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Frédéric Bayart

Summability of the coefficients of a multilinear form

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Abstract. Let T be an m-linear form defined on a product of ℓ_p-spaces and let $\Lambda \subset \mathbb{N}^m$. We investigate the best exponent s such that the sequence of coefficients of T belongs to $\ell_s(\Lambda)$. The cases $\Lambda = \mathbb{N}^m$ and Λ is the diagonal of $\mathbb{N}^m$ are already known. We study the intermediate cases using notions like combinatorial dimension of sets, multiple summing maps and random polynomials.

Keywords. Multiple summing operators, multilinear mappings, random polynomials, combinatorial dimension

1. Introduction

Let T be an m-linear form defined on a product of ℓ_p-spaces. We are interested in the sequence of its coefficients $(T(e(\mathbf{j})))_{\mathbf{j}\in\mathbb{N}^m}$, where $\mathbf{j}$ stands for $(j_1,\dots,j_m)$ and $e(\mathbf{j})$ for $(e_{j_1},\dots,e_{j_m})$. In particular, we are interested in which ℓ_s-space this sequence belongs. If $m=1$, the answer is given by the duality of the ℓ_p-spaces. For $m\geq 2$, this problem has attracted the attention of many mathematicians for one century since the seminal work of Littlewood [19]. For $\mathbf{p}=(p_1,\dots,p_m)\in[1,+\infty]^m$ and $\lambda\in\mathbb{R}$, we set

$$\lambda\mathbf{p} := (\lambda p_1,\dots,\lambda p_m) \quad\text{and}\quad \left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1}+\cdots+\frac{1}{p_m}.$$

We shall denote $Z_p=\ell_p$ for $1\leq p<+\infty$ and $Z_\infty = c_0$. The works of [19], [12], [17], [23] and [14] culminate in the following statement. Assume that $|1/\mathbf{p}|<1$. Then there exists a constant $C_{m,\mathbf{p}}>0$ such that, for all m-linear forms $T: Z_{p_1}\times\cdots\times Z_{p_m}\to\mathbb{C}$,

$$\Big(\sum_{\mathbf{j}\in\mathbb{N}^m} |T(e(\mathbf{j}))|^{\frac{2m}{m+1-|2/\mathbf{p}|}}\Big)^{\frac{m+1-|2/\mathbf{p}|}{2m}} \leq C_{m,\mathbf{p}}\|T\| \quad\text{provided } \left|\frac{1}{\mathbf{p}}\right|\leq\frac{1}{2}, \tag{1}$$

$$\Big(\sum_{\mathbf{j}\in\mathbb{N}^m} |T(e(\mathbf{j}))|^{\frac{1}{1-|1/\mathbf{p}|}}\Big)^{1-|1/\mathbf{p}|} \leq C_{m,\mathbf{p}}\|T\| \quad\text{provided } \left|\frac{1}{\mathbf{p}}\right|\geq\frac{1}{2}. \tag{2}$$

Moreover, the exponents in (1) and (2) are optimal.

Frédéric Bayart: Laboratoire de Mathématiques Blaise Pascal UMR 6620 CNRS, Université Clermont Auvergne, Campus universitaire des Cézeaux, 3 place Vasarely, 63178 Aubière Cedex, France; frederic.bayart@uca.fr

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It is also natural to ask what happens if we look at the summability of $(T(e(\mathbf{j})))_{\mathbf{j}\in\Lambda}$ for some subset $\Lambda \subset \mathbb{N}^m$. The most obvious case is $\Lambda = \mathrm{Diag}(\mathbb{N}^m) := \{(j,\dots,j);\ j \in \mathbb{N}\}$. Then, from the work of Defant-Voigt, Aron and Globevnik [4] and Zalduendo [25], we know that, provided $|1/\mathbf{p}| < 1$, for all m-linear forms $T : Z_{p_1} \times \cdots \times Z_{p_m} \to \mathbb{C}$,

$$\Bigg(\sum_{\mathbf{j}\in\mathrm{Diag}(\mathbb{N}^m)} |T(e(\mathbf{j}))|^{\frac{1}{1-|1/\mathbf{p}|}}\Bigg)^{1-|1/\mathbf{p}|} \leq \|T\| \tag{3}$$

(here, the constant may be taken equal to 1). Again, the exponent is optimal. For products of diagonals, namely when $\Lambda = \Lambda_1 \times \cdots \times \Lambda_p$ where each $\Lambda_i = \mathrm{Diag}(\mathbb{N}^{m_i})$ and $m_1 + \cdots + m_p = m$, the sharp exponent has been obtained in [3] for $\mathbf{p} = (\infty,\dots,\infty)$ and in [1] for the general case. We do not state the precise statement here because it is not so easy to write it and because we will get a simpler one soon.

In this paper our aim is to get similar inequalities for general subsets $\Lambda \subset \mathbb{N}^m$. The case $\mathbf{p} = (\infty,\dots,\infty)$ has already been considered by Blei [9] (see also a very nice account of that work in the book [10]). We consider the other cases. The exponent that we will get will depend on the size of Λ, more precisely on its combinatorial dimension. For $\Lambda \subset \mathbb{N}^m$ and $n \geq 0$, define

$$\psi_\Lambda(n) := \max\{\mathrm{card}((A_1 \times \cdots \times A_m) \cap \Lambda);\ A_i \subset \mathbb{N},\ \mathrm{card}(A_i) \leq n\}.$$

The *combinatorial dimension* of Λ, denoted by $\dim(\Lambda)$, is defined as

$$\dim(\Lambda) := \limsup_{n\to+\infty} \frac{\log \psi_\Lambda(n)}{\log n} = \inf\{s > 0;\ \exists C > 0,\ \psi_\Lambda(n) \leq Cn^s \text{ for all } n \in \mathbb{N}\}.$$

We will also say that $\dim(\Lambda)$ is *exact* if $\psi_\Lambda(n) \leq Cn^{\dim(\Lambda)}$ for some $C > 0$ and all $n \in \mathbb{N}$.

We introduce the *Hardy–Littlewood exponent* of Λ of index $\mathbf{p}$ as follows: $\mathcal{HL}(\Lambda,\mathbf{p})$ is the set of those $s \geq 1$ such that there exists $C > 0$ satisfying, for all m-linear forms $T : Z_{p_1} \times \cdots \times Z_{p_m} \to \mathbb{C}$,

$$\Bigg(\sum_{j\in\Lambda} |T(e(\mathbf{j}))|^s\Bigg)^{1/s} \leq C\|T\|. \tag{4}$$

The Hardy–Littlewood exponent $HL(\Lambda,\mathbf{p})$ is the infimum of $\mathcal{HL}(\Lambda,\mathbf{p})$.

Our first main theorem now reads:

Theorem 1.1. *Let $\Lambda \subset \mathbb{N}^m$ be infinite and $\mathbf{p} \in [1,+\infty]^m$ with $|1/\mathbf{p}| < 1$. Then*

(a) $$HL(\Lambda,\mathbf{p})^{-1} \geq \frac{\dim(\Lambda)+1}{2\dim(\Lambda)} - \left|\frac{1}{\dim(\Lambda)\mathbf{p}}\right| \quad \textit{provided} \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{2}.$$

Moreover, if $\dim(\Lambda)$ *is exact, then* $\left(\frac{\dim(\Lambda)+1}{2\dim(\Lambda)} - \left|\frac{1}{\dim(\lambda)\mathbf{p}}\right|\right)^{-1}$ *belongs to* $\mathcal{HL}(\Lambda,\mathbf{p})$.

(b) $$HL(\Lambda,\mathbf{p})^{-1} \geq 1 - \left|\frac{1}{\mathbf{p}}\right| \quad \textit{provided} \left|\frac{1}{\mathbf{p}}\right| \geq \frac{1}{2}.$$

Moreover, $(1 - |1/\mathbf{p}|)^{-1}$ *belongs to* $\mathcal{HL}(\Lambda,\mathbf{p})$.

We may observe that the two inequalities (a) and (b) coincide if $|1/\mathbf{p}| = 1/2$.

Since $\dim(\mathbb{N}^m) = m$ and $\dim(\mathrm{Diag}(\mathbb{N}^m)) = 1$, this result covers (1)–(3). It is not difficult to see that it also covers the case $\Lambda = \mathrm{Diag}(\mathbb{N}^{m_1}) \times \cdots \times \mathrm{Diag}(\mathbb{N}^{m_p})$ with a more pleasant-looking result. It should be noticed that case (b) only appears for aesthetic reasons. Indeed, it is already known, since $HL(\Lambda, \mathbf{p})^{-1} \geq HL(\mathbb{N}^m, \mathbf{p})$ and the result follows from (2).

Of course, it is natural to ask whether the inequalities on $HL(\Lambda, \mathbf{p})$ are optimal with respect to $\dim(\Lambda)$. At this level of generality, this cannot be the case except if $\mathbf{p} = (\infty, \dots, \infty)$. For instance, let $m = 2$ and $\mathbf{p} = (p, p)$ with $p \geq 2$, $\Lambda_1 = \mathrm{Diag}(\mathbb{N}^2)$ and $\Lambda_2 = \{(j, 1);\ j \in \mathbb{N}\}$. In both cases, $\dim(\Lambda_1) = \dim(\Lambda_2) = 1$ whereas the optimal value of s such that (4) holds for all 2-linear forms $T : Z_p \times Z_p \to \mathbb{C}$ is given by $1/s_1 = 1 - 2/p$ for Λ_1 (the optimality is shown in [25]) and by $1/s_2 = 1 - 1/p$ for Λ_2 (we can replace the bilinear T by the linear $S(\cdot) = T(\cdot, e_1)$).

Thus, the right question seems to be the following: for a fixed $d \in [1, m)$, does there exist $\Lambda \subset \mathbb{N}^m$ with $\dim(\Lambda) = d$ and such that the inequalities of Theorem 1.1 are optimal? For the sake of simplicity, we will assume that $\mathbf{p} = (p, \dots, p)$ for some $p \in [1, +\infty]$. Again, in case (b), the result is already known from [25]: the optimality of the exponent in (3) tells us that $HL(\mathrm{Diag}(\mathbb{N}^m), \mathbf{p})^{-1} \leq 1 - |1/\mathbf{p}|$. Taking for Λ_0 any subset of $\mathbb{N}^m$ of dimension d and setting $\Lambda = \Lambda_0 \cup \mathrm{Diag}(\mathbb{N}^m)$, we clearly have $HL(\Lambda, \mathbf{p})^{-1} \leq 1 - |1/\mathbf{p}|$ with $\dim(\Lambda) = d$.

We have been able to show the optimality of part (a) of Theorem 1.1 when the dimension d is sufficienly large or when m/d is an integer.

Theorem 1.2. *Let $m \geq 2$ and let $d \in [1, m]$. There exists $\Lambda \subset \mathbb{N}^m$ with $\dim(\Lambda) = d$ such that, for all $p \in [2, +\infty]$ with $m/p \leq 1/2$,*

$$HL(\Lambda, \mathbf{p})^{-1} = \frac{d+1}{2d} - \left|\frac{1}{d\mathbf{p}}\right|$$

in the following cases :

- *m is even and $d \geq 3/2$;*
- *m is odd and $d \geq 3/2 + \frac{1}{2\lfloor m/2 \rfloor}$;*
- *m/d is an integer.*

As is usual in this context, the reverse inequality is proved by using a random construction of the multilinear form. Nevertheless, a new difficulty arises: we also have to find the right subset Λ of $\mathbb{N}^m$ with prescribed dimension. This will be done using an argument that is partly probabilistic and partly deterministic, starting from the so-called fractional cartesian products.

The paper is organized as follows. In Section 2, we prove an extension of an inequality due to Bohnenblust and Hille on sequences indexed by $\mathbb{N}^m$ when we restrict summation to a subset $\Lambda \subset \mathbb{N}^m$. We apply this inequality in Section 3 to the sequence of coefficients of an m-linear form, using the notion of multiple summing maps. Section 4 is devoted to the proof of Theorem 1.2. Finally, in Section 5, we discuss some related problems.

Notations. For positive integers m, n,

$$\mathcal{M}(m,n) = \{\mathbf{j} = (j_1,\dots,j_m);\ 1 \le j_1,\dots,j_m \le n\}.$$

If $k \in \{1,\dots,m\}$, $j_k \in \mathbb{N}$ and $\widehat{\mathbf{j}_k} = (j_1,\dots,j_{k-1},j_{k+1},\dots,j_m) \in \mathbb{N}^{m-1}$ are given, then $\mathbf{j} = (j_1,\dots,j_k,\dots,j_m) \in \mathbb{N}^m$. If $A_1,\dots,A_m$ are sets and $k \in \{1,\dots,m\}$, then $\widehat{A_k}$ denotes $A_1 \times \cdots \times A_{k-1} \times A_{k+1} \times \cdots \times A_m$.

For $\mathbf{p} \in [1,+\infty]^m$, $B_{\ell_{\mathbf{p}}}$ denotes the product of the unit balls $B_{\ell_{p_1}} \times \cdots \times B_{\ell_{p_m}}$. For $x = (x^{(1)},\dots,x^{(m)}) \in B_{\ell_{\mathbf{p}}}$, and $\mathbf{j} \in \mathbb{N}^m$, $x_{\mathbf{j}}$ stands for the product $x^{(1)}_{j_1} \cdots x^{(m)}_{j_m}$. In a similar way, if $x^* = (x^*(1),\dots,x^*(m)) \in X_1^* \times \cdots \times X_m^*$ and $x = (x^{(1)},\dots,x^{(m)}) \in X_1 \times \cdots \times X_m$, then

$$x^*(x) = x^*(1)(x^{(1)}) \cdots x^*(m)(x^{(m)}).$$

Finally, for $p \in [1,+\infty]$, p^* is the conjugate exponent of p.

2. A Blei–Bohnenblust–Hille inequality

In their pioneering work on coefficients of polynomials [12], Bohnenblust and Hille showed the following inequality: for all sequences $u \in \ell_\infty(\mathbb{N}^m)$,

$$\Big(\sum_{\mathbf{j}\in\mathbb{N}^m} |u(\mathbf{j})|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le \sum_{k=1}^m \sum_{j_k\in\mathbb{N}} \Big(\sum_{\widehat{\mathbf{j}_k}\in\mathbb{N}^{m-1}} |u(\mathbf{j})|^2\Big)^{1/2}. \tag{5}$$

In modern developments, this inequality has been overpassed by variants of an inequality due to Blei which allow better constants (see e.g. [8]). Nevertheless, at our level of generality, we will need to extend (5) to sequences indexed by a subset of $\mathbb{N}^m$. This was inspired by [10, Chapter XIII].

Theorem 2.1. *Let $\Lambda \subset \mathbb{N}^m$, let $d \ge 1$ and let $C_\Lambda \ge 1$ satisfy $\psi_\Lambda(n) \le C_\Lambda n^d$ for all $n \in \mathbb{N}$. Let also $q \ge 1$. Then for all $a \in \ell_\infty(\Lambda)$ and all $\gamma \in [1,q]$,*

$$\Big(\sum_{\mathbf{j}\in\Lambda} |a(\mathbf{j})|^s\Big)^{1/s} \le C_\Lambda \sum_{k=1}^m \Big(\sum_{j_k\in\mathbb{N}} \Big(\sum_{\widehat{\mathbf{j}_k}\in\mathbb{N}^{m-1}} |a(\mathbf{j})|^q\Big)^{\gamma/q}\Big)^{1/\gamma}$$

where

$$\frac{1}{s} = \frac{1}{d\gamma} + \frac{d-1}{dq}.$$

We shall prove this result by duality. Hence, we need a kind of dual version of it.

Lemma 2.2. *Let $\Lambda \subset \mathbb{N}^m$ and let $C_\Lambda \geq 1$ with $\psi_\Lambda(n) \leq C_\Lambda n^d$ for all $n \in \mathbb{N}$. Let also $\rho \in (1, +\infty)$. Then for all $p \in [\rho, d\rho/(d-1)]$, all $u \in \ell_p(\Lambda)$ and all n-subsets $A_1, \ldots, A_m$ of $\mathbb{N}$, there is a partition $G_1, \ldots, G_m$ of $A_1 \times \cdots \times A_m$ such that, for all $k \in \{1, \ldots, m\}$,*

$$\Big(\sum_{j_k \in A_k} \Big(\sum_{\widehat{\mathbf{j}_k} \in \widehat{A_k}} |u(\mathbf{j})|^\rho \mathbf{1}_{G_k}(\mathbf{j}) \Big)^{t/\rho} \Big)^{1/t} \leq C_\Lambda \|u\|_p$$

where

$$\frac{1}{t} = \frac{d}{p} - \frac{d-1}{\rho}.$$

We prove the case $p = \rho d/(d-1)$ following an argument of [10]. The case $p = \rho$ is trivial. The general statement will follow by interpolation. However, we have not been able to use the standard interpolation theorems because the partition designed in the statement depends on the sequence u. The following lemma, first proved in [24, Lemma 5.1] is the starting point of our study. We formulate it as in [10, Lemma 21]. We recall that a finite set is called an *n-set* if its cardinality is n.

Lemma 2.3. *Let $\varphi : \mathbb{N}^m \to \mathbb{C}$ be such that there exists $D > 0$ satisfying, for all $n \in \mathbb{N}$ and all n-sets $A_1, \ldots, A_m \subset \mathbb{N}$,*

$$\sum_{\mathbf{j} \in A_1 \times \cdots \times A_m} |\varphi(\mathbf{j})| \leq Dn.$$

Then for any $n \in \mathbb{N}$ and any n-sets $A_1, \ldots, A_m \subset \mathbb{N}$, there exists a partition $G_1, \ldots, G_m$ of $A_1 \times \cdots \times A_m$ satisfying, for all $k \in \{1, \ldots, m\}$,

$$\max_{j_k \in A_k} \sum_{\widehat{\mathbf{j}_k} \in \widehat{A_k}} |\varphi(\mathbf{j})| \mathbf{1}_{G_k}(\mathbf{j}) \leq D.$$

Proof of Lemma 2.2. Let $p \in [\rho, d\rho/(d-1)]$, $u \in B_{\ell_p(\Lambda)}$, $n \in \mathbb{N}$, $A_1, \ldots, A_m$ n-subsets of $\mathbb{N}$ and $\theta \in [0, 1]$ be such that $1/p = (1-\theta)(d-1)/d\rho + \theta/\rho$. A small computation shows that $\theta = \rho/t$, so $1/t = (1-\theta)/\infty + \theta/\rho$. To simplify the notations, we set $p_0 = d\rho/(d-1)$, $p_1 = \rho$, $t_0 = \infty$ and $t_1 = \rho$. We also define $\varphi = |u|^{\rho p/p_0}$. Then, using Hölder's inequality with exponents d and $d/(d-1)$, we get

$$\begin{aligned}\sum_{\mathbf{j} \in A_1 \times \cdots \times A_m} |\varphi(\mathbf{j})| &= \sum_{\mathbf{j} \in A_1 \times \cdots \times A_m} |u(\mathbf{j})|^{p(d-1)/d} \times 1 \\ &\leq \operatorname{card}(\Lambda \cap (A_1 \times \cdots \times A_m))^{1/d} \leq C_\Lambda^{1/d} n.\end{aligned}$$

By Lemma 2.3, there exists a cover $(G_1, \ldots, G_m)$ of $A_1 \times \cdots \times A_m$ satisfying, for all $k \in \{1, \ldots, m\}$,

$$\max_{j_k \in A_k} \sum_{\widehat{\mathbf{j}_k} \in \widehat{A_k}} |u(\mathbf{j})|^{\rho p/p_0} \mathbf{1}_{G_k}(\mathbf{j}) \leq C_\Lambda^{1/d}. \tag{6}$$

We now fix $k \in \{1, \dots, m\}$. For $q_0, q_1 \in [1, +\infty]$, we shall denote by $\ell_{q_0}(\ell_{q_1})$ the space of sequences $(v(\mathbf{j}))_{\mathbf{j} \in A_1 \times \cdots \times A_m}$ such that if we set $V(j_k) = (v(\mathbf{j}))_{\widehat{\mathbf{j}_k} \in \widehat{A_k}}$ for all $j_k \in A_k$, the sequence $(\|V(j_k)\|_{q_1})_{j_k \in A_k}$ belongs to ℓ_{q_0}, endowed with the norm

$$\|v\|_{\ell_{q_0}(\ell_{q_1})} = \Big(\sum_{j_k \in A_k} \|V(j_k)\|_{q_1}^{q_0} \Big)^{1/q_0}.$$

Of course, $\ell_{q_0}(\ell_{q_1})$ depends on k and on $A_1 \times \cdots \times A_m$, but we prefer to avoid cumbersome notations. We intend to prove that $|u|\mathbf{1}_{G_k}$ belongs to $\ell_t(\ell_\rho)$ (we extend u on $A_1 \times \cdots \times A_m \backslash \Lambda$ by setting it equal to zero outside Λ). By duality we fix w in the unit ball of $\ell_{t^*}(\ell_{\rho^*})$ and we have to prove that

$$\Big| \sum_{\mathbf{j}} u(\mathbf{j}) w(\mathbf{j}) \mathbf{1}_{G_k}(\mathbf{j}) \Big| \leq C_\Lambda. \tag{7}$$

For $\Re e(z) \in [0, 1]$, we set

$$\begin{aligned} \frac{1}{p(z)} &= \frac{1-z}{p_0} + \frac{z}{p_1}, \\ u(\mathbf{j})(z) &= |u(\mathbf{j})|^{p/p(z)}, \\ w(\mathbf{j})(z) &= w(\mathbf{j}) \|W(j_k)\|_{\rho^*}^{t^*(1/t_0^* - 1/t_1^*)(\theta - z)}. \end{aligned}$$

We finally define

$$f(z) = \sum_{\mathbf{j}} u(\mathbf{j})(z) w(\mathbf{j})(z) \mathbf{1}_{G_k}(\mathbf{j}),$$

which is analytic in the open strip $0 < \Re e(z) < 1$ and bounded and continuous in the closed strip $0 \leq \Re e(z) \leq 1$ (recall that all the sums are finite). We now observe that, for all $y \in \mathbb{R}$,

$$\begin{aligned} \|w(\cdot)(iy)\|_{\ell_{t_0^*}(\ell_{\rho^*})}^{t_0^*} &= \sum_{j_k} \|W(j_k)\|_{\rho^*}^{t_0^*} \times \|W(j_k)\|_{\rho^*}^{t_0^* t^*(1/t_0^* - 1/t_1^*)\theta} \\ &= \sum_{j_k} \|W(j_k)\|_{\rho^*}^{t^*} = \|w\|_{\ell_{t^*}(\ell_{\rho^*})}^{t^*} \leq 1 \end{aligned}$$

where we have used $\theta/t_0^* - \theta/t_1^* = 1/t_0^* - 1/t^*$. Since $|u(\mathbf{j})(iy)| = |u(\mathbf{j})|^{p/p_0}$, (6) means that $\|u(\cdot)(iy)\|_{\ell_{t_0}(\ell_p)} \leq C_\Lambda^{1/(\rho d)}$. Hence duality implies that for all $y \in \mathbb{R}$,

$$|f(iy)| \leq C_\Lambda^{1/(\rho d)}.$$

In a similar way, we prove that

$$\|w(\cdot)(1+iy)\|_{\ell_{t_1^*}(\ell_{\rho^*})}^{t_1^*} = \|w(\cdot)(1+iy)\|_{\rho^*}^{\rho^*} = \|w\|_{\ell_{t^*}(\ell_{\rho^*})}^{t^*} \leq 1.$$

Moreover since $|u(\mathbf{j})(1+iy)|=|u(\mathbf{j})|^{p/\rho}$ we know that $(u(\cdot)(1+iy))$ belongs to $\ell_\rho(\Lambda)$ with $\|u(\cdot)(1+iy)\|_\rho\leq 1$. This yields, for all $y\in\mathbb{R}$,

$$|f(1+iy)|\leq 1.$$

The three-lines theorem allows us to conclude that $|f(\theta)|\leq C_\Lambda^{(1-\theta)/(d\rho)}\leq C_\Lambda$, which is exactly (7). ■

Proof of Theorem 2.1. Let $A_1,\dots,A_m$ be n-subsets of $\mathbb{N}$. Set $p=s^*$, $\rho=q^*$ and let u in the unit ball of $\ell_p(\Lambda\cap(A_1\times\cdots\times A_m))$ be such that

$$\Big(\sum_{\mathbf{j}\in\Lambda\cap A_1\times\cdots\times A_m}|a(\mathbf{j})|^s\Big)^{1/s}=\sum_{\mathbf{j}\in\Lambda}a(\mathbf{j})u(\mathbf{j}).$$

Let $G_1,\dots,G_m$ be the partition of $A_1\times\cdots\times A_m$ associated to u given by Lemma 2.2. Then

$$\sum_{\mathbf{j}\in\Lambda}a(\mathbf{j})u(\mathbf{j})=\sum_{k=1}^m\sum_{j_k\in A_k}\sum_{\widehat{\mathbf{j}_k}\in\widehat{A_k}}a(\mathbf{j})u(\mathbf{j})\mathbf{1}_{G_k}(\mathbf{j}).$$

A small computation shows that the three conditions $\gamma\in[1,q]$, $s\in[dq/(d+q-1),q]$ and $p\in[\rho,d\rho/(d-1)]$ are equivalent. Hence, by Hölder's inequality and Lemma 2.2,

$$\begin{aligned}\sum_{\mathbf{j}\in\Lambda}a(\mathbf{j})u(\mathbf{j})&\leq\sum_{k=1}^m\sum_{j_k\in A_k}\Big(\sum_{\widehat{\mathbf{j}_k}\in\widehat{A_k}}|a(\mathbf{j})|^q\Big)^{1/q}\Big(\sum_{\widehat{\mathbf{j}_k}\in\widehat{A_k}}|u(\mathbf{j})|^\rho\mathbf{1}_{G_k}(\mathbf{j})\Big)^{1/\rho}\\&\leq\sum_{k=1}^m\Big(\sum_{j_k\in A_k}\Big(\sum_{\widehat{\mathbf{j}_k}\in\widehat{A_k}}|a(\mathbf{j})|^q\Big)^{t^*/q}\Big)^{1/t^*}\Big(\sum_{j_k\in A_k}\Big(\sum_{\widehat{\mathbf{j}_k}\in\widehat{A_k}}|u(\mathbf{j})|^\rho\mathbf{1}_{G_k}(\mathbf{j})\Big)^{t/\rho}\Big)^{1/\rho}\\&\leq C_\Lambda\sum_{k=1}^m\Big(\sum_{j_k\in A_k}\Big(\sum_{\widehat{\mathbf{j}_k}\in\widehat{A_k}}|a(\mathbf{j})|^q\Big)^{t^*/q}\Big)^{1/t^*}\end{aligned}$$

where

$$\frac{1}{t}=\frac{d}{p}-\frac{d-1}{\rho}.$$

It is easy to check that $t^*=\gamma$, which concludes the proof. ■

Theorem 2.1 is optimal in a very strong sense.

Proposition 2.4. *Let $\Lambda\subset\mathbb{N}^m$, let $d\geq 1$ and let $C>0$ be such that $\psi_\Lambda(n)\geq Cn^d$ for infinitely many $n\in\mathbb{N}$. Let also $q\geq 1$ and $\gamma\in[1,q]$. The smallest $s>0$ such that there exists $D>0$ with*

$$\Big(\sum_{\mathbf{j}\in\Lambda}|a(\mathbf{j})|^s\Big)^{1/s}\leq D\sum_{k=1}^m\Big(\sum_{j_k\in\mathbb{N}}\Big(\sum_{\widehat{\mathbf{j}_k}\in\mathbb{N}^{m-1}}|a(\mathbf{j})|^q\Big)^{\gamma/q}\Big)^{1/\gamma}\tag{8}$$

for all $a \in \ell_\infty(\Lambda)$ *satisfies*

$$\frac{1}{s} \leq \frac{1}{d\gamma} + \frac{d-1}{dq}. \tag{9}$$

Proof. Let n be very large and let $A_1, \dots, A_m$ be n-subsets of $\mathbb{N}$ such that $\operatorname{card}(\Lambda \cap (A_1 \times \cdots \times A_m)) \geq Cn^d$. For $k = 1, \dots, m$, we write $A_k = \{j_1(k), \dots, j_n(k)\}$ and we denote by $u_i(k)$ the cardinality of $A_1 \times \cdots \times A_{k-1} \times \{j_i(k)\} \times A_k \times \cdots \times A_m$. Let $a \in \ell_\infty(\Lambda)$ be such that $a_j = 1$ if $j \in \Lambda \cap (A_1 \times \cdots \times A_m)$ and $a_j = 0$ otherwise, so that

$$\Big(\sum_{\mathbf{j}\in\Lambda} |a(\mathbf{j})|^s\Big)^{1/s} \geq C^{1/s} n^{d/s}. \tag{10}$$

On the other hand,

$$\sum_{k=1}^m \Big(\sum_{j_k\in\mathbb{N}} \Big(\sum_{\widehat{\mathbf{j}_k}\in\mathbb{N}^{m-1}} |a(\mathbf{j})|^q\Big)^{\gamma/q}\Big)^{1/\gamma} \leq \sum_{k=1}^m \Big(\sum_{i=1}^n u_i(k)^{\gamma/q}\Big)^{1/\gamma}.$$

Now elementary considerations show that, for any finite sequence of nonnegative real numbers $u_1, \dots, u_n$ satisfying $u_1 + \cdots + u_n \geq Cn^d$, we have

$$\Big(\sum_{i=1}^n u_i^{\gamma/q}\Big)^{1/\gamma} \leq C^{1/\gamma} n^{1/\gamma} n^{(d-1)/q}$$

(the optimal choice being $u_i = Cn^{d-1}$, recall that $\gamma/q < 1$). Therefore,

$$\sum_{k=1}^m \Big(\sum_{i=1}^n u_i(k)^{\gamma/q}\Big)^{1/\gamma} \leq mC^{1/\gamma} n^{1/\gamma} n^{(d-1)/q}. \tag{11}$$

In view of (10) and (11), (9) is a necessary condition for (8) to hold for all $a \in \ell_\infty(\Lambda)$. ■

3. Lifting summability

Theorem 1.1 has a natural statement in the context of multiple summing maps, more precisely in the context of Λ-multiple summing maps, a notion introduced independently in [7] and in [22]. Let $X_1, \dots, X_m, Y$ be Banach spaces, $T \in \mathcal{L}(X_1, \dots, X_m; Y)$, $r \in [1, +\infty)$ and $\mathbf{p} \in [1, +\infty]^m$. We say that T is Λ-$(r, \mathbf{p})$-*summing* if there exists a constant $C > 0$ such that for all sequences $x^{(j)} \subset X_j^{\mathbb{N}}$, $1 \leq j \leq m$,

$$\Big(\sum_{\mathbf{j}\in\Lambda} \|T(x(\mathbf{j}))\|^r\Big)^{1/r} \leq C w_{p_1}(x^{(1)}) \cdots w_{p_m}(x^{(m)})$$

where $T(x(\mathbf{j}))$ stands for $T(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)})$ and $\omega_p(x)$ stands for the weak ℓ^p-norm of $x \in X^{\mathbb{N}}$ defined by

$$\omega_p(x) = \sup_{\|x^*\|\leq 1} \Big(\sum_{j=1}^{+\infty} |x^*(x_j)|^p\Big)^{1/p}.$$

The least constant C for which the inequality holds is denoted by $\pi^{\Lambda}_{r,\mathbf{p}}(T)$. When $\Lambda = \mathbb{N}^m$ we recover the notion of a multiple (r, p)-summing map and we shall write simply $\pi_{r,\mathbf{p}}(T)$ instead of $\pi^{\mathbb{N}^m}_{r,\mathbf{p}}(T)$.

In [6], following the pioneering work of [13], it was studied for which $s \geq 1$ an m-linear map is multiple $(s, \mathbf{p})$-summing when the restriction of T to each X_k (fixing the other coordinates) is (r_k, p_k)-summing. We do the same now with $\mathbb{N}^m$ replaced by $\Lambda \subset \mathbb{N}^m$. The value of s will depend on the combinatorial dimension of Λ.

Definition 3.1. Let $T \in \mathcal{L}(^m X_1, \ldots, X_m; Y)$. We say that T is (r, p)*-summing in the k-th coordinate* if, for all $x = (x^{(1)}, \ldots, x^{(k-1)}, x^{(k+1)}, \ldots, x^{(m)}) \in \widehat{X_k}$, the linear map $T^{(k)}_x(y) = T(x^{(1)}, \ldots, x^{(k-1)}, y, x^{(k+1)}, \ldots, x^{(m)})$ is (r, p)-summing. In that case, we shall denote

$$\|T^{(k)}\|_{CW(r,p)} := \sup\left\{\pi_{r,p}(T^{(k)}_x(\cdot));\ \|x^{(i)}\| \leq 1,\ i \in \{1, \ldots, m\} \setminus \{k\}\right\}.$$

Theorem 3.2. *Let $T \in \mathcal{L}(^m X_1, \ldots, X_m; Y)$ with Y a cotype q space and let $\mathbf{p}, \mathbf{r} \in [1, +\infty)^m$. Assume that T is (r_k, p_k)-summing in the k-th coordinate and that there exists $\theta \leq 0$ such that $1/r_k - 1/p_k = \theta$ for all k. Set $1/\gamma = 1 + \theta - \sum_{k=1}^m 1/p_k^*$. Let finally $\Lambda \subset \mathbb{N}^m$ and $C_\Lambda \geq 1$ with $\psi_\Lambda(n) \leq C_\Lambda n^d$ for all $n \in \mathbb{N}$. If $\gamma \in (0, q)$, then T is Λ-$(s, \mathbf{p})$-summing with*

$$\frac{1}{s} = \frac{d-1}{dq} + \frac{1}{d\gamma}. \tag{12}$$

Proof. We use the results of [6]. In particular, in [6, proof of Theorem 2.1], it is shown that there exists $\kappa > 0$ depending only on $\mathbf{r}$ and on the cotype q constant of Y such that, for all sequences $x^{(j)} \subset X_j^{\mathbb{N}}$, $1 \leq j \leq m$, with $w_{p_j}(x^{(j)}) \leq 1$, and all $k = 1, \ldots, m$,

$$\left(\sum_{j_k \in \mathbb{N}} \left(\sum_{\widehat{\mathbf{j}_k} \in \mathbb{N}^{m-1}} \|T(x(\mathbf{j}))\|^q\right)^{\gamma/q}\right)^{1/\gamma} \leq \kappa \prod_{k=1}^m \|T^{(k)}\|^{1/m}_{CW(r_k,p_k)}.$$

Then we may apply Theorem 2.1 to get

$$\left(\sum_{\mathbf{j} \in \Lambda} \|T(x(\mathbf{j}))\|^s\right)^{1/s} \leq \kappa C_\Lambda \prod_{k=1}^m \|T^{(k)}\|^{1/m}_{CW(r_k,p_k)}.$$

with

$$\frac{1}{s} = \frac{d-1}{dq} + \frac{1}{d\gamma}.$$

This exactly means that T is Λ-$(s, \mathbf{p})$-summing. ∎

Remark 3.3. If $\gamma \geq q$, then we know that T is Λ-$(\gamma, \mathbf{p})$-summing since this is true for $\Lambda = \mathbb{N}^m$. Observe that if $\gamma = q$, then (12) implies that $s = q$.

To deduce Theorem 1.1 on the summation of coefficients of multilinear froms, it is convenient to use the following reformulation (see [21, Corollary 3.20] for the proof of a similar statement for multiple summability).

Lemma 3.4. *Let* $\mathbf{p} \in [1, +\infty]^m$, $\Lambda \subset \mathbb{N}^m$ *and* $s \geq 1$. *The following assertions are equivalent:*

(1) *for all* $T \in \mathcal{L}(^m Z_{p_1}, \dots, Z_{p_m}; \mathbb{C})$, $(T(e(\mathbf{j})))_{\mathbf{j}\in\Lambda}$ *belongs to* $\ell_s(\Lambda)$*;*

(2) *for all Banach spaces* $X_1, \dots, X_m$ *and all* $S \in \mathcal{L}(^m X_1, \dots, X_m; \mathbb{C})$, S *is* Λ-$(s, \mathbf{q})$-*summing where* $q_j = p_j^*$ *for all* $1 \leq j \leq m$.

We conclude by proving the following corollary, which itself easily implies Theorem 1.1.

Corollary 3.5. *Let* $\Lambda \subset \mathbb{N}^m$ *be infinite. Assume that there exist* $C_\Lambda > 0$ *and* $d \geq 1$ *such that* $\psi_\Lambda(n) \leq C_\Lambda n^d$ *for all* $n \in \mathbb{N}$. *Let also* $\mathbf{p} \in [1, +\infty]^m$ *with* $|1/\mathbf{p}| < 1$. *Then there exists a constant* $D_{\Lambda,\mathbf{p}}$ *such that, for all* m*-linear forms* $T : Z_{p_1} \times \cdots \times Z_{p_m} \to \mathbb{C}$,

$$\Big(\sum_{\mathbf{j}\in\Lambda} |T(e(\mathbf{j}))|^s\Big)^{1/s} \leq D_{\Lambda,\mathbf{p}} \|T\|$$

where $\frac{1}{s} = \frac{d+1}{2d} - \left|\frac{1}{d\mathbf{p}}\right|$ *provided* $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$, *and* $\frac{1}{s} = 1 - \left|\frac{1}{\mathbf{p}}\right|$ *provided* $\left|\frac{1}{\mathbf{p}}\right| \geq \frac{1}{2}$.

Proof. As pointed out in the introduction, we only have to consider the case $|1/\mathbf{p}| \leq 1/2$. Let $X_1, \dots, X_m$ be Banach spaces and let $S \in \mathcal{L}(^m X_1, \dots, X_m; \mathbb{C})$. Then S is (p_k^*, p_k^*)-summing with respect to the k-th coordinate. Applying Theorem 3.2 with $\gamma = 1 - |1/\mathbf{p}|$, we see that S is Λ-$(s, \mathbf{q})$-summing with

$$\frac{1}{s} = \frac{d-1}{2d} + \frac{1}{d} - \left|\frac{1}{d\mathbf{p}}\right| = \frac{d+1}{2d} - \left|\frac{1}{d\mathbf{p}}\right| \quad \text{and} \quad q_j = p_j^*.$$

An application of Lemma 3.4 gives the result. ■

Our method allows us to extend the results of Defant–Voigt, Aron–Globevnik and Zalduendo quoted in the introduction. Let us recall that we say that an m-linear map is absolutely $(r, \mathbf{p})$-summing if it is $\mathrm{Diag}(\mathbb{N}^m)$-$(r, \mathbf{p})$-summing. Using Lemma 3.4, Zalduendo's theorem may be reformulated by saying that each m-linear form in $\mathcal{L}(^m \ell_p)$ is absolutely $\left(\frac{p}{p-m}, p^*\right)$-summing for all $p > m$. Observe that any linear form $\ell_p \to \mathbb{C}$ is (p^*, p^*)-summing. We get the following abstract extension.

Corollary 3.6. *Let* $X_1, \dots, X_m, Y$ *be Banach spaces with* Y *of cotype* q. *Assume that each linear map* $X_k \to Y$ *is* (r, p)*-summing and that* $\gamma \in (0, q)$ *is defined by*

$$\frac{1}{\gamma} = 1 + \frac{1}{r} - \frac{1}{p} - \frac{m}{p^*}.$$

Then every m*-linear map in* $\mathcal{L}(X_1, \dots, X_m; Y)$ *is absolutely* (γ, p)*-summing.*

Proof. This follows from Theorem 3.2 with $d = 1$. ■

4. Optimality

4.1. General considerations

We begin with a general statement whose proof is a variant of [5, Theorem 3.1]. We recall that a *Young function* ψ is a convex increasing function on $\mathbb{R}_+$ with $\lim_{t\to\infty}\psi(t)=\infty$ and $\psi(0)=0$. The *Orlicz space* $L_\psi = L_\psi(\Omega,\mathcal{A},\mathbb{P})$ associated to ψ is defined as the space of all real valued random variables Z on $(\Omega,\mathcal{A},\mathbb{P})$ such that $\mathbb{E}(\psi(|Z|/c))<\infty$ for some $c>0$. Recall that it is a Banach space for the norm

$$\|Z\|_\psi = \inf\big\{c>0;\ \mathbb{E}\big(\psi(|Z|/c)\big)\le 1\big\}.$$

We shall use the following Young function ψ_s, with $s\ge 2$:

$$\psi_s(x)=\exp(x^s)-1.$$

Proposition 4.1. *Let $m\ge 2$, $\beta\in(0,1)$ and $s\ge 2$. There exists $C_{m,\beta,s}>0$ with the following property: for all $m\ge 1$, all Banach spaces $X_1,\dots,X_m$ of dimension n, all sequences of random variables $(\eta(\mathbf{j}))_{\mathbf{j}\in\mathbb{N}^m}$ defined on $(\Omega,\mathcal{A},\mathbb{P})$, and all $(x(\mathbf{j})^*)_{\mathbf{j}\in\mathbb{N}^m}\subset X_1^*\times\cdots\times X_m^*$, setting*

$$T(\omega,x)=\sum_{\mathbf{j}\in\mathbb{N}^m}\eta(\mathbf{j})(\omega)x(\mathbf{j})^*(x),$$

we have

$$\sup_{x\in B_{X_1}\times\cdots\times B_{X_m}}|T(\omega,x)|\le C_{m,\beta,s}n^{1/s}\sup_{x\in B_{X_1}\times\cdots\times B_{X_m}}\|T(\cdot,x)\|_{\psi_s}$$

for all ω in a set of probability greater than β.

Proof. Fix $\omega\in\Omega$ and let $x,y\in B_{X_1}\times\cdots\times B_{X_m}$. Then, writing

$$T(\omega,x)-T(\omega,y)=\sum_{k=1}^m T(\omega,y^{(1)},\dots,y^{(k-1)},x^{(k)}-y^{(k)},x^{(k+1)},\dots,x^{(m)})$$

we get

$$|T(\omega,x)-T(\omega,y)|\le m\varepsilon\|T(\omega,\cdot)\|$$

provided $\|x^{(i)}-y^{(i)}\|\le\varepsilon$ for all $i=1,\dots,m$. Setting $\varepsilon=1/(2m)$, and since each X_i has dimension n, we can find a finite ε-covering F of $B_{X_1}\times\cdots\times B_{X_m}$ with $\operatorname{card}(F)\le A_m^n$, where the constant A_m does not depend on n. Thus, for all $\omega\in\Omega$,

$$\|T(\omega,\cdot)\|\le 2\sup_{x\in F}|T(\omega,x)|.$$

Now, for $R > 0$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{x\in F}|T(\cdot,x)|>R\Big) &\le \sum_{x\in F}\mathbb{P}(|T(\cdot,x)|>R)\\ &\le \sum_{x\in F}\mathbb{P}\bigg(\psi_s\bigg(\frac{T(\cdot,x)}{\|T(\cdot,x)\|_{\psi_s}}\bigg)\ge\psi_s\bigg(\frac{R}{\|T(\cdot,x)\|_{\psi_s}}\bigg)\bigg)\\ &\le \frac{A_m^n}{\psi_s\big(\frac{R}{\|T(\cdot,x)\|_{\psi_s}}\big)}\le\frac{\exp(n\log A_m)}{\exp\big(\frac{R^s}{\|T(\cdot,x)\|_{\psi_s}^s}-1\big)}.\end{aligned}$$

It remains to choose $R^s = \lambda\|T(\cdot,x)\|_{\psi_s}^s n\log(A_m)$ for a sufficiently large λ. ■

As soon as finite subsets of $\mathcal{M}(m,n)$ satisfying certain properties can be exhibited, Proposition 4.1 can be used to prove the optimality of our estimation of $HL(\Lambda,\mathbf{p})$.

Proposition 4.2. *Let $m\ge 2$, $d\in[1,m]$, $\Gamma\subset[1,+\infty]$ and $C>0$. Assume that, for all $N\ge 1$, there exist $n\ge N$ and a set $\Lambda_n\subset\mathcal{M}(m,n)$ such that, for all $p\in\Gamma$, setting $\mathbf{p}=(p,\dots,p)$,*

$$\begin{aligned}&\psi_{\Lambda_n}(s)\le Cs^d\quad\text{for all } s\ge 1;\\ &\psi_{\Lambda_n}(n)\ge C^{-1}n^d;\\ &\sup_{x\in B_{\ell_{\mathbf{p}}}}\Big(\sum_{\mathbf{j}\in\Lambda_n}|x_{\mathbf{j}}|^2\Big)^{1/2}\le Cn^{d/2-|1/\mathbf{p}|}.\end{aligned}$$

Then there exists $\Lambda\subset\mathbb{N}^m$ with $\dim(\Lambda)=d$ and, for all $p\in\Gamma$,

$$HL(\Lambda,\mathbf{p})^{-1}\le\frac{d+1}{2d}-\left|\frac{1}{d\mathbf{p}}\right|.$$

Proof. By induction, it is easy to construct a set $\Lambda\subset\mathbb{N}^m$, a constant $C>0$ and an increasing sequence of integers (n_ℓ) such that

- $\psi_\Lambda(s)\le Cs^d$ for all $s\ge 1$;
- for all $\ell\ge 1$, there exist subsets $A_1,\dots,A_m$ of $\mathbb{N}$ with $\mathrm{card}(A_i)=n_\ell$ and

$$\begin{aligned}&\mathrm{card}(\Lambda\cap(A_1\times\cdots\times A_m))\ge C^{-1}n_\ell^d,\\ &\sup_{x\in B_{\ell_{\mathbf{p}}}}\Big(\sum_{\mathbf{j}\in\Lambda\cap(A_1\times\cdots\times A_m)}|x_{\mathbf{j}}|^2\Big)^{1/2}\le Cn_\ell^{d/2-|1/\mathbf{p}|}.\end{aligned}$$

Fix $\ell\ge 1$ and set $\Lambda_0=\Lambda\cap(A_1\times\cdots\times A_m)$. Consider a sequence $(\varepsilon(\mathbf{j}))_{\mathbf{j}\in\Lambda_0}$ of independent Bernoulli variables defined on the same probability space $(\Omega,\mathcal{A},\mathbb{P})$. Applying Proposition 4.1 with $s=2$, $\eta(\mathbf{j})=\varepsilon(\mathbf{j})$, $X_k=\ell_p(A_k)$, and $x_{\mathbf{j}}^*(x)=x_{\mathbf{j}}$, we get an m-linear form $T(x)=\sum_{j\in\Lambda_0}\delta(\mathbf{j})x_{\mathbf{j}}$ with $|\delta(\mathbf{j})|=1$ and $\|T\|\le Cn_\ell^{1/2+d/2-|1/\mathbf{p}|}$ (we recall that the ψ_2-norm of a Rademacher process is comparable to its L^2-norm [18]). Pick now $s \in \mathcal{HL}(\Lambda, \mathbf{p})$. Using this particular T, we get

$$n_\ell^{d/s} \le C n_\ell^{1/2+d/2-|1/\mathbf{p}|}.$$

leading to the result. ∎

4.2. Random sets

Proposition 4.2 shows the importance of producing subsets Λ of $\mathbb{N}^m$ with big combinatorial dimension and such that $\sup_{x\in B_{\ell_\mathbf{p}}} \sum_{\mathbf{j}\in\Lambda} |x_\mathbf{j}|^2$ is as small as possible. We will partly construct such sets using a probabilistic argument, which modifies a construction of Blei and Körner [11], who produce sets with arbitrary combinatorial dimension, to include the control of $\sup_{x\in B_{\ell_\mathbf{p}}} \sum_{\mathbf{j}\in\Lambda} |x_\mathbf{j}|^2$. This last property will be ensured using again Proposition 4.1, but not for a Bernoulli sequence.

We begin with a lemma allowing the modification of a set of precise combinatorial dimension into a subset of another combinatorial dimension, with additional properties.

Lemma 4.3. *Let $m \ge 1$, $\delta \in [1,m]$, $d \in [1,\delta)$. There exists $D > 0$ such that, for all $n \ge 2$, all $C > 0$, all $\Lambda_0 \subset \mathcal{M}(m,n)$, satisfying*

$$\psi_{\Lambda_0}(s) \le C s^\delta \quad \textit{for all } s \ge 1, \qquad \psi_{\Lambda_0}(n) \ge C^{-1} n^\delta,$$

all $\alpha, \beta, \gamma \ge 0$ and all $\mathbf{p} \in [2,+\infty)^m$ satisfying, with $q_j = p_j/2$,

$$\sup_{y\in B_{\ell_\mathbf{q}}} \sum_{j\in\Lambda_0} |y_\mathbf{j}| \le C n^\alpha, \tag{13}$$

$$\sup_{y\in B_{\ell_\mathbf{q}}} \Big(\sum_{j\in\Lambda_0} |y_\mathbf{j}|^2\Big)^{1/2} \le C n^\beta, \tag{14}$$

$$\sup_{y\in B_{\ell_\mathbf{p}}} \sum_{j\in\Lambda_0} |y_\mathbf{j}| \le C n^\gamma, \tag{15}$$

there exists $\Lambda \subset \mathcal{M}(m,n)$ with

$$\psi_\Lambda(s) \le CDs^d \quad \textit{for all } s \ge 1, \tag{16}$$

$$\psi_\Lambda(n) \ge C^{-1}D^{-1}n^d, \tag{17}$$

$$\sup_{x\in B_{\ell_\mathbf{p}}} \Big(\sum_{j\in\Lambda} |x_\mathbf{j}|^2\Big)^{1/2} \le CD\max\left(n^{(d-\delta+\alpha)/2}, \frac{n^{1/4+\beta/2}}{(\log n)^{1/4}}\right), \tag{18}$$

$$\sup_{x\in B_{\ell_\mathbf{p}}} \sum_{j\in\Lambda} |x_\mathbf{j}| \le CD\max\left(n^{d-\delta+\gamma}, \frac{n^{1/2+\alpha/2}}{(\log n)^{1/2}}\right). \tag{19}$$

Remark 4.4. It will be clear from the proof that if we only assume (13) and (14), we still get a set Λ satisfying (16)–(18).

Proof of Lemma 4.3. Let $(\xi(\mathbf{j}))_{\mathbf{j}\in\Lambda_0}$ be a sequence of independent random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ such that, for all $\mathbf{j} \in \Lambda_0$,

$$\mathbb{P}(\xi(\mathbf{j}) = 1) = n^{d-\delta}, \quad \mathbb{P}(\xi(\mathbf{j}) = 0) = 1 - n^{d-\delta}.$$

We set $\Lambda(\omega) = \{\mathbf{j} \in \Lambda_0;\ |\xi(\mathbf{j})| = 1\}$ and we show that, with high probability, $\Lambda(\omega)$ satisfies (16) and (17). We will need the following elementary result about the binomial distribution (see for instance [10, Chapter XIII]): if S follows the binomial distribution with parameters N and p and if $k \geq 2Np$, then

$$\mathbb{P}(S \geq k) \leq 2\binom{N}{k} p^k. \tag{20}$$

Let $s \in \{1, \ldots, n\}$ and let $A = A_1 \times \cdots \times A_n$ be an s-hypercube of $\mathcal{M}(m,n)$ (meaning that $\operatorname{card}(A_i) = s$ for all s). Let also $k = Bs^d$ for some $B \geq \max(2C, 1)$ whose precise value will be fixed later. Observe that $k \geq 2Cs^\delta n^{d-\delta}$. Hence (20) with $N = \operatorname{card}(\Lambda_0 \cap A)$ and $p = n^{d-\delta}$ implies that

$$\mathbb{P}\Big(\sum_{\mathbf{j}\in A\cap\Lambda_0} \xi(\mathbf{j}) \geq k\Big) \leq 2\binom{N}{k} p^k \leq \frac{2N^k}{k!} n^{(d-\delta)k} \leq \frac{2C^k s^{\delta k}}{k^k e^{-k} n^{(\delta-d)k}}.$$

We take the sum over all s-hypercubes of $\mathcal{M}(m,n)$. Since there are $\binom{n}{s}^m$ such hypercubes, we get

$$\begin{aligned}\mathbb{P}\Big(\sum_{\mathbf{j}\in A\cap\Lambda_0} \xi(\mathbf{j}) \geq k \text{ for some } s\text{-hypercube } A\Big) &\leq \binom{n}{s}^m \frac{2C^k s^{\delta k}}{k^k e^{-k} n^{(\delta-d)k}} \\ &\leq \frac{n^{ms}}{s^{ms} e^{-ms}} \times \frac{2C^k s^{\delta k}}{B^k s^{dk} e^{-k} n^{(\delta-d)k}} \leq \frac{2C^k e^{k+ms}}{B^k} \times \left(\frac{s}{n}\right)^{(\delta-d)k-ms}.\end{aligned}$$

If we choose $B \geq 2Ce^{m+2}$, then

$$\frac{2C^k e^{k+ms}}{B^k} \leq e^{-k} e^{m(s-k)} \leq e^{-s}$$

since $k \geq s$. On the other hand, if we choose $B \geq (m+1)/(\delta - d)$, then $(\delta - d)k - ms \geq s$ so that

$$\left(\frac{s}{n}\right)^{(\delta-d)k-ms} \leq \left(\frac{s}{n}\right)^s.$$

We deduce that

$$\mathbb{P}\Big(\sum_{\mathbf{j}\in A\cap\Lambda_0} \xi(\mathbf{j}) \geq k \text{ for some } s\text{-hypercube } A,\ s = 1, \ldots, n\Big) \leq \sum_{s=1}^n e^s \left(\frac{s}{n}\right)^s$$

and this last quantity goes to zero as $n \to +\infty$. Moreover, since

$$\mathbb{E}\Big(\sum_{\mathbf{j}\in\Lambda_0} \xi(\mathbf{j})\Big) \geq C^{-1} n^{\delta} n^{d-\delta} = C^{-1} n^d \quad \text{and} \quad \mathrm{Var}\Big(\sum_{\mathbf{j}\in\Lambda_0} \xi(\mathbf{j})\Big) \leq C n^d,$$

it follows from Chebyshev's inequality that, provided B is large enough, we can require $\mathbb{P}(\sum_{\mathbf{j}\in\Lambda_0} \xi(\mathbf{j}) \leq (BC)^{-1} n^d)$ to be as small as we want, independently of n. Hence, with large probability, the random set $\Lambda(\omega)$ satisfies (16) and (17).

Let us turn to the proof of (18) and (19). We first observe that

$$\sup_{x\in B_{\ell_{\mathbf{p}}}} \sum_{\mathbf{j}\in\Lambda(\omega)} |x_{\mathbf{j}}|^2 = \sup_{y\in B_{\ell_{\mathbf{q}}}} \sum_{\mathbf{j}\in\Lambda(\omega)} y_{\mathbf{j}} = \sup_{y\in B_{\ell_{\mathbf{q}}}} \sum_{\mathbf{j}\in\Lambda_0} \xi(\mathbf{j})(\omega) y_{\mathbf{j}}.$$

Set $\eta(\mathbf{j})(\omega) = \xi(\mathbf{j})(\omega) - n^{d-\delta}$ so that $(\eta(\mathbf{j}))_{\mathbf{j}\in\Lambda_0}$ is a family of independent and zero-mean random variables. Then

$$\sum_{\mathbf{j}\in\Lambda_0} \xi(\mathbf{j})(\omega) y_{\mathbf{j}} = n^{d-\delta} \sum_{\mathbf{j}\in\Lambda_0} y_{\mathbf{j}} + \sum_{\mathbf{j}\in\Lambda_0} \eta(\mathbf{j})(\omega) y_{\mathbf{j}}.$$

By Proposition 4.1 with $s = 2$ and (13) we get, with large probability,

$$\sup_{y\in B_{\ell_{\mathbf{q}}}} \Big|\sum_{\mathbf{j}\in\Lambda_0} \xi(\mathbf{j})(\omega) y_{\mathbf{j}}\Big| \leq C n^{d-\delta+\alpha} + C_m n^{1/2} \sup_{y\in B_{\ell_{\mathbf{q}}}} \|T(\cdot, y)\|_{\psi_2}$$

where $T(\omega, y) = \sum_{\mathbf{j}\in\Lambda_0} \eta(\mathbf{j})(\omega) y_{\mathbf{j}}$. Now it is well-known that

$$\|T(\cdot, y)\|_{\psi_2} = \sup_{r\geq 2} \frac{\|T(\cdot, y)\|_r}{r^{1/2}}.$$

Moreover, the L_r-norm of a sum of nonsymmetric Bernoulli variables has been estimated in [20]. With $\varepsilon = n^{d-\delta}$, Theorem 2.1 of [20] implies that

$$\|T(\cdot, y)\|_r \leq D_0 \sqrt{r}\, \|T(\cdot, y)\|_2 \times \begin{cases} \sqrt{\frac{1/\varepsilon}{\log(1/\varepsilon)}} & \text{if } r \geq \log(1/\varepsilon), \\ (1/\varepsilon)^{1/2-1/r} & \text{if } r \leq \log(1/\varepsilon). \end{cases}$$

Since

$$\|T(\cdot, y)\|_2^2 = \sum_{\mathbf{j}\in\Lambda_0} \big((1-\varepsilon)^2\varepsilon + \varepsilon^2(1-\varepsilon)\big)|y_{\mathbf{j}}|^2 \leq D_1 \varepsilon \sum_{\mathbf{j}\in\Lambda_0} |y_{\mathbf{j}}|^2,$$

it follows from the increase of the map $r \mapsto (1/\varepsilon)^{1/2-1/r}/\sqrt{r}$ on the interval $[2, \log(1/\varepsilon)]$ that

$$\|T(\cdot, y)\|_{\psi_2} \leq \frac{D_2}{\sqrt{\log(1/\varepsilon)}} \Big(\sum_{\mathbf{j}\in\Lambda_0} |y_{\mathbf{j}}|^2\Big)^{1/2} \leq \frac{D_2}{\sqrt{\log(1/\varepsilon)}} n^{\beta}$$

by (14). Hence we get, with large probability,

$$\sup_{y\in B_{\ell_{\mathbf{q}}}}\Big|\sum_{\mathbf{j}\in\Lambda_0}\xi(\mathbf{j})(\omega)y_{\mathbf{j}}\Big|\leq Cn^{d-\delta+\alpha}+\frac{CD_2}{\sqrt{\log n}}\cdot n^{1/2+\beta},$$

which gives in turn (with large probability) (18) for $\Lambda=\Lambda(\omega)$, by taking the square root. The proof of (19), again with $\Lambda=\Lambda(\omega)$ and with large probability, is completely similar and left to the reader. ■

Let us show now what happens if we apply the previous lemma, starting from $\Lambda_0=\mathcal{M}(m,n)$. From now on, we fix $p\in[1,+\infty]$ and consider $\mathbf{p}=(p,\dots,p)\in[1,+\infty]^m$.

Corollary 4.5. *Let $m\geq 2$ and let $d\in[(m+1)/2,m]$. There exists $\Lambda\subset\mathbb{N}^m$ with* $\dim(\Lambda)=d$ *such that, for all $p\in[2,+\infty]$ with $m/p\leq 1/2$,*

$$HL(\Lambda,\mathbf{p})^{-1}=\frac{d+1}{2d}-\left|\frac{1}{d\mathbf{p}}\right|.$$

Proof. We apply Lemma 4.3 with $p=4$, $\Lambda_0=\mathcal{M}(m,n)$, $\delta=m$, $\alpha=m-|2/\mathbf{p}|$, $\beta=m/2-|2/\mathbf{p}|=0$. Since $d\geq(m+1)/2$, we know that $d-\delta+\alpha\geq 1/2$. Hence we get for all $n\geq 1$ a set $\Lambda_n\subset\mathcal{M}(m,n)$ satisfying $\psi_{\Lambda_n}(s)\leq Cs^d$ for all $s\geq 1$, $\psi_{\Lambda_n}(n)\geq C^{-1}n^d$ and $\sup_{x\in B_{\ell_{\mathbf{p}}}}(\sum_{\mathbf{j}\in\Lambda_n}|x_{\mathbf{j}}|^2)^{1/2}\leq Cn^{d/2-|1/\mathbf{p}|}$. This last property is also true for $p=+\infty$ (it is a consequence of $\psi_{\Lambda_n}(n)\leq Cn^d$). Thus by interpolation, it is true for all $p\in[4,+\infty]$, in particular for all $p\in[2m,+\infty]$. We now conclude by applying Proposition 4.2. ■

To get the full range of d given by Theorem 1.1, we will need to iterate the construction and to start from sets different from $\mathcal{M}(m,n)$. Observe that in the proof of Corollary 4.5, we applied Lemma 4.3 without the assumption (15) (and thus we did not get the conclusion (19)). The full strength of Lemma 4.3 will be needed only when we iterate the construction.

4.3. *Fractional cartesian products*

Let $l\geq 1$, $1\leq k\leq l$ and let $U=\{S_1,\dots,S_m\}$ be a k-cover of $\{1,\dots,l\}$, i.e. each S_j is a subset of $\{1,\dots,l\}$ of cardinality k and their union is $\{1,\dots,l\}$. The cover is said to be *uniformly q-incident* if each $j\in\{1,\dots,l\}$ belongs to exactly q different sets in $S_1,\dots,S_m$.

If l is fixed and $U=\{S_1,\dots,S_m\}$ is a k-cover of $\{1,\dots,l\}$, we define

$$\mathbb{N}^U=\{(\Pi_{S_1}(\mathbf{j}),\dots,\Pi_{S_m}(\mathbf{j}));\ \mathbf{j}\in\mathbb{N}^l\}\subset\mathbb{N}^k\times\cdots\times\mathbb{N}^k\ (m\text{ times})$$

where $\Pi_S(\mathbf{j})=(j_k)_{k\in S}$. We may and will see $\mathbb{N}^U$ as a subset of $\mathbb{N}^m$ by identifying $\mathbb{N}^k$ with $\mathbb{N}$ through any bijection. For a single set S, $\mathbb{N}^S$ will simply denote $\mathbb{N}^{\mathrm{card}(S)}$. It is shown in [10, Theorem 14 and Corollary 16, Chapter XIII] that, provided U is uniformly incident, there exists $C > 0$ such that, for all $s \geq 1$,

$$C^{-1} s^{l/k} \leq \psi_{\mathbb{N}^U}(s) \leq C s^{l/k}.$$

In particular $\mathbb{N}^U$ has combinatorial dimension l/k.

We shall need the following variant of the left hand inequality.

Lemma 4.6. *Let $U = \{S_1, \dots, S_m\}$ be a k-cover of $\{1, \dots, l\}$. For all n large enough, the set $\Lambda_0 = \mathbb{N}^U \cap (\{1, \dots, n\}^k)^m$ satisfies $\psi_{\Lambda_0}(n^k) \geq n^l$.*

Proof. Setting, for $j = 1, \dots, m$, $A_j = \{1, \dots, n\}^k$, it suffices to observe that the map $\{1, \dots, n\}^l \ni \mathbf{j} \mapsto (\Pi_{S_1}(\mathbf{j}), \dots, \Pi_{S_m}(\mathbf{j})) \in (A_1 \times \dots \times A_m) \cap \mathbb{N}^U$ is a bijection. ■

To illustrate this part of the work, let us provide an example: $S_1 = \{1, 2\}$, $S_2 = \{3, 1\}$ and $S_3 = \{2, 3\}$ is a 2-cover of $\{1, 2, 3\}$ which is uniformly 2-incident. Let $U = \{S_1, S_2, S_3\}$. Then

$$\mathbb{N}^U = \{((i, j), (k, i), (j, k));\ (i, j, k) \in \mathbb{N}^3\} \subset \mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2.$$

If we fix a bijection $\phi : \mathbb{N}^2 \to \mathbb{N}$, $\mathbb{N}^U$ may be seen as a subset of $\mathbb{N}^3$:

$$\mathbb{N}^U = \big\{\big(\phi(i, j), \phi(k, i), \phi(j, k)\big);\ (i, j, k) \in \mathbb{N}^3\big\}.$$

With this point of view, $\mathbb{N}^U$ becomes a subset of $\mathbb{N}^3$ with combinatorial dimension $3/2$.

We will need a result allowing us to estimate $(\sum_{\mathbf{j} \in \mathbb{N}^U} |x_{\mathbf{j}}|^2)^{1/2}$ for all $x \in B_{\ell_{\mathbf{p}}}$. This is provided by the next lemma, where we do not need that the cover is made up of subsets of the same cardinality (this will be important during the proof). Because of the identification between $\mathbb{N}^S$ and $\mathbb{N}$ and to avoid the confusion with the product $x_{\mathbf{j}}$, an element $x \in \ell_\infty(\mathbb{N}^s)$ will be denoted $(x_j)_{j \in \mathbb{N}^S}$.

Lemma 4.7. *Let $l \geq 1$, and let $U = \{S_1, \dots, S_m\}$ be a cover of $\{1, \dots, l\}$ which is uniformly q-incident. For any $x = (x^{(1)}, \dots, x^{(m)}) \in \ell_\infty(\mathbb{N}^{S_1}) \times \dots \times \ell_\infty(\mathbb{N}^{S_m})$ with nonnegative entries,*

$$\sum_{\mathbf{j} \in \mathbb{N}^l} \prod_{k=1}^m x^{(k)}_{\Pi_{S_k}(\mathbf{j})} \leq \prod_{k=1}^m \Big(\sum_{j \in \mathbb{N}^{S_k}} (x^{(k)}_j)^q \Big)^{1/q}.$$

In our example, this inequality simply says that, for all $x, y, z \in \ell_\infty(\mathbb{N}^2)$ with nonnegative entries,

$$\sum_{(i,j,k) \in \mathbb{N}^3} x_{i,j} y_{k,i} z_{j,k} \leq \Big(\sum_{i,j} x^2_{i,j}\Big)^{1/2} \Big(\sum_{i,j} y^2_{i,j}\Big)^{1/2} \Big(\sum_{i,j} z^2_{i,j}\Big)^{1/2}.$$

Proof of Lemma 4.7. We use induction on l. The case $l = 1$ is easy: reordering the sets if necessary, we have $S_1 = \cdots = S_q = \{1\}$, $S_{q+1} = \cdots = S_m = \emptyset$ and the inequality

$$\sum_{j\in\mathbb{N}} \prod_{k=1}^{q} x_j^{(k)} \leq \prod_{k=1}^{q} \Big(\sum_{j\in\mathbb{N}} (x_j^{(k)})^q \Big)^{1/q}$$

is just Hölder's inequality. Assume now that the result has been shown up to $l-1$ and let us prove it for l. Reordering the sets S_k if necessary, we may assume that $l \in S_1, \dots, S_q$ (thus $l \notin S_{q+1}, \dots, S_m$). We then write

$$\sum_{\mathbf{j}\in\mathbb{N}^l} \prod_{k=1}^{m} x^{(k)}_{\Pi_{S_k}(\mathbf{j})} = \sum_{\widehat{\mathbf{j}_l}\in\mathbb{N}^{l-1}} \prod_{k=q+1}^{m} x^{(k)}_{\Pi_{S_k}(\widehat{\mathbf{j}_l})} \sum_{j_l\in\mathbb{N}} \prod_{k=1}^{q} x^{(k)}_{\Pi_{S_k}(\mathbf{j})}$$

(the notation $\Pi_{S_k}(\widehat{\mathbf{j}_l})$ is well-defined for $k = q+1, \dots, m$ because $l \notin S_k$). We then write

$$\sum_{\mathbf{j}\in\mathbb{N}^l} \prod_{k=1}^{m} x^{(k)}_{\Pi_{S_k}(\mathbf{j})} \leq \sum_{\widehat{\mathbf{j}_l}\in\mathbb{N}^{l-1}} \prod_{k=q+1}^{m} x^{(k)}_{\Pi_{S_k}(\widehat{\mathbf{j}_l})} \prod_{k=1}^{q} \Big(\sum_{j_l\in\mathbb{N}} (x^{(k)}_{\Pi_{S_k}(\mathbf{j})})^q \Big)^{1/q}$$

by Hölder's inequality. Set $T_k = S_k \setminus \{l\}$ if $k \in \{1, \dots, q\}$, $T_k = S_k$ if $k \in \{q+1, \dots, m\}$, $y_j^{(k)} = (\sum_{j_l} (x^{(k)}_{\Pi_{S_k(j,j_l)}})^q)^{1/q}$ if $k \in \{1, \dots, q\}$ and $j \in \mathbb{N}^{T_k}$, and $y_j^{(k)} = x_j^{(k)}$ if $k \in \{q+1, \dots, m\}$ and $j \in \mathbb{N}^{T_k}$. Then the previous inequality reads

$$\sum_{\mathbf{j}\in\mathbb{N}^l} \prod_{k=1}^{m} x^{(k)}_{\Pi_{S_k}(\mathbf{j})} \leq \sum_{\mathbf{j}\in\mathbb{N}^{l-1}} \prod_{k=1}^{m} y^{(k)}_{\Pi_{T_k}(\mathbf{j})}.$$

Now, $\{T_1, \dots, T_m\}$ is a uniformly q-incident cover of $\{1, \dots, l-1\}$ and the induction hypothesis leads to

$$\sum_{\mathbf{j}\in\mathbb{N}^l} \prod_{k=1}^{m} x^{(k)}_{\Pi_{S_k}(\mathbf{j})} \leq \prod_{k=1}^{m} \Big(\sum_{j\in\mathbb{N}^{T_k}} (y_j^{(k)})^q \Big)^{1/q} \leq \prod_{k=1}^{m} \Big(\sum_{j\in\mathbb{N}^{S_k}} (x_j^{(k)})^q \Big)^{1/q}. \qquad \blacksquare$$

In particular the previous lemma implies that if $U = \{S_1, \dots, S_m\}$ is a k-cover of $\{1, \dots, l\}$ which is uniformly q-incident, then the set $\Lambda = \mathbb{N}^U \subset \mathbb{N}^m$ satisfies $\dim(\Lambda) = l/k$ and $\sup_{x\in B_{\ell_p}} \sum_{\mathbf{j}\in\Lambda} |x_{\mathbf{j}}|^2 < +\infty$ for $p = 2q$. These sets are good candidates in order to apply Proposition 4.2. Therefore, it is important to exhibit this kind of sets. This can be done easily.

Lemma 4.8. *Let $m \geq 2$ and $q \in \{1, \dots, m\}$. There exists a q-cover $U = \{S_1, \dots, S_m\}$ of $\{1, \dots, m\}$ which is uniformly q-incident.*

Proof. For $1 \le k \le mq$, define $x_k = r$ with $r \in \{1,\dots,m\}$ and $k = r \pmod m$. For $j = 1,\dots,m$, let $S_j = \{x_{(j-1)q+1},\dots,x_{(j-1)q+q}\}$. Since each $r \in \{1,\dots,m\}$ appears q times in the sequence $x_1,\dots,x_{mq}$, $\{S_1,\dots,S_m\}$ is a q-cover of $\{1,\dots,m\}$ which is uniformly q-incident. ■

It could be observed that the sets S_j in the previous covering are not necessarily distinct. For instance, if m is even and $q = m/2$, then $S_{2j+1} = \{1,\dots,m/2\}$ for all j. We can also think of the sets $\mathbb{N}^U$ obtained thanks to the previous lemma as a way to generalize $\mathrm{Diag}(\mathbb{N}^m)$ to higher dimensions. The set $\mathrm{Diag}(\mathbb{N}^m)$ itself corresponds to the case $S_1 = \dots = S_m = \{1,\dots,m\}$. We can also observe that our example above corresponds to the proof of the lemma with $m = 3$ and $q = 2$.

As an immediate consequence of this construction, we get the last part of Theorem 1.2.

Corollary 4.9. *Let $m \ge 2$ and let $d \in [1,m]$ with m/d an integer. There exists $\Lambda \subset \mathbb{N}^m$ with $\dim(\Lambda) = d$ such that, for all $p \in [2,+\infty]$ with $m/p \le 1/2$,*

$$HL(\Lambda,\mathbf{p})^{-1} = \frac{d+1}{2d} - \left|\frac{1}{d\mathbf{p}}\right|.$$

Proof. Let $q = m/d$ and let U be the covering designed in Lemma 4.8. We set $\Lambda = \mathbb{N}^U \cap \mathcal{M}(m,n)$, where we have identified $\mathbb{N}^q$ and $\mathbb{N}$ in the definition of $\mathbb{N}^U$. Observe that for all $x \in B_{\ell_{\mathbf{2q}}}$, Lemma 4.7 ensures that

$$\Big(\sum_{\mathbf{j}\in\Lambda} |x_{\mathbf{j}}|^2\Big)^{1/2} \le \prod_{k=1}^{m} \Big(\sum_{j\in\mathbb{N}} |x_j^{(k)}|^{2q}\Big)^{1/2q} \le n^0 = n^{d/2-|1/(\mathbf{2q})|}.$$

By interpolation, for all $p \in [2q,+\infty]$, in particular for all $p \in [2m,+\infty]$, we get

$$\sup_{x\in B_{\ell_{\mathbf{p}}}} \Big(\sum_{\mathbf{j}\in\Lambda} |x_{\mathbf{j}}|^2\Big)^{1/2} \le n^{d/2-|1/\mathbf{p}|}.$$

We conclude by using Proposition 4.2. ■

4.4. *Mixing the arguments*

In this section, we iteratively apply the random methods of Section 4.2 in the fractional cartesian products $\mathbb{N}^U$ described in Section 4.3.

Proposition 4.10. *Let $m \ge 2$, $p_0 \in \{1,\dots,m\}$, $k \ge 1$, and $d \in [1+(m/p_0-1)/2^{k-1}, m/p_0]$. There exists $C > 0$ such that, for all $n \in \mathbb{N}$, there exists $\Lambda \subset \mathcal{M}(m,n)$ satisfying, for all $p \in [2^k p_0,+\infty]$,*

$$\psi_\Lambda(s) \le C s^d \quad \text{for all } s \ge 1, \quad \psi_\Lambda(n) \ge C^{-1} n^d,$$

$$\sup_{x\in B_{\ell_{\mathbf{p}}}} \sum_{\mathbf{j}\in\Lambda} |x_{\mathbf{j}}| \le C n^{d-|1/\mathbf{p}|}, \tag{21}$$

$$\sup_{x\in B_{\ell_{\mathbf{p}}}} \Big(\sum_{\mathbf{j}\in\Lambda} |x_{\mathbf{j}}|^2\Big)^{1/2} \le C n^{d/2-|1/\mathbf{p}|}. \tag{22}$$

Observe that the case $p_0=1, k=2$ has already appeared in the proof of Corollary 4.5.

Proof of Proposition 4.10. We fix $m\geq 2$, $p_0\in\{1,\dots,m\}$ and for all $k\geq 1$, set $d_k=1+(m/p_0-1)/2^{k-1}$, $p_k=2^kp_0$, $\alpha_k=d_k-m/p_k$, $\beta_k=d_k/2-m/p_k$, $\gamma_k=d_k-m/p_{k+1}$. We use induction on k. For the base case, we observe that $d_1=m/p_0$ and thus we only have to consider the case $d=d_1$. We consider the set $\mathbb{N}^U$ devised in the proof of Corollary 4.9 for $q=p_0$. Inequality (22) for $p=p_1$ has already been obtained in that proof, whereas (21) follows in the same vein from Lemma 4.7: for $x\in B_{\ell_{\mathbf{p}_1}}$,

$$\sum_{\mathbf{j}\in\Lambda}|x_{\mathbf{j}}|\leq\prod_{k=1}^m\Big(\sum_{j=1}^n|x_j^{(k)}|^{p_0}\Big)^{1/p_0}\leq(n^{1-p_0/p_1})^{m/p_0}\leq n^{m/p_0-m/p_1}=n^{\alpha_1}.$$

Since these inequalities are clear for $p=+\infty$, we obtain their validity for all $p\in[p_1,+\infty]$.

Assume now that the proof has been done until step k. Let $n\geq 1$ and let Λ_k be the set obtained at step k for $d=d_k$. Then the assumptions of Lemma 4.3 are satisfied with $\Lambda_0=\Lambda_k$, $p=p_{k+1}$, $\alpha=\alpha_k$, $\beta=\beta_k$ and $\gamma=\gamma_k$ (we apply the induction hypothesis to $p=p_k$ to get (13) and (14) and to $p=p_{k+1}$ to get (15)). Now, since $d_{k+1}=(d_k+1)/2$, for all $d\in[d_{k+1},d_k]$ we get

$$\begin{aligned}d-d_k+\alpha_k&\geq d_{k+1}-\frac{m}{p_k}\geq\frac12+\beta_k,\\ d-d_k+\gamma_k&\geq d_{k+1}-\frac{m}{p_{k+1}}\geq\frac{1+\alpha_k}{2}.\end{aligned}$$

Therefore, we obtain a set Λ_{k+1} (depending on d) satisfying $\psi_{\Lambda_{k+1}}(s)\leq DCs^d$ for all $s\geq 1$, $\psi_{\Lambda_{k+1}}(n)\geq D^{-1}C^{-1}n^d$ and

$$\begin{aligned}\sup_{x\in B_{\ell_{\mathbf{p}_{k+1}}}}\sum_{\mathbf{j}\in\Lambda_{k+1}}|x_{\mathbf{j}}|&\leq DCn^{d-d_k+\gamma_k}=DCn^{d-m/p_{k+1}},\\ \sup_{x\in B_{\ell_{\mathbf{p}_{k+1}}}}\Big(\sum_{\mathbf{j}\in\Lambda_{k+1}}|x_{\mathbf{j}}|^2\Big)^{1/2}&\leq DCn^{(d-d_k+\alpha_k)/2}=DCn^{d/2-m/p_{k+1}}.\end{aligned}$$

For $p\in[p_{k+1},+\infty]$ we conclude the proof by interpolation. ■

Combining Propositions 4.2 and 4.10, we immediately get the following ranges of m, d and p such that inequality (a) in Theorem 1.1 is optimal.

Corollary 4.11. *Let $m\geq 2$, $p_0\in\{1,\dots,m\}$ and $k\geq 1$ be such that $m\geq 2^{k-1}p_0$. Then for all $d\in[1+(m/p_0-1)/2^{k-1},m/p_0]$, there exists a set $\Lambda\subset\mathbb{N}^m$ such that* $\dim(\Lambda)=d$ *and, for all $p\geq 2m$,*

$$HL(\Lambda,\mathbf{p})^{-1}=\frac{d+1}{2d}-\left|\frac{1}{d\mathbf{p}}\right|. \tag{23}$$

Observe that the statement of Corollary 4.11 remains true for all $k \geq 1$ (without the assumption $m \geq 2^{k-1} p_0$) provided we assume $p \geq 2^k p_0$. Therefore, if we only want to prove the validity of (23) for p in a smaller interval than $[2m, +\infty]$, then we are allowed to choose smaller values for d.

We now prove the first two points of Theorem 1.2. Our strategy is to apply Corollary 4.11 to several values of p_0, hoping that the union of the intervals $[1 + (m/p_0 - 1)/2^{k-1}, m/p_0]$ for $p_0 \in \{1, \dots, m\}$ and $m \geq 2^{k-1} p_0$ covers a large part of $[1, m]$.

Proof of Theorem 1.1. We first apply Corollary 4.11 with $p_0 = 1$ (this means that we start the iteration with $\mathcal{M}(m, n)$). If $m = 2$ or $m = 3$, we may only choose $k = 1$. This shows that for $d \in [3/2, 2]$ if $m = 2$, for $d \in [2, 3]$ if $m = 3$, and for $p \geq 2m$, we may find a set $\Lambda \subset \mathbb{N}^m$ with $\dim(\Lambda) = d$ and $HL(\Lambda, \mathbf{p})^{-1} = \frac{d+1}{2d} - \left|\frac{1}{d\mathbf{p}}\right|$. This is exactly the content of Theorem 1.1 for $m = 2$ or $m = 3$.

Assume now that $m \geq 4$ and let $k \geq 1$ be such that $2^{k-1} \leq m < 2^k$. Then

$$1 + \frac{m-1}{2^{k-1}} \leq 1 + \frac{2(m-1)}{m} \leq 3.$$

This implies that for all $d \in [3, m]$, we may find $\Lambda \subset \mathbb{N}^m$ with $\dim(\Lambda) = d$ and $HL(\Lambda, \mathbf{p})^{-1} = \frac{d+1}{2d} - \left|\frac{1}{d\mathbf{p}}\right|$ for all $p \geq 2m$.

To allow d become smaller than 3, we will start from a fractional cartesian product $\mathbb{N}^U$ instead of $\mathcal{M}(m, n)$. We write $m = 2\lfloor m/2 \rfloor + u$, $u \in \{0, 1\}$ and we apply Corollary 4.11 with $p_0 = \lfloor m/2 \rfloor$ and $k = 2$ so that $2^{k-1} p_0 \leq m < 2^k p_0$. After a small computation, we find that the desired conclusion holds true provided d belongs to

$$I_m = \left[\frac{3}{2} + \frac{u}{2\lfloor m/2 \rfloor}, 2 + \frac{u}{\lfloor m/2 \rfloor}\right].$$

We then write $m = 2\lfloor m/4 \rfloor + v$, $v \in \{0, 1, 2, 3\}$, and we apply Corollary 4.11 for $p_0 = \lfloor m/4 \rfloor$ and $k = 3$. This time we find that the conclusion holds true for d in

$$J_m = \left[\frac{7}{4} + \frac{v}{4\lfloor m/4 \rfloor}, 4 + \frac{v}{\lfloor m/4 \rfloor}\right].$$

The proof is finished if we are able to prove that

$$I_m \cup J_m \supset \left[\frac{3}{2} + \frac{u}{2\lfloor m/2 \rfloor}, 3\right].$$

This is clear if $v = 0$ or $v = 1$, because in these cases the minimal element of J_m is less than or equal to 2. If $v = 2$, then $u = 0$ and we have to verify that

$$\frac{7}{4} + \frac{1}{2\lfloor m/4 \rfloor} \leq 2.$$

This inequality is true starting from $m=10$; unfortunately, it is false for $m=6$ where $I_6 \cup J_6 = [3/2,2] \cup [9/4,6]$. When $v=3$, we know that $u=1$ and we have to verify that

$$\frac{7}{4} + \frac{3}{4\lfloor m/4\rfloor} \leq 2 + \frac{1}{2\lfloor m/2\rfloor}.$$

This is true for $m \geq 11$, but false when $m=7$ where $I_7 = [5/3, 7/3]$ and $J_7 = [5/2, 7]$.

Thus it remains to handle the cases $m=6$ and $m=7$. We again apply Corollary 4.11, this time with $p_0 = 2$ and $k=2$. Then the result holds true for all d in K_m with $K_6 = [2,3]$ and $K_7 = [9/4, 7/2]$. The interval K_m fills the gap between I_m and J_m for $m=6$ and $m=7$. ■

We conclude this section by a result which may be seen as an extension of the Kahane–Salem–Zygmund strategy to produce multilinear forms with many unimodular coefficients and small norm. It follows immediately from the arguments given throughout this work.

Corollary 4.12. *Let $m \geq 2$, $p_0 \in \{1,\dots,m\}$, $k \geq 1$, $d \in [1+(m/p_0-1)/2^k, m/p_0]$ and let $p \geq 2^k p_0$. There exists $C>0$ such that, for all $n \geq 1$, there exist $\Lambda \subset \mathcal{M}(m,n)$ with* card$(\Lambda) \geq C^{-1}n^d$ *and $T = \sum_{\mathbf{j}\in\Lambda} \varepsilon(\mathbf{j})x_{\mathbf{j}}$ with $|\varepsilon(\mathbf{j})|=1$ satisfying*

$$\|T\|_{\mathcal{L}(^m\ell_p)} \leq C n^{1/2+d/2-|1/\mathbf{p}|}.$$

5. Steiner systems and different norms of multilinear forms

5.1. Steiner systems

In [16], the authors produce multilinear forms (more precisely, polynomials) with around n^{m-1} unimodular coefficients and small norm. More precisely, they show that there exists $\Lambda \subset \mathcal{M}(m,n)$ with card$(\Lambda) \geq C^{-1}n^{m-1}$ and $T(x) = \sum_{\mathbf{j}\in\Lambda} \varepsilon(\mathbf{j})x_{\mathbf{j}}$ with $|\varepsilon(\mathbf{j})|=1$ an m-linear form on ℓ_p satisfying

$$\|T\|_{\mathcal{L}(^m\ell_p)} \leq C(\log n)^{3/p} n^{m/2-m/p} \quad \text{for all } p \geq 2. \tag{24}$$

Their subset Λ satisfies a very special combinatorial property: Λ is a partial $(m-1, m, n)$ Steiner system, meaning that Λ is a collection of subsets of $\{1,\dots,n\}$ of size m such that every subset containing $m-1$ elements is contained in at most one element of Λ. They use this combinatorial property to produce (with a random method) such a multilinear form with spectrum in Λ.

Our results improve when $p \geq 4$ that of [16] by deleting a logarithmic factor. Indeed, provided $m \geq 3$ (to be sure that $m-1 \geq (m+1)/2$ but the case $m=2$ is easy by taking for Λ the diagonal of $\mathcal{M}(2,n)$), Corollary 4.12 gives us a set $\Gamma \subset \mathcal{M}(m,n)$ with card$(\Lambda) \geq C^{-1}n^{m-1}$ and an m-linear form $T(x) = \sum_{\mathbf{j}\in\Lambda} \varepsilon(\mathbf{j})x_{\mathbf{j}}$ on ℓ_p, with $|\varepsilon(\mathbf{j})|=1$ for all $\mathbf{j}\in\Lambda$, satisfying

$$\|T\|_{\mathcal{L}(^m\ell_p)} \leq C n^{m/2-m/p}. \tag{25}$$

Unfortunately, our method does not give a bound similar to (24) when $p \in [2,4)$. In particular, we do not recover the very interesting fact that for all $\varepsilon > 0$, there exists $C > 0$ such that, for all n, there exist $\Lambda \subset \mathcal{M}(m,n)$ with $\mathrm{card}(\Lambda) \geq C^{-1}n^{m-1}$, $T = \sum_{\mathbf{j}\in\Lambda} \varepsilon(\mathbf{j})x_{\mathbf{j}}$ with $|\varepsilon(\mathbf{j})| = 1$ and $\|T\|_{\mathcal{L}(^m\ell_2)} \leq Cn^{\varepsilon}$. We shall explain below why we think we cannot obtain this by using our arguments.

This leads us to the following problem.

Problem 5.1. *Let $m \geq 2$ and $d \in [1,m]$. Define*

$$\Gamma_{\mathrm{mult}}(m,d) = \Big\{p \geq 1; \textit{ for all } \varepsilon > 0, \textit{ there exists } C > 0 \textit{ such that, for all } n \in \mathbb{N}, \textit{ there exists } \Lambda \subset \mathcal{M}(m,n) \textit{ with } \mathrm{card}(\Lambda) \geq C^{-1}n^d \textit{ and } T = \sum_{\mathbf{j}\in\Lambda} \varepsilon(\mathbf{j})x_{\mathbf{j}} \textit{ with } |\varepsilon(\mathbf{j})| = 1 \textit{ satisfying } \|T\|_{\mathcal{L}(^m\ell_p)} \leq Cn^{\varepsilon}\Big\}$$

and $\gamma_{\mathrm{mult}}(m,d) = \sup \Gamma_{\mathrm{mult}}(m,d)$. What is the value of $\gamma_{\mathrm{mult}}(m,d)$?

The work of [16] shows that $\gamma_{\mathrm{mult}}(m,m-1) \geq 2$ by taking for Λ a partial $(m-1,m,n)$ Steiner system whereas $\gamma_{\mathrm{mult}}(m,1) \geq m$ by choosing for Λ the diagonal of $\mathcal{M}(m,n)$.

Problem 5.1 is related to the following one, which is reminiscent of Lemma 4.7.

Problem 5.2. *Let $m \geq 2$ and $d \in [1,m]$. Define $\gamma_{\mathrm{prod}}(m,d)$ as the supremum of those $p \geq 1$ for which there exists $\Lambda \subset \mathbb{N}^m$ with $\dim(\Lambda) = d$ and*

$$\sup_{x\in B_{\ell_p}} \sum_{\mathbf{j}\in\Lambda} |x_{\mathbf{j}}| < +\infty. \tag{26}$$

What is the value of $\gamma_{\mathrm{prod}}(m,d)$?

It is clear that $\gamma_{\mathrm{prod}}(m,d) \leq \gamma_{\mathrm{mult}}(m,d)$. Moreover, when $m/d = q$ is an integer, taking for U a uniformly q-incident q-cover of $\{1,\dots,m\}$ and setting $\Lambda = \mathbb{N}^U$, Lemma 4.7 tells us that $\gamma_{\mathrm{prod}}(m,d) \geq m/d$.

Although we do not know the answer to Problem 5.1, we can at least give upper bounds and lower bounds for $\gamma_{\mathrm{mult}}(m,d)$ which allow us to settle certain cases.

Proposition 5.3. *Let $m \geq 2$ and $d \in [1,m]$. Then*

$$\gamma_{\mathrm{mult}}(m,d) \leq \min\left(m - \lceil d\rceil + 1, \frac{2m}{d+1}\right).$$

Moreover, if d is an integer and $m = k(d+1)$ for some $k \in \mathbb{N}$, then $\gamma_{\mathrm{mult}}(m,d) = \frac{2m}{d+1}$.

Corollary 5.4. *Let $m \geq 2$.*

- *If $d \in (m-1,m]$, then $\gamma_{\mathrm{mult}}(m,d) = 1$.*
- *If $d \in (m-2,m-1]$, then $\gamma_{\mathrm{mult}}(m,d) = 2$.*

This corollary explains why our method does not seem clever enough to produce multilinear forms with norm less than n^ε on ℓ_p, with at least n^d unimodular coefficients and with an optimal relation between d, m and p. In particular, the map $d \mapsto \gamma_{\text{mult}}(m,d)$ is not continuous at $m-1$, a property which seems difficult to capture with a probabilistic construction.

We shall give a proof of Proposition 5.3 in the next subsection.

5.2. *The sup-norm vs norm of the coefficients of multilinear forms*

The proof of Proposition 5.3 is linked to the following general problem: given m, n, p, r, what is the best constant $A_{p,r}^m(n)$ such that, for all m-linear forms $T(x) = \sum_{\mathbf{j}\in\mathcal{M}(m,n)} T(e(\mathbf{j}))x_{\mathbf{j}}$ on ℓ_p, we have $|T|_r \leq A_{p,r}^m(n)\|T\|_{\mathcal{L}(^m\ell_p)}$ where

$$|T|_r := \Big(\sum_{\mathbf{j}\in\mathcal{M}(m,n)} |T(e(\mathbf{j}))|^r\Big)^{1/r}?$$

In particular, we are interested in the growth with respect to n of $A_{p,r}^m(n)$, m, p, r being kept fixed. The Hardy–Littlewood inequality gives conditions for $A_{p,r}^m(n)$ to be bounded. So far, the best known estimates of $A_{p,r}^m(n)$ come from [15] (see also [2]).

For two sequences (a_n) and (b_n) of real numbers, we will write $a_n \ll b_n$ if there exists $C>0$ such that $a_n \leq Cb_n$ for every n, and $a_n \sim_n b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$.

Theorem 5.5 ([15, Theorem 2.1]). *Let $m \geq 2$ and $p, r \geq 1$. Then*

(A) $A_{p,r}^m(n) \sim 1$ *when* $\left(\frac12 \leq \frac1r \leq \frac{m+1}{2m} - \frac1p\right)$ *or* $\left(\frac1r \leq \frac12 \text{ and } \frac mp \leq 1 - \frac1r\right)$.

(B) $A_{p,r}^m(n) \sim n^{m/p+1/r-1}$ *when* $\frac{1}{2m} \leq \frac1p \leq \frac1m$ *and* $1 - \frac mp \leq \frac1r \leq \frac12$.

(C) $A_{p,r}^m(n) \sim n^{m(1/p+1/r-1/2)-\frac12}$ *when* $\left(\frac{m+1}{2m} \leq \frac1r \text{ and } \frac1p \leq \frac12\right)$ *or* $\left(\frac12 \leq \frac1r \leq \frac{m+1}{2m} \leq \frac1r + \frac1p \text{ and } \frac1p \leq \frac12\right)$.

(D) $A_{p,r}^m(n) \sim n^{m/r+1/p-1}$ *when* $\frac12 \leq \frac1p$ *and* $1 - \frac1p \leq \frac1r$.

(E) $A_{p,r}^m(n) \ll n^{(m-1)/r}$ *when* $\frac12 \leq \frac1p \leq 1 - \frac1r$.

(F) $A_{p,r}^m(n) \sim n^{1/r}$ *when* $\frac{m-1}{p} \leq 1 - \frac1r$ *and* $\frac1m \leq \frac1p \leq \frac{1}{m-1}$.

Moreover, the power of n in (E) *cannot be improved.*

Observe in particular that the situation is completely clear for $m=2$ (in that case, the region (F) does not appear). We can now control $A_{p,r}^m(n)$ in some other regions.

Proposition 5.6. *Let $k \geq 1$ and $m \geq k+1$. Define*

$$F_k(m) = \left\{(p,r) \in [1,+\infty]^2;\ \frac{1}{m-k+1} \leq \frac1p \leq \frac{1}{m-k} \text{ and } \frac{m-k}{p} \leq 1 - \frac1r\right\}.$$

Then $A_{p,r}^m(n) \ll n^{k/r}$ for all $(p,r) \in F_k(m)$.

Proof. The result is already known for $k = 1$ ($F_1(m)$ is nothing other than the region (F) in Theorem 5.5). The other cases are proved by induction. Indeed, let us assume that the result has been proved until k for all $m \geq k+1$ and for all $(p,r) \in F_k(m)$. Let us prove it for $k+1$. Thus let $m \geq k+2$ and $(p,r) \in F_{k+1}(m)$. Let $T \in \mathcal{L}(^m\ell_p)$ and for $i = 1,\dots,n$, let $T_i \in \mathcal{L}(^{m-1}\ell_p)$ be defined by $T_i(x^{(2)},\dots,x^{(m)}) = T(e_i, x^{(2)},\dots,x^{(m)})$. Then

$$\begin{aligned}\sum_{\mathbf{j}\in\mathcal{M}(m,n)} |T(e(\mathbf{j}))|^r &= \sum_{i=1}^{n} \sum_{\widehat{\mathbf{j}_1}\in\mathcal{M}(m-1,n)} |T_i(e(\widehat{\mathbf{j}_1}))|^r \\ &\leq C\sum_{i=1}^{n} n^k \|T_i\|_{\mathcal{L}(^{m-1}\ell_p)} \leq Cn^{k+1}\|T\|_{\mathcal{L}(^m\ell_p)}\end{aligned}$$

where we use the induction hypothesis for the $(m-1)$-linear forms T_i, since $(p,r) \in F_k(m-1)$. ∎

We are now ready for the proof of Proposition 5.3.

Proof of Proposition 5.3. Let $m \geq 2$ and $d \in [1,m]$. We first prove that $\gamma_{\mathrm{mult}}(m,d) \leq m - \lceil d\rceil + 1$.

Let $p < \gamma_{\mathrm{mult}}(m,d)$ and assume that $p > m - \lceil d\rceil + 1$, so that $\frac{1}{p} < \frac{1}{m-\lceil d\rceil+1}$. If $1/p \geq 1/m$, there exists $k \in \{1,\dots,\lceil d\rceil - 1\}$ such that $\frac{1}{m-k+1} \leq \frac{1}{p} < \frac{1}{m-k}$. We then select a very large value of r so that $\frac{m-k}{p} \leq 1 - \frac{1}{r}$. Now, let $\varepsilon > 0$ be such that $k + \varepsilon r < d$. For any large integer n, one may find $\Lambda \subset \mathcal{M}(m,n)$ with $\mathrm{card}(\Lambda) \geq C^{-1}n^d$ and $T(x) = \sum_{\mathbf{j}\in\Lambda} \varepsilon(\mathbf{j})x_{\mathbf{j}}$ with $|\varepsilon(\mathbf{j})| = 1$ and $\|T\|_{\mathcal{L}(^m\ell_p)} \leq Cn^{\varepsilon}$. Furthermore, $|T|_r \geq C^{-1/r}n^{d/r}$. Applying Proposition 5.6, we obtain $n^{d/r} \ll n^{k/r+\varepsilon}$, a contradiction.

If we now assume $1/p < 1/m$, then we select $r \geq 2$ such that $m/p < 1 - 1/r$ and we get a similar contradiction by using case (A) of Theorem 5.5.

Let us now show that $\gamma_{\mathrm{mult}}(m,d) \leq \frac{2m}{d+1}$. Let $p < \gamma_{\mathrm{mult}}(m,d)$. For all $\varepsilon > 0$ and all $n \in \mathbb{N}$, we consider $\Lambda \subset \mathcal{M}(m,n)$ with $\mathrm{card}(\Lambda) \geq C^{-1}n^d$ and $T = \sum_{\mathbf{j}\in\Lambda}\varepsilon(\mathbf{j})x_{\mathbf{j}}$, $|\varepsilon(\mathbf{j})| = 1$, satisfying $\|T\|_{\mathcal{L}(^m\ell_p)} \leq Cn^{\varepsilon}$. Using Hölder's inequality, we find that

$$\|T\|_{\mathcal{L}(^m\ell_{2m})} \leq n^{m(\frac{1}{p}-\frac{1}{2m})}\|T\|_{\mathcal{L}(^m\ell_p)} \leq Cn^{m/p-1/2+\varepsilon}.$$

Now by Theorem 1.1 or by case (A) of Theorem 5.5, for $r^{-1} = 1/2 - \varepsilon$,

$$\Big(\sum_{\mathbf{j}\in\mathbb{N}^m} |T(e(\mathbf{j}))|^r\Big)^{1/r} \leq C\|T\|_{\mathcal{L}(^m\ell_{2m})},$$

which implies

$$n^{d(1/2-\varepsilon)} \leq Cn^{m/p-1/2+\varepsilon}.$$

Letting $n \to +\infty$ and then $\varepsilon \to 0$, we get the inequality $p \leq 2m/(d+1)$.

Finally, assume that d is an integer and $m = k(d+1)$. We modify the proof of [16, Theorem 2.5] and we refer to that paper for the details we do not state here.

Let $\Lambda_0 \subset \mathcal{M}(d+1,n)$ be a partial $(d,d+1,n)$ Steiner system with at least n^d coefficients. We define the random $m = k(d+1)$-linear form $T(\omega,x)$ on ℓ_{2k} by

$$T(\omega,x) = \sum_{\mathbf{j}\in\Lambda_0} \varepsilon(\mathbf{j}) x_{j_1}^{(1)} \cdots x_{j_{d+1}}^{(d+1)} x_{j_1}^{(d+2)} \cdots x_{j_{d+1}}^{(2d+2)} \cdots x_{j_{d+1}}^{(kd+k)}$$

where $(\varepsilon(\mathbf{j}))_{\mathbf{j}\in\Lambda_0}$ is a sequence of independent Bernoulli variables. The work done in [16] implies that it is sufficient to show that

$$\|T(\cdot,x) - T(\cdot,y)\|_2 \leq C \sup_{l=1,\dots,m} \|x^{(l)} - y^{(l)}\|_\infty$$

for any $x, y \in B_{\ell_{2\mathbf{k}}}$. Thus, let us fix $x, y \in B_{\ell_{2\mathbf{k}}}$ and compute

$$\begin{aligned}\|T(\cdot,x) - T(\cdot,y)\|_2 &= \left(\sum_{\mathbf{j}\in\Lambda_0} \left|\sum_{u=1}^{m} x_{j_1}^{(1)} \cdots (x_j^{(u)} - y_j^{(u)}) \cdots y_{j_{d+1}}^{(m)}\right|^2\right)^{1/2} \\ &\leq \sum_{u=1}^{m} \left(\sum_{\mathbf{j}\in\Lambda_0} |x_{j_1}^{(1)} \cdots (x_j^{(u)} - y_j^{(u)}) \cdots y_{j_{d+1}}^{(m)}|^2\right)^{1/2}\end{aligned}$$

(in these sums, if $l \in \{0,\dots,k-1\}$ is the single integer such that u belongs to $\{l(d+1)+1,\dots,l(d+1)+d+1\}$, then j is linked to u by $u = l(d+1)+j$). We shall prove that, for all $u \in \{1,\dots,m\}$,

$$\left(\sum_{\mathbf{j}\in\Lambda_0} |x_{j_1}^{(1)} \cdots (x_j^{(u)} - y_j^{(u)}) \cdots y_{j_{d+1}}^{(m)}|^2\right)^{1/2} \leq \|x^{(u)} - y^{(u)}\|_\infty.$$

To simplify the notations, assume that $u = 1$. Using Hölder's inequality, we have

$$\begin{aligned}&\left(\sum_{\mathbf{j}\in\Lambda_0} |(x_{j_1}^{(1)} - y_{j_1}^{(1)}) y_{j_2}^{(2)} \cdots y_{j_{d+1}}^{(m)}|^2\right)^{1/2} \\ &\qquad \leq \left(\sum_{\mathbf{j}\in\Lambda_0} |x_{j_1}^{(1)} - y_{j_1}^{(1)}|^{2k} |y_{j_2}^{(2)}|^{2k} \cdots |y_{j_{d+1}}^{(d+1)}|^{2k}\right)^{1/2k} \\ &\qquad\quad \times \prod_{l=1}^{k-1} \left(\sum_{\mathbf{j}\in\Lambda_0} |y_{j_1}^{(l(d+1)+1)} \cdots y_{j_{d+1}}^{((l+1)(d+1))}|^{2k}\right)^{1/(2k)}.\end{aligned}$$

Now, it is clear that, for all $l = 1,\dots,k-1$,

$$\begin{aligned}&\left(\sum_{\mathbf{j}\in\Lambda_0} |y_{j_1}^{(l(d+1)+1)} \cdots y_{j_{d+1}}^{((l+1)(d+1))}|^{2k}\right)^{1/(2k)} \\ &\qquad \leq \left(\sum_{\mathbf{j}\in\mathbb{N}^{d+1}} |y_{j_1}^{(l(d+1)+1)} \cdots y_{j_{d+1}}^{((l+1)(d+1))}|^{2k}\right)^{1/(2k)} \leq 1,\end{aligned}$$

whereas it is explained in [16] why, because Λ_0 is a partial Steiner system and $y^{(2)}, \dots, y^{(d+1)}$ are in ℓ_{2k},

$$\Big(\sum_{\mathbf{j}\in\Lambda_0} |x_{j_1}^{(1)} - y_{j_1}^{(1)}|^{2k} |y_{j_2}^{(2)}|^{2k} \cdots |y_{j_{d+1}}^{(d+1)}|^{2k}\Big)^{1/2k} \leq \|x^{(1)} - y^{(1)}\|_\infty. \qquad \blacksquare$$

Question 5.7. *Is it true that $\gamma_{\mathrm{mult}}(m,d)$ belongs to $\mathbb{N}$ for all m,d?*

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