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Ruelle–Pollicott resonances for manifolds with hyperbolic cusps

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Abstract. We present a method to construct a Ruelle–Pollicott spectrum for the geodesic flow on manifolds with strictly negative curvature and a finite number of hyperbolic cusps.

Keywords. Ruelle–Pollicott resonances, hyperbolic cusps, b-calculus

The spectrum of Ruelle–Pollicott resonances is a notion that was developed in the 1980s [51, 53, 54] to associate Axiom A flows [58] with a discrete set of complex numbers that describe its mixing properties. Let us recall their definition: if φ_t is a flow on some manifold M, $d\mu$ an invariant measure and $A, B \in C_c^\infty(M)$ two observables, then we can define the correlation function

$$\rho_{A,B}(t) := \int_M (A \circ \varphi_t) \cdot B \, d\mu,$$

as well as its Laplace transform $\hat{\rho}_{A,B}(s)$ which is holomorphic for $\mathrm{Re}(s) > 0$. Pollicott [51] and Ruelle [54] proved that, for Axiom A flows, this Laplace transform $\hat{\rho}_{A,B}$ extends meromorphically to a small strip $\mathrm{Re}(s) > -\varepsilon$ for a certain class of measures. Its poles are called Ruelle–Pollicott resonances of φ_t with respect to μ. In the following decade, several works [22, 29, 34, 55] were dedicated to obtaining sharp bounds on the maximal strip on which the continuation is possible in terms of the regularity of the flow. More recently, it has been understood that these resonances can be seen as the discrete spectrum in the usual sense of the generator of the flow on some carefully chosen Banach spaces. They appear as the poles of the meromorphic continuation of the resolvent kernel. See [10, 15, 17, 20, 23, 39], and also [2–5, 19, 25] for the related case of hyperbolic diffeomorphisms. A wide generality of dynamical systems is considered in these articles; however, all these

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results have in common that they assume the system has a compact *trapped set*

$$K(\varphi_t) := \{x \in M \mid \liminf_{t\to\pm\infty} d(x, \varphi_t(x)) < +\infty\}.$$

Since this is where the "non-trivial" part of the dynamics happen, one can crucially use Fredholm theory. In this paper, we consider a family of hyperbolic flows whose trapped set is not compact. For this class, we explain how Ruelle–Pollicott resonances can be defined. We are convinced that the methods developed in this article will also apply to more general settings and that they will lead to subsequent results such as meromorphic continuation of zeta functions, and decay of correlations results. However, the new arguments that we introduce to handle the non-compact trapped set are already a bit more involved than the usual ones. We have thus chosen to restrain ourselves to the following setting, where they can be cleanly developed.

The class of dynamical systems we are considering are geodesic flows on manifolds with cusps. We assume that (N, g) is a complete, smooth, $(d+1)$-dimensional Riemannian manifold, that decomposes into a compact core with strictly negative variable sectional curvature, and a finite union of hyperbolic cusps with constant negative curvature, that are attached to this core (see Definition 1.1 for more precision). We consider the geodesic flow φ_t acting on the cosphere bundle $M = S^*N$ and denote its vector field by X. When endowed with the Sasaki metric, M is a $(2d+1)$-dimensional Riemannian manifold. From the inclusion $M \subset T^*N$, M inherits the Liouville measure μ_L which is preserved by the geodesic flow, and gives finite volume to M. As N has strictly negative curvature, the geodesic flow is uniformly hyperbolic, and due to the particular structure of the cusps, its trapped set (in forward *and* backward times) has full measure in M.

As X is an antisymmetric unbounded operator on $L^2(M) := L^2(M, \mu_L)$, we deduce that its resolvent

$$\mathscr{R}(s) := (X - s)^{-1} : L^2(M) \to L^2(M)$$

is a holomorphic family of bounded operators for $\mathrm{Re}(s) > 0$. We prove the following theorem.

Theorem 1. *The resolvent has a meromorphic continuation as a family of continuous operators* $\mathscr{R}(s): C_c^\infty(M) \to \mathcal{D}'(M)$ *from* $\mathrm{Re}(s) > 0$ *to the whole complex plane, and for any pole of this meromorphic continuation, the residue is a finite rank operator.*

Note that, for $A, B \in C_c^\infty(M)$ and $\rho_{A,B}$ the correlation function with respect to μ_L, it is easy to check that, for $\mathrm{Re}(s) > 0$,

$$\hat{\rho}_{A,B}(s) = \langle \mathscr{R}(s)A, B\rangle_{\mathcal{D}', C_c^\infty}.$$

Thus the meromorphic continuation of the resolvent gives the continuation of $\hat{\rho}_{A,B}$ to the whole complex plane and its poles coincide with the poles of $\mathscr{R}(s)$ – when A and B vary in $C_c^\infty(M)$. Consequently, we call the poles of $\mathscr{R}(s)$ *Ruelle–Pollicott resonances* of the geodesic flow (with respect to the Liouville measure).

To the best of our knowledge, such a global definition of Ruelle–Pollicott resonances of the geodesic flow on cusp manifolds was so far not known, even in the case of constant negative curvature manifolds. We therefore want to mention another consequence.

Corollary 1. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a co-finite Fuchsian group such that there is a torsion-free, normal subgroup $\tilde{\Gamma} \trianglelefteq \Gamma$ of finite index*[1]*. Let us consider the orbifold*

$$M_\Gamma := S^*(\Gamma \setminus \mathbb{H}) \cong \Gamma \setminus \mathrm{PSL}(2,\mathbb{R}).$$

Then the resolvent $\mathscr{R}(s) := (X-s)^{-1}\colon L^2(M_\Gamma) \to L^2(M_\Gamma)$ has a meromorphic continuation $\mathscr{R}(s)\colon C_c^\infty(M_\Gamma) \to \mathscr{D}'(M_\Gamma)$.

Proof. Theorem 1 applies to the smooth manifold $M_{\tilde{\Gamma}} = \tilde{\Gamma} \setminus \mathrm{PSL}(2,\mathbb{R})$. The corollary then follows by definition of smooth functions and distributions on orbifolds and because the resolvent commutes with isometries. ∎

We now want to mention some results related to Theorem 1. In order to study eigenvalues of the Laplacian on moduli spaces, Avila and Gouëzel [1] develop a functional analytic framework for the Teichmüller flows which are also a class of dynamical systems with non-compact finite volume trapped set. They obtain a meromorphic continuation of the resolvent to a neighbourhood of zero (cf. [1, Proposition 3.3]). It would probably be possible to adapt their method to geodesic flows on cusp manifolds in order to obtain a continuation to a small strip along the imaginary axis (instead of $\mathbb{C}$ in our case). However, their functional analytic tools are quite different from ours.

Another series of related results have been obtained for the special case of surfaces of constant negative curvature with cusps. It has been shown in a series of articles by Mayer, Morita and Pohl [41, 45, 49, 50] that one can associate the geodesic flow with one-dimensional expanding maps, using a carefully chosen discretization. Out of this discretization, one can build transfer operators with discrete spectrum, and these spectra have interesting relations to number theory and the theory of Maass cusp forms [7, 38, 43]. One should be able to recover these spectra as a subset[2] of the resonances defined from Theorem 1. It will be subject to further research to establish this connection precisely.

As Ruelle–Pollicott resonances are an important tool to study decay of correlations, let us shortly mention that the question of mixing is not yet satisfactorily answered for our class of cusp manifolds: for constant curvature manifolds with cusps, exponential decay of correlations for the Liouville measure was proved in [44], while for variable curvature, only its mixing property is known [12]. Two other recent results on the mixing of Weil–Petersson geodesic flows on manifolds with cusp-like singularities[3] have been obtained

[1]Note that a particular example of this situation is $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ and $\tilde{\Gamma} = \Gamma(2)$ the principal congruence subgroup (see e.g. [32, Section 2.3]).

[2]More precisely, the connection should be to the so-called first band of Ruelle–Pollicott resonances (cf. [14, 27, 28]).

[3]Note that their notion of cusp singularities differs from ours: they consider singularities where the distance to the cusp is bounded, but the curvature is divergent.

in [8, 9]. We hope that the analytic tools that we develop in this article will prove to be helpful in the future for studying mixing properties of geodesic flows on manifolds with hyperbolic cusps.

The meromorphic continuation of dynamical zeta functions is another important field where meromorphically continued resolvents of flow vector fields have successfully been applied. If $\mathscr{P}$ denotes the set of primitive periodic orbits of a hyperbolic flow and $\ell(\gamma)$ their lengths, then the Ruelle zeta function is defined for $\mathrm{Re}(s) \gg 0$ by

$$\zeta_R(s) := \prod_{\gamma\in\mathscr{P}} (1 - e^{-s\ell(\gamma)}).$$

Smale [58] raised the question if, for Axiom A flows, the Ruelle zeta function[4] has a meromorphic continuation to $\mathbb{C}$? This question has recently been affirmatively answered by Dyatlov and Guillarmou [16] following a long series of precedent works that prove meromorphic continuation under additional assumptions [15, 17, 22, 23, 47, 52] (we refer to [2, 60] for a recent overview of the literature). In all the recent accounts [15–17, 21, 23] of these meromorphically continued zeta functions, a meromorphically continued resolvent was the central ingredient. Consequently, Theorem 1 indicates[5] that the Smale conjecture could hold for geodesic flows on cusp manifolds, i.e. beyond the class of Axiom A flows. So far, such a result is only known in the particular case of constant negative curvature where it is a rather direct consequence of Selberg's trace formula.

Contrary to the "classical" Ruelle–Pollicott resonances, the definition of "quantum" resonances of the Laplace–Beltrami operator Δ_N on a cusp manifold N has been established for a long time starting with works of Maaß [40] and Selberg [56]. See the introduction of [37] for the constant curvature case, and [11, 46] for the variable curvature case. In fact, the proof of our main result borrows ideas from the definition of quantum resonances (such as the compact Sobolev embedding, Lemma 4.12).

Let us shortly sketch the further ingredients for proving the meromorphic continuation of the resolvent to the whole complex plane (Theorem 1). As a first step, we construct a family of *anisotropic spaces* $H^{\gamma\mathbf{m}}$ that are adapted to the hyperbolic structure of the flow. These are Hilbert spaces of distributions on M, and $C_c^\infty(M)$ is dense in each $H^{\gamma\mathbf{m}}$. They are an adaptation of the spaces defined by Faure–Sjöstrand [20] and Dyatlov–Zworski [17]. Using a mix of their techniques, we obtain much in the same way a first parametrix, which inverts $X - s$ up to a smoothing remainder. However, this parametrix is – contrary to the compact case – not sufficient for a meromorphic resolvent. Therefore, it was necessary to introduce another technique. We chose to use ideas from Melrose's b-calculus to deal with the explicit form of the generator X in the cusp. From the very

[4] Actually, Smale considered a different version of a dynamical zeta function which is rather an analogue of Selberg's zeta function. The question of meromorphic continuation is however trivially related to ζ_R.

[5] In fact, the full statement of our result (Theorem 3) already provides several further ingredients necessary for the meromorphic continuation of ζ_R such as the extension to differential forms and wavefront estimates.

nature of these techniques, they work independently of the dimension of N. It is not entirely clear whether the technique could be applied or not to the case that the curvature is not exactly equal to -1 in the cusps, only *close* to -1. However, by analogy to the resonances of the Laplace operators, we conjecture that the meromorphic continuation to the full complex plane will not hold true when assuming only pinched negative curvature in the cusps.

Let us present the structure of the paper. In Section 1, we introduce the precise settings in which we are working and collect several properties of the geodesic flow on cusp manifolds, that will be crucial in the sequel. To prove our theorem, we then build a first parametrix in Section 2.2 following the arguments of [17,20]. The geometric construction of the escape function is presented; however, the technical microlocal lemmas are proved in Appendix A. Section 3 is devoted to introducing techniques adapted from b-calculus and proving the meromorphic continuation of the resolvent of a certain class of translation invariant operators. These operators show up precisely when restricting the geodesic flow to the zeroth Fourier mode in the cusp. In Section 4, such a resolvent is used for the construction of a parametrix (up to compact remainder) of the geodesic flow vector field. Then, using analytic Fredholm theory, we conclude on the meromorphic continuation announced in Theorem 1. In Sections 3 and 4, we work in a more general setting under a list of assumptions. This should allow for an easy generalization to more general settings (such as fibred or complex hyperbolic cusps) in the future. In Section 5, we finally compute explicitly the indicial roots for the b-operators associated to the geodesic flow on our class of cusp manifolds and we check that all the necessary assumptions in Section 3 and 4 are fulfilled.

Note that, in fact, we prove more general and more precise versions of Theorem 1. For example, we continue the resolvent for a certain class of derivations on vector bundles (cf. Definition 1.4) including the geodesic vector field with smooth potential, Lie derivatives on perpendicular k-forms and general associated vector bundles over constant curvature manifolds (cf. Examples 1.5–1.7). Furthermore, we give a precise description of the wavefront set of the resolvent. For a full statement, we refer the reader to Theorems 3 and 4.

1. Geometric preliminaries

1.1. The geodesic flow on cusp manifolds

Let us give a precise definition of the manifolds on which we are working.

Definition 1.1. A manifold N will be called an *admissible cusp manifold* if the following assumptions hold. First, (N, g) is a $(d+1)$-dimensional Riemannian manifold, connected and complete when endowed with the corresponding Riemannian distance. Second, it decomposes as the union $N_0 \cup Z_1 \cup \cdots \cup Z_\kappa$, where N_0 is a compact manifold whose boundary ∂N_0 is a finite disjoint union of d-dimensional tori. At each component $\ell = 1, \ldots, \kappa$ of ∂N_0 is glued the *hyperbolic cusp* Z_ℓ, which takes the form

$$Z_\ell = [a, +\infty[_y \times (\mathbb{R}^d/\Lambda_\ell)_\theta.$$

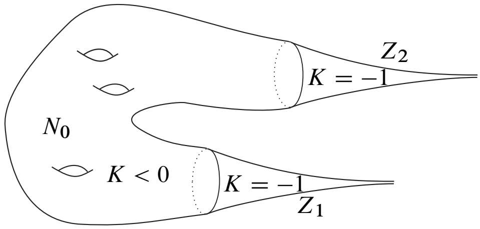

Fig. 1. Schematic sketch of a cusp manifold

(Here, Λ_ℓ is a lattice in $\mathbb{R}^d$, and we can impose the normalization condition that it is unimodular.) We require that the metric g has strictly negative curvature in the whole of N, and additionally, we fix, for each $\ell = 1, \ldots, \kappa$,

$$g_{|Z_\ell} = \frac{dy^2 + d\theta^2}{y^2}. \tag{1.1}$$

Then the sectional curvature is -1 in each cusp, and the volume of N is finite. Since the sectional curvature of N is pinched, we deduce that its geodesic flow φ_t is a *uniformly hyperbolic flow*[6] on its cosphere bundle. More precisely, we have the following proposition.

Proposition 1.2. *Let $M = S^*N$ be the cosphere bundle of an admissible cusp manifold. There is a splitting*

$$TM = E_0 \oplus E_s \oplus E_u$$

into $d\varphi_t$-invariant subbundles, which is Hölder continuous with uniform constants. Furthermore, the angle between any pair of the invariant bundles is bounded from below by a uniform constant. Finally, there are global constants $c, C, \beta, B > 0$ such that

$$\begin{aligned} ce^{-Bt}\|v\| \le \|(d\varphi_t)v\| \le Ce^{-\beta t}\|v\| \quad &\textit{for all } v \in E_s,\ t > 0,\\ ce^{-Bt}\|v\| \le \|(d\varphi_{-t})v\| \le Ce^{-\beta t}\|v\| \quad &\textit{for all } v \in E_u,\ t > 0. \end{aligned}$$

Proof. Let $\tilde{N}$ be the universal cover of N. It is a simply connected, complete Riemannian manifold with pinched negative sectional curvature ($-k_{\max}^2 < K < -k_{\min}^2 < 0$) because the non-compact ends Z_i are endowed with a constant negative curvature metric. For the same reason, all derivatives of the sectional curvature are bounded. Thus [48, Theorem 7.3 and Lemma 7.4] apply to this situation and they provide the splitting into invariant bundles over $S^*\tilde{N}$ with the above properties. As the invariant bundles are invariant under isometries, taking the quotient, we obtain the desired result. ∎

[6] Often, uniformly hyperbolic flows are also called *Anosov flows*. Several authors use the term "Anosov flow" however only in the more narrow setting of hyperbolic flows on *compact* manifolds. For this reason we refrain from using the term Anosov in our setting to avoid confusion.

For the proof of Theorem 1, it will be crucial to have a precise understanding of the geometry and the dynamics on the non-compact ends of S^*N. We therefore start by introducing explicit coordinates on S^*Z_ℓ. In order to simplify the notation, we will drop the indices $\ell = 1, \dots, \kappa$ that number the cusps.

Recall that a cusp is $Z = [a, \infty[\times \mathbb{R}^d/\Lambda$, and since we have assumed that Λ is *unimodular*, we have *canonical coordinates* $y \in [a, \infty[$, $\theta \in \mathbb{R}^d/\Lambda$. In many cases, it will be convenient to perform the change of variables $r = \log y \in [\log a, \infty[$, and the metric becomes

$$g = dr^2 + e^{-2r}\, d\theta^2.$$

A single cusp has the *local isometry pseudo-group* given by $\mathbb{R} \times \mathbb{R}^d$ which is realized by linear scaling and translations in the y, θ variables,

$$T_{\tau,\theta_0}(y, \theta) := (e^\tau y, e^\tau \theta + \theta_0), \tag{1.2}$$

or in r, θ-variables,

$$T_{\tau,\theta_0}(r, \theta) := (r + \tau, e^\tau \theta + \theta_0). \tag{1.3}$$

Using the y, θ variables, we can write $\xi \in T^*_{y,\theta} Z$ as $\xi = Y\, dy + J\, d\theta$ for $Y \in \mathbb{R}$ and $J \in \mathbb{R}^d$, and the Riemannian norm of such a cotangent vector is given by

$$|\xi|_g = y\sqrt{Y^2 + |J|^2_{\mathbb{R}^d}}.$$

Elements $\xi \in S^*_{y,\theta} Z$ of a cosphere fibre are thus in bijection with

$$\zeta := y(Y, J) \in \mathbb{S}^d \subset \mathbb{R}^{d+1}.$$

In particular, the cosphere bundle over the cusp is trivializable $S^*Z \cong Z_{(y,\theta)} \times \mathbb{S}^d_\zeta$. The usual metric on S^*Z, the Sasaki metric (see e.g. [26] for an easily accessible introduction), is not a product metric. However, one can check (see the expression of the Sasaki metric in [6, Section C.2]) that it is equivalent to the *product metric* $g_Z \otimes g_{\mathbb{S}^d}$, where $g_{\mathbb{S}^d}$ is the usual metric on the sphere. We will use the product metric in the sequel.

For the study of the geodesic flow, some more precise variables on the spheres are useful. We choose a orthonormal base of coordinates $\theta_1, \dots, \theta_d$ in $\mathbb{R}^d$. We fix

$$(yY = 1, J = 0) \simeq y^{-1}\, dy$$

to be *zenith*, and we fix $y^{-1}\, d\theta_1$ to be the *azimuthal reference*. With these conventions, a point $\zeta \in S^*_{y,\theta} Z$ is non-ambiguously determined by its *inclination* φ – the angle it makes with the zenith – and its azimuthal position, $u \in \mathbb{S}^{d-1}$, which is determined by the choice of base in $\mathbb{R}^d$. As a point in $\mathbb{R}^{d+1}$, $\zeta = (\cos\varphi, \sin\varphi u) \simeq y^{-1} \cos\varphi\, dy + y^{-1} \sin\varphi u \cdot d\theta$.

We single out two important points, the North Pole $\mathcal{N} \in \mathbb{S}^d$ with $\varphi = 0$ that corresponds to the cotangent element $y^{-1}\, dy = dr \in S^*Z$ pointing into the direction of the cusp and the South Pole $\mathcal{S}$ corresponding to $-y^{-1}\, dy = -dr$ pointing perpendicularly to the bottom of the cusp.

The geodesic flow is known to be the Hamiltonian flow with Hamiltonian

$$\hbar(x, \xi) = \tfrac{1}{2} g_x(\xi, \xi) = \tfrac{1}{2} y^2 (Y^2 + |J|^2_{\mathbb{R}^d}),$$

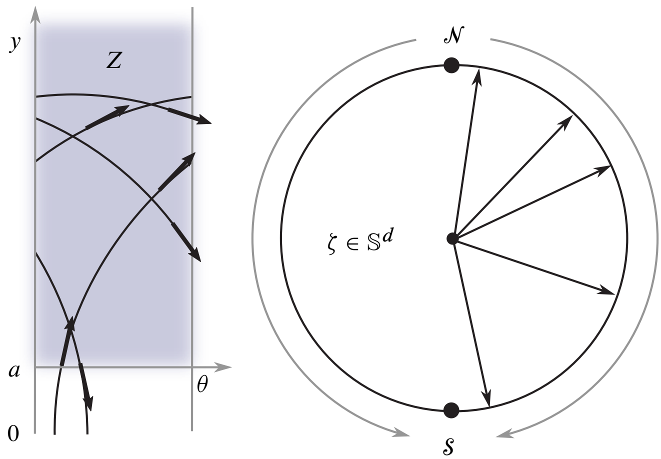

Fig. 2. This figure illustrates the dynamics of the geodesic flow on a cusp. The left part shows the fundamental domain of a cusp (for $d = 1$) in the y, θ variable. The black solid line is the trace of a geodesic projected from S^*Z to Z. The arrows indicate the direction of the flow and correspond to the cotangent vectors. On the right, each of these arrows is represented by its ζ coordinate in $\mathbb{S}^d$, evidencing the transient dynamics from $\mathcal{N}$ to $\mathcal{S}$.

and a straightforward calculation with the canonical symplectic structure on T^*Z gives the associated Hamiltonian vector field

$$y^2 Y \partial_y + y^2 J \cdot \partial_\theta - y(Y^2 + J^2)\partial_Y.$$

Restricting this vector field to S^*Z and using the spherical coordinates φ, u, we obtain an explicit expression for the geodesic vector field,

$$\begin{aligned} X &= y\cos(\varphi)\partial_y + y\sin(\varphi)u \cdot \partial_\theta + \sin(\varphi)\partial_\varphi \\ &= \cos(\varphi)\partial_r + e^r \sin(\varphi)u \cdot \partial_\theta + \sin(\varphi)\partial_\varphi. \end{aligned}$$

Note that $u \cdot \partial_\theta$ is understood after identifying $u \in \mathbb{S}^{d-1} \subset \mathbb{R}^d \cong T_\theta(\mathbb{R}^d/\Lambda)$.

The dynamics of the geodesic flow vector field is illustrated in Figure 2. Let us emphasize two important properties of the geodesic flow dynamics on S^*Z.

(A) The Hamiltonian h is independent of the θ variable, which implies that the corresponding momentum variable u is a constant of motion under the geodesic flow.

(B) The dynamics of the variable $\zeta \in \mathbb{S}^d \cong S^*_{y,\theta}Z$ is decoupled from the dynamics on Z. By property (A), this dynamics is even rotationally invariant around the axis through $\mathcal{N}$ and $\mathcal{S}$, and it is precisely the gradient flow on the sphere $\mathbb{S}^d$ with the obvious height function.

This has the following consequence for the dynamics of the geodesic flow on the cusp. Assume that trajectories stop when they reach the lower boundary $y = a$. Then the only *wandering* trajectories are those with $\zeta = \mathcal{N}$ or $\zeta = \mathcal{S}$. They correspond to the geodesics that leave or enter the cusp, parallel to the y-axis. All other trajectories only rise up to a finite height into the cusp and are thus "trapped". However, by choosing ζ arbitrary close to $\mathcal{N}$, this height can be made arbitrary large and the trapped set is non-compact. Since the non-compactness of the trapped set is the central problem in extending the techniques of [17, 20], these regions around $\mathcal{N}$ and $\mathcal{S}$ will become crucial in the analysis.

Finally, let us add a third remark that is not directly related to the dynamics of the geodesic flow, but rather to its action as a differential operator.

(C) As the geodesic vector field commutes with local isometries, it commutes in particular with the $\mathbb{R}^d$-action by translation and thus preserves the Fourier modes in the θ variable. If k is an element in the dual lattice $\Lambda^* \subset \mathbb{R}^d$, then restricting the geodesic flow vector field to the Fourier modes $e^{ik\theta}$ yields a differential operator

$$X_k = \cos(\varphi)\partial_r + \sin(\varphi)\partial_\varphi + i e^r \sin(\varphi) u \cdot k. \tag{1.4}$$

When $k = 0$, this is a vector field with coefficients that do not depend on r.

Remark 1.3. The structure of the flow restricted to the zeroth Fourier mode is essential to our proof. Indeed, since it is translation invariant, we can use techniques adapted from Melrose's b-calculus to find an exact inverse for the model flow on a "full" cusp (cf. Section 3). The fact that, in the other Fourier modes, the flow does not have such a nice structure is compensated by the fact that we have a compact injection for functions in H^1 whose zeroth Fourier mode vanishes in each cusp (cf. Lemma 4.12).

1.2. Admissible vector bundles

As mentioned in the introduction, we want to prove the meromorphy of the resolvent not only for the geodesic vector field acting on functions but also for a larger class of admissible vector bundles. In order to precisely define these admissible vector bundles, let us first recall how to write the non-compact ends S^*Z_ℓ as locally homogeneous spaces.

Given a cusp $Z_\ell = [a_\ell, \infty[\times \mathbb{R}^d/\Lambda_\ell$, we will consider the associated *full cusp* to be the space $Z_{\ell,f} = (\mathbb{R}_+)_y \times (\mathbb{R}^d/\Lambda_\ell)_\theta$ with the metric g defined in equation (1.1) extended to $Z_{\ell,f}$ in the obvious way. Let $\mathbb{G} = \mathrm{SO}(d+1, 1)$; then using the Iwasawa decomposition, we can write $\mathbb{G} = \mathbb{N}\,\mathbb{A}\,\mathbb{K}$, where $\mathbb{A} = (\mathbb{R}_+, \cdot)$, $\mathbb{N} = (\mathbb{R}^d, +)$ are abelian groups and $\mathbb{K} = \mathrm{SO}(d+1)$ is the maximal compact subgroup in $\mathbb{G}$. Then a full cusp is simply the double quotient $Z_{\ell,f} = \Lambda_\ell \setminus \mathbb{G} / \mathbb{K}$, where we consider $\Lambda_\ell \subset \mathbb{N} \cong \mathbb{R}^d$. The unit cosphere bundle can then be simply written as $S^*Z_{\ell,f} = \Lambda_\ell \setminus \mathbb{G} / \mathbb{M}$, where $\mathbb{M} = \mathrm{SO}(d)$ (see e.g. [27, 30] for more details). Recall furthermore that the Bruhat decomposition on the Lie algebra

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$$

is $\mathrm{Ad}\,\mathbb{M}$ invariant. Accordingly, $\mathbb{G}/\mathbb{M}$ is a reductive homogeneous space, and for any orthogonal representation (τ, V) of $\mathbb{M}$, the associated vector bundle $\mathbb{G} \times_\tau V$ is a homogeneous Riemannian vector bundle with a canonical compatible connection.

Now we can define admissible vector bundles.

Definition 1.4. Let $N = N_0 \cup (\bigcup_{\ell=1}^{\kappa} Z_\ell)$ be an admissible cusp manifold in the sense of Definition 1.1 and $M = S^*N$. Let $L \to M$ be a Riemannian bundle endowed with a compatible connection ∇. Moreover, L is an *admissible vector bundle* if, for each cusp Z_ℓ, $\ell = 1, \dots, \kappa$, there is an orthogonal $\mathbb{M}$-representation (τ_ℓ, V_ℓ) such that $L|_{Z_\ell}$ coincides with the associated vector bundle

$$L_{\ell,\tau_\ell} = \Lambda_\ell \setminus \mathbb{G} \times_{\tau_\ell} V_\ell.$$

(The Riemannian bundle metric and connection of L are also assumed to coincide with those of the associated bundle.)

Let $\mathcal{X}$ be a derivation on sections of L that lifts the geodesic flow vector field X. That is to say that it satisfies the Leibniz relation

$$\mathcal{X}(fs) = (Xf)s + f\mathcal{X}s \quad \text{for } f \in C^\infty(M),\ s \in C^\infty(M, L).$$

We say that $\mathcal{X}$ is an *admissible lift* of X if there is a fixed $A_\ell \in \mathrm{End}(V_\ell)^{\mathbb{M}}$ for each cusp Z_ℓ such that, when restricted to $L|_{Z_\ell}$, $\mathcal{X}$ acts as $\mathcal{X}_\ell := \nabla_X + A_\ell$.

Let us mention three important examples of admissible vector bundles and differential operators.

Example 1.5. Let $V \in C^\infty(M)$ be so that, in each cusp, V is just a constant. Then $X + V$ is an admissible operator on the trivial bundle.

Example 1.6. Let $\Gamma \subset \mathbb{G}$ be a non-uniform torsion-free lattice. Then $\Gamma \setminus \mathbb{G} / \mathbb{K}$ is a non-compact manifold of constant curvature whose ends are cusps in the sense we have defined; it is thus an admissible cusp manifold. There is a finite number of ends. Given an orthogonal representation τ of $\mathbb{M}$ on V, we can then construct globally the bundle $L_\tau = \Gamma \setminus \mathbb{G} \times_\tau V$ and the corresponding connection. Then the operator ∇_X is an admissible lift of the geodesic flow on $M = \Gamma \setminus \mathbb{G} / \mathbb{M}$.

Example 1.7. Let us take N an admissible cusp manifold, $M = S^*N$ and X the corresponding geodesic vector field. We can consider the Lie derivative $\mathcal{L}_X$ acting on $\Lambda(T^*M)$, the bundle of forms of arbitrary degree over M; it is an admissible lift of X. Further, define

$$\Lambda^\perp(T^*M) := \{\omega \in L(T^*M) \mid \iota_X \omega = 0\}.$$

This sub-bundle of $\Lambda(T^*M)$ is invariant under $\mathcal{L}_X$. Also, $\mathcal{L}_X$ preserves the Liouville one-form α, which is a *contact* one-form. In particular, we can identify the action of $\mathcal{L}_X$ on $\Lambda^\perp(T^*M)$ with the action of $\mathcal{L}_X$ on $\Lambda((\ker \alpha)^*)$. This is also an admissible lift of X.

2. Anisotropic space and first parametrix

The main idea that was presented in [20] was to resort to usual semi-classical techniques to prove the meromorphic continuation of the resolvent of the flow generator for Anosov flows on compact manifolds. This is not the only method available for compact manifolds – see [10] – but it is the one we will extend to our case. Another paper [17] used propagation of singularities to obtain the wavefront set of the resolvent, in order to simplify the proof of meromorphic continuation of the zeta functions. We will use a mixture of both, since we use the approach of [20] to continue the resolvent, and ideas from [17] to obtain the wavefront set of the resolvent.

We consider $L \to M = S^*N$ an admissible bundle and $\mathcal{X}$ an admissible lift of X the geodesic flow on M. Since we will use semi-classical techniques, we introduce a small parameter $0 < h \leq h_0$, and we let $\mathbf{X} := h\mathcal{X}$. We refer to Appendix A where we collect the definition of the notions of microlocal and semi-classical analysis (pseudo-differential operators, symbol classes, ...) which we will use in the sequel.

The first result in this section is the following proposition.

Proposition 2.1. *For each $\gamma > 0$ and $h > 0$, we can build a space of L-valued distributions $H^{\gamma\mathbf{m}}$ on M that contains $C_c^\infty(M, L)$, and a pseudo-differential operator Q microsupported in an arbitrarily small neighbourhood of the zero section in the fibres of T^*M, so that there exists $h_0 > 0$ so that, for $0 < h \leq h_0$ and for $|\operatorname{Im} s| < h^{-1/2}$ and $\operatorname{Re}(s) > -\gamma$,*

$$\mathbf{X} - Q - hs \text{ is invertible} \quad \text{and} \quad \|(\mathbf{X} - Q - hs)^{-1}\|_{H^{\gamma\mathbf{m}}} = \mathcal{O}(1/h).$$

As h varies, the spaces $H^{\gamma\mathbf{m}}$ remain the same as vector spaces, with equivalent norms.

The space $H^{\gamma\mathbf{m}}$ will take the form (see Definition 2.7)

$$\operatorname{Op}(e^{-\gamma G}) \cdot L^2(M, L).$$

In this formula, G denotes a so-called *escape function*, and Op a semi-classical quantization that we define in Appendix A (see equation (A.4)). The construction of G will be done first for X acting on functions. Then the general case is obtained by tensorizing $\operatorname{Op}(e^{-\gamma G})$ with the identity $\mathbb{1} \in \operatorname{End}(L)$.

Remark 2.2. As should be clear after reading the proof, the construction of the escape function is *local* in the sense that it can be done in the universal cover. In particular, Proposition 2.1 should hold in any geometrically finite negatively curved manifold whose universal cover has *bounded geometry*. We do not prove this general result because that would require the construction of an explicit quantization with uniform bounds on these non-compact spaces. This seemed too much a detour considering that our aim is to study cusp manifolds and that a suitable quantization in this setting has already been developed by the first author in [6].

2.1. *Building the escape function*

In this subsection, we want to construct an escape function in complete analogy to [20, Lemma 1.2]. As we deal with a non-compact situation, we however have to take care that the required uniform bounds hold.

The escape function G will be a function on the cotangent bundle T^*M, and we introduce the decomposition

$$T^*M = E_0^* \oplus E_u^* \oplus E_s^* \tag{2.1}$$

so that $E_0^* = \mathbb{R}\alpha$, where α is the Liouville one-form $-\xi \cdot dx$. Furthermore, we have $E_u^* = (E^u \oplus E^0)^\perp$ and $E_s^* = (E^s \oplus E^0)^\perp$.

We have to introduce some notation regarding the dynamics. We lift the geodesic flow φ_t symplectically to the flow

$$\Phi_t : (x, \xi) \mapsto \big(\varphi_t(x), (d_x\varphi_t^*)^{-1} \cdot \xi\big).$$

It is the Hamiltonian flow associated to the Hamiltonian

$$p(x, \xi) := \xi \cdot X(x), \tag{2.2}$$

which is the symbol of $-iX$, and we denote by X_Φ its Hamiltonian vector field. Decomposition (2.1) is preserved by the flow, and

$$(x, \xi) \in E_s^* \implies |\Phi_t(x, \xi)| \leq Ce^{-\beta t}|(x, \xi)| \quad \text{for } t > 0. \tag{2.3}$$

(Likewise in negative time for E_u^*.)

Lemma 2.3. *For any sufficiently small uniform conical neighbourhoods N_0, N_u, N_s of E_0^*, E_u^*, E_s^*, there are constants $C_G, R > 0$ such that, for any $\delta > 0$, there is an escape function $G \in C^\infty(T^*M)$ with*

(i) $X_\Phi G > 1$ *outside of* $\{|\xi| < R\delta\} \cup N_0$,

(ii) $X_\Phi G \geq 0$ *globally on* $\{|\xi| > \delta\}$,

(iii) *for* $|\xi| > R\delta$

$$G(x, \xi) = \begin{cases} +C_G \log|\xi| + \mathcal{O}(1), & \xi \in N_u, \\ -C_G \log|\xi| + \mathcal{O}(1), & \xi \in N_s, \\ \ 0, & \xi \in N_0, \end{cases}$$

(iv) $e^G \in S^{\mathbf{m}}_{\log}(M)$*: it is an anisotropic symbol of order* $\mathbf{m}(x, \xi)$*, with* $\mathbf{m} \in S^0_{\mathrm{cl}}(M)$ *being a* 0*-homogeneous classical symbol (see Definition A.6 for a definition of classical symbol classes) with*

$$\mathbf{m}(x, \xi/|\xi|) = \begin{cases} +C_G, & \xi \in N_u, \\ -C_G, & \xi \in N_s, \\ \ 0, & \xi \in N_0. \end{cases}$$

In order to prove Lemma 2.3, it will be helpful to restrict Φ_t to the unit sphere bundle S^*M. In order to do this, let us interpret $S^*M \cong (T^*M \setminus \{0\})/\mathbb{R}$, where $\mathbb{R}$ acts on each fibre by linear multiplication. Then, by homogeneity, Φ_t factors to a flow $\tilde{\Phi}_t : S^*M \to S^*M$ with vector field $X_{\tilde{\Phi}}$. By an abuse of notation, we can see E_0^*, E_u^* and E_s^* as subsets of S^*M. From the uniform estimates in Proposition 1.2, we obtain the following lemma.

Lemma 2.4. *For $\epsilon > 0$, let $U_u^\epsilon \subset S^*M$ be the ϵ-neighbourhood of E_u^* and likewise let $U_{0,s}^\epsilon \subset S^*M$ be the ϵ-neighbourhood of $E_0^* \oplus E_s^*$. Then there exists $\epsilon > 0$ such that U_u^ϵ and $U_{0,s}^\epsilon$ are disjoint. Furthermore, for any fixed ϵ as above, there is a finite maximal transition time $\tau_{\max} > 0$ such that, for $t \geq \tau_{\max}$,*

(1) *for all $(x,\xi) \in S^*M \setminus U_u^\epsilon$, $\tilde{\Phi}_{-t}(x,\xi) \in U_{0,s}^\epsilon$,*

(2) *for all $(x,\xi) \in S^*M \setminus U_{0,s}^\epsilon$, $\tilde{\Phi}_t(x,\xi) \in U_u^\epsilon$.*

Finally, for any $T > 0$, there is $\epsilon' > 0$ such that $U_u^{\epsilon'} \subset \tilde{\Phi}_T(U_u^\epsilon)$ and $U_{0,s}^{\epsilon'} \subset \tilde{\Phi}_{-T}(U_{0,s}^\epsilon)$. The same statement holds for $E_0^* \oplus E_u^*$ and E_s^*.

Proof of Lemma 2.3. Let us first construct the weight function $\mathbf{m}$. This decomposes into two symmetrical steps.

Take an $\epsilon > 0$ from Lemma 2.4 such that $U_u^{3\epsilon} \cap U_{0,s}^{3\epsilon} = \emptyset$. In a first step, we want to smooth the characteristic functions $\mathbb{1}_u^{2\epsilon}, \mathbb{1}_{0,s}^{2\epsilon} \in L^1_{\mathrm{loc}}(S^*M)$ on these two sets. Therefore, take $\tilde{\chi}_m \in C_c^\infty(]-\epsilon,\epsilon[)$, with $\tilde{\chi}_m \geq 0$ and $\tilde{\chi}_m(0) > 0$. Then we define a smoothing kernel $\chi_m \in C^\infty(S^*M \times S^*M)$ by

$$\chi_m(x,x') := \frac{\tilde{\chi}_m(d(x,x'))}{\int_{S^*M} \tilde{\chi}_m(d(x,x''))\,dx''},$$

and we denote by K_m the corresponding smoothing operator. Now we define the function $m_{0,s}^u := K_m(\mathbb{1}_u^{2\epsilon} - \mathbb{1}_{0,s}^{2\epsilon})$ which, by the construction of the smoothing operator, fulfils the following assumptions:

(1) $m_{0,s}^u \in \mathscr{C}^\infty(S^*M)$ – this means that all derivatives are bounded, see the discussion at the start of Appendix A;

(2) $m_{0,s}^u$ equals $+1$ on U_u^ϵ and -1 on $U_{0,s}^\epsilon$;

(3) $m_{0,s}^u$ takes values in $[-1,1]$.

Now take the time $T = 2\tau_{\max}$ (with $\tau_{\max}$ being the transition time from Lemma 2.4), and set

$$m_T^+ = \int_{-T}^{T} m_{0,s}^u \circ \tilde{\Phi}_t\,dt.$$

We have

$$X_{\tilde{\Phi}} m_T^+ = m_{0,s}^u \circ \tilde{\Phi}_T - m_{0,s}^u \circ \tilde{\Phi}_{-T}.$$

By Lemma 2.4, for any $(x,\xi) \in S^*M$, either $\tilde{\Phi}_T \in U_u^\epsilon$ or $\tilde{\Phi}_{-T} \in U_{0,s}^\epsilon$. Since $m_{0,s}^u$ takes values in $[-1,1]$, we deduce that everywhere

$$X_{\tilde{\Phi}} m_T^+ \geq 0.$$

Now let us define $V_u := \Phi_T(S^*M \setminus U_{0,s}^\epsilon) \subset U_u^\epsilon$ and $V_{0,s} := \Phi_{-T}(S^*M \setminus U_u^\epsilon) \subset U_{0,s}^\epsilon$. Then, for $(x,\xi) \notin (V_u \cup V_{0,s})$, $\tilde{\Phi}_T(x,\xi) \in U_u^\epsilon$ and $\tilde{\Phi}_{-T}(x,\xi) \in U_{0,s}^\epsilon$, and consequently,

$$X_{\tilde{\Phi}} m_T^+(x,\xi) = 2.$$

On the other side, if $(x,\xi) \in V_u$, then $m_{0,s}^u(\tilde{\Phi}_t(x,\xi)) = 1$ for $t > -\tau_{\max}$, and from the definition of T, we deduce that $m_T^+(x,\xi) \geq T$. For the same reason, $m_T^+ < -T$ on $V_{0,s}$.

Finally, with $\epsilon' > 0$, from Lemma 2.4, we deduce that m_T^+ is constantly equal to $2T$ (resp. $-2T$) on $U_u^{\epsilon'}$ (resp. $U_{0,s}^{\epsilon'}$).

The second step is to build a similar function m_T^- replacing E_u^* by E_s^*, and going through the same procedure. Taking

$$m = \frac{m_T^+ + m_T^-}{2},$$

we get

(a) $m \in \mathscr{C}^\infty(S^*M)$,

(b) $X_{\tilde{\Phi}} m \geq 0$ in S^*M,

(c) $X_{\tilde{\Phi}} m \geq 1$ on $S^*M \setminus (V_u \cup V_s \cup U_0^\epsilon)$,

(d) on $U_u^{\epsilon'}$ (resp. $U_s^{\epsilon'}$, $U_0^{\epsilon'}$), m equals $2T$ (resp. $-2T$, 0),

(e) $m > T$ on V_u and $m < -T$ on V_s.

The actual weight function $\mathbf{m}$ will be m multiplied by a constant that we will determine at the end.

Now comes the second part of the proof: building the symbol G. Choose N_u (resp. N_s, N_0) to be the cone in T^*M generated by $V_u \subset S^*M$ (resp. V_s, U_0^ϵ). We want to choose a symbol $f \in S^1(M)$ to be a positive *elliptic* symbol so that, outside of $|\xi| < \delta$, on N_0, it equals $|p|$. We also want that, on N_u (resp. N_s), it satisfies $X_\Phi \log f \geq \beta/2$ (resp. $\leq -\beta/2$). We would like to set f to be just the norm $|\xi|$ in a neighbourhood of $E_u^* \oplus E_s^*$, but this is not suitable because the constant C in estimate (2.3) is not necessarily 1. However, we find that, for $(x, \xi) \in E_s^*$,

$$X_\Phi \left(\frac{1}{2t} \int_{-t}^{t} \log |\Phi_s(x, \xi)| \, ds \right) \leq \frac{\log C}{2t} - \beta.$$

This suggests to pick $T' > 2\log(C)/\beta$ and define, for (x, ξ) in a fixed conical neighbourhood of $E_u^* \oplus E_s^*$,

$$f_{us}(x, \xi) := \exp\left(\frac{1}{2T'} \int_{-T'}^{T'} \log |\Phi_t(x, \xi)| \, dt \right).$$

This is not a norm anymore, but is still 1-homogeneous and smooth – except at 0. On E_s^*, $X_\Phi \log f_{us} \leq -3\beta/4$ so that, if $\epsilon > 0$ was chosen small enough, $X_\Phi \log f_{us} \leq -\beta/2$ in N_s. We also have the corresponding estimates in N_u. We can piece together f_{us} and $|p|$ around N_0 to obtain a globally defined elliptic 1-homogeneous symbol. Let c_f be its infimum on $\{|\xi| = 1\}$.

We have all the pieces to define

$$G(x, \xi) = C'_G \Big[1 - \chi_G \Big(\frac{|\xi|}{\delta} \Big) \Big] m \Big(x, \frac{\xi}{|\xi|} \Big) \log \frac{2 f(x, \xi)}{c_f \delta},$$

where $C'_G > 0$ is a constant fixed later and χ_G is a $C_c^\infty(]-1, 1[)$ function that equals 1 in $[-\frac{1}{2}, \frac{1}{2}]$ and takes values between 0 and 1. It is there to ensure that G is smooth at $\xi = 0$.

We can check that $G \geq 0$. By the properties of m from above, we directly deduce that Lemma 2.3 (iii) holds.

Now we can compute

$$X_\Phi G = -C'_G (X_\Phi \chi_G) m \log \frac{2f}{c_f \delta} + C'_G (1-\chi_G)\Big[(X_{\tilde{\Phi}} m) \log \frac{2f}{c_f \delta} + m \frac{X_\Phi f}{f}\Big].$$

Let us discuss the different terms: $X_\Phi \chi_G$ vanishes outside $\{|\xi| < \delta\}$; thus the first line is irrelevant for properties (i) and (ii) of Lemma 2.3. Let us consider the second line case by case.

- $(x,\xi) \notin (N_0 \cup N_u \cup N_s)$: Note that $|m|$ and $|X_\Phi f/f|$ are globally bounded by a constant C_0. By property (c) above, $X_{\tilde{\Phi}} m > 1$. By the fact that f is elliptic, there is a constant $R > 0$ such that, when $\{|\xi| > R\}$, $\log(2f/c_f) > 1 + C_0^2$. Then, for $|\xi| > R\delta$, we also have $\log(2f/c_f\delta) > 1 + C_0^2$; thus $X_\Phi G > C'_G$ for $|\xi| > R\delta$
- $(x,\xi) \in N_u$: Now we only know that $X_{\tilde{\Phi}} m \geq 0$, so we need a uniform lower bound for the second term. But, from the choice of f, it is precisely there that $X_\Phi f/f > \beta/2$. Together with property (e) of m above, we deduce $X_\Phi G > \beta T C'_G/2$ for $|\xi| > \delta$.
- $(x,\xi) \in N_s$: As in the previous case, we obtain $X_\Phi G > \beta T C'_G/2$ for $|\xi| > \delta$.
- $(x,\xi) \in N_0$: As f is a function of p on N_0 and X_Φ is the Hamiltonian flow of p, we have $X_\Phi f = 0$. Since $X_{\tilde{\Phi}} m \geq 0$, we conclude $X_\Phi G \geq 0$ for $|\xi| > \delta$.

Let N'_u (resp. N'_s) be the conical neighbourhood corresponding to $U_u^{\epsilon'}$ (resp $U_s^{\epsilon'}$). We have $N'_u \subset N_u$ and $N'_s \subset N_s$. On N'_u (resp. N'_s), $G = 2C'_G T \log|\xi| + \mathcal{O}(1)$ (resp. $-2C'_G T \log|\xi| + \mathcal{O}(1)$). So we choose $C'_G \geq \max(\frac{2}{\beta T}, 1)$. This gives $C_G = 2C'_G T$ and $\mathbf{m} = C'_G m$.

At last, we have to verify that $\mathbf{m}$ and G are symbols in the right class in the sense of Definition A.6. The weight was constructed as a $\mathscr{C}^\infty$ function on S^*M, and that is the definition of being in $S^0_{\mathrm{cl}}(M)$. For e^G to be elliptic in $S^{\mathbf{m}}_{\log}(M)$, it suffices then to check that $(1-\chi_G(|\xi|))f$ itself is elliptic in $S^1_{\mathrm{cl}}(M)$. By definition, this means that $f/|\xi|$ is a $\mathscr{C}^\infty$ function on S^*M. That is also a direct consequence of the construction. ∎

Actually, in our case, we can say something a little better, that will simplify the rest of the proof.

Lemma 2.5. *We can assume that, for $y > \mathbf{a}$ with $\mathbf{a}$ large enough, both G and $\mathbf{m}$ are invariant under the local isometries T_{τ,θ_0} defined in equation* (1.2).

Proof. Recall from the discussion in Section 1.2 that each cusp Z_ℓ can been seen as a subset of the full cusp $Z_{\ell,f} = \Lambda_\ell \setminus \mathbb{G} / \mathbb{K}$. The geodesic flow on the hyperbolic space $\mathbb{G} \setminus \mathbb{K}$ or rather on its sphere bundle $S(\mathbb{G} / \mathbb{K}) = \mathbb{G} / \mathbb{M}$ is known to be uniformly hyperbolic with analytic stable and unstable bundles $\tilde{E}^{s/u}$ which are invariant under all isometries of the hyperbolic space $\mathbb{G} / \mathbb{K}$, i.e. under the left $\mathbb{G}$ action. Consequently, these bundles descend to the full cusp $SZ_{\ell,f}$ and can thus be restricted to the cusps. We call them the stable and unstable bundles corresponding to constant curvature and denote them by $E^c_{u/s}$. By the invariance under isometries of hyperbolic space, the bundles $E^c_{u/s}$ are invariant under the local isometries T_{τ,θ_0} defined in equation (1.2).

Let us now explain that $E^c_{u/s}$ and $E_{u/s}$ become $\mathcal{O}(1/y)$-close high in the cusp. Let us do this for the example of E_s. First, note that $E_u \oplus E_s = E^c_u \oplus E^c_s$ (this is because the contact form of both flows coincides. Now, for trajectories whose past is included in the cusp, E^u and E^c_u have to coincide, so at the bottom of the cusp ($y = a$), directions that are close to the South Pole (i.e. incoming trajectories), E^s and E^c_u are transverse (by continuity of the bundles). Now, high in the cusp ($y \gg a$) in an arbitrary direction (except in $\mathcal{N}$), its trajectory, when it exits the cusp, has to be almost vertical, i.e. in the neighbourhood of the South Poles considered above; now the uniform hyperbolicity of both splittings implies that E^s and E^c_s are $\mathcal{O}(1/y)$-close as $y \to +\infty$.

As a consequence of the fact that the bundles $E^c_{u/s}$ and $E_{u/s}$ become close, when building the functions $m^u_{0,s}$ and $m^s_{0,u}$, we can actually choose them to be invariant by T_{τ,θ_0} high in the cusp – say $y > y_0$.

Since it takes at least a time $\sim \log y$ to go from height y in the cusp to the compact part N_0, and since all the constructions above make use only of propagation for a global finite time under the flow, we obtain that, for $y > y_0 e^T$, m is also invariant under T_{τ,θ_0}.

The last thing to check is the invariance of f. In the cusp, the vector field X is also invariant under local isometries of the hyperbolic space so that f also can be chosen to be T_{τ,θ_0} invariant for $y > y_0 e^{T'}$. ∎

Remark 2.6. We can choose **a** so that it coincides with the **a** in Definition 4.3. It will be smaller than the a of point (7) of Proposition A.8.

2.2. *A first parametrix*

Now that we have built our escape function, we focus on building an approximate inverse for $\mathbf{X} - hs$. Recall that we use semi-classical analysis: we had defined $\mathbf{X} = h\mathcal{X}$, and we will work with the semi-classical quantization $\mathrm{Op}^w_{h,L}$ acting on sections of L; see Appendix A, equation (A.4). For a simpler notation, we simply write Op in the sequel. A priori, for $\mathrm{Op}(\sigma)$ to make sense, we need that σ is valued in $\mathrm{End}(L)$. If σ is just a function, we can consider $\mathrm{Op}(\sigma \otimes \mathbb{1})$. This operator will be denoted by abuse of notation just as $\mathrm{Op}(\sigma)$.

Definition 2.7. Let $\delta > 0$, and let G_δ be the corresponding escape function given by Lemma 2.3. Let $\gamma > 0$. We denote by $H^{\gamma\mathbf{m}}_\delta$ the set of distributions

$$H^{\gamma\mathbf{m}}_\delta = \mathrm{Op}(e^{-\gamma G_\delta}) \cdot L^2(M, L).$$

It is endowed with the norm

$$\|f\|_{H^{\gamma\mathbf{m}}_\delta} = \|\mathrm{Op}(e^{-\gamma G_\delta})^{-1} f\|_{L^2(M,L)}.$$

The space actually does not depend on h or on δ, but the norm does. As a convention, we denote $H^{\mathbf{0}}_\delta = L^2(M, L)$.

We will drop the δ indices in the notation, to lighten a bit the presentation, and just write $H^{\gamma\mathbf{m}}(= H^{\gamma\mathbf{m}}_\delta)$. Only at the end of Section 4 in the proof of Theorem 3, we will let δ

go to 0. From the properties of G, we directly obtain the following regularity properties, which show that $H_\delta^{\gamma\mathbf{m}}$ is an *anisotropic space*[7].

Lemma 2.8. *For any $\gamma > 0$, $\delta > 0$, we have the continuous inclusions*

$$H^{+C_G\gamma} \subset H_\delta^{\gamma\mathbf{m}} \subset H^{-C_G\gamma}.$$

Furthermore, near E_u^*, $H_\delta^{\gamma\mathbf{m}}$ is microlocally equivalent to $H^{C_G\gamma}$, and near E_s^*, $H_\delta^{\gamma\mathbf{m}}$ is microlocally equivalent to $H^{-C_G\gamma}$. In particular, for $A \in S^0(M, L)$,

$$\mathrm{WF}_h(A) \in N_s \implies \|Au\|_{H_\delta^{\gamma\mathbf{m}}} \le C\|Au\|_{H^{-C_G\gamma}},$$
$$\mathrm{WF}_h(A) \in N_u \implies \|Au\|_{H^{C_G\gamma}} \le C\|Au\|_{H_\delta^{\gamma\mathbf{m}}}.$$

The differential operator $\mathbf{X}$, which is a priori defined on $C_c^\infty(M, L)$, has a unique closed extension [20, Lemma A.1] to the domain $D_\gamma := \{u \in H^{\gamma\mathbf{m}} : \mathbf{X}u \in D_\gamma\}$. The domain D_γ is naturally a Hilbert space with respect to the scalar product

$$\langle\cdot,\cdot\rangle_{D_\gamma} := \langle\cdot,\cdot\rangle_{H^{\gamma\mathbf{m}}} + \langle\mathbf{X}\cdot,\mathbf{X}\cdot\rangle_{H^{\gamma\mathbf{m}}}.$$

The action of $\mathbf{X} - hs$ on $H^{\gamma\mathbf{m}}$, is equivalent to the action on $H^0 = L^2$ of

$$\mathrm{Op}(e^{-\gamma G})^{-1}(\mathbf{X} - hs)\,\mathrm{Op}(e^{-\gamma G}) = \mathbf{X} - h\big(\gamma\,\mathrm{Op}(\{p, G\}) + s\big) + \mathcal{O}(h^2\Psi_{\log}^{-1^+}).$$

Since X_Φ is the Hamiltonian vector field of the Hamiltonian p defined in (2.2), we have $\{p, G\} = X_\Phi G$. We will need the following observation.

Lemma 2.9. *There are constants $C, C' > 0$ such that $\mathbf{X} - hs\colon D_\gamma \to H^{\gamma\mathbf{m}}$ is invertible for $\mathrm{Re}(s) > C(1 + \gamma)$. We denote the inverse by $\mathscr{R}(s)$, and its operator norm is bounded: $\|\mathscr{R}(s)\|_{H^{\gamma\mathbf{m}}\to H^{\gamma\mathbf{m}}} \le C'h^{-1}$.*

Proof. From the sharp Gårding inequality (Lemma A.10), we conclude that there are $C, \varepsilon > 0$ such that $\mathrm{Re}\langle(\mathbf{X} - hs)u, u\rangle_{H^{\gamma\mathbf{m}}} < -\varepsilon h\|u\|^2_{H^{\gamma\mathbf{m}}}$ for $\mathrm{Re}(s) > C(1 + \gamma)$ and all $u \in C_c^\infty(M)$ (C does not depend on γ).

We deduce that $\|(\mathbf{X} - hs)u\|_{H^{\gamma\mathbf{m}}} \ge \varepsilon h\|u\|_{H^{\gamma\mathbf{m}}}$. Consequently, the image of $(\mathbf{X} - hs)$ is closed. We deduce that it is the orthogonal of the kernel of the adjoint. We also get that the kernel of $(\mathbf{X} - hs)$ is empty. Additionally, we observe that the adjoint of $\mathbf{X} - hs$ satisfies the same sharp Gårding estimate so that it also is injective, and thus $(\mathbf{X} - hs)$ is surjective. We conclude that it is invertible. ∎

For each $\delta > 0$, we pick Q $(= Q_\delta)$, a self-adjoint semi-classical pseudo-differential operator, of the form $\mathrm{Op}(q)$, with $q \in S^0$ equal to 1 in $\{|\xi| \le 2R\delta\}$, everywhere positive, and supported in $\{|\xi| < 3R\delta\}$ – the constant R was given in Lemma 2.3. This is an absorbing potential. Let us define

$$\mathbf{X}_Q(s) = \mathbf{X} - Q - hs.$$

[7] The spaces that show up here are distributions that are regular in the E_u^* direction and irregular in the E_s^* direction. This is no contradiction to precedent works like e.g. [15] where the authors continue the resolvent of the operator $-\mathbf{X}$ and thus obtain the converse regularity properties.

Then we have the following key estimate.

Proposition 2.10. *Let $\delta > 0$; then there is a constant $C_\delta > 0$. Assume that s satisfies* $\mathrm{Re}(s) > C_\delta - \gamma + 1$ *and* $|\mathrm{Im}\, s| \le h^{-1/2}$. *Then, for sufficiently small h, the operator $\mathbf{X}_Q(s)$ is invertible on $H^{\gamma\mathbf{m}}$. Denoting by $\mathscr{R}_Q(s)$ its inverse, we get $\|\mathscr{R}_Q(s)\| = \mathcal{O}(h^{-1})$.*

Proof. We fix a tempered family of functions $u \in C_c^\infty(M, L)$. We consider the regions

$$\Omega_{\mathrm{ell}} := \{(x,\xi) \mid |\xi| < 3R\delta/2 \text{ or } |p(x,\xi)| > \epsilon\langle\xi\rangle\},$$
$$\Omega_{\text{Gårding}} := \{(x,\xi) \mid |\xi| > R\delta \text{ and } \xi/|\xi| \notin N_0\}.$$

If $\epsilon > 0$ is chosen small enough, then they overlap, so we can build a partition of unity $1 = A_{\mathrm{ell}} + A_{\text{Gårding}}$, with A_{ell} (resp. $A_{\text{Gårding}}$) microsupported in Ω_{ell} (resp. $\Omega_{\text{Gårding}}$), and both A's in Ψ^0.

In the region Ω_{ell}, the principal symbol of $\mathbf{X}_Q(s)$ is elliptic, so we deduce[8], from Proposition A.14,

$$\|A_{\mathrm{ell}} u\|_{H^{\gamma\mathbf{m}}} \le C\|\mathbf{X}_Q(s)u\|_{H^{\gamma\mathbf{m}}} + \mathcal{O}(h^\infty)\|u\|_{H^{\gamma\mathbf{m}}}.$$

Now we can concentrate on the region of interest $\Omega_{\text{Gårding}}$. By definition, the action of $\mathbf{X}_Q(s)$ on $H^{\gamma\mathbf{m}}$ is conjugated by $\mathrm{Op}(e^{-\gamma G})$ to the action on L^2 of

$$\widetilde{\mathbf{X}_Q(s)} = \mathbf{X} - Q - h\big(\gamma\,\mathrm{Op}(\{p + iq, G\} + s)\big) + \mathcal{O}(h^2\Psi_{\log}^{-1^+}).$$

We denote by $\widetilde{A_{\text{Gårding}}}$ the operator obtained after conjugation by $\mathrm{Op}(e^{-\gamma G})$, and define $\tilde{u} := \mathrm{Op}(e^{-\gamma G})^{-1}u \in L^2$. We consider

$$\begin{aligned}-\mathrm{Re}\langle\widetilde{\mathbf{X}_Q(s)}\widetilde{A_{\text{Gårding}}}\tilde{u}, \widetilde{A_{\text{Gårding}}}\tilde{u}\rangle_{L^2} &= \langle PA_2\widetilde{A_{\text{Gårding}}}\tilde{u}, \widetilde{A_{\text{Gårding}}}\tilde{u}\rangle_{L^2} \\ &\quad + h(\mathrm{Re}(s) + \gamma)\|\widetilde{A_{\text{Gårding}}}\tilde{u}\|_{L^2}^2,\end{aligned}$$

where A_2 is a microlocal cutoff in a slightly bigger neighbourhood of $\mathrm{WF}_h(A_{\text{Gårding}})$ and

$$P := -\mathrm{Re}\,\mathbf{X} + Q + h\gamma\,\mathrm{Op}(\{p, G\} - 1) + \mathcal{O}(h^2\Psi_{\log}^{-1^+}).$$

(Here, $\mathrm{Re}\,\mathbf{X}$ is the real part of $\mathbf{X}$ acting on L^2, and it is an $\mathcal{O}(h)$ order 0 operator.) By Lemma 2.3 (i) (recall that $\{p, G\} = X_\Phi G$), we conclude that $PA_2 \in \Psi^{0+}$ has non-negative principal symbol, and by the sharp Gårding inequality A.10, we deduce that

$$-\mathrm{Re}\langle\widetilde{\mathbf{X}_Q(s)}\widetilde{A_{\text{Gårding}}}\tilde{u}, \widetilde{A_{\text{Gårding}}}\tilde{u}\rangle_{L^2} \ge h\big(-C_\delta + \mathrm{Re}(s) + \gamma\big)\|\widetilde{A_{\text{Gårding}}}\tilde{u}\|_{L^2}^2.$$

[8] Note that Proposition A.14 (2) is stated in terms of ordinary Sobolev spaces and not in terms of anisotropic Sobolev spaces. The statement on anisotropic spaces can however be deduced by applying Proposition A.14 (2) to the conjugated operators $\mathrm{Op}(e^{-\gamma G})^{-1}A_{\mathrm{ell}}\,\mathrm{Op}(e^{-\gamma G})$ and $\mathrm{Op}(e^{-\gamma G})^{-1}(\mathbf{X} - \mathbf{Q} - hs)\,\mathrm{Op}(e^{-\gamma G})$ respectively. Note therefore that the conjugation does not affect the ellipticity.

The constant depends on Q, which depends itself on δ. Using Cauchy–Schwarz and our assumption $\mathrm{Re}(s) > C_\delta - \gamma + 1$, we get

$$\|\widetilde{A_{\text{Gårding}}\tilde{u}}\|_{L^2} \leq Ch^{-1}\|\widetilde{\mathbf{X}_Q(s)}\widetilde{A_{\text{Gårding}}}\tilde{u}\|_{L^2},$$

i.e.

$$\|A_{\text{Gårding}}u\|_{H^{\gamma\mathbf{m}}} \leq Ch^{-1}\|\mathbf{X}_Q(s)A_{\text{Gårding}}u\|_{H^{\gamma\mathbf{m}}}.$$

Gathering our estimates, we find that

$$\|u\|_{H^{\gamma\mathbf{m}}} \leq Ch^{-1}\|\mathbf{X}_Q(s)A_{\text{Gårding}}u\|_{H^{\gamma\mathbf{m}}} + C\|\mathbf{X}_Q(s)u\|_{H^{\gamma\mathbf{m}}} + \mathcal{O}(h^\infty)\|u\|_{H^{\gamma\mathbf{m}}}.$$

Now let us consider

$$\|\mathbf{X}_Q(s)A_{\text{Gårding}}u\|_{H^{\gamma\mathbf{m}}} \leq \|A_{\text{Gårding}}\mathbf{X}_Q(s)u\|_{H^{\gamma\mathbf{m}}} + \|[\mathbf{X}_Q(s), A_{\text{Gårding}}]u\|_{H^{\gamma\mathbf{m}}}.$$

We have $[\mathbf{X}_Q(s), A_{\text{Gårding}}] \in h\Psi^0$, and as $\mathrm{WF}_h([\mathbf{X}_Q(s), A_{\text{Gårding}}]) \subset \Omega_{\text{ell}} \cap \Omega_{\text{Gårding}}$, we get by elliptic regularity $\|[\mathbf{X}_Q(s), A_{\text{Gårding}}]u\|_{H^{\gamma\mathbf{m}}} \leq C\|\mathbf{X}_Q(s)u\|_{H^{\gamma\mathbf{m}}} + \mathcal{O}(h^\infty)\|u\|_{H^{\gamma\mathbf{m}}}$. By continuity of $A_{\text{Gårding}}$, we deduce $\|A_{\text{Gårding}}\mathbf{X}_Q(s)u\|_{H^{\gamma\mathbf{m}}} \leq C\|\mathbf{X}_Q(s)u\|_{H^{\gamma\mathbf{m}}}$, so altogether, we get $\|\mathbf{X}_Q(s)A_{\text{Gårding}}u\|_{H^{\gamma\mathbf{m}}} \leq C\|\mathbf{X}_Q(s)u\|_{H^{\gamma\mathbf{m}}}$, and consequently,

$$\|u\|_{H^{\gamma\mathbf{m}}} \leq \frac{C}{h}\|\mathbf{X}_Q(s)u\|_{H^{\gamma\mathbf{m}}} + \mathcal{O}(h^\infty)\|u\|_{H^{\gamma\mathbf{m}}}.$$

This estimate implies that, for sufficiently small h, the operator $X_Q(s)$ is injective and has closed range. Performing exactly the same estimates for the adjoint operator, we deduce that $X_Q(s)$ is surjective. ■

In the case of compact manifolds, the end of the proof of the equivalent of Theorem 1 is based on the fact that, by writing

$$(\mathbf{X} - hs)\mathscr{R}_Q(s) = \mathbb{1} + Q\mathscr{R}_Q,$$

we have that $\mathbf{X} - hs$ is invertible modulo a smoothing operator, and smoothing operators are compact on compact manifolds, so $\mathbf{X} - hs$ is invertible modulo *compact* operator. Hence it is Fredholm, of index 0, and its inverse is a meromorphic family of operators in the s parameter.

However, in our case, smoothing operators are *not* compact. We will present a special ingredient in the next section to overcome this problem. Before that, we consider wavefront sets.

Proposition 2.11. *Let Ω_+ be the subset of phase space*

$$\Omega_+ := \{((x,\xi); \Phi_t(x,\xi)) \mid p(x,\xi) = 0,\ t \geq 0\} \subset T^*M \times T^*M.$$

Recall that $\Delta(T^*M)$ is the diagonal in T^*M. The wavefront set of $\mathscr{R}_Q(s)$ satisfies

$$\mathrm{WF}'_h(\mathscr{R}_Q(s)) \cap (T^*M \times T^*M) \subset \Delta(T^*M) \cup \Omega_+.$$

Proof. First, by ellipticity in $\{p(x,\xi) \neq 0\} \cup \{|\xi| \leq 2R\delta\}$, the wavefront of $\mathscr{R}_Q(s)$ is contained in $\Delta(T^*M) \cup \{p(x,\xi) = p(x',\xi') = 0,\ |\xi|, |\xi'| > 2R\delta\}$.

Next, note that, by Lemma A.13, we have to prove that, for

$$((x,\xi),(x',\xi')) \in T^*M \times T^*M$$

fulfilling

$$p(x,\xi) = p(x',\xi') = 0, \quad |\xi|, |\xi'| > 2R\delta, \quad \text{and} \quad ((x,\xi),(x',\xi')) \notin \Omega_+,$$

there are $A, A' \in S^0$, elliptic in (x,ξ) (resp. (x',ξ')) such that

$$A\mathscr{R}_Q(s)A' = \mathcal{O}_{H^{-\infty}\to H^\infty}(h^\infty).$$

In order to achieve this, let $((x,\xi),(x',\xi')) \in T^*M \times T^*M$ be such a point. Recall that, as $t \to +\infty$, $|\Phi_t(x',\xi')|$ either goes to 0 or to ∞. Hence we can chose two relatively compact open sets $U, U' \subset T^*M$ such that $\Phi_t(U) \cap U' = \emptyset$ for all $t \geq 0$, and $(x,\xi) \in U$, $(x',\xi') \in U'$. Fix $A, A' \in \Psi^0$ microsupported in respectively U and U'.

Let us prove that $A\mathscr{R}_Q(s)A' = \mathcal{O}_{H^{-\infty}\to H^\infty}(h^\infty)$. Let u be a tempered family of distributions. Let $T > 0$, and B, B_1 elliptic on respectively $\Phi_T(U)$ and $\bigcup_{0\leq t\leq T} \Phi_t(U)$. Observe that $A\mathscr{R}_Q(s)A'u$ is in all Sobolev spaces because A, A' are compactly microsupported. Then we get by propagation of singularities (Lemma A.16) that, for $k \in \mathbb{R}$,

$$\|A\mathscr{R}_Q(s)A'u\|_{H^k} \leq C\|B\mathscr{R}_Q(s)A'u\|_{H^k} + \frac{C}{h}\|B_1A'u\|_{H^k} + \mathcal{O}_{k,u}(h^\infty).$$

By the assumption on the microsupport of A and A', by taking the microsupport of B_1 small enough, we can ensure that $B_1A'u = \mathcal{O}(h^\infty)$; hence

$$\|A\mathscr{R}_Q(s)A'u\|_{H^k} \leq C\|B\mathscr{R}_Q(s)A'u\|_{H^k} + \mathcal{O}_{k,u}(h^\infty). \tag{2.4}$$

Now we just have to consider what happens when the time T becomes larger. For

$$(x,\xi) \in \{p = 0\} \subset T^*M,$$

there are only two possibilities: either there is $T > 0$ such that

$$\Phi_T(x,\xi) \subset \mathrm{ell}_1(Q) = \{|\xi| \leq R\delta\},$$

or $\Phi_t(x,\xi)$ converges to $E_u^* \cap \partial\overline{T^*M}$ (see Definition A.6 for the radial compactification).

In the first case, take U sufficiently small such that $\Phi_T(U) \subset \mathrm{ell}_1(Q)$. Thus we can assume that B in the propagation estimate (2.4) is microsupported in $\mathrm{ell}_1(Q)$. Taking $B' \in \Psi^0$ elliptic on the microsupport of B, the elliptic estimate (Proposition A.14) gives

$$\|B\mathscr{R}_Q(s)A'u\|_{H^k} \leq C\|B'A'u\|_{H^k} + \mathcal{O}_{m,u}(h^\infty).$$

Since we can choose B' such that $\mathrm{WF}(B') \cap \mathrm{WF}(A') = \emptyset$, the right-hand side is $\mathcal{O}(h^\infty)$.

Now we turn to the second case which we will treat using the high regularity radial estimate (Proposition A.18). Note that $E_u^* \cap \partial\overline{T^*M}$ is a sink in the sense of Definition A.17. Next, let us choose $C \in \Psi^0$ such that $E_u^* \cap \partial\overline{T^*M} \subset \mathrm{ell}_1(C)$ and such that

$\mathrm{WF}_h(C) \cap \mathrm{WF}_h(A') = \emptyset$. Then Proposition A.18 provides us with an order 0 operator C_1 which is elliptic in a neighbourhood of $E_u^* \cap \partial\overline{T^*M}$. Furthermore, we can assume $\mathrm{WF}_h(C_1) \subset N_u$.

Since $C_1 \mathscr{R}_Q A'u \in H^{\gamma\mathbf{m}}$ and it is microsupported in N_u, by Lemma 2.8, we know that $C_1 \mathscr{R}_Q(s) A'u \in H^{\gamma C_G}$, and taking $\gamma C_G > k_0$, we have the necessary regularity for the sink estimate. We get, for any $k > k_0$,

$$\|C_1 \mathscr{R}_Q(s) A'u\|_{H^k} \leq \frac{C}{h}\|CA'u\|_{H^k} + \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty). \tag{2.5}$$

Finally, for U sufficiently small and by propagation of singularity for a long enough but finite time T, we can assume that $\Phi_T(U) \subset \mathrm{ell}_1(C)$. Combining (2.4) and (2.5), we obtain as desired $\|A\mathscr{R}_Q(s)A'u\|_{H^k} = \mathcal{O}(h^\infty)$. ■

We have a final remark for this section.

Definition-Proposition 2.12. *If $\gamma \geq 0$ and $N \in \mathbb{R}$, we say that $\mathbf{k} = \gamma\mathbf{m} + N$ is a* weight. *Such a weight is said to be large if γ is large and N/γ is small. We define*

$$H^{\mathbf{k}}_{(\delta)}(M, L) := \mathrm{Op}(e^{-\gamma G_\delta}) H^N(M, L).$$

Then the conclusion of Proposition 2.10 holds on the space $H^{\mathbf{k}}$ when $|\mathrm{Im}\, s| < h^{-1/2}$, $\mathrm{Re}\, s \geq C_\delta - \gamma + CN + 1$ for some constant C independent of γ, N, and for $h > 0$ small enough.

The proof is completely analogous to the proof of Proposition 2.10.

3. Continuation of the resolvent for translation invariant operators

In this section, we will be considering a vector bundle over some compact Riemannian manifold $\mathsf{L} \to \mathsf{F}$, endowed with a bundle metric and a compatible connection. We will always see the space $\mathbb{R} \times \mathsf{L}$ as a fibre bundle over $(\mathbb{R})_r \times (F)_\zeta$, endowed with the product structure. We will also use the natural measure $dr\, d\zeta$, and $L^2(\mathbb{R} \times \mathsf{L})$ will be understood as the space of square-integrable sections with respect to this measure.

Let us first see how bundles of this type can be naturally obtained from admissible vector bundles in the sense of Definition 1.4.

Example 3.1. Let $L \to M = S^*N$ be an admissible vector bundle, and fix a cusp Z_ℓ. Then, over this cusp, the bundle takes the form $L = \Lambda_\ell \backslash \mathbb{G} \times_{\tau_\ell} V_\ell$. Using the Iwasawa decomposition $\mathbb{G} = \mathbb{N}\,\mathbb{A}\,\mathbb{K}$ and identifying $\mathbb{A} \cong (\mathbb{R}, +)$, $\mathbb{N} \cong (\mathbb{R}^d, +)$, we obtain

$$L = (\mathbb{R}^d/\Lambda_\ell) \times \mathbb{R} \times (\mathbb{K} \times_{\tau_\ell} V_\ell).$$

In Section 4, we will study sections of these bundles that are independent on the variable $\theta \in (\mathbb{R}^d/\Lambda_\ell)$, and these sections are naturally identified with sections of $\mathbb{R} \times (\mathbb{K} \times_{\tau_\ell} V_\ell)$. This shows that studying θ-independent sections of admissible vector bundles $L_{|S^*Z_\ell}$ leads to the study of sections of $\mathbb{R} \times \mathsf{L}_\ell \to \mathbb{R} \times \mathsf{F}$ with $\mathsf{L}_\ell = \mathbb{K} \times_{\tau_\ell} V_\ell \to \mathbb{K}/\mathbb{M} \cong \mathbb{S}^d = \mathsf{F}$.

Remark 3.2. For the proof of Theorem 3 on vector bundles, one could restrict the discussion of the whole section to the special case in the example above. As all arguments, however, hold without any further complications in the general case of vector bundles $\mathsf{L} \to \mathsf{F}$ over general compact manifolds F, we announce and prove all results in this section in this setting. Additionally, we expect that this wider class is likely to show up when studying uniformly hyperbolic flows on fibred cusps.

3.1. b-Operators

We will consider a particular class of operators on $\mathbb{R} \times \mathsf{L} \to \mathbb{R} \times \mathsf{F}$. Recall that, by the Schwartz kernel theorem, any continuous linear operator $A: C_c^\infty(\mathbb{R} \times \mathsf{L}) \to \mathcal{D}'(\mathbb{R} \times \mathsf{L})$ is represented by its kernel $K_A \in \mathcal{D}'(\mathbb{R} \times \mathbb{R} \times \mathsf{L} \boxtimes \mathsf{L})$. We call such an operator A a convolution operator if there is $\tilde{K}_A \in \mathcal{D}'(\mathbb{R} \times \mathsf{L} \boxtimes \mathsf{L})$ such that $K_A = p^* \tilde{K}_A$, where

$$p: \mathbb{R} \times \mathbb{R} \times \mathsf{L} \boxtimes \mathsf{L} \ni (r, r', l \boxtimes l') \mapsto (r - r', l \boxtimes l') \in \mathbb{R} \times \mathsf{L} \boxtimes \mathsf{L}\,.$$

Definition 3.3. The set of semi-classical pseudo-differential operators acting on sections of $\mathbb{R} \times \mathsf{L}$ that are convolution operators in the r variable will be denoted by $\Psi_b(\mathbb{R} \times \mathsf{L})$. It is the set of *b-operators*.

Such operators that additionally are supported in $\{|r - r'| \le \log C\}$ will be denoted $\Psi_{b,C}(\mathbb{R} \times \mathsf{L})$. We say that they are b-operators with *precision* C. When $C = 1$, the kernels are supported on $\{r = r'\}$.

Remark 3.4. Our notion of b-operators is, as its name suggests, strongly inspired by Melrose's b-calculus (see e.g. [42]). However, in this article, we use a much more restrictive class of operators. Let us shortly explain the relation of our b-operators to the usual class of b-differential operators in the sense of Melrose. Let $[0, \infty[_x \times \mathbb{R}_\zeta$ be the simplest model of a manifold with boundary. Then the b-differential operators are those in the algebra of operators generated by b-vector fields that take the form $a(x, \zeta) x \partial_x + b(x, \zeta) \partial_\zeta$ with $a, b \in C^\infty([0, \infty[_x \times \mathbb{R}_\zeta)$. Using a Taylor expansion, the leading order near the boundary of these operators takes the form $a_0(\zeta) x \partial_x + b_0(\zeta) \partial_\zeta$. After a variable transformation $r = \log(x)$, these are in the form $a_0(\zeta) \partial_r + b_0(\zeta) \partial_\zeta$. Such operators are then translation invariant in the r variable, i.e. are convolution operators. Their kernels take the form

$$a_0(\zeta) \delta(r - r') + b_0(\zeta) \delta(\zeta - \zeta').$$

In some sense, our class of b-operators contains just those which are equal to their leading part in the asymptotic expansion near the boundary of the usual class of b-(pseudo-) differential operators. For our purpose, this is sufficient, and the restriction to this class allows us to concentrate on the difficulties that arise from the fact that we have to construct a parametrix for an operator that is not elliptic (even in a b-calculus sense).

Example 3.5. The generator of the geodesic flow acting on functions supported in a cusp and not depending on θ is a differential operator acting on the trivial bundle, i.e. on $L^2(\mathbb{R} \times \mathbb{S}^d, e^{-rd} dr\, d\zeta)$ given by (cf. equation (1.4))

$$X_b^0 = \cos\varphi \partial_r + \sin\varphi \partial_\varphi.$$

In order to make it a b-operator acting on $L^2(\mathbb{R}\times\mathbb{S}^d, dr\,d\zeta)$, we conjugate it with $e^{-rd/2}$ and get

$$X_b = \cos\varphi\partial_r + \frac{d}{2}\cos\varphi + \sin\varphi\partial_\varphi. \tag{3.1}$$

In order to work in the semi-classical calculus, we write $\mathbf{X}_b := hX_b$.

The aim of Section 3 is to show that the resolvent of $\mathbf{X}_b$ can be continued meromorphically from $\mathrm{Re}(s) > 0$ to $\mathbb{C}$. In fact, for the reasons discussed in Remark 3.2, we will treat a more general class of operators $\mathbf{X}_b \in \Psi_{b,0}(\mathbb{R}\times\mathsf{L})$ whose precise assumptions will be formulated in Section 3.2

Next, let us introduce symbols and quantizations that lead to b-operators.

Definition 3.6. Denote by g the metric on F and by $T(T^*\mathsf{F}) = H\oplus V$ the splitting into vertical and horizontal directions with respect to the Levi-Civita connection. We endow $T^*(\mathbb{R}\times\mathsf{F})$ with the metric described in Definition A.1. Consider its restriction $\overline{g}_b$ to $(T_0^*\mathbb{R})_\lambda\times(T^*\mathsf{F})_{(\zeta,\eta)}$. It can be expressed as

$$\begin{aligned}\overline{g}_{b,(\zeta;\eta,\lambda)}&(X^v+Y^h+\mu\partial_\lambda, W^v+Z^h+\mu'\partial_\lambda)\\ &= g_\zeta(Y,Z)+\frac{1}{1+g_\zeta(\eta,\eta)+\lambda^2}[g_\zeta(X,W)+\mu\mu'].\end{aligned}$$

By Lemma A.2, $\overline{g}_b$ has bounded geometry.

Definition-Proposition 3.7. *We denote by $S_b^0(\mathbb{R}\times\mathsf{L})$ the set of $\mathscr{C}^\infty$ sections of $T^*(\mathbb{R}\times\mathsf{F})\to\mathrm{End}(\mathsf{L})$ with uniformly bounded derivatives with respect to $\overline{g}_b$ which additionally are independent of the r variable. They are the translation invariant elements of $S^0(\mathbb{R}\times\mathsf{F},\mathbb{R}\times\mathsf{L})$ from Definition A.3. Similarly, we define $S^0_{b,\epsilon}$, $S^0_{b,\epsilon,\xi}$ and $S^0_{b,\log}$. We call them order* 0 b-symbols. *Given $m_b\in S_b^0(\mathbb{R}\times\mathsf{F})$, we can also define $S^{m_b}_{b,\log}(\mathbb{R}\times\mathsf{L})$ as $\langle\xi\rangle^{m_b}S^0_{b,\log}(\mathbb{R}\times\mathsf{L})$. It is the set of anisotropic symbols of order m_b.*

These symbol classes are stable by all the usual symbolic manipulations (because $\overline{g}_b$ has bounded geometry).

Consider a semi-classical Weyl quantization Op_h^w for sections of $\mathsf{L}\to\mathsf{F}$ (see e.g. [59, Theorem 14.1] or Appendix A.2). Given a finite open cover U_k of F and a trivialization $t_k\colon \mathrm{pr}^{-1}_{\mathsf{L}\to\mathsf{F}}(U)\to V\times\mathbb{R}^{\dim(L_x)}$ as well as a quadratic partition of unity $\sum_k\chi_k^2 = 1$, $\chi_k\in C_c^\infty(U_k,\mathbb{R}_{\geq 0})$, such a quantization can be written for $\sigma\in S^m(\mathsf{L})$ by

$$\mathrm{Op}^w_{h,\mathsf{L}}(\sigma) := \sum_k \chi_k t_k^*\,\mathrm{Op}^w_{h,\mathbb{R}^{\dim\mathsf{F}}}((t_k^{-1})^*\sigma)(t_k^{-1})^*\chi_k, \tag{3.2}$$

where $\mathrm{Op}^w_{h,\mathbb{R}^{\dim\mathsf{F}}}$ is the usual Weyl quantization on $\mathbb{R}^{\dim\mathsf{F}}$.

Now we can use $\mathrm{Op}^w_{h,\mathsf{L}}$ to define a quantization of b-symbols $\sigma_b\in S_b^m(\mathbb{R}\times\mathsf{L})$ on $\mathbb{R}\times\mathsf{L}$ by

$$(\mathrm{Op}^b(\sigma_b)f)(r,\zeta) := \frac{1}{2\pi h}\int e^{(i/h)\lambda(r-r')}[\mathrm{Op}^w_{h,\mathsf{L}}(\sigma_b(\cdot,\cdot;\lambda))f(r',\cdot)](\zeta)\,d\lambda\,dr', \tag{3.3}$$

which yields an element of $\Psi_b(\mathbb{R}\times L)$. If, additionally, we choose a smooth cutoff χ_C supported in $]-\log C, \log C[$, equal to 1 in $]-\log C^{1/2}, \log C^{1/2}[$, we can multiply the kernel of $\mathrm{Op}^b(\sigma_b)$ by $\chi_C(r-r')$ and obtain an operator $\mathrm{Op}^b(\sigma)_C$ in $\Psi_{b,C}(\mathbb{R}\times \mathsf{L})$.

It should be noted that plugging in (3.2) into (3.3) and writing

$$\tilde{t}_k : \mathbb{R}\times \mathrm{pr}^{-1}_{\mathsf{L}\to\mathsf{F}}(U) \ni (r,l) \mapsto (r, t_k(l)) \in \mathbb{R}\times V\times \mathbb{R}^{\dim(L_x)},$$

we get

$$(\mathrm{Op}^b(\sigma_b)f)(r,\zeta) := \sum_k \chi_k \tilde{t}_k^* \,\mathrm{Op}^w_{h,\mathbb{R}^{\dim \mathsf{F}+1}}((\tilde{t}_k^{-1})^*\sigma_b)(\tilde{t}_k^{-1})^*\chi_k .$$

From this expression, we see that all usual properties of quantizations, such as composition formulas, L^2 estimates, sharp Gårding inequalities, etc., that hold for the quantization on $\mathbb{R}^{\dim \mathsf{F}+1}$ (see e.g. [18, Appendix E]) directly transfer to $\mathrm{Op}^b(\sigma_b)$. The same holds for $\mathrm{Op}^b(\sigma_b)_C$ because the cutoff away from the diagonal modifies the operator only by an element of $h^\infty\Psi^{-\infty}$.

Remark 3.8. We will see in Proposition 4.11 that there will be a method to construct b-symbols from any symbol $\sigma\in S(L\to S^*Z)$ which is invariant by the local isometries of the cusp $T_{\tau,\theta}$. (Recall that e.g. the escape function had this property.)

3.2. *Approximate inverse*

Definition 3.9. Let $\mathbf{X}_b\in\Psi^1_{b,1}(\mathbb{R}\times\mathsf{L})$, $G_b\in S^{0+}_b(\mathbb{R}\times\mathsf{F})$ and $Q_b\in\Psi^{-\infty}_{b,C}(\mathbb{R}\times\mathsf{F})$. We will say that this triple is admissible if

- $-i\mathbf{X}_b$ and Q_b have scalar, real principal symbols,
- $e^{\gamma G}$ is elliptic in $S^{m_b}_{b,\log}$ for some $m_b\in S^0_b$,
- $\mathbf{X}_b = hX_b$, where X_b is a differential operator independent of h,
- letting ip_b be the principal symbol of $\mathbf{X}_b$, there is a $\delta'>0$ such that

$$|p_b|\le \delta'|\xi| \text{ and } |\xi|>\delta' \implies \{p_b, G_b\}>1,$$

- for the same $\delta'>0$, Q_b is elliptic on $|\xi|<2\delta'$ and microsupported in $|\xi|<3\delta'$.

Example 3.10. We will see in Section 4 that $\mathbf{X}$, G and Q defined in Section 2.2 will give rise to an admissible triple after restricting to θ-invariant sections. The constant δ' is just $R\delta$ when $\delta>0$ is small enough.

Definition 3.11. As in Definition-Proposition 2.12, we say that $k_b\in S^0_b$ is a *weight* if it is of the form $\gamma m_b + N$. When we say that a weight is *large*, it means that $\gamma>0$ is large and that N/γ is arbitrarily small.

Given a weight k_b and $\rho\in\mathbb{R}$, we will work with the space of L-valued distributions on $\mathbb{R}\times\mathsf{F}$,

$$\mathcal{H}^{k_b}_{b,\rho} := e^{\rho r}\,\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C L^2(\mathbb{R}\times L),$$

endowed with the corresponding norm

$$\|u\|_{\mathcal{H}^{k_b}_{b,\rho}} := \|\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C^{-1} e^{-\rho r}u\|^2_{L^2}.$$

(Note that, for $h > 0$ small enough, $\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C^{-1}$ exists because of the ellipticity of $e^{-\gamma G_b}\langle\xi\rangle^{-N}$ in $S^{-\gamma m_b - N}_{b,\log}(\mathbb{R}\times \mathsf{F})$.)

The main result in this subsection is the following.

Lemma 3.12. *Assume that* $(\mathbf{X}_b, G_b, Q_b)$ *is an admissible triple. Then there is a constant* $C > 0$ *such that, for* $\mathrm{Re}(s) > 1 + C_\delta + C(|\rho| + |N|) - \gamma$ *and* $|\mathrm{Im}\, s| \leq h^{-1/2}$, *and for small enough* $h > 0$, $\mathbf{X}_b - Q_b - hs$ *is invertible on* $\mathcal{H}^{\gamma m_b + N}_{b,\rho}$.

Proof. We can apply the same arguments as in the proof of Proposition 2.10. Note that, in the positive commutator part, which uses the sharp Gårding inequality, it is important that the real part of $\mathbf{X}_b - Q_b - hs$ on $\mathcal{H}^{\gamma m_b+N}_{b,\rho}$ is unitarily equivalent to the action on L^2 of

$$\begin{aligned}\mathrm{Re}\,\mathbf{X}_b - Q_b - h\big(&\mathrm{Re}(s) + \gamma\,\mathrm{Op}^b(\{p_b, G_b\})_C \\ &+ N\,\mathrm{Op}^b(\{p_b, \log\langle\xi\rangle\})_C - e^{-\rho r}[i\,\mathrm{Im}\,X_b, e^{\rho r}]\big) + \mathcal{O}_{L^2\to L^2}(h^2).\end{aligned} \tag{3.4}$$

It is crucial that the absolute value of the second line in (3.4) is bounded by $C(|\rho| + |N|)$ – it would not be the case a priori replacing $\log\langle\xi\rangle$ by $m_b'\log\langle\xi\rangle$, where $m_b' \in S^0_b$. ∎

3.3. The indicial family

For the following constructions, it is useful to bear in mind the elementary method of inversion of convolution operator on $\mathbb{R}$. Consider some $f \in \mathcal{D}'(\mathbb{R})$ compactly supported and the operator $T_f\colon g \mapsto f * g$. Obviously, the Fourier transform of $T_f g$ is just $\hat{f}\hat{g}$. To invert T_f, is suffices then to invert $\hat{g} \mapsto \hat{f}\hat{g}$. Our aim is to invert the b-operators introduced in Section 3.1, which motivates us to introduce an analogon to the above appearing Fourier transform.

Let $A \in \Psi_{b,C}(\mathbb{R}\times\mathsf{L})$, $f \in C^\infty(\mathsf{L})$. For $\lambda \in \mathbb{C}$, we consider $e^{\lambda r/h} f(\zeta) \in C^\infty(\mathbb{R}\times\mathsf{L})$. By the support properties of the kernel, A is a properly supported pseudo-differential operator and thus defines a continuous operator on $C^\infty(\mathbb{R}\times\mathsf{L})$. Moreover, by the fact that A is a convolution operator, $(r,\zeta) \mapsto e^{-\lambda r/h}(Ae^{\lambda\bullet/h} f(\bullet))(r,\zeta)$ is independent of r; thus $\zeta \mapsto e^{-\lambda r_0/h}(Ae^{\lambda\bullet/h} f(\bullet))(r_0,\zeta)$ is independent of r_0 and is a well defined smooth section of L.

Definition 3.13. Given $A \in \Psi_{b,C}(\mathbb{R}\times\mathsf{L})$ and $\lambda \in \mathbb{C}$, we define the *indicial family associated to* A as the family of operators $I(A,\lambda)\colon C^\infty(\mathsf{L}) \to C^\infty(\mathsf{L})$ given by

$$(I(A,\lambda)f)(\zeta) := e^{-\lambda r_0/h}(Ae^{\lambda\bullet/h} f(\bullet))(r_0,\zeta)$$

Note that, given a second operator $B \in \Psi_{b,C}(\mathbb{R}\times\mathsf{L})$, it follows from the definition that $I(AB,\lambda) = I(A,\lambda)I(B,\lambda)$.

Example 3.14. If $\mathbf{X}_b$ is obtained from the geodesic flow on a cusp, i.e. is the operator in equation (3.1), the corresponding indicial family is

$$I(\mathbf{X}_b,\lambda) = \lambda\cos\varphi + h\Big(\frac{d}{2}\cos\varphi + \sin\varphi\,\partial_\varphi\Big).$$

An equivalent description of the indicial family is the following: fix $\chi \in C_c^\infty(\mathbb{R})$ with $\int \chi(r)\,dr = 1$; then the indicial family is the family of operators

$$I(A,\lambda) \colon C^\infty(\mathsf{L}) \to \mathcal{D}'(\mathsf{L})$$

such that, for any $f_1, f_2 \in C^\infty(\mathsf{L})$,

$$\begin{aligned}\langle f_2, I(A,\lambda) f_1\rangle_{C^\infty(\mathsf{L}),\mathcal{D}'(\mathsf{L})} \\ = \int \chi(r) e^{-\lambda r/h} \langle f_2(\zeta), (A e^{\lambda \bullet/h} f_1(\bullet))(r,\zeta)\rangle_{\mathsf{L}_\zeta}\, dr. \end{aligned} \tag{3.5}$$

This expression is helpful for two purposes. First, by taking complex derivatives of the right-hand side with respect to λ, we conclude with the following lemma.

Lemma 3.15. *For $A \in \Psi_{b,C}(\mathbb{R}\times\mathsf{L})$, $I(A,\lambda)$ is holomorphic in λ as a family of operators $C^\infty(\mathsf{L}) \to \mathcal{D}'(\mathsf{L})$.*

Secondly, it allows to extend the definition of the indicial families to convolution operators on $\mathbb{R}\times\mathsf{L}$ that fail to be in $\Psi_{b,C}$. Note that we will work with non-elliptic problems; thus the appearing inverse operators (like for example $(\mathbf{X}_b - Q_b - hs)^{-1}$ from Lemma 3.12) will not be pseudo-differential operators, so it will be crucial to have the following extended definition.

Lemma 3.16. *Let A be a convolution operator $C_c^\infty(\mathbb{R}\times\mathsf{L}) \to \mathcal{D}'(\mathbb{R}\times\mathsf{L})$ such that, for some $N_1, N_2 \in \mathbb{R}$ and $\rho_0 < \rho_1$,*

$$\|A\|_{\mathcal{H}^{N_1}_{b,\rho_0} \to \mathcal{H}^{N_2}_{b,\rho_0}} < \infty \quad \text{and} \quad \|A\|_{\mathcal{H}^{N_1}_{b,\rho_1} \to \mathcal{H}^{N_2}_{b,\rho_1}} < \infty.$$

Then equation (3.5) defines for $\mathrm{Re}(\lambda) \in \,]\rho_0,\rho_1[$ a holomorphic family of linear operators $C^\infty(\mathsf{L}) \to \mathcal{D}'(\mathsf{L})$. Furthermore, for $\rho \in \,]\rho_0,\rho_1[$, we have

$$\|I(A,\rho + iw)\|_{H^{N_1}(\mathsf{L}) \to H^{N_2}(\mathsf{L})} \le C\langle w\rangle^{|N_1|+|N_2|}.$$

Given a second convolution operator B fulfilling $\|B\|_{\mathcal{H}^{N_2}_{b,\rho_{0/1}} \to \mathcal{H}^{N_3}_{b,\rho_{0/1}}} < \infty$, we have, for any $\mathrm{Re}(\lambda) \in \,]\rho_0,\rho_1[$,

$$I(AB,\lambda) = I(A,\lambda) I(B,\lambda). \tag{3.6}$$

Proof. We want to show that, for $f_1 \in H^{N_1}(\mathsf{L})$ and $\mathrm{Re}(\lambda) \in \,]\rho_0,\rho_1[$, $Ae^{\lambda\bullet/h} f_1(\bullet)$ is well defined. Let us choose a partition of unity $\Psi_1, \Psi_2 \in C^\infty(\mathbb{R})$, $\mathrm{supp}(\Psi_1) \in \,]-\infty, 1]$, $\mathrm{supp}(\Psi_2) \in \,]-1,\infty]$, $\Psi_1 + \Psi_2 = 1$. Choose $\rho \in \,]\rho_0,\rho_1[$, and set $\lambda = \rho + iw$. Then the maps

$$\begin{cases} H^{N_1}(\mathsf{L}) \ni f_1(\zeta) \mapsto \Psi_1(r) e^{\lambda r/h} f_1(\zeta) \in \mathcal{H}^{N_1}_{b,\rho_0}, \\ H^{N_1}(\mathsf{L}) \ni f_1(\zeta) \mapsto \Psi_2(r) e^{\lambda r/h} f_1(\zeta) \in \mathcal{H}^{N_1}_{b,\rho_1} \end{cases}$$

are continuous with operator norm bounded by $C\langle w\rangle^{|N_1|}$. By the compact support of the cutoff function χ appearing in (3.5), for any $\rho' \in \mathbb{R}$, the linear operator

$$H^{-N_2}(\mathsf{L}) \ni f_2(\zeta) \mapsto \chi(r) e^{-\lambda r/h} f_2(\zeta) \in \mathcal{H}^{-N_2}_{b,\rho'}$$

is well defined and bounded by $C\langle w\rangle^{|N_2|}$. Using the continuity of $A\colon \mathcal{H}^{N_1}_{b,\rho_0} \to \mathcal{H}^{N_2}_{b,\rho_0}$ and $A\colon \mathcal{H}^{N_1}_{b,\rho_1} \to \mathcal{H}^{N_2}_{b,\rho_1}$ respectively yields that

$$\begin{aligned}
&\langle f_2, I(A,\lambda) f_1\rangle_{C^\infty(\mathsf{L}),\mathcal{D}'(\mathsf{L})} \\
&\qquad = \langle \chi(r)e^{-\lambda r/h} f_2(\zeta), (A\Psi_1(\bullet)e^{\lambda\bullet/h} f_1(\bullet))\rangle_{C_c^\infty(\mathbb{R}\times\mathsf{L}),\mathcal{D}'(\mathbb{R}\times\mathsf{L})}, \\
&\qquad\qquad + \langle \chi(r)e^{-\lambda r/h} f_2(\zeta), (A\Psi_2(\bullet)e^{\lambda\bullet/h} f_1(\bullet))\rangle_{C_c^\infty(\mathbb{R}\times\mathsf{L}),\mathcal{D}'(\mathbb{R}\times\mathsf{L})}, \\
&\qquad \le C\langle w\rangle^{|N_1|+|N_2|}\|f_2\|_{H^{-N_2}(\mathsf{L})}\|f_1\|_{H^{N_1}(\mathsf{L})}.
\end{aligned}$$

This shows the well-definedness of $I(A,\lambda)$ for $\mathrm{Re}(\lambda) \in \,]\rho_0,\rho_1[$ and the bounds on the operator norm. The holomorphicity is again deduced from the fact that the right-hand side of (3.5) is holomorphic in λ. The above calculations also show that

$$A(e^{\lambda\bullet/h} f_1(\bullet)) = e^{\lambda\bullet/h} I(A,\lambda) f_1,$$

and from this equation, the composition property (3.6) follows directly. ■

Lemma 3.17. *Let $\sigma_b \in S_b(\mathbb{R}\times\mathsf{L})$. Then there is a holomorphic family $\lambda \to \sigma_{b,\lambda} \in S_b(\mathsf{L})$ such that*

$$I(\mathrm{Op}^b(\sigma_b)_C, \lambda) = \mathrm{Op}^w_{h\,\mathsf{L}}(\sigma_{b,\lambda}).$$

It is given by

$$\sigma_{b,\lambda}(\zeta,\eta) := \frac{1}{2\pi h}\int_{\mathbb{R}} \sigma_b(\zeta,\eta,\lambda')\hat{\chi}_C\Big(\frac{-i\lambda-\lambda'}{h}\Big)\,d\lambda'. \tag{3.7}$$

Furthermore, if $\sigma_b \in S^{m_b}_{b,\log}(\mathbb{R}\times\mathsf{L})$, then $\sigma_{b,\lambda} \in S^{m_b,0}_{\log}(\mathsf{L})$ and the leading asymptotics in the high frequency limit is given by $\sigma_b(\cdot,\cdot,0)$, i.e.

$$\sigma_{b,\lambda} - \sigma_b(\cdot,\cdot,0) \in (1+\log\langle\xi\rangle)S^{m_b,0-1}_{\log}(\mathsf{L}). \tag{3.8}$$

In particular, the leading asymptotics of $\sigma_{b,\lambda}$ in the high frequency regime is independent of λ.

Proof. We use Definition 3.13 and choose $r_0 = 0$ for simplicity. Then, using (3.3), we get

$$\begin{aligned}
I(\mathrm{Op}^b(\sigma_b)_C,\lambda)f &= \frac{1}{2\pi h}\int e^{(i/h)\lambda'(r-r')}\chi_C(-r')e^{\lambda r'/h}[\mathrm{Op}^w_{h,\mathsf{L}}(\sigma_b(\cdot,\cdot;\lambda'))f]\,d\lambda'\,dr' \\
&= \frac{1}{2\pi h}\int \hat{\chi}_C\Big(\frac{-i\lambda'-\lambda}{h}\Big)[\mathrm{Op}^w_{h,\mathsf{L}}(\sigma_b(\cdot,\cdot;\lambda'))f]\,d\lambda'.
\end{aligned}$$

Now the fact that the $d\lambda'$ integral can be interchanged with $\mathrm{Op}^w_{h,\mathsf{L}}$ is justified by the fact that $\mathrm{Op}^w_{h,\mathsf{L}}$ is defined in a finite number of charts (see (3.2)).

It remains to study the leading asymptotics of $\sigma_{b,\lambda}$. Let $\sigma_b \in S^{m_b}_{\log}(\mathbb{R}\times\mathsf{L})$, and choose $N > 0$ such that $-N \le m_b \le N$. By the symbol estimates, we deduce

$$|\partial^\alpha_\xi\partial^\beta_\eta\partial^k_\lambda\sigma_b(\zeta,\eta,\lambda)| \le C(1+\log\langle\xi\rangle)^{|\alpha|+|\beta|+k}(\langle\eta\rangle\langle\lambda\rangle)^{m_b(\zeta,\eta,\lambda)-|\beta|-k}. \tag{3.9}$$

In particular, we have $\partial_\lambda^k \sigma_b(\zeta,\eta,0) \in \log(2+\xi^2)^k S_{\log}^{m_b(\zeta,\eta,0)-k}(\mathsf{L})$. Now, by remainder estimates on the Taylor series in λ, for any $(\zeta,\eta) \in T^*\mathsf{F}$ and $\lambda \in \mathbb{R}$, there is $|p_{\zeta,\eta,\lambda}| \leq |\lambda|$ such that

$$\sigma_b(\zeta,\eta,\lambda) = \sum_{k=0}^{2N} \frac{1}{k!}\partial_\lambda^k \sigma_b(\zeta,\eta,0)\lambda^k + \frac{\lambda^{2N+1}}{(2N+1)!}(\partial_\lambda^{2N+1}\sigma_b)(\zeta,\eta,p_{\zeta,\eta,\lambda}).$$

Plugging this into the formula for $\sigma_{b,\lambda}$ yields

$$\begin{aligned}\sigma_{b,\lambda}(\zeta,\eta) &:= \sigma_b(\zeta,\eta,0) + \sum_{k=1}^{2N} c_k \cdot \partial_\lambda^k \sigma_b(\zeta,\eta,0) \\ &\quad + \frac{1}{2\pi h}\int_{\mathbb{R}} \frac{\lambda^{2N+1}}{(2N+1)!}(\partial_\lambda^{2N+1}\sigma_b)(\zeta,\eta,p_{\zeta,\eta,\lambda})\hat{\chi}_C\left(\frac{-i\lambda-\lambda'}{h}\right) d\lambda'.\end{aligned}$$

Now (3.9) assures that the last term is in $S^{-N-1}(\mathsf{L})$. Putting everything together, we conclude $\sigma_{b,\lambda} - \sigma_b(\cdot,\cdot,0) \in \log(2+\xi^2) S_{\log}^{m_b,0-1}(\mathsf{L})$. ∎

Now we can define spaces on $\mathsf{L} \to \mathsf{F}$.

Definition 3.18. Let $k_b = \gamma m_b + N$ be a weight. We denote by $\mathsf{H}_\lambda^{\gamma m_b + N}$ the space

$$I(\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C, \lambda) L^2(\mathsf{L}),$$

endowed with the corresponding norm.

Remark 3.19. The different letters are associated to functional spaces on different objects. First, $H^{\mathbf{k}}$ or $H^{\mathbf{k}}(M, L)$ is a space on the whole manifold. Then, $\mathscr{H}^{k_b}$ is the corresponding space "restricted" to the zeroth Fourier mode in a cusp. Finally, $\mathsf{H}_\lambda^{k_b}$ is the "Fourier transform" of $\mathscr{H}^{k_b}$.

Let us discuss the λ subscript in the notation of the spaces $\mathsf{H}_\lambda^{\gamma m_b + N}$. It may seem that these spaces depend on the parameter λ, and since we want to consider analytic families of operators depending on the parameter λ, this may be problematic – recall that, for the theory of Kato [33] to apply, we need that operators are of type (A), which basically means that they all act on the same domain. To address this problem, we start with the following lemma.

Lemma 3.20. *For any weight k_b, the space $\mathsf{H}_\lambda^{k_b}(\mathsf{L})$ does not depend on the λ parameter. Only the norm does, and it varies continuously with λ.*

Proof. Recall from Definition 3.11 that $\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C^{-1}$ exists for small enough $h > 0$, and by the fact that $I(A,\lambda)$ is an algebra homomorphism, we get

$$I(\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C, \lambda)^{-1} = I(\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C^{-1}, \lambda').$$

It suffices to check that the operators

$$\begin{aligned}&I(\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C, \lambda) I(\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C^{-1}, \lambda'),\\ &I(\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C^{-1}, \lambda) I(\mathrm{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C, \lambda')\end{aligned}$$

are bounded on L^2 for $\lambda, \lambda' \in \mathbb{C}$ (and depend continuously on λ, λ'). Since they are pseudo-differential and their symbols have the same asymptotics for large η (see (3.8)), this is a consequence of usual pseudo-differential arguments. ∎

Besides the indicial family, we will need the following inverse construction.

Definition-Proposition 3.21. *Let $\rho_0 < \rho_1$, and for $\operatorname{Re}\lambda \in]\rho_0, \rho_1[$, let $\lambda \mapsto I(\lambda)$ be a holomorphic family of continuous operators $I(\lambda): C^\infty(\mathsf{L}) \to \mathcal{D}'(\mathsf{L})$. Also assume that it is tempered, i.e. $\|I(\lambda)\|_{H^N(\mathsf{L}) \to H^{-N}(\mathsf{L})} \le C\langle \operatorname{Im}\lambda\rangle^N$, with $C, N > 0$ depending continuously on $\operatorname{Re}\lambda$.*

Then, for $\rho \in]\rho_0, \rho_1[$, there is a continuous operator

$$A(I, \rho): C_c^\infty(\mathbb{R} \times \mathsf{L}) \to \mathcal{D}'(\mathbb{R} \times \mathsf{L})$$

with kernel given by

$$e^{\rho(r-r')/h} \mathscr{F}_h^{-1}(I(\rho + i\,\cdot))(r - r').$$

The resulting operator does not depend on ρ, so we denote it by $A(I)$. In the case that $I(\lambda) = I(A, \lambda)$ for $A \in \Psi_{b,C}$ or some A as in Lemma 3.16, we get that $A(I) = A$.

Further, for two families $I_1(\lambda), I_2(\lambda)$ of operators holomorphic on $\operatorname{Re}(\lambda) \in]\rho_1, \rho_2[$ fulfilling

$$\|I_1(\rho + iw)\|_{H^{k_1}(\mathsf{L}) \to H^{k_2}(\mathsf{L})} \le C\langle w\rangle^{N_1}, \quad \|I_2(\rho + iw)\|_{H^{k_2}(\mathsf{L}) \to H^{k_3}(\mathsf{L})} \le C\langle w\rangle^{N_2},$$

one has

$$A(I_2 I_1) = A(I_2)A(I_1).$$

Proof. Let us first check that the given kernel defines a well defined continuous operator $A(I, \rho): C_c^\infty(\mathbb{R} \times \mathsf{L}) \to \mathcal{D}'(\mathbb{R} \times \mathsf{L})$. The expression of the kernel means that, for $f_1, f_2 \in C_c^\infty(\mathsf{L})$, $g_1, g_2 \in C_c^\infty(\mathbb{R})$, one has

$$\begin{aligned}
&\langle f_1 g_1, A(I, \rho) f_2 g_2\rangle_{C_c^\infty(\mathbb{R}\times\mathsf{L}), \mathcal{D}'(\mathbb{R}\times\mathsf{L})} \\
&\quad := \frac{1}{2\pi h} \int e^{iw(r-r')/h} e^{\rho(r-r')/h} g_1(r) g_2(r') \\
&\qquad\qquad \langle f_1, I(\rho + iw) f_2\rangle_{C_c^\infty(\mathsf{L}), \mathcal{D}'(\mathsf{L})}\, dr\, dr'\, dw, \\
&\quad = \int \mathscr{F}_h^{-1}(e^{\rho\bullet/h} g_1)(w) \mathscr{F}_h(e^{-\rho\bullet/h} g_2)(w) \\
&\qquad\qquad \langle f_1, I(\rho + iw) f_2\rangle_{C_c^\infty(\mathsf{L}), \mathcal{D}'(\mathsf{L})}\, dw.
\end{aligned} \tag{3.10}$$

As the Fourier transform of compactly supported functions extends holomorphically to $\mathbb{C}$, the independence from ρ follows from Cauchy's theorem. The fact that $A(I, \rho)$ can be extended continuously to arbitrary (non-product) elements of $C_c^\infty(\mathbb{R} \times \mathsf{L})$ can be seen by letting any of the $f_1, f_2 \in C^\infty(\mathsf{L})$ or $g_1, g_2 \in C_c^\infty(\mathbb{R})$ to zero in the corresponding topologies. Then the temperedness assumption of $I(\lambda)$ implies that (3.10) goes to zero. As to why $A(I(A)) = A$, this follows by Fourier inversion after plugging in the definitions (3.5) and (3.10) and a few lines of straightforward calculations. Also the multiplicativity $A(I_2 I_1) = A(I_2)A(I_1)$ follows from a straightforward calculation which is completely analogous to the calculations needed to show that the Fourier transform of a product is the convolution of Fourier transforms. ∎

Next, we have the following lemma on boundedness.

Lemma 3.22. *Consider $I(\lambda)$ as in Definition-Proposition 3.21. We have the following identities. For $h\rho \in]\rho_0, \rho_1[$ and two weights $k_b = \gamma m_b + N$ and $\ell_b = \gamma' m_b + N$,*

$$\|A(I, h\rho)\|_{\mathcal{H}^{k_b}_{b,\rho} \to \mathcal{H}^{\ell_b}_{b,\rho}} = \sup_{\operatorname{Re}\lambda = h\rho} \|I(\lambda)\|_{\mathsf{H}^{k_b}_{\lambda} \to \mathsf{H}^{\ell_b}_{\lambda}}. \tag{3.11}$$

Proof. The first step is to reduce to the case $\rho = 0$. Since

$$\|A(I)\|_{\mathcal{H}^{k_b}_{b,\rho} \to \mathcal{H}^{\ell_b}_{b,\rho}} = \|e^{-\rho r} A(I) e^{\rho r}\|_{\mathcal{H}^{k_b}_{b,0} \to \mathcal{H}^{\ell_b}_{b,0}},$$

the action of $A(I)$ is equivalent to the action of A_ρ on $\mathcal{H}^{k_b}_{b,0} \to \mathcal{H}^{\ell_b}_{b,0}$, where A_ρ is an operator whose kernel is that of A multiplied by $e^{\rho(r'-r)}$, i.e. it is $\mathscr{F}_h^{-1}(I(h\rho + i\,\cdot))(r - r')$. Let $I_\rho(\lambda) = I(\lambda + h\rho)$. We deduce that the action of $A(I, h\rho)$ is equivalent to the action of $A(I_\rho, 0)$ on $\mathcal{H}^{k_b}_{b,0} \to \mathcal{H}^{\ell_b}_{b,0}$. Next, we conjugate to an operator $L^2 \to L^2$. By definition,

$$\|A(I_\rho)\|_{\mathcal{H}^{k_b}_{b,0} \to \mathcal{H}^{\ell_b}_{b,0}} = \|\operatorname{Op}[e^{-\gamma' G_b}\langle\xi\rangle^{-N'}]_C^{-1} A(I_\rho) \operatorname{Op}[e^{-\gamma G_b}\langle\xi\rangle^{-N}]_C\|_{L^2 \to L^2},$$

$$\|I_\rho(\lambda)\|_{\mathsf{H}^{k_b}_{\lambda} \to \mathsf{H}^{\ell_b}_{\lambda}} = \|I(\operatorname{Op}(e^{-\gamma' G_b}\langle\xi\rangle^{-N'}), \lambda)^{-1} I(\lambda) I(\operatorname{Op}(e^{-\gamma G_b}\langle\xi\rangle^{-N}), \lambda)\|_{L^2 \to L^2}.$$

Now both maps $A \to I(A)$ and $I \to A(I)$ are multiplicative. We deduce that it suffices to prove the lemma in the case that $k_b = \ell_b = 0$.

After this additional reduction, we are left to prove that

$$\|A(I)\|_{L^2 \to L^2} = \sup_{\operatorname{Re}\lambda = 0} \|I(\lambda)\|_{L^2 \to L^2}.$$

This is just an avatar of the Plancherel formula. By the definition of $A(I)$ (see (3.10)), one has, for $f_1, f_2 \in C_c^\infty(\mathsf{L})$, $g_1, g_2 \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}&\langle f_1 g_1, A(I,\rho) f_2 g_2\rangle_{L^2(\mathbb{R}\times\mathsf{L})}\\ &\quad = \frac{1}{2\pi h} \int \overline{\mathscr{F}_h(g_1)(w)} \langle f_1, I(\rho + iw) f_2\rangle_{L^2(\mathsf{L})} \mathscr{F}_h(g_2)(w)\, dw,\end{aligned}$$

and from this, formula (3.11) can be read off directly. ∎

Finally, we get the following proposition.

Proposition 3.23. *Let $\mathbf{X}_b \in \Psi^1_C(\mathbb{R} \times \mathsf{L})$, and let k_b be a weight. Then each $I(\mathbf{X}_b, \lambda)$ has a unique extension as a closed operator on $\mathsf{H}^{k_b}_{\lambda}(\mathsf{L})$. The domain, as a subset of $\mathsf{H}^{k_b}_{\lambda}(\mathsf{L}) = \mathsf{H}^{k_b}_{0}(\mathsf{L}) \subset \mathcal{D}'(\mathsf{L})$, does not depend on λ.*

Proof. Since $\mathbf{X}_b$ is of order 1, $I(\mathbf{X}_b, \lambda)$ and $I(\mathbf{X}_b, \lambda')$ differ by an order 0 operator, which acts boundedly on each $\mathsf{H}^{k_b}_{\lambda}(\mathsf{L})$. So it suffices to check the case $\lambda = 0$. The operator $I(\mathbf{X}_b, 0)\colon C^\infty(\mathsf{L}) \subset \mathsf{H}^{k_b}_0(\mathsf{L}) \to \mathsf{H}^{k_b}_0(\mathsf{L})$ is unitarily equivalent to the operator

$$\begin{aligned}W = {}& I\big(\operatorname{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C, 0\big)^{-1} I(\mathbf{X}_b, 0)\\ & I\big(\operatorname{Op}^b(e^{-\gamma G_b}\langle\xi\rangle^{-N})_C, 0\big)\colon C^\infty(\mathsf{L}) \subset L^2(\mathsf{L}) \to L^2(\mathsf{L}),\end{aligned}$$

and W is a PDO of order one in L. Now the uniqueness of the closed extensions follows from the proof of [20, Lemma A.1]. ∎

As a consequence, the family $I(\mathbf{X}_b, \lambda)$ is a type (A) family so that we can apply the results from [33].

3.4. Fredholm indicial families

We now come back to admissible triples and will prove that their indicial families are Fredholm.

Lemma 3.24. *Assume* $\mathrm{Re}(s) > 1 + C_\delta + C(h^{-1}|\mathrm{Re}\,\lambda| + |N|) - \gamma$ *and* $|\mathrm{Im}\, s| \le h^{-1/2}$. *Then* $I(\mathbf{X}_b - Q_b - hs, \lambda)$ *is invertible with norm* $\mathcal{O}(1/h)$ *on* $\mathsf{H}^{\gamma m_b + N}_\lambda$, *uniformly in* λ.

Additionally, if either

$$\begin{cases} |\mathrm{Im}\, s| \le h^{-1/2}, \\ \mathrm{Re}(s) > 1 + C_\delta + C(h^{-1}|\mathrm{Re}\,\lambda| + |N|) - \gamma, \\ |\mathrm{Im}\,\lambda| > 4\delta', \end{cases} \tag{3.12}$$

or

$$\mathrm{Re}(s) > C(1 + \gamma + h^{-1}|\mathrm{Re}\,\lambda| + |N|),$$

then $I(\mathbf{X}_b - hs, \lambda)$ *is also invertible with norm* $\mathcal{O}(1/h)$ *on* $\mathsf{H}^{\gamma m_b + N}_\lambda$, *uniformly in* λ.

Proof. We start with the invertibility of $I(\mathbf{X}_b - Q_b - hs, \lambda)$. By Lemma 3.12, we deduce that $(\mathbf{X}_b - Q_b - hs)^{-1}$ is a well defined convolution operator for s in the announced domain. It furthermore fulfils all requirements of Definition 3.16, and thus we conclude that $I((\mathbf{X}_b - Q_b - hs)^{-1}, \lambda)$ is well defined. By the multiplicativity of $I(\cdot, \lambda)$, we conclude

$$I((\mathbf{X}_b - Q_b - hs), \lambda)^{-1} = I((\mathbf{X}_b - Q_b - hs)^{-1}, \lambda),$$

and Lemma 3.22 implies that it is $\mathcal{O}(1/h)$ uniformly in λ.

Now we turn to the case of $I(\mathbf{X}_b - hs, \lambda)$, in the region that $I(\mathbf{X}_b - Q_b - hs, \lambda)$ is invertible. First, we study $I(Q_b, \lambda)$. Since Q_b is microsupported for $|\xi| < 3\delta'$, we can use Lemma 3.17 and equation (3.7) to deduce that, when $|\mathrm{Im}\,\lambda| > 4\delta'$, $I(Q_b, \lambda) = \mathcal{O}(h^\infty)$ in $\Psi^{-\infty}$ uniformly in $\mathrm{Im}\,\lambda$ and locally uniformly in $\mathrm{Re}\,\lambda$. This implies that $I(\mathbf{X}_b - hs, \lambda)$ is invertible because

$$\begin{aligned} I(\mathbf{X}_b - hs, \lambda) I(\mathbf{X}_b - Q_b - hs, \lambda)^{-1} &= \mathbb{1} + I(Q_b, \lambda) I(\mathbf{X}_b - Q_b - hs, \lambda)^{-1} \\ &= \mathbb{1} + \mathcal{O}(h^\infty) \end{aligned} \tag{3.13}$$

(the remainder being bounded on the relevant spaces).

Finally, when $\mathrm{Re}(s) > C(1 + \gamma + h^{-1}|\mathrm{Re}\,\lambda| + |N|)$, recall formula (3.4) (removing the Q_b part). We deduce that the sharp Gårding inequality applies to show that $I(\mathbf{X} - hs, \lambda)$ is invertible with norm $\mathcal{O}(1/h)$ uniformly in λ, provided C is large enough (as in the proof of Lemma 2.9). ■

Now we get to the aim of this section. Recall from Definition 3.9 of $\mathbf{X}_b$ and Definition 3.13 that $I(\mathbf{X}_b - hs, h\lambda) = h(P_\lambda - s)$, where P_λ is an h-independent holomorphic family of differential operators on L. By Lemma 3.24, we know that

$$I(\mathbf{X}_b - hs, h\lambda)^{-1} \colon L^2(\mathsf{L}) \to L^2(\mathsf{L})$$

is well defined and holomorphic on $\{\mathrm{Re}(s) > C(1 + |\mathrm{Re}\,\lambda|)\} \subset \mathbb{C}^2$.

Proposition 3.25. *We have meromorphic extension of* $I(\mathbf{X}_b - hs, h\lambda)^{-1} \colon L^2(\mathsf{L}) \to L^2(\mathsf{L})$ *to* $\mathbb{C}^2$ *as operators* $I(\mathbf{X}_b - hs, h\lambda)^{-1} \colon C^\infty(\mathsf{L}) \to \mathcal{D}'(\mathsf{L})$.

Proof. All the work has already been done in some sense, since formula (3.13) shows that, up to an invertible operator, $I(\mathbf{X}_b - hs, \lambda)$ can be written as $\mathbb{1} + K(\lambda, s)$, where K is a holomorphic family of compact operators (recall that $I(Q_b, \lambda) \in \Psi^{-\infty}(\mathsf{F})$). The statement then follows from analytic Fredholm theory. ■

3.5. Effective continuation

In this last subsection of Section 3, we want to establish a meromorphic continuation of $(\mathbf{X}_b - hs)^{-1}$.

Before going on with the proof, let us come back to the convolution operator on the real line $T_f \colon g \mapsto f * g$, with $f \in \mathcal{D}'(\mathbb{R})$ compactly supported. In the language above, $I(T_f, h\lambda) = \hat{f}(-i\lambda)$. Since f is compactly supported, $\hat{f}$ is an entire function and acts by multiplication on the whole of $\mathbb{C}$. Given $s \in \mathbb{C}$, the function $(\hat{f} - s)^{-1}$ is a meromorphic function. So one can define, for $g \in C_c^\infty(\mathbb{R})$ and $\rho_0 \in \mathbb{R}$,

$$R_f(\rho_0, s)g(x) := \frac{1}{2\pi i} \int_{\mathrm{Re}\,\lambda = \rho_0} e^{\lambda x} \frac{\hat{g}(-i\lambda)}{\hat{f}(-i\lambda) - s}\, d\lambda.$$

Note that, in the general notation from Definition-Proposition 3.21, we can identify after setting $h = 1$, $R_f(\rho_0, s) = A((I(T_f, \lambda) - s)^{-1}, \rho)$. One finds that $(T_f - s)R_f(0, s) = \mathbb{1}$ when s is not in the closure of $\hat{f}(\mathbb{R})$. By Cauchy's theorem, $R_f(\rho_0, s) = R_f(\rho_1, s)$ when $\hat{f}(-i\lambda)$ does not take the value s in the region $\mathrm{Re}\,\lambda \in [\rho_0, \rho_1]$. Now consider $\lambda_1 \in \mathbb{C}$ such that $\hat{f}(-i\lambda_1) = s$, $\hat{f}'(-i\lambda_1) \neq 0$, and $\hat{f}(-i\,\cdot)$ does not take the value s another time in a region $\mathrm{Re}\,\lambda \in]\mathrm{Re}\,\lambda_1 - \epsilon, \mathrm{Re}\,\lambda_1 + \epsilon[$ for some $\epsilon > 0$. Another application of Cauchy's theorem gives

$$\big(R_f(\mathrm{Re}(\lambda_1) + \epsilon, s) - R_f(\mathrm{Re}(\lambda_1) - \epsilon, s)\big)g(x) = -ie^{\lambda_1 x} \frac{\hat{g}(-i\lambda_1)}{\hat{f}'(-i\lambda_1)}. \tag{3.14}$$

Using this argument, one can hope to obtain a meromorphic continuation of the resolvent of a translation invariant operator $\mathbf{X}_b$ from the meromorphicity of the resolvent of its indicial family $I(\mathbf{X}_b, \lambda)$. This is done by replacing "multiplication" by "action in the ζ variable". This heuristics is at the core of Melrose's b-calculus and will be pursued here. As one can expect, just as it is crucial to follow the solutions of $\hat{f}(i\lambda) = s$ for the convolution by f, we have to follow the (λ, s)'s such that $I(\mathbf{X}_b - hs, h\lambda)$ is not invertible.

Definition 3.26. Given an admissible triple $\mathbf{X}_b$, G_b, Q_b, let us consider the meromorphically continued family of indicial operators $I(\mathbf{X}_b - hs, h\lambda)^{-1}$ from Proposition 3.25.

(1) For fixed $s \in \mathbb{C}$, the set of $\lambda \in \mathbb{C}$ such that $I(\mathbf{X}_b - hs, h\lambda)^{-1}$ is singular is the set of (s-)*indicial roots* of $\mathbf{X}_b$. It will be denoted by $\mathrm{Spec_b}(s)$, and by construction, it is independent of h.

(2) If there are $a_k \in \mathbb{R} \setminus \{0\}$, $|a_k| \leq C$ uniformly in k, $b_k \in \mathbb{C}$ such that
$$\mathrm{Spec_b}(s) = \{a_k s + b_k\},$$
then we say that the *roots are affine*.

(3) For affine roots, say that a root $\lambda_k(s) = a_k s + b_k$ is positive if $a_k > 0$ (resp. negative if $a_k < 0$), and we denote the set of positive/negative roots by $\mathrm{Spec_b}^{\pm}(s)$.

(4) For any $-\infty \leq \rho < \rho' \leq \infty$, we define
$$\mathrm{Spec_b}^{(\pm)}(s, \rho, \rho') := \{\lambda \in \mathrm{Spec_b}^{(\pm)}(s) \mid \rho < \operatorname{Re}\lambda < \rho'\}.$$
In particular, we call elements of $\mathrm{Spec_b}^{+}(s, -\infty, 0)$ (resp. $\mathrm{Spec_b}^{-}(s, 0, +\infty)$) the positive (resp. negative) visible roots.

By the analytic Fredholm theorem, the set
$$\mathfrak{C} := \{(\lambda, s) \mid I(\mathbf{X}_b - hs, h\lambda) \text{ is not invertible}\}$$
is a complex analytic submanifold of $\mathbb{C}^2$, possibly with algebraic singularities – corresponding to intersection of indicial roots. The set $\mathrm{Spec_b}(s)$ is the intersection of $\mathfrak{C}$ with $\{(\lambda, s) \mid \lambda \in \mathbb{C}\}$. From Proposition 3.25, we deduce that the set of roots depends neither on the choice of Q_b nor on that of G_b. From now on, we work under the assumption that all indicial roots are affine. (This implies in particular that there are no algebraic singularities in $\mathfrak{C}$.)

Example 3.27. In Section 5, we will be able to explicitly compute the indicial roots for the geodesic flow vector field (and even for admissible lifts in the sense of Definition 1.4). In the scalar case, we get (see Proposition 5.5)
$$\mathrm{Spec_b}(s) = \left\{\pm\left(s + \left(\frac{d}{2} + n\right)\right),\, n \in \mathbf{N}\right\}.$$

Conjecture 3.28. *The roots of an admissible triple are always affine, and for* $\operatorname{Re} s = 0$*, no root is on the imaginary axis.*

The next theorem is the technical heart of our article. In order to formulate it, we introduce
$$\rho_{\max}(s) := \max\{0\} \cup \{|\operatorname{Re}\lambda| \mid \lambda \in \mathrm{Spec_b}^{+}(s, -\infty, 0) \text{ or } \lambda \in \mathrm{Spec_b}^{-}(s, 0, \infty)\}, \quad (3.15)$$
which encodes the maximal real part of the visible indicial roots. Note that, under the assumption that the roots are affine, we deduce that $\rho_{\max}(s)$ is continuous and depends only on $\operatorname{Re}(s)$. Furthermore, $\{\tau \mid \rho(\tau) \neq 0\}$ is a semi-bounded interval $]-\infty, \tau_0[$ on which $\rho_{\max}$ is strictly decreasing.

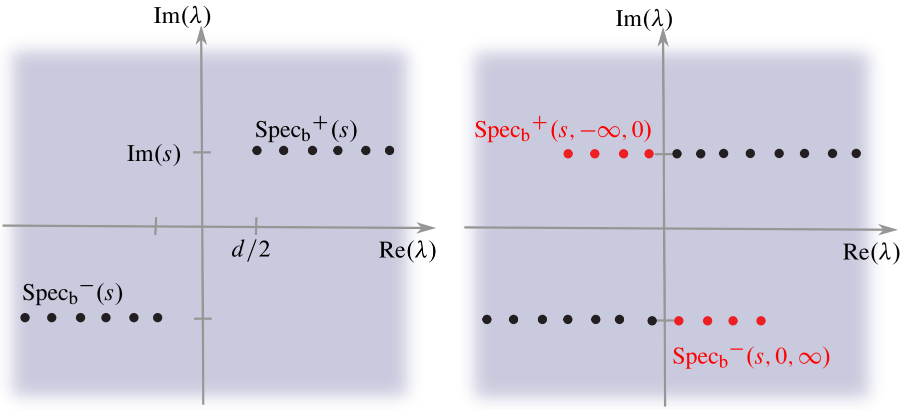

Fig. 3. Indicial roots for the geodesic flow on a $(d+1)$-dimensional cusp. On the left, the situation is depicted for $\mathrm{Re}(s) = 0$ and on the right for negative $\mathrm{Re}(s) = -(d/2 + 4.5)$. The visible positive and negative roots that have crossed the imaginary axis are marked in red.

Theorem 2. *Assume that, for some h-differential operator $\mathbf{X}_b$, there exist Q_b and G_b such that $(\mathbf{X}_b, G_b, Q_b)$ form an admissible triple with affine roots. Then the inverse $(\mathbf{X}_b - hs)^{-1}$, defined as a bounded operator on $L^2(\mathbb{R} \times \mathsf{L})$ for $\mathrm{Re}(s) > C$ for some constant $C > 0$, has a meromorphic extension $\mathbf{R}^{\mathbf{X}_b}(s)$ to $\mathbb{C}$, as an operator mapping $C_c^\infty(\mathbb{R} \times \mathsf{L})$ to $\mathcal{D}'(\mathbb{R} \times \mathsf{L})$.*

Additionally, given any $\tau, N \in \mathbb{R}$, $\gamma > 1 + C_\delta + C(|\rho_{\max}(\tau)| + |N|) - \tau$ (with the constants of Lemma 3.12), then $\mathbf{R}^{\mathbf{X}_b}(s)$ is a meromorphic family of bounded operators

$$\mathbf{R}^{\mathbf{X}_b}(s) \colon e^{-\rho_{\max}(\tau)\langle r\rangle} \mathcal{H}_{b,0}^{\gamma m_b + N} \to e^{\rho_{\max}(\tau)\langle r\rangle} \mathcal{H}_{b,0}^{\gamma m_b + N} \tag{3.16}$$

on the domain $\mathrm{Re}(s) > \tau$ and $|\mathrm{Im}\, s| \le h^{-1/2}$. At the eventual poles, the order is finite, and the rank of the Laurent expansion is also finite.

The remainder of this section is devoted to the proof of Theorem 2. We start with some observations. As a direct consequence of Lemma 3.24, we get the following lemma.

Lemma 3.29. *Let $\mathrm{Spec_b}^{\pm}(s) = \{a_{k,\pm}s + b_{k,\pm}\}$; then there is a constant $C \in \mathbb{R}$ such that $\pm\,\mathrm{Re}(b_{k,\pm}) > C$. Furthermore, for all $R > 0$,*

$$\sup\{|\mathrm{Im}\, b_k| : |\mathrm{Re}(b_k)| < R\} < \infty$$

Proof. For the first statement, we recall from Lemma 3.24 that there is a constant C such that, for $\mathrm{Re}(s) > C$, there are no indicial roots with $\mathrm{Re}(\lambda) = 0$. Consequently, all positive indicial roots satisfy $\mathrm{Re}(a_k s + b_k) > 0$ if $\mathrm{Re}(s) > C$. By the assumption that a_k are uniformly bounded, the assertion follows for positive roots and is completely analogous for negative ones.

The second statement follows directly from considering (3.12) in the case $s = 0$. ∎

We call $\rho \in \mathbb{R}$ s-regular if $\mathrm{Spec}_{\mathrm{b}}(s) \cap (\rho + i\mathbb{R}) = \emptyset$. The above bounds on a_k and b_k imply that, for any $s \in \mathbb{C}$, the set of s-regular $\rho \in \mathbb{R}$ is open and dense. Furthermore, for an s-regular ρ, Lemma 3.24 implies that

$$I(\mathbf{X}_b - hs, h\rho + iw)^{-1} \colon \mathsf{H}^{\gamma m_b + N}_{h\rho + iw} \to \mathsf{H}^{\gamma m_b + N}_{h\rho + iw}$$

is uniformly bounded in $w \in \mathbb{R}$ with norm $O(1/h)$ provided that

$$\gamma > 1 + C_g + C(\rho + |N|) - \mathrm{Re}(s).$$

Thus we can define

$$\mathbf{R}^{\mathbf{X}_b}_{\rho}(s) := A(I(\mathbf{X}_b - hs, \lambda)^{-1}, h\rho) \colon \mathcal{H}^{\gamma m_b + N}_{b,\rho} \to \mathcal{H}^{\gamma m_b + N}_{b,\rho}$$

which is again bounded with norm $O(1/h)$. Indeed, one directly checks that

$$(\mathbf{X}_b - hs)\, \mathbf{R}^{\mathbf{X}_b}_{\rho}(s) = \mathbf{R}^{\mathbf{X}_b}_{\rho}(s)(\mathbf{X}_b - hs) = \mathbb{1}.$$

However, $\mathbf{R}^{\mathbf{X}_b}_{\rho}(s)$ depends strongly on the choice of ρ due to the fact that $I(\mathbf{X}_b - hs, \lambda)^{-1}$ has singularities, i.e. that there exist indicial roots. In order to understand the meromorphic continuation, one has examine what happens if indicial roots cross the integration contours.

In order to shorten the notation in the sequel, it is convenient to define

$$F \colon (s, \lambda) \mapsto h e^{\lambda(r - r')} I(\mathbf{X}_b - hs, h\lambda)^{-1},$$

seen as a meromorphic function on $\mathbb{C}^2$ taking values in convolution operators on $\mathbb{R} \times \mathsf{L}$. The h factor is actually chosen such that it becomes h-independent, and using the definition of $A(I)$ (Definition-Proposition 3.21), we write

$$\mathbf{R}^{\mathbf{X}_b}_{\rho}(s) = \frac{1}{2\pi i h} \int_{\mathrm{Re}(\lambda) = \rho} F(s, \lambda)\, d\lambda. \tag{3.17}$$

When freezing the s variable, the poles of $F(s, \cdot)$ are precisely $\mathrm{Spec}_{\mathrm{b}}(s)$. We can integrate F over a small closed curve γ around a pole λ_0, enclosing only λ_0, and obtain its residue $\mathrm{Res}(F(s, \cdot), \lambda_0)$ in the λ variable. With this notation, we can state an equivalent to equation (3.14).

Lemma 3.30. *Let $-\infty < \rho < \rho' < \infty$ be s-regular for some $s \in \mathbb{C}$. Then* $\mathrm{Spec}_{\mathrm{b}}(s, \rho, \rho')$ *is finite and*

$$\mathbf{R}^{\mathbf{X}_b}_{\rho'}(s) - \mathbf{R}^{\mathbf{X}_b}_{\rho}(s) = h^{-1} \sum_{\lambda \in \mathrm{Spec}_{\mathrm{b}}(s, \rho, \rho')} \mathrm{Res}(F(s, \cdot), \lambda).$$

Proof. That the sets are finite follows from the uniform estimates on a_k, b_k. The identity is a consequence of Cauchy's theorem and (3.17). ∎

The following lemma is crucial in the proof.

Lemma 3.31. *Define* $B(s,\lambda) := h^{-1}\operatorname{Res}(F(s,\cdot),\lambda)$ *which is by definition a convolution operator on* $\mathbb{R}\times L$. *Consider a parametrized indicial root* $\lambda_k(s) = a_k s + b_k$ *and Then the map*

$$s \mapsto B(s,\lambda_k(s))$$

is a meromorphic function of s, *and the set of poles is contained in the set of* s*'s such that* $\lambda_k(s)$ *crosses another root.*

Proof. Since we already know that we can parametrize the roots without algebraic singularities – in the words of Kato, there is no branching point – this is a direct consequence of [33, Theorem 1.8, p. 70]. ∎

Now we can come back to the proof of our theorem.

Proof of Theorem 2. First, we focus on the meromorphic continuation of the Schwartz kernel of the resolvent. Recall from Lemma 3.24 that there is C such that $I(\mathbf{X}_b - hs, i\xi)$ is invertible for $\operatorname{Re}(s) > C$, and in this half plane, we define

$$\mathbf{R}^{\mathbf{X}_b}(s) = A\big(I(\mathbf{X}_b - hs,\lambda)^{-1},0\big).$$

If $\rho_1 < 0 < \rho_2$ are such that $\{\operatorname{Re}\lambda \in [\rho_1,\rho_2]\}$ does not intersect $\operatorname{Spec_b}(s)$, we deduce that $\mathbf{R}^{\mathbf{X}_b}(s)$ is bounded on all spaces $\mathscr{H}^{k_b}_{b,\rho}$ for $\rho \in [\rho_1,\rho_2]$, given that the weight k_b is large enough.

We want to construct a meromorphic continuation of $\mathbf{R}^{\mathbf{X}_b}(s)$ to all $\mathbb{C}$, and therefore, we have to take care of the indicial roots that cross the contour at $\operatorname{Re}(\lambda) = 0$. We define the set of *positive (resp. negative) visible roots at* s as $\operatorname{Spec_b}^+(s,-\infty,0)$ and $\operatorname{Spec_b}^-(s,0,\infty)$, respectively (see Figure 3 for the case of the geodesic flow for cusps).

By the uniform bounds on a_k, b_k, we deduce that, for any $s \in \mathbb{C}$, there are finitely many visible roots.

Let $\mathscr{U}$ be the set of $s \in \mathbb{C}$ such that 0 is s-regular, i.e. $\operatorname{Spec_b}(s) \cap i\mathbb{R} = \emptyset$. For $s \in \mathscr{U}$, we set

$$\mathbf{R}^{\mathbf{X}_b}_{\mathscr{U}}(s) := \mathbf{R}^{\mathbf{X}_b}_0(s) - \sum_{\lambda\in\operatorname{Spec_b}^+(s,-\infty,0)} B(s,\lambda) + \sum_{\lambda\in\operatorname{Spec_b}^-(s,0,\infty)} B(s,\lambda). \tag{3.18}$$

As $\mathbf{R}^{\mathbf{X}_b}_0(s)$ is holomorphic on any connected component of $\mathscr{U}$ and as $B(s,\lambda_k(s))$ are meromorphic by Lemma 3.31, this defines a meromorphic family on $\mathscr{U}$. It remains to prove that we can patch the different connected components of $\mathscr{U}$ (which are vertical strips because the roots are affine) together.

Therefore, take s_0 such that 0 is not s_0-regular; we consider $\rho < 0 < \rho'$ small enough such that $\operatorname{Spec_b}(s_0,\rho,\rho') \subset i\mathbb{R}$. Then, for s in a small vertical strip D around s_0, ρ and ρ' are still s-regular. For $s \in D$, we define

$$\begin{aligned}\mathbf{R}^{\mathbf{X}_b}_D(s) &:= \mathbf{R}^{\mathbf{X}_b}_\rho(s) + \sum_{\lambda\in\operatorname{Spec_b}^-(s,\rho,\infty)} B(s,\lambda) - \sum_{\lambda\in\operatorname{Spec_b}^+(s,-\infty,\rho)} B(s,\lambda)\\ &= \mathbf{R}^{\mathbf{X}_b}_{\rho'}(s) - \sum_{\lambda\in\operatorname{Spec_b}^+(s,-\infty,\rho')} B(s,\lambda) + \sum_{\lambda\in\operatorname{Spec_b}^-(s,\rho',\infty)} B(s,\lambda).\end{aligned}$$

The equality between the two expressions follows from Lemma 3.30. By construction and by Lemma 3.31, $\mathbf{R}_D^{\mathbf{X}_b}(s)$ defines a meromorphic operator on the strip D. It only remains to check that, on $\mathscr{U} \cap D$, both definitions of $\mathbf{R}_{\mathscr{U}}^{\mathbf{X}_b}(s)$ and $\mathbf{R}_{\mathscr{D}}^{\mathbf{X}_b}(s)$ coincide. But this is again a direct consequence of Lemma 3.30. We can thus patch the definitions to a globally meromorphic operator which we denote by $\mathbf{R}^{\mathbf{X}_b}(s)$.

Now we will determine on which functional spaces this meromorphic continuation acts. Let us focus on the structure of the residues B of F. If we assume that λ_0 is an s-indicial root and that, for $\epsilon > 0$, there are no other indicial roots in $\{\lambda, |\lambda - \lambda_0| \leq \epsilon\}$. In that case,

$$B(s, \lambda_0) = \frac{1}{2i\pi h} \int_{|\lambda-\lambda_0|=\epsilon} F(s, \lambda)\, d\lambda.$$

We will need the following lemma.

Lemma 3.32. *For $\epsilon > 0$ and $\rho \in \mathbb{R}$, we have the equality of spaces*

$$e^{\rho r+\epsilon\langle r\rangle} \operatorname{Op}^b(e^{-\gamma G}) H^N(\mathbb{R} \times \mathsf{L}) = \operatorname{Op}^b(e^{-\gamma G}) e^{\rho r+\epsilon\langle r\rangle} H^N(\mathbb{R} \times \mathsf{L}).$$

The corresponding norms are equivalent with $\mathcal{O}(1)$ constants as $h \to 0$.

Proof. It suffices to prove that both

$$e^{-\rho\langle r\rangle} \operatorname{Op}^b(e^{-\gamma G})^{-1} e^{\rho\langle r\rangle} \operatorname{Op}^b(e^{-\gamma G}), \quad \operatorname{Op}^b(e^{-\gamma G})^{-1} e^{-\rho\langle r\rangle} \operatorname{Op}^b(e^{-\gamma G}) e^{\rho\langle r\rangle}$$

are bounded on $L^2(\mathbb{R} \times \mathsf{L})$. However, since the quantization is properly supported, these operators are pseudo-differential with symbols in $1 + \mathcal{O}(hS^{-1^+})$. Hence they give rise to bounded operators on $L^2(\mathbb{R} \times \mathsf{L})$. ∎

With λ_0, ϵ as above, we deduce

$$\begin{aligned} \|B(s, \lambda_0)\|&_{e^{-2\epsilon\langle r\rangle}\mathcal{H}^{k_b}_{b,\operatorname{Re}\lambda_0} \to e^{2\epsilon\langle r\rangle}\mathcal{H}^{k_b}_{b,\operatorname{Re}\lambda_0}} \\ &\leq \frac{C_\epsilon}{h} \sup_{|\lambda-\lambda_0|=\epsilon} \|e^{-\operatorname{Re}\lambda_0 r - 2\epsilon\langle r\rangle} \operatorname{Op}(e^{-\gamma G - N\log\langle\xi\rangle})^{-1} F(s, \lambda) \\ &\qquad\qquad \operatorname{Op}(e^{-\gamma G - N\log\langle\xi\rangle}) e^{\operatorname{Re}\lambda_0 r - 2\epsilon\langle r\rangle}\|_{L^2\to L^2}. \end{aligned}$$

If W is the multiplication by $e^{-2\epsilon\langle r\rangle}$, the operator in the norm is the composition $WS_\lambda W$ so that S_λ is a convolution operator whose kernel takes the form

$$he^{(\lambda-\operatorname{Re}\lambda_0)(r-r')} I\big(\operatorname{Op}(e^{-\gamma G-N\log\langle\xi\rangle})^{-1}(\mathbf{X}_b - hs)\operatorname{Op}(e^{-\gamma G-N\log\langle\xi\rangle}), h\lambda\big)^{-1}.$$

Recall that $\operatorname{Re} s > 1 + C_\delta + C(|\rho_{\max}(s)| + |N|) - \gamma$ and $|\operatorname{Im} s| \leq h^{-1/2}$, so we can apply Lemma 3.24. In particular, the indicial operator in the last line is bounded on L^2 with norm $C(\epsilon)$. Since the kernel of S_λ decomposes as a product, we see directly that it is bounded from $e^{\operatorname{Re}(\lambda-\lambda_0)r-\epsilon\langle r\rangle} L^2$ to $e^{\operatorname{Re}(\lambda-\lambda_0)r+\epsilon\langle r\rangle} L^2$. But, since $|\lambda - \lambda_0| = \epsilon$, it is thus bounded from $e^{-2\epsilon\langle r\rangle} L^2$ to $e^{2\epsilon\langle r\rangle} L^2$ uniformly in λ. Finally, since W maps L^2 to $e^{-2\epsilon\langle r\rangle} L^2$ and $e^{2\epsilon\langle r\rangle} L^2$ to L^2, we obtain the desired result

$$\|B(s, \lambda_0)\|_{e^{-2\epsilon\langle r\rangle}\mathcal{H}^{k_b}_{b,\operatorname{Re}\lambda_0} \to e^{2\epsilon\langle r\rangle}\mathcal{H}^{k_b}_{b,\operatorname{Re}\lambda_0}} \leq \frac{C(s, \epsilon)}{h}, \tag{3.19}$$

for some $C(s,\epsilon) > 0$ locally uniform. On the other hand, using Lemma 3.22, we obtain that, when ρ is s-regular,

$$\|\mathbf{R}_\rho^{\mathbf{X}_b}(s)\|_{\mathcal{H}_{b,\rho}^{\gamma m_b}\to\mathcal{H}_{b,\rho}^{\gamma m_b}} \leq C_{s,\rho}. \tag{3.20}$$

If $\mathrm{Re}(s)$ is such that there are no visible roots, i.e.

$$\mathrm{Spec_b}^+(s,-\infty,0)\cup\mathrm{Spec_b}^-(s,0,\infty)=\emptyset,$$

then the boundedness estimate (3.16) follows directly from (3.18) and (3.20).

Else, if $s\in\mathbb{C}$ with $\mathrm{Re}(s)>\tau$ such that there are visible roots, let us choose $\varepsilon>0$ such that

$$\max_{\lambda\in\mathrm{Spec_b}^+(s,-\infty,0)\cup\mathrm{Spec_b}^-(s,0,\infty)}|\mathrm{Re}(\lambda)|+2\varepsilon<\rho_{\max}(\tau).$$

Note that this is possible because we are in the case $\rho(\tau)>0$, and thus, as was discussed after (3.15), ρ is strictly monotonous. Now, combining equations (3.18), (3.19) and (3.20), we deduce that

$$\|\mathbf{R}^{\mathbf{X}_b}(s)\|_{e^{-\rho_{\max}(\tau)\langle r\rangle}\mathcal{H}_{b,-\rho_{\max}}^{k_b}\to e^{\rho_{\max}(\tau)\langle r\rangle}\mathcal{H}_{b,\rho_{\max}}^{k_b}}\leq C_{s,\rho}.$$

To obtain the boundedness for $s\in\mathbb{C}\setminus\mathcal{U}$, one can use similar arguments.

Consider a pole s of $\mathbf{R}^{\mathbf{X}_b}(s)$ corresponding to an indicial root crossing λ_0. From the considerations above, it follows that the Laurent expansion has its image contained in the direct sum of

$$e^{\lambda_0 r}H_0\oplus\cdots\oplus r^k e^{\lambda_0 r}H_k,$$

where $H_0,\ldots,H_k$ are finite-dimensional subspaces of $\mathsf{H}_{\lambda_0}^{k_b}(\mathsf{L})$, related to the images of the Laurent expansion of $I(\mathbf{X}_b-hs,\lambda)^{-1}$ around λ_0. In particular, this is finite-dimensional. ■

Note that, in the case of a geodesic flow, we will see in Section 5 that the resonant states of $\mathbf{X}_b$ coming from the indicial resolvent can be explicitly expressed by Dirac distributions and homogeneous distributions on the North and South Pole of $\mathsf{F}=\mathbb{S}^d$.

4. Black box formalism and main theorem

In this section, we introduce a black box formalism in the spirit of [57]. For the same reason as in Section 3, we work in a geometric setting that is more general than the admissible bundles $L\to S^*N$ from Definition 1.4. Again, this bigger generality comes without any additional effort in the proofs. Let us define the geometric setting of this section.

Definition 4.1. Let us consider a cusp $Z=[a,+\infty)\times\mathbb{R}^d/\Lambda$ and a product $Z\times\mathsf{F}$ with $(\mathsf{F},g_{\mathsf{F}})$ a compact connected Riemannian manifold. The product $(Z\times\mathsf{F},g_Z+g_{\mathsf{F}})$ is a *trivial fibred cusp*.

Definition 4.2. Let (M, g') be a complete connected Riemannian manifold, and assume that it can be decomposed as the union of a compact manifold M_0 and several ends $M_1, \dots, M_\kappa$ that are trivial fibred cusps. Then we say that M is an *admissible manifold*.

Observe that if M is admissible, then its curvature tensor is $\mathscr{C}^\infty$ bounded.

Definition 4.3. Let (M, g') be an admissible manifold. Let $L \to M$ be a vector bundle with Riemannian bundle metric $\|\cdot\|_L$ and compatible connection ∇. We say that L is a *general admissible bundle* if, over each cusp $Z_\ell \times \mathsf{F}$, for $y > \mathbf{a}$, L has a product structure $L_{|Z_\ell} \simeq Z_\ell \times \mathsf{L}_\ell$, where L_ℓ is a Riemannian bundle over F_ℓ.

Again, if L is general admissible, its curvature and derivatives are bounded.

Example 4.4. Let (N, g) be an admissible cusp manifold, and let $L \to M = SN \to N$ be an admissible bundle. Then $L \to M$ is a general admissible bundle and the fibre F is just the sphere $\mathbb{S}^d$.

Let $L \to M$ be a general admissible bundle, with κ cusps $Z_1, \dots, Z_\kappa$. Take $\mathsf{a} > \mathbf{a}$, and let

$$L^2_{\mathsf{a}}(M, L) = \left\{ f \in L^2(M, L) \;\middle|\; \int f_{|y>\mathsf{a}}\, d\theta = 0 \right\}.$$

We have the orthogonal decomposition

$$L^2(M, L) = L^2_{\mathsf{a}}(M, L) \oplus \left(\bigoplus_{\ell=1}^{\kappa} L^2(]\log \mathsf{a}, +\infty[\times \mathsf{L}_\ell, e^{-rd}\, dr\, d\zeta) \right). \tag{4.1}$$

In Section 3, we used the measure $dr\, d\zeta$ instead of $e^{-rd}\, dr\, d\zeta$. In particular,

$$L^2(e^{-rd}\, dr\, d\zeta) = e^{rd/2} L^2(dr\, d\zeta).$$

In equation (4.1), the first term will be regarded as a *black box* and the second one as the *free space*. In the black box, we will use the variable y (more appropriate for geometric purposes), and in the free space the r variable (more appropriate for analysis). In the case of elliptic operators, one can really isolate the black box because it can be embedded in another space where the relevant operator – mostly the Laplacian – has compact resolvent. However, in our case, since being uniformly hyperbolic is a global property, such surgery cannot be performed a priori. It is the fact that the flow is *exactly* translation invariant that will save us.

We can define extension and restriction operators. Let $\phi \in C^\infty(M, L)$. We let $\mathscr{P}^{\mathsf{a}}_\ell \phi$ be the function in $C^\infty([\log \mathsf{a}, +\infty[_r \times \mathsf{F}_\zeta, \mathsf{L})$ obtained by restriction to the cusp Z_ℓ and averaging in the θ variable. Conversely, let $\phi \in C^\infty_c(]\log \mathsf{a}, +\infty[_r \times \mathsf{F}_\zeta, \mathsf{L})$. We consider it as a function $\mathscr{E}^{\mathsf{a}}_\ell \phi$ supported in cusp Z_ℓ, not depending on θ. We have $\mathscr{P}^{\mathsf{a}}_\ell \mathscr{E}^{\mathsf{a}}_\ell = \mathbb{1}$. We extend these definitions to distributions by duality: for distributions $v \in \mathcal{D}'(M, L)$ and $u \in \mathcal{D}'([\log \mathsf{a}, +\infty[\times \mathsf{F}, \mathsf{L})$,

$$\langle \mathscr{E}^{\mathsf{a}}_\ell u, \phi \rangle := \langle u, \mathscr{P}^{\mathsf{a}}_\ell \phi \rangle \quad \text{and} \quad \langle \mathscr{P}^{\mathsf{a}}_\ell v, \phi \rangle := \langle v, \mathscr{E}^{\mathsf{a}}_\ell \phi \rangle.$$

Note that, after this extension, we can apply $\mathscr{E}^{\mathsf{a}}_\ell$ equally to $C^\infty([\log \mathsf{a}, \infty[)$ and the composition $\mathscr{E}^{\mathsf{a}}_\ell \mathscr{P}^{\mathsf{a}}_\ell$ is well defined. Given a function $\chi \in C^\infty([\log \mathsf{a}, +\infty[)$ that is constant near a, we can define the associated *black box multiplication operator* as the operator

$$\mathbf{B}(\chi) = \chi(\log \mathsf{a}) + \sum_{\ell} \mathscr{E}_\ell^{\mathsf{a}}(\chi(r) - \chi(\log \mathsf{a}))\mathscr{P}_\ell^{\mathsf{a}}. \tag{4.2}$$

(Here $\chi(r)$ is the multiplication operator.) Note that, with this definition, the operator $\mathbf{B}(\chi)$ acts on $L^2_{\mathsf{a}}(M, L)$ simply by multiplication with the constant $\chi(\log \mathsf{a})$ and on the free spaces $L^2(]\log \mathsf{a}, \infty[\times \mathsf{L}_\ell)$ as a multiplication operator with $\chi(r)$.

In this section, we will define a class of operators that preserve this structure, and review some of their properties. Then we will conclude on the meromorphic extension of the resolvent of admissible such operators.

4.1. The class of cusp-b-pseudors

Now that we have added some structure to our space $L^2(M, L)$, we need to determine a reasonable class of operators that will preserve the structure. First, consider a differential operator P that commutes with $y\partial_\theta$ and $y\partial_y$ in each cusp, for $y > \mathsf{a}$. It thus acts on the space of smooth functions supported in a cusp that do not depend θ. We denote by $P^0_{b,\ell}$ that restriction for each cusp Z_ℓ. Then we find that, for $i = 1, \dots, \kappa$, acting on $\mathcal{D}'(M, L)$,

$$\mathscr{P}_\ell^{\mathsf{a}} P = P^0_{b,\ell}\mathscr{P}_\ell^{\mathsf{a}}.$$

We also have the dual statement, acting on $C^\infty_c(]\mathsf{a}, +\infty[\times \mathsf{F}_\ell, \mathsf{L}_\ell)$,

$$\mathscr{E}_\ell^{\mathsf{a}} P^0_{b,\ell} = P\mathscr{E}_\ell^{\mathsf{a}}.$$

Since we want to use anisotropic spaces that can only be defined using *pseudo-differential* operators, we have to accept slightly different relations. Indeed, pseudo-differential operators cannot be *exactly* supported on the diagonal.

Definition 4.5 (Cusp-b-operators). Let A be an operator $C^\infty_c(M, L) \to \mathcal{D}'(M, L)$. We say that A is a *black box operator* with precision $C \geq 1$ at height a if, acting on $C^\infty_c(M, L)$,

$$\mathscr{P}_\ell^{C\mathsf{a}} A(\mathbb{1} - \mathscr{E}_\ell^{\mathsf{a}}\mathscr{P}_\ell^{\mathsf{a}}) = 0 \quad \text{for all } \ell = 1, \dots, \kappa, \tag{4.3}$$

and acting on $C^\infty_c(]\log(C\mathsf{a}), +\infty[\times \mathsf{F}, \mathsf{L})$,

$$(\mathbb{1} - \mathscr{E}_\ell^{\mathsf{a}}\mathscr{P}_\ell^{\mathsf{a}})A\mathscr{E}_\ell^{C\mathsf{a}} = 0 \quad \text{for all } \ell = 1, \dots, \kappa. \tag{4.4}$$

If additionally, for each $\ell = 1, \dots, \kappa$, $\mathscr{P}_\ell^{\mathsf{a}} A\mathscr{E}_\ell^{C\mathsf{a}}$ acts on $C^\infty_c(]\log(C\mathsf{a}), +\infty[\times \mathsf{F}, \mathsf{L})$ as the restriction of a translation invariant operator $A^0_{b,\ell}$ on sections of $\mathbb{R} \times \mathsf{L}$, that is supported for $|r - r'| \leq \log C$, we say that A is a *cusp-b-operator*.

We define

$$A_{b,\ell} := e^{-rd/2} A^0_{b,\ell} e^{rd/2},$$

which is again translation invariant. In this way, while $A^0_{b,\ell}$ acts naturally on

$$L^2(\mathbb{R} \times \mathsf{L}, e^{-rd}\, dr\, d\zeta),$$

$A_{b,\ell}$ acts on

$$L^2(\mathbb{R} \times \mathsf{L}, dr\, d\zeta).$$

Finally, if $A \in \Psi(M, L)$ is also a pseudo-differential operator, we say that A is a *cusp-b-pseudor* and write $A \in \Psi_{b,C}(M, L)$.

Example 4.6. In the case of the geodesic flow $M = S^*N$, the vector field of the geodesic flow X is a cusp-b-operator with precision 1. We also have

$$X^0_{b,\ell} = \cos\varphi \partial_r + \sin\varphi \partial_\varphi \quad \text{and} \quad X_{b,\ell} = \cos\varphi \partial_r + \frac{d}{2}\cos\varphi + \sin\varphi\partial_\varphi.$$

In what follows, the constant a will be fixed a priori, it is a geometric data of the problem, and we will mostly not mention it. Let us give a word of explanation. Condition (4.3) implies that if f has zero mean value in the θ variable in each cusp for $y > \mathsf{a}$, then the mean value of Af in the θ variable vanishes when $y > C\mathsf{a}$. Condition (4.4) is the dual version of the assumption: it means that if f was supported only in cusps for $y > C\mathsf{a}$ and did not depend on θ, then Af would be supported in $y > \mathsf{a}$ and also not depend on θ.

Proposition 4.7. *Let $A \in \Psi_{b,C}(M, L)$. Then, for $\ell = 1, \dots, \kappa$, the operator $A_{b,\ell}$ defined from A by Definition 4.5 is an element in $\Psi_{b,C}(\mathbb{R} \times \mathsf{L}_\ell)$ – see Definition 3.3.*

This follows directly from the definition. Recall from Section 1.1 that covectors decompose as $\xi = Y\,dy + J\,d\theta + \eta\,d\zeta$.

Definition 4.8. Let $\sigma \in S^0(M, L)$. Assume that, in each cusp, $\partial_\theta\sigma = 0$ for $y > \mathsf{a}$, and for $r > \log \mathsf{a}$, $\ell = 1, \dots, \kappa$, let

$$\sigma_{b,\ell}(r, \zeta; \lambda, \eta) := \sigma|_{Z_\ell}(e^r, \theta, \zeta; e^{-r}\lambda, J = 0, \eta).$$

Assume that $\sigma_{b,\ell}$ does not depend on r for $\ell = 1, \dots, \kappa$. Then we say that σ is a *b-symbol* of order 0 and write $\sigma \in S^0_b(M, L)$. Given a cusp-b-symbol m of order 0, we correspondingly define $S^m_b(M, L)$ the set of cusp-b-symbols of order m.

By a direct computation, one gets the following proposition.

Proposition 4.9. *For a general admissible bundle $L \to M \to N$, any $\sigma \in S(M, L)$ that is invariant under the action of local isometries $T_{\tau,\theta}$ (see equation (1.3)) in each cusp is a cusp-b-symbol.*

We also get the following lemma.

Lemma 4.10. *Let $\sigma \in S_b(M, L)$. Then, for $\ell = 1, \dots, \kappa$, we have $\sigma_{b,\ell} \in S_b(\mathbb{R} \times \mathsf{L})$ – see Definition-Proposition 3.7.*

Proof. From the considerations in Section A.1.1, we deduce that $\sigma_{b,\ell}$ satisfies usual symbol estimates on $\mathbb{R} \times \mathsf{L}$. The r-invariance follows from the definition. ∎

Let us consider $\sigma \in S_b(M, L)$ and the corresponding operator $\mathrm{Op}(\sigma)$. According to Proposition A.8, by adjusting the parameter $\mathsf{a} \geq \mathbf{a}$, we get that $\mathrm{Op}(\sigma)$ satisfies equation (4.4). It is not difficult to check that it also satisfies equation (4.3) for similar reasons. We now consider its restriction to functions f supported in the cusp Z_ℓ and not depending on θ. Actually, we want to compute directly $\{\mathrm{Op}(\sigma)\}_{b,\ell}$ instead of $\{\mathrm{Op}(\sigma)\}^0_{b,\ell}$. Thus we take a function of the form $e^{rd/2} f(r,\zeta)$ so that the action of $\mathrm{Op}(\sigma)$ on $L^2(L)$ – with the measure $e^{-rd}\, dr\, d\zeta$ – will correspond to the action on $L^2(\mathbb{R}\times \mathsf{L}, dr\, d\zeta)$. By definition of the quantization – see equations (A.2) and (A.3) – and already replacing

$$(2\pi h)^{-d} \int e^{i\langle \theta-\theta', J\rangle/h}\, d\theta'$$

by $\delta_{J=0}$, we get, for $e^{-rd/2}\,\mathrm{Op}(\sigma) e^{rd/2} f$,

$$\frac{1}{(2\pi h)^{1+k}} \int \chi^{\mathrm{Op}}\Big(\log \frac{y}{y'}\Big) e^{i/h(\langle y-y', Y\rangle + \langle \zeta-\zeta', \eta\rangle)} \\ \sigma\Big|_{Z_\ell}\Big(\frac{y+y'}{2}, \frac{\zeta+\zeta'}{2}, Y, J=0, \eta\Big) f(y',\zeta') \sqrt{\frac{y}{y'}}\, dy'\, d\zeta'\, dY\, d\eta$$

(recall k is the dimension of F). We take the coordinate change

$$r = \log y \quad \text{and} \quad \lambda = (y+y')\frac{Y}{2}.$$

The volume form becomes

$$\frac{2e^{r'+(r-r')/2}}{e^r + e^{r'}}\, dr'\, d\lambda\, d\zeta'\, d\eta,$$

and the phase

$$\Phi(r,\lambda,\zeta,\eta) = \langle \zeta-\zeta', \eta\rangle + 2\lambda \tanh \frac{r-r'}{2}.$$

The symbol under the integral giving $\mathrm{Op}(\sigma) f$ is now in the form

$$\chi^{\mathrm{Op}}(r-r')\tilde{f}(r',\zeta')\sigma_{b,\ell}\Big(r + \log \frac{1+e^{r'-r}}{2}, \frac{\zeta+\zeta'}{2}, \lambda, \eta\Big),$$

where $\tilde{f} = \mathscr{P}_\ell f$. Since $\sigma_{b,\ell}$ does not depend on r, we deduce that

$$\{\mathrm{Op}(\sigma)\}_{b,\ell} \tilde{f}(r,\zeta) := \int e^{i/h \Phi(r,\lambda,\zeta,\eta)} \chi^{\mathrm{Op}}(r-r')\tilde{f}(r',\zeta') \\ \sigma_{b,\ell}\Big(\frac{\zeta+\zeta'}{2}, \lambda, \eta\Big) \frac{2e^{(r+r')/2}}{e^r+e^{r'}} \frac{dr'\, d\lambda\, d\zeta'\, d\eta}{(2\pi h)^{1+k}}.$$

Provided that the support of the cutoff χ_C chosen after equation (3.3) is slightly larger than the support of χ^{Op}, we can find a symbol $\widetilde{\sigma_{b,\ell}} \in S_b(\mathbb{R}\times\mathsf{L})$ such that

$$\{\mathrm{Op}(\sigma)\}_{b,\ell} = \mathrm{Op}^b(\widetilde{\sigma_{b,\ell}})_C.$$

We have proved the following proposition.

Proposition 4.11. *Let $\sigma \in S_b(M,L)$. Then $\mathrm{Op}(\sigma) \in \Psi_{b,C}(M,L)$, where $C > 1$ is a constant chosen in the construction of the quantization and there is a symbol $\widetilde{\sigma}_{b,\ell} \in S_b(M,L)$ such that $\{\mathrm{Op}(\sigma)\}_{b,\ell} = \mathrm{Op}^b(\widetilde{\sigma_{b,\ell}})_C$. When the height a at which σ starts being invariant varies, we can change the quantization and keep the same constant C.*

4.2. Meromorphic continuation of resolvents of admissible b-operators

In this section, we will need the following crucial compactness lemma.

Lemma 4.12. *Let $L \to M$ be a general admissible bundle as in Definition 4.3. Let* $\mathsf{a} > \mathbf{a}$ *and*

$$H^1_{\mathsf{a}}(M, L) := \{f \in H^1(M, L) \mid \mathscr{P}^{\mathsf{a}} f = 0\}.$$

This is a closed subspace of $H^1(M, L)$, and the injection $H^1_{\mathsf{a}}(M, L) \hookrightarrow L^2(M, L)$ is compact.

Proof. We can adapt the argument of Lax–Phillips [36]. Let $\chi \in C^\infty_c(\mathbb{R}, [0, 1])$ be equal to one in a neighbourhood of 0, and set $\chi_n(y) := \chi(y/n)$. Consider the multiplication operator $\chi_n(y)$ in each cusp Z_ℓ of M. From Rellich's theorem, the multiplication operator χ_n is compact for all n from $H^1_{\mathsf{a}}(M, L)$ to $L^2(M, L)$. Now assume that, as $n \to +\infty$, χ_n restricted to $H^1_{\mathsf{a}}(M, L)$ has the injection $H^1_{\mathsf{a}}(M, L) \hookrightarrow L^2(M, L)$ as norm limit. Then that injection has to be compact also.

To show that it is a norm limit, we have to show that, for $f \in H^1_{\mathsf{a}}(M, L)$,

$$\|f\|_{L^2(M,L), y>n} \le C_n \|f\|_{H^1(M,L)},$$

with a constant $C_n \to 0$ as $n \to +\infty$. We use the Poincaré inequality: consider a unimodular lattice $\Lambda \subset \mathbb{R}^d$ and $\mathbb{T}_\Lambda = \mathbb{R}^d/\Lambda$. For $\tilde{f} \in H^1(\mathbb{T}_\Lambda)$ with $\int \tilde{f} = 0$, we have

$$\|\tilde{f}\|_{L^2} \le C_\Lambda \|\nabla \tilde{f}\|_{L^2}.$$

Now, with κ the number of cusps,

$$\begin{aligned}
\|f\|^2_{L^2(M,L), y>n} &= \sum_{\ell=1}^{\kappa} \int_{y>n} \frac{dy\, d\zeta}{y^{d+1}} \|f(y, \zeta, \cdot)\|^2_{L^2(\mathbb{T}_{\Lambda_\ell})} \\
&\le C \sum_{\ell=1}^{\kappa} \int_{y>n} \frac{dy\, d\zeta}{y^{d+1}} \|\partial_\theta f(y, \zeta, \cdot)\|^2_{L^2(\mathbb{T}_{\Lambda_\ell})} \\
&\le C \frac{1}{n^2} \|f\|^2_{H^1(M,L)}. \qquad \blacksquare
\end{aligned}$$

We have a statement for general weights.

Definition-Proposition 4.13. *Pick a smooth function $r'(r)$ equal to* $\log \mathsf{a}$ *for $r \le$* $\log C\mathsf{a}$, *and equal to r when $r >$* $\log C^2\mathsf{a}$. *Then, given $\gamma, N, \rho \in \mathbb{R}$ and the corresponding black box multiplication operator* $\mathbf{B}(e^{\rho r'})$ *from* (4.2)*, we define*

$$H^{\gamma\mathbf{m}+N}_\rho(M, L) := \mathbf{B}(e^{\rho r'}) H^{\gamma\mathbf{m}+N}(M, L).$$

Let $\rho < \rho'$. Then the injection $H^{\gamma\mathbf{m}+N+1}_\rho(M, L) \hookrightarrow H^{\gamma\mathbf{m}+N}_{\rho'}(M, L)$ is compact.

Proof. From the choice of r' and pseudo-differential operator symbol calculus, we can reduce directly to the case of $\gamma = 0$ and $N = 0$. Applying $\mathbf{B}(e^{(\rho-\rho')r'})$, we can also reduce to the case $\rho < 0 = \rho'$. Then we can adapt the argument from before, adding a contribution from the zeroth Fourier mode that decays as $e^{\rho \log n}\|f\|_{L^2}$. $\blacksquare$

Definition 4.14. Let $L \to M$ be a general admissible bundle. Let $\mathcal{X}$ be a derivation on sections of L extending a vector field X on M. Also assume that $\mathbf{X} := h\mathcal{X} \in \Psi_b(M, L)$. Assume that the flow generated by X is uniformly hyperbolic and that we can construct escape functions $G \in S_b(M, L)$ for any $\delta > 0$ satisfying conclusions (i)–(iv) of Lemma 2.3 as well as the invariance properties from Lemma 2.5. Then we say that $\mathbf{X}$ is a *general admissible operator*. We denote by $E^{u,s}$ and $E^*_{u,s}$ the corresponding stable and unstable bundles.

Given a general admissible operator, the proofs of Propositions 2.10 and 2.11 apply, so we get a first parametrix $\mathscr{R}_Q(s) := (\mathbf{X} - Q - hs)^{-1}$ with norm $\mathcal{O}(h^{-1})$. Furthermore, from Definition 4.5 and Proposition 4.11, we deduce straightforwardly the following proposition.

Proposition 4.15. *Let $L \to M$ be a general admissible bundle, and let $\mathbf{X}$ a general admissible operator. Let $\delta > 0$. Let G be a corresponding escape function. Moreover, let $Q \in \Psi_b^{-\infty}(M, L)$ be microsupported in $|\xi| < 3R\delta$ and elliptic in $|\xi| < 2R\delta$ – as the Q used in Proposition 2.10. Then, for each ℓ, $(\mathbf{X}_{b,\ell}, G_{b,\ell}, Q_{b,\ell})$ is an admissible triple in the sense of Definition 3.9.*

Consequently, from Lemma 3.12 and Theorem 2, we deduce that

$$\mathbf{R}_{Q,\ell}(s) := (\mathbf{X}_{b,l} - Q_{b,\ell} - hs)^{-1}$$

and that $\mathbf{R}^{\mathbf{X}_{b,\ell}}(s)$ are analytic, respectively meromorphic families of operators on the appropriate anisotropic spaces. We now choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(r) = \begin{cases} 0 & \text{for } r < \log(C\mathsf{a}), \\ 1 & \text{for } r > \log(C^2\mathsf{a}) \end{cases}$$

and define

$$\mathscr{R}'_Q(s) := \mathscr{R}_Q(s) + \sum_\ell \mathscr{E}_\ell \chi [\mathbf{R}^{\mathbf{X}_{b,\ell}}(s) - \mathbf{R}_{Q,\ell}(s)] \chi \mathscr{P}_\ell.$$

Next, let us define, for $\tau \in \mathbb{R}$,

$$\rho_{\max}(\tau) := \sup_\ell \rho_{\max,\ell}(\tau).$$

Recall that $\rho_{\max,\ell}$ was defined in equation (3.15). Also keep in mind that weights are functions of the form $\mathbf{k} = \gamma\mathbf{m} + N$, and they are large when γ is large and so is $\gamma/|N|$.

Lemma 4.16. *Let $\tau < 0$, and let $\mathbf{k}$ be sufficiently large. Then, for*

$$\mathrm{Re}(s) > \tau \quad \textit{and} \quad |\mathrm{Im}(s)| \le h^{-1/2},$$

the operator family $\mathscr{R}'_Q(s)$ is a meromorphic family of bounded operators from $H^{\mathbf{k}}_{-\rho'_{\max}(\tau)}$ to $H^{\mathbf{k}}_{\rho'_{\max}(\tau)}$. Additionally, we can write

$$(\mathbf{X} - hs)\mathscr{R}'_Q(s) = \mathbb{1} + K(s),$$

where $K(s)$ is a meromorphic family of compact bounded operators on $H^{\mathbf{k}}_{-\rho'_{\max}(\tau)}$. Additionally, $\mathbb{1} + K(s)$ is invertible for $\mathrm{Re}(s)$ *large enough.*

As a consequence, we get the main theorem of this article.

Theorem 3. *Let $\mathbf{X} = h\mathcal{X}$ be a general admissible operator (see Definition 4.14) on a general admissible bundle $L \to M$ (see Definition 4.3), and assume that the indicial roots are affine in the sense of Definition 3.26. The Schwartz kernel of $\mathscr{R}(s) := (\mathcal{X} - s)^{-1}$ has a meromorphic continuation to $\mathbb{C}$. The corresponding poles are finite order, finite rank. We also have the wavefront set statements*

$$\mathrm{WF}'(\mathscr{R}(s)) = \mathrm{WF}'_h(\mathscr{R}(s)) \cap T^*(M \times M) \subset \Delta(T^*M) \cup \Omega_+ \cup E_s^* \times E_u^*. \tag{4.5}$$

Furthermore, if s_0 is a pole and

$$\mathscr{R}(s) = \sum_{j=1}^{J} \frac{A_j}{(s - s_0)^j} + \mathscr{R}_H(s)$$

is the Laurent expansion, with holomorphic part $\mathscr{R}_H(s)$, then

$$\mathrm{WF}'(A_j) \subset E_s^* \times E_u^* \quad \textit{and} \quad \mathrm{WF}'(\mathscr{R}_H(s_0)) \subset \Delta(T^*M) \cup \Omega_+ \cup E_s^* \times E_u^*. \tag{4.6}$$

Proof of Lemma 4.16. Recall that

$$K(s) = (\mathbf{X} - hs)\mathscr{R}'_Q(s) - \mathbb{1}.$$

The meromorphy of $\mathscr{R}'_Q(s)$ and K has already been proved, and so has the invertibility for large $\mathrm{Re}\, s > 0$ of $1 + K(s)$. It suffices now to show that $K(s)$ is compact on the appropriate space. We will use the fact that if $\mathbf{k}$ is large, then so is $\mathbf{k} \pm 1$.

The first observation is that, from a standard resolvent identity, for $\ell = 1, \dots, \kappa$, we have

$$\mathbf{R}^{\mathbf{X}_{b,\ell}}(s) - \mathbf{R}_{Q,\ell}(s) = \mathbf{R}^{\mathbf{X}_{b,\ell}}(s) Q_{b,\ell} \mathbf{R}_{Q,\ell}(s).$$

This is a bounded operator from $e^{-\rho_{\max,\ell}(\tau)\langle r \rangle} \mathcal{H}_b^{\mathbf{k}-1}$ to $e^{\rho_{\max,\ell}(\tau)\langle r \rangle} \mathcal{H}_b^{\mathbf{k}+1}$ (it is smoothing).

Now we compute $(\mathbf{X} - hs)\mathscr{R}'_Q(s)$ and find that the operator $K(s) = K_1(s) + K_2(s)$ writes as the sum of two terms. The first one is

$$K_1(s) := \sum_\ell \mathscr{E}_\ell [\mathbf{X}_{b,\ell}, \chi] \mathbf{R}^{\mathbf{X}_{b,\ell}}(s) Q_{b,\ell} \mathbf{R}_{Q,\ell}(s) \chi \mathscr{P}_\ell.$$

This operator is compact on $H^{\mathbf{k}}_{-\rho_{\max}(\tau)}$ since it maps it continuously to $\mathbb{1}_{y<C} H^{\mathbf{k}+1}$ – here we are applying Theorem 2 crucially.

The other term in $K(s)$ is

$$K_2(s) := Q\mathscr{R}_Q(s) - \sum_\ell \mathscr{E}_\ell \chi Q_{b,\ell} \mathbf{R}_{Q,\ell}(s) \chi \mathscr{P}_\ell.$$

Applying $(\mathbf{X} - Q - hs)$ on the right, we obtain

$$\underbrace{Q - \sum_\ell \mathscr{E}_\ell \chi Q_{b,\ell} \chi \mathscr{P}_\ell}_{:=K_3} + \underbrace{\sum_\ell \mathscr{E}_\ell \chi Q_{b,\ell} \mathbf{R}_{Q,\ell}(s) [\chi, \mathbf{X}_{b,\ell} - Q_{b,\ell}] \mathscr{P}_\ell}_{:=K_4}.$$

Since $\chi(y) = 1$ when $y > C^2\mathsf{a}$, we get that

$$\mathscr{P}^{C^3\mathsf{a}}\Big[Q - \sum_{\ell} \mathscr{E}_\ell \chi Q_{b,\ell} \chi \mathscr{P}_\ell\Big] = 0.$$

Using that Q is smoothing together with Lemma 4.12, we deduce that $K_3(\mathbf{X} - Q - hs)^{-1}$ is a compact operator on $H^{\mathbf{k}}_{-\rho'_{\max}(\tau)}$. According to Lemma 3.12, provided $\mathbf{k}$ is large enough, $\mathbf{R}_{Q,\ell}(s)$ is bounded on spaces $\mathscr{H}^{\mathbf{k_b}}_{b,\rho}$ with $\rho < -\rho_{\max}(\tau)$. Recall that χ was chosen to be constant outside a compact set, so $[\chi, \mathbf{X}_{b,\ell} - Q_{b,\ell}]\colon \mathscr{H}^{\mathbf{k_b}}_{b,\rho_1} \to \mathscr{H}^{\mathbf{k_b}}_{b,\rho_2}$ is bounded for arbitrary ρ_1, ρ_2. We deduce that $K_4(\mathbf{X} - Q - hs)^{-1}$ maps $H^{\mathbf{k}}_{-\rho'_{\max}(\tau)}$ to $H^{\mathbf{k}+1}_{\rho}$ for some $\rho < -\rho_{\max}(\tau)$, and by Lemma 4.13, it is compact. This concludes the proof. ■

Proof of Theorem 3. From Lemma 4.16, using the Gohberg–Sigal theorem [24] – see [18, Theorem C.7] for a version in English – we deduce that $\mathscr{R}'_Q(s)(1 + K(s))^{-1}$ is a meromorphic right inverse to $(\mathbf{X} - hs)$, bounded on $H^{\mathbf{k}}_{-\rho_{\max}(\tau)} \to H^{\mathbf{k}}_{\rho_{\max}(\tau)}$ for $\operatorname{Re} s > \tau$ and $\mathbf{k}$ large enough. As, for $\operatorname{Re}(s) > 0$, $\mathbf{X} - hs$ is invertible, it has to coincide with the inverse there, and we deduce it is a meromorphic continuation of $(\mathbf{X} - hs)^{-1}$. Since $C^\infty_c(M, L)$ is contained and dense in all spaces $H^{\gamma\mathbf{m}+N}_{\rho}$, we deduce the meromorphic extension of the Schwartz kernel. In particular, the poles do not depend on the choice of space.

It remains to show the announced property on the wavefront set. We can use the arguments from [17, page 18 of the arXiv version] again as in the end of the proof of Theorem 2. We reproduce the argument here. We have by the second resolvent identity

$$\mathscr{R}(s) = h\mathscr{R}_Q(s) - h\mathscr{R}_Q(s)Q\mathscr{R}_Q(s) + \mathscr{R}_Q(s)Q\mathscr{R}(s)Q\mathscr{R}_Q(s). \tag{4.7}$$

(One can check that all the terms in the equation are well defined.) The wavefront set of the first term in the right-hand side is contained in the announced wavefront set for $\mathscr{R}(s)$, so we concentrate on the second and third term. For both of them, their $\mathrm{WF}'_h \cap T^*(M \times M)$ is a subset of

$$\begin{aligned}\{(x, \xi, x', \xi') \mid \text{there exists } (x_1, \xi_1, x'_1, \xi'_1) \text{ such that } &(x, \xi, x_1, \xi_1) \in \mathrm{WF}'_h(\mathscr{R}_Q(s)Q),\\ &(x'_1, \xi'_1, x', \xi') \in \mathrm{WF}'_h(Q\mathscr{R}_Q(s))\}.\end{aligned}$$

This is contained in $E^+_\delta \times E^-_\delta$, where

$$E^\pm_\delta = \{(x, \xi) \in T^*M \mid \text{there exists } T > 0 \text{ such that } |\Phi_{\pm T}(x, \xi)| \le 3R\delta\}.$$

Since the wavefront set of $\mathscr{R}(s)$ does not depend on δ, we can let it go to 0. The intersection of the $E^+_\delta \times E^-_\delta$ for $\delta \ge 0$ is exactly $E^*_s \times E^*_u$.

For the wavefront set at a pole s_0, we consider (4.7). Comparing the Laurent coefficients, we obtain

$$A_J = \mathscr{R}_Q(s)QA_JQ\mathscr{R}_Q(s).$$

We can apply the same argument as above and obtain $\mathrm{WF}'(A_J) \subset E^*_s \times E^*_u$. For the other coefficients as well as $\mathscr{R}_H(s_0)$, we can argue inductively. Indeed, formula (4.7) will provide us with a formula for the Laurent coefficients that will involve other Laurent coefficients A_j of higher order and derivatives of $\mathscr{R}_Q(s)$ in the s parameter. But, as $\partial_s\mathscr{R}_Q(s) = -\mathscr{R}_Q(s)^2$, the wavefront set of its derivatives is contained in the same set. ■

5. Explicit computations for the geodesic flow

In this section, we come back to the case of admissible bundles over S^*N with N an admissible cusp manifold. Let us denote by $A_{\max}$ the maximum of $\mathrm{Re}(\lambda)$ when λ ranges in the eigenvalues of the endomorphisms A_ℓ. Then we define

$$\rho_{\max,L}(\tau) = \max\Big(0, A_{\max} - \tau - \frac{d}{2}\Big).$$

Note that, for functions, i.e. L being the trivial bundle, we have $A_{\max} = 0$. We prove the following theorem.

Theorem 4. *Let N be an admissible cusp manifold, $L \to M = S^*N$ an admissible bundle and $\mathcal{X}$ an admissible lift of the geodesic flow vector field (see Definitions 1.1 and 1.4).*

Then the resolvent $\mathscr{R}(s) := (\mathcal{X} - s)^{-1}$ which is defined on $L^2(M, L)$ for $\mathrm{Re}\, s \gg 0$ *has a meromorphic continuation to $\mathbb{C}$ as a family of continuous operators*

$$\mathscr{R}(s)\colon C_c^\infty(M, L) \to \mathscr{D}'(M, L).$$

More precisely, for any $\tau < 0$ and $N \in \mathbb{R}$, there is a sufficiently large γ such that, on $\mathrm{Re}(s) > \tau$, $|\mathrm{Im}(s)| \le h^{1/2}$, *the resolvent is a meromorphic family of bounded operators*

$$\mathscr{R}(s)\colon H^{\gamma\mathbf{m}+N}_{-\rho_{\max,L}(\tau)} \to H^{\gamma\mathbf{m}+N}_{\rho_{\max,L}(\tau)}.$$

Finally, the wavefront set of $\mathscr{R}(s)$ satisfies estimate (4.5)*, and its polar part satisfies* (4.6) *as in Theorem 3.*

According to the proof of Theorem 3, it suffices to show that the roots are affine in the sense of Definition 3.26. This will be shown in Lemma 5.11.

We will explicitly calculate the indicial roots for an admissible lift of the geodesic flow in the sense of Definition 1.4. We do this in three steps. First, we compute the family of indicial operators for admissible lifts. Then we determine the indicial roots for the scalar case, and finally deduce the precise formula for the indicial roots of an admissible vector bundle.

5.1. The indicial operator for admissible lifts

From now on, let $M = S^*N$ be the sphere bundle over an admissible cusp manifold, $L \to M$ an admissible vector bundle and $\mathcal{X}$ an admissible lift in the sense of Definition 1.4. Set $\mathbf{X} = h\mathcal{X}$, and fix a cusp Z_ℓ. Then, as a first step towards the indicial family, we want to calculate the b-operator $\mathbf{X}_{b,\ell}$ acting on sections of $\mathbb{R} \times \mathsf{L}_\ell \to \mathbb{R} \times \mathsf{F}$. Recall that, in Example 3.1, we have already determined that $\mathsf{L}_\ell = \mathbb{K} \times_{\tau_\ell} V_\ell \to \mathsf{F} = \mathbb{K} / \mathbb{M} = \mathbb{S}^d$. In order to give an explicit expression of the operator, we use the coordinates $r \in \mathbb{R}$ and spherical coordinates $(\varphi, u) \in [0, \pi] \times \mathbb{S}^{d-1}$ on $\mathbb{S}^d$ as introduced in Section 1.1.

Lemma 5.1. $\mathbf{X} \in \Psi_{b,1}(M, L)$ *is a cusp-b-operator, and its associated b-operator* $\mathbf{X}_{b,\ell}$ *in* $\Psi_{b,1}(\mathbb{R} \times \mathsf{L}_\ell)$ *as defined in Definition 4.5 is given by*

$$\mathbf{X}_{b,\ell} = h\Big[\cos(\varphi)\partial_r + \frac{d}{2}\cos(\varphi) + \nabla^{(\ell)}_{X_{\mathrm{gr}}} + A_\ell\Big], \tag{5.1}$$

where $\nabla^{(\ell)}$ *is the canonical connection on* L_ℓ*,* $A_\ell \in \mathrm{End}(V_\ell)^{\mathbb{M}}$ *is given by Definition 1.4 and acts as a zeroth order operator on* L_ℓ*, and* $X_{\mathrm{gr}} = \sin\varphi\partial_\varphi$ *is the vector field of the gradient flow on* $\mathbb{S}^d$.

Proof. Let us fix a cusp Z_ℓ and consider a section $f \in C^\infty(S^*Z_\ell, L)$ supported in $\{y > \mathsf{a}\}$. Recall that $L_{|S^*Z_\ell} = \Lambda_\ell \setminus \mathbb{G} \times_{\tau_\ell} V_\ell$; thus we can identify f with a function $\tilde{f}: \Lambda_\ell \setminus \mathbb{G} \to V_\ell$ that is right $\mathbb{M}$-equivariant, i.e. $\tilde{f}(\Lambda_\ell g m) = \tau_\ell(m^{-1})\tilde{f}(\Lambda_\ell g)$. Note that the geodesic flow on $S^*Z_{\ell,f} \cong \Lambda_\ell \setminus \mathbb{G} / \mathbb{M}$ is given by the right $\mathbb{A}$-action and we can write[9]

$$(\mathbf{X}\tilde{f})(\Lambda_\ell g) = h\Big[\frac{d}{dt}_{|t=0}\tilde{f}(\Lambda_\ell g e^{Ht}) + A_\ell \tilde{f}(\Lambda_\ell g)\Big] \tag{5.2}$$

for a suitably normalized $H \in \mathfrak{a} = \mathrm{Lie}(\mathbb{A})$. Let us check that $\mathbf{X}$ preserves sections that are independent of the θ variable. Note that, with respect to the $\mathbb{N}\,\mathbb{A}\,\mathbb{K}$ decomposition, this means that $\tilde{f}(\Lambda_\ell n g) = \tilde{f}(\Lambda_\ell g)$ (cf. Section 1.2). That such functions are preserved under $\mathbf{X}$ is obvious by (5.2). Consequently, $\mathbf{X}$ is a black box operator according to Definition 4.5.

Let us thus remove the dependencies in $\theta \in \Lambda_\ell \setminus \mathbb{N}$ and consider the operator $\mathbf{X}^0_{b,\ell}$ acting on sections $f \in C^\infty(\mathbb{R} \times \mathsf{F}, \mathbb{R} \times \mathsf{L}_\ell)$. Further, identify these sections with right $\mathbb{M}$-invariant functions $\tilde{f}: \mathbb{A} \times \mathbb{K} \to V_\ell$. By the $\mathbb{N}\,\mathbb{A}\,\mathbb{K}$-Iwasawa decomposition, we can write any $g \in \mathbb{G}$ in a unique way as $g = n_{NAK}(g)a_{NAK}(g)k_{NAK}(g)$. With this notation, we can write

$$\begin{aligned}\mathbf{X}^0_{b,\ell}\tilde{f}(a,k) &= h\Big[\frac{d}{dt}_{|t=0}\tilde{f}\big(a_{NAK}(ake^{Ht}), k_{NAK}(ake^{Ht})\big) + A_\ell\tilde{f}(a,k)\Big]\\ &= h\Big[\frac{d}{dt}_{|t=0}\tilde{f}\big(aa_{NAK}(ke^{Ht}), k_{NAK}(ke^{Ht})\big) + A_\ell\tilde{f}(a,k)\Big],\end{aligned}$$

where we used the identities $a_{NAK}(ag) = a \cdot a_{NAK}(g)$ and $k_{NAK}(ag) = k_{NAK}(g)$. This formula shows directly that $\mathbf{X}^0_{b,\ell}$ commutes with translations in the $\mathbb{A}$ direction, and we have thus shown that $\mathbf{X}$ is a cusp-b-operator according to Definition 4.5. It finally remains to express $\mathbf{X}^0_{b,\ell}$ in the coordinates r, φ, u of $\mathbb{R} \times \mathbb{S}^d \cong \mathbb{A} \times \mathbb{K} / \mathbb{M}$ as introduced above. In particular, we have to identify the differential operators

$$\frac{d}{dt}_{|t=0} a a_{NAK}(ke^{Ht}) \text{ on } \mathbb{A} \cong \mathbb{R} \quad \text{and} \quad \frac{d}{dt}_{|t=0} k_{NAK}(ke^{Ht}) \text{ on } \mathbb{K} / \mathbb{M} \cong \mathbb{S}^d.$$

[9] The identification of the canonical connection on reductive homogeneous spaces can be found in many geometry textbooks. For a short exposition in the context of geodesic flows on vector bundles over locally symmetric spaces, we refer to [35, Section 1.1.5].

As these differential operators are independent of the choice of the vector bundle, we can simply restrict to the scalar case and compare to the expression of the geodesic flow vector field in coordinates that have been calculated in Example 3.5 (cf. also equation (1.4)). This yields $\frac{d}{dt}_{|t=0} a a_{NAK}(ke^{Ht}) \cong \cos\varphi\partial_r$ and $\frac{d}{dt}_{|t=0} k_{NAK}(ke^{Ht}) \cong \sin\varphi\partial_\varphi = X_{\mathrm{gr}}$. Taking into account the definition of the canonical connection on $\mathsf{L}_\ell = \mathbb{K} \times_{\tau_\ell} V_\ell$, we get

$$\mathbf{X}^0_{b,\ell} = h[\cos\varphi\partial_r + \nabla^{\mathsf{L}_\ell}_{X_{\mathrm{gr}}} + A_\ell].$$

In order to pass from $\mathbf{X}^0_{b,\ell}$ to $\mathbf{X}_{b\ell}$, one simply has to conjugate the differential operator by $e^{-rd/2}$, which creates the additional $d/2\cos\varphi$ term in (5.1). ∎

Now, from equation (5.1) and the definition of the indicial family (Definition 3.13), we directly obtain the following corollary.

Corollary 5.2. *For $\mathbf{X}_{b\ell}$ as in Lemma 5.1, one has*

$$I(\mathbf{X}_{b,\ell}, \lambda) = \lambda\cos\varphi + h\Big[\frac{d}{2}\cos\varphi + \nabla^{\mathsf{L}_\ell}_{X_{\mathrm{gr}}} + A_\ell\Big]. \tag{5.3}$$

5.2. *Finding the indicial roots for functions*

In this section, we focus on the action on functions. In that case, $\mathcal{X} = X$ and $\mathbf{X} = hX$. Since the flow is the same for each cusp, we can safely drop the dependence in the index ℓ. We compute the indicial roots of $I(\mathbf{X}_b, \lambda) - hs$. As this operator will frequently show up in the sequel, we introduce the shorter notation

$$P_\lambda := I(\mathbf{X}_b, \lambda) = h\sin\varphi\partial_\varphi + \Big[\lambda + h\frac{d}{2}\Big]\cos\varphi. \tag{5.4}$$

Le us introduce some notation which we will need to formulate the spectral properties of P_λ. Recall that we have introduced the coordinates $(\varphi, u) \in [0,\pi] \times \mathbb{S}^{d-1}$ on $\mathbb{S}^d$. Consider the projection of $\mathbb{S}^d$ to the equatorial plane. It is a smooth chart on both strict hemispheres. We denote these smooth restrictions by

$$\kappa_{\mathcal{N}}\colon \Big\{(\varphi,u) \in \mathbb{S}^d \;\Big|\; \varphi < \frac{\pi}{2}\Big\} \to \{x \in \mathbb{R}^d \mid \|x\| < 1\},$$
$$\kappa_{\mathcal{S}}\colon \Big\{(\varphi,u) \in \mathbb{S}^d \;\Big|\; \varphi > \frac{\pi}{2}\Big\} \to \{x \in \mathbb{R}^d \mid \|x\| < 1\}.$$

Note that $(\rho, u) := (\sin\varphi, u) \in [0,1] \times \mathbb{S}^{d-1}$ are exactly the radial coordinates in both charts.

For further reference, we recall that the Taylor expansion in radial coordinates at 0 for $f \in C^n(\mathbb{R}^d)$ can be written in the following fashion:

$$f(\rho,u) = \sum_{|\mu|\le n} \frac{\partial_x^\mu f(0)}{\mu!} \cdot \rho^{|\mu|} \cdot \Upsilon_\mu(u) + o(\rho^n) \quad \text{as } \rho \to 0. \tag{5.5}$$

Here $\mu \in \mathbb{N}^d$ is a multi-index, $\Upsilon_\mu \in C^\infty(\mathbb{S}^{d-1})$ is the monomial x^μ, $x \in \mathbb{R}^d$, of degree $|\mu|$ restricted to the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$.

Let us come back to P_λ. According to Lemma 3.20, to determine the indicial roots, it suffices to consider the action of P_λ on $\mathsf{H}_0^{\gamma m_b}(\mathbb{S}^d)$, that we denote just $\mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$. Inspecting formula (5.4), we see that it is a gradient vector field plus a complex potential. Ruelle–Pollicott resonances for Morse–Smale gradient flows were studied in detail by Dang and Rivière [13]. In particular, the spaces they defined are quite similar to $\mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$. Recall that C_G was defined in Lemma 2.3

Lemma 5.3. *There is an $\epsilon > 0$ such that the following holds.*

- *Let $f \in \mathcal{D}'(\mathbb{S}^d)$ be supported in the ϵ-neighbourhood of the North Pole. Then*
$$f \in \mathsf{H}^{\gamma m_b}(\mathbb{S}^d) \iff f \in H^{-C_G\gamma}(\mathbb{S}^d).$$
- *Let $f \in \mathcal{D}'(\mathbb{S}^d)$ be supported in the ϵ-neighbourhood of the South Pole. Then*
$$f \in \mathsf{H}^{\gamma m_b}(\mathbb{S}^d) \iff f \in H^{C_G\gamma}(\mathbb{S}^d).$$

Proof. Let us prove the first assertion. By Definition 3.18 of $\mathsf{H}^{\gamma m_b}$ and standard pseudo-differential operator arguments, it is enough to prove that $m_{b,0} \in S^0(\mathsf{L})$ is constantly equal to $-C_G$ in a neighbourhood of $\mathcal{N}$ modulo some lower order terms $S^{-1+\varepsilon}(\mathsf{L})$. By Lemma 3.17 and Proposition 4.11, the leading term is given by

$$m_{b,0}(\zeta,\eta) = m(r,\theta,\zeta,\lambda = 0, J = 0,\eta) \mod S^{-1+\varepsilon}(\mathsf{L}),$$

where $m \in S^0(M,L)$ is the order function constructed in Lemma 2.3. By construction of m, we know that, high enough in the cusp, $m = -C_G$ in a neighbourhood of

$$E^*_{c,s} := (E^s_c \oplus E^0_c)^\perp.$$

Here E^s_c and E^0_c are the stable and neutral bundles corresponding to constant curvature (see discussions in the proof of Lemma 2.5). If we consider some point $(r,\theta,\mathcal{N}) \in SZ$ and write $T(r,\theta,\mathcal{N})(SZ) \cong T_r\mathbb{R} \oplus T_\theta(\mathbb{R}^d/\Lambda) \oplus T_{\mathcal{N}}\mathbb{S}^d$, then by standard hyperbolic geometry, we have $(E^0_c)_{(r,\theta,\mathcal{N})} = T_r\mathbb{R}$ and $(E^s_c) = T_\theta(\mathbb{R}^d/\Lambda)$. Consequently, $E^*_{c,s}$ is precisely given by $\lambda = 0$, $J = 0$. Putting everything together, we know that, at leading order, $m_{b,0}$ is constantly equal to $-C_G$ around $\mathcal{N}$, which implies the first assertion.

The second statement follows from similar arguments. ∎

The following lemma shows that, in the charts $\kappa_{\mathcal{N},\mathcal{S}}$, the operator P_λ takes a particularly simple form – recall that here $\rho = \sin\varphi$.

Lemma 5.4. *On the northern hemisphere, the function $2\tan(\varphi/2)/\sin\varphi$ is an analytic, non-zero function, and expressed in the (ρ,u)-charts, defined above, we have*

$$\Big(\frac{2\tan(\varphi/2)}{\sin\varphi}\Big)^{-s}(P_\lambda - hs)\Big(\frac{2\tan(\varphi/2)}{\sin\varphi}\Big)^{s} = \sqrt{1-\rho^2}\Big(h\rho\partial_\rho - hs + \lambda + h\frac{d}{2}\Big).$$

On the southern hemisphere, the function $2\tan(\varphi/2)\sin\varphi$ is an analytic, non-zero function, and expressed in the (ρ,u)-charts, defined above, we have

$$\Big(2\tan\Big(\frac{\varphi}{2}\Big)\sin\varphi\Big)^{-s}(P_\lambda - hs)\Big(2\tan\Big(\frac{\varphi}{2}\Big)\sin\varphi\Big)^{s}$$
$$= \sqrt{1-\rho^2}\Big(-h\rho\partial_\rho - hs - \lambda - h\frac{d}{2}\Big).$$

Proof. The results follow from a straightforward calculation using standard trigonometric identities. ∎

Let $\mu \in \mathbb{N}^d$ be a multi-index; then we define the standard Dirac distributions on $\mathbb{R}^d$ by $\delta_0^{(\mu)}\colon C_c^\infty(\mathbb{R}^d) \ni f \mapsto (\partial_x^\mu f)(0)$. Recall that all distributions on $\mathbb{R}^d$, supported in 0 are linear combinations of finitely many $\delta_0^{(\mu)}$. Furthermore,

$$\rho\partial_\rho\delta_0^{(\mu)} = -(|\mu| + d)\delta_0^{(\mu)}. \tag{5.6}$$

If $\kappa_{\mathcal{N}}$ is the chart of the northern hemisphere, then we define for any $\lambda \in \mathbb{C}$ the distribution

$$\delta_{\mathcal{N},\lambda}^{(\mu)} := \Big(\frac{2\tan(\varphi/2)}{\sin\varphi}\Big)^{\lambda/h-(|\mu|+d/2)} \kappa_{\mathcal{N}}^*\delta_0^{(\mu)} \in \mathcal{D}'(\mathbb{S}^d),$$

and for $\kappa_{\mathcal{S}}$ the chart of the southern hemisphere, we define the distribution

$$\delta_{\mathcal{S},\lambda}^{(\mu)} := \Big(2\tan\Big(\frac{\varphi}{2}\Big)\sin\varphi\Big)^{-\lambda/h+(|\mu|+d/2)} \kappa_{\mathcal{S}}^*\delta_0^{(\mu)} \in \mathcal{D}'(\mathbb{S}^d).$$

Combining (5.6) with Lemma 5.4, we obtain

$$\begin{aligned} \Big[P_\lambda - \Big(\lambda - h\Big(|\mu| + \frac{d}{2}\Big)\Big)\Big]\delta_{\mathcal{N},\lambda}^{(\mu)} &= 0,\\ \Big[P_\lambda + \Big(\lambda - h\Big(|\mu| + \frac{d}{2}\Big)\Big)\Big]\delta_{\mathcal{S},\lambda}^{(\mu)} &= 0, \end{aligned} \tag{5.7}$$

and up to linear combinations, these are the only eigendistributions of P_λ supported in the North or South Pole.

We next want to study the kernels of the operators P_λ on $\mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$. According to Proposition 3.23, each P_λ has a unique closed extension, and the domain $D^{\gamma m_b}(\mathbb{S}^d)$ does not depend on λ, so that $\lambda \mapsto P_\lambda$ is a type (A) family as in [33]. Further, to prove Proposition 3.25, we proved that $P_\lambda - hs$ is Fredholm of index 0 when

$$\operatorname{Re}(s) > -\gamma C_G + \frac{d}{2} + \Big|\operatorname{Re}\Big(\frac{\lambda}{h}\Big)\Big|.$$

The rest of this section is devoted to the proof of the following proposition.

Proposition 5.5. *The indicial roots of $\mathbf{X}_b$ acting on functions are affine, and they are given by*

$$\operatorname{Spec}_b(s) = \Big\{\pm\Big(s + \Big(\frac{d}{2} + n\Big)\Big),\ n \in \mathbf{N}\Big\}.$$

We start with the following lemma.

Lemma 5.6. *Let $\gamma > 0$, $\lambda \in \mathbb{C}$. Then, for $\operatorname{Re}(s) > -\gamma C_G + d/2 + |\operatorname{Re}(\lambda/h)|$, the operator $(P_\lambda - hs)\colon D^{\gamma m_b}(\mathbb{S}^d) \to \mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$ is injective unless*

$$\lambda = \pm h\Big[s + \Big(\frac{d}{2} + n\Big)\Big]$$

for some $n \in \mathbb{N}$.

Proof. Assume that $w \in \mathcal{D}'(\mathbb{S}^d)$ is a distribution that fulfils $(P_\lambda - hs)w = 0$. Then we can distinguish two cases: either $w_{|\mathbb{S}^d \setminus \{\mathcal{N},\mathcal{S}\}} = 0$ or not.

In the first case, w must be a linear combination of $\delta^{(\mu)}_{\mathcal{N}/\mathcal{S}}$, and from (5.7), we deduce that the possible solutions are either $\lambda = h(s + n + d/2)$ and w is a linear combination of $\delta^{(\mu)}_{\mathcal{N},\lambda}$ with $|\mu| = n$, or $\lambda = -h(s + n + d/2)$ and w is a linear combination of $\delta^{(\mu)}_{\mathcal{S},\lambda}$, again with $|\mu| = n$.

Whether these distributional solutions belong to $\mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$, or not, depends on γ. Suppose that $\gamma C_G > n + d/2$, $n \in \mathbb{N}$; then locally around the South Pole, according to Lemma 5.3, distributions in $\mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$ have to be of positive Sobolev order, so none of the Dirac distributions $\delta^{(\mu)}_{\mathcal{S},\lambda}$ are allowed. Near the North Pole, distributions are allowed to be in $H^{-n-d/2-\varepsilon}(\mathbb{S}^d)$, again from Lemma 5.3, and consequently, all the distributions $\delta^{(\mu)}_{\mathcal{N},\lambda}$ with $|\mu| \le n$ are contained in $\mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$.

In the second case, i.e. $w_{|\mathbb{S}^d \setminus \{\mathcal{N},\mathcal{S}\}} \neq 0$, we work in $\mathbb{S}^d \setminus \{\mathcal{N}, \mathcal{S}\}$ with the coordinates $(\varphi, u) \in]0, \pi[\times \mathbb{S}^{d-1}$. As P_λ is independent of u, we can choose a product form $w_{|\mathbb{S}^d \setminus \{\mathcal{N},\mathcal{S}\}} = f \otimes g$ with $f \in \mathcal{D}'(]0, \pi[)$ and $g \in \mathcal{D}'(\mathbb{S}^{d-1})$. Thus the PDE reduces to the (ordinary) differential equation $(P_\lambda - hs)f = 0$. By ellipticity, f has to be a smooth function on $]0, \pi[$, and for every λ, s, there is a unique solution

$$f(\varphi) := (\sin\varphi)^{-d/2-\lambda/h}\Big(2\tan\Big(\frac{\varphi}{2}\Big)\Big)^s.$$

We now have to discuss under what conditions $f \otimes g$ can be extended to a distribution in $\mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$. For $d/2 > \epsilon > 0$, let $\alpha = \gamma C_G - d/2 - \epsilon$; then from Lemma 5.3 and the Sobolev embedding theorem, distributions have to be $C^\alpha(\mathbb{S}^d)$ in a neighbourhood of the South Pole. Going to the charts $\kappa_{\mathcal{S}}$, we obtain

$$\Big(2\tan\Big(\frac{\varphi}{2}\Big)\sin\varphi\Big)^{-s} w = \rho^{-d/2-\lambda/h-s} \otimes g.$$

As, in a neighbourhood of the South Pole, $(2\tan(\varphi/2)\sin\varphi)^{-s}$ is a smooth non-vanishing $C^\infty(\mathbb{S}^d)$ function, we have to extend $\rho^{-d/2-\lambda/h-s} \otimes g$ to a C^α-function on $\mathbb{R}^d$. According to (5.5), this is possible if either $\mathrm{Re}(-d/2 - \lambda/h - s) \ge \alpha$ or if $-d/2 - \lambda/h - s = n$ for some $n \in \mathbb{N}$ and g is a linear combination of Υ_μ with $|\mu| = n$ (or in other words, g is a homogeneous polynomial of degree n). Note that the first case is ruled out since we assumed that $-\mathrm{Re}(s + \lambda/h) < -d/2 + C_G\gamma$ so that we would have $\alpha < C_G\gamma - d$, and $\epsilon > d/2$, contrary to our assumption. ∎

Now, to complete the proof of Proposition 5.5, we have to check that, for

$$\lambda = \pm h\Big(s + \frac{d}{2} + n\Big),$$

the kernel is not empty. We have already done this in the proof for $\lambda = h(s + d/2 + n)$ for $n \in \mathbb{N}$, so we concentrate on the case that $\lambda = -h(s + d/2 + n)$.

The question is whether the functions $f \otimes \Upsilon_\mu$, $|\mu| = n$, can be extended to distributions over the whole sphere $\mathbb{S}^d$. We have $hs = -\lambda - h(n + d/2)$. Let $\Upsilon \in \mathbb{R}_k(\mathbb{S}^{d-1})$, where $\mathbb{R}_k(\mathbb{S}^{d-1})$ denotes the space of homogeneous polynomial of degree n restricted to the unit sphere. We introduce the notation

$$f^0_{\mathcal{S},\Upsilon,\lambda} = (\sin\varphi)^{-d/2-\lambda/h}\Big(2\tan\Big(\frac{\varphi}{2}\Big)\Big)^{-(n+d/2+\lambda/h)} \otimes \Upsilon(u). \tag{5.8}$$

If we express these functions in the (ρ, u) coordinates in a neighbourhood of the North Pole, we get

$$\Big(\frac{2\tan(\varphi/2)}{\sin\varphi}\Big)^{n+d/2+\lambda/h} f^0_{\mathcal{S},\Upsilon,\lambda} = \rho^{-d-2\lambda/h-n}\Upsilon(u).$$

Since $2\tan(\varphi/2)/\sin\varphi$ is a non-vanishing $C^\infty(\mathbb{S}^d)$ function near $\mathcal{N}$, we conclude that $f^0_{\mathcal{S},\Upsilon,\lambda}$ is in $L^1_{\mathrm{loc}}(\mathbb{S}^d)$ and a legitimate distribution whenever $n + 2\operatorname{Re}\lambda/h < 0$. Now, using the ideas of Hadamard regularization as in [31, Theorem 3.2.4], we can show that $f^0_{\mathcal{S},\Upsilon,\lambda}$ extends from $\operatorname{Re}\lambda < -hn/2$ to the whole of $\mathbb{C}$ as meromorphic family of distributions $F_{\mathcal{S},\Upsilon,\lambda}$. When $\psi \in C^\infty(\mathbb{S}^d)$ is not supported around $\mathcal{N}$, the value of $F_{\mathcal{S},\Upsilon,\lambda}(\psi)$ is given by $f^0_{\mathcal{S},\Upsilon,\lambda}(\psi)$, so we can concentrate on the case of ψ supported around $\mathcal{N}$, and consider a smooth function ψ supported in $\{\rho < \epsilon\}$ in $\mathbb{R}^d$ such that, when $\operatorname{Re}\lambda < -hn/2$,

$$\Big(\frac{2\tan(\varphi/2)}{\sin\varphi}\Big)^{n+d/2+\lambda/h} f^0_{\mathcal{S},\Upsilon,\lambda}(\psi) = \int_0^\epsilon \int_{\mathbb{S}^{d-1}} \rho^{-2\lambda/h-n-1}\psi(\rho u)\Upsilon(u)\,du\,d\rho.$$

Integrating by parts in the ρ variable, N times when $\operatorname{Re}\lambda < -hn/2$, we obtain that

$$\begin{aligned}&\rho^{-2\lambda/h-n-1}\Upsilon(\psi)\\ &\quad= \prod_{j=0}^{N-1}\frac{1}{2\lambda/h+n-j}\int_0^\epsilon\int_{\mathbb{S}^{d-1}}\rho^{-2\lambda/h-n+N-1}\Upsilon(u)(\partial_\rho^N\psi(\rho u))\,du\,d\rho.\end{aligned}$$

The expression in the right-hand side is obviously meromorphic for $\operatorname{Re}\lambda < h(N-n)/2$. The poles are situated at $\lambda = h(j-n)/2$, with $j = 0,\dots,N-1$, and they are of order 1. At such a point, we find $s = -(n+d+j)/2$ and $\lambda = h(s+d/2+j)$. In other words, the poles of $F_{\mathcal{S},\Upsilon,\lambda}$ correspond to root crossings. This is sufficient to ensure that the indicial roots are exactly the $\pm h(s+d/2+n)$ with $n \in \mathbb{N}$ and finishes the proof of Proposition 5.5. ■

Now, while not necessary for the proof of the main theorem, we want here to describe the Jordan block structure at the root crossings. Since the residue of $F_{\mathcal{S},\Upsilon,\lambda}$ at a pole does not depend on the level of regularization N (as long as $N \geq j+1$), we can choose $N = j+1$. Then the residue is given by

$$\frac{h}{2j!}\int_0^\epsilon\int_{\mathbb{S}^{d-1}}\Upsilon(u)(\partial_\rho^{j+1}\psi(\rho u))\,d\rho\,du.$$

But, as ψ is supported in $\{\rho < \epsilon\}$, this is just

$$\frac{h}{2j!}\int_{\mathbb{S}^{d-1}}\Upsilon(u)\Big(\Big(\frac{\partial}{\partial\rho}\Big)^j_{|\rho=0}\psi(\rho u)\Big)\,du = \frac{h}{2}\sum_{|\mu|=j}\frac{1}{\mu!}\Big(\int_{\mathbb{S}^d}\Upsilon_\mu(u)\Upsilon(u)\,du\Big)\delta_0^{(\mu)}(\psi),$$

where the equality can be read off (5.5). Writing $a_\mu = 1/\mu! \int_{\mathbb{S}^{d-1}} \Upsilon(u)\Upsilon_\mu(u)\,du$, the residue of $F_{\mathcal{S},\Upsilon,\lambda}$ at $\lambda = h(j-n)/2$ is thus given by

$$\frac{h}{2}\sum_{|\mu|=j} a_\mu \delta^{(\mu)}_{\mathcal{N},\lambda}.$$

The finite part of $F_{\mathcal{S},\Upsilon,\lambda}$ at such a point is a distribution A_j such that A_j coincides with $f_{\mathcal{S},\Upsilon,h(j-n)/2}$ in $\mathbb{S}^d \setminus \mathcal{N}$. Additionally, since we have, for all λ,

$$\Big(P_\lambda + \lambda + h\Big(n + \frac{d}{2}\Big)\Big)F_{\mathcal{S},\Upsilon,\lambda} = 0,$$

differentiating in the parameter λ, we deduce that

$$\Big(P_{h(j-n)/2} + h\frac{n+j+d}{2}\Big)A_j = -(1+\cos\varphi)\frac{h}{2}\sum_{|\mu|=j} a_\mu \delta^{(\mu)}_{\mathcal{N},h(j-n)/2}.$$

Consequently, the finite part of $F_{\mathcal{S},\Upsilon,\lambda}$ is an eigendistribution of P_λ if all the a_μ vanish. If Υ is chosen such that this is not the case, we can however modify A_j in order to get a generalized eigendistribution. Consider E_j the space of distributions supported in $\{\mathcal{N}\}$, of order smaller than j. Choosing a basis of such distributions of decreasing order, we find that $P_{h(j-n)/2}$ acts on E_j in an upper triangular fashion, and the diagonal coefficients are non-singular. We deduce that $P_{h(j-n)/2}$ is invertible on E_j. In particular, since

$$\Big(P_{h(j-n)/2} + h\frac{n+j+d}{2}\Big)^2 A_j \in E_j,$$

we can find $e_j \in E_j$ such that

$$\Big(P_{h(j-n)/2} + h\frac{n+j+d}{2}\Big)^2 (A_j + e_j) = 0.$$

In particular, the kernel is non-empty, and there is an order 2 Jordan block.

Definition 5.7. When $\lambda \neq h(j-n)/2$ with j, n some integers, and $\Upsilon \in \mathbb{R}_n(\mathbb{S}^d)$, we denote by $f_{\mathcal{S},\Upsilon,\lambda}$ the continuation $F_{\mathcal{S},\Upsilon,\lambda}$. When $\lambda = h(j-n)/2$, $f_{\mathcal{S},\Upsilon,\lambda}$ will instead refer to the distribution $A_j + e_j$ thus defined.

Before we proceed, it will be useful to introduce some notation. Given real-valued $g, f \in C^\infty(\mathbb{S}^{d-1})$, we let

$$\langle f, g\rangle = \int_{\mathbb{S}^{d-1}} fg.$$

We recall that $\mathbb{R}_n(\mathbb{S}^{d-1})$ is the set of functions on the sphere that are restrictions of real homogeneous polynomials of order n on $\mathbb{R}^d$.

As a consequence of the proof of Lemma 5.6, we get the following explicit description of the generalized eigenstates of P_λ.

Lemma 5.8. *Let $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$ and $\gamma C_G > d + n + 2|\mathrm{Re}(\lambda/h)|$; consider the operator $P_\lambda \colon D^{\gamma m_b}(\mathbb{S}^d) \to \mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$ and the kernel of $P_\lambda + h(d/2+n) \pm \lambda$.*

- *If $\lambda \notin h\mathbb{Z}/2$, then for any $n \in \mathbb{N}$, there are no Jordan Blocks, and*

$$\ker\Big(P_\lambda + h\Big(\frac{d}{2}+n\Big)+\lambda\Big) = \mathrm{span}\{f_{\mathcal{S},\Upsilon_\mu,\lambda} \mid |\mu| = n\},$$
$$\ker\Big(P_\lambda + h\Big(\frac{d}{2}+n\Big)-\lambda\Big) = \mathrm{span}\{\delta^{(\mu)}_{\mathcal{N},\lambda} \mid |\mu| = n\}.$$

- *If $\lambda = hk/2$, $k \in \mathbb{N}$, and $n = 0,\dots,k-1$, then for $P_\lambda + h(d/2+n)-\lambda$, there are no Jordan Blocks, and one has*

$$\ker\Big(P_\lambda + h\frac{d+2n-k}{2}\Big) = \mathrm{span}\{\delta^{(\mu)}_{\mathcal{N},\lambda} \mid |\mu| = n\}.$$

- *For $\lambda = hk/2$, $k \in \mathbb{N}$, and $n = k, k+1,\dots,$ one has Jordan Blocks of index 2 for $P_\lambda + h(d/2+n)-\lambda$, and*

$$\ker\Big(P_\lambda + h\frac{d+2n-k}{2}\Big)^2 = \mathrm{span}\{\delta^{(\mu)}_{\mathcal{N},\lambda}, f_{\mathcal{S},\Upsilon_\nu,\lambda} \mid |\mu| = n,\ |\nu| = n-k\},$$
$$\begin{aligned}\ker\Big(P_\lambda + h\frac{d+2n-k}{2}\Big) = {} & \mathrm{span}\{\delta^{(\mu)}_{\mathcal{N},\lambda} \mid |\mu| = n\} \\ & \cup \{f_{\mathcal{S},\Upsilon,\lambda} \mid \Upsilon \in \mathbb{R}_{n-k}(\mathbb{S}^{d-1}),\ \langle \Upsilon, \Upsilon_\nu\rangle = 0 \\ & \quad \textit{for all } |\nu| = n\}.\end{aligned}$$

- *If $\lambda = -hk/2$, $k \in \mathbb{N}$, and $n = 0,\dots,k-1$, then for $P_\lambda + h(d/2+n)+\lambda$, there are no Jordan Blocks, and one has*

$$\ker(P_\lambda + h(d+2n-k)/2) = \mathrm{span}\{f_{\mathcal{S},\Upsilon_\mu,\lambda} \mid |\mu| = n\}.$$

- *For $\lambda = -hk/2$, $k \in \mathbb{N}$ and $n = k, k+1,\dots,$ one has Jordan Blocks of index 2 for $P_\lambda + h(d/2+n)+\lambda$, and*

$$\ker\Big(P_\lambda + h\frac{d+2n-k}{2}\Big)^2 = \mathrm{span}\{f_{\mathcal{S},\Upsilon_\mu,\lambda}, \delta^{(\nu)}_{\mathcal{N},\lambda} \mid |\mu| = n,\ |\nu| = n-k\},$$
$$\begin{aligned}\ker\Big(P_\lambda + h\frac{d+2n-k}{2}\Big) = {} & \mathrm{span}\{\delta^{(\nu)}_{\mathcal{N},\lambda} \mid |\nu| = n-k\} \\ & \cup \{f_{\mathcal{S},\Upsilon,\lambda} \mid \Upsilon \in \mathbb{R}_{n}(\mathbb{S}^{d-1}),\ \langle \Upsilon, \Upsilon_\mu\rangle = 0 \\ & \quad \textit{for all } |\mu| = n-k\}.\end{aligned}$$

5.3. *Indicial roots for fibre bundles*

After this study of the action on functions, we come back to the action on admissible vector bundles $\mathsf{L} = \mathbb{K} \times_{\tau_\ell} V_\ell \to \mathbb{K} / \mathbb{M} \cong \mathbb{S}^d$. For the moment, let us fix a cusp and drop the index ℓ. Note that Definition 1.4 does not assume that τ is an irreducible $\mathbb{M}$ representation. However, we can reduce the problem to the irreducible case. Consider the complexified representation $(\tau, V_\mathbb{C})$ which decomposes into irreducible unitary representations (σ_i, W_i). We then get

$$L^2(\mathbb{S}^d, \mathbb{K} \times_\tau V) = \bigoplus_{i=1}^{n} L^2(\mathbb{S}^d, \mathbb{K} \times_{\sigma_i} W_i). \tag{5.9}$$

Furthermore, using the explicit form of $I(\mathbf{X}_b, \lambda)$ from (5.3) and the fact that, according to Definition 1.4, the non-scalar zero order term A is $\mathbb{M}$ equivariant, we conclude that $I(\mathbf{X}_b, \lambda)$ preserves this splitting. Finally, we have to take into account that we do not want to study the operator acting on L^2 but rather on the anisotropic spaces $\mathsf{H}^{\gamma m_b}(\mathbb{S}^d, \mathsf{L})$. Recall that the escape function is a purely scalar symbol. We can pick the quantization so that scalar symbols are mapped to operators that preserve the decomposition (5.9) – that is a lower order term condition. In particular, $I(\mathrm{Op}^b(e^{-rG_b}), \lambda)$ acts on $L^2(\mathbb{S}^d, \mathbb{K} \times_{\sigma_i} W_i)$ as a principally scalar operator. Thus we assume from now on that we have fixed a cusp and that (τ, V) is unitary and irreducible. Since it is irreducible, the $\mathbb{M}$ equivariant term A has to be scalar by Schur's lemma and the indicial operator

$$I(\mathbf{X}_b, \lambda) = \lambda \cos\varphi + h\Big[\frac{d}{2}\cos\varphi + \nabla^{\mathsf{L}}_{X_{\mathrm{gr}}} + A\Big]$$

becomes the sum of a covariant derivative and a scalar term. It will thus be convenient to study its action on local trivializations by orthogonal parallel frames. Let $b_1^{\mathcal{N}}, \dots, b_{\dim V}^{\mathcal{N}}$ be an orthonormal basis of the fibre $\mathsf{L}_{\mathcal{N}}$ over the North Pole $\mathcal{N} \in \mathbb{S}^d$. Any point

$$(\varphi, u) \in \mathbb{S}^d \setminus \{\mathcal{S}\}$$

can be connected to $\mathcal{N}$ by a path $[0,1] \ni t \mapsto (t\varphi, u)$ in a unique way, and via parallel transport along these paths, we can define the orthonormal basis $b_i^{\mathcal{N}}(\zeta)$ of the fibre over $\zeta \in \mathbb{S}^d \setminus \{\mathcal{S}\}$. By definition, this means

$$\nabla^{\mathsf{L}}_{X_{\mathrm{gr}}} b_i^{\mathcal{N}} = 0.$$

Similarly we chose a orthonormal parallel frame $b_i^{\mathcal{S}}(\zeta)$ on $\mathbb{S}^d \setminus \{\mathcal{N}\}$. Comparing these two orthonormal frames on the equator $\varphi = \pi/2$, $u \in \mathbb{S}^{d-1}$, we get a smooth gluing function $g\colon \mathbb{S}^{d-1} \to U(V)$ such that

$$g(u) b_i^{\mathcal{N}}\Big(\frac{\pi}{2}, u\Big) = b_i^{\mathcal{S}}\Big(\frac{\pi}{2}, u\Big) \quad \text{for } i = 1, \dots, \dim V.$$

With this gluing function, we can express the transformation under the change of trivialization for $w \in \mathcal{D}'(\mathbb{S}^d \setminus \{\mathcal{N}, \mathcal{S}\}, \mathsf{L})$ as follows:

$$w = \sum_{k=1}^{\dim V} w_k^{\mathcal{S}} b_k^{\mathcal{S}}(\varphi, u) = \sum_{l=1}^{\dim V}\Bigg(\sum_{k=1}^{\dim V} \underbrace{\Big\langle g(u) b_k^{\mathcal{N}}\Big(\frac{\pi}{2}, u\Big), b_l^{\mathcal{N}}\Big(\frac{\pi}{2}, u\Big)\Big\rangle_V}_{=: g_{l,k}(u)} w_k^{\mathcal{S}}\Bigg) b_l^{\mathcal{N}}(u, \varphi).$$

Having introduced these orthonormal frames, we can prove the following lemma.

Lemma 5.9. *If we fix a cusp and consider $\mathsf{L} = \mathbb{K} \times_\tau V$ for an irreducible unitary $\mathbb{M}$ representation (τ, V), then the operator $I(X_b - hs, \lambda)\colon D^{\gamma m_b}(\mathbb{S}^d, \mathsf{L}) \to \mathsf{H}^{\gamma m_b}(\mathbb{S}^d, \mathsf{L})$ is injective unless*

$$\lambda = \pm h\Big[s - A + \Big(\frac{d}{2} + n\Big)\Big],$$

where $A \in \mathrm{End}(V)^{\mathbb{M}}$ has been identified with a scalar by Schur's lemma and $n \in \mathbb{N}$.

Proof. Let us reduce the problem to the case of functions, dealt with by Lemma 5.6. Suppose that $w \in D^{\gamma m_b}(\mathbb{S}^d, \mathsf{L}) \setminus \{0\}$ with $I(\mathbf{X}_b - hs, \lambda)w = 0$. Then one of the following cases holds.

First case: $\mathrm{supp}(w) \setminus \{\mathcal{N}, \mathcal{S}\} \neq \emptyset$. Then we can expand the restriction of w to $\mathbb{S}^d \setminus \{\mathcal{N}\}$ in the orthonormal trivialization $b_k^{\mathcal{S}}$ and get

$$w_{\mathbb{S}^d \setminus \{\mathcal{N}\}} = \sum w_k^{\mathcal{S}} b_k^{\mathcal{S}}(\varphi, u)$$

for scalar distributions $w_k^{\mathcal{S}} \in \mathcal{D}'(\mathbb{S}^d \setminus \{\mathcal{N}\})$. From the fact that $\nabla_{X_{\mathrm{gr}}} b_k^{\mathcal{S}} = 0$, we deduce that

$$\Big[h\Big(X_{\mathrm{gr}} + \frac{d}{2}\cos\varphi\Big) + \lambda\cos\varphi - h(s-A)\Big] w_k^{\mathcal{S}} = 0.$$

Next, using that $w \in \mathsf{H}^{\gamma m_b}(\mathbb{S}^d, \mathsf{L})$ and Lemma 5.3, we conclude that $w_k^{\mathcal{S}} \in H^{C_G\gamma}(\mathbb{S}^d)$, in a small neighbourhood around $\mathcal{S}$. Furthermore, at least one $w_k^{\mathcal{S}}$ must be non-vanishing on $\mathbb{S}^d \setminus \{\mathcal{N}, \mathcal{S}\}$. We are thus precisely in the setting of the second case in Lemma 5.6, and we deduce with the same arguments that such a distribution only exists if

$$s - A = -\frac{d}{2} - \frac{\lambda}{h} - n \quad \text{for some } n \in \mathbb{N}$$

and the eigendistributions are precisely given by a linear combination of $f_{\mathcal{S},\lambda,\Upsilon_\mu}$ with $|\mu| = n$.

Second case: $\mathrm{supp}(w) = \{\mathcal{S}\}$. Then we use the same trivialization as above. This would require distributions $w_k^{\mathcal{S}} \in \mathsf{H}^{\gamma m_b}(\mathbb{S}^d)$ with $\mathrm{supp}\, w_k^{\mathcal{S}} = \mathcal{S}$. But, as Lemma 5.3 requires these distributions to have positive Sobolev regularity, they have to be zero.

Third case: $\mathrm{supp}(w) = \mathcal{N}$. Then, using the trivialization on $\mathbb{S}^d \setminus \{S\}$, we write

$$w = \sum w_k^{\mathcal{N}} b_k^{\mathcal{N}}(\varphi, u) \quad \text{with } w_k^{\mathcal{N}} \in \mathsf{H}^{\gamma m_b}(\mathbb{S}^d),\ \mathrm{supp}(w_k^{\mathcal{N}}) = \mathcal{N},$$

and

$$\Big[h\Big(X_{\mathrm{gr}} + \frac{d}{2}\cos\varphi\Big) + \lambda\cos\varphi - h(s-A)\Big] w_k^{\mathcal{N}} = 0.$$

We are thus precisely in the setting of the first case in Lemma 5.6, and we deduce that such distributions only exist if $h(s-A) = \lambda - h(n + d/2)$ for some $n \in \mathbb{N}$ and they are precisely given by linear combinations of $\delta_{\mathcal{N},\lambda}^{(\mu)}$ with $|\mu| = n$. ∎

As in the case of functions, we have to care about the extension of those distributions coming from $f_{\mathcal{S},\Upsilon_\mu,\lambda}$ and check which still remain in the kernel of the indicial operator. Therefore, the following notation is convenient: given

$$\underline{\Upsilon} = (\Upsilon^{(1)}, \ldots, \Upsilon^{(\dim V)}) \in (\mathbb{R}_n(\mathbb{S}^{d-1}))^{\dim V},$$

define the section $F_{\mathcal{S},\underline{\Upsilon},\lambda} := \sum_{l=1}^{\dim V} f_{\mathcal{S},\Upsilon^{(l)},\lambda} b_l^{\mathcal{S}}$. In order to understand the extension in the sense of homogeneous distributions at the North Pole, we use the definition of $f_{\mathcal{S},\Upsilon,\lambda}$ (equation (5.8)) and pass to the trivialization $b_l^{\mathcal{N}}$:

$$F_{\mathcal{S},\underline{\Upsilon},\lambda} = (\sin\varphi)^{-d/2-\lambda/h}\Big(2\tan\Big(\frac{\varphi}{2}\Big)\Big)^{-(n+d/2+\lambda/h)} \sum_{l=1}^{\dim V}\Bigg(\sum_{i=1}^{\dim V} g_{l,i}(u)\Upsilon^{(i)}(u)\Bigg) b_l^{\mathcal{N}}.$$

We see that each coefficient in front of $b_l^{\mathcal{N}}$ is again a homogeneous distribution around $\mathcal{N}$ of degree $\rho^{-d-2\lambda/h-n}$, and we can apply the discussion before Definition 5.7 to extend each of the coefficient distributions. We conclude that the extension remains in the kernel of the indicial operator if and only if one of the following condition holds:

- $\lambda \notin h\mathbb{Z}/2$,
- $(2\,\mathrm{Re}\,\lambda/h + n) < 0$,
- $\lambda = -hk/2$, $k \in \mathbb{Z}$, $k \le n$ and

$$\int_{\mathbb{S}^{d-1}} \left(\Upsilon_\mu(u) \sum_{i=1}^{\dim V} g_{l,i}(u) \Upsilon^{(i)}(u) \right) du = 0 \tag{5.10}$$

  for all $l = 1, \dots, \dim V$, $|\mu| = n - k$, and $\Upsilon_\mu \in \mathbb{R}_{n-k}(\mathbb{S}^{d-1})$.

We will denote the set of all $\underline{\Upsilon} \in (\mathbb{R}_n(\mathbb{S}^{d-1}))^{\dim V}$ that fulfil (5.10) by $\mathcal{N}_{n,n-k}$. Obviously, $\mathcal{N}_{n,n-k} \subset (\mathbb{R}_n(\mathbb{S}^{d-1}))^{\dim V}$ is a subvectorspace.

Lemma 5.10. *Fix a cusp and a unitary irreducible representation (τ, V). Consider*

$$\mathsf{L} = \mathbb{K} \times_\tau V \to \mathbb{S}^d.$$

Let $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$ and $\gamma C_G > d + n + 2|\mathrm{Re}(\lambda/h)|$; consider the operator

$$I(\mathbf{X}_b, \lambda) \colon D^{\gamma m_b}(\mathbb{S}^d, \mathsf{L}) \to \mathsf{H}^{\gamma m_b}(\mathbb{S}^d, \mathsf{L}),$$

and identify $A \in \mathrm{End}(V)^{\mathbb{M}}$ with a complex number by Schur's lemma. We give the following description of the generalized eigenspaces:

$$\mathcal{K}^j_{\lambda,n,\pm} := \ker\Big(I(\mathbf{X}_b, \lambda) + h\Big(\frac{d}{2} + n - A\Big) \pm \lambda \Big)^j$$

by distinguishing the following cases.

- *If $\lambda \notin h\mathbb{Z}/2$, then for all $n \in \mathbb{N}$, there are no Jordan Blocks, i.e.*

$$\mathcal{K}^2_{\lambda,n,\pm} = \mathcal{K}^1_{\lambda,n,\pm}$$

  and

$$\mathcal{K}^1_{\lambda,n,+} = \mathrm{span}\{ f_{\mathcal{S},\Upsilon_\mu,\lambda} b_l^{\mathcal{S}} \mid l = 1, \dots, \dim V,\ |\mu| = n \},$$
$$\mathcal{K}^1_{\lambda,n,-} = \mathrm{span}\{ \delta^{(\mu)}_{\mathcal{N},\lambda} b_l^{\mathcal{N}} \mid l = 1, \dots, \dim V,\ |\mu| = n \}.$$

- *If $\lambda = hk/2$, $k \in \mathbb{N}$, and $n = 0, \dots, k-1$, there are no Jordan Blocks, i.e.*

$$\mathcal{K}^2_{\lambda,n,-} = \mathcal{K}^1_{\lambda,n,-}$$

  and

$$\mathcal{K}^1_{\lambda,n,-} = \mathrm{span}\{ \delta^{(\mu)}_{\mathcal{N},\lambda} b_l^{\mathcal{N}} \mid l = 1, \dots, \dim V,\ |\mu| = n \}.$$

- *For $\lambda = hk/2$ and $n = k, k+1, \dots$, one has Jordan Blocks of index 2, i.e.*

$$\mathcal{K}^3_{\lambda,n,-} = \mathcal{K}^2_{\lambda,n,-}$$

and

$$\mathscr{K}^2_{\lambda,n,-} = \operatorname{span}\{\delta^{(\mu)}_{\mathcal{N},\lambda} b^{\mathcal{N}}_l,\, F_{\mathcal{S},\underline{\Upsilon},\lambda} \mid |\mu| = n,\, l \le \dim V,\, \underline{\Upsilon} \in \mathbb{R}_{n-k}(\mathbb{S}^{d-1})^{\dim V}\},$$
$$\mathscr{K}^1_{\lambda,n,-} = \operatorname{span}\{\delta^{(\mu)}_{\mathcal{N},\lambda} b^{\mathcal{N}}_l,\, F_{\mathcal{S},\underline{\Upsilon},\lambda} \mid |\mu| = n,\, l \le \dim V,\, \underline{\Upsilon} \in \mathscr{N}_{n-k,n}\}.$$

- *If* $\lambda = -hk/2$, $k \in \mathbb{N}$, *and* $n = 0, \dots, k-1$, *there are no Jordan Blocks, i.e.*

$$\mathscr{K}^2_{\lambda,n,+} = \mathscr{K}^1_{\lambda,n,+}$$

and

$$\mathscr{K}^1_{\lambda,n,+} = \operatorname{span}\{F_{\mathcal{S},\underline{\Upsilon},\lambda} \mid \underline{\Upsilon} \in \mathbb{R}_n(\mathbb{S}^d)^{\dim V}\}.$$

- *For* $\lambda = -hk/2$ *and* $n = k, k+1, \dots$, *one has Jordan Blocks of index* 2, *i.e.*

$$\mathscr{K}^3_{\lambda,n,+} = \mathscr{K}^2_{\lambda,n,+}$$

and

$$\mathscr{K}^2_{\lambda,n,+} = \operatorname{span}\{\delta^{(\mu)}_{\mathcal{N},\lambda} b^{\mathcal{N}}_l,\, F_{\mathcal{S},\underline{\Upsilon},\lambda} \mid |\mu| = n-k,\, l \le \dim V,\, \underline{\Upsilon} \in \mathbb{R}_n(\mathbb{S}^{d-1})^{\dim V}\},$$
$$\mathscr{K}^1_{\lambda,n,+} = \operatorname{span}\{\delta^{(\mu)}_{\mathcal{N},\lambda} b^{\mathcal{N}}_l,\, F_{\mathcal{S},\underline{\Upsilon},\lambda} \mid |\mu| = n-k,\, l \le \dim V,\, \underline{\Upsilon} \in \mathscr{N}_{n,n-k}\}.$$

Taking into account that an admissible cusp manifolds has only finitely many cusps and that, over each cusp, the finite-dimensional unitary representation τ_ℓ, V_ℓ that describes the admissible vector bundle over the cusp splits into finitely many irreducible subrepresentations, we obtain the following corollary.

Corollary 5.11. *For an admissible cusp manifold and an admissible vector bundle in the sense of Definition* 1.4, *the indicial roots are affine. Their multiplicities are finite and can be calculated by Lemma* 5.10.

Appendix A. Quantization on manifolds with cusps and propagation of singularities

A.1. Symbols on non-compact spaces

Since we are working with pseudo-differential operators acting on fibre bundles over non-compact manifolds, it is important to clarify what notion of symbols we are using. We want to use symbols in the usual Kohn–Nirenberg class, but we have to be slightly careful to take into account the lack of compactness of the manifold. Throughout our arguments, we refer to C^k functions as functions with C^k regularity, and $\mathscr{C}^k$ functions as elements of the corresponding Banach space. The notation $\mathscr{C}^k$ implies the use of a metric to measure the size of the derivatives. Given a Riemannian or Hermitian vector bundle $L \to M$ over a Riemannian manifold, endowed with a compatible connection, we can also define $\mathscr{C}^k(M, L)$ spaces as well as Sobolev spaces $H^s(M, L)$. We introduce the following definition.

Definition A.1 (Kohn–Nirenberg metric). Assume that (M, g) is a Riemannian manifold. Then its cotangent bundle decomposes as

$$T(T^*M) = H \oplus V,$$

where $V = \ker d\pi$, with $\pi\colon T^*M \to M$ the usual projection. The so-called *horizontal* space H is given by the Levi-Civita connection. We have natural identifications

$$V \simeq H \simeq TM,$$

so we can define horizontal and vertical lifts – see [26]. We define the metric $\overline{g}$ on T^*M by

$$\overline{g}_{(x,\xi)}(X^v + Y^h, W^v + Z^h) = g_x(Y, Z) + \frac{1}{1 + g(\xi, \xi)} g_x(X, W).$$

Lemma A.2. *Assume that the curvature tensor of (M, g) is bounded and so are all its covariant derivatives. Then the same holds for $(T^*M, \overline{g})$.*

This can be proved using the expressions for the curvature tensor of such a metric presented in [26]. From now on, whenever M is a Riemannian manifold, its cotangent bundle will be endowed with $\overline{g}$.

Definition A.3 (Symbol classes). Let $(L, \|\cdot\|) \to (M, g)$ be a Riemannian or Hermitian vector bundle over M with compatible connection ∇. Assume that both the curvatures of L and M are bounded, as are all their covariant derivatives. Then the *semi-classical Kohn–Nirenberg symbols $S^n(M, L)$* on L of order n are family of sections $\sigma_h\colon T^*M \mapsto \mathscr{L}(L, L)$ parametrized by a parameter $0 < h \leq h_0$ such that, for all $k \in \mathbb{N}$, there is C_k independent of h such that

$$\|\nabla^k \sigma_h(x, \xi)\| \leq C_k \langle \xi \rangle^n.$$

Note that $\nabla^k \sigma_h$ is a section of the bundle $(T^*(T^*M))^{\otimes k} \otimes \mathcal{L}(L, L) \to T^*M$ and the norm $\|\bullet\|$ is constructed by the operator norm on $\mathcal{L}(L, L)$ and the Kohn–Nirenberg metric $\overline{g}$ on $T^*(T^*M)$.

For the definition of the anisotropic Sobolev spaces, we need the following class of anisotropic symbol classes.

Definition A.4. Let $m \in S^0(M)$ be an order zero Kohn–Nirenberg symbol which we call an *order function*. The space of *anisotropic symbols* $S^m_{\log}(M, L)$ consists of those sections $\sigma_h\colon T^*M \mapsto \mathscr{L}(L, L)$ parametrized by a parameter $0 < h \leq h_0$ such that, for all $k \in \mathbb{N}$, there is C_k independent of h such that

$$\|\nabla^k \sigma_h\| \leq C_k |\log(1 + \langle \xi \rangle)|^k \langle \xi \rangle^{m(x,\xi)}.$$

Note that the loss of $\log\langle \xi \rangle$ is necessary for the space of anisotropic symbols to contain sufficiently interesting elements such as for example $\langle \xi \rangle^{m(x,\xi)}$.

In the sequel, we will usually drop the parameters h to simplify the notation unless we want to emphasize dependence on h.

Consider two symbols $\sigma, \varsigma \in S(M, L)$. We say that σ is *scalar* if it takes the form $\sigma' \mathbb{1}$ with $\sigma' \in S(M, \mathbb{R})$. In this case, we define the Poisson bracket

$$\{\sigma, \varsigma\} := \nabla_{H_{\sigma'}} \varsigma,$$

where $H_{\sigma'}$ is the Hamiltonian vector field of $\sigma' \in S(M, L) \subset C^\infty(T^*M)$.

Proposition A.5. *Under the assumptions of the definition, the sets of symbols, i.e. the union $S(M, L) := \bigcup_n S^n(M, L)$ and $S_{\log}(M, L) := \bigcup_n S^n_{\log}(M, L)$, satisfy all the usual properties. They are stable by product, sum, division by elliptic symbols, and graded by the order in the usual sense. They are also stable under the Poisson bracket provided one of the symbols is scalar.*

It is important to notice that the set of symbols $S(N, \mathbb{R})$, where N is an admissible cusp manifold, is exactly the same class of symbols as was described in the paper [6]. It is straightforward to show that the proofs there apply to $S_{\log}$ (the largest of all classes here).

To close this section, we consider the radial compactification of the cotangent space.

Definition A.6. Let $\overline{T^*M}$ be the radial compactification of the cotangent space. It has a structure of continuous manifold, but not of C^∞ manifold a priori. We consider the map

$$\mathsf{comp}\colon (x, \xi) \to \Big(x, \frac{\xi}{1 + \langle \xi \rangle}\Big) = (x, \xi').$$

This is a homeomorphism of $\overline{T^*M}$ to $\overline{B(0,1)}$ in T^*M, and it endows $\overline{T^*M}$ with the structure of a smooth manifold with boundary. Let $\tilde{g} = \mathsf{comp}^* \, \overline{g}$, and define $\mathscr{C}^k_{\tilde{g}}$ norms on $\overline{T^*M}$ using $\tilde{g}$. Then we define the *classical symbols* as

$$S^0_{\mathrm{cl}}(M) := \mathscr{C}^\infty_{\tilde{g}}(\overline{T^*M}),$$

and for $k \in \mathbb{Z}$, we set $S^k_{\mathrm{cl}}(M) := \langle \xi \rangle^k \mathscr{C}^\infty_{\tilde{g}}(\overline{T^*M})$.

Note that, for the prescribed smooth structure on $\overline{T^*M}$, $\langle \xi \rangle^{-1}$ is a boundary defining function. In particular, because classical symbols are smooth up to the boundary, they have a homogeneous expansion as $\xi \to \infty$.

Proposition A.7. *We have the inclusion $S^0_{\mathrm{cl}}(M) \subset S^0(M)$.*

Proof. We set $\overline{g}' = \mathsf{comp}_* \, \overline{g}$ – it is a metric on the open ball $B(0,1)$. Then, close to $|\xi'| = 1$,

$$\overline{g}' = \frac{1}{1 - |\xi'|_x} (\xi' d\xi')^2 + g',$$

where g' is a smooth symmetric 2-form, and $\overline{g} \le C \overline{g}'$. Passing back to $\overline{T}^* M$, we deduce $\tilde{g} \le C \overline{g}$. This is sufficient to deduce that the $\mathscr{C}^k$ norms of $\tilde{g}$ control those of $\overline{g}$.

Since $\tilde{g}$ has bounded curvature, and derivatives thereof, one can estimate its $\mathscr{C}^k$ norms using flat derivatives in exponential coordinates in balls of size ~ 1. In other words, we can restrict our attention to the open unit ball in $\mathbb{R}^n$, and assume that $\overline{g} \ge 1$. We then have to show that the exponential map $\exp_0^{\overline{g}}$ has uniformly bounded derivatives (on the unit ball for $\overline{g}$).

We start by observing that it maps the unit ball for $\overline{g}$ inside the standard unit ball (because geodesics for $\overline{g}$ travel at speed $\sim 1/\|g\|^{1/2} \leq 1$). The derivative of the exponential map can be expressed in terms of Jacobi fields along geodesics through 0, and these fields satisfy equations involving only the curvature tensor of $\overline{g}$. For the higher derivatives, we also have a description using fields satisfying a Jacobi equation, with a forced term this time. The forcing term is itself a covariant derivative of the curvature tensor along Jacobi fields. In this way, one sees that the derivatives of the exponential map are all controlled using only the curvature of $\overline{g}$ and its covariant derivatives. Since those are bounded with respect to $\overline{g}$, they are also bounded with respect to 1, and we are done.

This proves that $C^\infty_{\tilde{g}} \subset C^\infty_{\overline{g}}$, and thus the inclusion

$$S^0_{\mathrm{cl}}(M) = \mathscr{C}^\infty_{\tilde{g}}(\overline{T^*M}) \subset \mathscr{C}^\infty_{\overline{g}}(T^*M) = S^0(M). \qquad \blacksquare$$

A.1.1. Symbols on cusps. Symbols on cusps have a particular structure that is central to all the arguments of the article. Consider a cusp Z and a symbol $\sigma \in S^n(Z)$. For the extension to trivially fibred cusps, nothing different happens, so we concentrate on $S^n(Z)$. The symbol estimates take the form (recall that $\xi = Y\,dy + J\,d\theta$)

$$|(y\partial_y)^\alpha (y\partial_\theta)^\beta (y^{-1}\partial_\xi)^{\alpha'} (y^{-1}\partial_J)^{\beta'}\sigma| \leq C\big(1 + y^2(Y^2 + J^2)\big)^{\frac{n-\alpha'-|\beta'|}{2}}.$$

We change variables to $r = \log y$, $yY = \lambda$. We get that

$$|(\partial_r)^\alpha (e^r\partial_\theta)^\beta (\partial_\lambda)^{\alpha'} (e^{-r}\partial_J)^{\beta'}\sigma| \leq C(1 + \lambda^2 + e^{2r}J^2)^{\frac{n-\alpha'-|\beta'|}{2}}.$$

Now a case of special importance will be symbols that do not depend on θ. When that is the case, we deduce from the estimate above that

$$\sigma = \widetilde{\sigma}(r; \lambda, e^r J), \tag{A.1}$$

where $\widetilde{\sigma}$ is a symbol on $\mathbb{R}_r \times \mathbb{R}^2_\xi$ in the usual sense that

$$|\partial^\alpha_r \partial^\beta_\xi \widetilde{\sigma}| \leq C\langle\xi\rangle^{n-|\beta|}.$$

A.2. *Quantization on fibred cusps*

We will use a quantization procedure similar to that presented in [6]. For most of the technical details, we refer to that article; we will only clarify a few points.

We want to obtain operators on trivially fibred cusp $Z \times \mathsf{F}$ (see Definition 4.1). The Schwartz kernels will be understood as taken with reference to the *Euclidean* volume form on the cusp, $dy\,d\theta\,d\mathrm{vol}_{\mathsf{F}}(\zeta)$.

First off, let us write $k = \dim \mathsf{F}$; given an open set $U \subset \mathbb{R}^k$, and a symbol σ on $T^*(Z \times U)$, we define an operator $\mathrm{Op}^{w,0}_{h,Z\times U}(\sigma)$ on $\mathbb{H}^{d+1}_{(y,\theta)} \times U_\zeta$ by the kernel

$$\frac{1}{(2\pi h)^{d+1+k}} \int e^{i/h\Phi}\sigma\Big(\frac{y+y'}{2}, \frac{\theta+\theta'}{2}, \frac{\zeta+\zeta'}{2}, Y, J, \eta\Big)\,d\xi\,\Big(\frac{y}{y'}\Big)^{(d+1)/2}, \tag{A.2}$$

where Φ is the usual phase function

$$\langle y - y', Y\rangle + \langle \theta - \theta', J\rangle + \langle \zeta - \zeta', \eta\rangle.$$

Here, we have identified the symbol σ with the corresponding Λ_Z-periodic function on $T^*(\mathbb{H}^{d+1} \times U)$. The operator we obtain is well defined on Λ_Z-periodic functions, preserves them, and so we obtain an operator acting on $Z \times U$ – as was explained in [6, page 319].

Actually, there is the slight inconvenience that we do not know exactly how the kernel of the quantization $\mathrm{Op}^{w,0}_{h,Z\times U}$ decays far from the diagonal. To avoid this discussion altogether, we take a function $\chi^{\mathrm{Op}} \in C^\infty_c(]-C, C[)$ equal to 1 around 0, and we let $\mathrm{Op}^{w}_{h,Z\times U}(\sigma)$ be the operator on $Z \times U$ whose kernel is

$$K_{\mathrm{Op}^{w,0}_{h,Z\times U}(\sigma)}\chi^{\mathrm{Op}}\Big(\log\frac{y}{y'}\Big). \tag{A.3}$$

Taking local charts on F, we can thus build a quantization $\mathrm{Op}^{w}_{h,Z\times\mathsf{F}}$ on a trivially fibred cusp.

We next want to define a quantization on general admissible vector bundles $L \to M$ in the sense of Definition 4.3. Given a relatively compact open set $U \subset M$, we can take a coordinate patch to $\mathbb{R}^{d+1} \times \mathbb{R}^k$ that maps the volume form to the standard volume form of $\mathbb{R}^{d+1+k}$. Such a chart will be called a *compact chart*, and we will use the ordinary Weyl quantization on these compact charts (see e.g. [59, § 4.1.1 and Theorem 14.1]). Now we have another type of charts: they are supported on open sets of the form $U^Z_{\mathsf{a}} \times \mathsf{U}$ with $U^Z_{\mathsf{a}} = \{z \in Z \mid y(z) > \mathsf{a}\}$ and $\mathsf{U} \subset \mathsf{F}$ with $\mathsf{a} \geq \mathbf{a}$. They can be mapped to open sets of the form $U^Z_{\mathsf{a}} \times U$ with U relatively compact in $\mathbb{R}^k$. We will also impose that the volume form on the fibres F is sent to the standard volume of $\mathbb{R}^k$, which is possible because the metric takes a product form. Such a chart will be called a *cusp chart*.

On any such open chart, we can define a quantization for sections of L by tensorizing a quantization on functions with a local orthogonal frame for L. This can be done over cusp charts because of the product structure of L.

In particular, we can choose $\mathsf{a} \geq \mathbf{a}$ and find a corresponding finite cover U_ℓ of M by compact charts or cusp charts and a corresponding partition of unity $\sum \chi_\ell^2 = 1$. Then we define for $\sigma \in S(M, L)$ its quantization $\mathrm{Op}^w_{h,L}(\sigma)\colon C^\infty_c(M, L) \to C^\infty(M, L)$ by

$$\mathrm{Op}^w_{h,L}(\sigma) f := \sum_\ell \chi_\ell \,\mathrm{Op}^w_{h,U_\ell,L}(\sigma)\chi_\ell f. \tag{A.4}$$

The notation will soon be shortened to just Op, and we obtain the following proposition.

Proposition A.8. *Let L be an admissible bundle, and consider (for $j = 1, 2$),*

$$\sigma_j \in S^{n_j}(M, L) \quad \textit{for } n_j \in \mathbb{R}$$

(respectively in $S^{n_j}_{\log}$ with $n_j \in S^0(M, L)$). We have several properties, valid in the limit $h \to 0$.

(1) *There exists a third symbol $\varpi \in S^{n_1+n_2}(M, L)$ (respectively $S^{n_1+n_2}_{\log}(M, L)$) such that*

$$\mathrm{Op}(\sigma_1)\,\mathrm{Op}(\sigma_2) = \mathrm{Op}(\varpi) + R,$$

where the remainder $\|R\|_{H^{-N}\to H^N} = \mathcal{O}(h^\infty)$ *for all* $N \in \mathbb{N}$ *and* R *is uniformly properly supported. Furthermore, we have*

$$\varpi - \sigma_1\sigma_2 \in hS^{n_1+n_2-1} \quad (\textit{resp. } h\log(1+\langle\xi\rangle)S^{n_1+n_2-1}_{\log}).$$

(2) *Assume that* σ_1 *is scalar. Then we have the commutator formula*

$$[\mathrm{Op}(\sigma_1), \mathrm{Op}(\sigma_2)] = \frac{h}{i}\mathrm{Op}(\{\sigma_1, \sigma_2\}) + \mathcal{O}(h^2 S^{n_1+n_2-2}).$$

The remainder worsens to $h^2\log(1+\langle\xi\rangle)^2 S^{n_1+n_2-2}_{\log}$ *in the case of exotic symbols.*

(3) *If* σ *is hermitian valued,* $\mathrm{Op}(\sigma)$ *is symmetric.*

(4) *For* $\sigma \in S^0_{\log}(M, L)$*,* $s \in \mathbb{R}$*,* $\mathrm{Op}(\sigma)\colon H^s(M, L) \to H^s(M, L)$ *is bounded uniformly in* $h > 0$*.*

(5) *For* $n \in S^0(M, L)$*, if* $\sigma \in S^n_{\log}(M, L)$ *is elliptic, i.e. if* $\sigma^{-1} \in S^{-n}_{\log}(M, L)$*, then* $\mathrm{Op}(\sigma)$ *is invertible on Sobolev spaces for h small enough so that we can let*

$$H^n(M, L) := \mathrm{Op}(\sigma)^{-1}L^2(M, L).$$

(6) *Assume that* $\partial_\theta\sigma = 0$ *for* $y \geq \mathsf{a}$*. Consider* f *supported in some fibred cusp end* $M_\ell = Z_\ell \times \mathsf{F}$*, and assume that* f *takes the form* $e^{ik\theta}g(y, \zeta)$*, where* $k \in \Lambda'_\ell$ *and* g *is supported in* $\{y > C\mathsf{a}\}$*. Taking* $C > 1$ *large enough depending only on the partition of unity appearing in* (A.4)*,* $\mathrm{Op}(\sigma)f$ *has the same form except that it is now supported in* $\{y > \mathsf{a}\}$*. In particular,* $\mathrm{Op}(\sigma)$ *preserves Fourier modes exactly.*

The stabilization of Fourier modes is a nice feature from which we profit because we have assumed that the curvature is constantly -1 in the cusps. In a more general case of curvature tending to -1, one would have to look for more subtle estimates.

Remark A.9. The remainder R in the product formula can actually be written as a $\mathrm{Op}'(r)$, with r an $\mathcal{O}(h^\infty S^{-\infty})$ symbol, if Op' is another quantization built in the same fashion, but where the cutoff away from the diagonal has been changed to another one with sufficiently larger support.

We will also need a sharp Gårding lemma.

Lemma A.10 (Sharp Gårding). *Let* $\sigma \in S^1(M, L)$*. Assume that* $\mathrm{Re}(\sigma) \geq 0$*. Then*

$$\mathrm{Re}(\langle \mathrm{Op}(\sigma)u, u\rangle) \geq -Ch\|u\|^2_{L^2}.$$

We will prove together Proposition A.8 and Lemma A.10.

Proof. Proofs for (1)–(5) can be found in [6] for non-exotic symbols. The arguments, however, all transfer to exotic symbols. Note that the key argument in [6] is that, for a symbol on a cusp, in an interval at height y_0, the cusps can be rescaled such that one transfers the problem to an Euclidean cylinder. The crucial point is that the symbols transform uniformly in y_0 under this rescaling (see [6, Section 1.3]). Furthermore, since we introduced a cutoff away from the diagonal in our quantization (A.3), we can rescale the whole operator from a neighbourhood of $y = y_0$ to a fixed Euclidean cylinder with uniform estimates. Much as in [6], the proof of boundedness and other estimates follow from the estimates holding on $\mathbb{R}^n$. It is also the case for the sharp Gårding estimate.

Let us say a word on property (6) in Proposition A.8. Inspecting formula (A.2), we observe that, in the θ variable, the kernel is just a Fourier transform of σ in the J variable. Such an operator commutes with ∂_θ and thus preserves Fourier modes. To be able to use this formula, we just need that the support of f does not intersect the support of the cutoff function χ_ℓ corresponding to compact charts, hence the condition that g is supported in $\{y \geq C\mathsf{a}\}$. ∎

Following [6, Lemma 1.8], we can prove that our operators actually act as pseudo-differential operators, and that our quantization is a quantization in the usual sense.

Definition-Proposition A.11. *Take $m \in S^0(M, \mathbb{R})$ scalar. Let $\Psi^m(M, L)$ be the algebra of operators generated by operators of the form* $\mathrm{Op}(\sigma)$ *with $\sigma \in S^m_{\log}(M, L)$. We call them the algebra of* semi-classical pseudo-differential operators *(or also just* pseudo-differential operators *in short).*

On Ψ^m, we have a principal symbol map σ^0_m which is defined independently of the choice of quantization Op *as a map $\sigma^0_m \colon \Psi^m \to S^m_{\log}/hS^{m-1}_{\log}$, with $\sigma^0(\mathrm{Op}(\sigma)) = [\sigma]$.*

Once we have fixed a quantization, we obtain by iterations a full symbol map

$$\sigma \colon \Psi^m \to S^m/h^\infty S^{-\infty}.$$

A.3. Semi-classical ellipticity and wavefront sets

Let us recall the notions of wavefront set and ellipticity.

Definition A.12. Let $A \in \Psi^m(M, L)$. We say that $(x, \xi) \in \overline{T^*M}$ is not in the wavefront set $\mathrm{WF}_h(A)$ of A if and only if $\|\sigma(A)(x', \xi')\| = \mathcal{O}(h^\infty \langle \xi' \rangle^{-\infty})$ in an open neighbourhood of (x, ξ). We say that A is *microsupported* in a set $S \subset \overline{T^*M}$ if and only if $\mathrm{WF}_h(A) \subset S$.

For an order m pseudor $A \in \Psi^m(M, L)$, we also define the δ-elliptic set for some $\delta > 0$,

$$\mathrm{ell}_\delta(A) = \{(x, \xi) \mid \|\langle \xi \rangle^m \sigma(A)^{-1}\| < \delta^{-1}\},$$

and we define the set of elliptic points by $\mathrm{ell}(A) = \bigcup_{\delta > 0} \mathrm{ell}_\delta(A)$.

A family of distributions $u_h \in \mathcal{D}'(M, L)$ parametrized by $0 < h \leq h_0$ is called h-tempered if there is $N \in \mathbb{N}$ such that $\|u_h\|_{H^{-N}} = \mathcal{O}(h^{-N})$. For any h-tempered family of distributions u_h, we say that (x, ξ) is *not* in $\mathrm{WF}_h(u)$ if and only if there is $A \in \Psi^0(M, L)$, $\delta > 0$, such that $(x, \xi) \in \mathrm{ell}_\delta(A)$ and

$$Au = \mathcal{O}_{H^\infty}(h^\infty).$$

We let $\mathrm{WF}(u) = \mathrm{WF}_h(u) \cap \partial \overline{T^*M}$. As usual, we call WF the *classical wavefront set*, and WF_h the *semi-classical wavefront set*. The first measures the regularity in terms of C^k spaces, while the second additionally measures a finer regularity as $h \to 0$.

Finally, we also have a concept of wavefront set for operators. Given an operator K with kernel $K(x, x')$, we let

$$\mathrm{WF}'_h K := \{(x, \xi; x', -\xi') \mid (x, \xi; x', \xi') \in \mathrm{WF}_h(K(\cdot, \cdot))\} \subset \overline{T^*}(M \times M).$$

When $A \in \Psi(M, L)$, $\mathrm{WF}'_{(h)}(A)$ is the image of $\mathrm{WF}_{(h)}(A)$ under the diagonal embedding $\overline{T^*}M \to \overline{T^*}(M \times M)$.

Lemma A.13. *Let K be an operator on sections of $L \to M$. Then*

$$(x, \xi; x', \xi') \in T^*(M \times M) \text{ is not in } \mathrm{WF}'_h(K)$$

if and only if there are pseudors A and B, 1-elliptic respectively at (x, ξ) and (x', ξ') so that

$$AKB = \mathcal{O}_{H^{-\infty} \to H^{\infty}}(h^{\infty}).$$

Proof. See e.g. [17, Lemma 2.3]. ∎

Proposition A.14 (Elliptic regularity). *Take $P \in \Psi^k(M, L)$, $A \in \Psi^0(M, L)$ and u a tempered family of distributions. Then the following statements hold.*

(1) *Let $\delta > 0$ and $\mathrm{WF}_h(A) \subset \mathrm{ell}_\delta(P)$; then there is*

$$Q \in \Psi^{-k}(M, L) \quad \text{with } \mathrm{WF}_h(Q) \subset \mathrm{WF}_h(A)$$

such that

$$A = QP + \mathcal{O}(h^\infty \Psi^{-\infty}).$$

(2) *Let $\delta > 0$, $r \in \mathbb{R}$ and $\mathrm{WF}_h(A) \subset \mathrm{ell}_\delta(P)$; then there is a constant C such that*

$$\|Au\|_{H^r} \leq C_\delta \|Pu\|_{H^{r-k}} + \mathcal{O}_{H^\infty}(h^\infty).$$

(3) *As a consequence,*

$$\mathrm{WF}_h(u) \cap \mathrm{ell}(P) \subset \mathrm{WF}_h(Pu).$$

Proof. (1) follows from a standard inductive parametrix construction (see e.g. [18, Proposition E.32]). The notion of δ-elliptic set has been introduced precisely to assure that the construction yields a symbol in the uniform symbol classes.

(2) follows from $\|Au\|_{H^r} = \|QPu\|_{H^r} + \mathcal{O}(h^\infty)$ after applying the uniform operator norm estimate (Proposition A.8 (4)) to Q.

For (3), assume that $(x, \xi) \in \mathrm{ell}(P)$; then this particular point is also in $\mathrm{ell}_\delta(P)$ for some $\delta > 0$. Assume further $(x, \xi) \notin \mathrm{WF}_h(Pu)$. By definition, there is $B \in \Psi^0(M, L)$ with $(x, \xi) \in \mathrm{ell}_\delta(B)$ such that $BPu \in \mathcal{O}_{H^\infty}(h^\infty)$. Now we apply (2) to the operator B and use that $(x, \xi) \in \mathrm{ell}_\delta(BP)$. We thus get $A \in \Psi^0$ with $(x, \xi) \in \mathrm{ell}_\delta(A)$ fulfilling $Au \in \mathcal{O}_{H^\infty}(h^\infty)$. ∎

A.4. Propagation of singularities and other estimates

Throughout the paper, to obtain results on the wavefront sets of several operators, we have used lemmas that were almost identical to some lemmas in [17]. In this section, we give the versions on admissible cusp manifolds.

For the most part, the proofs given in [17, Appendix] are also valid in our case. As a consequence, this is a cursory review of some special technicalities, destined to the reader already acquainted with the detail of the arguments in [17].

Remark A.15. The only difference in our setting from the usual quantization on compact manifolds is that we do *not* have the conclusion of Beal's theorem, i.e. we cannot incorporate smoothing $\mathcal{O}(h^\infty)$ remainders that are not properly supported in the symbols. However, that is not a problem because Beal's theorem is not invoked in [17].

Since we will refer to [17] for details, we explain the correspondence between our lemmas and theirs. Proposition 2.11 is where the following lemmas will be used. It is the equivalent of the wavefront set part of [17, Proposition 3.4]. Its proof employs Lemma A.13 and Propositions A.14, A.16 and A.18. Lemma A.13 is equivalent to [17, Lemma 2.3]; Proposition A.14 is similar to [17, Proposition 2.4], and Proposition A.18 to [17, Proposition 2.6]. Now we turn to the most involved one, the propagation of singularities (Lemma A.16), equivalent to [17, Lemma 2.5].

Lemma A.16 (Propagation of singularities). *Let $\mathbf{X} \in \Psi^1(M, L)$ have a scalar principal symbol of the form*

$$[ip - q] \in S^1/hS^0,$$

with p, q real, and $q \geq 0$. Also assume that $p \in S^1_{\mathrm{cl}}(M)$. Take a tempered family u_h, and $\delta > 0$.

(1) *Consider $A, B, B_1 \in \Psi^0(M, L)$ such that B, B_1 δ-control A in time T_0. That is, whenever $(x, \xi) \in \mathrm{WF}_h(A)$, there exists $0 < T < T_0$ such that $e^{TH_p}(x, \xi) \in \mathrm{ell}_\delta(B)$ and $e^{tH_p}(x, \xi) \in \mathrm{ell}_\delta(B_1)$ for all $t \in [0, T]$. Then, for each weight $m \in S^0(M, \mathbb{R})$,*

$$\|Au\|_{H^m(M,L)} \leq C_\delta \|Bu\|_{H^m(M,L)} + \frac{C_\delta}{h} \|B_1 \mathbf{X} u\|_{H^m(M,L)} + \mathcal{O}(h^\infty).$$

(2) *As a consequence, if $(x, \xi) \notin \mathrm{WF}(u)$ and $e^{-tH_p}(x, \xi) \notin \mathrm{WF}(\mathbf{X}u)$ for $t \in [0, T]$, then $e^{-TH_p}(x, \xi) \notin \mathrm{WF}(u)$.*

The constants are $\mathcal{O}(1)e^{\mathcal{O}(T_0)}$, but we will not need this fact. One can mimic the proof in [17] step by step. Be mindful that $\operatorname{Re} \mathbf{P}$ has to be replaced by $-\operatorname{Im} \mathbf{X}$, and $\operatorname{Im} \mathbf{P}$ by $\operatorname{Re} \mathbf{X}$.

Proof. In the whole proof, when working on subsets of $\overline{T^*M}$, we work with the notion of distance obtained on $\overline{T^*M}$ obtained by pulling back the distance on $B(0,1) \subset T^*M$ by the map comp defined in Definition A.6. Since $p \in S^1_{\mathrm{cl}}$, e^{tH_p} is a smooth flow for this structure. Additionally, we can always assume that the symbols of A, B and B_1 are in S^0_{cl}, i.e. smooth up to the boundary of $\overline{T^*M}$.

To start with, applying a partition of unity argument, we can assume that A is microsupported in a ball with small radius $\epsilon_0 > 0$. Then we can also assume that B is microsupported in a $3\epsilon_0$-neighbourhood of the image $e^{TH_p}(\mathrm{WF}_h(A))$ for some $T \in [0, T_0]$, and B_1 is microsupported in a $3\epsilon_0$-neighbourhood of the union $\bigcup_{t\in[0,T]} e^{tH_p}(\mathrm{WF}_h(A))$.

Since the proof in [17] is based on local considerations along the trajectories of the flow in bounded time, and we are *not* seeking to determine the behaviour of the constants when the time T_0 goes to infinity, we already observe that the estimate holds if A is supposed to be microsupported in a fixed compact set of M, with constants that depend on the compact set. As a consequence, we can restrict our attention to the case when A, B, B_1 are supported in a fibred cusp end M_ℓ, above a set of the form $\{y > y_0\}$ with y_0 arbitrary large, and satisfy symbol estimates with constants *not* depending on y_0.

From the structure of symbols estimates in the cusps – see equation (A.1) – we deduce that we can find operators $\widetilde{B}$, $\widetilde{B}_1$ such that

$$\mathrm{WF}_h(\widetilde{B}) \subset \mathrm{ell}_{\delta/2}(B), \quad e^{TH_p}(\mathrm{WF}_h(A)) \subset \mathrm{ell}_{\delta/2}(\widetilde{B}) \quad \text{and} \quad [\partial_\theta, \widetilde{B}] = 0,$$

and similarly for $\widetilde{B}_1$ and B_1. The idea is that the symbols of B and B_1 are almost invariant under rotations in the θ variable high in the cusp so that we can forget that variable altogether. Indeed, then we replace A by $\tilde{A}$ such that $|A| \leq \tilde{A}$, and $\tilde{A}$ also is invariant under rotations, and its wavefront set takes the form

$$\mathrm{WF}_h(\tilde{A}) = \{(y', \theta, \zeta') \mid (y', \theta_0, \zeta') \in B((y, \theta_0, \zeta); 2\epsilon_0),\ \theta \in \mathbb{R}^d/\Lambda_Z\}.$$

(ζ designates a generic point in the generic fibre M of $M \to N$).

Let us do some more reduction. The vector field H_p acts in a uniform C^∞ fashion on $\overline{T^*M}$, and as such, $|\nabla H_p|_{L^\infty} < \infty$. Additionally, by symbol estimates, we know that $\partial_\theta p = \mathcal{O}(y^{-\infty})$. As a consequence, for y large enough, an escape function for $\overline{p} = \int p\, d\theta$ is also an escape function for p. In other words, we can assume that p does not depend on θ. Then H_p commutes with ∂_θ.

Consider that, in the cusp, we have an additional fibre structure. Indeed, write

$$M_\ell = (\mathbb{R}^d/\Lambda_\ell)_\theta \times \mathbb{R}_r \times \mathsf{F}_\zeta\,.$$

Then we can see T^*M_ℓ as a fibre bundle

$$\mathsf{Proj}\colon T^*M_\ell = [(\mathbb{R}^d/\Lambda_\ell)_\theta \times \mathbb{R}^d] \times T^*[\mathbb{R}_r \times \mathsf{F}_\zeta] \to \mathsf{M}^0 := \mathbb{R}^d \times T^*(\mathbb{R} \times \mathsf{F})$$

by forgetting the θ variable. Seeing M^0 as a vector bundle over $\mathbb{R} \times \mathsf{F}$, we can also extend Proj as a map $\overline{T^*M_\ell} \to \overline{\mathsf{M}^0}$. Since H_p commutes with ∂_θ, it projects to a vector field H_p^0 on the base M^0. Then, for $\delta' > 0$, let

$$U_{\delta'} := \{(x, \xi) \in \overline{T^*M_\ell} \mid |H_p^0(x, \xi)| < \delta' e^{-CT_0}\},$$

with $C/|\nabla H_p|_{L^\infty} > 1$. These are θ invariant sets.

Provided C was chosen large enough, when $(x, \xi) \in U_{\delta'}$, $e^{tH_p}(x, \xi) \in U_{e^{CT_0}\delta'}$ for $t \in [0, T_0]$ so that

$$d\big(\mathsf{Proj}(x, \xi), \mathsf{Proj}(e^{tH_p}(x, \xi))\big) = \mathcal{O}(\delta').$$

Since the symbol estimates on the symbol of B are uniform over the whole manifold, we deduce that, when $e^{TH_p}(x, \xi) \in \mathrm{ell}_\delta(B)$ for some $T \in [0, T_0]$ and $(x, \xi) \in U_{\delta'}$, then $(x, \xi) \in \mathrm{ell}_{\delta/2}(B)$, provided δ' is small enough – and smaller and smaller as the symbol of B is allowed to become more singular. In such a case, we can apply directly the elliptic estimate (Proposition A.14) to conclude.

Now we can concentrate on the case when $\mathrm{WF}_h(A) \cap U_{\delta'} = \emptyset$. But the injectivity radius of $\mathbb{R} \times \mathsf{F}$ is positive, and the vector field H_p^0 is $\mathscr{C}^\infty$. As a consequence, on the complement of $U_{\delta'}$, we can apply a formal form for non-vanishing vector fields to obtain tubular coordinates.

We can build a local section of the flow $z \to (x(z), \xi(z))$ from a small open set $U_{\text{tube}} \subset \mathbb{R}^\ell$ around 0 to $\overline{\mathsf{M}^0}$ with $(x(0), \xi(0)) = (x, \xi)$, and a local diffeomorphism

$$\mathsf{Coord}\colon (z, \tau) \in U_{\text{tube}} \times \left]-\tfrac{1}{2}, T_0 + \tfrac{1}{2}\right[\mapsto e^{\tau H_p^0}(x'(z)) \in \mathsf{Proj}(U^c_{\delta' e^{-CT_0}}).$$

We can choose these coordinates so that they satisfy $\mathscr{C}^k$ estimates that do not depend on the central point (x, ξ), and the size of the open set U_{tube} is fixed also independently of (x, ξ). If the point (x, ξ) is close to a periodic orbit, this map is not injective, but the map is injective on each set of the form $\{|\tau - \tau_0| < \delta''\}$ for $\delta'' > 0$ small enough.

Consider a function $\chi \in C_c^\infty(U_{\text{tube}})$ equal to 1 around 0, and $\psi \in C_c^\infty(]-\frac{1}{2}, T + \frac{1}{2}[)$ such that $\psi(\tau) > 1$ for $\tau \in [0, T_0]$, $\psi \geq 0$ everywhere, and

$$\psi'(\tau) \geq C\psi(\tau) \quad \text{for } \tau \in \left[-\tfrac{1}{2}, T - \tfrac{1}{2}\right].$$

Finally, let

$$f(x', \xi') := \sum_{\mathsf{Coord}(z,\tau) = (x', \xi')} \chi(z)\psi(\tau). \tag{A.5}$$

When the trajectory of (x, ξ) is sufficiently far from periodic trajectories, the sum is reduced to 1 element, but there *may* be periodic points. Now we need to check that f thus defined satisfies symbol estimates independently of (x, ξ). There are two things to verify. First, since the tubular coordinates were constructed with uniform $\mathscr{C}^n$ norms, each branch in equation (A.5) satisfies uniform $\mathscr{C}^n$ estimates. Then we need to check that there are a finite number of such branches. But, from the local injectivity of the tubular coordinates, the sum has at most T/δ'' non-vanishing terms.

Now that we have an escape function adapted to the problem, the rest of the proof in [17] follows through. ∎

Before going to the equivalent of [17, Proposition 2.6], let us recall the definition of radial sinks.

Definition A.17. Let L be a conic subset of $T^*M \setminus \{0\}$. Assume that it is invariant under Φ_t. Also assume that, for some $\epsilon > 0$, its ϵ-conic neighbourhood U_ϵ is such that if κ is the projection on $\partial\overline{T^*M}$,

$$d(\kappa(e^{tH_p}U), \kappa(L)) \to 0 \quad \text{as } t \to +\infty,$$

and for some constant $C_0 > 0$, $|e^{tH_p}(x, \xi)| > Ce^{C_0 t}|\xi|_x$ whenever $(x, \xi) \in U$. Then L is a *radial sink*.

Note that $E_u^* \subset T^*M$ is a radial sink (cf. Lemma 2.4). Now we can state the high regularity radial sink estimate analogous to [17, Proposition 2.6] (note that their terminology of sink and source is reversed compared to ours, as they propagate in the opposite time direction). We will not introduces sources since we will not use them.

Proposition A.18 (Sink estimate). *Let* $\mathbf{X}$ *be as in Lemma* A.16*. Let* L *be a radial sink. Then there exists* $k_0 > 0$ *such that, for some* $\epsilon > 0$*,*

(1) *for all* $C \in \Psi^0$*, with* $\kappa(L) \subset ell_\epsilon(C)$*, there exists* $C_1 \in \Psi^0$ *also* ϵ*-elliptic around* $\kappa(L)$ *such that, whenever* u *is tempered and* $k \geq k_0$*,*

$$C_1 u \in H^{k_0} \implies \|C_1 u\|_{H^k} \leq C h^{-1} \|C\mathbf{X}u\|_{H^k} + \mathcal{O}(h^\infty),$$

(2) *as a consequence, if* $Cu \in H^{k_0}$ *and* $\mathrm{WF}(\mathbf{X}u) \cap \kappa(L) = \emptyset$*, then* $\mathrm{WF}(u) \cap \kappa(L) = \emptyset$*.*

Proof. Inspecting the proof in [17], the arguments are very similar to those in the proof of Lemma A.16. The only novelty is the introduction of a lemma "C.1" on the construction of escape functions. These escape functions are simplified versions of the escape function we built in Section 2.1, which itself is adapted from [20]. Since we have put in the definition of sinks that the neighbourhood U is actually a uniform ϵ-neighbourhood, the constructions are valid. ∎

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