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Divergent on average directions of Teichmüller geodesic flow

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Abstract. The set of directions from a finite area quadratic differential on a Riemann surface of finite type that diverge on average under Teichmüller geodesic flow has Hausdorff dimension exactly equal to one-half.

Keywords. Teichmüller geodesic flow, flat surfaces

1. Introduction

1.1. Background on Teichmüller dynamics

Before stating the main result we give some background and terminology. Let $S_{g,n}$ be an orientable surface of genus g closed except for n punctures. A marked Riemann surface structure on $S_{g,n}$ is a homeomorphism from $S_{g,n}$ to a Riemann surface. Two markings $f_1 : S_{g,n} \to X_1$ and $f_2 : S_{g,n} \to X_2$ are said to be equivalent if there is a conformal map $g : X_1 \to X_2$ so that $g \circ f_1$ is isotopic to f_2. The Teichmüller space $T_{g,n}$ is a complex manifold whose points are in bijection with equivalence classes of marked Riemann surface structures on $S_{g,n}$, or equivalently, by the uniformization theorem, marked hyperbolic surfaces. The mapping class group $\mathrm{Mod}(S)$ of S – i.e. the group of isotopy classes of orientation-preserving diffeomorphisms of S – acts properly discontinuously on $T_{g,n}$ and the quotient is $\mathcal{M}_{g,n}$ the moduli space of Riemann surface structures on $S_{g,n}$.

A meromorphic quadratic differential q on X assigns to each local coordinate z on X a meromorphic function $f^z(z)$, holomorphic except for possibly simple poles at the punctures, which in an overlapping coordinate w, transforms by

$$f^w(w)\left(\frac{dw}{dz}\right)^2 = f^z(z).$$

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A quadratic differential q has zeros and poles of order $(k_1, \ldots, k_n)$ with

$$\sum_{i=1}^{n} k_i = 4g - 4.$$

Since all quadratic differentials will be assumed to be finite area, the poles are at most simple poles.

In a more geometric fashion one can also describe a meromorphic quadratic differential q as a union of polygons embedded in $\mathbb{C}$ with pairs of sides identified by translations or translations followed by rotations by angle π. Since each polygon is embedded in $\mathbb{C}$, letting z be the local coordinate on the polygon, defines a quadratic differential on the surface made up of polygons by defining the quadratic differential to be dz^2 in each polygon away from the vertices. This associates a quadratic differential to the union of polygons and conversely every quadratic differential can be formed from this construction. This justifies the synonym “half-translation surface” for “quadratic differential”. If the sides of the polygons are only identified by translation, then the surface is called a translation surface and is usually denoted (X, ω). Translation surfaces correspond exactly to quadratic differentials that are squares of holomorphic one-forms ω on compact Riemann surfaces X. They are classically called Abelian differentials.

A quadratic differential q defines a metric $|q^{\frac{1}{2}}||dz|$ on X which is flat except at the singularities, which have concentrated negative curvature. In the polygon version one takes the Euclidean metric $|dz|^2$ in each polygon. Translations and half translations preserve the metric. Moreover, slopes of lines are preserved under the side identifications. A line segment joining singularities without singularities in its interior is called a saddle connection. If a geodesic β joins a nonsingular point to itself without passing through a singularity it is the core curve of a cylinder C_β, i.e. the isometric image of a Euclidean cylinder $[0, a] \times (0, b)/(0, y) \sim (a, y)$ for some positive real numbers a and b, into the flat metric on the surface. The cylinder is swept out by closed parallel loops homotopic to β. We will suppose throughout that all cylinders are maximal, i.e. in the notation of the previous sentence that b is as large as possible.

Suppose one has a translation surface defined by a holomorphic 1-form ω. Then given an oriented line segment γ one defines the holonomy of γ by $\mathrm{hol}(\gamma) := \int_\gamma \omega$. By identifying $\mathbb{C}$ with $\mathbb{R}^2$, the holonomy is a vector in $\mathbb{R}^2$. In the quadratic differential case the holonomy is defined up to multiplication by ± 1. In fact, letting Σ denote the singularities of the flat metric defined by a quadratic differential q on X, there is a homomorphism $\pi_1(X - \Sigma) \to \mathbb{Z}/2\mathbb{Z}$ that sends curves with well-defined holonomy to 0 and those with holonomy only defined up to sign to 1. The associated double cover of $X - \Sigma$ induces a holomorphic branched double cover from a compact Riemann surface Y – called the holonomy double cover of (X, q) – to X. By construction, every curve on Y has well-defined holonomy on the pullback of q, and hence the pullback of q is an Abelian differential. This construction often allows one to reduce the study of quadratic differentials to the study of Abelian differentials.

A half translation surface has a well defined vertical direction. These are the geodesics along which $q(z)dz^2$ is real and nonpositive or equivalently in the polygon version simply the vertical lines. Therefore given any interval I of angles on the unit circle we say that the direction of a holonomy vector lies in I if the argument of the holonomy vector, taken now as a complex number, belongs to I.

The cotangent space of $T_{g,n}$ at $X \in T_{g,n}$ is naturally identified with the vector space $QD(X)$ of quadratic differentials on X. The cotangent bundle of $T_{g,n}$, which we will denote $TQ_{g,n}$, is the moduli space of marked hyperbolic surfaces together with a quadratic differential. The mapping class group acts on $TQ_{g,n}$ and its quotient – denoted $Q_{g,n}$ – is the moduli space of Riemann surfaces together with a quadratic differential.

To every complex manifold, one may associate the Kobayashi semi-metric – the largest semi-metric so that any holomorphic map from the hyperbolic plane into the manifold is distance non-increasing. For $T_{g,n}$, the Kobayashi semi-metric is in fact a metric (though not necessarily a Riemannian one) and the geodesic flow in this metric induces an $\mathbb{R}$-action on its cotangent bundle of $T_{g,n}$. Since $TQ_{g,n}$ is the cotangent bundle to a complex manifold, it also admits a $\mathbb{C}^\times$-action by scalar multiplication. These two actions generate an $\mathrm{SL}(2,\mathbb{R})$ action on $TQ_{g,n}$, which is invariant under the mapping class group, and hence descends to an $\mathrm{SL}(2,\mathbb{R})$ action on $Q_{g,n}$. In the polygon description, the action of $\mathrm{SL}(2,\mathbb{R})$ is by the linear action of the matrix on the polygons.

We will be interested in the action of the subgroups containing the elements

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{for } t \in \mathbb{R}, \qquad r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{for } \theta \in [0, 2\pi).$$

The first action is the Teichmuller geodesic flow. It contracts each polygon along vertical lines and expands along horizontal lines. The second rotates the quadratic differential. See the survey paper of Zorich [26] for a more thorough introduction to Teichmüller dynamics.

1.2. *Statement of results*

Now $Q_{g,n}$ has an $\mathrm{SL}(2,\mathbb{R})$ invariant stratification into subsets $Q_{g,n}(\kappa)$ whose singularities are prescribed by the vector κ. If all the quadratic differentials in a stratum are squares of Abelian differential, we will denote the stratum $\mathcal{H}(\kappa)$. Finally, we remark that the $\mathrm{SL}(2,\mathbb{R})$ action preserves the area, in the flat metric, of any quadratic differential. We will therefore also make the tacit assumption in the sequel that our strata parameterize unit-area quadratic differentials. Again we remark that this means we allow at most simple poles. We are interested in the points in a stratum of quadratic differentials that diverge on average.

Suppose generally that $(g_t)_{t\in\mathbb{R}}$ is a flow on a noncompact topological space Ω. We say that $p \in \Omega$ is *divergent* if $\{g_t p\}_{t\ge 0}$ eventually leaves every compact subset of Ω. We say that p is *divergent on average* if $\{g_t p\}_{t\ge 0}$ spends asymptotically zero percent of its time in any compact set. More formally, p is divergent on average under the flow $(g_t)_{t\ge 0}$, if for any compact subset $K \subset \Omega$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \chi_K(g_t p)\, dt = 0,$$

where χ_K is the indicator function on K.

In the sequel, we will be interested in divergence (on average) of Teichmüller geodesics in spaces of quadratic differentials. One way for a Teichmüller geodesic to diverge is that its projection to $\mathcal{M}_{g,n}$ diverges. Given a Riemann surface $X \in \mathcal{M}_{g,n}$, there is a unique hyperbolic metric on X that is in the conformal class of the complex structure induced by the Riemann surface structure. Let $\mathrm{sys}_{\mathrm{hyp}} : \mathcal{M}_{g,n} \to \mathbb{R}_{\geq 0}$ be the function that associates to a Riemann surface X the length of its systole; the shortest simple essential closed curve in the hyperbolic metric. By Mumford's compactness theorem, a path in $\mathcal{M}_{g,n}$ diverges if and only if the limit of $\mathrm{sys}_{\mathrm{hyp}}$ exists and is zero along the path. Colloquially, a path diverges in $\mathcal{M}_{g,n}$ if and only if for all large times there is some curve that is hyperbolically short. For a Teichmüller geodesic, this is the only way that it may diverge in $Q_{g,n}$.

A Teichmüller geodesic determined by a quadratic differential q is *divergent on average* in $Q_{g,n}$ if and only if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{sys}_{\mathrm{hyp}}(g_t q)\, dt = 0,$$

where we abuse notation to let $\mathrm{sys}_{\mathrm{hyp}}(g_t p)$ mean the length of the systole on the underlying Riemann surface.

However, since Teichmüller geodesic flow preserves strata $Q_{g,n}(\kappa)$ we may also ask about divergence on average in strata of quadratic differentials. Since strata are subsets of $Q_{g,n}$, any Teichmüller geodesic that diverges in $Q_{g,n}$ also diverges in its stratum. However, there is another mechanism for divergence in strata. Recall that we defined a saddle connection on a quadratic differential to be a geodesic in the flat metric defined by q that begins and ends at (potentially distinct) singularities and contains no singularities in its interior. Define $\mathrm{sys}_{\mathrm{flat}} : Q_{g,n}(\kappa) \to \mathbb{R}_{\geq 0}$ to be the length of the shortest saddle connection on the quadratic differential. Similar in spirit to Mumford's compactness theorem, a path in a stratum diverges if and only if $\mathrm{sys}_{\mathrm{flat}}$ has a limit along the path and that limit is zero. A Teichmüller geodesic determined by a quadratic differential q is divergent on average in a stratum $Q_{g,n}(\kappa)$ if and only if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{sys}_{\mathrm{flat}}(g_t q)\, dt = 0.$$

For any holomorphic quadratic differential q it is a consequence of Chaika and Eskin [4, Theorem 1.1] that the set of directions θ such that the Teichmüller geodesic determined by $r_\theta q$ diverges on average in its stratum has measure zero. We prove the following result.

Theorem 1. *For a quadratic or Abelian differential q the set of directions $\theta \in [0, 2\pi]$ such that the Teichmüller geodesic $\{g_t r_\theta q\}_{t\geq 0}$ determined by $r_\theta q$ diverges on average (either in its stratum or in $Q_{g,n}$) has Hausdorff dimension exactly equal to $\frac{1}{2}$.*

Given a noncompact Hausdorff topological space Ω, let $C_0(\Omega)$ denote the space of continuous functions from Ω to $\mathbb{R}$ that vanish at infinity (i.e. that tend to zero along any sequence that leaves all compact sets; equivalently that has a continuous extension to the one-point compactification of Ω by sending the point at infinity to 0).

As a corollary, we have the following.

Corollary 1. *Let $\mathcal{H}$ be any stratum of quadratic or Abelian differentials and $f \in C_0(\mathcal{H})$. Then for any $q \in \mathcal{H}$ the set of $\theta \in [0, 2\pi)$ such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(g_t r_\theta q)\, dt = 0$$

has Hausdorff dimension at least $\frac{1}{2}$.

Remark 1. Notice that Corollary 1 is on the face of it stronger than the lower bound in Theorem 1 since $\mathrm{sys}_{\mathrm{flat}}$ and $\mathrm{sys}_{\mathrm{hyp}}$ belong to $C_0(\mathcal{H})$ for any stratum $\mathcal{H}$.

Remark 2. For any quadratic differential the set of directions that diverge on average in $Q_{g,n}$ is contained in the set of directions that diverge on average in the stratum. In al-Saqban, Apisa, Erchenko, Khalil, Mirzadeh and Uyanik [1], the authors adapted the techniques of Kadyrov, Kleinbock, Lindenstrauss and Margulis [12] to show that the latter set has Hausdorff dimension at most $\frac{1}{2}$ (this result improves on results of Masur [17, 18]). Therefore, the novelty of the current work is establishing the lower bound of Hausdorff dimension $\frac{1}{2}$ for the set of directions that diverge on average in $Q_{g,n}$.

Remark 3. The methods of [1] in fact show that the Hausdorff dimension of the set of directions that diverge on average in any open $\mathrm{SL}(2,\mathbb{R})$ invariant subset of a stratum is at most $\frac{1}{2}$. Therefore, Theorem 1 remains true when divergence on average is considered on any open $\mathrm{SL}(2,\mathbb{R})$ invariant subset of a stratum of quadratic differentials.

Remark 4. In the classical case of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$, which is the genus 1 case, it is known that the set of directions that diverge on average has Hausdorff dimension $\frac{1}{2}$. In fact, the behavior of a geodesic is determined by the continued fraction expansion of its endpoint $x = [a_0, a_1, a_2, \ldots]$ (see for instance Dani [9]). The geodesic diverges on average if and only if $(\prod_{i=1}^n a_i)^{\frac{1}{n}}$ goes to ∞ as $n \to \infty$ (see Choudhuri [8, Theorem 1.2]) and this set has Hausdorff dimension $\frac{1}{2}$ by [11, Theorem 1.2]. In Cheung [6], it is shown that the set of real numbers for which a_n tends to ∞ at a certain prescribed rate has Hausdorff dimension $\frac{1}{2}$. Our construction in higher genus Teichmüller space is modeled on this construction.

1.3. Connection with previous results

For any holomorphic quadratic differential q on a Riemann surface X, every direction specifies a foliation of the underlying Riemann surface. By Masur [18, Theorem 1.1], if the Teichmüller geodesic $\{g_t r_\theta q\}_{t\geq 0}$ is recurrent, then the foliation in the θ-direction is uniquely ergodic. A foliation is said to be uniquely ergodic if there is a unique up to scaling invariant transverse measure to the foliation. In particular, the non-uniquely ergodic directions – $\mathrm{NUE}(q)$ – are divergent directions.

By results of Strebel [22] and Katok and Zemlyakov [25], the collection of directions with non-minimal flow – $\mathrm{NM}(q)$ – is countable. In [19], the main theorem is that outside finitely many exceptional strata of quadratic differentials (the exceptions being the ones where every flat structure induced by a holomorphic quadratic differential has a holonomy double cover that is a translation covering of a flat torus), there is a constant $\delta > 0$

depending on the stratum so that for almost-every quadratic differential q in the stratum the set of directions with non-ergodic flow with respect to Lebesgue measure – $\mathrm{NE}(q)$ – has Hausdorff dimension exactly δ. The sequence of inclusions is then

$$\mathrm{NM}(q) \subseteq \mathrm{NE}(q) \subseteq \mathrm{NUE}(q) \subseteq D(q) \subseteq DA(q),$$

where $D(q)$ and $DA(q)$ are the set of directions that diverge (resp. diverge on average). The set $D(q)$ was shown to have measure zero in [13, Theorem 4].

The set $\mathrm{NUE}(q)$ was shown to have Hausdorff dimension at most $\frac{1}{2}$ by the main theorem of Masur [18]. Recently, Athreya and Chaika [2] showed that this inequality is actually an equality for almost every Abelian differential in $\mathcal{H}(2)$; the space of Abelian differentials in genus 2 with a single zero of order 2 and Chaika and Masur [5] showed that for hyperelliptic components of strata of Abelian differentials this inequality is actually an equality for almost every Abelian differential.

Problem 1. Is it the case that the Hausdorff dimension of $\mathrm{NUE}(q)$ is either 0 or $\frac{1}{2}$ for all quadratic differentials q?

For all known examples, the dimension is either 0 or $\frac{1}{2}$. Despite the fact that $\mathrm{NE}(q)$ has positive Hausdorff dimension for a full measure set of quadratic differentials (outside of finitely many exceptional strata), in each stratum there is a dense set of Veech surfaces – quadratic differentials whose stabilizer in $\mathrm{SL}(2,\mathbb{R})$ is a lattice – for which $D(q) = \mathrm{NM}(q)$ and hence is countable. The fact that $D(q)$ is positive-dimensional for a full measure set of q and zero-dimensional for a dense set of q shows that an analogue of Theorem 1 for divergent directions does not exist in general.

2. Proof of Theorem 1

In this section we will provide a proof of Theorem 1 modulo Propositions 1, 2, 3, 4, 5, and 6. These propositions are, respectively, the main results of Sections 3, 4, 6, 5, 7, and 8. This section provides both the strategy of the proof and an outline of the subsequent sections.

In the sequel, given a quadratic or Abelian differential q, we will say that θ is a divergent on average direction if $r_\theta q$ diverges on average with respect to the g_t flow on the moduli space of quadratic differentials.

2.1. A mechanism for certifying divergence on average

The certificate that a Teichmüller geodesic diverges on average in moduli space (and not just the stratum) will be large modulus cylinders. By a result of Maskit [15], large modulus cylinders have hyperbolically short core curves.

Definition 1. Given a cylinder β on a quadratic differential, let $|\beta|$ be the length of its core curve in the flat metric and let θ_β be the angle that the holonomy vector of its core curve makes with the horizontal. We will occassionally abuse notation and let “the holonomy vector of a cylinder" mean the "holonomy vector of its core curve". Let I_β be the interval of angles centered at θ_β with radius $\frac{1}{|\beta|^2 \log|\beta|}$.

Definition 2. Fix positive constants c and M. Let β be a cylinder. Another cylinder β' is called a (c, M)-potential child of β if the following hold:

(1) The area of β' is at least c.

(2) $|\beta| \log|\beta| \leq |\beta'| \leq M|\beta| \log|\beta|$.

(3) $I_{\beta'} \subseteq I_\beta$.

Lemma 2.1. *Fix positive constants c, M, and $0 < \epsilon < 1$. Let q be a quadratic differential. Suppose β' is a (c, M)-potential child of β and $\theta \in I_{\beta'} \subset I_\beta$. Suppose too that $\frac{4}{\epsilon^2} < \log|\beta|$. Then for all*

$$t \in [\log|\beta|, \log|\beta'|],$$

except for a subset of size at most $\log \frac{4M}{\epsilon^2}$, β has flat length at most ϵ on $g_t r_\theta q$.

Proof. Suppose without loss of generality that we have rotated q so that θ is the vertical direction. Let $h(t)$ and $v(t)$ be the horizontal (resp. vertical) component of the holonomy vector of β on $g_t r_\theta q$. When $t = \log|\beta| + \log(\frac{2}{\epsilon})$, we see that

$$v(t) \leq \frac{\epsilon}{2} \quad \text{and} \quad h(t) \leq \left(\frac{2|\beta|}{\epsilon}\right)|\beta| \sin\left(\frac{1}{|\beta|^2 \log|\beta|}\right) \leq \frac{2\epsilon^{-1}}{\log|\beta|} \leq \frac{\epsilon}{2}.$$

Similarly, when $t = \log|\beta'| - \log(\frac{2M}{\epsilon})$, we see that

$$v(t) \leq \frac{\epsilon}{2} \quad \text{and} \quad h(t) \leq \left(M|\beta| \log|\beta|\right)\left(\frac{\epsilon}{2M}\right)|\beta| \sin\left(\frac{1}{|\beta|^2 \log|\beta|}\right) \leq \frac{\epsilon}{2}.$$

Therefore, for all times

$$t \in \left[\log|\beta| + \log\left(\frac{2}{\epsilon}\right), \log|\beta'| - \log\left(\frac{2M}{\epsilon}\right)\right]$$

the curve β has length at most ϵ on $g_t r_\theta q$. ∎

Definition 3. Fix positive numbers c and M and a quadratic differential q. Notice that if $(\beta_n)_{n \geq 0}$ is a sequence of cylinders so that β_n is a (c, M)-potential child of β_{n-1}, then $(I_{\beta_n})_{n \geq 0}$ is a nested sequence of intervals whose diameter is tending to zero. By the nested interval theorem there is an angle θ so that $\bigcap_n I_{\beta_n} = \{\theta\}$. Let $\mathcal{D}$ be the collection of angles that can be written this way.

We will use computations of extremal length to determine when a quadratic differential is in the thin part of moduli space. We remind the reader of the definition.

Definition 4. A curve family Γ is a collection of curves on a Riemann surface X. The extremal length of Γ is

$$\sup_\rho \frac{1}{A(\rho)} \left(\inf_{\gamma \in \Gamma} \int_\gamma \rho(z)\,|dz|\right)^2,$$

where the supremum is taken over all conformal metrics ρ and where $A(\rho)$ is the area of ρ. The extremal length of a curve γ is defined to be the extremal length of the collection of curves freely homotopic to γ; it is also the reciprocal of the modulus of the largest topological annulus embedded in the hyperbolic surface whose waist curve is freely homotopic to γ.

Corollary 2.2. *Fix a quadratic differential q. Any angle $\theta \in \mathcal{D}$ is a divergent on average direction in the moduli space of quadratic differentials $Q_{g,n}$ (not just in a stratum).*

Proof. Let $\epsilon > 0$. By the Mumford compactness theorem, $\mathcal{M}_{g,n}$ has a compact exhaustion by sets K_ϵ of Riemann surfaces on which all simple closed essential curves have hyperbolic length at least ϵ. By Maskit [15], for sufficiently small ϵ, there is an $\epsilon' > 0$ so that K_ϵ is contained in the set of Riemann surfaces on which all simple closed curves have extremal length at least ϵ'.

Let $\epsilon'' := \sqrt{c\epsilon'}$ and let $t_n := \log|\beta_n|$. Since $(|\beta_n|)_n$ is an increasing sequence that tends to ∞ let N be an integer such that

$$\frac{4}{(\epsilon'')^2} < \log|\beta_n| \quad \text{for } n > N.$$

By Lemma 2.1, for all $n > N$ and for all but at most $\log \frac{4M}{(\epsilon'')^2}$ times in $[t_n, t_{n+1}]$ the half-translation surfaces $\{g_t r_\theta q\}_{t=t_n}^{t_{n+1}}$ contain a cylinder with core curve β_n of length less than ϵ'' and of area at least c. For these times β_n has extremal length at most

$$\frac{(\epsilon'')^2}{c} = \epsilon'$$

and hence the underlying Riemann surface lies outside of K_ϵ. Since $t_{n+1} - t_n$ tends to ∞ as $n \to \infty$ whereas the amount of time spent in K_ϵ for times in $[t_n, t_{n+1}]$ is at most $\log \frac{4M}{(\epsilon'')^2}$, we see that $\{g_t r_\theta q\}$ spends asymptotically zero percent of its time in K_ϵ as desired. ∎

The goal in the sequel will be to produce a set of Hausdorff dimension exactly $\frac{1}{2}$ in $\mathcal{D}$.

2.2. *Reduction to the case of translation surfaces*

In this subsection we will show that for studying divergence on average for quadratic differentials it suffices to study abelian differentials.

Lemma 2.3. *Let (X, q) be a quadratic differential with holonomy double cover (Y, ω). Any direction that belongs to $\mathcal{D}$ for (Y, ω) is a divergent on average direction for (X, q).*

Proof. A cylinder on the translation surface projects to a cylinder on the quadratic differential whose modulus is either equal to or half that of the original cylinder. The proof is now identical to that of Corollary 2.2. ∎

In the sequel, we will now only consider Abelian differentials. For the remainder of the paper we fix the following notation. Fix a translation surface (X, ω) in a stratum $\mathcal{H}$ of Abelian differentials on genus g Riemann surfaces. Let $|\Sigma|$ denote the number of zeros of ω.

Remark 5. In the sequel, all lengths and angles will be measured on (X, ω) unless otherwise stated.

2.3. Set-up for the child selection process

Given a cylinder, we hope to produce child cylinders (as in Definition 2) so that the angle of the holonomy vectors of their core curves can be used to build a Cantor set of divergent on average directions as in Section 2.1. Suppose the holonomy vector of the cylinder we start with is in the vertical direction. To construct child cylinders we will follow geodesics corresponding to directions in I_β for a period of time t where $t \in [\log|\beta|, M \log|\beta|]$. We then find a cylinder on this new surface whose length is approximately one, that has definite area, and that is approximately vertical and pull it back by the geodesic flow to the original surface. We hope to produce many child cylinders this way. However, there is at least one large obstacle – not every translation surface contains a definite area cylinder, of bounded length, near the vertical direction. This leads us to the following definition.

Definition 5. Given $\delta > 0$ and $0 < c < 1$, we say a cylinder is (δ, c)-thin if its circumference is at most δ and its area is at least c. A translation surface is said to belong to the (δ, c)-thick part of a stratum if it contains no (δ, c)-thin cylinders.

Remark 6. We remark that the (δ, c)-thick set is *not* compact. One can have a sequence of surfaces containing cylinders of circumferences going to 0 and areas less than c that lie in the (δ, c)-thick part. These sequences enter what is usually referred to as the thin set, i.e. the set where a curve is short with no reference to area.

Throughout the paper there will be technical conditions on the constants that appear. We will call these "Conditions on Constants". The final section will establish that constants can be chosen to satisfy all of these conditions. We also note here that specific integers will appear in the paper in the discussion of upper bounds. While they do not have significance in their own right, the authors felt that by contrast using the $O(\ )$ notation was not sufficient in the discussion of constants.

We defer the proof of the following proposition to Section 3. It allows us to find new cylinders under a thickness hypothesis.

Conditions on Constants 1. *For positive constants c_1, c_2, θ_1, and δ we require that*

(1) $c_1 < \frac{1}{3g-3}$,

(2) $c_2 < \frac{\lambda}{g(2g+|\Sigma|-2)}$, *where Σ is the singular set and $\lambda = 1 - (3g-3)c_1$*

(3) $c_1 \le c_2$

(4) $\cot\theta_1 < \frac{c_1}{16}$.

(5) *δ is smaller than the Margulis constant, i.e. any two curves on X that have hyperbolic length less than δ can be homotoped to not intersect.*

(6) *The base surface (X, ω) is (δ, c_1) thick.*

Proposition 1. *Let constants c_1, c_2, θ_1, and δ be chosen so that Conditions on Constants 1 holds. Then there are positive constants L and θ_0 so that any (δ, c_1)-thick trans-*

lation surface has a cylinder β so that the following hold:

(1) *Its circumference is strictly less than L.*

(2) *Its area is at least c_2.*

(3) *The angle θ_β the holonomy vector of the core curve of β makes with the horizontal is at least θ_1.*

(4) *If β' is any shorter cylinder satisfying the previous three properties, then*

$$|\theta_\beta - \theta_{\beta'}| > \theta_0.$$

We will refer to the constants L, θ_0 as determined by Proposition 1. Now for each integer m set

$$M = \frac{2^{m+2}L}{\delta}.$$

Conditions on Constants 2. *We choose c_1, c_2 and δ small enough, and m large enough so*

(1) *$c_1 = c_2$ satisfies Conditions on Constants 1.*

Calling this common value c and setting $\nu = \frac{\delta(192\sqrt{2}g - 192\sqrt{2})}{c}$:

(2) $\nu < \frac{1}{4}$.

(3) $m > 6L(\log(\frac{2}{1+2\nu}))^{-1}$.

(4) $M > 21$.

2.4. *The child selection process*

To recap, given a cylinder we need to produce many (c, M)-potential children (Definition 2) so that we can build a Cantor set of divergent on average directions contained in $\mathcal{D}$ (see Section 2.1). So given a parent cylinder β_0, we will flow in a prescribed set of directions, and if we are lucky and after flowing under geodesic flow for time approximately $\log |\beta_0|$ the surface is (δ, c)-thick we will pull back the cylinder described in Proposition 1 to produce a (c, M)-potential child. It is this process that we now wish to make precise and which will be the main object of study in the sequel.

Definition 6 (Child selection process). Consider the following process:

(1) Let (X_0, ω_0) be a (δ, c)-thick translation surface that contains a cylinder β_0 of area at least c. Let γ be a straight line that is contained entirely in β_0, that intersects the core curve of β_0 orthogonally, and which joins one boundary of the cylinder, say at point p, to the other. For any real number s, let γ_s be the straight arc joining the two boundaries that begins at p and has holonomy $\mathrm{hol}(\gamma) + s\,\mathrm{hol}(\beta_0)$ – we think of these arcs as partial Dehn twists of γ about the core curve of β_0. We will restrict s so

$$s \in \left(\frac{2\log|\beta_0|}{\delta}, \frac{M\log|\beta_0|}{2L}\right) = \left(\left(\frac{2\log|\beta_1|}{\delta}\right), 2^m\left(\frac{2\log|\beta_1|}{\delta}\right)\right).$$

The arcs γ_s for these values of s will be called *proto-children of β_0*.

(2) Fix s and rotate the translation surface so γ_s is vertical. Flow by g_t so that γ_s has unit length. The new surface – call it (X_1, ω_1) – will be called the *protochild surface* associated to γ_s.
(3) If (X_1, ω_1) – the protochild surface associated to γ_s – is (δ, c)-thick, then by Proposition 1 there is a cylinder β_1 on (X_1, ω_1) whose circumference is at most L, area is at least c, and such that the holonomy of the core curve makes an angle of at least θ_1 with the horizontal direction on (X_1, ω_1). Call this cylinder the *child* of β_0 corresponding to γ_s. Similarly, call β_0 the *parent*. We will use β_1 to refer to the cylinder on both (X_0, ω_0) and (X_1, ω_1).

The previous process therefore gives a means of associating "child" cylinders to a "parent" cylinder. However, at the moment, several questions remain – the biggest is how many child cylinders a parent cylinder has. Since child cylinders are only produced when the proto-child surface is (δ, c)-thick it is a priori possible that the child selection process produces no child cylinders for certain parents. This question will be addressed in the following subsection. Another question, which we address now, is whether the children produced by this procedure are, as the notation suggests, (c, M)-potential children for the parent cylinder.

The main result of Section 4 is the following.

Definition 7. Let $C := \frac{Lc}{16M}$. Given a cylinder β, define
$$N_\beta := \log|\beta| \quad \text{and} \quad \rho_\beta := \frac{C}{\log|\beta|}.$$
Fix a constant R_0, upon which conditions will be put in the sequel, so that "sufficiently large" will mean "larger than R_0". We will demand that $\log(R_0) > 1$.

Proposition 2. *Suppose that Conditions on Constants 1 holds. For a sufficiently large parent cylinder β, if β' and β'' are distinct children of β corresponding to indices s' and s'' so that $|s' - s''| \geq 1$, then*

(1) $|\beta| \log|\beta| \leq |\beta'| \leq M|\beta| \log|\beta|$,
(2) $I_{\beta'} \subseteq I_\beta$,
(3) *the distance between $I_{\beta'}$ and $I_{\beta''}$ is at least $\rho_\beta |I_\beta| = \frac{C}{N_\beta^2 |\beta|^2}$,*
(4) $\frac{N_\beta |\beta|}{N_{\beta'} |\beta'|} \geq \frac{1}{6Ls'}$.

The first two properties imply that the children constructed in the child selection process are (c, M)-potential children of the parent cylinder. The final two properties on the spacing of the intervals and the size of an interval associated to a parent interval relative to one associated to its child, will be used in the computation of Hausdorff dimension. A crucial issue to be addressed is to show there are enough children to make the desired computations of Hausdorff dimension.

2.5. *How to iterate the child selection process*

The following proposition is the main result of Section 6. It allows us to begin the child selection process.

Proposition 3. *Suppose that Conditions on Constants 3, 4, and 5 hold. Then there are arbitrarily long cylinders β of area at least c that contain a protochild whose protochild surface is (δ, c)-thick.*

The following proposition is the main result of Section 5. It says that we can find many protochildren whose protochild surfaces are thick unless thinness is caused by cylinders that are short on the base surface.

Proposition 4. *Suppose that Conditions on Constants 1, 2, 3, and 4 hold. For any sufficiently long parent cylinder β_0 and any interval of the form*

$$[s, 2s] \subseteq \left(\frac{2 \log |\beta_1|}{\delta}, 2^m \frac{2 \log |\beta_1|}{\delta} \right),$$

there are at least $(1-\nu)s - 1$ points in $[s, 2s]$ that are all separated by at least unit distance and whose protochild surfaces are either (δ, c)-thick or that contain a (δ, c)-thin cylinder β whose circumference satisfies

$$|\beta| \leq \frac{|\beta_0|}{2\sqrt{2}}.$$

As a reminder, while we let β_* denote cylinders on both the original surface – (X, ω) – and the protochild surface; the child selection process is only ever applied to cylinders on (X, ω). The following proposition is the main result of Section 7. It says that in fact that the second possibility in the conclusion of Proposition 4 does not hold so the first conclusion must hold which therefore allows us to find many (δ, c)-thick protochildren.

Proposition 5. *Suppose that Conditions on Constants 1, 2, 3, 4, 6, and 7 hold. Let β_0 be a sufficiently large cylinder on (X, ω) with a child β_1. If β_1 has a protochild σ_1 whose protochild surface has a (δ, c)-thin cylinder β_2, then on (X, ω) we have*

$$\frac{|\beta_1|}{2\sqrt{2}} \leq |\beta_2|.$$

In the final short section – Section 8 – we show

Proposition 6. *It is possible to choose constants so that all Conditions on Constants are satisfied.*

2.6. Constructing a Cantor set in $\mathfrak{D}$

By Proposition 6, pick constants that satisfy all the Conditions on Constants. By Proposition 3, there is a cylinder β_0 on (X, ω) whose circumference is at least R_0, area at least c, and which contains a protochild σ_0 whose protochild surface (X_1, ω_1) is (δ, c)-thick. Let β_1 be the child cylinder chosen in the child-selection process (Definition 6).

We will associate a collection of children to β_1. To each child cylinder constructed in this way we will associate a new collection of child cylinders and so on. We describe this iterative process. Let β be a cylinder constructed in this process. Define its collection of child cylinders D_β as follows.

Consider the set of protochildren of β, indexed by $(\frac{2\log|\beta|}{\delta}, 2^m\frac{2\log|\beta|}{\delta})$. By Proposition 5 any cylinder that is responsible for (δ, c)-thinness of a protochild surface has circumference of size at least $\frac{1}{2\sqrt{2}}|\beta|$. Divide the set of protochildren into sets

$$I_k := \left(2^k\frac{2\log|\beta|}{\delta}, 2^{k+1}\frac{2\log|\beta|}{\delta}\right)$$

for $k \in \{0, \dots, m-1\}$. By Proposition 4, there are at least $(1-\nu)2^k\frac{2\log|\beta|}{\delta} - 1$ points, call them J_k', in I_k that are unit distance apart and whose corresponding protochild surface is (δ, c)-thick. Let J_k be the subcollection of J_k' with the largest and smallest points deleted. This is done so that any two distinct points in $J_\beta := \bigcup_{k=0}^{m-1} J_k$ are unit distance apart. Then

$$|J_k| \geq (1-\nu)2^k\frac{2\log|\beta|}{\delta} - 3 > (1-2\nu)2^k\frac{2\log|\beta|}{\delta};$$

the last inequality holds since $\nu < \frac{1}{4}$ and since by definition of constants

$$\log|\beta| \geq \log(R_0) > 1 > c.$$

The set of children D_β will then be the children constructed in the child-selection process whose indices correspond to the indices in J_β.

Let $\mathcal{D}'$ be the collection of angles θ that can be written as $\{\theta\} = \bigcap_n I_{\beta_n}$ for some sequence $(\beta_n)_{n\geq 0}$, where β_n belongs to the set $D_{\beta_{n-1}}$. By Proposition 2, the elements of $\mathcal{D}'$ belong to $\mathcal{D}$, which in turn is contained in the set of directions that diverge on average (by Corollary 2.2).

2.7. *Proof of Theorem 1*

By Cheung [6, Theorem 3.3], given the set $\mathcal{D}'$ constructed in the previously described way and given that the children satisfy the four enumerated conditions in Proposition 2, then if s is some real number so that for every cylinder β constructed in the above process

$$\sum_{\beta'\in D_\beta}\frac{\rho_{\beta'}^s|I_{\beta'}|^s}{\rho_\beta^s|I_\beta|^s} > 1,$$

then the Hausdorff dimension of $\mathcal{D}'$ is at least s. We have already seen that these directions are divergent on average in the moduli space of quadratic differentials $Q_{g,n}$.

Proof of Theorem 1. Setting $s = \frac{1}{2}$, we see that

$$\sum_{\beta'\in D_\beta}\frac{\rho_{\beta'}^s|I_{\beta'}|^s}{\rho_\beta^s|I_\beta|^s} = \sum_{\beta'\in D_\beta}\left(\frac{CN_\beta}{CN_{\beta'}}\right)^{\frac{1}{2}}\left(\frac{N_\beta|\beta|^2}{N_{\beta'}|\beta'|^2}\right)^{\frac{1}{2}} = \sum_{\beta'\in D_\beta}\frac{N_\beta|\beta|}{N_{\beta'}|\beta'|}.$$

By (4) of Proposition 2 the sum on the right is greater than

$$\frac{1}{6L}\sum_{k=0}^{m-1}\sum_{s'\in J_k}\frac{1}{s'}.$$

For each k the smallest value of the inner sum occurs when the $(1-2\nu)2^k\frac{N_\beta}{\delta}$ values of s' in the interval $[2^k\frac{N_\beta}{\delta}, 2^{k+1}\frac{N_\beta}{\delta}]$ are all exactly distance 1 apart and lie in the interval

$[(1+2\nu)2^k \frac{N_\beta}{\delta}, 2^{k+1}\frac{N_\beta}{\delta}]$. But then

$$\sum_{s' \in J_k} \frac{1}{s'} \geq \log\left(2^{k+1}\frac{N_\beta}{\delta}\right) - \log\left((1+2\nu)2^k \frac{N_\beta}{\delta}\right) = \log\left(\frac{2}{1+2\nu}\right).$$

Since there are m such sums, we see that

$$\sum_{\beta' \in D_\beta} \frac{\rho_{\beta'}^s |I_{\beta'}|^s}{\rho_\beta^s |I_\beta|^s} > \frac{m \log\left(\frac{2}{1+2\nu}\right)}{6L} > 1,$$

where the final inequality holds by choice of constants in Conditions on Constants 2. By Cheung [6, Theorem 3.3], the Hausdorff dimension of $\mathcal{D}$ is at least $\frac{1}{2}$. Therefore, the Hausdorff dimension of the set of directions that diverge on average is exactly equal to $\frac{1}{2}$ by [1]. ∎

3. Cylinders and the thick-thin decomposition – Proof of Proposition 1

In this section, we will establish Proposition 1, which roughly says that any (δ, c)-thick translation surface contains a cylinder of bounded length and definite area whose core curve has a holonomy vector that is close to vertical. Recall that we have chosen δ to be smaller than the Margulis constant.

3.1. *Summary of the δ'-thick-thin decomposition of Rafi and the Geometric Compactification Theorem of Eskin–Kontsevich–Zorich*

Given a translation surface, there are two natural metrics on the underlying Riemann surface – the hyperbolic metric and the flat metric.

Definition 8. The *δ'-thick-thin decomposition* of the translation surface (Y, η) – where Y is a Riemann surface and η is an Abelian differential – is defined as follows. Let Γ be the simple closed hyperbolic geodesics on Y whose hyperbolic length is less than δ'. For each $\gamma \in \Gamma$, there is a geodesic representative of γ in the flat metric on (Y, η). Either the flat-geodesic is unique or it is contained in a flat cylinder. In the first case, cut out the unique flat-geodesic from (Y, η) and in the second excise the entire cylinder. Do this for each $\gamma \in \Gamma$. The resulting connected components are called δ'-thick-pieces.

The *size of a thick-piece* is defined as follows. If the thick piece is not a pair of pants, its size is the smallest flat length of a simple closed curve in the thick piece that is not homotopic to a boundary curve. If the thick piece is a pair of pants, then the size is the maximal flat length of a boundary curve.

These definitions are due to Rafi [21] who showed that there is a constant $C(g, |\Sigma|, \delta')$ so that in a thick piece of size λ and for any essential curve α contained in the thick piece,

$$\frac{\lambda}{C(g, |\Sigma|, \delta')} \ell_{\text{hyp}}(\alpha) \leq \ell_{\text{flat}}(\alpha) \leq C(g, |\Sigma|, \delta') \lambda \ell_{\text{hyp}}(\alpha),$$

where ℓ_{hyp} and ℓ_{flat} denote lengths in the hyperbolic and flat metrics, respectively.

Now we will state a result of Eskin, Kontsevich and Zorich [10, Geometric Compactification Theorem (Theorem 10)], in the case of Abelian differentials. The theorem studies a sequence of translation surfaces (X_n, ω_n) that converges to a stable differential on a compact nodal Riemann surface (Y, η). A compact nodal Riemann surface always admits a surjection from a smooth compact Riemann surface, called its normalization, that is one-to-one outside of cofinitely many points – called preimages of nodes. This makes precise the intuition that a nodal Riemann surface is a smooth Riemann surface after gluing together finitely many finite collections of points.

The desingularization of a compact nodal Riemann surface will be defined to be the normalization punctured at preimages of nodes. Since this is a (potentially disconnected) Riemann surface, there is a unique hyperbolic metric on the desingularization. The injectivity radius of this metric is what we will refer to with the phrase "the injectivity radius of the desingularization of a nodal Riemann surface".

Theorem 3.1 ([10, Theorem 10]). *Let (X_n, ω_n) be any sequence of unit-area translation surfaces that are not contained in a compact subset of a stratum of Abelian differentials. By passing to a subsequence assume that (X_n, ω_n) converges to a stable differential ω on a nodal Riemann surface Y. Let δ_0 be less than half the injectivity radius of the hyperbolic metric on the desingularization of Y. Then there is a subsequence of (X_n, ω_n) so that each thick component converges to a nonzero meromorphic quadratic differential when the flat metric on the thick component is renormalized so that its size is one.*

In the sequel we will also use a construction of Eskin, Kontsevich and Zorich [10, Geometric Compactification Theorem (Theorem 10)], to build a triangulation. The content of the statement and proof are found entirely in [10] (specifically in Lemma 4.2 and the first three paragraphs of the proof of Theorem 10) but we record them here since there is not a more convenient reference to these facts in [10].

Lemma 3.2. *Using the notation of Theorem 3.1, for sufficiently large n and after passing to a subsequence there is an identification of thick pieces of (X_n, ω_n) with those of (X_{n+1}, ω_{n+1}) for all n. Moreover, there is a triangulation of the thick pieces so that for constants C_1 and C_2:*

(1) *The sizes of the thick pieces converge.*
(2) *The triangulations have the same combinatorial type for all n.*
(3) *There are fewer than C_1 edges of the triangulation.*
(4) *The edges of the triangulation are saddle connections.*
(5) *The saddle connections that do not belong to the boundary of the thick piece have length bounded below by $\frac{\lambda}{2}$ and above by $C_2\lambda$ where λ is the size of the thick piece.*
(6) *The holonomy vectors of the edges of the triangulation, divided by the size of the thick piece, converge as n tends to infinity. In particular, this implies that the holonomy vectors of the edges of the triangulation converge whenever the size of the thick piece does not tend to zero.*

Proof. Since (X_n, ω_n) converges to (Y, η), for sufficiently large n, Y punctured at its nodes is homeomorphic to the δ_0-thick pieces of (X_n, ω_n). Identify the thick pieces of

(X_n, ω_n) with those on (X_{n+1}, ω_{n+1}) by identifying homeomorphic pieces with each other. (There is not a canonical way of doing this and the identification is merely a convenient way of fixing a way of talking about a single thick piece along the sequence.) By passing to a subsequence assume that the sizes of the thick pieces converge. By [10, Lemma 4.2], there are constants C_1 and C_2 depending only on the stratum so that every thick component of (X_n, ω_n) can be triangulated with fewer than C_1 saddle connections satisfying the length estimate in (5).

Now consider a thick piece of (X_n, ω_n) whose flat metric is rescaled so that its size is one. By [10, Lemma 4.2], the triangulation previously described has at most C_1 edges, all of which now have length between $\frac{1}{2}$ and C_2. There are finitely many combinatorial types of triangulations with this property. Passing to a subsequence therefore allows one to assume that the combinatorial type of a triangulation of a thick piece is constant for all n. This provides a means of identifying edges of a triangulation of a thick piece on (X_n, ω_n) with the edges of the triangulation of the corresponding thick piece on (X_{n+1}, ω_{n+1}). Therefore, we may again pass to a subsequence to ensure that the holonomy vectors (scaled by the size of the thick piece) of the edges of the triangulation converge along the subsequence. ∎

Definition 9. Suppose that A is an annulus around a curve γ on X. The annulus A is called regular if it is of the form $\{p : d(p, \gamma) < r\}$ for some r where $d(\cdot, \cdot)$ denotes distance in the flat metric. The annulus is primitive if additionally it contains no singularities in its interior. If A is a primitive regular annulus that is not a flat cylinder, then define $\mu(A) := \log(\frac{|\gamma_o|}{|\gamma_i|})$, where $|\cdot|$ denotes flat length and γ_o (resp. γ_i) is the longer (resp. shorter) boundary curve of A and is called the outer (resp. inner) curve of A. This definition agrees with the one made in Minsky [20] up to a multiplicative constant that only depends on the stratum containing (X, ω).

3.2. Proof of Proposition 1

Lemma 3.3. *Under the hypotheses of Theorem 3.1, every flat cylinder in (X_n, ω_n) around a δ_0-hyperbolically short curve has the length of its core curve tend to 0 as $n \to \infty$.*

Proof. By the result of Maskit [15, Corollary 2], since the δ_0-hyperbolically short curves have lengths tending to zero in the hyperbolic metric, their extremal length also tends to zero as $n \to \infty$.

Therefore, each δ_0-hyperbolically short curve γ is contained in a topological annulus whose modulus tends to ∞ as $n \to \infty$. By a result of Minsky [20, Theorems 4.5 and 4.6] (note that the inequality $\leq m_0$ should be $\geq m_0$ in the statement of the Theorem 4.6), either γ is contained in a flat cylinder whose modulus is unbounded in n or there is a primitive regular annulus $A_n \subseteq (X_n, \omega_n)$ contained in a thick piece whose core curve is homotopic to γ and so that $\mu(A_n)$ tends to ∞ as n increases.

Notice that if the modulus of the flat cylinder containing γ tends to ∞, then the flat length of γ tends to zero since each (X_n, ω_n) is unit-area. Therefore, suppose that for each n there is a primitive regular annulus A_n whose core curve is homotopic to γ and so that $\mu(A_n)$ is unbounded in n.

Let ℓ_n be the flat length of γ on (X_n, ω_n) and let a_n be the area of the thick piece containing A_n. The flat distance across A_n is at most $h_n := \frac{a_n}{\ell_n}$. The flat length of the outer curve A_n in the flat metric is at most $2\ell_n + 2\pi M h_n$, where M is some integer only depending on the stratum. Therefore, $\mu(A_n) \leq \log(2 + \frac{2\pi M}{\ell_n})$. Since $\mu(A_n)$ is unbounded in n, $\ell_n \to 0$ as $n \to \infty$. ∎

A similar argument to the one above is given in [7, Corollary 5.4].

Lemma 3.4. *Fix a stratum $\mathcal{H}$ of Abelian differentials, an open set I on the unit circle, the positive constant δ, the positive constants c_1 and c_2 satisfying Conditions on Constants 1. Then there is an L so that for any (δ, c_1)-thick unit area surface in $\mathcal{H}$ there is a cylinder of area at least c_2 whose core curve has length at most L and such that the direction of the holonomy vector lies in I.*

Remark 7. Only the first two conditions of Conditions on Constants 1 are needed for the proof. The same holds for Proposition 1.

Proof. Let $\mathcal{H}_{(\delta,c_1)}$ be the locus of (δ, c_1)-thick translation surfaces in $\mathcal{H}$. By Vorobets [23, Theorem 1.5], for every unit area translation surface in $\mathcal{H}$ there is a cylinder of area at least $\frac{1}{2g+|\Sigma|-2}$ whose core curve has holonomy vector whose direction lies in I. Let $\mathcal{C}$ be the set of cylinders of area at least c_2 and whose core curve has the direction of its holonomy vector lying in I. Notice that c_2 is less than $\frac{1}{2g+|\Sigma|-2}$ by Conditions on Constants 1. Let $\ell_{\mathcal{C}} : \mathcal{H} \to \mathbb{R}$ be the function that records the shortest length of a cylinder in $\mathcal{C}$. (Notice that this function is different from the function ℓ which is often used to denote the length of the shortest saddle connection). Since cylinders persist on open subsets of $\mathcal{H}$, it follows that $\ell_{\mathcal{C}}$ is bounded on compact subsets of $\mathcal{H}$.

Arguing by contradiction assume the lemma does not hold. It follows that there is a sequence (X_n, ω_n) of translation surfaces in $\mathcal{H}_{(\delta,c_1)}$ so that the $\ell_{\mathcal{C}}(X_n, \omega_n) \to \infty$. Since $\ell_{\mathcal{C}}$ is bounded on compact subsets of $\mathcal{H}$, it follows that (X_n, ω_n) leaves all compact subsets of $\mathcal{H}$. By passing to a subsequence, we suppose that the sequence converges to (X, ω) in the geometric compactification.

Let δ_0 be less than half the injectivity radius of the desingularized hyperbolic metric on X, and suppose that the δ_0 thick pieces converge as in Theorem 3.1. We claim that there is a thick piece that has definite area on each (X_n, ω_n). In the thick-thin decomposition, the only positive-area subsurfaces that are not contained in a thick piece are flat cylinders around δ_0-hyperbolically short curves. However, by Lemma 3.3 these cylinders have the length of their core curves tend to zero along the sequence (X_n, ω_n). By truncating an initial segment of the sequence we may suppose that these core curves are always less than length δ in flat length. Since (X_n, ω_n) are (δ, c_1)-thick surfaces, we have that the thin part has area at most $(3g-3)c_1$. Therefore, the thick part has area at least $\lambda := 1 - (3g-3)c_1$, which is positive by Conditions on Constants 1. Moreover, the thick part has at most g components. In particular, it contains some translation surface of area $\frac{\lambda}{g}$.

By Vorobets [23, Theorem 1.5], this thick piece contains a cylinder whose core curve has a holonomy vector whose direction lies in the interval I, of area at least $\frac{\lambda}{g(2g+|s|-2)}$. Therefore, pulling it back to (X_n, ω_n) (after again truncating a finite initial subsequence), produces a cylinder in $\mathcal{C}$ along the subsequence of bounded length, which is a contradiction. ∎

For convenience we restate Proposition 1 here.

Proposition (Proposition 1). *There are positive constants L, θ_0 so that any (δ, c_1)-thick translation surface has a cylinder β so that the following hold:*

(1) *Its circumference is strictly less than L.*

(2) *Its area is at least c_2.*

(3) *The angle θ_β its holonomy vector makes with the horizontal is at least θ_1.*

(4) *If β' is any shorter cylinder satisfying the previous three properties, then*

$$|\theta_\beta - \theta_{\beta'}| > \theta_0.$$

Proof of Proposition 1. Choose $\epsilon > 0$ so that $\theta_1 + \epsilon < \pi$ and let I be the collection of angles on the circle that are at least $\theta_1 + \epsilon$ from the horizontal. Let L be the length produced by Lemma 3.4. Given a translation surface, we will let $\mathcal{C}$ be the collection of cylinders on the surface satisfying items one through three in the statement of the proposition. This collection is nonempty for any (δ, c_1)-thick surface by Lemma 3.4.

Arguing by contradiction suppose that (X_n, ω_n) is a sequence along which the conclusion fails. Suppose without loss of generality that the sequence has a limit (X, ω) in the geometric compactification. Let δ_0 be half the injectivity radius of X in the hyperbolic metric. By passing to a subsequence assume that we have a triangulation of the thick pieces as in Lemma 3.2.

Step 1. *For sufficiently large n, a cylinder in $\mathcal{C}$ does not intersect a thick piece whose size tends to zero.*

Each cylinder in $\mathcal{C}$ has height at least $\frac{c_2}{L}$. Take n sufficiently large so that for each thick piece whose size is tending to zero it is triangulated by saddle connections of length less than $\frac{c_2}{L}$. Then $\mathcal{C}$ cannot intersect this thick piece since it cannot cross a saddle connection of length less than $\frac{c_2}{L}$.

Step 2. *For sufficiently large n, a cylinder in $\mathcal{C}$ does not intersect a positive area thin piece.*

These thin pieces are exactly flat cylinders around δ_0-hyperbolically short curves. By Lemma 3.3, the circumference of these flat cylinders tend to zero. Since each (X_n, ω_n) is (δ, c_1)-thick, these cylinders must have area strictly less than $c_1 \leq c_2$ for sufficiently large n. Hence no cylinder in $\mathcal{C}$ can coincide with one of these cylinders. Moreover, the heights of the cylinders in $\mathcal{C}$ are bounded below and so no cylinder in $\mathcal{C}$ can cross them either (since the circumferences tend to zero).

Step 3. *For sufficiently large n, a cylinder in $\mathcal{C}$ is contained in a single thick piece.*

Consider a thick piece whose size does not tend to zero. By Eskin, Kontsevich and Zorich [10, Geometric Compactification Theorem (Theorem 10)], when this piece is rescaled by its size it converges to a meromorphic quadratic differential. However, since the size is bounded away from zero this quadratic differential has finite area, no boundary, and trivial linear holonomy, i.e. it is an Abelian differential on a closed Riemann surface. Therefore, the boundary of the thick piece necessarily consisted of saddle connections whose holonomy tended to zero as n tended to ∞. Since cylinders in $\mathcal{C}$ have height that is bounded below, it must be the case that $\mathcal{C}$ cannot cross the boundary of a thick piece when n is sufficiently large (i.e. when all the saddle connections in the boundary are sufficiently small).

Step 4. *There is a finite collection of curves S defined only in terms of the combinatorial type of the triangulation so that any cylinder belonging to $\mathcal{C}$ on (X_n, ω_n) for sufficiently large n, has core curve homotopic to a curve in S.*

Recall that the triangulations of the thick pieces of (X_n, ω_n) all have the same combinatorial type and that all edges converge in length. Let ℓ be the supremum of the edge lengths that appear in these triangulations for all (X_n, ω_n). Since the smallest that the height of a cylinder in $\mathcal{C}$ can be is $\frac{c_2}{L}$, a cylinder in $\mathcal{C}$ can only intersect an edge of the triangulation $\frac{\ell L}{c_2}$ times. Consider the finite collection S of all paths through the triangulations of these thick parts that are (1) straight lines in each triangle, (2) connect midpoints of edges of the triangulation, and (3) cross any edge at most $\frac{\ell L}{c_2}$ times. For sufficiently large n, the core curves of the cylinders in $\mathcal{C}$ are homotopic to a curve in S and moreover, since the curves in S are only defined in terms of the combinatorial type of the triangulation, they define piecewise geodesic curves on all (X_n, ω_n). Moreover, since the holonomy vectors of the edges of the triangulation converge as n tends to ∞, so does the holonomy of each element of S.

Step 5. *One may choose a cylinder for each n to derive a contradiction.*

Let C_n be the shortest cylinder in $\mathcal{C}$ on (X_n, ω_n) whose holonomy belongs to I (such a cylinder exists by Lemma 3.4). After passing to a subsequence, we may assume that C_n corresponds to a fixed element $s_0 \in S$ for all n. Let $\mathrm{hol}_n(s_0)$ be the holonomy of s_0 on (X_n, ω_n). If the angle of $\mathrm{hol}_n(s_0)$ with the horizontal converges to an angle θ in the interior of I, then on each (X_n, ω_n) we notice that by choosing C_n on (X_n, ω_n) we have produced a cylinder in $\mathcal{C}$, whose holonomy vector makes an angle of θ_1 from the horizontal, and which, for sufficiently large n, makes an angle of $\frac{d}{2}$ from the angle of the holonomy vector of the core curve of any shorter cylinder in $\mathcal{C}$ where d is the distance from θ to the boundary of I. However, (X_n, ω_n) was chosen so that along the sequence no such cylinder could be found. This is a contradiction.

Suppose now that the angle $\mathrm{hol}_n(s_0)$ makes with the horizontal converges to the boundary of I. Without loss of generality suppose that it converges to $\theta_1 + \epsilon$. Let S' be the subset of S of paths whose holonomy has its angle with the horizontal converges to $\theta_1 + \epsilon$. Define

$$\Lambda := \left\{ \lim_{n \to \infty} \theta(\mathrm{hol}_n(s)) \right\}_{s \in S},$$

where $\theta(\mathrm{hol}_n(s))$ is the angle $\mathrm{hol}_n(s)$ makes with the horizontal. Let d be the distance from $\theta_1 + \epsilon$ to the nearest distinct point in Λ (and let it be 2π if there are no other distinct points). For each n, let C_n be the shortest cylinder in $\mathcal{C}$ whose core curve is homotopic to an element of S'. Notice that unlike in the previous case, the angle with the horizontal of the holonomy vector of the core curve of C_n might lie outside of I. However, for sufficiently large n, the angle of the holonomy vector of the core curve of C_n will be bounded away the horizontal by θ_1 and from any other element of $\mathcal{C}$ by at least $\frac{d}{2}$. Again this is a contradiction. ■

4. Elementary facts about the child selection process – Proof of Proposition 2

In this section, we prove Proposition 2, which states that children constructed in the child selection process (Definition 6) are (c, M)-potential children (Definition 2) of the parent cylinder and that the associated intervals have the size and spacing requirements needed in the Hausdorff dimension computation in Section 2.

Let β_0, β_1, and γ_s be defined as in the definition of the child selection process. Recall that R_0 was defined in Definition 7 to be a constant so that when the circumference of a cylinder is sufficiently large, it will mean that the circumference is larger than R_0. Moreover, by Definition 7, we have set $C = \frac{Lc}{16M}$. In the sequel, it will be necessary to put several constraints on R_0. We begin with the following.

Conditions on Constants 3. *Suppose that* $R_0 > \max(e^{\frac{4}{C}}, 21, e^{2\delta}, e^M)$.

Make the following definition.

Definition 10. Given two straight line segments γ and γ' in (X, ω), we will let $\theta(\gamma, \gamma')$ denote the angle between them. We will always take this number to be positive. By associating a cylinder β with its core curve, we will also write $\theta(\gamma, \beta)$ to mean the angle $\theta(\gamma, \gamma_\beta)$ where γ_β is the core curve of the cylinder β.

Lemma 4.1 (Length facts). *The following facts hold for the child selection process.*

(1) *If* $|\beta_0| > 1$, *then it is immediate that*

$$s|\beta_0| \leq |\gamma_s| \leq (s+1)|\beta_0|.$$

Since $s \in (\frac{2\log|\beta_0|}{\delta}, \frac{M\log|\beta_0|}{2L})$, *it follows that*

$$\frac{2}{\delta}|\beta_0|\log|\beta_0| \leq |\gamma_s| \leq \frac{M}{L}|\beta_0|\log|\beta_0|.$$

If $e^{\frac{\delta}{2}} < |\beta_0|$, *then* $|\beta_0| < |\gamma_s|$.

(2) *Since the length of* β_1 *on* (X_1, ω_1) *is between* δ *and* L *and since its holonomy vector makes an angle of at least* $\frac{\pi}{4}$ *with the horizontal, when* β_1 *is pulled back to* (X_0, ω_0) *by* $g_{-|\gamma_s|}$ *we have*

$$|\gamma_s|\frac{\delta}{2} \leq |\beta_1| \leq |\gamma_s|L.$$

Combined with the previous estimate this yields

$$|\beta_0|\log|\beta_0| \leq |\beta_1| \leq M|\beta_0|\log|\beta_0|.$$

(3) *If* $|\beta_0| > e^M$, *then*

$$\frac{|\beta_0|\log|\beta_0|}{|\beta_1|\log|\beta_1|} \geq \frac{1}{6Ls}.$$

Proof. Only the proof of (3) remains to be given. First,

$$\frac{|\beta_0|}{|\beta_1|} \geq \frac{|\beta_0|}{L|\gamma_s|} \geq \frac{|\beta_0|}{2Ls|\beta_0|} = \frac{1}{2Ls},$$

where the first inequality is from (2) and the second is from (1). Finally, by (2) we have

$$\frac{\log|\beta_1|}{\log|\beta_0|} \leq \frac{M}{\log|\beta_0|} + \frac{\log|\beta_0|}{\log|\beta_0|} + \frac{\log\log|\beta_0|}{\log|\beta_0|} \leq 3,$$

where the final inequality comes from the fact that each summand is less than 1. ∎

Lemma 4.2 (Angle facts 1). *The following facts hold for the child selection process.*

(1) *Since* $|\beta_0 \times s_t| = \mathrm{area}(\beta_0)$,

$$\frac{c}{|\beta_0||\gamma_s|} \leq \sin\theta(\beta_0, \gamma_s) = \frac{\mathrm{area}(\beta_0)}{|\beta_0||\gamma_s|} \leq \frac{1}{|\beta_0||\gamma_s|}.$$

(2) *The largest angle that* β_1 *makes with the vertical is when it lies in the direction of* $(\cos\theta_1, \sin\theta_1)$, *which pulls back to* $(\frac{\cos\theta_1}{|\gamma_s|}, |\gamma_s|\sin\theta_1)$ *on* (X_0, ω_0). *Therefore,*

$$|\tan\theta(\beta_1, \gamma_s)| \leq \frac{\cot\theta_1}{|\gamma_s|^2} \leq \frac{(\delta/2)^2}{|\beta_0|^2\log^2|\beta_0|}.$$

(3) *Since*

$$|\gamma_s \times \gamma_{s'}| = |(s_0 + s\beta_0) \times (s_0 + s'\beta_0)| = \mathrm{area}(\beta_0)|s - s'|,$$

it follows that

$$|\sin\theta(\gamma_s, \gamma_{s'})| = \frac{\mathrm{area}(\beta_0)|s - s'|}{|\gamma_s||\gamma_{s'}|}.$$

If additionally, $s, s' \in [\frac{2\log|\beta_0|}{\delta}, \frac{M\log|\beta_0|}{2L}]$ *and* $|s - s'| \geq 1$, *then*

$$\frac{c(2L/M)^2}{|\beta_0|^2\log^2|\beta_0|} \leq |\sin\theta(\gamma_s, \gamma_{s'})|.$$

(4) *Suppose that* β' *and* β'' *are children of* β_0 *corresponding to* $\gamma_{s'}$ *and* $\gamma_{s''}$, *respectively. Suppose* $|s'' - s'| \geq 1$. *Suppose also* $|\beta_0| > e^{2\delta}$. *Then*

$$\frac{c|s'' - s'|}{16|\beta_0|^2|s's''|} \leq \theta(\beta', \beta'').$$

Proof. Only the proof of (4) remains to be given. By the triangle inequality,

$$\theta(\beta'', \beta') \geq \theta(\gamma_{s''}, \gamma_{s'}) - \theta(\gamma_{s''}, \beta'') - \theta(\gamma_{s'}, \beta').$$

Since $\sin\theta \leq \theta \leq \tan\theta$ for all $0 \leq \theta \leq \frac{\pi}{2}$, we have

$$\theta(\beta'', \beta') \geq \sin\theta(\gamma_{s''}, \gamma_{s'}) - \tan\theta(\gamma_{s''}, \beta'') - \tan\theta(\gamma_{s'}, \beta').$$

By (2) and (3) we have

$$\theta(\beta'', \beta') \geq \frac{\operatorname{area}(\beta_0)|s'' - s'|}{|\gamma_{s''}||\gamma_{s'}|} - \frac{\cot\theta_1}{|\gamma_{s'}|^2} - \frac{\cot\theta_1}{|\gamma_{s''}|^2}.$$

Assume without loss of generality that $s' > s''$. Using the estimate that $\cot\theta_1 < \frac{c}{16}$,

$$\theta(\beta'', \beta') \geq \frac{c}{4|\beta_0|^2}\left(\frac{s' - s''}{s''s'} - \frac{(1/4)}{s'^2} - \frac{(1/4)}{s''^2}\right).$$

Now since $s' \geq s'' + 1$,

$$4(s' - s'')s's'' - s'^2 - s''^2 \geq 2(s' - s'')s''s' - (s' - s'')(s' + s'').$$

The right-hand side is equal to

$$(s' - s'')(2s''s' - s' - s'') = (s' - s'')(s''(s' - 1) + s'(s'' - 1)).$$

Therefore, we have

$$\theta(\beta'', \beta') \geq \frac{c|s' - s''|}{16|\beta_0|^2 s''s'}\left(\frac{s' - 1}{s'} + \frac{s'' - 1}{s''}\right).$$

If $|\beta_0| > e^{2\delta}$, it follows that s' and s'' are greater than 1, hence

$$\theta(\beta'', \beta') \geq \frac{c|s' - s''|}{16|\beta_0|^2 s''s'}. \qquad \blacksquare$$

Lemma 4.3 (Angle facts 2). *Suppose that $|\beta_0| > \max(21, e^{\frac{4}{c}})$. The following facts hold for the child selection process.*

(1) *Suppose β_1 is a child of β_0. Then $I_{\beta_1} \subseteq I_{\beta_0}$.*

(2) *Suppose that β' and β'' are distinct children of β_0 corresponding to $\gamma_{s'}$ and $\gamma_{s''}$ respectively and suppose that $|s' - s''| \geq 1$. Then the distance between $I_{\beta'}$ and $I_{\beta''}$ is at least $\rho_{\beta_0}|I_{\beta_0}|$.*

Proof. (1) The radius of I_{β_1} is $\frac{1}{|\beta_1|^2\log|\beta_1|}$ and so the largest angle between an element of I_{β_1} and β_0 is bounded by

$$\frac{1}{|\beta_1|^2\log|\beta_1|} + \theta(\gamma_s, \beta_0) + \theta(\gamma_s, \beta_1)$$

for any γ_s. If $|\beta_0| > 4$, it follows that from Lemma 4.2 (1) that $\sin\theta(\beta_0, \gamma_s) \leq \frac{1}{4}$ and hence that $\theta(\beta_0, \gamma_s) \leq 2\sin\theta(\beta_0, \gamma_s)$. From this observation and from Lemma 4.2 (2), we have that the largest angle between an element of I_{β_1} and β_0 is at most

$$\frac{1}{|\beta_1|^2\log|\beta_1|} + \frac{2}{|\beta_0||\gamma_s|} + \frac{(\delta/2)^2}{|\beta_0|^2\log^2|\beta_0|}.$$

By Lemma 4.1 this is at most

$$\frac{1}{|\beta_0|^2\log^2|\beta_0|} + \frac{\delta}{|\beta_0|^2\log|\beta_0|} + \frac{(\delta/2)^2}{|\beta_0|^2\log^2|\beta_0|}.$$

If $|\beta_0| > 21$, $\frac{1}{\log|\beta_0|} < \frac{1}{3}$. Since $\delta < \frac{1}{3}$, the largest angle between an element of I_{β_1} and the holonomy vector of β_0 is less than $\frac{1}{|\beta_0|^2\log|\beta_0|}$ as desired.

(2) By definition, the interval $I_{\beta'}$ with center $\theta_{\beta'}$ has radius $\frac{1}{|\beta'|^2 \log|\beta'|}$ whereas, since $|s'-s''| \geq 1$, we have by Lemma 4.2 (4) that

$$\theta(\beta', \beta'') \geq \frac{c|s'-s''|}{16|\beta_0|^2 s's''}.$$

The distance between two distinct intervals $I_{\beta'}$ and $I_{\beta''}$ is at least

$$\frac{1}{|\beta_0|^2 \log^2|\beta_0|}\left(\frac{Lc}{8M} - \frac{2}{\log|\beta_0|}\right).$$

If $|\beta_0| > e^{\frac{4}{C}}$, the distance is at least $\rho_{\beta_0}|I_{\beta_0}|$ as desired. ■

Proof of Proposition 2. If $|\beta_0| \geq R_0$, then all estimates in this section that require $|\beta_0|$ to be of a certain length hold. The first and final items of Proposition 2 are items (2) and (3) of Lemma 4.1, while the remaining two are the two items of Lemma 4.3. ■

5. A definite proportion of protochild surfaces are (δ, c)-thick – Proof of Proposition 4

Throughout this section we will suppose that β_1 is a parent cylinder on a (δ, c)-thick surface (X, ω). The main result of the section is that a definite proportion of the proto-children of β_1 either have proto-child surfaces that are (δ, c)-thick (which is the outcome we want) or admit a (δ, c)-thin cylinder which pulls back to a short cylinder on (X, ω) (a possibility that we rule out in Section 7).

Conditions on Constants 4. *One has*

$$\delta < \frac{c}{768\sqrt{2}(g-1)}.$$

Lemma 5.1. *Suppose that γ_s is a protochild of β_1 and that β_2 is a (δ, c)-thin cylinder on the corresponding protochild surface. If on (X, ω) we have*

$$\frac{|\beta_1|}{2\sqrt{2}} \leq |\beta_2|,$$

then β_2 cannot be parallel to β_1.

Proof. We have

$$\sin\theta(\beta_2, \gamma_s) \leq \frac{\delta}{|\beta_2||\gamma_s|} \leq \frac{2\sqrt{2}\delta}{|\beta_1||\gamma_s|} < \frac{c}{|\beta_1||\gamma_s|} \leq \sin\theta(\beta_1, \gamma_s).$$

This implies β_2 cannot be parallel to β_1 ■

Definition 11. If β is a (δ, c)-thin cylinder on the protochild surface of γ_s and $\frac{|\beta_1|}{2\sqrt{2}} \leq |\beta|$ then make the following definitions,

(1) By Lemma 5.1, let s_0 be the real number such that β points in the direction of γ_{s_0}.

(2) Let $I(\beta, r)$ (resp. $I^h(\beta, r)$, $I^v(\beta, r)$) be the collection of s for which $\mathrm{hol}(\beta)$ (resp. its horizontal, vertical component) has length less than r on the protochild surface corresponding to γ_s.

(3) Define $I_1(\beta) := I(\beta, \delta)$, i.e. the collection of s so that β is (δ, c)-thin on the protochild surface corresponding to γ_s. Define $I_2(\beta) := I(\beta, \frac{c}{32})$. Define I_1^h, I_1^v, I_2^h, and I_2^v analogously.

Note that it is possible that $s_0 \notin I_1(\beta_1)$.

Lemma 5.2. *Using the same notation as in Definition 11,*

$$I_1^h(\beta) = \left\{ s : |s - s_0| < \frac{\delta |\gamma_{s_0}|}{\mathrm{area}(\beta_1)|\beta|} \right\}$$

and similarly

$$I_2^h(\beta) = \left\{ s : |s - s_0| < \frac{c|\gamma_{s_0}|}{32\mathrm{area}(\beta_1)|\beta|} \right\}.$$

Proof. Let γ_s be a protochild. Rotate so γ_s is vertical and let h be the horizontal component of $\mathrm{hol}(\beta)$ – recall that we define this to mean the holonomy vector of the core curve of β. Now

$$\frac{h}{|\beta|} = \sin\theta(\gamma_s, \beta) = \sin\theta(\gamma_{s_0}, \gamma_s) = \frac{\mathrm{area}(\beta_1)|s - s_0|}{|\gamma_{s_0}||\gamma_s|}.$$

We now apply g_t until γ_s has unit length. The number s belongs to $I_1^h(\beta)$ if and only if $h|\gamma_s| < \delta$, equivalently,

$$\frac{\mathrm{area}(\beta_1)|s - s_0||\beta|}{|\gamma_{s_0}|} < \delta.$$

This proves the first statement. The proof of the second is identical. ∎

Remark 8. Notice that when $|\beta_1| > 1$, which follows from Condition on Constants 3, we have

$$s|\beta_1| \le |\gamma_s| \le (s+1)|\beta_1|.$$

If $\frac{1}{2\sqrt{2}}|\beta_1| \le |\beta|$, then by Lemma 5.2 the radius of $I^h(\beta, r)$ is bounded above by $\frac{4(s_0+1)r}{\sqrt{2}c}$. If $s_0 > 1$ then the bounds simplify to

$$|\gamma_{s_0}| \le 2s_0|\beta_1| \quad \text{and} \quad |I^h(\beta, r)| \le 2\left(\frac{8s_0 r}{\sqrt{2}c}\right).$$

Lemma 5.3. *If $\frac{|\beta_1|}{2\sqrt{2}} \le |\beta|$ and $I_1(\beta)$ intersects the interval $(\frac{2\log|\beta_1|}{\delta}, 2^m\frac{2\log|\beta_1|}{\delta})$, then $I_2^h(\beta) \subseteq I_2^v(\beta)$.*

Proof. Let $v_s(\cdot)$ and $h_s(\cdot)$ denote respectively the vertical and horizontal parts of a holonomy vector when (X, ω) is rotated so that γ_s is vertical. We will allow cylinders as arguments by identifying them with the holonomy vector of their core curve.

If $I_1(\beta)$ intersects the interval of times s in which protochildren are chosen, then so does $I_1^h(\beta)$. By Remark 8, it follows that, since $I_1(\beta)$ is centered at s_0 and has radius bounded above by $\frac{4(s_0+1)\delta}{\sqrt{2}c}$,

$$\frac{2\log|\beta_1|}{\delta} < s_0 + \frac{4\delta(s_0+1)}{\sqrt{2}c} < (s_0+1)\left(1 + \frac{1}{144}\right),$$

where the second inequality follows from Condition on Constants 4. Since $\log|\beta_1| > 1$ and $\delta < 1$ (see Condition on Constants 3 and 4), it follows that $s_0 > 1$, so we can use the simpler bounds in Remark 8.

Notice that

$$\lim_{s\to\infty} \frac{v_s(\beta)}{|\gamma_s|} = 0.$$

Therefore, it suffices to show that if s' is a point so that

$$\frac{v_{s'}(\beta)}{|\gamma_{s'}|} = \frac{c}{32},$$

then s' is less than the left endpoint of $I_2^h(\beta)$, which is at least $s_0(1 - \frac{1}{4\sqrt{2}})$ by Remark 8. Arguing by contradiction suppose that there is a point s' satisfying $s' > s_0(1 - \frac{1}{4\sqrt{2}})$ such that

$$\frac{v_{s'}(\beta)}{|\gamma_{s'}|} = \frac{c}{32}.$$

Since $I_1(\beta)$ is nonempty, there is some point s'' so that $\frac{v_{s''}(\beta)}{|\gamma_{s''}|} = \delta$ and s'' is smaller than the left endpoint of $I_1^v(\beta)$, i.e.

$$s'' \le s_0\left(1 + \frac{8\delta}{\sqrt{2}c}\right) < s_0\left(1 + \frac{1}{144}\right).$$

Notice that

$$\sin\theta(\gamma_{s''}, \gamma_{s_0}) = \frac{|s'' - s_0|\text{area}(\beta)}{|\beta_1||\gamma_{s_0}|} < \frac{1}{144}$$

which implies that s'' is close enough to s_0 that $v_{s''}(\beta) \ge \frac{|\beta|}{2}$.

Since $\frac{v_{s'}(\beta)}{|\gamma_{s'}|} = \frac{c}{32}$ and $\frac{v_{s''}(\beta)}{|\gamma_{s''}|} = \delta$, we have

$$|\gamma_{s'}|v_{s''}(\beta) = \frac{32\delta}{c}|\gamma_{s''}|v_{s'}(\beta).$$

This implies that

$$\frac{s'|\beta_1||\beta|}{2} < \frac{64\delta s''|\beta_1||\beta|}{c}.$$

In other words,

$$s' < \frac{128\delta s''}{c} < s_0\left(\frac{1 + \frac{1}{144}}{2}\right) < s_0\left(1 - \frac{1}{4\sqrt{2}}\right)$$

which is a contradiction. Therefore, we have that every point in $I_2^h(\beta)$ is also contained in $I_2^v(\beta)$ as desired. ∎

Corollary 5.4. *If* $\frac{|\beta_1|}{2\sqrt{2}} \le |\beta|$ *and* $I_1(\beta)$ *intersects the interval* $(\frac{2\log|\beta_1|}{\delta}, 2^m\frac{2\log|\beta_1|}{\delta})$*, then*

$$I^h\left(\beta, \frac{c}{32\sqrt{2}}\right) \subseteq I_2(\beta) \subseteq I_2^h(\beta).$$

It follows that

$$\frac{|I_1(\beta)|}{|I_2(\beta)|} \le \frac{32\sqrt{2}\delta}{c}.$$

Proof. Notice that $I_2(\beta)$ contains the set

$$I^h\left(\beta, \frac{c}{32\sqrt{2}}\right) \cap I^v\left(\beta, \frac{c}{32\sqrt{2}}\right).$$

The proof of Lemma 5.3 with c replaced by $\frac{c}{\sqrt{2}}$ shows that this intersection is exactly $I^h(\beta, \frac{c}{32\sqrt{2}})$, which establishes the first inclusion.

The second inclusion is immediate from Lemma 5.3. ∎

Lemma 5.5. *For any interval of the form $[s, 2s] \subseteq (\frac{2\log|\beta_1|}{\delta}, 2^m\frac{2\log|\beta_1|}{\delta})$, the subset that is contained in some $I_1(\beta)$, for some cylinder β satisfying*

$$\frac{|\beta_1|}{2\sqrt{2}} \leq |\beta|,$$

has length at most $\frac{\delta(192\sqrt{2}g-192\sqrt{2})}{c}s$

Note our choice of δ in Conditions on Constants 4 says the above quantity is at most $\frac{s}{2}$.

Proof. We will proceed in three steps.

Step 1. *Any $3g-2$ intervals of the form $I_2(\beta)$ have empty intersection.*

Such an intersection would contain $(3g-2)$ cylinders that have length at most $\frac{c}{32}$ and area at least c, hence height at least 32. These cylinders cannot cross each other and hence such an intersection would contradict the fact that there are at most $3g-3$ disjoint simple closed curves on a surface of genus g.

Step 2. *If $I_1(\beta)$ intersects $[s, 2s]$, then $I_2(\beta)$ is no longer than $\frac{s}{4}$.*

If $I_1(\beta)$ intersects $[s, 2s]$, then the s_0 corresponding to β must satisfy

$$s_0 - \frac{4\sqrt{2}\delta s_0}{c} \leq 2s, \quad \text{that is} \quad s_0 \leq \frac{2s}{1-\frac{4\sqrt{2}\delta}{c}}.$$

This shows that $s_0 \leq 2\sqrt{2}s$ and so by the simplified estimates in Remark 8, the radius of $I_2(\beta)$ is at most $\frac{s}{2}$.

Step 3. *The subset of $[s, 2s]$ that is contained in some $I_1(\beta)$ has length at most*

$$\frac{\delta(192\sqrt{2}g-192\sqrt{2})}{c}s.$$

Let J be the collection of s_0 from cylinders β so that $I_1(\beta)$ intersects the interval $[s, 2s]$. For any such β, $I_2(\beta) \subseteq [\frac{s}{2}, \frac{5s}{2}]$ and each element in the interval $[\frac{s}{2}, \frac{5s}{2}]$ may lie in at most $3g-3$ intervals of the form $I_2(\beta)$. Therefore,

$$\sum_{s_0\in J} |I_1(\beta)| \leq \frac{32\sqrt{2}\delta}{c}\sum_{s_0\in J} |I_2(\beta)| \leq \frac{(192\sqrt{2}g-192\sqrt{2})\delta}{c}s,$$

where the first inequality holds by Corollary 5.4. ∎

For convenience we restate an equivalent form of Proposition 4 here.

Proposition. *For any interval of the form* $[s, 2s] \subseteq (\frac{2\log|\beta_1|}{\delta}, 2^m \frac{2\log|\beta_1|}{\delta})$*, there are at least* $(1-\nu)s - 1$ *points in* $[s, 2s]$ *that are not contained in any* $I_1(\beta)$*, for some cylinder* β *satisfying*

$$\frac{|\beta_1|}{2\sqrt{2}} \leq |\beta|,$$

and that are separated by at least unit distance.

Proof of Proposition 4. Recall $\nu = \frac{\delta(192\sqrt{2}g-192\sqrt{2})}{c}$. Let s_1 be the first point in $[s, 2s]$ not contained in some I_1 and set $a_1 = s_1 - s$. Let s_2 be the next point that lies beyond $s_1 + 1$ and set $a_2 = s_2 - (s_1 + 1)$. Iterate this procedure until s_n lies within unit distance of $2s$. Let the leftover distance at the end be $\rho = 2s - s_n$. By Lemma 5.5, $\sum_{i=1}^n a_i < \nu s$. Since

$$n + \sum_{i=1}^{n} a_i + \rho = s,$$

we must have

$$n \geq s - 1 - \nu s = (1-\nu)s - 1. \qquad \blacksquare$$

6. Getting started – Proof of Proposition 3

We will continue to assume that all Conditions on Constants from previous sections hold in this section. Make the following definitions.

Definition 12. Let $\mathrm{Sys}(X, \omega)$ be the length of the shortest saddle connection on (X, ω). Set

$$T_1 := 2g + |\Sigma| - 2 \quad \text{and} \quad T_0 = 2^{(2^{4T_1})}.$$

Let $\Theta(R) \subseteq [\frac{-\pi}{2}, \frac{\pi}{2}]$ be the collection of the angles of holonomy vectors of core curves of cylinders whose circumference is at most R and whose area is at least $\frac{1}{T_1}$.

We will use the following theorem about the distribution of cylinders on translation surfaces, which is based on work of Chaika [3] and Vorobets [24].

Theorem 6.1 (Marchese, Treviño and Weil [14, Theorem 1.9 (4)]). *Fix* $K \geq \frac{\sqrt{2}T_0^2}{\mathrm{Sys}(X,\omega)}$ *and an integer* $n \geq 1$*. For any interval* $I \subseteq [\frac{-\pi}{2}, \frac{\pi}{2}]$ *such that*

$$|I| \geq \frac{1}{2T_1 \mathrm{Sys}(X, \omega) K^{n-1}}$$

at least half of the points in I *are within* $\frac{\sqrt{3}K}{K^{2n}}$ *of the angle of the holonomy vector of an element of* $\Theta(R)$*.*

We will only use the following immediate consequence.

Corollary 6.2. *There are positive constants* R_0', d_1, d_2, d_3 *so that for any* $R > R_0'$ *there are* $d_1 R^2$ *cylinders of circumference at most* R*, area bounded below by* d_2*, and whose holonomy vectors make angles at least* $\frac{d_3}{R^2}$ *apart.*

Proof. Set

$$\lambda := \frac{\sqrt{2}T_0^2}{\mathrm{Sys}(X,\omega)}.$$

For any $r > \lambda$ there is some $\ell \in [\lambda, \lambda^2]$ so that $r = \ell^n$ for some positive integer n. Take the interval I to be $[\frac{-\pi}{2}, \frac{\pi}{2}]$. Theorem 6.1 states that half of all points in I are within $\frac{\sqrt{3}\lambda^2}{r^2}$ of an element of $\Theta(r)$ when $r > \lambda$.

Fix $r > \lambda$ and divide the circle into intervals of equal size that are as close as possible to radius $\frac{1}{r^2}$. There will be at least $\frac{\pi}{2}r^2 - 1 > \frac{r^2}{2}$ intervals. Let N be the least integer greater than or equal to $\sqrt{3}\lambda^2$. If the angle corresponding to the holonomy element of $\Theta(r)$ is contained in one of the intervals, then the ball of radius $\sqrt{3}\lambda^2$ about it is contained in $2N + 1$ intervals. The ball of radius $\sqrt{3}\lambda^2$ about any point in those $2N + 1$ intervals is contained in $4N + 1$ intervals.

Let S be a maximal collection of points in $\Theta(r)$ whose corresponding angles are all pairwise distance $\frac{4N+2}{r^2}$ apart. Theorem 6.1 implies that

$$|S|(4N + 1) \geq \frac{r^2}{4}.$$

Therefore, we have found $\frac{r^2}{16N+4}$ cylinders of circumference less than r, area at least $\frac{1}{T_1}$, and whose angle of holonomy vectors are separated by a distance of $\frac{4N+2}{r^2}$. ∎

Remark 9. We see that all constants are explicit, that is, we may take

$$R_0' = \frac{\sqrt{2}T_0^2}{\mathrm{Sys}(X,\omega)}, \qquad d_1 = \frac{1}{16\sqrt{3}(R_0')^2 + 20},$$

$$d_2 = \frac{1}{2g + |\Sigma| - 2}, \quad d_3 = 4\sqrt{3}(R_0')^2 + 6.$$

We will also use the quadratic asymptotics of cylinders,

Theorem 6.3 (Masur [16, Theorem 1]). *There is also a constant d_4 such that for any R there are at most $d_4 R^2$ cylinders of circumference at most R.*

Set

$$D := \max\left(2, \sqrt{\frac{2d_4}{d_1}}\right).$$

Conditions on Constants 5. *We make the following additional assumptions on the constants c, δ, and R_0:*

(1) $c < d_2$,

(2) $\delta < \frac{c}{512D^4}$,

(3) $R_0 > \max(R_0', \exp(\frac{4}{d_3}), D, \frac{1}{\mathrm{Sys}(X,\omega)})$.

For convenience we record an equivalent version of Proposition 3.

Proposition. *For $R \geq R_0$ there is a cylinder β of circumference at least R, area at least c, and that contains a protochild whose protochild surface is (δ, c)-thick.*

Proof of Proposition 3. By Corollary 6.2, let Cyl be a collection of $d_1(DR)^2 \geq 2d_4R^2$ cylinders, of circumference at most DR, area bounded below by d_2, and whose holonomy vectors make angles at least $\frac{d_3}{(DR)^2}$ apart. By Theorem 6.3, there are at most d_4R^2 cylinders of circumference less than R. Let $\mathcal{C}$ be the subcollection of at least d_4R^2 cylinders in Cyl whose circumference is in $[R, DR]$.

Step 1. *Two distinct cylinders in $\mathcal{C}$ have disjoint sets of protochildren and the sets of protochildren have length at least $\frac{\delta c}{16(DR)^2\log(DR)}$.*

Given a cylinder $\beta \in \mathcal{C}$, the collection of angles of holonomy vectors of protochildren has the form $[N, 2^m N]$, where $N = \frac{2\log|\beta|}{\delta}$. Therefore,

$$\sin\theta(\gamma_N, \gamma_{2^m N}) = \frac{\mathrm{area}(\beta)|2^m N - N|}{|\gamma_N||\gamma_{2^m N}|}.$$

Using that $|\gamma_N| \leq 2N|\beta|$ and a similar estimate for $|\gamma_{2^m N}|$, it follows that

$$\frac{\delta c}{16|\beta|^2\log|\beta|} \leq \sin\theta(\gamma_N, \gamma_{2^m N}) \leq \theta(\gamma_N, \gamma_{2^m N}).$$

Since $|\beta|$ is large. it follows that $\sin\theta(\gamma_N, \beta)$ is small and so,

$$\theta(\gamma_N, \beta) \leq 2\sin\theta(\gamma_N, \beta) \leq \frac{2}{|\beta|^2 N} = \frac{\delta}{|\beta|^2\log|\beta|}.$$

Since s_N is the furthest point from β, we have that the largest distance from the angle of β to an angle of its protochild set is

$$\frac{\delta}{|\beta|^2\log|\beta|} \leq \frac{\delta}{R^2\log R}.$$

If β and β' are two distinct cylinders in $\mathcal{C}$, then the angles are separated by a distance of at least

$$\frac{d_3}{(DR)^2}.$$

Therefore, the distance between the two sets of angles of protochildren of β and β' is at least

$$\frac{d_3}{(DR)^2} - \frac{2\delta}{R^2\log R}.$$

Using the estimate that

$$\delta < \frac{1}{512D^4} \quad \text{and} \quad \log R > \frac{4}{d_3},$$

we see that the distance between the angles of two sets of protochildren is at least $\frac{d_3}{2(DR)^2}$. Hence, the sets of protochildren are disjoint.

Let $\mathcal{J}$ be the collection of all protochildren of cylinders in $\mathcal{C}$. Partition the interval $[1, DR]$ into subintervals

$$I_j := \left[\frac{R}{D^j}, \frac{R}{D^{j-1}}\right]$$

with $j \geq 0$. We analyze the following cases.

Step 2. *Cylinders of circumference greater than DR can only cause (δ, c)-thinness for half of all protochildren in $\mathcal{I}$.*

This step follows immediately from Condition on Constants 4 and Lemma 5.5.

Step 3. *Cylinders of circumference less than DR cause (δ, c)-thinness for at most a quarter of all protochildren in $\mathcal{I}$.*

Suppose that β' is a cylinder whose circumference belongs to I_j and which is thin for the protochild surface of γ_s of β. Then

$$|\beta'| \sin\theta(\beta', \gamma_s)|\gamma_s| \leq \delta.$$

In other words,

$$\sin\theta(\beta', \gamma_s) \leq \frac{\delta}{|\gamma_s||\beta'|} \leq \frac{\delta^2 D^j}{2R^2 \log R}.$$

Notice that we may assume that $D^j < R^2$ because $\frac{1}{R} < \mathrm{Sys}(X, \omega)$. This implies that

$$\sin\theta(\beta', \gamma_s) < \frac{\delta^2}{2\log R},$$

i.e. the sine of the angle is so small that we can use the estimate $\frac{\theta}{2} < \sin\theta$. Therefore, the length of the collection of angles for which β' is (δ, c)-thin on the corresponding protochild surface is at most

$$\frac{2\delta^2 D^j}{R^2 \log R}.$$

By Theorem 6.3, there are at most

$$\frac{d_4 R^2}{D^{2j-2}}$$

cylinders with circumference in I_j. Since these cylinders are (δ, c)-thin on protochild surfaces corresponding to an interval of angles of length at most

$$\frac{2\delta^2 D^j}{R^2 \log R},$$

the total length of the collection of angles for which a cylinder with circumference in I_j is (δ, c)-thin is at most

$$\left(\frac{d_4 R^2}{D^{2j-2}}\right)\left(\frac{2\delta^2 D^j}{R^2 \log R}\right) = \left(\frac{2d_4 D^2 \delta^2}{\log R}\right)\frac{1}{D^j}.$$

However, there are at least $d_4 R^2$ cylinders in $\mathcal{C}$ and each has a collection of protochildren whose corresponding angles of holonomy vectors has length at least

$$\frac{\delta c}{16(DR)^2 \log(DR)}.$$

Therefore, the length of the interval of angles from the collection of protochildren associated to cylinders in $\mathcal{C}$ is at least

$$(d_4 R^2)\left(\frac{\delta c}{16(DR)^2 \log(DR)}\right) = \left(\frac{\delta c d_4}{16 D^2 \log(DR)}\right).$$

Therefore, the largest proportion of $\mathcal{J}$ whose protochild surface contains a (δ, c)-thin cylinder that has circumference smaller than DR on (X, ω) is

$$\frac{\left(\frac{2d_4 D^2 \delta^2}{\log R}\right)}{\left(\frac{\delta c d_4}{16 D^2 \log(DR)}\right)} \sum_{j=0}^{\infty} \frac{1}{D^j} < \left(\frac{32\delta D^4}{c}\right)\left(\frac{\log(DR)}{\log(R)}\right)\left(\frac{D}{D-1}\right).$$

Since $2 \leq D \leq R$, this ratio is bounded above by $(\frac{128 D^4 \delta}{c})$, which is at most $\frac{1}{4}$ by Conditions on Constants 5. Combining these steps we conclude that a fourth of all protochildren have protochild surfaces that are (δ, c)-thick. ■

7. Cylinders that cause thinness are comparable in size to parent cylinders – Proof of Proposition 5

We continue to assume that all previous Conditions on Constants hold and keep the notation of Definition 6. For reasons that will become apparent set $(X_0, \omega_0) := (X, \omega)$. We add another Condition on R_0.

Conditions on Constants 6. *Set* $\theta_2 := \operatorname{arccot}(10 \cot \theta_1)$ *and then let*

$$R_0 > \max\left(\exp\left(\frac{4}{\delta \cot \theta_1}\right), \sqrt{\frac{12L^2 \cot \theta_1}{\pi}}, \exp\left(\frac{1}{\cot \theta_1}\right), \sqrt{\frac{4L}{\pi \sin \theta_2}}, \sqrt{\frac{8}{\delta \pi}}\right).$$

The main result of this section is the following.

Proposition. *For* $|\beta_0| \geq R_0$*, if* σ_1 *is a protochild of* β_1 *whose protochild surface has a* (δ, c)*-thin cylinder* β_2*, then on* (X_0, ω_0) *we have*

$$\frac{|\beta_1|}{2\sqrt{2}} \leq |\beta_2|.$$

In this section we will adopt the following notation. Let σ_1 be a protochild of β_1 whose protochild surface (X_2, ω_2) has a (δ, c)-thin cylinder β_2. Rename the protochild of β_0 as σ_0 and suppose without loss of generality, that it is vertical on (X_0, ω_0). Let (X_1, ω_1) be the protochild surface associated to σ_1.

Remark 10. Note here that the subscripts 0, 1, and 2 do not refer to times but to labeling. Also, recall the convention that all angles and lengths will be measured on the (X_0, ω_0) unless otherwise mentioned.

Lemma 7.1. *There is some constant* c_3 *depending only on* δ *and* θ_1 *so that for* $|\beta_0| \geq R_0$*, the angle* ϕ *between the holonomy vectors of* β_1 *and* β_2 *satisfies*

$$\phi \leq \frac{c_3}{|\beta_1|^2 \log |\beta_1|}.$$

Proof. Let $\theta = \theta(\sigma_0, \sigma_1)$ be the angle between σ_0 and σ_1. Let $\theta' = \theta_{\beta_2}$ be the angle that the holonomy vector of β_2 makes with the horizontal on (X_1, ω_1). We proceed in three steps.

Step 1. *Since* $|\beta_0| \geq \frac{4}{\delta \cot\theta_1}$, *one has* $\theta(\sigma_0,\sigma_1) \leq \frac{3\cot\theta_1}{|\sigma_0|^2}$.

By Lemma 4.2 (2),

$$\theta(\beta_1,\sigma_0) \leq \tan\theta(\beta_1,\sigma_0) \leq \frac{\cot\theta_1}{|\sigma_0|^2}.$$

Similarly, by Lemma 4.2 (1) and that fact that $\theta(\beta_1,\sigma_1)$ is less than $\frac{\pi}{2}$,

$$\theta(\beta_1,\sigma_1) \leq 2\sin\theta(\beta_1,\sigma_1) \leq \frac{2}{|\sigma_1||\beta_1|}.$$

The triangle inequality now implies that

$$\theta = \theta(\sigma_0,\sigma_1) \leq \frac{2\cot\theta_1}{|\sigma_0|^2} + \frac{2}{|\sigma_1||\beta_1|} = \frac{1}{|\sigma_0|^2}\left(2\cot\theta_1 + 2\frac{|\sigma_0|}{|\sigma_1|}\frac{|\sigma_0|}{|\beta_1|}\right).$$

Again by Lemma 4.1, the ratio $\frac{|\sigma_0|}{|\beta_1|} \leq \frac{2}{\delta}$ and the ratio $\frac{|\sigma_0|}{|\sigma_1|} \leq \frac{1}{\log|\beta_0|}$ so our choice of $|\beta_0|$ gives

$$\theta \leq \frac{3\cot\theta_1}{|\sigma_0|^2}.$$

Step 2. *Since* $|\beta_0| \geq \sqrt{\frac{12L^2\cot\theta_1}{\pi}}$ *and* $\log|\beta_0| \geq \frac{1}{\cot\theta_1}$, *we have that* $|\cot\theta'| \leq 10\cot\theta_1$.

The matrix that passes from (X_1,ω_1) to (X_2,ω_2) is

$$g := g_{\log|\sigma_1|} r_\theta g_{-\log|\sigma_0|} = \begin{pmatrix} \frac{|\sigma_1|}{|\sigma_0|}\cos\theta & -|\sigma_0||\sigma_1|\sin\theta \\ \frac{\sin\theta}{|\sigma_0||\sigma_1|} & \frac{|\sigma_0|}{|\sigma_1|}\cos\theta \end{pmatrix}.$$

Therefore, if some vector (h,v) has length less than δ after applying g, it follows that

$$\left| h\frac{|\sigma_1|}{|\sigma_0|}\cos\theta - v|\sigma_0||\sigma_1|\sin\theta \right| \leq \delta.$$

By the triangle inequality,

$$|h|\frac{|\sigma_1|}{|\sigma_0|}|\cos\theta| \leq |v||\sigma_0||\sigma_1||\sin\theta| + \delta.$$

Since $|\sigma_0| > \frac{|\beta_0|}{L}$, Step 1 implies

$$\theta \leq \frac{3L^2\cot\theta_1}{|\beta_0|^2}$$

so that by our choice of $|\beta_0|$ we have $\cos\theta > \frac{1}{2}$. This implies that

$$|h| \leq 2\delta\frac{|\sigma_0|}{|\sigma_1|} + 2|v||\sigma_0|^2\theta.$$

By Step 1 we have

$$|h| \leq 2\delta\frac{|\sigma_0|}{|\sigma_1|} + 6\cot\theta_1|v|.$$

Assume now that (h, v) is the holonomy vector of β_2 on (X_1, ω_1). Dividing through the previous equation by $|v|$, we see that

$$\cot\theta' = \frac{|h|}{|v|} \leq \left(\frac{2\delta}{|v|}\right)\left(\frac{|\sigma_0|}{|\sigma_1|}\right) + 6\cot\theta_1.$$

Since $(\frac{|\sigma_0|}{|\sigma_1|}) < \frac{1}{\log|\beta_0|} < \cot\theta_1$,

$$\cot\theta' = \frac{|h|}{|v|} \leq \left(\frac{2\delta}{|v|} + 6\right)\cot\theta_1.$$

Recall that, by Conditions on Constants 1 and 2, $\cot\theta_1 < \frac{c_1}{16} < \frac{1}{16}$. Therefore, if $|v| < \frac{\delta}{2}$, the previous displayed inequality implies that $h < \frac{5\delta}{16}$ and so (h, v) has length less than δ. However, since (X_1, ω_1) is (δ, c)-thick, it implies that (h, v) has length at least δ. In other words, $|v| \geq \frac{\delta}{2}$, which now implies that $\cot\theta' \leq 10\cot\theta_1$ as desired.

Step 3. *For* $|\beta_0|^2 \geq \max(\frac{4L}{\pi\sin\theta_2}, \frac{8}{\delta\pi})$ *there is some constant* c_3 *depending only* δ, θ_1 *so that* $\phi \leq \frac{c_3}{|\beta_1|^2\log|\beta_1|}$.

By Step 2, the vertical part of the holonomy vector of β_2 on (X_1, ω_1) is at least $\delta\sin\theta_2$. Therefore, on (X_0, ω_0),

$$|\beta_2| \geq \delta|\sigma_0|\sin\theta_2.$$

Let α_1 be the angle between the holonomy vectors of σ_1 and β_2. Let (X_0', ω_0') be the surface (X_0, ω_0) rotated so that σ_1 is vertical. On this surface the horizontal part of the holonomy vector of β_2 is at most $\frac{\delta}{|\sigma_1|}$ since β_2 has length less than δ on (X_2, ω_2). Therefore, by the above lower bound on $|\beta_2|$

$$\sin\alpha_1 \leq \frac{\delta}{|\beta_2||\sigma_1|} \leq \frac{1}{\sin\theta_2|\sigma_0||\sigma_1|} \leq \frac{L}{\sin\theta_2|\beta_0|^2\log|\beta_0|}.$$

Let α_2 be the angle between the holonomy vectors of σ_1 and β_1. We have

$$\sin\alpha_2 \leq \frac{1}{|\beta_1||\sigma_1|} \leq \frac{2}{\delta|\beta_0|^2(\log|\beta_0|)^2}.$$

Now our choice of $|\beta_0|$ says

$$\frac{\alpha_i}{2} \leq \sin\alpha_i$$

for $i \in \{1, 2\}$.

By the triangle inequality,

$$\phi \leq 2\left(\frac{1}{\sin\theta_2|\sigma_0||\sigma_1|} + \frac{1}{|\beta_1||\sigma_1|}\right).$$

By Lemma 4.1 $|\beta_1|$ is comparable to $|\sigma_0|$ and $|\sigma_1| \geq |\beta_1|\log|\beta_1|$. It follows that there is a constant c_3 only depending on δ and θ_1 such that

$$\phi \leq \frac{c_3}{|\beta_1|^2\log|\beta_1|}.$$

This finishes the proof of the lemma. ■

For Proposition 5 to hold we impose one final condition on constants.

Conditions on Constants 7. *Suppose that*

$$R_0 > \max\left(\exp\left(\frac{c_3}{2\sqrt{2}\delta^2 \sin\theta_0}\right), \frac{2c_3}{\delta}\right).$$

Proof of Proposition 5. Arguing by contradiction assume that $\frac{|\beta_1|}{2\sqrt{2}} \geq |\beta_2|$. The proof will be divided into two steps.

Step 1. *For* $\log|\beta_0| \geq \frac{c_3}{2\sqrt{2}\delta^2 \sin\theta_0}$, *on* (X_1, ω_1) *we have* $|\beta_2| \geq |\beta_1|$.

Now changing notation from the previous lemma, let α_i be the angle between β_1 and β_2 on (X_i, ω_i) for $i \in \{0, 1\}$. Let $|\cdot|_i$ denote lengths on (X_i, ω_i) for $i \in \{0, 1\}$. We have

$$\delta^2 \sin\alpha_1 \leq |\beta_1|_1 |\beta_2|_1 \sin\alpha_1 = |\beta_1||\beta_2| \sin\alpha_0 \leq \frac{c_3|\beta_2|}{|\beta_1| \log|\beta_1|},$$

where the left-hand inequality follows from the fact that (X_1, ω_1) is (δ, c_1)-thick and the righthand inequality follows from Lemma 7.1. By assumption, we have

$$\sin\alpha_1 \leq \frac{c_3}{2\sqrt{2}\delta^2 \log|\beta_1|}.$$

By our choice of $|\beta_0|$, since $|\beta_1| > |\beta_0|$ it follows that

$$\sin\alpha_1 < \sin\theta_0.$$

However, if $|\beta_2| \leq |\beta_1|$ on (X_1, ω_1), then by Proposition 1 (4), the angle α_1 between the holonomy vectors of β_1 and β_2 on (X_1, ω_1) is bounded below by θ_0, which is a contradiction to the above inequality.

Step 2. *For* $|\beta_0| \geq \frac{2c_3}{\delta}$, *on* (X_0, ω_0) *we have* $|\beta_2| \geq \frac{|\beta_1|}{2\sqrt{2}}$.

We again argue by contradiction. The holonomy vector of the core curve of β_2 on (X_0, ω_0) is

$$(|\beta_2|\cos(\varphi + \alpha_0), |\beta_2|\sin(\varphi + \alpha_0)),$$

where φ is the angle that the holonomy vector of β_1 on (X_0, ω_0) makes with the horizontal. The holonomy of β_2 on (X_1, ω_1) is

$$\left(|\sigma_0||\beta_2|\cos(\varphi + \alpha_0), \frac{|\beta_2|}{|\sigma_0|}\sin(\varphi + \alpha_0)\right).$$

Therefore (using the comparison of L^1 and L^2 norms on $\mathbb{R}^2$: $\frac{1}{\sqrt{2}}\|\cdot\|_1 \leq \|\cdot\|_2 \leq \|\cdot\|_1$),

$$\frac{1}{\sqrt{2}}\left(|\sigma_0||\beta_1|\cos(\varphi) + \frac{|\beta_1|}{|\sigma_0|}\sin(\varphi)\right) \leq |\sigma_0||\beta_2|\cos(\varphi + \alpha_0) + \frac{|\beta_2|}{|\sigma_0|}\sin(\varphi + \alpha_0).$$

The right-hand side is bounded above by

$$|\sigma_0||\beta_2|\cos\varphi + |\sigma_0||\beta_2|\sin\alpha_0 + \frac{|\beta_2|}{|\sigma_0|}\sin\varphi + \frac{|\beta_2|}{|\sigma_0|}\sin\alpha_0.$$

Subtracting

$$\frac{1}{2\sqrt{2}}(|\sigma_0||\beta_1|\cos(\varphi) + \frac{|\beta_1|}{|\sigma_0|}\sin(\varphi))$$

from both sides of the inequality yields (along with the estimate $|\beta_2| \le \frac{|\beta_1|}{2\sqrt{2}}$),

$$\frac{1}{2\sqrt{2}}\left(|\sigma_0||\beta_1|\cos(\varphi) + \frac{|\beta_1|}{|\sigma_0|}\sin(\varphi)\right) \le |\sigma_0||\beta_2|\sin\alpha_0 + \frac{|\beta_2|}{|\sigma_0|}\sin\alpha_0.$$

Again using the comparison of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^2$ and the fact that β_1 has length at least δ on (X_1, ω_1), we have

$$\frac{\delta}{2\sqrt{2}} \le |\beta_2|\sin\alpha_0 + \frac{|\beta_2|}{|\sigma_0|^2}\sin\alpha_0.$$

It now follows from Lemma 7.1 that

$$\frac{\delta}{2\sqrt{2}} \le \frac{c_3|\beta_2|}{|\beta_1|^2\log|\beta_1|} + \frac{c_3|\beta_2|}{|\sigma_0|^2|\beta_1|^2\log|\beta_1|}.$$

Applying the estimate $|\beta_2| \le \frac{|\beta_1|}{2\sqrt{2}}$,

$$\delta \le \frac{c_3}{|\beta_1|\log|\beta_1|} + \frac{c_3}{|\sigma_0|^2|\beta_1|\log|\beta_1|}.$$

By our condition on $|\beta_0|$ both terms on the right are smaller than $\frac{\delta}{2}$ which yields a contradiction. ∎

8. Selecting constants – Proof of Proposition 6

For convenience we recall the statement of Proposition 6.

Proposition. *It is possible to choose constants so that all Conditions on Constants are satisfied.*

Proof of Proposition 6. Choose constants as follows.

(1) Let d_1, d_2, d_3, d_4 be the constants associated to (X, ω) as in Section 6.

(2) Choose

$$c < \min\left(d_2, \frac{1}{2g(2g + |\Sigma| - 2)}\right).$$

Being less than the second quantity implies that we may choose $c = c_1 = c_2$, where c_1 and c_2 satisfy Conditions on Constants 1. Being less than the first quantity is required to satisfy Conditions on Constants 5.

(3) Define

$$D := \max\left(2, \sqrt{\frac{2d_4}{d_1}}\right).$$

(4) Choose

$$\delta < \min\left(\frac{c}{512D^4}, \frac{c}{768\sqrt{2}(g-1)}, \mathrm{Sys}(X,\omega), \epsilon_2\right),$$

where ϵ_2 is the Margulis constant. Being less than the first quantity is required to satisfy Conditions on Constants 5. Being less than the second quantity is required to satisfy Conditions on Constants 4. Being less than the third and fourth is required to satisfy Conditions on Constants 1.

(5) Choose $\theta_1 \in (0,\pi)$ so that $\cot\theta_1 < \frac{c}{16}$. Recall that $c = c_1$. Conditions on Constants 1 is now completely satisfied.

(6) Choose

$$\nu = \frac{\delta(192\sqrt{2}g - 192\sqrt{2})}{c}.$$

This choice of ν and the above choice of δ says $\nu < \frac{1}{4}$

(7) Let L and θ_0 be as in Proposition 1.

(8) Choose

$$M = \frac{2^{m+2}L}{\delta}$$

for a positive integer m such that

$$m > 6L\left(\log\left(\frac{2}{1+2\nu}\right)\right)^{-1}$$

and so that $M > 21$. Conditions on Constants 2 are now completely satisfied.

(9) Define

$$T_1 := 2g + |\Sigma| - 2, \qquad T_0 := 2^{(2^{4T_1})}, \quad R_0' := \frac{\sqrt{2}T_0^2}{\mathrm{Sys}(X,\omega)},$$

$$\theta_2 := \operatorname{arccot}(10\cot\theta_1), \quad C := \frac{Lc}{16M}.$$

(10) Define

$$R_0'' := \max\left(R_0', \exp\left(\frac{4}{d_3}\right), D, \frac{1}{\mathrm{Sys}(X,\omega)}, \exp\left(\frac{4}{C}\right), e^M, \exp\left(\frac{4}{\delta\cot\theta_1}\right),\right.$$
$$\left.\sqrt{\frac{12L^2\cot\theta_1}{\pi}}, \exp\left(\frac{1}{\cot\theta_1}\right), \sqrt{\frac{4L}{\pi\sin\theta_2}}, \sqrt{\frac{8}{\delta\pi}}\right).$$

Now $M > 21$ and $\delta < 1$ imply $e^M > 21 > e^{2\delta}$. Thus the fact that

$$R_0 \geq \max\left\{e^M, \exp\left(\frac{4}{C}\right)\right\}$$

shows Conditions on Constants 3 is satisfied. Moreover, being larger than the final five quantities implies that Conditions on Constants 6 is completely satisfied. Finally, being larger than the first four terms is required to satisfy Conditions on Constants 5, which is now completely satisfied.

(11) For cylinders of circumference at least R_0'', Lemma 7.1 produces a constant c_3.

(12) Set

$$R_0 := \max\left(R_0'', \exp\left(\frac{c_3}{2\sqrt{2}\delta^2 \sin\theta_0}\right), \frac{2c_3}{\delta}\right).$$

Conditions on Constants 7 is now completely satisfied. At this point all conditions on constants are satisfied

∎

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