-dimensional sheaf and s is surjective in dimension 1.

For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, let

$$P_n(X, \beta) \tag{14}$$

be the moduli space of stable pairs (F, s) on X such that $[F] = \beta$ and $\chi(F) = n$. It is a projective scheme parametrizing two-term complexes

$$I = (\mathcal{O}_X \xrightarrow{s} F) \in \text{D}^b(\text{Coh}(X))$$

in the derived category of coherent sheaves on X .

Similar to moduli spaces of stable sheaves, the stable pair moduli space (14) admits a deformation-obstruction theory, whose tangent, obstruction and ‘higher’ obstruction spaces are given by

$$\text{Ext}^1(I, I)_0, \quad \text{Ext}^2(I, I)_0, \quad \text{Ext}^3(I, I)_0,$$

where $(-)_0$ denotes the trace-free part. Note that Serre duality gives an isomorphism $\text{Ext}^1_0 \cong (\text{Ext}^3_0)^\vee$ and a non-degenerate quadratic form on Ext^2_0 . Moreover, we have

Lemma 2.3. *The stable pair moduli space $P_n(X, \beta)$ can be given the structure of a (-2) -shifted symplectic derived scheme in the sense of Pantev–Töen–Vaquié–Vezzosi [37].*

Proof. By [34, Thm. 2.7], $P_n(X, \beta)$ is a disjoint union of connected components of the moduli stack of perfect complexes of coherent sheaves of trivial determinant on X , whose (-2) -shifted symplectic structure is constructed by [37, Thm. 0.1] (see [37, Sect. 3.2, p. 48] for pull-back to determinant fixed substack). ■

Let $\mathbb{I} = (\mathcal{O}_{X \times P_n(X, \beta)} \rightarrow \mathbb{F})$ be the universal pair. The determinant line bundle

$$\mathcal{L} := \det(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}, \mathbb{I})_0) \in \text{Pic}(P_n(X, \beta))$$

is endowed with a non-degenerate quadratic form Q defined by Serre duality, where $\pi_P: X \times P_n(X, \beta) \rightarrow P_n(X, \beta)$ is the projection. As before, the orientability issue for the pair moduli space $P_n(X, \beta)$ is whether the structure group of the quadratic line bundle (\mathcal{L}, Q) can be reduced from $O(1, \mathbb{C})$ to $SO(1, \mathbb{C}) = \{1\}$. By [7], these moduli spaces are always orientable.

Theorem 2.4. *Let X be a CY 4-fold, $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$. Then $P_n(X, \beta)$ has a virtual class*

$$[P_n(X, \beta)]^{\text{vir}} \in H_{2n}(P_n(X, \beta), \mathbb{Z}), \tag{15}$$

in the sense of Borisov–Joyce [4], depending on the choice of orientation.

Proof. By Lemma 2.3, $P_n(X, \beta)$ has a (-2) -shifted symplectic structure. By [7], $P_n(X, \beta)$ is orientable in the sense stated above. Then we may apply [4, Thm. 1.1] to $P_n(X, \beta)$. ■

When $n = 0$, the virtual dimension of the virtual class (15) is zero. We define the stable pair invariant

$$P_{0,\beta} := \int_{[P_0(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Z},$$

as the degree of the virtual class.

When $n = 1$, the (real) virtual dimension of the virtual class (15) is 2, so we consider insertions as follows. For integral classes $\gamma_i \in H^{m_i}(X, \mathbb{Z})$, $1 \leq i \leq n$, let

$$\begin{aligned} \tau: H^m(X) &\rightarrow H^{m-2}(P_1(X, \beta)), \\ \tau(\gamma) &:= (\pi_P)_*(\pi_X^* \gamma \cup \text{ch}_3(\mathbb{F})), \end{aligned}$$

where π_X, π_P are the projections from $X \times P_1(X, \beta)$ to the corresponding factors, $\mathbb{I} = (\pi_X^* \mathcal{O}_X \rightarrow \mathbb{F})$ is the universal pair and $\text{ch}_3(\mathbb{F})$ is the Poincaré dual to the fundamental cycle of \mathbb{F} .

We define the stable pair invariant

$$P_{1,\beta}(\gamma_1, \dots, \gamma_n) := \int_{[P_1(X,\beta)]^{\text{vir}}} \prod_{i=1}^n \tau(\gamma_i).$$

2.4. Relations to GW/GV conjecture on CY_4

We use the stable pair invariants defined in Section 2.3 to give a sheaf-theoretic approach to the GW/GV conjecture of Section 2.1.

Conjecture 2.5 (Genus 0). *For a suitable choice of orientation, we have*

$$P_{1,\beta}(\gamma_1, \dots, \gamma_n) = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ \beta_1, \beta_2 \geq 0}} n_{0,\beta_1}(\gamma_1, \dots, \gamma_n) \cdot P_{0,\beta_2},$$

where the sum is over all possible effective classes, and we set $n_{0,0}(\gamma_1, \dots, \gamma_n) := 0$ and $P_{0,0} := 1$.

Conjecture 2.6 (Genus 1). *For a suitable choice of orientation, we have*

$$\sum_{\beta \geq 0} P_{0,\beta} q^\beta = \prod_{\beta > 0} M(q^\beta)^{n_{1,\beta}},$$

where $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function and $P_{0,0} := 1$.

2.5. *Heuristic approach to conjectures*

In this subsection, we give a heuristic argument to explain why we expect Conjectures 2.5 and 2.6 (and equality (6)) to be true. Even in this heuristic discussion, we ignore questions of orientation.

Let X be an ‘ideal’ CY_4 in the sense that all curves of X deform in families of expected dimensions, and have expected generic properties, i.e.:

1. Any rational curve in X is a chain of smooth \mathbb{P}^1 's with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$, and moves in a compact 1-dimensional smooth family of embedded rational curves, whose general member is smooth with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$.
2. Any elliptic curve E in X is smooth, super-rigid, i.e. the normal bundle is $L_1 \oplus L_2 \oplus L_3$ for general degree 0 line bundle L_i on E satisfying $L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_E$. Furthermore any two elliptic curves are disjoint and disjoint from all families of rational curves on X .
3. There is no curve in X with genus $g \geq 2$.

$P_0(X, \beta)$ and genus 1 conjecture. Under our ideal assumptions, a one-dimensional Cohen–Macaulay scheme C supported in one of our families of rational curves has $\chi(\mathcal{O}_C) \geq 1$, so for any stable pair $I = (\mathcal{O}_X \rightarrow F) \in P_0(X, \beta)$, the sheaf F can only be supported on some rigid elliptic curves in X . For a rigid elliptic curve E with $[E] = \beta$ and ‘general’ normal bundle (i.e. direct sum of three degree 0 general line bundles on E), its contribution to the pair invariant is

$$\sum_{m \geq 0} P_{0,m[E]} q^m = M(q), \quad \text{where} \quad M(q) = \prod_{k \geq 1} (1 - q^k)^{-k},$$

by a localization calculation (see Theorem 5.10). Similarly, if we have $n_{1,\beta}$ ($\beta \in H_2(X, \mathbb{Z})$) many such elliptic curves, then they contribute to pair invariants according to the formula

$$\sum_{\beta \geq 0} P_{0,\beta} q^\beta = \prod_{\beta > 0} M(q^\beta)^{n_{1,\beta}}.$$

$P_1(X, \beta)$ and genus 0 conjecture. Given a stable pair $I = (\mathcal{O}_X \rightarrow F) \in P_1(X, \beta)$, F may be supported on a union of rational curves and elliptic curves. Let $C := \text{supp}(F)$. Then $C = C_1 \sqcup C_2$ is a disjoint union of ‘rational curve components’ and ‘elliptic curve components’. Note that a Cohen–Macaulay scheme D in $\text{Tot}_{\mathbb{P}^1}(-1, -1, 0)$ (resp. in $\text{Tot}_E(L_1 \oplus L_2 \oplus L_3)$, where E a smooth elliptic curve and L_i 's are degree 0 general line bundles on E) satisfies $\chi(\mathcal{O}_D) \geq 1$ (resp. $\chi(\mathcal{O}_D) \geq 0$).

Thus from the exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow F \rightarrow Q \rightarrow 0,$$

we know that if $C_1 \neq \emptyset$, then $Q = 0$ and $F \cong \mathcal{O}_{C_1 \sqcup C_2}$ (with $\chi(\mathcal{O}_{C_1}) = 1, \chi(\mathcal{O}_{C_2}) = 0$). Note that when $C_1 = \emptyset$, i.e. when F is supported on elliptic curves, once we include

insertions, these stable pairs do not contribute to the invariant

$$\int_{[P_1(X, \beta)]^{\text{vir}}} \tau(\gamma).$$

So we only consider the case when $F \cong \mathcal{O}_{C_1 \sqcup C_2}$ with C_1 supported on rational curves in a one-dimensional family $\{C_t\}_{t \in T}$. We may further assume the support of C_1 is smooth with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$ due to the presence of insertions, at which point it must have multiplicity 1 as well.

Since the families of rational curves are disjoint from the elliptic curves, the moduli space $P_1(X, \beta)$ of stable pairs is a disjoint union of products of rational curve families (with curve class β_1) and $P_0(X, \beta_2)$ (where $\beta_1 + \beta_2 = \beta$). And a direct calculation shows the corresponding virtual class factors as the product of the fundamental class of those rational curve families and $[P_0(X, \beta_2)]^{\text{vir}}$. For $\gamma \in H^4(X)$, we then have

$$\int_{[P_1(X, \beta)]^{\text{vir}}} \tau(\gamma) = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ \beta_1, \beta_2 \geq 0}} n_{0, \beta_1}(\gamma) \cdot P_{0, \beta_2}.$$

$P_n(X, \beta)$ and generating series. For the moduli space $P_{n, \beta}(X)$ of stable pairs with $n \geq 1$, we want to compute

$$\int_{[P_n(X, \beta)]^{\text{vir}}} \tau(\gamma)^n, \quad \gamma \in H^4(X, \mathbb{Z}),$$

when X is an ideal CY 4-fold. Let $\{Z_i\}_{i=1}^n$ be 4-cycles which represent the class γ . For dimensional reasons, we may assume for any $i \neq j$ the rational curves which meet Z_i are disjoint from those which meet Z_j . The insertions cut out the moduli space and pick up stable pairs whose support intersects all $\{Z_i\}_{i=1}^n$. We denote the moduli space of such ‘incident’ stable pairs by

$$Q_n(X, \beta; \{Z_i\}_{i=1}^n) \subseteq P_n(X, \beta).$$

Then we claim that

$$Q_n(X, \beta; \{Z_i\}_{i=1}^n) = \coprod_{\beta_0 + \beta_1 + \dots + \beta_n = \beta} P_0(X, \beta_0) \times Q_1(X, \beta_1; Z_1) \times \dots \times Q_1(X, \beta_n; Z_n), \quad (16)$$

where $Q_1(X, \beta_i; Z_i)$ is the moduli space of stable pairs supported on rational curves (in class β_i) which meet Z_i .

Indeed, take a stable pair $(\mathcal{O}_X \rightarrow F)$ in $Q_n(X, \beta; \{Z_i\}_{i=1}^n)$. Then F decomposes into a direct sum $\bigoplus_{i=0}^n F_i$, where F_0 is supported on elliptic curves and each F_i for $1 \leq i \leq n$ is supported on rational curves which meet Z_i . As explained before, a Cohen–Macaulay scheme C supported in the family of rational curves (resp. elliptic curves)

satisfies $\chi(\mathcal{O}_C) \geq 1$ (resp. $\chi(\mathcal{O}_C) \geq 0$), so $\chi(F_0) \geq 0$ and $\chi(F_i) \geq 1$ for $1 \leq i \leq n$. Hence $\chi(F_0) = 0$ and $\chi(F_i) = 1$ for $1 \leq i \leq n$. Therefore (16) holds.

Moreover each $Q_1(X, \beta_i; Z_i)$ consists of finitely many rational curves which meet Z_i , whose number is exactly $n_{0,\beta_i}(\gamma)$. By counting the number of points in $P_0(X, \beta_0)$ and $Q_1(X, \beta_i; Z_i)$'s, we obtain

$$P_{n,\beta}(\gamma) := \int_{[P_n(X,\beta)]^{\text{vir}}} \tau(\gamma)^n = \int_{[Q_n(X,\beta;\gamma)]^{\text{vir}}} 1 = \sum_{\substack{\beta_0+\beta_1+\dots+\beta_n=\beta \\ \beta_0,\beta_1,\dots,\beta_n \geq 0}} P_{0,\beta_0} \cdot \prod_{i=1}^n n_{0,\beta_i}(\gamma).$$

The above arguments give a heuristic explanation for the formula

$$\sum_{n,\beta} \frac{P_{n,\beta}(\gamma)}{n!} y^n q^\beta = \prod_{\beta} (\exp(yq^\beta)^{n_{0,\beta}(\gamma)} \cdot M(q^\beta)^{n_{1,\beta}})$$

mentioned in Section 1.7.

3. Compact examples

In this section, we verify Conjectures 2.5 and 2.6 for certain compact Calabi–Yau 4-folds.

3.1. Sextic 4-folds

Let X be a smooth sextic 4-fold, i.e. a smooth degree 6 hypersurface of \mathbb{P}^5 . By the Lefschetz hyperplane theorem, $H_2(X, \mathbb{Z}) \cong H_2(\mathbb{P}^5, \mathbb{Z}) \cong \mathbb{Z}$. In order to verify our conjectures, we may use deformation invariance and assume X is general in the (projective) space $\mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(6)))$ of degree 6 hypersurfaces.

Genus 0. For the genus 0 conjecture, we have:

Proposition 3.1. *Let X be a smooth sextic 4-fold and $[l] \in H_2(X, \mathbb{Z})$ be the line class. Then Conjecture 2.5 is true for $\beta = [l], 2[l]$.*

Proof. In such cases, $P_{0,\beta}(X) = 0$ by Proposition 3.2. So we only need to show

$$P_{1,\beta}(X)(\gamma_1, \dots, \gamma_n) = n_{0,\beta}(\gamma_1, \dots, \gamma_n).$$

We consider $\beta = 2[l]$, since the degree 1 case follows from the same argument. A Cohen–Macaulay curve C in X with $[C] = \beta$ has $\chi(\mathcal{O}_C) = 1$. For a stable pair $(\mathcal{O}_X \rightarrow F)$ in $P_1(X, \beta)$, there is an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow F \rightarrow Q \rightarrow 0,$$

where C is the support of F . Since $1 = \chi(F) = \chi(\mathcal{O}_C) + \chi(Q)$, we must have $Q = 0$ and $F \cong \mathcal{O}_C$.

When X is a general sextic, C is either a smooth conic or a pair of distinct intersecting lines (see e.g. [5, Prop. 1.4]). The morphism²

$$P_1(X, \beta) \rightarrow M_{1,\beta}(X), \quad I = (\mathcal{O}_X \rightarrow F) \mapsto F,$$

to the moduli space $M_{1,\beta}(X)$ of one-dimensional stable sheaves, with $[F] = \beta$ and $\chi(F) = 1$, is an isomorphism. Furthermore, under the isomorphism, we have identifications

$$\text{Ext}^1(I, I)_0 \cong \text{Ext}^1(F, F) \cong \mathbb{C}, \quad \text{Ext}^2(I, I)_0 \cong \text{Ext}^2(F, F) = 0,$$

of deformation and obstruction spaces [5, Prop. 2.2]). So one can identify the virtual classes

$$[P_1(X, \beta)]^{\text{vir}} = [M_{1,\beta}(X)]^{\text{vir}},$$

for a certain choice of orientation. Then Conjecture 2.5 reduces to our previous conjecture [13, Conjecture 0.2], which has been verified in this setting in [5, Thm. 2.4]. ■

Genus 1. From [24, Table 2, p. 33], we know genus 1 GV type invariants of X are 0 for degree 1 and 2 classes. In these cases, pair invariants are obviously 0.

Proposition 3.2. *Let X be a smooth sextic 4-fold and $[l] \in H_2(X, \mathbb{Z})$ be the line class. Then Conjecture 2.6 is true for $\beta = [l], 2[l]$.*

Proof. Let $\beta = [l], 2[l]$. For a stable pair $(\mathcal{O}_X \rightarrow F) \in P_0(X, \beta)$, there is an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow F \rightarrow Q \rightarrow 0,$$

where C is the support of F and Q is zero-dimensional. A Cohen–Macaulay curve C in X with $[C] = \beta$ has $\chi(\mathcal{O}_C) \geq 1$, contradicting $\chi(F) = 0$. So $P_0(X, \beta) = \emptyset$. ■

3.2. Elliptic fibration

For $Y = \mathbb{P}^3$, we take general elements

$$u \in H^0(Y, \mathcal{O}_Y(-4K_Y)), \quad v \in H^0(Y, \mathcal{O}_Y(-6K_Y)).$$

Let X be a CY 4-fold with an elliptic fibration

$$\pi: X \rightarrow Y \tag{17}$$

given by the equation

$$zy^2 = x^3 + uxz^2 + vz^3$$

²The map is well-defined as \mathcal{O}_C is stable [5, Prop. 2.2].

in the \mathbb{P}^2 -bundle

$$\mathbb{P}(\mathcal{O}_Y(-2K_Y) \oplus \mathcal{O}_Y(-3K_Y) \oplus \mathcal{O}_Y) \rightarrow Y,$$

where $[x : y : z]$ is the homogeneous coordinate of the above projective bundle. A general fiber of π is a smooth elliptic curve, and any singular fiber is either a nodal or a cuspidal plane curve. Moreover, π admits a section ι whose image corresponds to the fiber point $[0 : 1 : 0]$.

Let h be a hyperplane in \mathbb{P}^3 , let f be a general fiber of $\pi: X \rightarrow Y$ and set

$$B = \pi^*h, \quad E = \iota(\mathbb{P}^3) \in H_6(X, \mathbb{Z}). \tag{18}$$

Genus 0. We consider the stable pair moduli space $P_1(X, [f])$ for the fiber class of π and verify Conjecture 2.5 in this case.

Lemma 3.3. *Let $[f]$ be the fiber class of the elliptic fibration (17). Then we have an isomorphism*

$$P_1(X, [f]) \rightarrow X,$$

under which the virtual class satisfies

$$[P_1(X, [f])]^{\text{vir}} = \pm \text{PD}(c_3(X)),$$

where the sign corresponds to the choice of orientation in defining the LHS.

Proof. Since $[f]$ is irreducible, we have a morphism

$$\phi : P_1(X, [f]) \rightarrow M_{1,[f]}(X) \cong X$$

to the moduli space $M_{1,[f]}(X)$ of one-dimensional stable sheaves on X with Chern character $(0, 0, 0, [f], 1)$ (which is isomorphic to X by [13, Lem. 2.1]). The fiber of ϕ over F is $\mathbb{P}(H^0(X, F))$ [36, p. 270].

By [13, Lem. 2.2], any $F \in M_{1,[f]}(X)$ is scheme-theoretically supported on a fiber, and $F = (i_t)_* m_x^\vee$ for some $x \in X_t := \pi^{-1}(t)$, where $i_t : X_t \rightarrow X$ is the inclusion and m_x is the maximal ideal sheaf of x in X_t . By Serre duality, we have

$$H^1(X, F) \cong H^1(X_t, m_x^\vee) \cong H^0(X_t, m_x)^\vee = 0.$$

Hence $H^0(X, F) \cong \mathbb{C}$, and ϕ is an isomorphism.

Next, we compare the obstruction theories. Let $I = (\mathcal{O}_X \rightarrow F) \in P_1(X, [f])$ be a stable pair. By applying $\mathbf{R}\text{Hom}_X(-, F)$ to $I \rightarrow \mathcal{O}_X \rightarrow F$, we obtain a distinguished triangle

$$\mathbf{R}\text{Hom}_X(F, F) \rightarrow \mathbf{R}\text{Hom}_X(\mathcal{O}_X, F) \rightarrow \mathbf{R}\text{Hom}_X(I, F),$$

whose cohomology gives an exact sequence

$$0 = H^1(X, F) \rightarrow \text{Ext}_X^1(I, F) \rightarrow \text{Ext}_X^2(F, F) \rightarrow H^2(X, F) = 0. \tag{19}$$

From the distinguished triangle

$$F \rightarrow I[1] \rightarrow \mathcal{O}_X[1],$$

we have the diagram

$$\begin{array}{ccccc}
 & & \mathbf{R}\Gamma(\mathcal{O}_X)[1] & \xlongequal{\quad} & \mathbf{R}\Gamma(\mathcal{O}_X)[1] \\
 & & \downarrow & & \downarrow \\
 \mathbf{R}\mathrm{Hom}_X(I, F) & \longrightarrow & \mathbf{R}\mathrm{Hom}_X(I, I)[1] & \longrightarrow & \mathbf{R}\mathrm{Hom}_X(I, \mathcal{O}_X)[1] \\
 & & \downarrow & & \downarrow \\
 & & \mathbf{R}\mathrm{Hom}_X(I, I)_0[1] & & \mathbf{R}\mathrm{Hom}_X(F, \mathcal{O}_X)[2]
 \end{array}$$

where the horizontal and vertical arrows are distinguished triangles. By taking cones, we obtain a distinguished triangle

$$\mathbf{R}\mathrm{Hom}_X(I, F) \rightarrow \mathbf{R}\mathrm{Hom}_X(I, I)_0[1] \rightarrow \mathbf{R}\mathrm{Hom}_X(F, \mathcal{O}_X)[2],$$

whose cohomology gives an exact sequence

$$0 \rightarrow \mathrm{Ext}_X^1(I, F) \rightarrow \mathrm{Ext}_X^2(I, I)_0 \rightarrow H^1(X, F)^\vee = 0. \tag{20}$$

Combining (19) and (20), we can identify the obstruction spaces

$$\mathrm{Ext}_X^2(I, I)_0 \cong \mathrm{Ext}_X^2(F, F).$$

Then under the isomorphism ϕ , their virtual classes can be identified. The identification of the virtual class of $M_{1,[f]}(X)$ with the Poincaré dual of the third Chern class of X can be found in [13, Lem. 2.1]. ■

Then by [13, Prop. 2.3], we have the following

Proposition 3.4. *Let $\pi : X \rightarrow Y$ be the elliptic fibration (17). Then Conjecture 2.5 is true for the fiber class $\beta = [f]$ and $\gamma = B^2, B \cdot E$ (see (18)).*

Genus 1. We consider the stable pair moduli space $P_0(X, r[f])$ for multiple fiber classes $r[f]$ ($r \geq 1$) of π and confirm Conjecture 2.6 in this case.

Lemma 3.5. *For any $r \in \mathbb{Z}_{\geq 1}$, there exists an isomorphism*

$$P_0(X, r[f]) \cong \mathrm{Hilb}^r(\mathbb{P}^3),$$

under which the virtual class is given by

$$[P_0(X, r[f])]^{\mathrm{vir}} = (-1)^r \cdot [\mathrm{Hilb}^r(\mathbb{P}^3)]^{\mathrm{vir}}$$

for a certain choice of orientation in defining the LHS, where $[\mathrm{Hilb}^r(\mathbb{P}^3)]^{\mathrm{vir}}$ is the DT_3 virtual class [40].

Proof. The proof is similar to the one in [41, Prop. 6.8]. We show that the natural morphism

$$\pi^* : \text{Hilb}^r(\mathbb{P}^3) \rightarrow P_0(X, r[f]) \tag{21}$$

is an isomorphism. Let $(s : \mathcal{O}_X \rightarrow F) \in P_0(X, r[f])$ be a stable pair. By the Harder–Narasimhan and Jordan–Hölder filtrations, we have

$$0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = F,$$

where the quotients $E_i = F_i / F_{i-1}$ are non-zero stable sheaves with decreasing slopes

$$\frac{\chi(E_1)}{r_1} \geq \dots \geq \frac{\chi(E_n)}{r_n}.$$

Here the slope of a zero-dimensional sheaf is defined to be infinity.

Since F is a pure one-dimensional sheaf, so $E_1 = F_1$ cannot be zero-dimensional ($r_1 \geq 1$). Therefore $\text{ch}(E_i) = (0, 0, 0, r_i[f], \chi(E_i))$ for some $r_i \geq 1$. The stability of E_i implies that it is scheme-theoretically supported on some fiber $X_{p_i} = \pi^{-1}(p_i)$ of π , i.e. $E_i = (\iota_{p_i})_*(E'_i)$ for some $\iota_{p_i} : X_{p_i} \hookrightarrow X$ and stable sheaf $E'_i \in \text{Coh}(X_{p_i})$.

Since $s : \mathcal{O}_X \rightarrow F$ is surjective in dimension 1, so is the composition $\mathcal{O}_X \rightarrow F \twoheadrightarrow E_n$. By adjunction, there is an isomorphism

$$\text{Hom}_X(\mathcal{O}_X, E_n) \cong \text{Hom}_{X_{p_n}}(\mathcal{O}_{X_{p_n}}, E'_n) \neq 0,$$

which implies that $\chi(E'_n) \geq 0$, hence $\chi(E_n) \geq 0$. Then

$$0 = \chi(F) = \sum_{i=1}^n \chi(E_i) \geq 0 \tag{22}$$

implies that $\chi(E_i) = 0$ for any i , and hence $E'_n \cong \mathcal{O}_{X_{p_n}}$ [19, Prop. 1.2.7].

By diagram chasing, we obtain a morphism $I_{X_{p_n}} \rightarrow F_{n-1}$ for the ideal sheaf $I_{X_{p_n}} \subseteq \mathcal{O}_X$ of X_{p_n} , which is surjective in dimension 1. Then so is the composition

$$I_{X_{p_n}} \rightarrow F_{n-1} \twoheadrightarrow E_{n-1}. \tag{23}$$

We have the isomorphism

$$\text{Hom}_X(I_{X_{p_n}}, E_{n-1}) \cong \text{Hom}_{X_{p_{n-1}}}(\iota_{p_{n-1}}^* I_{X_{p_n}}, E'_{n-1}) \neq 0.$$

Notice that $I_{X_{p_n}} \cong \pi^* I_{p_n}$ for the ideal sheaf $I_{p_n} \subseteq \mathcal{O}_{\mathbb{P}^3}$ of $p_n \in \mathbb{P}^3$ by the flatness of π , so

$$\iota_{p_{n-1}}^* I_{X_{p_n}} \cong \begin{cases} \pi^* N_{\{p_{n-1}\}/\mathbb{P}^3}^\vee \cong (\mathcal{O}_{X_{p_{n-1}}})^{\oplus 3} & \text{if } p_{n-1} = p_n, \\ \mathcal{O}_{X_{p_{n-1}}} & \text{if } p_{n-1} \neq p_n. \end{cases}$$

In either case, as before, we have $E'_{n-1} \cong \mathcal{O}_{X_{p_{n-1}}}$. Moreover the morphism (23) is the pull-back of a surjection $I_{p_n} \rightarrow \mathcal{O}_{p_{n-1}}$ by π^* .

By repeating the above argument, we see that each E_i is isomorphic to $\mathcal{O}_{X_{p_i}}$, $s: \mathcal{O}_X \rightarrow F$ is surjective and given by the pull-back of a surjection $\mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_Z$ by π^* for some zero-dimensional subscheme $Z \subset \mathbb{P}^3$ with length n . Using the section ι of $\pi: X \rightarrow \mathbb{P}^3$, we have the morphism $\iota^*: P_0(X, r[f]) \rightarrow \text{Hilb}^r(\mathbb{P}^3)$, which gives an inverse of (21). Therefore the morphism (21) is an isomorphism.

It remains to compare the virtual classes. We take $I_Z \in \text{Hilb}^r(\mathbb{P}^3)$ and use the spectral sequence

$$\text{Ext}_{\mathbb{P}^3}^*(I_Z, I_Z \otimes R^* \pi_* \mathcal{O}_X) \Rightarrow \text{Ext}_X^*(\pi^* I_Z, \pi^* I_Z),$$

where $R^* \pi_* \mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^3} \oplus K_{\mathbb{P}^3}[-1]$. This gives canonical isomorphisms

$$\begin{aligned} \text{Ext}_X^1(\pi^* I_Z, \pi^* I_Z) &\cong \text{Ext}_{\mathbb{P}^3}^1(I_Z, I_Z), \\ \text{Ext}_X^2(\pi^* I_Z, \pi^* I_Z) &\cong \text{Ext}_{\mathbb{P}^3}^2(I_Z, I_Z) \oplus \text{Ext}_{\mathbb{P}^3}^2(I_Z, I_Z)^\vee. \end{aligned}$$

Furthermore, Kuranishi maps for deformations of $\pi^* I_Z$ on X can be identified with Kuranishi maps for deformations of I_Z on \mathbb{P}^3 . Similar to [12, Thm. 6.5], we are done. ■

Proposition 3.6. *Let $\pi : X \rightarrow Y$ be the elliptic fibration (17) and $[f]$ be the fiber class. Then Conjecture 2.6 is true for $\beta = r[f]$ ($r \geq 1$), i.e.*

$$\sum_{r=0}^{\infty} P_{0,r[f]} q^r = M(q)^{-20}$$

for a certain choice of orientation in defining the LHS, where $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function and we define $P_{0,0[f]} = 1$.

Proof. Combining Lemma 3.5 (where we choose the sign to be $(-1)^r$ according to the parity of r) and the generating series for zero-dimensional DT invariants [27, 28, 32], we obtain the formula. Notice that from [24, Table 7], we have $n_{1,[f]} = -20$ and $n_{1,k[f]} = 0$ for $k \neq 1$ (which can also be checked from GW theory). ■

3.3. Quintic fibration

We consider a compact Calabi–Yau 4-fold X which admits a quintic 3-fold fibration structure $\pi : X \rightarrow \mathbb{P}^1$, i.e. π is a proper morphism whose general fiber is a smooth quintic 3-fold $Y \subseteq \mathbb{P}^4$. Examples of such CY 4-folds include a resolution of degree 10 orbifold hypersurface in $\mathbb{P}^5(1, 1, 2, 2, 2, 2)$ and hypersurface of bidegree (2, 5) in $\mathbb{P}^1 \times \mathbb{P}^4$ (see [24, pp. 33–37]).

In this section, we discuss the irreducible curve class in a quintic fiber for these two examples. Here we only consider genus 1 invariants.

Genus 1. Conjecture 2.6 predicts that for an irreducible class β and a suitable choice of orientation, we have

$$P_{0,\beta} = n_{1,\beta} := \text{GW}_{1,\beta} + \frac{1}{24} \text{GW}_{0,\beta}(c_2(X)).$$

Note that the genus 1 invariants $n_{1,\beta}$ for irreducible β are 0 for both quintic fibration examples in [24], where the computations of $\text{GW}_{1,\beta}$ are based on BCOV theory [3]. The pair invariant $P_{0,\beta}$ is obviously 0 in this case since we have

Lemma 3.7. *Let $\beta \in H_2(X, \mathbb{Z})$ be an irreducible class. The pair moduli space $P_0(X, \beta)$ is empty if and only if any curve $C \in \text{Chow}_\beta(X)$ in the Chow variety is a smooth rational curve.*

Proof. \Leftarrow) Given a stable pair $(s : \mathcal{O}_X \rightarrow F) \in P_0(X, \beta)$, F is a torsion-free sheaf (in fact a line bundle) over a curve $C \cong \mathbb{P}^1$. Since $\chi(F) = 0$, we have $F = \mathcal{O}_C(-1)$, contradicting the surjectivity of s in dimension 1.

\Rightarrow) For $C \in \text{Chow}_\beta(X)$ in an irreducible class β , the restriction map $(\mathcal{O}_X \rightarrow \mathcal{O}_C)$ gives a stable pair. Since $P_0(X, \beta)$ is empty, we have

$$\chi(\mathcal{O}_C) = 1 - h^1(C, \mathcal{O}_C) > 0,$$

i.e. $h^1(C, \mathcal{O}_C) = 0$, which implies that C is a smooth rational curve. ■

With this lemma, we can verify Conjecture 2.6 for an irreducible classes in more examples.

Proposition 3.8. *Conjecture 2.6 is true for irreducible class $\beta \in H_2(X, \mathbb{Z})$ when X is either*

- (1) *one of the quintic fibrations in [24];*
- (2) *a smooth complete intersection in a projective space;*
- (3) *one of the complete intersections in Grassmannian varieties in [17].*

Proof. In all the above cases, any curve C in an irreducible class $[C] = \beta$ is a smooth \mathbb{P}^1 , by Lemma 3.7, $P_{0,\beta}(X) = \emptyset$ and hence $P_{0,\beta} = 0$. Meanwhile for those examples in (1) and (3), Klemm–Pandharipande [24] and Gerhardus–Jockers [17] used BCOV theory [3] to compute genus 1 GW invariants and found that $n_{1,\beta} = 0$. As for (2), we have Popa’s computation of genus 1 GW invariants using hyperplane principle developed by Li–Zinger [38, 43]. ■

3.4. Product of an elliptic curve and a CY 3-fold

In this subsection, we consider a CY 4-fold of type $X = Y \times E$, where Y is a projective CY 3-fold and E is an elliptic curve.

Genus 0. We study Conjecture 2.5 for an irreducible curve class of $X = Y \times E$. If $\beta = [E]$, $P_{1,\beta} = 0$, the conjecture is obviously true (in fact for any $r \geq 1$, one can show Conjecture 2.5 is true for $\beta = r[E]$). Below we consider curve classes coming from the CY 3-fold.

Lemma 3.9. *Let $\beta \in H_2(Y, \mathbb{Z})$ be an irreducible curve class on a CY 3-fold Y . Then the pair deformation-obstruction theory of $P_n(Y, \beta)$ is perfect in the sense of [1, 29]. Hence we have an algebraic virtual class*

$$[P_n(Y, \beta)]_{\text{pair}}^{\text{vir}} \in A_{n-1}(P_n(Y, \beta), \mathbb{Z}).$$

Proof. For any stable pair $I_Y = (s : \mathcal{O}_Y \rightarrow F) \in P_n(Y, \beta)$ with β irreducible, we know F is stable [36, p. 270], hence

$$\text{Ext}_Y^3(F, F) \cong \text{Hom}_Y(F, F)^\vee \cong \mathbb{C}.$$

Applying $\mathbf{R}\text{Hom}_Y(-, F)$ to $I_Y \rightarrow \mathcal{O}_Y \rightarrow F$, we obtain a distinguished triangle

$$\mathbf{R}\text{Hom}_Y(F, F) \rightarrow \mathbf{R}\text{Hom}_Y(\mathcal{O}_Y, F) \rightarrow \mathbf{R}\text{Hom}_Y(I_Y, F), \tag{24}$$

whose cohomology gives an exact sequence

$$0 = H^2(Y, F) \rightarrow \text{Ext}_Y^2(I_Y, F) \rightarrow \text{Ext}_Y^3(F, F) \rightarrow 0 \rightarrow \text{Ext}_Y^3(I_Y, F) \rightarrow 0.$$

Hence $\text{Ext}_Y^i(I_Y, F) = 0$ for $i \geq 3$ and $\text{Ext}_Y^2(I_Y, F) \cong \text{Ext}_Y^3(F, F) \cong \mathbb{C}$. By truncating $\text{Ext}_Y^2(I_Y, F) = \mathbb{C}$, the pair deformation theory is perfect. ■

In particular, when $n = 1$, the virtual class $[P_1(Y, \beta)]_{\text{pair}}^{\text{vir}}$ has zero degree. We show the following virtual push-forward formula.

Proposition 3.10. *Let $\beta \in H_2(Y, \mathbb{Z})$ be an irreducible curve class on a CY 3-fold Y . Then*

$$\int_{[P_1(Y, \beta)]_{\text{pair}}^{\text{vir}}} 1 = \int_{[M_{1, \beta}(Y)]^{\text{vir}}} 1,$$

where $M_{1, \beta}(Y)$ is the moduli scheme of one-dimensional stable sheaves on Y with Chern character $(0, 0, 0, \beta, 1)$.

Proof. Since β is irreducible, there is a morphism

$$f : P_1(Y, \beta) \rightarrow M_{1, \beta}(Y), \quad (\mathcal{O}_Y \rightarrow F) \mapsto F, \tag{25}$$

whose fiber over $[F]$ is $\mathbb{P}(H^0(Y, F))$. Let $\mathbb{F} \rightarrow M_{1, \beta}(Y) \times Y$ be the universal sheaf. Then the above map identifies $P_1(Y, \beta)$ with $\mathbb{P}(\pi_{M*}\mathbb{F})$ where $\pi_M : M_{1, \beta}(Y) \times Y \rightarrow M_{1, \beta}(Y)$ is the projection. Then the universal stable pair is given by

$$\mathbb{I} = (\mathcal{O}_{Y \times P_1(Y, \beta)} \xrightarrow{s} \mathbb{F}^\dagger), \quad \mathbb{F}^\dagger := (\text{id}_Y \times f)^*\mathbb{F} \otimes \mathcal{O}(1),$$

where $\mathcal{O}(1)$ is the tautological line bundle on $\mathbb{P}(\pi_{M*}\mathbb{F})$ and s is the tautological map.

Let $\pi_P : P_1(Y, \beta) \times Y \rightarrow P_1(Y, \beta)$ be the projection. There exists a distinguished triangle

$$(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1])^\vee \rightarrow (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}, \mathbb{F}^\dagger))^\vee \rightarrow (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathcal{O}_{Y \times P_1(Y, \beta)}, \mathbb{F}^\dagger))^\vee. \tag{26}$$

By considering a derived extension of the morphism f of (25), the first two terms in (26) are the restriction of cotangent complexes of the corresponding derived schemes to the classical underlying schemes. They are obstruction theories (see [39, Sect. 1.2]), which fit into a commutative diagram

$$\begin{array}{ccccc}
 (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1])^\vee & \longrightarrow & (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}, \mathbb{F}^\dagger))^\vee & \longrightarrow & (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathcal{O}_{Y \times P_1(Y, \beta)}, \mathbb{F}^\dagger))^\vee \\
 \downarrow & & \downarrow & & \downarrow \\
 f^* \mathbb{L}_{M_{1, \beta}(Y)} & \longrightarrow & \mathbb{L}_{P_1(Y, \beta)} & \longrightarrow & \mathbb{L}_{P_1(Y, \beta)/M_{1, \beta}(Y)} \\
 \downarrow & & \downarrow & & \downarrow \\
 \tau^{\geq -1}(f^* \mathbb{L}_{M_{1, \beta}(Y)}) & \longrightarrow & \tau^{\geq -1} \mathbb{L}_{P_1(Y, \beta)} & \longrightarrow & \tau^{\geq -1} \mathbb{L}_{P_1(Y, \beta)/M_{1, \beta}(Y)}
 \end{array}$$

where the bottom vertical arrows are truncation functors.

Note the above obstruction theories are not perfect. To kill h^{-2} , as in [20, Sect. 4.4], we consider the top part of the trace map

$$t : \mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1] \rightarrow \mathbf{R}^3 \pi_{P*}(\mathcal{O}_{Y \times P_1(Y, \beta)})[-2],$$

whose cone is $(\tau^{\leq 1}(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1]))[1]$. Then we have a commutative diagram $(\mathbf{R}^3 \pi_{P*}(\mathcal{O}_{Y \times P_1(Y, \beta)})[-2])^\vee = (\mathbf{R}^3 \pi_{P*}(\mathcal{O}_{Y \times P_1(Y, \beta)})[-2])^\vee$

$$\begin{array}{ccccc}
 & \downarrow t^\vee & & \downarrow \alpha & \\
 (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1])^\vee & \longrightarrow & (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}, \mathbb{F}^\dagger))^\vee & \longrightarrow & (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathcal{O}_{Y \times P_1(Y, \beta)}, \mathbb{F}^\dagger))^\vee \\
 \downarrow & & \downarrow & & \downarrow \\
 (\tau^{\leq 1}(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1]))^\vee & \longrightarrow & \text{Cone}(\alpha) & &
 \end{array}$$

By taking cones, we obtain a distinguished triangle

$$(\tau^{\leq 1}(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1]))^\vee \rightarrow \text{Cone}(\alpha) \rightarrow (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathcal{O}_{Y \times P_1(Y, \beta)}, \mathbb{F}^\dagger))^\vee.$$

Since $(\mathbf{R}^3 \pi_{P*}(\mathcal{O}_{Y \times P_1(Y, \beta)})[-2])^\vee$ is a vector bundle concentrated in degree -2 and $\tau^{\geq -1}(-)$ has cohomology in degree greater than -2 , so we have a commutative diagram

$$\begin{array}{ccccc}
 (\tau^{\leq 1}(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1]))^\vee & \longrightarrow & \text{Cone}(\alpha) & \longrightarrow & (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathcal{O}_{Y \times P_1(Y, \beta)}, \mathbb{F}^\dagger))^\vee \\
 \downarrow & & \downarrow & & \downarrow \\
 \tau^{\geq -1}(f^* \mathbb{L}_{M_{1, \beta}(Y)}) & \longrightarrow & \tau^{\geq -1} \mathbb{L}_{P_1(Y, \beta)} & \longrightarrow & \tau^{\geq -1} \mathbb{L}_{P_1(Y, \beta)/M_{1, \beta}(Y)}
 \end{array}$$

To kill h^1 of the left upper term, we consider the inclusion

$$\mathcal{O}_{P_1(Y, \beta)}[1] \rightarrow \tau^{\leq 1}(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1]),$$

whose restriction to a closed point $I = (\mathcal{O}_Y \rightarrow F)$ induces an isomorphism $\mathbb{C} \rightarrow \text{Hom}(F, F)$. The cone of the inclusion is $\tau^{[0,1]}(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1])$.

Then we have a commutative diagram

$$\begin{array}{ccc}
 \text{Cone}(\beta) & \longrightarrow & (\tau^{[0,1]}(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1]))^\vee \\
 \downarrow & & \downarrow \\
 (\mathbf{R}\mathcal{H}om_{\pi_P}(\mathcal{O}_{Y \times P_1(Y,\beta)}, \mathbb{F}^\dagger))^\vee[-1] & \longrightarrow & (\tau^{\leq 1}(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1]))^\vee \longrightarrow \text{Cone}(\alpha) \\
 \beta \downarrow & & \downarrow \\
 (\mathcal{O}_{P_1(Y,\beta)}[1])^\vee & \xlongequal{\hspace{2cm}} & (\mathcal{O}_{P_1(Y,\beta)}[1])^\vee
 \end{array}$$

As $(\mathcal{O}_{P_1(Y,\beta)}[1])^\vee$ is a vector bundle concentrated in degree 1, we get a commutative diagram

$$\begin{array}{ccccc}
 (\tau^{[0,1]}(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}^\dagger, \mathbb{F}^\dagger)[1]))^\vee & \longrightarrow & \text{Cone}(\alpha) & \longrightarrow & \text{Cone}(\beta)[1] \\
 \phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow \\
 \tau^{\geq -1}(f^*\mathbb{L}_{M_{1,\beta}(Y)}) & \longrightarrow & \tau^{\geq -1}\mathbb{L}_{P_1(Y,\beta)} & \longrightarrow & \tau^{\geq -1}\mathbb{L}_{P_1(Y,\beta)/M_{1,\beta}(Y)}
 \end{array}$$

It is easy to see that ϕ_1 and ϕ_2 define perfect obstruction theories. By diagram chasing on cohomology, ϕ_3 defines a perfect relative obstruction theory. Then we apply Manolache’s virtual push-forward formula [30]:

$$f_*[P_1(Y, \beta)]_{\text{pair}}^{\text{vir}} = c \cdot [M_{1,\beta}(Y)]^{\text{vir}},$$

where the coefficient c is the degree of the virtual class of the relative obstruction theory ϕ_3 and can be shown to be 1 by base-change to a closed point. ■

Now we come back to the CY 4-fold $X = Y \times E$ and show the virtual class $[P_1(Y, \beta)]_{\text{pair}}^{\text{vir}}$ defined using pair deformation-obstruction theory naturally arises in this setting.

Proposition 3.11. *Let $X = Y \times E$ be a product of a CY 3-fold Y with an elliptic curve E . For an irreducible curve class $\beta \in H_2(Y, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$, we have an isomorphism*

$$P_n(X, \beta) \cong P_n(Y, \beta) \times E.$$

The virtual class of $P_n(X, \beta)$ satisfies

$$[P_n(X, \beta)]^{\text{vir}} = [P_n(Y, \beta)]_{\text{pair}}^{\text{vir}} \otimes [E]$$

for a certain choice of orientation in defining the LHS. Here $[P_n(Y, \beta)]_{\text{pair}}^{\text{vir}} \in A_{n-1}(P_n(Y, \beta), \mathbb{Z})$ is the virtual class defined in Lemma 3.9.

Proof. As β is irreducible, for $I_X = (s : \mathcal{O}_X \rightarrow E) \in P_n(X, \beta)$, E is stable [36, p. 270], hence E is scheme-theoretically supported on some $Y \times \{t\}$, $t \in E$ (e.g. [13, Lem. 2.2]).

Let $i_t : Y \times \{t\} \rightarrow X$ be the inclusion. Then $E = (i_t)_*F$ for some $F \in \text{Coh}(Y)$. By adjunction, we have

$$\text{Hom}_X(\mathcal{O}_X, E) \cong \text{Hom}_Y(\mathcal{O}_Y, F).$$

Hence, the morphism

$$\begin{aligned} P_n(Y, \beta) \times E &\rightarrow P_n(X, \beta), \\ (I_Y := (s : i_t^* \mathcal{O}_X \rightarrow F), t) &\mapsto (s : \mathcal{O}_X \rightarrow (i_t)_*F) =: I_X \end{aligned} \tag{27}$$

is bijective on closed points. Next, we compare their deformation-obstruction theories.

Denote $i = i_t$. From the distinguished triangle

$$i_*F \rightarrow I_X[1] \rightarrow \mathcal{O}_X[1], \tag{28}$$

we have the diagram

$$\begin{array}{ccccc} & & \mathbf{R}\Gamma(\mathcal{O}_X)[1] & \xlongequal{\quad\quad\quad} & \mathbf{R}\Gamma(\mathcal{O}_X)[1] \\ & & \downarrow & & \downarrow \\ \mathbf{R}\text{Hom}_X(I_X, i_*F) & \longrightarrow & \mathbf{R}\text{Hom}_X(I_X, I_X)[1] & \longrightarrow & \mathbf{R}\text{Hom}_X(I_X, \mathcal{O}_X)[1] \\ & & \downarrow & & \downarrow \\ & & \mathbf{R}\text{Hom}_X(I_X, I_X)_0[1] & & \mathbf{R}\text{Hom}_X(i_*F, \mathcal{O}_X)[2] \end{array}$$

where the horizontal and vertical arrows are distinguished triangles. By taking cones, we obtain a distinguished triangle

$$\mathbf{R}\text{Hom}_X(I_X, i_*F) \rightarrow \mathbf{R}\text{Hom}_X(I_X, I_X)_0[1] \rightarrow \mathbf{R}\text{Hom}_X(i_*F, \mathcal{O}_X)[2]. \tag{29}$$

On the other hand, from the distinguished triangle

$$I_X \rightarrow \mathcal{O}_X \rightarrow i_*F$$

and the isomorphism (see e.g. [13, Proposition-Definition 3.3])

$$\mathbf{L}i^*i_*F \cong F \oplus (F \otimes N_{Y \times \{t\}/X}^\vee)[1], \quad \text{where } N_{Y \times \{t\}/X} = \mathcal{O}_{Y \times \{t\}},$$

we can obtain the isomorphism

$$\mathbf{L}i^*I_X \cong I_Y \oplus F, \tag{30}$$

which implies that

$$\mathbf{R}\text{Hom}_X(I_X, i_*F) \cong \mathbf{R}\text{Hom}_Y(I_Y, F) \oplus \mathbf{R}\text{Hom}_Y(F, F).$$

Therefore by (29), we have the distinguished triangle

$$\mathbf{RHom}_Y(I_Y, F) \oplus \mathbf{RHom}_Y(F, F) \rightarrow \mathbf{RHom}_X(I_X, I_X)_0[1] \rightarrow \mathbf{RHom}_X(i_*F, \mathcal{O}_X)[2].$$

It follows that we have the distinguished triangle

$$\mathbf{RHom}_Y(I_Y, F) \rightarrow \mathbf{RHom}_X(I_X, I_X)_0[1] \rightarrow T, \tag{31}$$

where T fits into the distinguished triangle

$$\mathbf{RHom}_Y(F, F) \rightarrow T \rightarrow \mathbf{RHom}_X(i_*F, \mathcal{O}_X)[2]. \tag{32}$$

By Serre duality, adjunction and degree shift, (32) becomes

$$T \rightarrow \mathbf{RHom}_Y(\mathcal{O}_Y, F)^\vee[-2] \rightarrow \mathbf{RHom}_Y(F, F)^\vee[-2],$$

whose dual gives a distinguished triangle

$$\mathbf{RHom}_Y(F, F)[2] \rightarrow \mathbf{RHom}_Y(\mathcal{O}_Y, F)[2] \rightarrow T^\vee. \tag{33}$$

Combining (24) and (33), we obtain

$$T \cong \mathbf{RHom}_Y(I_Y, F)^\vee[-2].$$

Combining this with (31) and taking the cohomological long exact sequence, we have

$$\rightarrow \text{Ext}_Y^1(I_Y, F) \rightarrow \text{Ext}_X^2(I_X, I_X)_0 \rightarrow \text{Ext}_Y^1(I_Y, F)^\vee \rightarrow .$$

We claim that the above exact sequence breaks into short exact sequences

$$\begin{aligned} 0 \rightarrow \text{Ext}_Y^0(I_Y, F) \rightarrow \text{Ext}_X^1(I_X, I_X)_0 \rightarrow \mathbb{C} \rightarrow 0, \\ 0 \rightarrow \text{Ext}_Y^1(I_Y, F) \rightarrow \text{Ext}_X^2(I_X, I_X)_0 \rightarrow \text{Ext}_Y^1(I_Y, F)^\vee \rightarrow 0, \end{aligned}$$

since $\text{Ext}_Y^2(I_Y, F) \cong \mathbb{C}$ (see the proof of Lemma 3.9) and in view of a dimension counting by Riemann–Roch. The first exact sequence above implies that the map (27) induces an isomorphism on tangent spaces. The second exact sequence implies that the obstructions to deforming stable pairs on LHS of (27) vanish if and only if those on RHS of (27) vanish. Therefore, the map (27) induces an isomorphism on formal completions of structure sheaves of both sides at any closed point. So (27) must be a scheme-theoretical isomorphism.

Next, we show $\text{Ext}_Y^1(I_Y, F) \subseteq \text{Ext}_X^2(I_X, I_X)_0$ is a maximal isotropic subspace with respect to the Serre duality pairing on $\text{Ext}_X^2(I_X, I_X)_0$. For $u \in \text{Ext}_Y^1(I_Y, F)$, the corresponding element in $\text{Ext}_X^2(I_X, I_X)_0$ is given by the composition

$$I_X \xrightarrow{\alpha} i_*I_Y \xrightarrow{i_*u} i_*F[1] \xrightarrow{\beta[1]} I_X[2],$$

where α is the canonical morphism and β is given by (28). For another $u' \in \text{Ext}_Y^1(I_Y, F)$, it is enough to show the vanishing of the composition

$$I_X \xrightarrow{\alpha} i_* I_Y \xrightarrow{i_* u} i_* F[1] \xrightarrow{\beta[1]} I_X[2] \xrightarrow{\alpha[2]} i_* I_Y[2] \xrightarrow{i_* u'[2]} i_* F[3] \xrightarrow{\beta[3]} I_X[4]. \tag{34}$$

Since $\text{Ext}_Y^0(F, I_Y \otimes K_Y) \cong \text{Ext}_Y^3(I_Y, F)^\vee = 0$ (see the proof of Lemma 3.9), the composition $i_* F[1] \xrightarrow{\beta[1]} I_X[2] \xrightarrow{\alpha[2]} i_* I_Y[2]$ can be written as $i_* \gamma$. Therefore the composition

$$i_* I_Y \xrightarrow{i_* u} i_* F[1] \xrightarrow{\beta[1]} I_X[2] \xrightarrow{\alpha[2]} i_* I_Y[2] \xrightarrow{i_* u'[2]} i_* F[3]$$

vanishes, again because $\text{Ext}_Y^3(I_Y, F) = 0$.

Moreover, a local Kuranishi map of $P_n(X, \beta)$ at I_X can be identified as

$$(\kappa_{I_Y}, 0) : \text{Ext}_Y^0(I_Y, F) \times T_t E \rightarrow \text{Ext}_Y^1(I_Y, F),$$

where κ_{I_Y} is a local Kuranishi map of $P_n(Y, \beta)$ at I_Y . Similarly to [12, Thm. 6.5], we have the desired equality on virtual classes. ■

Combining the above result with Proposition 3.10, our genus 0 conjecture can be reduced to Katz’s conjecture [22].

Corollary 3.12. *Let $X = Y \times E$ be a product of a CY 3-fold Y with an elliptic curve E . Then Conjecture 2.5 holds for an irreducible curve class $\beta \in H_2(Y, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$ if and only if Katz’s conjecture holds for β .*

Proof. To have non-trivial invariants, we only need to consider insertions of the form

$$\gamma = (\gamma_1, [\text{pt}]) \in H^2(Y, \mathbb{Z}) \otimes H^2(E, \mathbb{Z}).$$

By Propositions 3.10 and 3.11, we have

$$P_{1,\beta}(\gamma) = (\gamma_1 \cdot \beta) \int_{[P_1(Y,\beta)]_{\text{pair}}^{\text{vir}}} 1 = (\gamma_1 \cdot \beta) \int_{[M_{1,\beta}(Y)]^{\text{vir}}} 1.$$

Thus Conjecture 2.5 reduces to Katz’s conjecture. ■

Katz’s conjecture has been verified for primitive classes in complete intersection CY 3-folds [13, Cor. A.6]. So we obtain

Theorem 3.13. *Let Y be a complete intersection CY 3-fold in a product of projective spaces, and $X = Y \times E$ be the product of Y with an elliptic curve E . Then Conjecture 2.5 is true for an irreducible curve class $\beta \in H_2(Y, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$.*

Genus 1. Similar to Lemma 3.5, for $X = Y \times E$ and $\beta = r[E]$, we have

Lemma 3.14. *For any $r \in \mathbb{Z}_{\geq 1}$, there exists an isomorphism*

$$P_0(X, r[E]) \cong \text{Hilb}^r(Y),$$

under which the virtual class is given by

$$[P_0(X, r[E])]^{\text{vir}} = (-1)^r \cdot [\text{Hilb}^r(Y)]^{\text{vir}}$$

for a certain choice of orientation in defining the LHS. Furthermore, their degrees fit into the generating series

$$\sum_{r=0}^{\infty} P_{0,r[E]} q^r = M(q)^{\chi(Y)},$$

where $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function and we define $P_{0,0[E]} = 1$.

We check Conjecture 2.6 for this case.

Theorem 3.15. *Let $X = Y \times E$ be the product of a CY 3-fold Y with an elliptic curve E . Then Conjecture 2.6 is true for $\beta = r[E] \in H_2(X, \mathbb{Z})$ for any $r \geq 1$.*

Proof. By Lemma 3.14, it remains to show $n_{1,[E]} = \chi(Y)$ and $n_{1,r[E]} = 0$ if $r \geq 2$. Since genus 0 Gromov–Witten invariants $\text{GW}_{0,r[E]}(X)$ are 0 for any $r \geq 1$, this is equivalent to

$$\sum_{r=1}^{\infty} \text{GW}_{1,r[E]}(X) q^r = \chi(Y) \cdot \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^d,$$

where $\sigma(d) = \sum_{i|d} i$. We have an isomorphism

$$\overline{M}_{1,0}(X, r[E]) \cong \overline{M}_{1,0}(E, r[E]) \times Y$$

for the moduli space $\overline{M}_{1,0}(X, r[E])$ of genus 1 stable maps to X . Note that $\overline{M}_{1,0}(E, r[E])$ is smooth of expected dimension and consists of $\sigma(r)/r$ points (modulo automorphisms) (see e.g. [33]). And the genus 1 invariant for the constant map to Y is $\chi(Y)$. So $\text{GW}_{1,r[E]}(X) = \chi(Y) \cdot \sigma(r)/r$. ■

When the curve class $\beta \in H_2(Y) \subseteq H_2(X)$ comes from Y , we have

Lemma 3.16. *Let $X = Y \times E$ be the product of a CY 3-fold Y with an elliptic curve E . Then for $\beta \in H_2(Y) \subseteq H_2(X)$, we have*

$$\text{GW}_{0,\beta}(\gamma) = \text{deg}[\overline{M}_{0,0}(Y, \beta)]^{\text{vir}} \cdot \int_{\beta} \gamma_1 \cdot \int_E \gamma_2 \quad \text{if } \gamma = \gamma_1 \otimes \gamma_2 \in H^2(Y) \otimes H^2(E);$$

$$\text{GW}_{0,\beta}(\gamma) = 0 \quad \text{if } \gamma \in H^4(Y) \subseteq H^4(X); \quad \text{GW}_{1,\beta} = 0.$$

Proof. We have an isomorphism

$$\overline{M}_{0,1}(X, \beta) \cong \overline{M}_{0,1}(Y, \beta) \times E, \tag{35}$$

under which the virtual class satisfies

$$[\overline{M}_{0,1}(X, \beta)]^{\text{vir}} \cong [\overline{M}_{0,1}(Y, \beta)]^{\text{vir}} \otimes [E].$$

By the divisor equation, one can compute genus 0 GW invariants. $\overline{M}_{1,0}(X, \beta)$ has a similar product structure to (35). The obstruction sheaf has a trivial factor $TE = \mathcal{O}_E$ in E direction. So genus 1 GW invariants vanish. ■

Then it is easy to show the following:

Proposition 3.17. *Let $X = Y \times E$ be the product of a CY 3-fold Y with an elliptic curve E . Then Conjecture 2.6 is true for any irreducible class $\beta \in H_2(Y) \subseteq H_2(X)$.*

Proof. By Lemma 3.16, we know $n_{1,\beta} = 0$. By Proposition 3.11, the virtual dimension of $[P_0(Y, \beta)]_{\text{pair}}^{\text{vir}}$ is negative, so $P_{0,\beta} = 0$. ■

3.5. Hyperkähler 4-folds

When the CY 4-fold X is hyperkähler, GW invariants on X vanish as they are deformation-invariant and there are no holomorphic curves for generic complex structures in the \mathbb{S}^2 -twistor family. Another way to see the vanishing is via the cosection localization technique developed by Kiem–Li [23].

Roughly speaking, given a perfect obstruction theory [1, 29] on a Deligne–Mumford moduli stack M , the existence of a cosection

$$\varphi : Ob_M \rightarrow \mathcal{O}_M$$

of the obstruction sheaf Ob_M makes the virtual class of M localize to the closed subspace $Z(\varphi) \subseteq M$ where φ is not surjective. In particular, if φ is surjective everywhere (in GW theory this is guaranteed by the existence of holomorphic symplectic forms), then the virtual class of M vanishes. Moreover, by truncating the obstruction theory to remove the trivial factor \mathcal{O}_M , one can define a reduced obstruction theory and reduced virtual class.

To verify Conjectures 2.5 and 2.6 for hyperkähler 4-folds, we only need to show the vanishing of stable pair invariants of $P_0(X, \beta)$ and $P_1(X, \beta)$.

Cosection and vanishing of DT_4 virtual classes. Fix a stable pair $I \in P_n(X, \beta)$. By taking wedge product with the square $\text{At}(I)^2$ of the Atiyah class and contraction with the holomorphic symplectic form σ , we get a surjective map

$$\phi : \text{Ext}^2(I, I)_0 \xrightarrow{\wedge^{\frac{\text{At}(I)^2}{2}}} \text{Ext}^4(I, I \otimes \Omega_X^2) \xrightarrow{-\sigma} \text{Ext}^4(I, I) \xrightarrow{\text{tr}} H^4(X, \mathcal{O}_X).$$

In fact, we have

Proposition 3.18. *Let X be a projective hyperkähler 4-fold, I be a perfect complex on X and Q be the Serre duality quadratic form on $\text{Ext}^2(I, I)_0$. Then the composition map*

$$\phi : \text{Ext}^2(I, I)_0 \xrightarrow{\wedge^{\frac{\text{At}(I)^2}{2}}} \text{Ext}^4(I, I \otimes \Omega_X^2) \xrightarrow{-\sigma} \text{Ext}^4(I, I) \xrightarrow{tr} H^4(X, \mathcal{O}_X)$$

is surjective if either $\text{ch}_3(I) \neq 0$ or $\text{ch}_4(I) \neq 0$. Moreover,

(1) if $\text{ch}_4(I) \neq 0$, then we have a Q -orthogonal decomposition

$$\text{Ext}^2(I, I)_0 = \text{Ker}(\phi) \oplus \mathbb{C}\langle \text{At}(I)^2 \lrcorner \sigma \rangle,$$

where Q is non-degenerate on each subspace;

(2) if $\text{ch}_4(I) = 0$ and $\text{ch}_3(I) \neq 0$, then we have a Q -orthogonal decomposition

$$\begin{aligned} \text{Ext}^2(I, I)_0 &= \mathbb{C}\langle \text{At}(I)^2 \lrcorner \sigma, \kappa_X \circ \text{At}(I) \rangle \\ &\oplus (\mathbb{C}\langle \text{At}(I)^2 \lrcorner \sigma, \kappa_X \circ \text{At}(I) \rangle)^\perp, \end{aligned}$$

where Q is non-degenerate on each subspace. Here κ_X is the Kodaira–Spencer class which is Serre dual to $\text{ch}_3(I)$.

Proof. See [13, proof of Prop. 2.9]. ■

We claim that the surjectivity of cosection maps leads to the vanishing of virtual classes for stable pair moduli spaces (it also applies to other moduli spaces, e.g. Hilbert schemes of curves/points used in DT/PT correspondence [8, 9]).

Claim 3.19. *Let X be a projective hyperkähler 4-fold and $P_n(X, \beta)$ be the moduli space of stable pairs with $n \neq 0$ or $\beta \neq 0$. Then the virtual class satisfies*

$$[P_n(X, \beta)]^{\text{vir}} = 0.$$

At the moment, a Kiem–Li type theory of cosection localization for D-manifolds is not available in the literature. We believe that when such a theory is established, our claim should follow automatically. Nevertheless, we have the following evidence for the claim.

1. At least when $P_n(X, \beta)$ is smooth, Proposition 3.18 gives the vanishing of the virtual class.
2. If there is a complex analytic version of (-2) -shifted symplectic geometry [37] and the corresponding construction of virtual classes [4], one could prove the vanishing result as in GW theory, i.e. taking a generic complex structure in the \mathbb{S}^2 -twistor family of the hyperkähler 4-fold which does not support coherent sheaves and then vanishing of virtual classes follows from their deformation invariance.

4. Non-compact examples

4.1. Irreducible curve classes on local Fano 3-folds

Let Y be a Fano 3-fold. When Y embeds into a CY 4-fold X , the normal bundle of $Y \subseteq X$ is the canonical bundle K_Y of Y . By the negativity of K_Y , there exists an analytic neighborhood of Y in X which is isomorphic to an analytic neighborhood of Y in K_Y . Here we simply consider non-compact CY 4-folds of the form $X = K_Y$.

Similar to Lemma 3.9, we have

Lemma 4.1. *Let $\beta \in H_2(Y, \mathbb{Z})$ be an irreducible curve class on a Fano 3-fold Y . Then the pair deformation-obstruction theory of $P_n(Y, \beta)$ is perfect in the sense of [1, 29]. Hence we have an algebraic virtual class*

$$[P_n(Y, \beta)]_{\text{pair}}^{\text{vir}} \in A_n(P_n(Y, \beta), \mathbb{Z}).$$

Proof. For any stable pair $I_Y = (s : \mathcal{O}_Y \rightarrow F) \in P_n(Y, \beta)$ with β irreducible, we know F is stable [36, p. 270], hence

$$\text{Ext}_Y^3(F, F) \cong \text{Hom}_Y(F, F \otimes K_Y)^\vee = 0.$$

Applying $\mathbf{R}\text{Hom}_Y(-, F)$ to $I_Y \rightarrow \mathcal{O}_Y \rightarrow F$, we obtain a distinguished triangle

$$\mathbf{R}\text{Hom}_Y(F, F) \rightarrow \mathbf{R}\text{Hom}_Y(\mathcal{O}_Y, F) \rightarrow \mathbf{R}\text{Hom}_Y(I_Y, F), \tag{36}$$

whose cohomology gives an exact sequence

$$0 = H^2(Y, F) \rightarrow \text{Ext}_Y^2(I_Y, F) \rightarrow \text{Ext}_Y^3(F, F) \rightarrow 0 \rightarrow \text{Ext}_Y^3(I_Y, F) \rightarrow 0.$$

Hence $\text{Ext}_Y^i(I_Y, F) = 0$ for $i \geq 2$. Then we can apply the construction of [1, 29]. ■

When $n = 1$, similar to Proposition 3.10, we have

Proposition 4.2. *Let $\beta \in H_2(Y, \mathbb{Z})$ be an irreducible curve class on a Fano 3-fold Y . Then*

$$f_*[P_1(Y, \beta)]_{\text{pair}}^{\text{vir}} = [M_{1,\beta}(Y)]^{\text{vir}},$$

where $f : P_1(Y, \beta) \rightarrow M_{1,\beta}(Y)$, $(\mathcal{O}_X \rightarrow F) \mapsto F$, is the morphism forgetting the section, $M_{1,\beta}(Y)$ is the moduli scheme of one-dimensional stable sheaves E on Y with $[E] = \beta$ and $\chi(E) = 1$.

Now we come back to the CY 4-fold $X = K_Y$. Similar to Proposition 3.11, we have

Proposition 4.3. *Let Y be a Fano 3-fold and $X = K_Y$. For an irreducible curve class $\beta \in H_2(X, \mathbb{Z}) \cong H_2(Y, \mathbb{Z})$, we have an isomorphism*

$$P_n(X, \beta) \cong P_n(Y, \beta).$$

The virtual class of $P_n(X, \beta)$ satisfies

$$[P_n(X, \beta)]^{\text{vir}} = [P_n(Y, \beta)]_{\text{pair}}^{\text{vir}}$$

for a certain choice of orientation in defining the LHS, where $[P_n(Y, \beta)]_{\text{pair}}^{\text{vir}}$ is the virtual class defined in Lemma 4.1.

Proof. The proof is the same as the proof of Proposition 3.11. Just note that as in (29), there is a distinguished triangle

$$\mathbf{RHom}_X(I_X, i_*F) \rightarrow \mathbf{RHom}_X(I_X, I_X)_0[1] \rightarrow \mathbf{RHom}_X(i_*F, \mathcal{O}_X)[2],$$

where the cohomology of $\mathbf{RHom}_X(I_X, I_X)_0[1]$ is finite-dimensional as F has compact support (although X is non-compact) and we may work with a compactification of X . ■

Genus 0. Combining Propositions 4.2 and 4.3 shows that Conjecture 2.5 for irreducible curve classes on K_Y is equivalent to the genus 0 GV/DT₄ conjecture [13, Conjecture 0.2] on K_Y (see also [6, Conjecture 1.2]), which has been verified in the following cases [6, Prop. 2.1, 2.3, Thm. 2.7].

Proposition 4.4. *Conjecture 2.5 is true for any irreducible curve class $\beta \in H_2(K_Y, \mathbb{Z}) \cong H_2(Y, \mathbb{Z})$ provided that (i) $Y \subseteq \mathbb{P}^4$ is a smooth hypersurface of degree $d \leq 4$, or (ii) $Y = S \times \mathbb{P}^1$ for a toric del Pezzo surface S .*

Genus 1. When any curve C in an irreducible class $\beta \in H_2(Y)$ is a smooth rational curve, $P_0(Y, \beta) = \emptyset$ by Lemma 3.7, so $P_{0,\beta}(X) = 0$ (by Proposition 4.3). In this case, to verify Conjecture 2.6, we are reduced to computing GW invariants and show $n_{1,\beta} = 0$.

Proposition 4.5. *Let $Y = \mathbb{P}^3$ and $X = K_Y$. Then Conjecture 2.6 is true for any irreducible curve class $\beta \in H_2(X, \mathbb{Z}) \cong H_2(Y, \mathbb{Z})$.*

Proof. When $Y = \mathbb{P}^3$, $n_{1,\beta} = 0$ by [24, Table 1, p. 31]. ■

4.2. Irreducible curve classes on local surfaces

Let $(S, \mathcal{O}_S(1))$ be a smooth projective surface and

$$\pi: X = \text{Tot}_S(L_1 \oplus L_2) \rightarrow S \tag{37}$$

be the total space of the direct sum of two line bundles L_1, L_2 on S . If we assume that

$$L_1 \otimes L_2 \cong K_S, \tag{38}$$

then X is a non-compact CY 4-fold. For a curve class

$$\beta \in H_2(X, \mathbb{Z}) \cong H_2(S, \mathbb{Z}),$$

we can consider the moduli space $P_n(X, \beta)$ of stable pairs on X , which is in general non-compact. In this section, we restrict to the case when the curve class β is irreducible such that $L_i \cdot \beta < 0$, in which case $P_n(X, \beta)$ is compact and smooth.

Lemma 4.6. *Let S be a smooth projective surface and $\beta \in H_2(S, \mathbb{Z})$ be an irreducible curve class such that $K_S \cdot \beta < 0$. Then the moduli space $P_n(S, \beta)$ of stable pairs on S is smooth.*

Proof. Similar to the proof of Lemma 4.1, for any stable pair $I_S = (s : \mathcal{O}_S \rightarrow F) \in P_n(S, \beta)$ with β irreducible, F is stable, hence

$$\text{Ext}_S^2(F, F) \cong \text{Hom}_S(F, F \otimes K_S) = 0.$$

Applying $\mathbf{R}\text{Hom}_S(-, F)$ to $I_S \rightarrow \mathcal{O}_S \rightarrow F$, we obtain a distinguished triangle

$$\mathbf{R}\text{Hom}_S(F, F) \rightarrow \mathbf{R}\text{Hom}_S(\mathcal{O}_S, F) \rightarrow \mathbf{R}\text{Hom}_S(I_S, F), \tag{39}$$

whose cohomology gives an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_S(F, F) \rightarrow H^0(F) \rightarrow \text{Hom}_S(I_S, F) \rightarrow \text{Ext}_S^1(F, F) \\ \rightarrow H^1(F) \rightarrow \text{Ext}_S^1(I_S, F) \rightarrow \text{Ext}_S^2(F, F) = 0, \end{aligned}$$

and $\text{Ext}_S^i(I_S, F) = 0$ for $i \geq 2$. We claim the map

$$\text{Ext}_S^1(F, F) \rightarrow H^1(F)$$

above is surjective; then $\text{Ext}_S^1(I_S, F) = 0$ follows from the exact sequence (so the smoothness of moduli follows).

In fact, we only need to show the surjectivity of

$$H^1(\mathcal{O}_C) \xrightarrow{\text{id}} H^1(\mathcal{H}om(F, F)) \subseteq \text{Ext}_S^1(F, F) \rightarrow H^1(F),$$

where C is the scheme-theoretical support of F . However, the above map is simply multiplication by the section s , which fits into an exact sequence

$$H^1(\mathcal{O}_C) \xrightarrow{s} H^1(F) \rightarrow H^1(Q) = 0,$$

where $Q \cong F/s(\mathcal{O}_S)$ is zero-dimensional. ■

Proposition 4.7. *Let S be a smooth projective surface and L_1, L_2 be line bundles on S such that $L_1 \otimes L_2 \cong K_S$. Then for any irreducible curve class $\beta \in H_2(X, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$ such that $L_i \cdot \beta < 0$ ($i = 1, 2$), we have an isomorphism*

$$P_n(X, \beta) \cong P_n(S, \beta).$$

And the virtual class satisfies

$$[P_n(X, \beta)]^{\text{vir}} = [P_n(S, \beta)] \cdot e(-\mathbf{R}\mathcal{H}om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1))$$

for a certain choice of orientation in defining the LHS. Here

$$\mathbb{I}_S = (\mathcal{O}_{S \times P_n(S, \beta)} \rightarrow \mathbb{F}) \in \mathbf{D}^b(S \times P_n(S, \beta))$$

is the universal stable pair and $\pi_{P_S}: S \times P_n(S, \beta) \rightarrow P_n(S, \beta)$ is the projection.

Proof. Under assumption $L_i \cdot \beta < 0$ and β is irreducible, as in the proof of [13, Prop. 3.1], one can show, for the zero section $i : S \rightarrow X$, that the morphism

$$\begin{aligned} P_n(S, \beta) &\rightarrow P_n(X, \beta), \\ I_S := (s : i^* \mathcal{O}_X \rightarrow F) &\mapsto (s : \mathcal{O}_X \rightarrow i_* F) =: I_X, \end{aligned} \tag{40}$$

is bijective on closed points. And we have distinguished triangles

$$\begin{aligned} i_* F &\rightarrow I_X[1] \rightarrow \mathcal{O}_X[1], \\ \mathbf{RHom}_X(I_X, i_* F) &\rightarrow \mathbf{RHom}_X(I_X, I_X)_0[1] \rightarrow \mathbf{RHom}_X(i_* F, \mathcal{O}_X)[2], \\ \mathbf{L}i^* I_X &\cong I_S \oplus (F \otimes L_1^{-1}) \oplus (F \otimes L_2^{-1}) \oplus (F \otimes K_S^{-1})[1], \end{aligned} \tag{41}$$

where the last isomorphism is deduced similarly to (30).

It follows that we have a distinguished triangle

$$\mathbf{RHom}_S(I_S, F) \oplus \mathbf{RHom}_S(F, F \otimes L_1) \rightarrow \mathbf{RHom}_X(I_X, I_X)_0[1] \rightarrow T, \tag{42}$$

where T fits into the distinguished triangle

$$\mathbf{RHom}_S(F, F \otimes L_2) \oplus \mathbf{RHom}_S(F, F \otimes K_S)[-1] \rightarrow T \rightarrow \mathbf{RHom}_X(i_* F, \mathcal{O}_X)[2]. \tag{43}$$

By Serre duality, degree shift and taking duals, (43) becomes

$$\mathbf{RHom}_S(F, F \otimes L_1)[1] \oplus \mathbf{RHom}_S(F, F)[2] \rightarrow \mathbf{RHom}_S(\mathcal{O}_S, F)[2] \rightarrow T^\vee.$$

Combining this with (39), we obtain a distinguished triangle

$$\mathbf{RHom}_S(F, F \otimes L_1)[1] \rightarrow \mathbf{RHom}_S(I_S, F)[2] \rightarrow T^\vee,$$

whose dual is

$$T \rightarrow \mathbf{RHom}_S(I_S, F)^\vee[-2] \rightarrow \mathbf{RHom}_S(F, F \otimes L_1)^\vee[-1]. \tag{44}$$

By taking cohomology of (44), we obtain exact sequences

$$\begin{aligned} 0 \rightarrow H^0(T) &\rightarrow \text{Ext}_S^2(I_S, F)^\vee \rightarrow \text{Ext}_S^1(F, F \otimes L_1)^\vee \rightarrow H^1(T) \\ &\rightarrow \text{Ext}_S^1(I_S, F)^\vee \rightarrow \text{Hom}_S(F, F \otimes L_1)^\vee = 0, \end{aligned}$$

where $\text{Ext}_S^{i \geq 1}(I_S, F) = 0$ by the proof of Lemma 4.6. Hence

$$H^0(T) = 0, \quad H^1(T) \cong \text{Ext}_S^1(F, F \otimes L_1)^\vee.$$

By taking cohomology of (42), we obtain

$$\begin{aligned} \text{Ext}_S^0(I_S, F) &\cong \text{Ext}_X^1(I_X, I_X)_0, \\ 0 \rightarrow \text{Ext}_S^1(F, F \otimes L_1) &\rightarrow \text{Ext}_X^2(I_X, I_X)_0 \rightarrow H^1(T) \rightarrow \text{Ext}_S^2(I_S, F) \oplus \text{Ext}_S^2(F, F \otimes L_1) = 0, \end{aligned}$$

hence also the exact sequence

$$0 \rightarrow \text{Ext}_S^1(F, F \otimes L_1) \rightarrow \text{Ext}_X^2(I_X, I_X)_0 \rightarrow \text{Ext}_S^1(F, F \otimes L_1)^\vee \rightarrow 0.$$

By the first isomorphism above, we know the map (40) induces an isomorphism on tangent spaces. Moreover since $P_n(S, \beta)$ is smooth (Lemma 4.6) and (40) is bijective on closed points, the map (40) is an isomorphism.

As in Proposition 4.3, we can show $\text{Ext}_S^1(F, F \otimes L_1)$ is a maximal isotropic subspace of $\text{Ext}_X^2(I_X, I_X)_0$ with respect to the Serre duality pairing on $\text{Ext}_X^2(I_X, I_X)_0$.

Since $\text{Ext}_S^0(F, F \otimes L_1) = \text{Ext}_S^2(F, F \otimes L_1) = 0$, $\text{Ext}_S^1(F, F \otimes L_1)$ is constant over $P_n(S, \beta)$, it forms a maximal isotropic subbundle of the obstruction bundle of $P_n(X, \beta)$ whose fiber over $I_X \in P_n(X, \beta)$ is $\text{Ext}_X^2(I_X, I_X)_0$. Then the virtual class has the desired property [12]. ■

It is easy to check Conjectures 2.5 and 2.6 for irreducible curve classes on $\text{Tot}_S(L_1 \oplus L_2)$ in the following setting.

Proposition 4.8. *Let S be a del Pezzo surface and L_1^{-1}, L_2^{-1} be ample line bundles on S such that $L_1 \otimes L_2 \cong K_S$. Let $\beta \in H_2(X, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$ be an irreducible curve class on $X = \text{Tot}_S(L_1 \oplus L_2)$. Then Conjectures 2.5 and 2.6 are true for β .*

Proof. We claim that S does not contain any (-1) curve. In fact, if C is a (-1) curve, then

$$-2 \geq \deg(L_1|_C) + \deg(L_2|_C) = \deg(K_S|_C) = -1.$$

So S is either \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$, and any curve in an irreducible class is a smooth rational curve. By Lemma 3.7, $P_0(S, \beta) = \emptyset$, so $[P_0(X, \beta)]^{\text{vir}} = 0$ by Proposition 4.7. This matches Klemm–Pandharipande’s computation [24, pp. 22, 24], i.e. Conjecture 2.6 is true for β .

As for the genus 0 conjecture, for any stable pair $(s : \mathcal{O}_S \rightarrow F) \in P_1(S, \beta)$, F is stable and supported on some $C \cong \mathbb{P}^1$ in S . Then $F = \mathcal{O}_C$ and the morphism

$$\phi : P_1(S, \beta) \xrightarrow{\cong} M_{1,\beta}(S), \quad (\mathcal{O}_S \rightarrow F) \mapsto F,$$

to the moduli space $M_{1,\beta}(S)$ of one-dimensional stable sheaves F on S with $[F] = \beta$ and $\chi(F) = 1$ is an isomorphism.

As for the moduli space $\overline{M}_{0,0}(X, \beta)$ of stable maps, we have isomorphisms

$$\overline{M}_{0,0}(X, \beta) \cong \overline{M}_{0,0}(S, \beta) \cong M_{1,\beta}(S),$$

where the first isomorphism is by the negativity of L_i ($i = 1, 2$) and the second one is defined by mapping $f : \mathbb{P}^1 \rightarrow S$ to $\mathcal{O}_{f(\mathbb{P}^1)}$.

Next, we compare obstruction theories. By Proposition 4.7, the ‘half’ obstruction space of $P_1(X, \beta)$ at $(s : \mathcal{O}_X \rightarrow \mathcal{O}_C)$ is $\text{Ext}_S^1(\mathcal{O}_C, \mathcal{O}_C \otimes L_1)$ which fits into the exact sequence

$$0 \rightarrow H^1(C, L_1|_C) \rightarrow \text{Ext}_S^1(\mathcal{O}_C, \mathcal{O}_C \otimes L_1) \rightarrow H^0(C, L_1|_C \otimes N_{C/S}) \rightarrow 0.$$

Since S is either \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$ and β is irreducible, all stable maps are embeddings. The obstruction space of $\overline{M}_{0,0}(X, \beta)$ at $f : \mathbb{P}^1 \rightarrow S$ is $H^1(C, N_{C/X}) \cong H^1(\mathbb{P}^1, f^*TX)$ with $C = f(\mathbb{P}^1)$, which fits into the exact sequence

$$0 = H^1(\mathbb{P}^1, f^*TS) \rightarrow H^1(\mathbb{P}^1, f^*TX) \rightarrow H^1(\mathbb{P}^1, f^*(L_1 \oplus L_2)) \rightarrow 0.$$

Note that

$$H^1(\mathbb{P}^1, f^*(L_1 \oplus L_2)) \cong H^1(C, L_1|_C) \oplus H^0(C, L_1|_C \otimes N_{C/S})^*.$$

The family version of these computations shows the virtual classes satisfy

$$[P_1(X, \beta)]^{\text{vir}} = [M_{0,0}(X, \beta)]^{\text{vir}}$$

up to sign (for each connected component of the moduli space). It is easy to match the insertions and then verify Conjecture 2.5. More specifically, when $S = \mathbb{P}^2$, $P_{1,1}([\text{pt}]) = n_{0,1}([\text{pt}]) = -1$ and when $S = \mathbb{P}^1 \times \mathbb{P}^1$, $P_{1,(1,0)}([\text{pt}]) = n_{0,(1,0)}([\text{pt}]) = P_{1,(0,1)}([\text{pt}]) = n_{0,(0,1)}([\text{pt}]) = 1$ for a certain choice of orientation. ■

4.3. Small degree curve classes on local surfaces

We learned from discussions with Kool and Monavari [26] (see also [10]) that by using relative Hilbert schemes and techniques developed in Kool–Thomas [25], one can do explicit computations of pair invariants in small degrees for non-compact CY 4-folds

$$\text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)), \quad \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1)).$$

We list the results as follows (where pair invariants are defined with respect to certain choices of orientation).

If $X = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$, then

- $P_{0,1} = P_{0,2} = 0, P_{0,3} = -1, P_{0,4} = 2,$
- $P_{1,1}([\text{pt}]) = -1, P_{1,2}([\text{pt}]) = 1, P_{1,3}([\text{pt}]) = -1, P_{1,4}([\text{pt}]) = 3.$

If $X = \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1))$, then

- $P_{0,(2,2)} = 1, P_{0,(2,3)} = 2, P_{0,(2,4)} = 5, P_{0,(3,3)} = 10,$
- $P_{1,(2,2)}([\text{pt}]) = 2, P_{1,(2,3)}([\text{pt}]) = 5.$

Comparing with [24, pp. 22, 24], we see that our Conjectures 2.5 and 2.6 hold in all the above cases.

5. Local curves

Let C be a smooth projective curve of genus $g(C) = g$, and

$$p: X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3) \rightarrow C \tag{45}$$

be the total space of a split rank 3 vector bundle on it. Assuming that

$$L_1 \otimes L_2 \otimes L_3 \cong \omega_C, \tag{46}$$

the variety (45) is a non-compact CY 4-fold. Below we set $l_i := \deg L_i$ and may assume that $l_1 \geq l_2 \geq l_3$ without loss of generality.

Let $T = (\mathbb{C}^*)^3$ be the three-dimensional complex torus which acts on the fibers of X . Its restriction to the subtorus

$$T_0 = \{t_1 t_2 t_3 = 1\} \subset T$$

preserves the CY 4-form on X and also the Serre duality pairing on $P_n(X, \beta)$. In this section, we aim to define equivariant virtual classes of $P_n(X, \beta)$ using a localization formula with respect to the T_0 -action [12, 13], and investigate their relations to equivariant GW invariants.

Let \bullet be the point $\text{Spec } \mathbb{C}$ with trivial T -action, $\mathbb{C} \otimes t_i$ be the one-dimensional T -representation with weight 1, and $\lambda_i \in H_T^*(\bullet)$ be its first Chern class. They are generators of equivariant cohomology rings:

$$H_T^*(\bullet) = \mathbb{C}[\lambda_1, \lambda_2, \lambda_3], \quad H_{T_0}^*(\bullet) = \frac{\mathbb{C}[\lambda_1, \lambda_2, \lambda_3]}{(\lambda_1 + \lambda_2 + \lambda_3)} \cong \mathbb{C}[\lambda_1, \lambda_2]. \tag{47}$$

5.1. Localization for GW invariants

Let $j: C \hookrightarrow X$ be the zero section of the projection (45). We have

$$H_2(X, \mathbb{Z}) = \mathbb{Z}[C],$$

where $[C]$ is the fundamental class of $j(C)$. For $m \in \mathbb{Z}_{>0}$, we consider the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & C \\ \downarrow \pi & & \\ \overline{M}_h(C, m[C]) & & \end{array}$$

where \mathcal{C} is the universal curve and f is the universal stable map.

The T -equivariant GW invariant of X is defined by

$$\text{GW}_{h,d[C]}(X) = \text{GW}_{h,d}(X) := \int_{[\overline{M}_h(C, d[C])]^{\text{vir}}} e(-\mathbf{R}h_* f^* N) \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3),$$

where N is the T -equivariant normal bundle of $j(C) \subset X$:

$$N = (L_1 \otimes t_1) \oplus (L_2 \otimes t_2) \oplus (L_3 \otimes t_3). \tag{48}$$

If $g(C) > 0$, the vanishing of genus 0 GW invariants

$$\text{GW}_{0,d}(X) = 0, \quad g(C) > 0, \quad d \in \mathbb{Z}_{>0}$$

follows from $\overline{M}_0(C, d[C]) = \emptyset$.

If $g(C) = 0$, we have

$$GW_{0,d}(X) = \int_{[\overline{M}_0(\mathbb{P}^1,d)]} e(-\mathbf{R}h_* f^*(\mathcal{O}_{\mathbb{P}^1}(l_1)t_1 \oplus \mathcal{O}_{\mathbb{P}^1}(l_2)t_2 \oplus \mathcal{O}_{\mathbb{P}^1}(l_3)t_3)).$$

For example in the $d = 1$ case, $\overline{M}_0(\mathbb{P}^1, 1)$ is one point and

$$GW_{0,1}(X) = \lambda_1^{-l_1-1} \lambda_2^{-l_2-1} \lambda_3^{-l_3-1}. \tag{49}$$

In the $d = 2$ case, a straightforward localization calculation with respect to the $(\mathbb{C}^*)^2$ -action on \mathbb{P}^1 gives

$$\begin{aligned} GW_{0,2}(X) = & \frac{1}{8} \lambda_1^{-2l_1-1} \lambda_2^{-2l_2-1} \lambda_3^{-2l_3-1} \{(\bar{l}_1^2 - (\bar{l}_1 - 1)^2 + \dots) \lambda_1^{-2} \\ & + (\bar{l}_2^2 - (\bar{l}_2 - 1)^2 + \dots) \lambda_2^{-2} + (\bar{l}_3^2 - (\bar{l}_3 - 1)^2 + \dots) \lambda_3^{-2} \\ & + l_1 l_2 \lambda_1^{-1} \lambda_2^{-1} + l_2 l_3 \lambda_2^{-1} \lambda_3^{-1} + l_1 l_3 \lambda_1^{-1} \lambda_3^{-1}\}. \end{aligned} \tag{50}$$

Here we write $\bar{l} = l$ for $l \geq 0$ and $\bar{l} = -l - 1$ for $l < 0$.

5.2. Localization for stable pairs

Similarly, for $m \in \mathbb{Z}_{\geq 0}$, we want to define an (equivariant) stable pair invariant

$$P_{n,m[C]}(X) = [P_n(X, m[C])^{T_0}]^{\text{vir}} \cdot e(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}, \mathbb{I})_0^{\text{mov}})^{1/2}, \tag{51}$$

where $\mathbb{I} = (\mathcal{O}_{X \times P_n(X, m[C])} \rightarrow \mathbb{F}) \in D^b(X \times P_n(X, m[C]))$ is the universal stable pair and $\pi_P : X \times P_n(X, m[C]) \rightarrow P_n(X, m[C])$ is the projection. Of course, the above equality is not a definition as the virtual class of the fixed locus as well as the square root needs justification. We will make this precise in specific cases where we compare with GW invariants of X .

Let us first describe stable pairs $(s: \mathcal{O}_X \rightarrow F) \in P_n(X, m[C])^T$ which are fixed by the full torus T : decompose F into T -weight spaces,

$$p_* F = \bigoplus_{(i_1, i_2, i_3) \in \mathbb{Z}^3} F^{i_1, i_2, i_3},$$

where the T -weight of F^{i_1, i_2, i_3} is (i_1, i_2, i_3) . We define an index set

$$\Delta := \{(i_1, i_2, i_3) \in \mathbb{Z}_{\geq 0}^3 : F^{-i_1, -i_2, -i_3} \neq 0\}. \tag{52}$$

We also have the decomposition

$$p_* \mathcal{O}_X = \bigoplus_{(i_1, i_2, i_3) \in \mathbb{Z}_{\geq 0}^3} L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3}$$

into the direct sum of weight $(-i_1, -i_2, -i_3)$ factors $L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3}$.

The T -equivariance of s induces morphisms

$$s^{i_1, i_2, i_3}: L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3} \rightarrow F^{-i_1, -i_2, -i_3}$$

in $\text{Coh}(C)$ which are surjective in dimension 1. It follows that each $F^{-i_1, -i_2, -i_3}$ is either 0 or can be written as

$$F^{-i_1, -i_2, -i_3} = L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3} \otimes \mathcal{O}_C(Z_{i_1, i_2, i_3})$$

for some effective divisor $Z_{i_1, i_2, i_3} \subset C$. Moreover, the $p_*\mathcal{O}_X$ -module structure on F gives a morphism

$$F^{-i_1, -i_2, -i_3} \otimes L_1^{-1} \rightarrow F^{-i_1-1, -i_2, -i_3}$$

which commutes with s^{i_1, i_2, i_3} and s^{i_1+1, i_2, i_3} . Similar morphisms with L_1 replaced by L_2 and L_3 exist and have similar commuting properties. Hence, for $(i_1, i_2, i_3) \in \Delta$, we have

$$Z_{i_1-1, i_2, i_3}, Z_{i_1, i_2-1, i_3}, Z_{i_1, i_2, i_3-1} \leq Z_{i_1, i_2, i_3},$$

as divisors in C . So the set Δ of (52) is a three-dimensional Young diagram, which is finite by the coherence of F .

In general, it is difficult to explicitly determine T_0 -fixed stable pairs. In fact, a T_0 -fixed stable pair is not necessarily T -fixed. Nevertheless, for a T_0 -fixed stable pair $(s: \mathcal{O}_X \rightarrow F)$, $\mathcal{O}_{C_F} := \text{Im } s$ and the corresponding ideal sheaf I_{C_F} are actually T -fixed.

Lemma 5.1. *Let $I = (s: \mathcal{O}_X \rightarrow F) \in P_n(X, m[C])^{T_0}$ be a T_0 -fixed stable pair and $\mathcal{O}_{C_F} := \text{Im } s \subseteq F$. Then the ideal sheaf $I_{C_F} \subseteq \mathcal{O}_X$ is T -fixed.*

Proof. Since I_{C_F} equals $\mathcal{H}^0(I)$, it is T_0 -fixed. For $t \in T$, we have the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_{C_F} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{C_F} \longrightarrow 0 \\ & & & & \downarrow \cong & & \\ 0 & \longrightarrow & t^*I_{C_F} & \longrightarrow & t^*\mathcal{O}_X & \longrightarrow & t^*\mathcal{O}_{C_F} \longrightarrow 0 \end{array}$$

The above diagram induces a morphism $u \in \text{Hom}(I_{C_F}, t^*\mathcal{O}_{C_F})$. It is enough to show $u = 0$. For a general point $c \in C$, let $X_c = p^{-1}(c) = \mathbb{C}^3$ be the fiber of p at c . Then $I_{C_F}|_{X_c}$ is an ideal sheaf of T_0 -fixed zero-dimensional subscheme of \mathbb{C}^3 . Then it is also T -fixed by [2, Lemma 4.1]. This implies that the morphism restricted to X_c is a zero map. Then $\text{Im } u \subset t^*\mathcal{O}_{C_F}$ is 0 on the general fiber of p , hence $\text{Im } u = 0$ by the purity of C_F . ■

Another convenient way to determine T_0 -fixed stable pairs is when $P_n(X, m[C])^T$ is smooth and $\text{Hom}_X(I, F)^{T_0} = \text{Hom}_X(I, F)^T$ for any $I = (\mathcal{O}_X \rightarrow F) \in P_n(X, m[C])^T$ (see e.g. [35, Sect. 3.3] on toric 3-folds). Then one has $P_n(X, m[C])^T = P_n(X, m[C])^{T_0}$. In the examples below, we will explicitly determine the T_0 -fixed locus, mainly using Lemma 5.1.

5.3. $P_{1,m[C]}(X)$ and genus 0 conjecture

Let $C = \mathbb{P}^1$ be a smooth rational curve and $X = \mathcal{O}_{\mathbb{P}^1}(l_1, l_2, l_3)$ with $l_1 + l_2 + l_3 = -2$. This serves as the local model for a neighborhood of a rational curve in a CY 4-fold.

For some special choice of (l_1, l_2, l_3) , we can determine $P_{1,m[C]}(X)$ for all m .

Proposition 5.2. *If $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$, then $P_{1,m[\mathbb{P}^1]}(X)$ is well-defined and satisfies*

$$P_{1, [\mathbb{P}^1]}(X) = \pm \lambda_3^{-1}, \quad P_{1,m[\mathbb{P}^1]}(X) = 0 \quad \text{when } m > 1.$$

If $X = \mathcal{O}_{\mathbb{P}^1}(-2, 0, 0)$, then $P_{1,m[\mathbb{P}^1]}(X)$ is well-defined and satisfies

$$P_{1, [\mathbb{P}^1]}(X) = \pm \frac{\lambda_1}{\lambda_2 \lambda_3}, \quad P_{1,m[\mathbb{P}^1]}(X) = 0 \quad \text{when } m > 1.$$

Proof. Let $(s : \mathcal{O}_X \rightarrow F)$ be a T_0 -fixed stable pair and $\mathcal{O}_{C_F} = \text{Im}(s)$. Then

$$1 = \chi(F) = \chi(\mathcal{O}_{C_F}) + \chi(F/\mathcal{O}_{C_F}).$$

So $\chi(\mathcal{O}_{C_F}) = 1$ or 0 . By Lemma 5.1, $(s : \mathcal{O}_X \rightarrow \mathcal{O}_{C_F})$ is T -fixed. From the characterization of T -fixed stable pairs, it is of the form

$$\mathcal{O}_X \twoheadrightarrow \bigoplus_{(i_1, i_2, i_3) \in \Delta} \mathcal{O}_{\mathbb{P}^1}(l_1)^{-i_1} \otimes \mathcal{O}_{\mathbb{P}^1}(l_2)^{-i_2} \otimes \mathcal{O}_{\mathbb{P}^1}(l_3)^{-i_3} \otimes \mathcal{O}_{\mathbb{P}^1}(Z_{i_1, i_2, i_3}).$$

If $(l_1, l_2, l_3) = (-1, -1, 0)$ or $(-2, 0, 0)$, it is obvious that the only possibility is $\chi(\mathcal{O}_{C_F}) = 1$ (so $F \cong \mathcal{O}_{C_F}$) and C_F is the zero section of X . So $P_1(X, m[\mathbb{P}^1]) = \emptyset$ unless $m = 1$.

By (51), we have

$$\begin{aligned} P_{1, [\mathbb{P}^1]}(X) &= \pm \frac{\sqrt{(-1)^{\frac{1}{2}} \text{ext}_X^2(I_{\mathbb{P}^1}, I_{\mathbb{P}^1})_0} \cdot e_{T_0}(\text{Ext}_X^2(I_{\mathbb{P}^1}, I_{\mathbb{P}^1})_0)}{e_{T_0}(\text{Ext}_X^1(I_{\mathbb{P}^1}, I_{\mathbb{P}^1})_0)} \\ &= \pm \frac{e_{T_0}(H^1(\mathbb{P}^1, L_1 \otimes t_1 \oplus L_2 \otimes t_2 \oplus L_3 \otimes t_3))}{e_{T_0}(H^0(\mathbb{P}^1, L_1 \otimes t_1 \oplus L_2 \otimes t_2 \oplus L_3 \otimes t_3))}. \end{aligned}$$

Then the calculation is straightforward. ■

By comparing the above computations with the corresponding GW invariants, we obtain the following equivariant analogue of Conjecture 2.5 (note that from the above proof, we know $P_{0,m[C]}(X) = 0$ for $m \geq 1$ since $P_0(X, m[C]) = \emptyset$).

Corollary 5.3. *Let X be $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$ or $\mathcal{O}_{\mathbb{P}^1}(-2, 0, 0)$. Then*

$$\text{GW}_{0,m}(X) = \sum_{k|m, k \geq 1} \frac{1}{k^3} P_{1,(m/k)[\mathbb{P}^1]}(X)$$

for suitable choices of orientation in defining the RHS.

Proof. If $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$, by the Aspinwall–Morrison formula we have

$$\text{GW}_{0,m}(X) = \frac{1}{m^3} \lambda_3^{-1}.$$

If $X = \mathcal{O}_{\mathbb{P}^1}(-2, 0, 0)$, from GW invariants of $K_{\mathbb{P}^1}$ (e.g. [31, Thm 1.1]) we can conclude that

$$\text{GW}_{0,m}(X) = \frac{1}{m^3} \cdot \frac{\lambda_1}{\lambda_2 \lambda_3}.$$

Comparing this with Proposition 5.2, we are done. ■

For a general local curve $X = \mathcal{O}_{\mathbb{P}^1}(l_1, l_2, l_3)$, we study $P_1(X, m[\mathbb{P}^1])$ for $m = 1, 2$ as follows.

Degree 1 class. When $m = 1$, it is easy to show that the canonical section

$$(s : \mathcal{O}_X \rightarrow \mathcal{O}_{\mathbb{P}^1})$$

gives the only T_0 -fixed stable pair in $P_1(X, [\mathbb{P}^1])$. Similar to Proposition 5.2, we have

$$\begin{aligned} P_{1, [\mathbb{P}^1]}(X) &= \frac{e_{T_0}(H^1(\mathbb{P}^1, L_1 \otimes t_1 \oplus L_2 \otimes t_2 \oplus L_3 \otimes t_3))}{e_{T_0}(H^0(\mathbb{P}^1, L_1 \otimes t_1 \oplus L_2 \otimes t_2 \oplus L_3 \otimes t_3))} \\ &= \lambda_1^{-l_1-1} \lambda_2^{-l_2-1} \lambda_3^{-l_3-1}, \end{aligned}$$

which coincides with the corresponding GW invariant (49). Here we have chosen the plus sign in defining $P_{1, [\mathbb{P}^1]}(X)$.

Degree 2 class. When $m = 2$, let

$$(s : \mathcal{O}_X \rightarrow F) \in P_1(X, 2[\mathbb{P}^1])$$

be a T_0 -fixed stable pair. Then F is thickened in one of the L_i -directions, i.e.

$$p_* F = F_0 \oplus (F_i \otimes t_i^{-1}),$$

where F_0, F_i are line bundles on \mathbb{P}^1 , hence F is also T -fixed. As the T -weight of F_i is not of the form (l, l, l) , T_0 -invariant sections of F are also T -invariant. So we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^1} \otimes L_i^{-1} & \xrightarrow{s^0} & F_0 \otimes L_i^{-1} \\ \downarrow = & & \downarrow \phi \\ L_i^{-1} & \xrightarrow{s^i} & F_i \otimes t_i^{-1} \end{array}$$

where s^0 and s^i are injective, and surjective in dimension 1, ϕ defines the $p_* \mathcal{O}_X$ -module structure (which is also injective, and surjective in dimension 1 by the diagram).

Denote $F_0 = \mathcal{O}_{\mathbb{P}^1}(d_0)$ and $F'_i = F_i \otimes L_i = \mathcal{O}_{\mathbb{P}^1}(d_i)$. Then the above diagram is equivalent to a commutative diagram

$$\begin{CD} \mathcal{O}_{\mathbb{P}^1} @>s^0>> \mathcal{O}_{\mathbb{P}^1}(d_0) \\ @= @VV\phi V \\ \mathcal{O}_{\mathbb{P}^1} @>s^i>> \mathcal{O}_{\mathbb{P}^1}(d_i) \end{CD}$$

where s^0, s^i and ϕ are injective. These impose conditions

$$0 \leq d_0 \leq d_i, \quad d_0 + d_i = l_i - 1,$$

where the last equality is because $\chi(F) = 1$. It is not hard to show

Lemma 5.4. *We have an isomorphism*

$$P_1(X, 2[\mathbb{P}^1])^{T_0} \cong \coprod_{i=1}^3 \coprod_{\substack{(d_0, d_i) \in \mathbb{Z}^2 \\ d_0 + d_i = l_i - 1 \\ 0 \leq d_0 \leq (l_i - 1)/2}} \text{Pic}^{(d_0, d_i)}(\mathbb{P}^1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d_0))), \quad (53)$$

where $\text{Pic}^{(a,b)}(\mathbb{P}^1)$ denotes the moduli space of triples

$$(L, L', \iota), \quad (L, L') \in \text{Pic}^a(\mathbb{P}^1) \times \text{Pic}^b(\mathbb{P}^1), \quad \iota: L \hookrightarrow L',$$

and ι is an inclusion of sheaves.

To determine the virtual class $[P_1(X, 2[\mathbb{P}^1])^{T_0}]^{\text{vir}}$ and the square root in (51), we take a T_0 -fixed stable pair $I = (s : \mathcal{O}_X \rightarrow F)$ and view it as an element in the T_0 -equivariant K -theory of X . Then

$$\chi(I, I)_0 = \chi(F, F) - \chi(\mathcal{O}_X, F) - \chi(F, \mathcal{O}_X) \in K_{T_0}(\text{pt}),$$

where both sides can be written using grading into T_0 -weight spaces.

Similar to [13, Sect. 4.4], we set

$$\begin{aligned} \chi(F, F)^{1/2} &:= \chi(j_* F_0, j_* F_0) + \chi(j_* F_0, j_* F_i) t_i^{-1}, \\ \chi(I, I)_0^{1/2} &:= \chi(F, F)^{1/2} - \chi(\mathcal{O}_X, F), \end{aligned} \quad (54)$$

where $F = F_0 + F_i \otimes t_i^{-1} \in K_{T_0}(X)$ and j is inclusion of the zero section of X .

The T_0 -fixed and movable part satisfies

$$\begin{aligned} \chi(I, I)_0^{1/2, \text{fix}} &= \chi(F, F)^{1/2, \text{fix}} - \chi(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(d_0)), \\ \chi(I, I)_0^{1/2, \text{mov}} &= \chi(F, F)^{1/2, \text{mov}} - \chi(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(d_i - l_i)) \cdot t_i^{-1}, \end{aligned}$$

where $\chi(F, F)^{1/2, \text{fix}}$ and $\chi(F, F)^{1/2, \text{mov}}$ were computed in [13, Sect. 4.4]. In particular,

$$\dim_{\mathbb{C}} \chi(F, F)^{1/2, \text{fix}} = 1 - d_i + d_0, \quad \dim_{\mathbb{C}} \chi(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(d_0)) = d_0 + 1.$$

So $\dim_{\mathbb{C}}(-\chi(I, I)_0^{1/2, \text{fix}}) = d_i$ is the dimension of $P_1(X, 2[\mathbb{P}^1])^{T_0}$. Thus the virtual class of the associated T_0 -fixed locus $P_1(X, 2[\mathbb{P}^1])^{T_0}$ may be defined to be its usual fundamental class.

We can now give a definition of $P_{1,2[\mathbb{P}^1]}(X) \in \mathbb{Q}(\lambda_1, \lambda_2)$ based on the localization formula (51) and the above discussion. Denote by

$$(\mathcal{F}_0, \mathcal{F}'_i, \iota), \quad \iota: \mathcal{F}_0 \hookrightarrow \mathcal{F}'_i,$$

the universal object on $\text{Pic}^{(d_0, d_i)}(\mathbb{P}^1) \times \mathbb{P}^1$, where $\mathcal{F}_0, \mathcal{F}'_i$ are line bundles on $\text{Pic}^{(d_0, d_i)}(\mathbb{P}^1) \times \mathbb{P}^1$ and ι is the universal injection. Let $\mathcal{F}_i := \mathcal{F}'_i \boxtimes L_i^{-1}$, and consider its push-forward

$$j_* \mathcal{F}_i \in \text{Coh}(\text{Pic}^{(d_0, d_i)}(\mathbb{P}^1) \times X), \quad i = 1, 2, 3.$$

From (51) and (54), we define $P_{1,2[\mathbb{P}^1]}(X)$ as an element in $\mathbb{Q}(\lambda_1, \lambda_2)$ by

$$P_{1,2[\mathbb{P}^1]}(X) := \sum_{i=1}^3 \sum_{\substack{(d_0, d_i) \in \mathbb{Z}^2 \\ d_0 + d_i = l_i - 1 \\ 0 \leq d_0 \leq (l_i - 1)/2}} \left(\int_{\text{Pic}^{(d_0, d_i)}(\mathbb{P}^1)} e_{T_0}(\mathcal{N}_1) \cdot \int_{\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d_0)))} e_{T_0}(\mathcal{N}_2) \right), \tag{55}$$

where

$$\begin{aligned} \mathcal{N}_1 &:= \mathbf{R}\mathcal{H}om_{\pi_1}(j_* \mathcal{F}_0, j_* \mathcal{F}_0)^{\text{mov}} + \mathbf{R}\mathcal{H}om_{\pi_1}(j_* \mathcal{F}_0, j_* \mathcal{F}_i \cdot t_i^{-1})^{\text{mov}}, \\ \mathcal{N}_2 &:= -\mathbf{R}(\pi_2)_*(\mathcal{O}_{\mathbb{P}^{d_0} \times \mathbb{P}^1}(1, d_i - l_i) \cdot t_i^{-1}). \end{aligned}$$

Here $\pi_1 : \text{Pic}^{(d_0, d_i)}(\mathbb{P}^1) \times X \rightarrow \text{Pic}^{(d_0, d_i)}(\mathbb{P}^1)$ and $\pi_2 : \mathbb{P}^{d_0} \times \mathbb{P}^1 \rightarrow \mathbb{P}^{d_0}$ are the natural projections. and we have used the isomorphism (53).

In the above definition, the second integral can be easily shown to be 1 and the first one has been explicitly determined before [13, Corollary 4.9]. So we obtain

Proposition 5.5. *Let $X = \mathcal{O}_{\mathbb{P}^1}(l_1, l_2, l_3)$ with $l_1 + l_2 + l_3 = -2$ and $l_1 \geq l_2 \geq l_3$. Then*

$$P_{1,2[\mathbb{P}^1]}(X) = -\lambda_1^{-2l_1-2} \lambda_2^{-2l_2-2} (\lambda_1 + \lambda_2)^{-2l_3-2} \cdot \left(\sum_{\substack{1 \leq k \leq l_1, \\ k \equiv l_1 \pmod{2}}} A(l_1, l_2, l_3, k) + \sum_{\substack{1 \leq k \leq l_2, \\ k \equiv l_2 \pmod{2}}} B(l_1, l_2, l_3, k) \right),$$

where

$$\begin{aligned} A(l_1, l_2, l_3, k) &:= \text{Res}_{h=0} \{ h^{-k} (-\lambda_1 + h)^2 (\lambda_2 + h)^{k+l_2} (-\lambda_1 - \lambda_2 + h)^{k+l_3} \\ &\quad \cdot (-\lambda_1 + \lambda_2 + h)^{l_1-l_2-k} (-2\lambda_1 - \lambda_2 + h)^{l_1-l_3-k} (-2\lambda_1 + h)^{k-2-2l_1} \}, \\ B(l_1, l_2, l_3, k) &:= \text{Res}_{h=0} \{ h^{-k} (-\lambda_2 + h)^2 (\lambda_1 + h)^{k+l_1} (-\lambda_2 - \lambda_1 + h)^{k+l_3} \\ &\quad \cdot (-\lambda_2 + \lambda_1 + h)^{l_2-l_1-k} (-2\lambda_2 - \lambda_1 + h)^{l_2-l_3-k} (-2\lambda_2 + h)^{k-2-2l_2} \}. \end{aligned}$$

We pose the following equivariant version of Conjecture 2.5 (note in this case $P_{0, [\mathbb{P}^1]}(X) = 0$ as $\chi(\mathcal{O}_{\mathbb{P}^1}) > 0$). It is consistent with our previous conjecture on one-dimensional stable sheaves [13, Conj. 4.10].

Conjecture 5.6. *Let $X = \mathcal{O}_{\mathbb{P}^1}(l_1, l_2, l_3)$ for $l_1 + l_2 + l_3 = -2$. Then*

$$\text{GW}_{0,2}(X) = P_{1,2[\mathbb{P}^1]}(X) + \frac{1}{8}P_{1, [\mathbb{P}^1]}(X).$$

Combining Proposition 5.5 and [13, Thm. 4.12], we can verify the conjecture in a large number of examples.

Theorem 5.7. *Conjecture 5.6 is true if $|l_1| \leq 10$ and $|l_2| \leq 10$.*

5.4. $P_{0,m[C]}(X)$ and genus 1 conjecture

To complete the heuristic argument for our genus 1 conjecture in Section 2.5, we consider $X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3)$ where C is an elliptic curve and $L_1 \otimes L_2 \otimes L_3 \cong \omega_C \cong \mathcal{O}_C$.

Lemma 5.8. *Let $I \subset \mathcal{O}_X$ be the ideal sheaf of a closed subscheme $Z \subset X$ with $\dim Z \leq 1$. Then we have canonical isomorphisms*

$$\begin{aligned} \text{Ext}_X^1(I, I)_0 &\cong H^0(X, \mathcal{E}xt_X^1(I, I)), \\ \text{Ext}_X^2(I, I)_0 &\cong H^0(X, \mathcal{E}xt_X^2(I, I)) \oplus H^1(X, \mathcal{E}xt_X^1(I, I)). \end{aligned} \tag{56}$$

Furthermore, if $p_*\mathcal{E}xt_X^1(I, I)$ and $p_*\mathcal{E}xt_X^2(I, I)$ are locally free, then

$$H^1(X, \mathcal{E}xt_X^1(I, I)) \cong H^0(X, \mathcal{E}xt_X^2(I, I))^\vee.$$

And $H^0(X, \mathcal{E}xt_X^2(I, I))$ and $H^1(X, \mathcal{E}xt_X^1(I, I))$ are maximal isotropic subspaces of $\text{Ext}_X^2(I, I)_0$ with respect to the Serre duality pairing.

Proof. We have the local-to-global spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt_X^q(I, I)_0) \Rightarrow \text{Ext}_X^{p+q}(I, I)_0.$$

And

$$\mathcal{E}xt_X^0(I, I)_0 = 0, \quad \mathcal{E}xt_X^{\geq 1}(I, I)_0 \cong \mathcal{E}xt_X^{\geq 1}(I, I)$$

are supported on Z . Therefore $E_2^{p,0} = 0$ and $E_2^{p,q} = 0$ for $p \geq 2, q \geq 1$. Then the above spectral sequence degenerates and (56) holds. The latter statement follows from the adjunction

$$\text{Ext}_X^i(p^*\mathcal{O}_C, \mathcal{E}xt_X^j(I, I)) = \text{Ext}_C^i(\mathcal{O}_C, p_*\mathcal{E}xt_X^j(I, I)),$$

and the Grothendieck duality

$$\mathbf{R}p_*\mathbf{R}\mathcal{H}om_X(I, I)_0[4] \cong \mathbf{R}\mathcal{H}om_C(\mathbf{R}p_*\mathbf{R}\mathcal{H}om_X(I, I)_0, \omega_C[1])$$

for the projection $p : X \rightarrow C$ of (45). ■

We describe the torus fixed locus $P_0(X, m[C])^{T_0}$ as follows.

Lemma 5.9. *Let C be an elliptic curve and $L_i \in \text{Pic}^0(C)$. Then $(\mathcal{O}_X \rightarrow F) \in P_0(X, m[C])$ is T_0 -fixed if and only if it is of the form*

$$\mathcal{O}_X \rightarrow \bigoplus_{(i_1, i_2, i_3) \in \Delta} L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3} \tag{57}$$

for some three-dimensional Young diagram $\Delta \subset \mathbb{Z}_{\geq 0}^3$. In particular, in this case

$$P_0(X, m[C])^T = P_0(X, m[C])^{T_0}.$$

Proof. The stable pair (57) is obviously T -fixed, hence T_0 -fixed. Conversely, for a T_0 -fixed stable pair $(s: \mathcal{O}_X \rightarrow F)$ with $\chi(F) = 0$, we denote $\mathcal{O}_Z = \text{Im } s$ and then I_Z is T -fixed by Lemma 5.1. It follows that $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ is of the form

$$\mathcal{O}_X \twoheadrightarrow \bigoplus_{(i_1, i_2, i_3) \in \Delta} L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3} \otimes \mathcal{O}_C(Z_{i_1, i_2, i_3}).$$

Since $c_1(L_i) = 0$ and F/\mathcal{O}_Z is zero-dimensional, we have

$$0 = \chi(F) = \chi(\mathcal{O}_Z) + \chi(F/\mathcal{O}_Z) \geq \chi(F/\mathcal{O}_Z) \geq 0.$$

So $F \cong \mathcal{O}_Z$ and $Z_{i_1, i_2, i_3} = 0$. ■

We determine stable pair invariants for $X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3)$ when the line bundles $L_i \in \text{Pic}^0(C)$ over the elliptic curve C are general.

Theorem 5.10. *Let C be an elliptic curve, $L_i \in \text{Pic}^0(C)$ ($i = 1, 2, 3$) general line bundles satisfying $L_1 \otimes L_2 \otimes L_3 \cong \omega_C$ and $X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3)$. Then the stable pair invariants $P_{0, m[C]}(X)$ of (51) are well-defined and fit into a generating series*

$$\sum_{m \geq 0} P_{0, m[C]}(X) q^m = M(q),$$

where $M(q) := \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function.

Proof. By Lemma 5.9, $P_0(X, m[C])^{T_0}$ is the finite set of three-dimensional partitions of m , and any $(s: \mathcal{O}_X \rightarrow F) \in P_0(X, m[C])^{T_0}$ satisfies $F \cong \mathcal{O}_W$ for some Cohen-Macaulay curve W in X . We denote by I the ideal sheaf of W .

Let $U \subset C$ be an open subset on which the L_i are trivial. Then $p^{-1}(U) \cong U \times \mathbb{C}^3$ and $I|_{p^{-1}(U)}$ is isomorphic to $\pi^* I_Z$ for the T -fixed zero-dimensional subscheme $Z \subset \mathbb{C}^3$ corresponding to Δ . Therefore we have an isomorphism of T -equivariant sheaves on U ,

$$p_* \text{Ext}_X^k(I, I)|_U \cong \text{Ext}_{\mathbb{C}^3}^k(I_Z, I_Z) \otimes_{\mathbb{C}} \mathcal{O}_U. \tag{58}$$

Let

$$\text{Ext}_{\mathbb{C}^3}^k(I_Z, I_Z) = \bigoplus_{(i_1, i_2, i_3) \in \mathbb{Z}^3} V_{i_1, i_2, i_3}^k \otimes t_1^{i_1} t_2^{i_2} t_3^{i_3}$$

be the decomposition into T -weight spaces. By (58), we have

$$p_* \mathcal{E}xt_X^k(I, I) \cong \bigoplus_{(i_1, i_2, i_3) \in \mathbb{Z}^3} V_{i_1, i_2, i_3}^k \otimes L_1^{i_1} \otimes L_2^{i_2} \otimes L_3^{i_3} \otimes t_1^{i_1} t_2^{i_2} t_3^{i_3}.$$

The relation (46) and Lemma 5.8 imply that

$$\text{Ext}_X^1(I, I)_0^{T_0} = \bigoplus_{i \in \mathbb{Z}} V_{i, i, i}^1, \quad \text{Ext}_X^2(I, I)_0^{T_0} = \bigoplus_{i \in \mathbb{Z}} V_{i, i, i}^1 \oplus V_{i, i, i}^2.$$

By [2, Lemma 4.1], we have $V_{i, i, i}^1 = V_{i, i, i}^2 = 0$ (note $(V_{i, i, i}^2)^\vee \cong V_{-i, -i, -i}^1$). Therefore

$$[P_0(X, m[C])^{T_0}]^{\text{vir}} = [P_0(X, m[C])].$$

For the movable part, there are decompositions

$$\text{Ext}_X^1(I, I)_0^{\text{mov}} = \bigoplus_{(i_1 - i_3, i_2 - i_3) \neq (0, 0)} V_{i_1, i_2, i_3}^1 \otimes H^0(L_1^{i_1 - i_3} \otimes L_2^{i_2 - i_3}) \otimes t_1^{i_1 - i_3} t_2^{i_2 - i_3},$$

$$\begin{aligned} \text{Ext}_X^2(I, I)_0^{\text{mov}} &= \bigoplus_{(i_1 - i_3, i_2 - i_3) \neq (0, 0)} (V_{i_1, i_2, i_3}^1 \oplus V_{i_1, i_2, i_3}^2) \otimes H^0(L_1^{i_1 - i_3} \otimes L_2^{i_2 - i_3}) \otimes t_1^{i_1 - i_3} t_2^{i_2 - i_3}. \end{aligned}$$

For a general choice of (L_1, L_2) , we have $H^0(L_1^a \otimes L_2^b) = H^1(L_1^a \otimes L_2^b) = 0$ for any $(a, b) \neq (0, 0)$, so the movable part also vanishes. Thus

$$P_{0, m[C]}(X) = \sharp(P_0(X, m[C])^{T_0}),$$

which is the number of three-dimensional partitions of m . ■

6. Appendices

6.1. Stable pairs and one-dimensional sheaves for irreducible curve classes

When $\beta \in H_2(X, \mathbb{Z})$ is an irreducible curve class on a smooth projective CY 4-fold X , we have a morphism

$$\phi_n : P_n(X, \beta) \rightarrow M_{n, \beta}(X)$$

to the moduli scheme of one-dimensional stable sheaves with Chern character $(0, 0, 0, \beta, n)$ (e.g. [36, p. 270]), whose fiber over $[F]$ is $\mathbb{P}(H^0(X, F))$. Note that $M_{n, \beta}(X)$ is in general a stack instead of scheme when β is arbitrary. The virtual dimension of $M_{n, \beta}(X)$ satisfies

$$\text{vir. dim}_{\mathbb{R}}(M_{n, \beta}(X)) = 2,$$

by [4, 13]. One could use the virtual class to define invariants.

For integral classes $\gamma_i \in H^{m_i}(X, \mathbb{Z})$, $1 \leq i \leq l$, let

$$\tau : H^m(X) \rightarrow H^{m-2}(M_{n, \beta}(X)), \quad \tau(\gamma) = \pi_{P*}(\pi_X^* \gamma \cup \text{ch}_3(\mathbb{F})),$$

where π_X, π_M are the projections from $X \times M_{n,\beta}(X)$ to the corresponding factors, $\mathbb{F} \rightarrow X \times M_{n,\beta}(X)$ is the universal sheaf, and $\text{ch}_3(\mathbb{F})$ is the Poincaré dual to the fundamental cycle of \mathbb{F} .

Then we define the DT_4 invariant by

$$\text{DT}_4(n, \beta \mid \gamma_1, \dots, \gamma_l) := \int_{[M_{n,\beta}(X)]^{\text{vir}}} \prod_{i=1}^l \tau(\gamma_i).$$

We propose the following conjecture.

Conjecture 6.1. *For an irreducible class $\beta \in H_2(X, \mathbb{Z})$, the invariants*

$$\text{DT}_4(n, \beta \mid \gamma_1, \dots, \gamma_l)$$

are independent of the choice of n for a certain choices of orientation in defining them.

In all compact examples studied in this paper, one can check that Conjecture 6.1 holds. In particular, when $X = Y \times E$ is the product of a CY 3-fold Y with an elliptic curve E and the irreducible class $\beta \in H_2(Y, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$ sits inside Y , then Conjecture 6.1 reduces to a special case of the multiple cover formula ([21, Conjecture 6.20], [41, Conjecture 6.3])

$$N_{n,\beta} = \sum_{k \geq 1, k \mid (n,\beta)} \frac{1}{k^2} N_{1,\beta/k}$$

for any β in a CY 3-fold Y , where $N_{n,\beta} \in \mathbb{Q}$ is the generalized DT invariant [21] which counts one-dimensional semistable sheaves E on Y with $[E] = \beta, \chi(E) = n$. The above formula is proved when β is primitive in [42, Lemma 2.12] (see also [13, Appendix A]).

It is an interesting question to define a ‘generalized DT_4 type invariant’ counting semistable sheaves on CY 4-folds and search for a similar multiple cover formula on CY 4-folds.

6.2. An orientability result for moduli spaces of stable pairs on CY 4-folds

Let X be a smooth projective CY 4-fold and $c \in H^{\text{even}}(X)$. For a moduli stack M_c of coherent sheaves on X with Chern character c , we define

$$\mathcal{L} := \det(\mathbf{R}(p_M)_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathbb{F}))$$

to be the determinant line bundle, where $\mathbb{F} \rightarrow M_c \times X$ is the universal sheaf of M_c and $p_M : M_c \times X \rightarrow M_c$ is the projection. By Serre duality, we have a non-degenerate pairing

$$Q : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{O}_{M_c},$$

which defines an $O(1, \mathbb{C})$ -structure on \mathcal{L} . The quadratic line bundle (\mathcal{L}, Q) is called *orientable* if its structure group can be reduced to $SO(1, \mathbb{C}) = \{1\}$. An orientability result is recently proved on arbitrary CY 4-folds [7]; the proof uses involved tools like

semi-topological K -theory. To be self-contained, we include here a simpler proof of an orientability result for CY 4-folds with the technical assumptions $\text{Hol}(X) = SU(4)$ and $H^{\text{odd}}(X, \mathbb{Z}) = 0$.

Lemma 6.2. *Let X be a CY 4-fold with $\text{Hol}(X) = SU(4)$ and $H^{\text{odd}}(X, \mathbb{Z}) = 0$. Let M_c be a finite type open substack of the moduli stack of coherent sheaves with Chern character $c \in H^{\text{even}}(X)$. Then the quadratic line bundle (\mathcal{L}, Q) is orientable.*

Proof. By the work of Joyce–Song [21, Thm. 5.3], the moduli stack M_c is 1-isomorphic to a finite type moduli stack of holomorphic vector bundles on X via Seidel–Thomas twists, under which the universal family can be identified (so is the determinant line bundle and Serre duality pairing). Thus we may assume M_c to be a moduli stack of (rank n) holomorphic bundles without loss of generality.

Fix a base point $x_0 \in X$. A framing ϕ of a vector bundle E is an isomorphism

$$\phi : E|_{x_0} \cong \mathbb{C}^n.$$

There is a natural $GL(n, \mathbb{C})$ -action on ϕ changing the framing.

Let M_c^{framed} denote the moduli stack of framed holomorphic bundles with Chern character c , on which $GL(n, \mathbb{C})$ acts by changing framings. Note that M_c^{framed} is a scheme as the stabilizer is trivial and we have a 1-isomorphism

$$[M_c^{\text{framed}}/GL(n, \mathbb{C})] \cong M_c, \quad (E, \phi) \mapsto E,$$

of Artin stacks. The universal family

$$\mathcal{E} \rightarrow M_c^{\text{framed}} \times X$$

descends to the universal sheaf \mathbb{F} of M_c . Let

$$\mathcal{L} := \det(\mathbf{R}p_*\mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}))$$

be the determinant line bundle of M_c^{framed} , where $p : M_c^{\text{framed}} \times X \rightarrow M_c^{\text{framed}}$ is the projection. One may reduce the orientability problem of M_c to the orientability of (\mathcal{L}, Q) , where Q is the quadratic form on \mathcal{L} defined by Serre duality.

We view a holomorphic bundle as an integrable $\bar{\partial}$ connection on its underlying topological bundle. Then there is a natural embedding of M_c^{framed} (with induced complex analytic topology)

$$M_c^{\text{framed}} \hookrightarrow \widetilde{\mathcal{B}}_E$$

into the space $\widetilde{\mathcal{B}} := \mathcal{A} \times_{\mathcal{G}} E_{x_0}$ of framed (not necessarily integrable) connections on the underlying topological bundle E . The determinant line bundle \mathcal{L} on M_c^{framed} is the pull-back of a line bundle $\mathcal{L}_{\widetilde{\mathcal{B}}}$ on $\widetilde{\mathcal{B}}_E$ defined as the determinant of the index bundle of certain twisted Dirac operators, and the quadratic form Q on \mathcal{L} extends to $\mathcal{L}_{\widetilde{\mathcal{B}}}$ defined using the spin structure of X (see [11, pp. 50–51]). By [11, Thm. 1.3], the quadratic line bundle $(\mathcal{L}_{\widetilde{\mathcal{B}}}, Q)$ is orientable. Hence we are done. ■

Then orientability for moduli spaces of stable pairs follows from the orientability of moduli stacks of one-dimensional sheaves.

Theorem 6.3. *Let X be a CY 4-fold with $\text{Hol}(X) = SU(4)$ and $H^{\text{odd}}(X, \mathbb{Z}) = 0$. Then for any $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the quadratic line bundle (\mathcal{L}, Q) over $P_n(X, \beta)$ is orientable.*

Proof. There is a morphism

$$\phi : P_n(X, \beta) \rightarrow M(0, 0, 0, \beta, n), \quad \phi(F, s) = F,$$

to the moduli stack of one-dimensional sheaves on X with Chern character $(0, 0, 0, \beta, n)$.

Parallel to the quadratic line bundle $(\mathcal{L} = \det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathbb{I}, \mathbb{I})_0), Q)$ over $P_n(X, \beta)$, there exists a determinant line bundle

$$\mathcal{L}_M = \det(\mathbf{R}(\pi_M)_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathbb{F}))$$

with a quadratic form Q_M over $M(0, 0, 0, \beta, n)$, where $\pi_M : M(0, 0, 0, \beta, n) \times X \rightarrow M(0, 0, 0, \beta, n)$ is the projection, and we use \mathbb{F} to denote the universal sheaf for both $P_n(X, \beta)$ and $M(0, 0, 0, \beta, n)$.

Via the morphism ϕ , we have an isomorphism

$$\phi^* \mathcal{L}_M \cong \det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathbb{F})), \tag{59}$$

where $\pi_P : X \times P_n(X, \beta) \rightarrow P_n(X, \beta)$ is the projection.

Since $\mathbb{I} = (\mathcal{O}_{X \times P_n(X, \beta)} \rightarrow \mathbb{F})$, we have a distinguished triangle

$$\mathbf{R}\mathcal{H}om(\mathbb{F}, \mathbb{F}) \rightarrow \mathbf{R}\mathcal{H}om(\mathcal{O}_{X \times P_n(X, \beta)}, \mathbb{F}) \rightarrow \mathbf{R}\mathcal{H}om(\mathbb{I}, \mathbb{F}),$$

which gives an isomorphism

$$\begin{aligned} \det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathbb{F})) \otimes \det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathbb{I}, \mathbb{F})) \\ \cong \det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathcal{O}_{X \times P_n(X, \beta)}, \mathbb{F})) \end{aligned} \tag{60}$$

between determinant line bundles.

Similarly, from the distinguished triangle

$$\mathbf{R}\mathcal{H}om(\mathbb{I}, \mathbb{F}) \rightarrow \mathbf{R}\mathcal{H}om(\mathbb{I}, \mathbb{I})_0[1] \rightarrow \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathcal{O}_{X \times P_n(X, \beta)})[2],$$

we have an isomorphism

$$\begin{aligned} \det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathbb{I}, \mathbb{F})) \otimes \det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathcal{O}_{X \times P_n(X, \beta)})) \\ \cong (\det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathbb{I}, \mathbb{I})_0))^{-1}. \end{aligned} \tag{61}$$

Combining (60), (61) and Serre duality, we obtain

$$\det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathbb{I}, \mathbb{I})_0) \cong \det(\mathbf{R}(\pi_P)_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathbb{F})),$$

under which the natural quadratic forms on them are identified.

By Lemma 6.2, the structure group of the quadratic line bundle (\mathcal{L}_M, Q_M) can be reduced to $SO(1, \mathbb{C})$, and so can (\mathcal{L}, Q) via the pull-back (59). ■

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