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The Ramsey property for Banach spaces and Choquet simplices

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Abstract. We show that the Gurarij space $\mathbb{G}$ has extremely amenable automorphism group. This answers a question of Melleray and Tsankov. We also compute the universal minimal flow of the automorphism group of the Poulsen simplex $\mathbb{P}$ and we prove that it consists of the canonical action on $\mathbb{P}$ itself. This answers a question of Conley and Törnquist. We show that the pointwise stabilizer of any closed proper face of $\mathbb{P}$ is extremely amenable. Similarly, the pointwise stabilizer of any closed proper biface of the unit ball of the dual of the Gurarij space (the Lusky simplex) is extremely amenable.

These results are obtained via several Kechris–Pestov–Todorcevic correspondences, by establishing the approximate Ramsey property for several classes of finite-dimensional Banach spaces and function systems and their versions with distinguished contractions. This is the first direct application of the Kechris–Pestov–Todorcevic correspondence in the setting of metric structures. The fundamental combinatorial principle that underpins the proofs is the Dual Ramsey Theorem of Graham and Rothschild.

Keywords. Gurarij space, Poulsen simplex, extreme amenability, Ramsey property, Banach space, Choquet simplex, function systems, oscillation stability, Dual Ramsey Theorem

1. Introduction

Given a topological group G, a compact G-space or G-flow is a compact Hausdorff space X endowed with a continuous action of G. Such a G-flow X is called *minimal* when

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every orbit is dense. There is a natural notion of morphism between G-flows, given by a G-equivariant continuous map (factor). A minimal G-flow is *universal* if it factors onto any minimal G-flow. It is a classical fact that any topological group G admits a unique (up to isomorphism of G-flows) universal minimal flow, usually denoted by $M(G)$ [16, 29]. For any locally compact non-compact Polish group G, the universal minimal G-flow is non-metrizable. At the opposite end, non-locally-compact topological groups often have metrizable universal minimal flows, or even reduced to a single point. A topological group for which $M(G)$ is a singleton is called *extremely amenable*. (Amenability of G is equivalent to the assertion that every compact G-space has an invariant Borel measure. Thus any extremely amenable group is in particular amenable.)

The universal minimal flow has been explicitly computed for a number of topological groups, typically given as automorphism groups of naturally arising mathematical structures. Examples of extremely amenable Polish groups include the group of order automorphisms of $\mathbb{Q}$ [49], the group of unitary operators on the separable infinite-dimensional Hilbert space [27], the automorphism group of the hyperfinite II_1 factor and of infinite type UHF C^*-algebras [14, 18], or the isometry group of the Urysohn space [50]. Examples of non-trivial metrizable universal minimal flows include the universal minimal flow of the group of orientation preserving homeomorphisms of the circle, which is equivariantly homeomorphic to the circle itself [49], the universal minimal flow of the group S_∞ of permutations of $\mathbb{N}$, which can be identified with the space of linear orders on $\mathbb{N}$ [20], and the universal minimal flow of the homeomorphism group $\mathrm{Homeo}(2^{\mathbb{N}})$ of the Cantor set $2^{\mathbb{N}}$, which can be seen as the canonical action of $\mathrm{Homeo}(2^{\mathbb{N}})$ on the space of maximal chains of closed subsets of $2^{\mathbb{N}}$ [21, 31, 54].

There are essentially two known ways to establish extreme amenability of a given topological group. The first method involves the phenomenon of *concentration of measure*, and can be applied to topological groups that admit an increasing sequence of compact subgroups with a dense union [27, 51, Chapter 4]. The second method applies to automorphism groups of discrete ultrahomogeneous structures or, more generally, approximately ultrahomogeneous metric structures [51, Chapter 6]. A metric structure is *approximately ultrahomogeneous* if any partial isomorphism between finitely generated substructures is the pointwise limit of maps that are restrictions of automorphisms. It is worth noting that any Polish group can be realized as the automorphism group of an approximately ultrahomogeneous metric structure [43, Theorem 6]. For the automorphism group $\mathrm{Aut}(M)$ of an approximately ultrahomogeneous structure M, extreme amenability is equivalent to the approximate Ramsey property of the class of finitely generated substructures of M. This criterion is known as the *Kechris–Pestov–Todorcevic* (KPT) *correspondence*, first established in [31] for discrete structures, and recently generalized to the metric setting in [45]. The discrete KPT correspondence has been extensively used in the last decade. In this paper the KPT correspondence is directly used for the first time to obtain new natural extreme amenability results.

In all the known examples of computations of metrizable universal minimal flows, the argument hinges on extreme amenability of a suitable subgroup and the following result due to Nguyen Van Thé [48] based on previous work of Pestov [49]. Suppose that G is a topological group with an extremely amenable closed subgroup H. If the completion X

of the homogeneous space G/H endowed with the quotient of the right uniformity on G is a minimal compact G-space, then X is the universal minimal flow of G. It was recently shown in [10, 44] that whenever the universal minimal flow of G is metrizable, it can be realized as the completion of G/H for a suitable closed subgroup H of G.

In this paper we compute the universal minimal flows of the automorphism groups of structures coming from functional analysis and Choquet theory: the Gurarij space $\mathbb{G}$ and the Poulsen simplex $\mathbb{P}$. Recall that the Gurarij space is the unique separable *approximately ultrahomogeneous* Banach space that contains ℓ_n^∞ for every $n \in \mathbb{N}$ [39], while $\mathbb{P}$ is the unique non-trivial metrizable Choquet simplex with dense extreme boundary [36]. The group $\mathrm{Aut}(\mathbb{G})$ of surjective linear isometries of the Gurarij space is shown to be extremely amenable by establishing the approximate Ramsey property of the class of finite-dimensional Banach spaces. This answers a question of Melleray and Tsankov from [45]. Similarly, the stabilizer $\mathrm{Aut}_p(\mathbb{P})$ of an extreme point p of $\mathbb{P}$ is proven to be extremely amenable by establishing the approximate Ramsey property of the class of Choquet simplices with a distinguished point. It is then deduced from this that the universal minimal flow of $\mathrm{Aut}(\mathbb{P})$ is $\mathbb{P}$ itself, endowed with the canonical action of $\mathrm{Aut}(\mathbb{P})$. This answers Question 4.4 from [13]. More generally, we prove that for any closed face F of $\mathbb{P}$, the pointwise stabilizer $\mathrm{Aut}_F(\mathbb{P})$ is extremely amenable. The analogous result holds in the Banach space setting as well. A *Lazar simplex* is a compact absolutely convex set that arises as the unit ball of the dual of a Lindenstrauss space. The *Lusky simplex* $\mathbb{L}$ is the Lazar simplex that arises in this fashion from the Gurarij space. The group $\mathrm{Aut}(\mathbb{G})$ can be identified with the group $\mathrm{Aut}(\mathbb{L})$ of symmetric affine homeomorphisms of $\mathbb{L}$. It is proven in [37, Theorem 1.2] that $\mathbb{L}$ plays the same role among Lazar simplices as the Poulsen simplex plays in the class of Choquet simplices, where closed faces are replaced with closed bifaces. We prove that, for any closed proper biface H of $\mathbb{L}$, the corresponding pointwise stabilizer $\mathrm{Aut}_H(\mathbb{L})$ is extremely amenable. In the particular case when H is the trivial biface, this recovers the extreme amenability of $\mathrm{Aut}(\mathbb{G})$.

Recall that a *function system* is a closed subspace V of the space $C(T)$ of continuous $\mathbb{C}$-valued functions on some compact Hausdorff space T containing the function constantly equal to 1 and such that if $f \in V$ then the function f^* defined by $f^*(t) = \overline{f(t)}$ also belongs to V. In particular, when K is a compact convex set, the space $A(K)$ of continuous complex-valued affine functions on K is a function system, and in fact any function system $V \subseteq C(T)$ arises in this way from a suitable compact convex set K. Precisely, K is the compact convex set of *states* of V, that is, the contractive functionals on V that are *unital*, i.e. map the unit of $C(T)$ to 1 [1, Theorem II.1.8]. Furthermore, the map $K \mapsto A(K)$ is a contravariant isomorphism of categories from the category of compact convex sets and continuous affine maps to the category of function systems and unital linear contractions (*Kadison correspondence*). A metrizable compact convex set K is a simplex if and only if $A(K)$ is a separable Lindenstrauss space, which means that the identity map of $A(K)$ is the pointwise limit of a sequence of unital completely contractive maps that factor through finite-dimensional (abelian) C^*-algebras. The function system $A(\mathbb{P})$ corresponding to the Poulsen simplex is the unique separable approximately ultrahomogeneous function system that contains unital copies of ℓ_n^∞ for $n \in \mathbb{N}$ [37, Theorem 1.1]. The automorphism group $\mathrm{Aut}(A(\mathbb{P}))$ can be identified with the group of affine homeomorphisms of $\mathbb{P}$. The Poulsen simplex $\mathbb{P}$ is then equivariantly homeomorphic to the state space of $A(\mathbb{P})$.

The main tool to establish the results mentioned above will be the Dual Ramsey Theorem of Graham and Rothschild [26]. This is a powerful pigeonhole principle known to imply many other results, such as the Hales–Jewett theorem and the Ramsey theorem. It can be seen to be equivalent to a factorization result for colorings of Boolean matrices, which implies the celebrated Graham–Leeb–Rothschild theorem on Grassmannians over a finite field [25]. In fact, it is shown in [5, 6] that this is again a particular case of a factorization result for colorings of matrices over a finite field, stating that the coloring of matrices only depends on the invertible matrix needed to transform a given matrix into one in reduced column echelon form. In [6] we provide factorization theorems for colorings of matrices and Grassmannians over the real or complex numbers, and we prove in particular that colorings of matrices depend only on the canonical norm that a given matrix determines, while colorings of Grassmannians are determined by the Banach–Mazur type of the given subspace.

The paper is organized as follows. We start in Subsection 2.1 by recalling some basic concepts such as extreme amenability. In Subsection 2.2 we recall and introduce different versions of ultrahomogeneity and Ramsey properties for Banach spaces, and we prove a version of the KPT correspondence in this setting (Theorem 2.12). In Subsection 2.3 we prove the approximate Ramsey property (ARP) of the class $\{\ell_\infty^n\}_n$. This has as a consequence the extreme amenability of the group of isometries of the Gurarij space. In Subsection 2.4 we prove the (ARP) of the class of polyhedral finite-dimensional spaces, and the class of all finite-dimensional Banach spaces. Using this, in Subsection 2.5 we give a direct proof of the (ARP) for the class of finite metric spaces. This provides a new proof of extreme amenability of the isometry group of the Urysohn space [50]. Subsection 2.6 studies closed bifaces of Lusky simplices. We prove that the group stabilizers of closed proper bifaces of the Lazar simplex are extremely amenable. This is done by establishing the corresponding (ARP) and a (KPT)-correspondence, introduced in §2.6.1. In Section 3 we study Choquet simplices (with a distinguished face), and we prove that the pointwise stabilizer of any closed proper face of the Poulsen simplex is extremely amenable. We conclude in Subsection 3.4 where we prove that the universal minimal flow of the group of affine homeomorphisms of the Poulsen simplex $\mathbb{P}$ is the canonical action on $\mathbb{P}$.

2. The Ramsey property of Banach spaces

The goal of this section is to introduce different notions of "Ramsey property" for several classes of structures. We show that in the setting we are interested in, such notions are equivalent to each other. We furthermore establish an analogue of the Kechris–Pestov–Todorcevic correspondence. We then establish the (stable) Ramsey property for the class of Banach spaces $\{\ell_\infty^n\}_n$. From this, we infer that that the group of isometries of the Gurarij space is extremely amenable.

2.1. Colorings and extreme amenability

We introduce some terminology to be used in the following. A *metric coloring* of a pseudo-metric space M is a 1-Lipschitz map from M to a metric space (K, d_K). A metric coloring with target space (K, d_K) will also be called a K-*coloring*. Following [45], a *continuous coloring* is a metric coloring whose target space is the closed unit interval $[0, 1]$. A *compact coloring* is a metric coloring whose target space is a compact metric space. For a subset X of a compact metric space (K, d_K) and $\varepsilon > 0$, the ε-*fattening* K_ε is the set of elements of K at distance at most ε from some element of X.

The *oscillation* $\operatorname{osc}_F(c)$ of a compact coloring $c : M \to (K, d_K)$ on a subset F of M is the supremum of $d_K(c(y), c(y'))$ where y, y' range within F. If $\operatorname{osc}_F(c) \leq \varepsilon$, then we say that c ε-*stabilizes* on F, or that F is ε-*monochromatic* for c. A *finite* coloring of M is a function from M to a finite set X. When the target space is a natural number r (identified with $\{0, 1, \ldots, r-1\}$), we will say that c is an r-*coloring*. A subset F of M is *monochromatic* for c if $c(p) = c(q)$ for all $p, q \in F$, and ε-*monochromatic* for c if there exists $x \in X$ such that for every $p \in F$ there is $q \in M$ such that $c(q) = x$ and $d_M(p, q) \leq \varepsilon$. If F is ε-monochromatic, then we also say that c ε-*stabilizes* on F.

Given a Polish group G and a continuous action $G \curvearrowright M$ of G on a metric space (M, d_M), we write $[p]_G$ for the closure of the G-orbit of $p \in M$, and $M /\!/ G$ for the space of closures of G-orbits of M. Since G acts by isometries, the formula

$$\widehat{d}_{G,M}([p], [q]) := \inf\{d_M(\bar{p}, \bar{q}) : \bar{p} \in [p],\ \bar{q} \in [q]\}$$

defines the quotient pseudometric induced by the quotient map $\pi_{M,G} : M \to M /\!/ G$, and since we consider *closures* of orbits, $\widehat{d}_{G,M}$ is a metric. It is easy to see that $\widehat{d}_{G,M}$ is complete when d_M is complete.

When M is endowed with an action of a Polish group G we say that M is a *metric* G-*space*. A compact coloring $c : (M, d_M) \to (K, d_K)$ is *finitely* G-*factorizable* when there is a K-coloring $\widehat{c} : M /\!/ G \to K$ defined on the space $M /\!/ G$ of closed G-orbits of M such that for every $\varepsilon > 0$ and every compact subset $F \subseteq M$ there is some $g \in G$ such that $d_K(c(p), \widehat{c}([p]_G)) \leq \varepsilon$ for every $p \in g \cdot F$, where $[p]_G$ is the closed G-orbit of p. Similarly, c is *finitely oscillation stable* [51, Definition 1.1.8] if for every compact subset F of M and $\varepsilon > 0$ there exists $g \in G$ such that c ε-stabilizes on $g \cdot F$. We say that the action of G on M is *finitely oscillation stable* if every continuous coloring of M is finitely oscillation stable [51, Definition 1.1.11].

Given a compact metric space (K, d_K), we let $\operatorname{Lip}((M, d_M), (K, d_K))$ be the collection of all K-colorings of M; with the topology of pointwise convergence, it is a compact space, which is metrizable when (M, d_M) is separable. A continuous action $G \curvearrowright (M, d_M)$ induces a natural continuous action $G \curvearrowright \operatorname{Lip}((M, d_M), (K, d_K))$, defined by setting $(g \cdot c)(p) := c(g^{-1} \cdot p)$ for all $c \in \operatorname{Lip}((M, d_M), (K, d_K))$ and $p \in M$.

Lemma 2.1. *Suppose that* G *is a Polish group, and that* M *is a metric* G*-space. Let* $\mathcal{F}$ *be a* $\subseteq$*-directed family of compact subsets of* M *whose union is* M*. The following assertions are equivalent:*

(1) *Every compact coloring of* M *is finitely* G*-factorizable.*

(2) *For every $F \in \mathcal{F}$, every compact metric space K and every $\varepsilon > 0$ there is an $H \in \mathcal{F}$ such that for every coloring $c : H \to K$ there is a coloring $\widehat{c} : H /\!/ G \to K$ and $g \in G$ such that $gF \subseteq H$ and $d_K(c(p), \widehat{c}([p]_G)) \leq \varepsilon$ for every $p \in g \cdot F$.*

Proof. Suppose that (1) holds but not (2). Fix a counterexample K, $\mathcal{F}$, M, $\varepsilon > 0$ and $F \in \mathcal{F}$ and for each $H \in \mathcal{F}$ containing F fix a bad coloring $c_H : H \to K$. For each $V \in \mathcal{F}$, let $\langle V \rangle$ be the collection of those $W \in \mathcal{F}$ containing V. Choose a non-principal ultrafilter $\mathcal{U}$ on $\mathcal{F}$ containing each $\langle V \rangle$. This is possible since $\mathcal{F}$ is $\subseteq$-directed. Define $c_{\mathcal{U}} : M \to K$ by declaring $c_{\mathcal{U}}(p) := \mathcal{U}\text{-}\lim c_H(p)$. This is well defined because there is $H \in \mathcal{F}$ such that $p \in H$. Let $\widehat{c} : M /\!/ G \to K$ be the corresponding factorization, and let g be such that $d_K(\widehat{c}([p]_G), c_{\mathcal{U}}(p)) \leq \varepsilon/2$ for every $p \in g \cdot F$. Choose $H \in \mathcal{F}$ such that $p \cdot F \subseteq H$ and $d_K(c_H(p), c_{\mathcal{U}}(p)) \leq \varepsilon/2$ for every $p \in g \cdot F$. Then the restriction $\widehat{c} : H /\!/ G \to K$ disproves that c_H is a bad color. Suppose now that (2) holds but not (1). This means that there is some $c : M \to K$ that cannot be finitely G-factorized, so we fix the corresponding $\varepsilon > 0$. For every $F \in \mathcal{F}$ we use (2) for it, K, and for $\varepsilon/2$ to find the corresponding $H_F \in \mathcal{F}$, and then we apply the property of it to the restriction $c : H \to K$ to find $e_F : H /\!/ G \to K$. Now define $\widehat{c} : M /\!/ G \to K$ as the $\mathcal{U}$-limit of $(e_F)_F$. Since $\widehat{c}$ does not finitely G-factorize c there must be a bad compact A witnessing this. Without loss of generality we may assume that A is a finite set. Let $F \in \mathcal{F}$ be such that $A \subseteq F$, and let $H \in \{V \in \langle F \rangle \ : \ d_K(\widehat{c}([p]_G), e_V([p]_G)) \leq \varepsilon/2 \text{ for every } p \in F\} \in \mathcal{U}$. Let $g \in G$ be such that $d_K(c(p), e_H([p]_G)) \leq \varepsilon/2$ for every $p \in g \cdot F$, and consequently $d_K(c(p), \widehat{c}([p]_G)) \leq \varepsilon$ for every $p \in g \cdot A$, contradicting the defining property of A. ∎

Recall that a topological group G is called *extremely amenable* if every continuous action of G on a compact Hausdorff space has a fixed point. The following characterization of extreme amenability will be used extensively in this paper.

Proposition 2.2. *Suppose that G is a Polish group. The following assertions are equivalent:*

(1) *G is extremely amenable.*

(2) *For every left-invariant compatible metric d_G on G, the left translation of G on (G, d_G) is finitely oscillation stable.*

(3) *Every compact coloring of a metric G-space is finitely G-factorizable.*

(4) *Let M be a metric G-space, and let $\mathcal{F}$ be a $\subseteq$-directed family of compact subsets of M whose union is M. For every $F \in \mathcal{F}$, every compact metric space K and every $\varepsilon > 0$ there is an $H \in \mathcal{F}$ such that for every coloring $c : H \to K$ there is a coloring $\widehat{c} : H /\!/ G \to K$ and $g \in G$ such that $gF \subseteq H$ and $d_K(c(p), \widehat{c}([p]_G)) \leq \varepsilon$ for every $p \in g \cdot F$.*

Proof. The equivalence of (1) and (2) can be found in [51, Theorem 2.1.11]. The implication (3)⇒(2) is immediate, since the orbit space $G /\!/ G$ is one point. We now establish (1)⇒(3): Fix a 1-Lipschitz $c : (M, d_M) \to (K, d_K)$. Let L be the closure of the G-orbit of c in $\mathrm{Lip}((M, d_M), (K, d_K))$. By the extreme amenability of G, there is $c_\infty \in L$ such that $G \cdot c_\infty = \{c_\infty\}$, so we can define $\widehat{c} : M /\!/ G \to K$ by $\widehat{c}([p]_G) := c_\infty(p)$.

It is clear that $\widehat{c}$ is 1-Lipschitz. Given a compact subset F of M, let $g \in G$ be such that $\max_{p\in F} d_K(c_\infty(p), c(g \cdot p)) < \varepsilon$. If $x \in F$, then $d_K(c(g \cdot x), \widehat{c}([g \cdot x]_G)) = d_K(c(g \cdot x), c_\infty(x)) < \varepsilon$. The equivalence of (3) and (4) follows from Lemma 2.1. ■

2.2. *The Ramsey property and the KPT correspondence for Banach spaces*

In this section, we provide a characterization of extreme amenability of the isometry group of a Banach space (endowed with the topology of pointwise convergence). This can be seen as an analogue in this context of the Kechris–Pestov–Todorcevic from [31]. A more general KPT correspondence for arbitrary metric structures is the topic of [45].

We introduce some basic terminology on Banach spaces. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Given $n \in \mathbb{N}$ and $1 \le p < \infty$, let ℓ_p^n be the normed space $(\mathbb{F}^n, \|\cdot\|_p)$ where $\|(a_j)_{j<n}\|_p := (\sum_{j<n} |a_j|^p)^{1/p}$ is the p-norm; similarly, let $\ell_\infty^n = (\mathbb{F}^n, \|\cdot\|_\infty)$ where $\|(a_j)_{j<n}\|_\infty := \max_{j<n} |a_j|$. Given a Banach space $(X, \|\cdot\|)$, let $\mathrm{Ball}(X) := \{x \in X : \|x\| \le 1\}$ and $\mathrm{Sph}(X) := \{x \in X : \|x\| = 1\}$ be the unit ball and the unit sphere of X. Recall that given Banach spaces X, Y, a *contraction* $T : X \to Y$ is a bounded linear mapping $T : X \to Y$ such that $\|T\| := \max_{\|x\|\le 1} \|T(x)\| \le 1$. Given $\delta \ge 0$, let $\mathrm{Emb}_\delta(X, Y)$ be the space of contractions $T : X \to Y$ such that $\|Tx\| \ge \|x\|/(1 + \delta)$, endowed with the norm metric, $d(T, U) := \|T - U\| := \max_{\|x\|\le 1} \|T(x) - U(x)\|$; when $\delta = 0$, $\mathrm{Emb}(X, Y) := \mathrm{Emb}_0(X, Y)$ is the space of *isometric embeddings* from X into Y. Dually, when X and Y are finite-dimensional, a *quotient map* $T : X \to Y$ is a linear mapping such that $T(\mathrm{Ball}(X)) = \mathrm{Ball}(Y)$. The space of those quotient maps is denoted by $\mathrm{Quo}(X, Y)$. It is well-known that $T \in \mathrm{Emb}(X, Y)$ if and only if the dual operator $T^* : Y^* \to X^*$ is a quotient map, and this assignment is an isometry. Finally, given a Banach space E, let $\mathrm{Iso}(E)$ be the group of surjective isometries of E, endowed with the strong operator topology (SOT), and observe that $\mathrm{Iso}(E)$ acts continuously on $\mathrm{Emb}_\delta(X, E)$ by left composition, $g \cdot T := g \circ T$.

In particular, suppose that X is a finite-dimensional subspace of E. Given a finite-dimensional subspace Y of E containing X we can canonically identify $\mathrm{Emb}(X, Y)$ with the collection of those isometric embeddings $T : X \to E$ such that $\mathrm{Im}\, T \subseteq Y$, so in this way $\mathrm{Emb}(X, E) = \bigcup_{X\subseteq Y\subseteq E} \mathrm{Emb}(X, Y)$, where each $\mathrm{Emb}(X, Y)$ is a compact subset of $\mathrm{Emb}(X, E)$. Suppose that $\mathrm{Iso}(E)$ is extremely amenable. By applying Proposition 2.2 we find that given such an $X \subseteq Y$, a compact metric (K, d_K) and $\varepsilon > 0$ we can find a finite-dimensional subspace Z_0 of E such that for every coloring $c : \mathrm{Emb}(X, Z_0) \to K$ there is $g \in \mathrm{Iso}(E)$ such that

$$\begin{aligned}&\text{there is a coloring } \widehat{c} : \mathrm{Emb}(X, Z_0)/\!/\,\mathrm{Iso}(E) \to K \text{ with}\\ &\max_{\gamma\in\mathrm{Emb}(X,Y)} d_K(c(g \circ \gamma), \widehat{c}([\gamma]_{\mathrm{Iso}(E)})) < \varepsilon/2.\end{aligned} \tag{2.1}$$

We consider on $\mathrm{Lip}(\mathrm{Emb}(X, Z_0), K)$ the compatible metric defined for K-colorings c_1 and c_2 by $d(c_1, c_2) := \max_{\gamma\in\mathrm{Emb}(X,Z_0)} d_K(c_1(\gamma), c_2(\gamma))$. Since $\mathrm{Lip}(\mathrm{Emb}(X, Z_0), K)$ is compact, we can find a finite $\varepsilon/2$-dense subset D of it, and for each $c \in D$ we choose some $g_c \in \mathrm{Iso}(E)$ witnessing (2.1). Let Z be a finite-dimensional subspace of E containing Y and $\bigcup_{c\in D} g_c Y$. Then for every coloring $c : \mathrm{Emb}(X, Z) \to K$ there are $g \in \mathrm{Iso}(E)$ and $\hat{c} : \mathrm{Emb}(X, Z) /\!/ \mathrm{Iso}(E) \to K$ with the property that $gY \subseteq Z$ and $d_K(c(g \circ \gamma), \hat{c}([g]_{\mathrm{Iso}(E)})) \le \varepsilon$ for every $\gamma \in \mathrm{Emb}(X, Y)$. This means in particular that the oscillation of c in $g \circ \mathrm{Emb}(X, Y)$ is determined by the diameter of $\mathrm{Emb}(X, E) /\!/ \mathrm{Iso}(E)$. Recall that an action $G \curvearrowright M$ of a group G on a metric space (M, d) is *ε-transitive*, for some $\varepsilon > 0$, when the diameter of $M /\!/ G$ is at most ε, that is, for all $x, y \in M$ there is $g \in G$ such that $d(g \cdot x, y) \le \varepsilon$. $G \curvearrowright M$ is *approximately transitive* when it is ε-transitive for every $\varepsilon > 0$, or equivalently, when $M /\!/ G$ consists of one point.

Definition 2.3. A Banach space E is called *approximately ultrahomogeneous* when for every finite-dimensional subspace X of E the action $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}(X, E)$ is approximately transitive.

Hence, we obtain the following.

Corollary 2.4. *Suppose that E is approximately ultrahomogeneous and* $\mathrm{Iso}(E)$ *is extremely amenable. Then for any finite-dimensional subspaces $X \subseteq Y$ of E and every compact metric space (K, d_K) there is a finite-dimensional subspace Z of E containing Y such that every coloring $c : \mathrm{Emb}(X, Z) \to K$ ε-stabilizes in some set of the form $\gamma \circ \mathrm{Emb}(X, Y)$.* ■

Up to now the list of known approximately ultrahomogeneous (real or complex) Banach spaces includes:

- Hilbert spaces (indeed, they are ultrahomogeneous, i.e. the algebraic quotients $\mathrm{Emb}(X, E)/G$ are singletons);
- the Lebesgue spaces $L_p[0, 1]$ when $p \notin 2\mathbb{N}$, as proved by W. Lusky [41];
- the Gurarij space $\mathbb{G}$.

The original characterization of the Gurarij space considered by Gurarij [28] and Lusky [39,40,42] is as the unique separable Banach space with the following extension property: for all finite-dimensional Banach spaces $E \subseteq F$, any linear contraction $\varphi : E \to \mathbb{G}$, and $\varepsilon > 0$, there exists an extension $\hat{\varphi} : F \to \mathbb{G}$ satisfying $\|\hat{\varphi}\| < 1 + \varepsilon$. The fact that such a space is indeed approximately ultrahomogeneous in the sense of Definition 2.3 is proved by I. Ben Yaacov [8].

The isometry groups (endowed with the strong operator topology) of the Banach spaces in the list above have very special topological dynamical properties. The groups $\mathrm{Iso}(L_p(0, 1))$ are extremely amenable for every $1 \le p < \infty$, which was proved in the case of $p = 2$ by M. Gromov and V. D. Milman [27] and for $p \ne 2$ by T. Giordano and V. Pestov [18]. Both cases use the method of concentration of measure. In this paper we prove the following.

Theorem 2.5. *The group of isometries of the Gurarij space endowed with the strong operator topology is extremely amenable.*

Our proof is not based on concentration of measure, but on a combinatorial property, the approximate Ramsey property, that characterizes the extreme amenability of certain isometry groups. With a similar approach, this has been extended in [5] to the context of operator spaces. We now introduce several variants of the Ramsey property for Banach spaces.

Definition 2.6 (Approximate Ramsey property). Let $\mathcal{F}$ be a family of finite-dimensional Banach spaces.

(a) $\mathcal{F}$ has the *approximate Ramsey property* (ARP) if for any $X, Y \in \mathcal{F}$ and $\varepsilon > 0$ there exists $Z \in \mathcal{F}$ such that any continuous coloring of $\mathrm{Emb}(X, Z)$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(X, Y)$ for some $\gamma \in \mathrm{Emb}(Y, Z)$.

(b) $\mathcal{F}$ has the *compact approximate Ramsey property* when for any $X, Y \in \mathcal{F}$, $\varepsilon > 0$ and any compact metric space (K, d_K) there exists $Z \in \mathcal{F}$ such that any K-coloring of $\mathrm{Emb}(X, Z)$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(X, Y)$ for some $\gamma \in \mathrm{Emb}(Y, Z)$.

(c) $\mathcal{F}$ has the *discrete approximate Ramsey property* when for any $X, Y \in \mathcal{F}$, $r \in \mathbb{N}$ and $\varepsilon > 0$ there is some $Z \in \mathcal{F}$ such that any r-coloring of $\mathrm{Emb}(X, Z)$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(X, Y)$ for some $\gamma \in \mathrm{Emb}(Y, Z)$.

So, rephrasing Corollary 2.4, if E is an approximately ultrahomogeneous Banach space whose isometry group is extremely amenable, then the class $\mathrm{Age}(E)$ of finite-dimensional subspaces of E has the approximate Ramsey property. Conversely, we will see in Theorem 2.12 that in fact the (ARP) of $\mathrm{Age}(E)$ characterizes the extreme amenability of $\mathrm{Iso}(E)$ for approximately ultrahomogeneous spaces E. Now we show that the different versions of the Ramsey property are in fact equivalent.

Proposition 2.7. *The following are equivalent for a class $\mathcal{F}$ of finite-dimensional Banach spaces:*

(1) *$\mathcal{F}$ satisfies the* (*ARP*).

(2) *$\mathcal{F}$ satisfies the compact* (*ARP*).

(3) *$\mathcal{F}$ satisfies the discrete* (*ARP*).

Proof. The compact (ARP) obviously implies the (ARP). Suppose that $\mathcal{F}$ satisfies the (ARP), and let us prove that $\mathcal{F}$ satisfies the discrete (ARP). This is done by induction on $r \in \mathbb{N}$. The case $r = 1$ is trivial. Suppose that we have shown that $\mathcal{F}$ satisfies the discrete (ARP) for r-colorings. Consider $X, Y \in \mathcal{F}$ and $\varepsilon > 0$. Then by the inductive hypothesis, there is $Z_0 \in \mathcal{F}$ such that every r-coloring of $\mathrm{Emb}(X, Z_0)$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(X, Y)$ for some $\gamma \in \mathrm{Emb}(Y, Z_0)$. Since by assumption $\mathcal{F}$ satisfies the continuous (ARP), there is $Z \in \mathcal{F}$ such that every continuous coloring of $\mathrm{Emb}(X, Z)$ $\varepsilon/2$-stabilizes on $\gamma \circ \mathrm{Emb}(X, Z_0)$ for some $\gamma \in \mathrm{Emb}(Z_0, Z)$. We claim that Z witnesses that $\mathcal{F}$ satisfies the discrete (ARP) for $(r + 1)$-colorings. Indeed, suppose that c is an $(r + 1)$-coloring of $\mathrm{Emb}(X, Z)$. Define $f : \mathrm{Emb}(X, Z) \to [0, 1]$ by $f(\phi) := \frac{1}{2} d(\phi, c^{-1}(r))$. This is a continuous coloring, so by the choice of Z there exists $\gamma \in \mathrm{Emb}(Z_0, Z)$ such that f $\varepsilon/2$-stabilizes on $\gamma \circ \mathrm{Emb}(X, Z_0)$. Now, if there is some $\phi \in \mathrm{Emb}(X, Z_0)$ such that $c(\gamma \circ \phi) = r$, then $\gamma \circ \mathrm{Emb}(X, Z_0) \subseteq (c^{-1}(r))_\varepsilon$, so choosing an arbitrary $\bar{\gamma} \in \mathrm{Emb}(Y, Z_0)$ we find that c ε-stabilizes on $\gamma \circ \bar{\gamma} \circ \mathrm{Emb}(X, Y)$. Otherwise, $(\gamma \circ \mathrm{Emb}(X, Z_0)) \cap c^{-1}(r) = \emptyset$, so defining $\bar{c}(\phi) := c(\gamma \circ \phi)$ for $\phi \in \mathrm{Emb}(X, Z_0)$ gives an r-coloring of $\mathrm{Emb}(X, Z_0)$. By the choice of Z_0 there exists $\bar{\gamma} \in \mathrm{Emb}(Y, Z_0)$ such that $\bar{c}$ ε-stabilizes on $\bar{\gamma} \circ \mathrm{Emb}(X, Y)$.

Therefore c ε-stabilizes on $\gamma \circ \bar{\gamma} \circ \mathrm{Emb}(X, Y)$. This concludes the proof that the continuous (ARP) implies the discrete (ARP).

Finally, the discrete (ARP) implies the compact (ARP). In fact, given $\varepsilon > 0$ and a compact metric space K, one can find a finite ε-dense subset $D \subseteq K$. Thus if $Z \in \mathcal{F}$ witnesses the discrete (ARP) for X, Y, ε and D, then given a 1-Lipschitz $f : \mathrm{Emb}(X, Z) \to K$ we can define a coloring $c : \mathrm{Emb}(X, Z) \to D \subseteq K$ such that $d_K(c(\phi), f(\phi)) \leq \varepsilon$ for every $\phi \in \mathrm{Emb}(X, Z)$. In this way, if c ε-stabilizes on $\gamma \circ \mathrm{Emb}(X, Y)$, then f 3ε -stabilizes on $\gamma \circ \mathrm{Emb}(X, Y)$. ∎

We are going to see that when E is approximately ultrahomogeneous, the extreme amenability of $\mathrm{Iso}(E)$ is equivalent to the (ARP) of $\mathrm{Age}(E)$ and, in fact, also to a stronger version of the Ramsey property for a rich subfamily of $\mathrm{Age}(E)$. To state this property we recall that for k-dimensional Banach spaces X, Y, the *Banach–Mazur* (pseudo)distance is defined by

$$d_{\mathrm{BM}}(X, Y) := \log\bigl(\min \{\|T\| \cdot \|T^{-1}\| \,:\, T : X \to Y \text{ is a linear isomorphism}\}\bigr).$$

Definition 2.8. Given a family $\mathcal{F}$ of finite-dimensional Banach spaces, let $[\mathcal{F}]$ be the class of all separable Banach spaces E such that $\mathcal{F} \subseteq \mathrm{Age}(E)$ and every finite-dimensional subspace of E is the d_{BM}-limit of a sequence of subspaces of elements of $\mathcal{F}$.

For example, the spaces c_0, $C[0, 1]$ or the Gurarij space are in the class $[\{\ell_\infty^n\}_n]$, where each ℓ_∞^n is the (real or complex) vector space $\mathbb{F}^n$ endowed with the sup norm, $\|(a_1, \ldots, a_n)\|_\infty := \max_i |a_i|$. In general $[\{\ell_\infty^n\}_n]$ is the class of separable Lindenstrauss spaces. In what follows, by a *modulus of stability* we mean a function $\varpi : [0, \infty[\to [0, \infty[$ that is increasing and continuous at zero with value zero.

Definition 2.9 (Fraïssé properties). Let E be a separable Banach space, and let $\mathcal{F}$ be a family of finite-dimensional spaces.

(a) E has the *stable homogeneity property* with respect to $\mathcal{F}$ with modulus of stability ϖ if $\mathrm{Emb}(X, E)$ is non-empty for every $X \in \mathcal{F}$ and if for all $X \in \mathcal{F}$, $\delta \geq 0$, $\varepsilon > 0$, the canonical action $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X, E)$ is $(\varpi(\delta) + \varepsilon)$-transitive.

(b) E is a *stable Fraïssé* Banach space with modulus of stability ϖ when E has the stable homogeneity property with respect to $\mathrm{Age}(E)$.

(c) $\mathcal{F}$ has the *stable amalgamation property* (SAP) with modulus ϖ when for all $X, Y, Z \in \mathcal{F}$, $\varepsilon > 0$, $\delta \geq 0$, $\gamma \in \mathrm{Emb}_\delta(X, Y)$ and $\eta \in \mathrm{Emb}_\delta(X, Z)$ there are $V \in \mathcal{F}$, $I \in \mathrm{Emb}(Y, V)$ and $J \in \mathrm{Emb}(Z, V)$ such that $\|I \circ \gamma - J \circ \eta\| \leq \varpi(\delta) + \varepsilon$.

(d) $\mathcal{F}$ is a *stable amalgamation class* with modulus ϖ when it has the (SAP) with modulus ϖ and the *joint embedding property* (JEP), that is, for all $X, Y \in \mathcal{F}$ there is $Z \in \mathcal{F}$ such that $\mathrm{Emb}(X, Z), \mathrm{Emb}(Y, Z)$ are non-empty.

(e) $\mathcal{F}$ is a *stable Fraïssé class* with modulus ϖ when $\mathcal{F}$ is a stable amalgamation class with modulus ϖ and it is hereditary, that is, if $X \in \mathcal{F}$ and $\mathrm{Emb}(Y, X) \neq \emptyset$, then $Y \in \mathcal{F}$.

It is easy to see that if $\mathcal{F}$ satisfies the (SAP) and it has a least element with respect to inclusion, then $\mathcal{F}$ has the (JEP), and consequently $\mathcal{F}$ is a stable amalgamation class. Using the fact that $\{\ell_\infty^n\}_n$ is a stable amalgamation class with modulus $\varpi(\delta) = \delta$ (see Proposition 2.18), it is proved in [37, §6.1] that the Gurarij space is a stable Fraïssé Banach space with modulus $\varpi(\delta) = \delta$. In fact, this approximate ultrahomogeneity is a direct consequence of the fact that the Gurarij space is the "generic" direct limit of the class of all finite-dimensional Banach spaces, an instance of the following Fraïssé correspondence for Banach spaces (see for instance [37, §2.6], [17, §2.1]).

Proposition 2.10. *Suppose that $\mathcal{F}$ is a class of finite-dimensional Banach spaces, and E is a separable Banach space.*

(a) *If E is a Fraïssé space with modulus ϖ, then* $\mathrm{Age}(E)$ *is a stable Fraïssé class with modulus ϖ.*

(b) *If $\mathcal{F}$ is a stable amalgamation class with modulus ϖ, then there is a unique separable $E \in [\mathcal{F}]$ that has the stable homogeneity property with respect to $\mathcal{F}$ with modulus ϖ. This space is called the* Fraïssé limit of $\mathcal{F}$ *and denoted by* $\mathrm{FLim}\,\mathcal{F}$. ∎

Consequently, the class of all finite-dimensional Banach spaces is stable with modulus δ. The classes $\{\ell_p^n\}_n$ for $1 \le p \le \infty$ are also stable amalgamation classes: The case $p = \infty$ is rather easy (see Proposition 2.18), as well as the case $p = 2$, where one can use the polar decomposition; for $1 < p < \infty$, $p \neq 2$, one can use a result of G. Schechtman [53] on approximation of δ-embeddings by isometric embeddings. Also, it is proved in [17] that for $p \neq 4, 6, 8, \ldots$, the class $\mathrm{Age}(L_p(0,1))$ has a weaker form of stable approximate ultrahomogeneity, namely one that may depend on dimension. Several other examples of Fraïssé classes of structures in functional analysis are studied in [37].

As mentioned before, we will see that for an approximately ultrahomogeneous space E, the (ARP) of its age is equivalent to the extreme amenability of the isometry group of E. Furthermore, when $E = [\mathcal{F}]$ for some stable amalgamation class $\mathcal{F}$, a stronger form of the (ARP) of $\mathcal{F}$ is also equivalent to the extreme amenability of the isometry group of E.

Definition 2.11. A class $\mathcal{F}$ of finite-dimensional Banach spaces has the *stable approximate Ramsey property* (SRP) with stability modulus ϖ if for any $X, Y \in \mathcal{F}$, $\varepsilon > 0$, $\delta \ge 0$ there exists $Z \in \mathcal{F}$ such that every 1-Lipschitz mapping $c : \mathrm{Emb}_\delta(X, Z) \to [0, 2(1+\delta)]$ $(\varpi(\delta) + \varepsilon)$-stabilizes on $\gamma \circ \mathrm{Emb}_\delta(X, Y)$ for some $\gamma \in \mathrm{Emb}(Y, Z)$.

The *compact* (SRP) and *discrete* (SRP) are defined like the (ARP), by replacing continuous colorings with compact and finite colorings, respectively.

Theorem 2.12 (KPT correspondence for Banach spaces). *Let E be an approximately ultrahomogeneous Banach space. Then the following are equivalent:*

(1) $\mathrm{Iso}(E)$ *is extremely amenable.*

(2) $\mathrm{Age}(E)$ *has the approximate Ramsey property.*

(3) *For any $X, Y \in \mathrm{Age}(E)$, every $\varepsilon > 0$ and every continuous coloring c of* $\mathrm{Emb}(X, E)$ *there is $g \in \mathrm{Iso}(E)$ such that* $\mathrm{Osc}(c \restriction g \circ \mathrm{Emb}(X, Y)) \le \varepsilon$.

If in addition $\mathcal{F}$ has the stable amalgamation property with $E \in [\mathcal{F}]$ and $\mathcal{F} \preceq \mathrm{Age}(E)$, that is, every space in $\mathcal{F}$ can be isometrically embedded into E, then (1)–(3) *above are also equivalent to*

(4) *$\mathcal{F}$ satisfies the* (*SRP*).

The equivalence of (1) and (2) is a particular instance a more general characterization of extreme amenability in terms of an approximate Ramsey property when Banach spaces are regarded as *metric structures* [9] as in [23, Appendix B] or [37, §8.1]. Before we give a proof of the correspondence, we compare these Ramsey properties.

Proposition 2.13. *Suppose that $\mathcal{F}$ is a class of finite-dimensional spaces with the* joint embedding embedding property, *that is, for any $X, Y \in \mathcal{F}$ there is $Z \in \mathcal{F}$ such that* $\mathrm{Emb}(X, Z), \mathrm{Emb}(Y, Z) \neq \emptyset$. *Then the following assertions are equivalent:*

(1) *$\mathcal{F}$ satisfies the* (*ARP*) *and the* (*SAP*) *with modulus ϖ.*

(2) *$\mathcal{F}$ satisfies the* (*SRP*) *with modulus ϖ.*

(3) *$\mathcal{F}$ satisfies the discrete* (*SRP*) *with modulus ϖ.*

(4) *$\mathcal{F}$ satisfies the compact* (*SRP*) *with modulus ϖ.*

Proof. Trivially, the compact (SRP) with modulus ϖ implies the discrete (SRP) with modulus ϖ, and a simple modification of the proof of Proposition 2.7 shows that the discrete (SRP) with modulus ϖ implies the (SRP) with modulus ϖ. Trivially, the (SRP) with modulus ϖ implies the (ARP). In addition, we have the following

Claim 2.13.1. *If $\mathcal{F}$ has the* (*SRP*) *with modulus ϖ then $\mathcal{F}$ has the* (*SAP*) *with modulus ϖ.*

Proof of Claim. Fix $X, Y, Z \in \mathcal{F}$, $\varepsilon > 0$, $\delta \geq 0$ and $\gamma \in \mathrm{Emb}_\delta(X, Y)$ and $\eta \in \mathrm{Emb}_\delta(X, Z)$. Find $V \in \mathcal{F}$ such that $\mathrm{Emb}(X, V), \mathrm{Emb}(Y, V)$ and $\mathrm{Emb}(Z, V)$ are non-empty. Find $W \in \mathcal{F}$ witnessing the (SRP) for initial parameters $X, V \in \mathcal{F}$, ε, δ. We claim that W also witnesses the (SAP) for γ, η, ε and δ. Choose θ_Y in $\mathrm{Emb}(Y, V)$ and θ_Z in $\mathrm{Emb}(Z, V)$. Let $I \in \mathrm{Emb}(V, W)$ be such that $\mathrm{Osc}(c \restriction I \circ \mathrm{Emb}_\delta(X, V)) \leq \varpi(\delta) + \varepsilon$, where $c : \mathrm{Emb}_\delta(X, W) \to [0, 2 + \delta]$ is defined by $c(\xi) := d(\xi, \mathrm{Emb}(V, W) \circ \theta_Y \circ \gamma)$. Since $c(I \circ \theta_Y \circ \gamma) = 0$, $d(I \circ \theta_Z \circ \eta, \mathrm{Emb}(V, W) \circ \theta_Y \circ \gamma) \leq \varpi(\delta) + \varepsilon$, there is $J \in \mathrm{Emb}(V, W)$ such that $\|I \circ \theta_Z \circ \eta - J \circ \theta_Y \circ \gamma\| \leq \varpi(\delta) + \varepsilon$, as desired. ■

Suppose that $\mathcal{F}$ has the (ARP) and the (SAP) with modulus ϖ; we will prove that $\mathcal{F}$ has the compact (SRP). The next claim is not difficult to prove.

Claim 2.13.2. *$\mathcal{F}$ has the* (*SAP*) *with modulus ϖ if and only if for any $X, Y \in \mathcal{F}$, $\delta \geq 0$ and $\varepsilon > 0$ there exist $Z \in \mathcal{F}$ and $I \in \mathrm{Emb}(Y, Z)$ such that for all $\phi, \psi \in \mathrm{Emb}_\delta(X, Y)$ there is $J \in \mathrm{Emb}(Y, Z)$ such that $\|I \circ \phi - J \circ \psi\| \leq \varpi(\delta) + \varepsilon$.* ■

Fix $X, Y \in \mathcal{F}$, $\delta, \varepsilon > 0$ and a compact metric space K. We use the previous claim to find $Y_0 \in \mathcal{F}$ such that for any $\phi, \psi \in \mathrm{Emb}_\delta(X, Y)$ there are $i, j \in \mathrm{Emb}(Y, Y_0)$ such that $\|i \circ \phi - j \circ \psi\|_{\mathrm{cb}} \leq \varpi(\delta) + \varepsilon$. We consider the space $\mathcal{L} := \mathrm{Lip}(\mathrm{Emb}_\delta(X, Y), K)$

of 1-Lipschitz maps from $\mathrm{Emb}_\delta(X,Y)$ to K as a compact metric space, endowed with the metric $d(f,g) := \sup\{d_K(f(\phi), g(\phi)) : \phi \in \mathrm{Emb}_\delta(X,Y)\}$. By Proposition 2.7, $\mathcal{F}$ satisfies the compact (ARP). Thus there exists $Z \in \mathcal{F}$ such that every $\mathcal{L}$-coloring of $\mathrm{Emb}(Y,Z)$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(Y,Y_0)$ for some $\gamma \in \mathrm{Emb}(Y_0,Z)$. We claim that Z works, so let $c : \mathrm{Emb}_\delta(X,Z) \to K$ be 1-Lipschitz. We can define a coloring $\widehat{c} : \mathrm{Emb}(Y,Z) \to \mathcal{L}$ by setting, for $\xi \in \mathrm{Emb}(Y,Z)$, $\widehat{c}(\xi) : \mathrm{Emb}_\delta(X,Y) \to K$, $\phi \mapsto c(\xi \circ \phi)$. By the choice of Z, there exists $\bar{\gamma} \in \mathrm{Emb}(Y_0,Z)$ such that $\widehat{c}$ ε-stabilizes on $\bar{\gamma} \circ \mathrm{Emb}(Y,Y_0)$. Choose an arbitrary $\varrho \in \mathrm{Emb}(Y,Y_0)$. We claim that c $(\varpi(\delta)+3\varepsilon)$-stabilizes on $\bar{\gamma} \circ \varrho \circ \mathrm{Emb}_\delta(X,Y)$. Let $\phi, \psi \in \mathrm{Emb}_\delta(X,Y)$. By the choice of Y_0 there are $i, j \in \mathrm{Emb}(Y,Y_0)$ such that $\|i \circ \phi - j \circ \psi\| \le \varpi(\delta)+\varepsilon$. Since $d_{\mathcal{L}}(\widehat{c}(\bar{\gamma}\circ\varrho), \widehat{c}(\bar{\gamma}\circ i)) \le \varepsilon$ and $d_{\mathcal{L}}(\widehat{c}(\bar{\gamma}\circ\varrho), \widehat{c}(\bar{\gamma}\circ j)) \le \varepsilon$, it follows that $d_K(c(\bar{\gamma}\circ\varrho\circ\phi), c(\bar{\gamma}\circ i\circ\phi)) \le \varepsilon$, $d_K(c(\bar{\gamma}\circ\varrho\circ\psi), c(\bar{\gamma}\circ j\circ\psi)) \le \varepsilon$. Furthermore, from $\|i\circ\phi - j\circ\psi\| \le \varpi(\delta)+\varepsilon$ and the fact that c is 1-Lipschitz we deduce that $d_K(c(\bar{\gamma}\circ\varrho\circ\phi), c(\bar{\gamma}\circ\varrho\circ\psi)) \le \varpi(\delta)+3\varepsilon$. ■

Proof of Theorem 2.12. Corollary 2.4 gives that (1) implies (2). Let us prove the converse. Suppose that $\mathrm{Age}(E)$ has the (ARP). Let $(X_n)_n$ be an increasing sequence of finite-dimensional subspaces of E whose union is dense in E, and let d be the metric on $\mathrm{Iso}(E)$ defined by $d(g,h) := \sum_n 2^{-n-1}\|g\restriction X_n - h\restriction X_n\|$. Observe that d is a left-invariant compatible metric on $\mathrm{Iso}(E)$. In order to prove the extreme amenability of $\mathrm{Iso}(E)$ we prove (2) in Proposition 2.2 for the distance d, that is, that the left translation of $\mathrm{Iso}(E)$ on $(\mathrm{Iso}(E),d)$ is finitely oscillation stable. We fix a 1-Lipschitz mapping $c : \mathrm{Iso}(E) \to [0,1]$, a finite subset $F \subseteq \mathrm{Iso}(E)$ and $\varepsilon > 0$. Let n be such that $2^{n-2}\varepsilon \ge 1$ and let $Y \subseteq E$ be a finite-dimensional subspace of E such that $X_n \cup \bigcup_{g\in F} g(X_n) \subseteq Y$. Let $Y \subseteq Z \subseteq E$ be a finite-dimensional space witnessing the (ARP) of $\mathrm{Age}(E)$ for the parameters X_n, Y and $\varepsilon/8$. For each $\gamma \in \mathrm{Emb}(X_n,Z)$ we choose $g_\gamma \in \mathrm{Iso}(E)$ such that $\|\gamma - g_\gamma\restriction X_n\| \le \varepsilon/8$, and now we define the (discrete) coloring $\widehat{c} : \mathrm{Emb}(X_n,Z) \to \{1,\dots,2^{n+1}\}$ by $\widehat{c}(\gamma) := j$ when j is the first integer i such that $c(g_\gamma) \in J_i$, where $J_i := [(i-1)/2^{n+1}, i/2^{n+1}]$. There are $\xi \in \mathrm{Emb}(Y,Z)$ and $j \in \{1,\dots,2^{n+1}\}$ such that $\xi \circ \mathrm{Emb}(X_n,Y) \subseteq (\widehat{c}^{-1}(j))_{\varepsilon/8}$. Choose $h \in \mathrm{Iso}(E)$ such that $\|\xi - h\restriction Y\| \le \varepsilon/16$. We claim that $\mathrm{Osc}(c\restriction h\cdot F) \le \varepsilon$: given $g_0, g_1 \in F$, there are $f_0, f_1 \in \mathrm{Iso}(E)$ such that $(j-1)/2^{n+1} \le c(f_0), c(f_1) \le j/2^{n+1}$ and such that $\|\xi\circ g_0\restriction X_n - f_0\restriction X_n\|, \|\xi\circ g_1\restriction X_n - f_1\restriction X_n\| \le \varepsilon/4$. Hence $d(h\circ g_0, f_0), d(h\circ g_1, f_1) \le 7\varepsilon/16$, and since c is 1-Lipschitz,

$$|c(h\circ g_0) - c(h\circ g_1)| \le d(h\circ g_0, f_0) + |c(f_0) - c(f_1)| + d(h\circ g_1, f_1) \le \varepsilon.$$

(2) and (3) are equivalent by Claim 2.13.2, under the hypothesis that E is approximately ultrahomogeneous.

Suppose that $\mathcal{F}$ is a family such that $\mathcal{F} \preceq \mathrm{Age}(E)$, $E \in [\mathcal{F}]$ and suppose that it satisfies the stable amalgamation property. We suppose first that (2) holds, that is, $\mathrm{Age}(E)$ has the (ARP), and we prove (4): By Propositions 2.13 and 2.7, it suffices to show that $\mathcal{F}$ satisfies the discrete (ARP). Fix X, Y in $\mathcal{F}$, $r \in \mathbb{N}$, and $\varepsilon > 0$. We know by the hypothesis and Proposition 2.7 that $\mathrm{Age}(E)$ satisfies the discrete (ARP). Thus, we can find $Z_0 \in \mathrm{Age}(E)$ containing a copy of Y and such that every r-coloring of $\mathrm{Emb}(X,Z_0)$ has an ε-monochromatic subset of the form $\gamma \circ \mathrm{Emb}(X,Y)$ for some $\gamma \in \mathrm{Emb}(Y,Z_0)$.

Let $\delta \le \varepsilon$ be such that $\varpi(\delta) < \varepsilon$. Pick $Z_1 \in \mathcal{F}$ for which there exists an δ-embedding $\theta : Z_0 \to Z_1$. By the (SAP) of $\mathcal{F}$ we can find $Z \in \mathcal{F}$ and $I \in \mathrm{Emb}(Z_1, Z)$ such that for every $\phi \in \mathrm{Emb}_\delta(X, Z_1)$ there is $\bar{\phi} \in \mathrm{Emb}(X, Z)$ such that $\|I \circ \phi - \bar{\phi}\| \le \varepsilon$, and similarly for the elements of $\mathrm{Emb}_\delta(Y, Z_1)$.

We claim that Z witnesses the discrete (ARP) for the given X, Y, ε, r. Fix a coloring $c : \mathrm{Emb}(X, Z) \to r$. Define $b : \mathrm{Emb}(X, Z_0) \to r$ by choosing for each $\phi \in \mathrm{Emb}(X, Z_0)$ an element $\bar{\phi} \in \mathrm{Emb}(X, Z)$ such that $\|I \circ \theta \circ \phi - \bar{\phi}\| \le \varepsilon$ and declaring $b(\phi) := c(\bar{\phi})$. By the choice of Z_0 from the discrete (ARP) of $\mathrm{Age}(E)$, there exist $\alpha \in \mathrm{Emb}(Y, Z_0)$ and $j < r$ such that $\alpha \circ \mathrm{Emb}(X, Y) \subseteq (b^{-1}(j))_\varepsilon$. Let $\bar{\alpha} \in \mathrm{Emb}(Y, Z)$ be such that $\|I \circ \theta \circ \alpha - \bar{\alpha}\| \le \varepsilon$. We claim that $\bar{\alpha} \circ \mathrm{Emb}(X, Y) \subseteq (c^{-1}(j))_{3\varepsilon}$: Fix $\phi \in \mathrm{Emb}(X, Y)$. Let $\sigma \in \mathrm{Emb}(X, Z_0)$ be such that $b(\sigma) = j$ and $d_{\mathrm{cb}}(\alpha \circ \phi, \sigma) \le \varepsilon$. By definition, we can find $\bar{\sigma} \in \mathrm{Emb}(X, Z)$ such that $c(\bar{\sigma}) = j$ and $\|I \circ \theta \circ \sigma - \bar{\sigma}\| \le \varepsilon$. Then

$$\|\bar{\alpha} \circ \phi - \bar{\sigma}\| \le \|\bar{\alpha} \circ \phi - I \circ \theta \circ \alpha \circ \phi\| + \|I \circ \theta \circ \alpha \circ \phi - I \circ \theta \circ \sigma\| + \|I \circ \theta \circ \sigma - \bar{\sigma}\| \le 3\varepsilon.$$

Finally, suppose that (4) holds, that is, $\mathcal{F}$ has the stable approximate Ramsey property with modulus ϖ, and let us prove (3): Let $\mathcal{F}_E$ be the collection of subspaces of E that are isometric to some element of $\mathcal{F}$. Obviously, $\mathcal{F}_E$ also has the (ARP). Fix $X, Y \in \mathrm{Age}(E)$ and $\varepsilon > 0$. We consider $0 < \delta \le 1$ such that $\varpi(\delta) < \varepsilon$ and $X_0 \in \mathcal{F}_E$ such that there is $\theta \in \mathrm{Emb}_\delta(X, X_0)$. Choose also a finite ε-dense subset D of $\mathrm{Emb}(X, Y)$, and for each $\gamma \in D$ some $g_\gamma \in \mathrm{Iso}(E)$ such that $\|g_\gamma \restriction X - \gamma\| \le \varepsilon$. Let now $X_1 \in \mathcal{F}_E$ be such that for every $\gamma \in D$ there is $\eta \in \mathrm{Emb}_\delta(X_0, X_1)$ such that $\|g_\gamma \restriction X_0 - \eta\| \le \varepsilon$. Let $Y_0 \in \mathcal{F}_E$ and $\iota \in \mathrm{Emb}(X_1, Y_0)$ be such that $\iota \circ \mathrm{Emb}_\delta(X_0, X_1) \subseteq (\mathrm{Emb}(X_0, Y_0))_\varepsilon$. We now apply the (ARP) of $\mathcal{F}_E$ to X_0, Y_0 and $\varepsilon/2$ to find the corresponding $Z \in \mathcal{F}_E$. Fix a continuous coloring $c : \mathrm{Emb}(X, E) \to [0, 1]$, and we define a continuous coloring $e : \mathrm{Emb}(X_0, Z) \to [0, 1]$ as follows: Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Given $\gamma \in \mathrm{Emb}(X_0, Z)$ we choose a sequence $(g_n)_n$ in $\mathrm{Iso}(E)$ such that $\|g_n \restriction X_0 - \gamma\| \le 1/2^n$. Let $e(\gamma) := \mathcal{U}\text{-}\lim (c(g_n \restriction X))$. It is easy to see that e is $(1 + \delta)$-Lipschitz. There is some $\gamma \in \mathrm{Emb}(Y_0, Z)$ such that $\mathrm{Osc}((e/(1 + \delta)) \restriction \gamma \circ \mathrm{Emb}_\delta(X_0, Y_0)) \le \varepsilon/2$, hence $\mathrm{Osc}(e \restriction \gamma \circ \mathrm{Emb}_\delta(X_0, Y_0)) \le \varepsilon$. Let $h \in \mathrm{Iso}(E)$ be such that $\|h \restriction X_1 - \gamma \circ \iota\| \le \varepsilon$.

We claim that $\mathrm{Osc}(h \circ \mathrm{Emb}(X, Y)) \le 23\varepsilon$: Fix $\gamma_0, \gamma_1 \in D$. Then $\|g_{\gamma_j} \restriction X - \gamma_j\| \le \varepsilon$ for $j = 0, 1$. Choose $\eta_0, \eta_1 \in \mathrm{Emb}_\delta(X_0, X_1)$ such that $\|g_{\gamma_j} \restriction X_0 - \eta_j\| \le \varepsilon$ for $j = 0, 1$. Choose $\xi_0, \xi_1 \in \mathrm{Emb}(X_0, Y_0)$ such that $\|\xi_j - \iota \circ \eta_j\| \le \varepsilon$, $j = 0, 1$. Then $|e(\gamma \circ \xi_0) - e(\gamma \circ \xi_1)| \le \varepsilon$. Choose $f_0, f_1 \in \mathrm{Iso}(E)$ such that $|e(\gamma \circ \xi_j) - c(f_j \restriction X)| \le \varepsilon$ and $\|f_j \restriction X_0 - \gamma \circ \xi_j\| \le \varepsilon$ for $j = 0, 1$. Then

$$\begin{aligned}
|c(h \circ \gamma_0) - c(h \circ \gamma_1)| &\le \|h \circ \gamma_0 - f_0 \restriction X\| + \|h \circ \gamma_1 - f_1 \restriction X\| + 3\varepsilon \\
&\le \|f_0 \restriction X - h \circ g_{\gamma_0} \restriction X\| + \|f_1 \restriction X - h \circ g_{\gamma_1} \restriction X\| + 5\varepsilon \\
&\le (1 + \delta)(\|f_0 \restriction X_0 - h \circ g_{\gamma_0} \restriction X_0\| + \|f_1 \restriction X_0 - h \circ g_{\gamma_1} \restriction X_0\|) + 5\varepsilon \\
&\le (1 + \delta)(\|f_0 \restriction X_0 - \gamma \circ \xi_0\| + \|f_1 \restriction X_0 - \gamma \circ \xi_1\| + 6\varepsilon) + 5\varepsilon \le 21\varepsilon. \quad \blacksquare
\end{aligned}$$

Since D is ε-dense, it follows from the previous inequality that $\mathrm{Osc}(h \circ \mathrm{Emb}(X, Y)) \le 23\varepsilon$.

2.3. *The approximate Ramsey property of* $\{\ell_\infty^n\}_n$

The content of this part is the proof of the approximate Ramsey property of the family $\{\ell_\infty^n\}_n$, and consequently of the class of all finite-dimensional Banach spaces, over $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Our proof is based on the *Dual Ramsey Theorem* (DRT) of R. L. Graham and B. L. Rothschild [26]. For convenience, we present its formulation in terms of rigid surjections between finite linear orderings. Given two linear orderings $(R, <_R)$ and $(S, <_S)$, a surjective map $f : R \to S$ is called a *rigid surjection* when $\min_R f^{-1}(s_0) < \min_R f^{-1}(s_1)$ for all $s_0, s_1 \in S$ such that $s_0 <_S s_1$. Let $\mathrm{Epi}(R, S)$ be the collection of rigid surjections from R to S.

Theorem 2.14 ((DRT) [26]). *For any finite linear orderings R and S such that $|R| < |S|$ and every $r \in \mathbb{N}$ there exists an integer $n > |S|$ such that, considering n naturally ordered, every r-coloring of* $\mathrm{Epi}(n, R)$ *has a monochromatic set of the form* $\mathrm{Epi}(S, R) \circ \gamma := \{\sigma \circ \gamma : \sigma \in \mathrm{Epi}(S, R)\}$ *for some* $\gamma \in \mathrm{Epi}(n, S)$.

We prove the following.

Theorem 2.15. *The class $\{\ell_\infty^n\}_{n\in\mathbb{N}}$ satisfies the (SRP) with modulus $\varpi(\delta) = \delta$.*

The KPT correspondence in Theorem 2.12 and Proposition 2.13 yield the announced result and a corollary.

Theorem 2.5. *The group of isometries of the Gurarij space endowed with the strong operator topology is extremely amenable.* ∎

Corollary 2.16. *The class of finite-dimensional Banach spaces satisfies the (SRP) with modulus $\varpi(\delta) = \delta$.*

We will give a direct proof of the (ARP) of the class of all finite-dimensional Banach spaces later. Coming back to Theorem 2.15, by means of Proposition 2.13 we need to prove that $\{\ell_\infty^n\}$ satisfies the stable amalgamation property with modulus δ, and that it has the (ARP). Observe that a linear map $\gamma : \ell_\infty^d \to \ell_\infty^n$ is a δ-isometric embedding if and only if its dual operator $\gamma^* : \ell_1^n \to \ell_1^d$ satisfies $\gamma^*(\mathrm{Ball}(\ell_1^n)) \subseteq \mathrm{Ball}(\ell_1^d) \subseteq \gamma^*((1+\delta)\,\mathrm{Ball}(\ell_1^n))$. When $\delta = 1$ such an operator $\sigma : \ell_1^n \to \ell_1^d$ satisfying $\sigma(\mathrm{Ball}(\ell_1^n)) = \mathrm{Ball}(\ell_1^d)$ is called a *quotient map*. A simple argument using extreme points shows that this is equivalent to saying that $\{u_j\}_{j<d} \subseteq S^1(\mathbb{F}) \cdot \{\sigma(u_j)\}_{j<n}$, where $S^1(\mathbb{F}) := \{a \in \mathbb{F} \ : \ |a| = 1\}$, and where u_j is the j^{th} unit vector whose only non-zero coordinate is 1 and it is on the j^{th} position. Let $\mathrm{Quo}(\ell_1^n, \ell_1^d)$ be the metric space of quotients. Finally, observe that the dual functor $\mathrm{Emb}(\ell_\infty^d, \ell_\infty^n) \ni \gamma \mapsto \gamma^* \in \mathrm{Quo}(\ell_1^n, \ell_1^d)$ is an isometric bijection. This means that the (ARP) of $\{\ell_\infty^n\}_n$ is equivalent to the assertion of the following lemma.

Lemma 2.17. *For any $d, m \in \mathbb{N}$ and $\varepsilon > 0$ there is some $n \in \mathbb{N}$ such that every continuous coloring of* $\mathrm{Quo}(\ell_1^n, \ell_1^d)$ *ε-stabilizes on* $\mathrm{Quo}(\ell_1^m, \ell_1^d) \circ \varrho$ *for some* $\varrho \in \mathrm{Quo}(\ell_1^n, \ell_1^m)$.

Lemma 2.17 will be proved later using the Dual Ramsey Theorem.

Proposition 2.18. *$\{\ell_\infty^n\}_n$ is a stable amalgamation class with modulus δ.*

Proof. Suppose that $\gamma : \ell_\infty^d \to \ell_\infty^m$ and $\eta : \ell_\infty^d \to \ell_\infty^n$ are δ-isometric embeddings. This means that the dual operators $\gamma^* : \ell_1^m \to \ell_1^d$ and $\eta^* : \ell_1^n \to \ell_1^d$ satisfy $\gamma^*(\mathrm{Ball}(\ell_1^m)) \subseteq \mathrm{Ball}(\ell_1^d) \subseteq \gamma^*((1+\delta)\,\mathrm{Ball}(\ell_1^m))$, and $\eta^*(\mathrm{Ball}(\ell_1^n)) \subseteq \mathrm{Ball}(\ell_1^d) \subseteq \eta^*((1+\delta)\,\mathrm{Ball}(\ell_1^n))$. We define $\sigma : \ell_1^{m+n} \to \ell_1^m$ and $\tau : \ell_1^{m+n} \to \ell_1^n$ as follows. For each $j < m$, choose $y_j \in \ell_1^n$ with $1 \le \|y_j\| \le 1+\delta$ such that $\eta^*(y_j) = \gamma^*(u_j)$, and for $k < n$ choose $x_k \in \ell_1^m$ with $1 \le \|x_k\| \le 1+\delta$ such that $\gamma^*(x_k) = \eta^*(u_k)$ Now for each $j < m$, let $\sigma(u_j) := u_j$ and $\tau(u_j) := y_j/\|y_j\|$, and for $k < n$, let $\sigma(u_{m+k}) := x_k/\|x_k\|$ and $\tau(u_{m+k}) := u_k$. Then clearly we have $\sigma(\mathrm{Ball}(\ell_1^{m+n})) = \mathrm{Ball}(\ell_1^m)$ and $\tau(\mathrm{Ball}(\ell_1^{m+1})) = \mathrm{Ball}(\ell_1^n)$ and $\|\gamma^* \circ \sigma - \eta^* \circ \tau\|_{\ell_1^{m+n},\ell_1^d} \le \delta$. ■

Our proof of the (ARP) of $\{\ell_\infty^n\}_n$ crucially uses the Dual Ramsey Theorem. The case $d = 1$ was first proved by Gowers [24], indirectly, as it follows easily via a compactness argument from the oscillation stability of the space c_0. We start by presenting a simple proof of this result for *positive* embeddings in the real case. Given integers k and n, let $\mathrm{FIN}_k(n)$ be the collection of all mappings from n into $k+1 = \{0,1,\dots,k-1,k\}$ with range including k. Let $T : \mathrm{FIN}_k(n) \to \mathrm{FIN}_{k-1}(n)$ be the *tetris* operation defined pointwise for $f \in \mathrm{FIN}_k(n)$ by $T(f)(i) := \max\{f(i)-1, 0\}$. Given disjointly supported $f_0,\dots,f_{l-1}$ in $\mathrm{FIN}_j(n)$, the combinatorial space $\langle f_i\rangle_{i<l}$ is the collection of all combinations $\sum_{i<l} T^{k-j_i}(f_i)$ where $(j_i)_{i<l} \in \mathrm{FIN}_k(l)$.

Proposition 2.19 (Gowers). *For any k, m and every r there is some n such that every r-coloring of $\mathrm{FIN}_k(n)$ has a monochromatic set of the form $\langle f_i\rangle_{i<m}$ for some disjointly supported sequence $(f_i)_{i<m}$ in $\mathrm{FIN}_k(n)$.*

In what follows, let $\mathbf{GR}(d,m,r)$ be the minimal n so that (DRT) holds for the parameters d, m and r.

Proof of Proposition 2.19. Fix k, m and r. We claim that $n = \mathbf{GR}(k+1, km+1, r)$ works. Fix an r-coloring c of $\mathrm{FIN}_k(n)$. We consider $k+1$, $mk+1$, and n canonically ordered. For a subset A of n, we let $\mathbb{1}_A$ be the indicator function of A. Let $\Phi : \mathrm{Epi}(n, k+1) \to \mathrm{FIN}_k(n)$ be defined by $\Phi(\sigma) := \sum_{i\le k} i \cdot \mathbb{1}_{\sigma^{-1}(i)}$. By the Ramsey property of n there is some rigid surjection $\varrho : n \to mk+1$ such that $c \circ \Phi$ is constant on $\mathrm{Epi}(mk+1, k+1) \circ \varrho$ with value $\hat{r}$. For each $j < m$, let

$$f_j := \sum_{1\le i\le k} i \cdot \mathbb{1}_{\varrho^{-1}(jk+i)}.$$

Then c is constant on $\langle f_j\rangle_{j<m}$. To see this, given $f = \sum_{l<m} T^{k-j_l} f_l \in \langle f_j\rangle_{j<m}$ we define $\sigma : mk+1 \to k+1$ by $\sigma(0) := 0$ and $\sigma(lk+i) := \max\{i-k+j_l, 0\}$ for $l < m$ and $1 \le i \le k$. Then for $0 < i_0$ one has $\min \sigma^{-1}(i_0) = kl_0 + (i_0 + k - j_{l_0})$ where $l_0 = \min\{l < m : i_0 \le j_l\}$, so σ is a rigid surjection. It is not difficult to see that $\Phi(\sigma \circ \varrho) = f$, so $c(f) = \hat{r}$. ■

Proof of Lemma 2.17. We start from the following simple fact.

Claim 2.19.1. *There is a finite ε-dense subset $\mathcal{D}$ of $\mathrm{Ball}(\ell_1^d)$ containing $\{u_j\}_{j<d}$ such that for every non-zero $x \in \mathrm{Ball}(\ell_1^d)$ there is $y \in \mathcal{D}$ with $\|y-x\|_1 \le \varepsilon$ and $\|y\|_1 < \|x\|_1$.*

Proof of Claim. Let D be a finite $\varepsilon/2$-dense subset of the unit sphere of ℓ_1^d containing $\{u_j\}_{j<d}$, and let $0 = \lambda_0 < \cdots < \lambda_{p-1} = 1$ be such that $\max_{0\le j\le p-2}(\lambda_{j+1} - \lambda_j) \le \varepsilon/2$. Then $\mathcal{D} := \bigcup_{k<p} \lambda_k \cdot D$ satisfies what we want. ∎

Fix such an ε-dense set $\mathcal{D}$, and let $\prec$ be any linear ordering of $\mathcal{D}$ such that if $\|x\|_1 < \|y\|_1$ then $x \prec y$. Let $\mathrm{emb}(d, m)$ be the collection of all 1-1 mappings $f : d \to m$, and let S be a finite ε-dense subset of $S^1(\mathbb{F})$. For each $(f, \theta) \in \mathrm{emb}(d, m) \times S^d$, let $h_{f,\theta} : \ell_1^d \to \ell_1^m$ be the linear map obtained by setting $h_{f,\theta}(u_j) := \theta_j \cdot u_{f(j)}$. Then clearly $h_{f,\theta}$ is an isometric embedding from ℓ_1^d into ℓ_1^m.

Claim 2.19.2. *For every $T \in \mathrm{Quo}(\ell_1^m, \ell_1^d)$ there is a pair $(f, \theta) \in \mathrm{emb}(d, m) \times S^d$ such that $\|T \circ h_{f,\theta} - \mathrm{Id}_{\ell_1^d}\|_{\ell_1^d,\ell_1^d} \le \varepsilon$.*

Proof of Claim. For each $k < d$ choose $f(k) < m$ such that $T(u_{f(k)}) = a_k u_k$ where $|a_k| = 1$. Clearly $d > k \mapsto f(k)$ is an injection from d into m. For each $k < d$, let $\theta_k \in S$ be such that $|1/a_k - \theta_k| \le \varepsilon$, and let $\theta := (\theta_k)_k$. Then

$$\begin{aligned}\|T \circ h_{f,\theta} - \mathrm{Id}_{\ell_1^d}\|_{\ell_1^d,\ell_1^d} &= \max_{k<d} \|T \circ h_{f,\theta}(u_k) - u_k\|_1 = \max_{k<d} \|T(\theta_k u_{f(k)}) - u_k\|_1 \\ &= \max_{k<d} \|a_k\theta_k u_k - u_k\|_1 \le \max_{k<d} |a_k\theta_k - 1| \le \varepsilon.\end{aligned}$$ ∎

Let $\Delta := \mathcal{D} \times \mathrm{emb}(d, m) \times S^d$ be ordered by the lexicographical ordering induced from $\mathcal{D}$ ordered by $\prec$, and $\mathrm{emb}(d, m) \times S^d$ ordered arbitrarily. We claim that $n := \mathbf{GR}(|\mathcal{D}|, |\Delta|, r)$ works. Indeed, let c be an r-coloring of $\mathrm{Quo}(\ell_1^n, \ell_1^d)$. We define an injection $\Phi : \mathrm{Epi}(n, \mathcal{D}) \to \mathrm{Quo}(\ell_1^n, \ell_1^d)$ by assigning to each $\sigma \in \mathrm{Epi}(n, \mathcal{D})$ the operator $T := \Phi(\sigma) : \ell_1^n \to \ell_1^d$ such that for each $\xi < n$ one has $T(u_\xi) := \sigma(\xi)$. Equivalently the ξ^{th} column vector of the matrix corresponding to $\Phi(\sigma)$ in the respective unit bases is $\sigma(\xi)$. It is easily verified that T is always a quotient map. It follows by the Dual Ramsey Theorem applied to the coloring $\hat{c} := c \circ \Phi$ that there is $\gamma_0 \in \mathrm{Epi}(n, \Delta)$ such that

$$\hat{c} \text{ is constant on } \mathrm{Epi}(\Delta, \mathcal{D}) \circ \gamma_0 \text{ with value } r_0 < r.$$

Let $R \in \mathrm{Quo}(\ell_1^n, \ell_1^m)$ be the quotient such that, for every $\xi < n$, one has $R(u_\xi) = h_{f,\theta}(v)$, where $(v, f, \theta) = \gamma_0(\xi)$. The proof is finished once we establish the following.

Claim 2.19.3. *For every $T \in \mathrm{Quo}(\ell_1^m, \ell_1^d)$ there exists $\phi \in \mathrm{Epi}(\Delta, \mathcal{D})$ such that*

$$\|\Phi(\phi \circ \gamma_0) - T \circ R\|_{\ell_1^n,\ell_1^d} \le \varepsilon.$$

Proof of Claim. Fix $T \in \mathrm{Quo}(\ell_1^m, \ell_1^d)$, and use Claim 2.19.2 to choose $(\bar{f}, \bar{\theta})$ in $\mathrm{emb}(d, m) \times S$ such that $\|T \circ h_{\bar{f},\bar{\theta}} - \mathrm{Id}_{\ell_1^d}\|_{\ell_1^d,\ell_1^d} \le \varepsilon$. Now we define $\phi : \Delta \to \mathcal{D}$ as follows. Fix $(v, f, \theta) \in \Delta$.

(i) If $T(h_{f,\theta}(v)) = 0$, then we set $\phi(v, f, \theta) := 0$.

(ii) Suppose that $T(h_{f,\theta}(v)) \ne 0$; if $(f, \theta) = (\bar{f}, \bar{\theta})$, then we set $\phi(v, f, \theta) := v$; otherwise, we set $\phi(v, f, \theta) := w$ where $w \in \mathcal{D}$ is such that $\|T(h_{f,\theta}(v)) - w\|_{\ell_1} \le \varepsilon$ and $\|w\|_1 < \|T(h_{f,\theta}(v))\|_1$.

We see that $\phi : \Delta \to D$ is a rigid surjection. First, $\min \phi^{-1}(0) = (0, f_0, \theta_0)$, where (f_0, θ_0) is the minimum of $\mathrm{emb}(d, m) \times S^d$. Now suppose that $v \in \mathcal{D}$ is a non-zero vector. We prove that $\min \phi^{-1}(v) = (v, \bar{f}, \bar{\theta})$: Suppose that $\phi(u, f, \theta) = v$ and $(f, \theta) \neq (\bar{f}, \bar{\theta})$. By the definition of ϕ, $\|v\|_1 < \|T(h_{f,\theta}(u))\|_1 \leq \|T\|_{\ell_1^m, \ell_1^d} \cdot \|h_{f,\theta}(u)\|_1 \leq \|u\|_1$, because T is a contraction and $h_{f,\theta}$ is an isometric embedding. Hence, $v \prec u$, and since in Δ we are considering the lexicographic ordering, $(v, \bar{f}, \bar{\theta}) < (u, f, \theta)$. Since obviously $\phi(v, \bar{f}, \bar{\theta}) = v$, we find that $\min \phi^{-1}(v) = (v, \bar{f}, \bar{\theta})$. Hence, if $0 \neq v < w$, then $\min \phi^{-1}(v) < \min \phi^{-1}(w)$.

Finally, we estimate $\|\Phi(\phi \circ \gamma_0) - T \circ R\|_{\ell_1^n, \ell_1^d} = \max_{\xi < n} \|\Phi(\phi \circ \gamma_0)(u_\xi) - T(R(u_\xi))\|_{\ell_1^d}$. Fix $\xi < n$, and suppose that $\gamma_0(\xi) = (v, f, \theta)$. Then by definition, $(\Phi(\phi \circ \gamma_0))(u_\xi) = \phi(\gamma_0(\xi))$, and $T(R(u_\xi)) = T(h_{f,\theta}(v))$. Now we have:

(a) If $T(h_{f,\theta}(v)) = 0$, then $0 = \phi(v, f, \theta) = \Phi(\phi \circ \gamma_0)(u_\xi)$ and $0 = T(h_{f,\theta}(v)) = T(R(u_\xi))$.

(b) If $T(h_{f,\theta}(v)) \neq 0$ and $(f, \theta) = (\bar{f}, \bar{\theta})$, then $\Phi(\phi \circ \gamma_0)(u_\xi) = \phi(v, f, \theta) = v$ while $T(R(u_\xi)) = T(h_{f,\theta}(v)) = w$ is such that $\|w - v\|_1 \leq \varepsilon$.

(c) If $T(h_{f,\theta}(v)) \neq 0$ and $(f, \theta) \neq (\bar{f}, \bar{\theta})$, then $\Phi(\phi \circ \gamma_0)(u_\xi) = \phi(v, f, \theta) = w$ is chosen such that $\varepsilon \geq \|w - T(h_{f,\theta}(v))\|_1 = \|w - T(R(u_\xi))\|_1$. ∎

2.4. *(ARP) of polyhedral spaces and finite-dimensional spaces*

We give an explicit proof of the approximate Ramsey property of the class of finite-dimensional polyhedral spaces. This is done by using injective envelopes of polyhedral spaces, and then by reducing colorings of polyhedral spaces to colorings of ℓ_∞^n-spaces. We also use this to explicitly prove the (ARP) of the class of all finite-dimensional Banach spaces. In this way, knowing the number of extreme points of the dual unit ball of given spaces, one can estimate upper bounds of the corresponding Ramsey numbers. For simplicity, we present the proof in the case of real Banach spaces. Thus, all the Banach spaces are assumed to be real in this section.

Definition 2.20. A finite-dimensional space F is called *polyhedral* when its unit ball $\mathrm{Ball}(F)$ is a polyhedron, i.e., the set $\partial_e(\mathrm{Ball}(F))$ of *extreme* points of $\mathrm{Ball}(F)$ is finite.

The spaces ℓ_∞^n and ℓ_1^n are polyhedral. In fact, a finite-dimensional space is polyhedral if and only if its dual ball is polyhedral. It follows from this, a separation argument, and the Milman theorem, that a finite-dimensional space F is polyhedral if and only if there is a finite set $A \subseteq \mathrm{Sph}(F^*)$ such that $\|x\| = \max_{f \in A} f(x)$ for every $x \in F$. Also, every subspace of a polyhedral space is polyhedral, and every finite-dimensional polyhedral space embeds into ℓ_∞^n for some $n \in \mathbb{N}$.

Definition 2.21 (Polyhedral spaces). Given an integer d, let Pol_d be the class of all polyhedral spaces F such that $\#\partial_e(B_{F^*}) = 2d$. Given $d, m \in \mathbb{N}$, $r \in \mathbb{N}$ and $\varepsilon > 0$, let $\mathbf{n_{pol}}(d, m, r, \varepsilon)$ be the minimal integer $n \geq m$ such that for every $F \in \mathrm{Pol}_d$ and $G \in \mathrm{Pol}_m$, every r-coloring of $\mathrm{Emb}(F, \ell_\infty^n)$ has an ε-monochromatic set of the form $T \circ \mathrm{Emb}(F, G)$ for some $T \in \mathrm{Emb}(G, \ell_\infty^n)$.

Definition 2.22 (Injective envelope of a polyhedral space). The *injective envelope* of a polyhedral space F is a pair (n_F, Ψ_F), where n_F is an integer and $\Psi_F \in \mathrm{Emb}(F, \ell_\infty^{n_F})$ is such that for every isometric embedding $T : F \to \ell_\infty^n$ there is an isometric embedding $U : \ell_\infty^{n_F} \to \ell_\infty^n$ such that $T = U \circ \Psi_F$.

Proposition 2.23. $\mathbf{n_{pol}}(d, m, r, \varepsilon) = \mathbf{n}_\infty(d, m, r, \varepsilon)$.

Proof. First of all, $\ell_\infty^k \in \mathrm{Pol}_k$, so $\mathbf{n_{pol}}(d, m, r, \varepsilon) \geq \mathbf{n}_\infty(d, m, r, \varepsilon)$. Fix now an r-coloring c of $\mathrm{Emb}(F, \ell_\infty^n)$. Let $\widehat{c} : \mathrm{Emb}(\ell_\infty^d, \ell_\infty^n) \to r$ be defined for $U \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^n)$ by $\widehat{c}(U) := c(U \circ \Psi_F)$. Let $\widehat{T} \in \mathrm{Emb}(\ell_\infty^m, \ell_\infty^n)$ and $\widehat{r} < r$ be such that

$$\widehat{T} \circ \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m) \subseteq (\widehat{c}^{-1}\{\widehat{r}\})_\varepsilon. \tag{2.2}$$

Let $T := \widehat{T} \circ \Psi_G$. We claim that $T \circ \mathrm{Emb}(F, G) \subseteq (c^{-1}\{\widehat{r}\})_\varepsilon$. Let $U \in \mathrm{Emb}(F, G)$, and let $W \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)$ be such that $\Psi_G \circ U = W \circ \Psi_F$. From the inclusion in (2.2) there exists $V \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)$ such that $\widehat{c}(V) = \widehat{r}$ and $\|V - \widehat{T} \circ W\| < \varepsilon$. Let $\widehat{V} := V \circ \Psi_F$. Then $c(V \circ \Psi_F) = \widehat{c}(V) = \widehat{r}$, while

$$\begin{aligned}\|\widehat{V} - T \circ U\| &= \|V \circ \Psi_F - \widehat{T} \circ \Psi_G \circ U\| = \|V \circ \Psi_F - \widehat{T} \circ W \circ \Psi_F\| \\ &\leq \|V - \widehat{T} \circ W\| < \varepsilon.\end{aligned}$$

∎

2.4.1. Approximate Ramsey property for finite-dimensional normed spaces. We give an explicit, constructive proof of approximate Ramsey property of arbitrary finite-dimensional normed spaces. The proof is based on the approximate Ramsey property of polyhedral spaces and the well known fact that the finite-dimensional polyhedral spaces are dense in the class of finite-dimensional normed spaces with respect to the Banach–Mazur distance. In fact, we have the following.

Proposition 2.24. *Suppose that* $\dim X = k$. *For every* $0 < \varepsilon < 1$ *there is a polyhedral space* $X_0 \in \mathrm{Pol}_d$ *such that* $d_{\mathrm{BM}}(X, X_0) \leq \varepsilon$, *where* $d \leq ((2 + 3\varepsilon)/\varepsilon)^k$.

Proof. Let $\delta := \varepsilon(1 + \varepsilon)^{-1}$. Let $D \subseteq \mathrm{Sph}(X^*)$ be a finite δ-dense subset of S_{X^*} of cardinality $\leq (1 + 2\delta^{-1})^k = ((2 + 3\varepsilon)/\varepsilon)^k$ (see for example [46, Lemma 2.6]). On X we define the polyhedral norm $N(x) := \max_{f \in D} |f(x)|$. It follows that $X_0 := (X, N) \in \mathrm{Pol}_d$ with $d \leq \#D$, and $d_{\mathrm{BM}}(X, X_0) \leq \varepsilon$. ∎

Definition 2.25. Given X of finite dimension and $\theta \geq 1$, let $\mathcal{E}\mathrm{mb}_\theta(X, Y)$ be the collection of all 1-1 mappings $T : X \to Y$ such that $1 \leq \|T\|, \|T^{-1}\|$ and $\|T\| \cdot \|T^{-1}\| \leq \theta$.

Let $\bar{X} = (X_i)_{i \leq n}$ be a sequence of Banach spaces. We say that a pair (Y, J) of a Banach space Y and $J \in \mathrm{Emb}(X_n, Y)$ is (θ, τ)-*correcting* for $\bar{X}$ $(1 < \theta < \tau)$ when every X_i isometrically embeds into Y, and for every $j < n$ and every $\gamma \in \mathcal{E}\mathrm{mb}_\theta(X_j, X_n)$ there exists $I_\gamma \in \mathrm{Emb}(X_j, Y)$ such that $\|J \circ \gamma - I_\gamma\| < \tau - 1$.

Proposition 2.26. *Every finite sequence of finite-dimensional spaces* $(X_i)_{i \leq n}$ *and every* $1 < \theta < \tau$ *has a* (θ, τ)-*correcting pair* (Y, J). *Moreover, if each* X_j *is polyhedral, then* Y *can be taken polyhedral.*

Proof. The proof is by induction on $n \geq 1$. Suppose first that $n = 1$. A simple inductive argument, where the case $\#\mathcal{N} = 1$ is proved by Kubiś and Solecki [33, Lemma 2.1], gives the following.

Claim 2.26.1. *Suppose that $\mathcal{N} \subseteq \mathcal{E}\mathrm{mb}_\theta(X_0, X_1)$ is finite. Then there exist a finite-dimensional space Y and $\Theta \in \mathrm{Emb}(X_1, Y)$ such that for every $T \in \mathcal{N}$ there is $I \in \mathrm{Emb}(X_0, Y)$ such that $\|I - \Theta \circ T\| < \theta - 1$.*

Let $\mathcal{N}$ be a finite $(\tau - \theta)$-net of $\mathcal{E}\mathrm{mb}_\theta(X_0, X_1)$. Then the pair (X, I) obtained by applying Claim 2.26.1 to $\mathcal{N}$ is (θ, τ)-correcting for (X_0, X_1). Now suppose that $n > 1$. Find a (θ, τ)-correcting pair (Y_0, Θ_0) for $(X_j)_{j=1}^n$. Let $\mathcal{N}$ be a finite $(\tau - \theta)$-net of $\mathcal{E}\mathrm{mb}_\theta(X_0, X_n)$. Let (Y, Θ_1) be a pair obtained by applying Claim 2.26.1 to $\Theta_0 \circ \mathcal{N}$. It can be easily verified that $(Y, \Theta_1 \circ \Theta_0)$ is a (θ, τ)-correcting pair for $(X_j)_{j \leq n}$. ∎

Theorem 2.27. *The class* FdBa *of all finite-dimensional Banach spaces has the* (*SRP*).

Proof. We know that FdBa is a stable Fraïssé class, so we only have to prove that it satisfies the discrete (ARP). Fix finite-dimensional spaces F, G, $r \in \mathbb{N}$, $\varepsilon > 0$, and set $\delta := \varepsilon/5$. Let $F_0 \in \mathrm{Pol}_d$, G_0 be polyhedral, and let $\Phi_F : F \to F_0$ and $\Phi_G : G \to G_0$ be surjective isomorphisms such that $\|\Phi_F\| = \|\Phi_G\| = 1$ and $\|\Phi_F^{-1}\|, \|\Phi_G^{-1}\| < 1 + \varepsilon/5$. Notice that d can be taken such that $d \leq ((10 + 3\varepsilon)/\varepsilon)^{\dim F}$. Let

(i) (H_0, Θ_0) be a $(1 + \varepsilon/5, 1 + \varepsilon/4)$-correcting pair for (F_0, G_0) with $H_0 \in \mathrm{Pol}_m$, and let

(ii) (H, Θ_1) be a $(1 + \varepsilon/5, 1 + \varepsilon/4)$-correcting pair for the triple (F, G, ℓ_∞^n) where $n := \mathbf{n}_{\mathrm{pol}}(d, m, r, \varepsilon/4)$.

We claim that H works. Fix $c : \mathrm{Emb}(F, H) \to r$. Let $\widehat{c} : \mathrm{Emb}(F_0, \ell_\infty^n) \to r$ be the induced coloring defined for $\gamma \in \mathrm{Emb}(F_0, \ell_\infty^n)$ by choosing $I_\gamma \in \mathrm{Emb}(F, H)$ such that $\|I_\gamma - \Theta_1 \circ \gamma \circ \Phi_F\| < \varepsilon/4$ and declaring $\widehat{c}(\gamma) := c(I_\gamma)$. By the Ramsey property of n, there exist $\varrho \in \mathrm{Emb}(H_0, \ell_\infty^n)$ and $\widehat{r} < r$ such that $\varrho \circ \mathrm{Emb}(F_0, H_0) \subseteq (\widehat{c}^{-1}\{\widehat{r}\})_{\varepsilon/4}$. Let $S \in \mathrm{Emb}(G, H)$ be such that

$$\|S - \Theta_1 \circ \varrho \circ \Theta_0 \circ \Phi_G\| < \varepsilon/4. \tag{2.3}$$

Claim 2.27.1. $S \circ \mathrm{Emb}(F, G) \subseteq (c^{-1}(\widehat{r}))_\varepsilon$.

Proof of Claim. Fix $T \in \mathrm{Emb}(F, G)$. We can choose $\tau \in \mathrm{Emb}(F_0, H_0)$ such that $\|\tau - \Theta_0 \circ \Phi_G \circ T \circ \Phi_F^{-1}\| < \varepsilon/4$, because $\Phi_G \circ T \circ \Phi_F^{-1} \in \mathcal{E}\mathrm{mb}_{1+\varepsilon/5}(F_0, G_0)$. Let now $\gamma \in \mathrm{Emb}(F_0, \ell_\infty^n)$ be such that $\widehat{c}(\gamma) = \widehat{r}$ and $\|\gamma - \varrho \circ \tau\| < \varepsilon/4$. Then $c(I_\gamma) = \widehat{r}$ and $\|S \circ T - I_\gamma\| < \varepsilon$. It follows from (2.3) and the fact that $\|T\| = 1$ that

$$\|S \circ T - \Theta_1 \circ \varrho \circ \Theta_0 \circ \Phi_G \circ T\| < \varepsilon/4.$$

This is the diagram:

$$\begin{tikzcd}
F \arrow[r, "T"] \arrow[d, "\Phi_F"'] \arrow[dr, phantom, "\circlearrowright"] \arrow[drrrr, bend right=70, "I_\gamma"'] & G \arrow[d, "\Phi_G"] \arrow[drrr, bend left, "S"] \arrow[drr, phantom, "\circlearrowright\, \varepsilon/4"] & & & \\
F_0 \arrow[r, "t"'] \arrow[rr, bend right=50, "\tau"'] \arrow[rrr, bend right=60, "\gamma"'] & G_0 \arrow[r, "\Theta_0"] & H_0 \arrow[r, "\varrho"] & \ell_\infty^n \arrow[r, "\Theta_1"] & H
\end{tikzcd}$$

Consequently, $\|S \circ T - I_\gamma\| < \varepsilon$. ∎

2.5. *Finite metric spaces*

Recall that the Urysohn space $\mathbb{U}$ is the unique (up to isometry) ultrahomogeneous universal separable complete metric space. Pestov proved in [50] that the group $\mathrm{Iso}(\mathbb{U})$ of surjective isometries of $\mathbb{U}$ is extremely amenable, using the method of concentration of measure. There is also a version of the (KPT) correspondence for $\mathrm{Iso}(\mathbb{U})$, which gives as a consequence the (ARP) of finite metric spaces.

Theorem 2.28. *For any finite metric spaces M and N, $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite metric space P such that every r-coloring* $\mathrm{emb}(M, P)$ *has an ε-monochromatic set of the form $\sigma \circ \mathrm{emb}(M, N)$ for some $\sigma \in \mathrm{emb}(N, P)$.*

In the previous statement $\mathrm{emb}(M, P)$ is the collection of all isometric embeddings from (M, d_M) into (N, d_N), endowed with the uniform metric $d(\sigma, \tau) := \max_{x \in M} d_N(\sigma(x), \tau(x))$. Later, Nešetřil established the (exact) Ramsey property of finite ordered metric spaces [47], that is, for any finite *ordered* metric spaces X and Y and every $r \in \mathbb{N}$ there exists a finite ordered metric space Z such that for every r-coloring of the set $\binom{Z}{X}_<$ of order isometric copies of X in Z there exists an order isometric copy Y_0 of Y in Z such that $\binom{Y_0}{X}_<$ is monochromatic. This gives another proof of the extreme amenability of $\mathrm{Iso}(\mathbb{U})$. We present here a third proof, which uses the approximate Ramsey property of the class of finite-dimensional polyhedral spaces.

Recall that a *pointed* metric space (X, d, p) is a metric space (X, d) with a distinguished point $p \in X$. Given two pointed metric spaces (M, p) and (N, q), let $\mathrm{emb}_0(M, N)$ be the set of pointed isometric embeddings, that is, all isometric embeddings from M into N sending p to q. Recall that when X and Y are normed spaces, we use $\mathrm{Emb}(X, Y)$ to denote *linear* isometric embeddings.

Definition 2.29. Given a pointed metric space (M, d, p), let $\mathrm{Lip}_0(M, p)$ be the Banach space of all Lipschitz maps $f : M \to \mathbb{R}$ with $f(p) = 0$ endowed with the *Lipschitz* norm,

$$\|f\| := \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} \;:\; x \neq y \in X\right\}.$$

Let $\mathcal{F}(M,p)$ be the (*Lipschitz*) *free space* over the pointed metric space (M,p) defined as the closed linear span of the *molecules* $\{\delta_x-\delta_p\}_{x\in M}$ in the dual space $\mathrm{Lip}_0(M,p)^*$, where δ_x for $x\in X$ denotes the *evaluation functional* at x. It turns out that $\mathcal{F}(M,p)^*$ is isometric to $\mathrm{Lip}_0(M,p)$.

It is well-known that $\mathrm{Lip}_0(M,p)$ does not depend, isometrically, on the choice of the point p, so the corresponding predual will be denoted by $\mathcal{F}(M)$. The space $\mathcal{F}(M)$ is also known as the *Arens–Eells space*. More information on Arens–Eells spaces can be found in [55, §2.2]. It is easy to see that the mapping $M\ni x\mapsto\delta_x\in\mathcal{F}(M)$ is an isometric embedding. Given finite metric spaces M and N such that M isometrically embeds into N, let $M_\infty:=M\cup\{p_\infty\}$, $N_\infty:=N\cup\{q_\infty\}$ be one-point extensions of M and N with the distance $d(p_\infty,x)=d(q_\infty,y):=\min_{z\neq t\in N}d(z,t)$ for all $z\in M$ and $y\in N$. Clearly M_∞ and N_∞ are metric spaces.

Proposition 2.30. *Suppose that M and N are metric spaces. Then every isometric embedding $\sigma:M\to N$ extends to a unique linear isometric embedding $T_\sigma:\mathcal{F}(M_\infty,p_\infty)\to\mathcal{F}(N_\infty,q_\infty)$.* ■

The proof is a straightforward use of a standard duality argument, the McShane–Whitney Extension Theorem for Lipschitz functions [55, Theorem 1.5.6], and the fact that $\delta_{p_\infty}=0$ in $\mathcal{F}(M_\infty,p_\infty)$ and $\delta_{q_\infty}=0$ in $\mathcal{F}(N_\infty,p_\infty)$.

Proposition 2.31. *If M is a finite metric space, then $\mathcal{F}(M)$ is a finite-dimensional polyhedral space.*

Proof. Observe that for all $x\neq y$ in M, $\mu_{x,y}:=(\delta_x-\delta_y)/d(x,y)$ has norm 1 in $\mathrm{Lip}_0(M)$ since clearly $\|\mu_{x,y}\|\le 1$, and the mapping $d_x(t):=d(x,t)$ for each $t\in M$ is 1-Lipschitz and $\mu_{x,y}(d_x)=1$. It follows from the definition of the Lipschitz norm that the convex hull of $\{\mu_{x,y}\}_{x\neq y\text{ in }M}$ is equal to $B_{\mathcal{F}(M)}$. ■

Lemma 2.32. *Suppose that M and N are finite metric spaces, $r\in\mathbb{N}$, and $\varepsilon>0$. Let $\varrho:=\mathrm{diam}(N)$. Then there exists $n\in\mathbb{N}$ such that every r-coloring of $\mathrm{emb}(M,\varrho\cdot B_{\ell_\infty^n})$ has an ε-monochromatic set of the form $\sigma\circ\mathrm{emb}(M,N)$ for some $\sigma\in\mathrm{emb}(N,\varrho\cdot B_{\ell_\infty^n})$.*

Proof. Fix finite pointed metric spaces (M,p), (N,q), r and $\varepsilon>0$. We assume that M isometrically embeds into N since otherwise the statement above is trivially true. Let d,m be such that $\mathcal{F}(M_\infty)\in\mathrm{Pol}_d$ and $\mathcal{F}(N_\infty)\in\mathrm{Pol}_m$. Then $n:=\mathbf{n}_{\mathrm{pol}}(d,m,r,\varepsilon_0)$ for $\varepsilon_0=\varepsilon/\mathrm{diam}(M)$ works. Fix a coloring $c:\mathrm{emb}(M,\varrho\cdot B_{\ell_\infty^n})\to r$. Define $\widehat{c}:\mathrm{Emb}(\mathcal{F}(M_\infty),\ell_\infty^n)\to r$ by $\widehat{c}(\gamma):=c(\sigma_\gamma)$, where $\sigma_\gamma:M\to\varrho\mathrm{Ball}(\ell_\infty^n)$ is defined by $\sigma_\gamma(x):=\gamma(\delta_x)$ for every $x\in M$. This is well defined since $\|\delta_x\|=\|\delta_x-\delta_p\|\le d(x,p)\le\mathrm{diam}(M)\le\mathrm{diam}(N)$, where the last inequality holds since $\mathrm{Emb}(M,N)\neq\emptyset$. Let $\bar{\alpha}\in\mathrm{Emb}(\mathcal{F}(N_\infty),\ell_\infty^n)$ and $\bar{r}<r$ be such that $\bar{\alpha}\circ\mathrm{Emb}(\mathcal{F}(M_\infty),\mathcal{F}(N_\infty))\subseteq(\widehat{c}^{-1}(\bar{r}))_{\varepsilon_0}$. Let $\bar{\tau}:N\to\varrho\mathrm{Ball}(\ell_\infty^n)$ be the embedding defined by $\bar{\tau}(x)=\bar{\alpha}(\delta_x)$. We claim that $\bar{\tau}$ works. In fact, $\bar{\tau}\circ\mathrm{emb}(M,N)\subseteq(c^{-1}(\bar{r}))_\varepsilon$. Let $\sigma\in\mathrm{emb}(M,N)$. Then there exists a unique extension $\gamma_\sigma\in\mathrm{Emb}(\mathcal{F}(M_\infty),\mathcal{F}(N_\infty))$. Let $\psi\in\mathrm{Emb}(\mathcal{F}(M_\infty),\ell_\infty^n)$ be such that $\widehat{c}(\psi)=\bar{r}$ and $\|\psi-\bar{\alpha}\circ\gamma_\sigma\|<\varepsilon_0$. Then $\sigma_\psi(x):=\psi(\delta_x)$ for every $x\in M$ satisfies

$c(\sigma_\psi) = \bar{r}$ and

$$d(\sigma_\psi, \bar{\tau} \circ \sigma) = \max_{x \in M} \|\psi(\delta_x) - \bar{\alpha}(\delta_{\sigma(x)})\|_\infty = \|\psi(\delta_x) - \bar{\alpha}(\gamma_\sigma(\delta_x)\|_\infty$$
$$< \varepsilon_0 \mathrm{diam}(M) = \varepsilon. \qquad \blacksquare$$

Proof of Theorem 2.28. This is a consequence of Lemma 2.32, via a compactness argument. Fix M, N, r and $\varepsilon > 0$. Let n be obtained from M, N, r and $\varepsilon/3$ by applying Lemma 2.32. Let $\varrho := \mathrm{diam}(N)$. Since M and N are finite and $\varrho\mathrm{Ball}(\ell_\infty^n)$ is compact, there exists $P \subseteq \varrho\mathrm{Ball}(\ell_\infty^n)$ finite such that

$$\mathrm{emb}(M, \varrho\mathrm{Ball}(\ell_\infty^n)) \subseteq (\mathrm{emb}(M, P))_{\varepsilon/3} \quad \text{and} \quad \mathrm{emb}(N, \varrho\mathrm{Ball}(\ell_\infty^n)) \subseteq (\mathrm{emb}(N, P))_{\varepsilon/3}.$$

We claim that (P, d_∞) works. To see this, let $c : \mathrm{emb}(P, A) \to r$. Define $\widetilde{c}$: $\mathrm{emb}(M, \varrho\mathrm{Ball}(\ell_\infty^n)) \to r$ by $\widetilde{c}(\gamma) := c(\sigma_\gamma)$ where $\sigma_\gamma \in \mathrm{emb}(M, A)$ is chosen such that $d(\gamma, \sigma_\gamma) < \varepsilon/3$. By the property of n, there are $\gamma \in \mathrm{emb}(N, \varrho\mathrm{Ball}(\ell_\infty^n))$ and $\bar{r} < r$ such that $\gamma \circ \mathrm{emb}(M, N) \subseteq (\widetilde{c}^{-1}(\bar{r}))_{\varepsilon/3}$. Let $\bar{\gamma} \in \mathrm{emb}(N, P)$ be such that $d(\gamma, \bar{\gamma}) < \varepsilon/3$. It takes a simple computation to see that $\bar{\gamma} \circ \mathrm{emb}(M, N) \subseteq (c^{-1}(r))_\varepsilon$. $\blacksquare$

2.6. *The closed bifaces of the Lusky simplex and R-Banach spaces*

There is a natural correspondence between Banach spaces and those compact spaces which are *absolutely convex*. In the real case, by a *compact absolutely convex set* we mean a compact subset of a locally convex topological real vector space that is closed under absolutely convex combinations of the form $\mu x + \lambda y$ for $\lambda, \mu \in \mathbb{R}$ such that $|\lambda| + |\mu| \leq 1$. Any compact absolutely convex set K has a canonical involution σ mapping x to $-x$. A real-valued continuous function f on K is *symmetric* if $f \circ \sigma = -f$. Similarly, a continuous affine function between compact absolutely convex sets is symmetric if it commutes with the given involutions. So, given a Banach space X, the unit ball $\mathrm{Ball}(X^*)$ of the dual space of X is a compact absolutely convex set when endowed with the w*-topology. Any compact absolutely convex set K is of this form, where X is the Banach space $A_\sigma(K)$ of real-valued symmetric affine continuous functions on K endowed with the supremum norm. Each contraction $T : X \to Y$ induces a symmetric affine continuous function $T^* : B_{Y^*} \to B_{X^*}$, and vice versa, a given symmetric affine continuous function $\xi : K \to L$ induces a contraction $\widehat{\xi} : A_\sigma(L) \to A_\sigma(K)$ by composition. Furthermore, such a correspondence is functorial, and induces an equivalence of categories. The following definition has been introduced in [37, §6.1].

Definition 2.33. A *Lazar simplex* is any compact absolutely convex set that is affinely homeomorphic to the unit ball of the dual of a Lindenstrauss space.

Lazar simplices have been internally characterized by A. J. Lazar [34] in terms of a uniqueness assertion for boundary representing measures, reminiscent of the analogous characterization of Choquet simplices due to Choquet [1, §II.3]; see also Subsection 3.1 below. The Lazar simplex corresponding to the Gurarij space is denoted by $\mathbb{L}$ and called the *Lusky simplex* in [37, §6.1]. It is proved in [37, 39, 42] that $\mathbb{L}$ plays the same role in the category of metrizable Lazar simplices as the Poulsen simplex $\mathbb{P}$ plays in the category of metrizable Choquet simplices (see Section 3). Recall that a closed subset H of a Lazar simplex is a *biface* or *essential face* if it is the absolutely convex hull of a (not necessarily closed) face [35]. This is equivalent to the linear span of H inside $A_\sigma(K)^*$ being a w*-closed L-ideal [2, 3]. Here are some properties of $\mathbb{L}$:

- The Lusky simplex is the *unique* non-trivial metrizable Lazar simplex with dense extreme boundary (Lusky [39]).
- The Lusky simplex is *universal* among metrizable Lazar simplices, in the sense that any metrizable Lazar simplex is symmetrically affinely homeomorphic to a closed biface of $\mathbb{L}$ (Lusky [42]).
- The Lusky simplex is *homogeneous*: any symmetric affine homeomorphism between proper closed bifaces of $\mathbb{L}$ extends to a symmetric affine homeomorphism of $\mathbb{L}$ (Lupini [37, §6.1]).

Our intention is to prove the following:

Theorem 2.34. *Suppose that H is a closed biface of the Lusky simplex $\mathbb{L}$. Then the group* $\mathrm{Aut}_H(\mathbb{L})$ *of symmetric affine homeomorphisms α of $\mathbb{L}$ such that $\alpha(p) = p$ for every $p \in H$ is extremely amenable.*

Remark 2.35. A similar result holds for complex Banach spaces. In this setting, one considers compact convex sets endowed with a continuous action of the circle group $\mathbb{T}$ (*compact convex circled sets*). The compact convex circled sets corresponding to complex Lindenstrauss spaces (*Effros simplices*) have been characterized by Effros [15]. Again, the unit ball of the dual space of the complex Gurarij space has canonical uniqueness, universality, and homogeneity properties within the class of Effros simplices [37, §6.2]. Here one considers the natural complex analog of the notion of a closed biface (*circled face*). The same argument as above shows that, in the complex case, the pointwise stabilizer of any closed circled face of $\mathrm{Ball}(\mathbb{G}^*)$ is extremely amenable.

Observe that in the particular case when H is the trivial biface $\{0\}$, such a statement recovers extreme amenability of the group of surjective linear isometries of $\mathbb{G}$. Observe also that given a closed biface H of a Lazar simplex L, we find that $g \in \mathrm{Aut}_H(L)$ if and only if $\widehat{g} \in \mathrm{Iso}_{\widehat{i}}(A_\sigma(L))$, where $i : H \to L$ is the inclusion map and where, in general, given Banach spaces X and Y and an operator $\sigma : X \to Y$, we denote by $\mathrm{Iso}_\sigma(X)$, we denote the subgroup of isometries g of X such that $\sigma \circ g = \sigma$. This motivates our study of such pairs (X, σ).

Definition 2.36 (R-Banach space). Given a Lindenstrauss space R, an *R-Banach space* is a couple $\mathbf{X} := (X, \sigma)$ where $\sigma : X \to R$ is a linear contraction, called an *R-functional*.

In this category, given R-spaces $\mathbf{X_0} := (X_0, \sigma_0)$, $\mathbf{X_1} := (X_1, \sigma_1)$ and $\delta > 0$, let $\mathrm{Emb}_\delta(\mathbf{X_0}, \mathbf{X_1})$ be the collection of δ-isometric embeddings $\gamma : X_0 \to X_1$ such that $\|\sigma_1 \circ \gamma - \sigma_0\| \leq \delta$, and in particular, let $\mathrm{Aut}(\mathbf{X}) = \mathrm{Iso}_\sigma(X)$ be the space of surjective isometries such that $\sigma \circ g = \sigma$. We write $(X_0, \sigma_0) \subseteq (X_1, \sigma_1)$ if $X_0 \subseteq X_1$ and $\sigma_1 \restriction X_0 = \sigma_0$.

The following result is established in [37, Section 5].

Theorem 2.37. *Given a separable Lindenstrauss space R there exists an onto contraction $\Omega_R : \mathbb{G} \to R$ such that the R-Banach space $\mathbf{G}_R := (\mathbb{G}, \Omega_R)$ is*

(1) universal *for separable R-Banach spaces, that is,* $\mathrm{Emb}(\mathbf{X}, (\mathbb{G}, \Omega_R)) \neq \emptyset$ *for every such space* $\mathbf{X}$*;*

(2) *a* stable Fraïssé *R-Banach space with modulus of stability $\varpi(\delta) = 2\delta$, that is, for every finite-dimensional R-space $\mathbf{X} := (X, \sigma) \subseteq (\mathbb{G}, \Omega_R)$, every $\delta > 0$ and every $\gamma \in \mathrm{Emb}_\delta(\mathbf{X}, (\mathbb{G}, \Omega_R))$ there is an isometry $g \in \mathrm{Iso}_{\Omega_R}(\mathbb{G})$ such that $\|g \restriction X - \gamma\| \leq 2\delta$.*

One can consider the similar category for an arbitrary separable Banach space R, not necessarily a Lindenstrauss space, and obtain a Fraïssé limit $(\mathbb{G}(R), \Omega_R)$ with the corresponding properties (1) and (2) but since R is 1-complemented in $\mathbb{G}(R)$, this space is Gurarij only when R is Lindenstrauss (see [11, Theorem 6.5]). Note that a classical result of Wojtaszczyk [56] asserts that the separable Lindenstrauss spaces are precisely the separable Banach spaces that are isometric to the range of a contractive projection on the Gurarij space $\mathbb{G}$. The R-functional Ω_R is called the *generic* contractive R-functional on $\mathbb{G}$. The name is justified by the fact that the $\mathrm{Iso}(\mathbb{G})$-orbit of Ω_R is a dense G_δ subset of the space of contractive R-functionals on $\mathbb{G}$. The universality and homogeneity properties of $\mathbb{L}$ can be seen as consequences of the following result, established in [37, §6.1] using the theory of M-ideals in Banach spaces developed by Alfsen and Effros [2, 3], and the Choi–Effros lifting theorem from [12].

Proposition 2.38. *Suppose that R is a separable Lindenstrauss space. A contraction $s : \mathbb{G} \to R$ belongs to the* $\mathrm{Iso}(\mathbb{G})$*-orbit of Ω_R if and only if s is a* non-trivial facial quotient*, that is,* $\ker s \neq 0$*, and s^* is an isometric embedding such that* $s^*(\mathrm{Ball}(R^*))$ *is a closed biface of* $\mathrm{Ball}(\mathbb{G}^*)$.

In particular, suppose that H is a proper closed biface of $\mathbb{L}$, $i : H \to \mathbb{L}$ is the canonical inclusion and we canonically identify $\mathbb{G}$ and $A_\sigma(\mathbb{L})$. Then $\widehat{i} : A_\sigma(\mathbb{L}) \to A_\sigma(H)$ is a non-trivial facial quotient, hence $\widehat{i} \in \mathrm{Iso}(\mathbb{G}) \cdot \Omega_{A_\sigma(H)}$. This implies that $\mathrm{Iso}_{\widehat{i}}(\mathbb{G}) = \mathrm{Iso}_{\Omega_{A_\sigma(H)}}(\mathbb{G})$, and Theorem 2.34 can be rephrased as follows.

Theorem 2.39. *The stabilizer of the generic contractive R-functional on the Gurarij space is extremely amenable for any separable Lindenstrauss Banach space R.*

When $R = \{0\}$, we recover the extreme amenability of $\mathrm{Iso}(\mathbb{G})$. In fact, the proof of this extension is based on the approximate Ramsey property of finite-dimensional R-Banach spaces, by means of the KPT correspondence. The corresponding non-commutative version of the previous theorem is established in [5, 7].

2.6.1. KPT correspondence and (ARP) of R-Banach spaces. We give a proof of Theorem 2.34. By the correspondence between the categories of Lazar simplices and that of R-Banach spaces, Theorem 2.34 is equivalent to the fact that $\mathrm{Aut}(\mathbf{G}_R)$ is extremely amenable, which will be proved by means of a KPT correspondence and an appropriate approximate Ramsey property. Given an R-space $\mathbf{X} = (X, s)$, let $\mathrm{Age}(\mathbf{X})$ be the collection of pairs $(F, s \restriction F)$, where $F \in \mathrm{Age}(X)$. Given a family $\mathcal{F}$ of finite-dimensional R-Banach spaces, let $[\mathcal{F}]$ be the collection of all separable R-Banach spaces $\mathbf{X}$ such that for every $\mathbf{F} \in \mathrm{Age}(\mathbf{X})$ and every $\delta > 0$ there is some $\mathbf{G} \in \mathcal{F}$ such that $\mathrm{Emb}_\delta(\mathbf{F}, \mathbf{G}) \neq \emptyset$.

Theorem 2.40 (KPT correspondence for stable Fraïssé R-Banach spaces). *Suppose that $\mathbf{E} = (E, \Omega)$ is an approximately ultrahomogeneous R-Banach space. Then the following are equivalent:*

(1) $\mathrm{Aut}(\mathbf{E})$ *is extremely amenable.*

(2) $\mathrm{Age}(\mathbf{E})$ *satisfies the (ARP), that is, for any $\mathbf{X}, \mathbf{Y} \in \mathrm{Age}(\mathbf{E})$ and $\varepsilon > 0$ there is $\mathbf{Z} \in \mathrm{Age}(\mathbf{E})$ such that every continuous coloring of $\mathrm{Emb}(\mathbf{X}, \mathbf{Z})$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(\mathbf{X}, \mathbf{Y})$ for some $\gamma \in \mathrm{Emb}(\mathbf{Y}, \mathbf{Z})$.*

Suppose that $\mathcal{F}$ is a family such that $\mathcal{F} \preceq \mathrm{Age}(\mathbb{E})$, $\mathbb{E} \in [\mathcal{F}]$. Then (1)–(3) are equivalent to

(3) *$\mathcal{F}$ satisfies the* stably approximate Ramsey property (SRP) *with modulus $\varpi(\delta)$, that is, for any $\mathbf{X}, \mathbf{Y} \in \mathcal{F}$, $\varepsilon > 0$ and $\delta \geq 0$ there is $\mathbf{Z} \in \mathcal{F}$ such that every continuous coloring of $\mathrm{Emb}_\delta(\mathbf{X}, \mathbf{Z})$ $(\varpi(\delta) + \varepsilon)$-stabilizes on $\gamma \circ \mathrm{Emb}_\delta(\mathbf{X}, \mathbf{Y})$ for some $\gamma \in \mathrm{Emb}(\mathbf{Y}, \mathbf{Z})$.*

The proof of Theorem 2.40 is a straightforward modification of that of Theorem 2.12; we leave its details to the reader.

Theorem 2.41. *The following classes have the (SRP) with modulus of stability 2δ:*

(a) *For every $k \in \mathbb{N}$, the class of ℓ_∞^k-Banach spaces (X, s) where $X = \ell_\infty^n$ for some $n \in \mathbb{N}$.*

(b) *For every separable Lindenstrauss space R, the class of all finite-dimensional R-Banach spaces.*

Proof. As for the case of Banach spaces in Proposition 2.13, a class of R-finite-dimensional spaces has the (SRP) with modulus ϖ if and only if it satisfies the (ARP) and it has the corresponding stable amalgamation property with modulus ϖ.

We now handle case (a).

Claim 2.41.1. *The family $\mathcal{F}$ of ℓ_∞^k-spaces of the form (ℓ_∞^n, s) for some n has the stable amalgamation property with modulus 2δ.*

Proof of Claim. Fix ℓ_∞^k-spaces $\mathbf{X} = (\ell_\infty^d, s)$, $\mathbf{Y} = (\ell_\infty^m, t)$ and $\mathbf{Z} = (\ell_\infty^n, u)$, $\delta > 0$, and $\gamma \in \mathrm{Emb}_\delta(\mathbf{X}, \mathbf{Y})$ and $\eta \in \mathrm{Emb}_\delta(\mathbf{X}, \mathbf{Z})$. Let $I \in \mathrm{Emb}(\ell_\infty^m, \ell_\infty^{m+n})$ and $J \in \mathrm{Emb}(\ell_\infty^n, \ell_\infty^{m+n})$ be such that $\|I \circ \gamma - J \circ \eta\| \leq \delta$ (see Proposition 2.18). Then $I_0 := (I, t) : \ell_\infty^m \to \ell_\infty^{m+n+k}$ and $J_0 := (J, u) : \ell_\infty^n \to \ell_\infty^{m+n+k}$ satisfies $I_0 \in \mathrm{Emb}(\mathbf{Y}, (\ell_\infty^{m+n+k}, \pi))$, $J_0 \in \mathrm{Emb}(\mathbf{Z}, (\ell_\infty^{m+n+k}, \pi))$ and $\|I_0 \circ \gamma - J_0 \circ \eta\| = \max\{\|I \circ \gamma - J \circ \eta\|, \|t \circ \gamma - u \circ \eta\|\} \leq 2\delta$, where $\pi : \ell_\infty^{m+n+k} \to \ell_\infty^k$ is the projection $\pi((a_j)_{j<m+n+k}) = (a_j)_{j=m+n}^{m+n+k-1}$. ■

We prove now the (ARP) of $\mathcal{F}$. Fix ℓ_∞^k-spaces $\mathbf{X} := (\ell_\infty^d, s)$ and $\mathbf{Y} := (\ell_\infty^m, u)$, and $\varepsilon > 0$. Let $n \in \mathbb{N}$ witness the (ARP) of $\{\ell_\infty^r\}_r$ for the initial parameters d, m, and ε. Let $\pi : \ell_\infty^{n+k} \to \ell_\infty^k$ be the canonical second projection $\pi((a_j)_{j<n+k}) := (a_j)_{j=n}^{n+k-1}$. We

claim that $\mathbf{Z} := (\ell_\infty^{n+k}, \pi)$ works: Indeed, suppose that $c : \mathrm{Emb}(\mathbf{X}, \mathbf{Z}) \to [0, 1]$ is a continuous coloring. Let $\widehat{c} : \mathrm{Emb}(\ell_\infty^d, \ell_\infty^n) \to [0, 1]$ be defined for $\gamma \in \mathrm{Emb}_\delta(\ell_\infty^d, \ell_\infty^n)$ by $\widehat{c}(\gamma) := c(\gamma, s)$. Observe that $\widehat{c}$ is 1-Lipschitz, so there is $I \in \mathrm{Emb}(\ell_\infty^m, \ell_\infty^n)$ such that $\mathrm{Osc}(\widehat{c} \restriction I \circ \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)) \le \varepsilon$. Let $J := (I, t) \in \mathrm{Emb}(\mathbf{Y}, \mathbf{Z})$. Notice that given $\gamma \in \mathrm{Emb}(\mathbf{X}, \mathbf{Y})$, we have $J \circ \gamma = (I \circ \gamma, t \circ \gamma) = (I \circ \gamma, s)$. Hence, $\mathrm{Osc}(c \restriction J \circ \mathrm{Emb}(\mathbf{X}, \mathbf{Y})) \le \mathrm{Osc}(\widehat{c} \restriction I \circ \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)) \le \varepsilon$.

For (b), fix a Lindenstrauss space R, and choose an increasing sequence $(R_n)_n$ of subspaces whose union is dense in R and such that each R_n is isometric to ℓ_∞^n.

Let $\mathcal{F}$ be the class of R-Banach spaces (X, s) where X is isometric to some ℓ_∞^d and such that $\mathrm{Im}\, s \subseteq \bigcup_n R_n$. It follows easily from (a) that $\mathcal{F}$ has the (SRP) with modulus 2δ. We know from Theorem 2.37 that $\mathbb{G}_R = (\mathbb{G}, \Omega_R)$ is a stable Fraïssé R-Banach space such that $\mathrm{age}(\mathbb{G}_R)$ consists of all finite-dimensional R-Banach spaces. On the other hand, $\mathbb{G}_R \in [\mathcal{F}]$, so it follows from (a) and the KPT correspondence in Theorem 2.40 that $\mathrm{age}(\mathbb{G}_R)$ satisfies the (SRP) with modulus 2δ. ■

Theorem 2.41 and the characterization of extreme amenability in Theorem 2.40 give the previously announced result.

Theorem 2.39. *The stabilizer of the generic contractive R-functional on the Gurarij space is extremely amenable for any separable Lindenstrauss Banach space R.* ■

3. The Ramsey property of Choquet simplices and function systems

The main goal of this section is to establish the approximate (dual) Ramsey property for Choquet simplices with a distinguished point. We will then apply this to compute the universal minimal flow of the automorphism group of the Poulsen simplex $\mathbb{P}$. We will prove that the minimal compact $\mathrm{Aut}(\mathbb{P})$-space is the Poulsen simplex $\mathbb{P}$ itself endowed with the canonical action of $\mathrm{Aut}(\mathbb{P})$, answering [13, Question 4.4] (the fact that such an action is minimal is a result of Glasner [19]). This will be done by studying function systems with a distinguished unital positive map to a fixed separable Lindenstrauss function system R. Similarly to the case of Banach spaces (§2.6), we will also consider function systems X with a distinguished *state*, a unital linear contraction $s : X \to R$ where R is a fixed separable Lindenstrauss function system.

3.1. Choquet simplices and function systems

Recall that a *compact convex set* K is a compact convex subset of some locally convex topological vector space. In a compact convex set one can define in the usual way the notion of convex combination. The *extreme boundary* $\partial_e K$ of K is the set of *extreme points* of K, that is, points that cannot be written in a non-trivial way as a convex combination of points of K. When K is metrizable the boundary $\partial_e K$ is a G_δ subset. In this case, a *boundary measure* on K is a Borel probability measure on K that vanishes off the boundary of K. Choquet's representation theorem asserts that any point in a compact convex set can be realized as the barycenter of a boundary measure on K (*representing* *measure*). A compact convex set K where every point has a *unique* representing measure is called a *Choquet simplex*. In particular, any standard finite-dimensional simplex Δ_n for $n \in \mathbb{N}$ is a Choquet simplex.

The class of standard finite-dimensional simplices Δ_n for $n \in \mathbb{N}$ naturally forms a *projective* Fraïssé class in the sense of [30]; see [32]. The corresponding Fraïssé limit is the *Poulsen simplex* $\mathbb{P}$. Initially constructed by Poulsen [52], $\mathbb{P}$ is a non-trivial metrizable Choquet simplex with dense extreme boundary. It was later shown in [36] that there exists a *unique* non-trivial metrizable Choquet simplex with this property up to affine homeomorphism. Furthermore $\mathbb{P}$ is *universal* among metrizable Choquet simplices, in the sense that any metrizable Choquet simplex is affinely homeomorphic to a closed proper face of $\mathbb{P}$. Also, the Poulsen simplex is ultrahomogeneous: any affine homeomorphism between closed proper faces of $\mathbb{P}$ extends to an affine homeomorphism of $\mathbb{P}$.

The Poulsen simplex $\mathbb{P}$ can also be studied from the perspective of direct Fraïssé theory by considering the natural dual category to compact convex sets. For a compact convex set K, let $A(K)$ be the space of complex-valued continuous affine functions on K. This is a closed subspace of the space $C(K)$ of complex-valued continuous functions on K, endowed with the supremum norm. Furthermore, $A(K)$ contains a distinguished element, its *unit*, that corresponds to the constant function equal to 1. In general, recall that a *function system* is a closed subspace V of $C(T)$ for some compact Hausdorff space T containing the function constantly equal to 1 and such that if $f \in V$ then the function f^* defined by $f^*(t) := \overline{f(t)}$ also belongs to V. So, $A(K)$ is a function system, and in fact any function system $V \subseteq C(T)$ arises in this way from a suitable compact convex set K. Precisely, K is the compact convex set of *states* of V, that is, the contractive functionals on V that are *unital*, i.e., map the unit of $C(T)$ to 1.

As mentioned in the introduction, the assignment $K \mapsto A(K)$ establishes a contravariant equivalence of categories from the category of compact convex sets and continuous affine maps to the category of function systems and unital contractive linear maps. The finite-dimensional function systems that are *injective* in such category are precisely the function systems $A(\Delta_n) = \ell_\infty^n$ corresponding to the standard finite-dimensional simplices Δ_n. The function systems that correspond to Choquet simplices are precisely those that are Lindenstrauss as Banach spaces, or equivalently, the function systems whose identity map is the pointwise limit of *unital* contractive linear maps that factor through finite-dimensional injective function systems.

The function systems approach has been adopted in the work of Conley and Törnquist [13], and independently in [37, 38], where it is shown that the class of finite-dimensional function systems is a Fraïssé class. Its limit can be identified with the function system $A(\mathbb{P})$ corresponding to the Poulsen simplex, which we will call the *Poulsen system*. The model-theoretic properties of $A(\mathbb{P})$ and their non-commutative analogues have been studied in [22].

Suppose that X is a function system. Recall that a *state* on X is a unital contractive linear map from X to $\mathbb{C}$. More generally, if R is any separable Lindenstrauss function system, we call a unital contractive linear map from X to R an *R-state* on X. Let $\mathrm{UC}(X, R)$ be the space of R-states on X. It is a Polish space endowed with a canonical continuous action of $\mathrm{Aut}(X)$. An *R-function system* is a pair $\mathbf{X} = (X, s_X)$ consisting of a function system X and an R-state s_X on X. In the following, we regard $\mathrm{UC}(X, R)$ as an $\mathrm{Aut}(X)$-space with respect to the canonical action $\mathrm{Aut}(X) \curvearrowright \mathrm{UC}(X, R)$ given by $(\alpha, s) \mapsto s \circ \alpha^{-1}$.

Given R-function systems $\mathbf{X} = (X, s_X)$ and $\mathbf{Y} = (Y, s_Y)$, and $\delta \geq 0$, let $\mathrm{Emb}_\delta(\mathbf{X}, \mathbf{Y})$ be the collection of unital δ-isometric embeddings $\gamma : X \to Y$ such that $\|s_Y \circ \gamma - s_X\|_{X,R} \leq \delta$. Given an R-function system $\mathbf{X} = (X, s_X)$, let $\mathrm{Age}(\mathbf{X})$ be the collection of all finite-dimensional *R-function subsystems* $\mathbf{Y} = (Y, s_Y)$ of $\mathbf{X}$, that is, $Y \subseteq X$ and $s_Y = s_X \restriction Y$. Given a class $\mathcal{F}$ of R-function systems, let $[\mathcal{F}]$ be the class of all separable R-function systems $\mathbf{X} = (X, s_X)$ such that for every $\mathbf{Y}$ and every $\delta > 0$ there is $\mathbf{Z} \in \mathcal{F}$ such that $\mathrm{Emb}_\delta(\mathbf{Y}, \mathbf{Z}) \neq \emptyset$. Let $\mathrm{Aut}(X, s_X)$ be the stabilizer of $s_X \in \mathrm{UC}(X, R)$ in $\mathrm{Aut}(X)$. Given a family $\mathcal{A}$ of function systems, let $\mathcal{A}^R$ be the collection of R-function systems (X, s_X) where $X \in \mathcal{A}$.

The following result is established in [37, §6.3].

Proposition 3.1. *Let R be a separable Lindenstrauss function system. Then the class* FdBa^R *of finite-dimensional R-function systems is a stable Fraïssé class with stability modulus $\varpi(\delta) = 2\delta$ and $\boldsymbol{A}(\mathbb{P})_R := (A(\mathbb{P}), \Omega_R)$ is its* Fraïssé limit, *that is, $\boldsymbol{A}(\mathbb{P})_R$ is a stable Fraïssé R-function system such that* $\mathrm{Age}(\boldsymbol{A}(\mathbb{P})_R) = \mathrm{FdBa}^R$.

As in the case of operator spaces, the R-state Ω_R as in Proposition 3.1 is called the *generic* R-state on $A(\mathbb{P})$. This is the unique R-state on $A(\mathbb{P})$ whose $\mathrm{Aut}(A(\mathbb{P}))$-orbit is a dense G_δ subset of the space $\mathrm{UC}(A(\mathbb{P}), R)$. The elements of the $\mathrm{Aut}(A(\mathbb{P}))$-orbit of Ω_R can be characterized as follows (see [37, §6.3]).

Proposition 3.2. *Suppose that R is a separable Lindenstrauss function system. A unital quotient map $s : A(\mathbb{P}) \to R$ belongs to the* $\mathrm{Aut}(A(\mathbb{P}))$*-orbit of Ω_R if and only if s is a* unital facial quotient, *i.e., s is unital and s^* is an isometric embedding such that* $s^*(\mathrm{Ball}(R^*))$ *is a closed proper face of* $\mathbb{P}$.

Our intention is to prove the following

Theorem 3.3. *For every metrizable Choquet simplex F the stabilizer* $\mathrm{Aut}(\boldsymbol{A}(\mathbb{P})_{A(F)})$ *of the generic $A(F)$-state $\Omega_{A(F)}$ on the Poulsen system $A(\mathbb{P})$ is extremely amenable.*

3.2. Approximate Ramsey property and extreme amenability

The following result provides a correspondence between extreme amenability and Ramsey properties in the context of R-function systems. The proof is analogous to the one for Banach spaces, and is left to the reader.

Theorem 3.4 (KPT correspondence for (aUH) and stable Fraïssé R-function systems). *Suppose that $\mathbf{X} = (X, \Omega)$ is an approximately ultrahomogeneous R-function system. Then the following are equivalent:*

(1) $\mathrm{Aut}(\mathbf{X})$ *is extremely amenable.*

(2) $\mathrm{Age}(\mathbf{X})$ *has the approximate Ramsey property.*

If in addition $\mathcal{F}$ is a family that has the stable amalgamation property such that $E \in [\mathcal{F}]$ and $\mathcal{F} \preceq \mathrm{Age}(\mathbf{X})$, that is, every R-function system in $\mathcal{F}$ can be isometrically R-embedded into E, then the previous are equivalent to

(3) *$\mathcal{F}$ satisfies the (SRP).*

Theorem 3.5. *Suppose that $(R_k)_k$ is a sequence of function subsystems of R, each R_k isometric to ℓ_∞^k, and with a dense union. The following classes of R-function systems have the (SRP) with modulus 2δ:*

(1) *For every $k \in \mathbb{N}$, the class of R_k-function systems $(R_n \oplus_\infty R_k, \pi_n^{(k)})$, where $\pi_n^{(k)} : R_n \oplus_\infty R_k \to R_k$ is the canonical second projection.*
(2) *For every $k \in \mathbb{N}$, the class of R_k-function systems (X, s) where X is isometric to some ℓ_∞^n.*
(3) *The class $\mathcal{B}_R$ of R-function systems (X, s) where X is isometric to some ℓ_∞^n and $s : X \to \bigcup_k R_k$.*
(4) *The class of all R-function systems.*

To prove this theorem we will use (and prove) the (ARP) of the class $\{\ell_\infty^n\}_n$ with respect to positive embeddings. Its proof is similar to that of Lemma 2.17. We present the details for the reader's convenience. Let $\mathrm{Emb}^+(\ell_\infty^d, \ell_\infty^n)$ be the space of *positive* isometric embeddings from ℓ_∞^d into ℓ_∞^n. Dually, let $\mathrm{Quo}^+(\ell_1^n, \ell_1^d)$ be the space of corresponding positive quotient mappings.

Lemma 3.6. *For every $d, m, r \in \mathbb{N}$, and $\varepsilon > 0$ there is some n such that every r-coloring of $\mathrm{Emb}^+(\ell_\infty^d, \ell_\infty^n)$ has an ε-monochromatic set of the form $\gamma \circ \mathrm{Emb}^+(\ell_\infty^d, \ell_\infty^n)$ for some $\gamma \circ \mathrm{Emb}^+(\ell_\infty^d, \ell_\infty^m)$.*

We write $\mathrm{Ball}^+(\ell_1^k)$ and $\mathrm{Sph}^+(\ell_1^k)$ for the positive part of the unit ball and of the unit sphere of ℓ_1^k. Recall that a linear map $\gamma : \ell_\infty^d \to \ell_\infty^n$ is unital if and only if its dual $\gamma^* : \ell_1^n \to \ell_1^d$ is trace-preserving, that is, $\mathrm{Tr}_d(\gamma^*((a_j)_{j<n})) = \mathrm{Tr}_n((a_j)_{j<n})$, where $\mathrm{Tr}_k((a_j)_{j<k}) := \sum_{j<k} a_j$ is the canonical trace. When in addition γ (equiv. γ^*) is a contraction, then γ and γ^* must be positive. Thus γ is a unital isometric embedding if and only if γ^* is a trace-preserving quotient mapping, or equivalently if each $\gamma^*(u_j)$ belongs to $\mathrm{Sph}^+(\ell_1^d)$ and $\{u_j\}_{j<d} \subseteq \{\gamma^*(u_j)\}_{j<n}$. Given R-function systems (ℓ_∞^d, s) and (ℓ_∞^n, t), let $\mathrm{Quo}((\ell_1^n, t^*), (\ell_1^d, s^*))$ be the space of trace-preserving quotients $\sigma : \ell_1^n \to \ell_1^d$ such that $\sigma \circ t^* = s^*$. Before proving Lemma 3.6, we use it.

Proof of Theorem 3.5. All the four classes considered have the stable amalgamation property with modulus 2δ: For the first three ones, the proof of Claim 2.41.1 can be easily adjusted to give the desired property, and for the last class, as mentioned above, the proof can be found in [37, §6.3]. So, we have to prove that in addition all those classes have the (ARP).

For (1), we consider the equivalent class, and easy to work with, $\{(\ell_\infty^{n+k}, \pi_n)\}_n$, where $\pi_n^{(k)} : \ell_\infty^{n+k} = \ell_\infty^n \oplus_\infty \ell_\infty^k \to \ell_\infty^k$ is the second projection. We prove the dual approximate Ramsey statement for the corresponding dual class: Write $\mathbf{X}_n$ for $(\ell_\infty^{n+k}, \pi_n^{(k)})$, and $\mathbf{X}_n^* :=$

$(\ell_1^{n+k}, (\pi_n^{(k)})^*)$. Notice that given d and m, we have $\sigma \in \mathrm{Quo}(\mathbf{X}_m^*, \mathbf{X}_d^*)$ exactly when σ is a trace-preserving quotient such that $\sigma(u_{m+j}) = u_{d+j}$ for all $j < k$. We prove that for all $d, m, r \in \mathbb{N}$ and $\varepsilon > 0$ there is some n such that every r-coloring of $\mathrm{Quo}(\mathbf{X}_n^*, \mathbf{X}_d^*)$ has an ε-monochromatic set of the form $\mathrm{Quo}(\mathbf{X}_m^*, \mathbf{X}_d^*) \circ \varrho$ for some $\varrho \in \mathrm{Quo}(\mathbf{X}_n^*, \mathbf{X}_m^*)$. Fix parameters d, m, r and ε.

We claim that the number n obtained by applying Lemma 3.6 to $d+k-1, m+k-1$, r,$\delta = 0$, and ε works. Suppose that $c : \mathrm{Quo}(\mathbf{X}_n^*, \mathbf{X}_d^*) \to r$. We define the auxiliary coloring $\widehat{c} : \mathrm{Quo}^+(\ell_1^n, \ell_1^{d+k-1}) \to r$ by declaring $\widehat{c}(\sigma) := c(\widehat{\sigma})$, where $\widehat{\sigma} \in \mathrm{Quo}(\mathbf{X}_n^*, \mathbf{X}_d^*)$ is such that $\widehat{\sigma}(u_j) = i_d(\sigma(u_j)) + (1 - \|\sigma(u_j)\|_1) u_{d+k-1}$, for $i_d : \ell_1^{k+d-1} \to \ell_1^{(k+d}$ being the canonical embedding $i_d((a_j)_{j<k+d-1}) := (a_0, \dots, a_{k+d-2}, 0)$. Notice that $\|\sigma - \eta\| \le \|\widehat{\sigma} - \widehat{\eta}\| \le 2\|\sigma - \eta\|$. By the choice of n, and the dual version of Lemma 3.6, we can find $\varrho \in \mathrm{Quo}^+(\ell_1^n, \ell_1^{m+k-1})$ and $\widehat{r} < r$ such that $\mathrm{Quo}^+(\ell_1^{m+k-1}, \ell_1^{d+k-1}) \circ \varrho \subseteq (\widehat{c}^{-1}(\widehat{r}))_\varepsilon$. Let $\widehat{\varrho} \in \mathrm{Quo}(\mathbf{X}_n^*, \mathbf{X}_m^*)$ be defined linearly for $j < n$ by $\widehat{\varrho}(u_j) := i_m(\varrho(u_j)) + (1 - \|\varrho(u_j)\|_1) u_{m-1}$. We claim that $\mathrm{Quo}(\mathbf{X}_m^*, \mathbf{X}_d^*) \circ \widehat{\varrho} \subseteq c^{-1}(\widehat{r})_{2\varepsilon}$. To see this, fix $\sigma \in \mathrm{Quo}(\mathbf{X}_m^*, \mathbf{X}_d^*)$. Let $\underline{\sigma} : \ell_1^{m+k-1} \to \ell_1^{d+k-1}$, $\underline{\sigma}(a_0, \dots, a_{m+k-2}) = \pi(\sigma(a_0, \dots, a_{m+k-2}, 0))$, where $\pi(b_0, \dots, b_{d+k-2}, b_{d+k-1}) = (b_0, \dots, b_{d+k-2})$. It follows that $\underline{\sigma} \in \mathrm{Quo}^+(\ell_1^{m+k-1}, \ell_1^{d+k-1})$ and $\underline{\sigma} \,\widehat{\circ}\, \varrho = \sigma \circ \widehat{\varrho}$. Let $\eta \in \mathrm{Quo}^+(\ell_1^n, \ell_1^{d+k-1})$ be such that $\|\underline{\sigma} \circ \varrho - \eta\| \le \varepsilon$ and $c(\widehat{\eta}) = \widehat{r}$. Then $\|\underline{\sigma} \,\widehat{\circ}\, \varrho - \widehat{\eta}\| \le 2\varepsilon$, or equivalently $\|\sigma \circ \widehat{\varrho} - \widehat{\eta}\| \le 2\varepsilon$.

For (2), we prove the (ARP) of the equivalent class of ℓ_∞^k-function systems (ℓ_∞^n, s) for some $n \in \mathbb{N}$. Fix $\mathbf{X} = (\ell_\infty^d, s)$, $\mathbf{Y} = (\ell_\infty^m, t)$, $r \in \mathbb{N}$ and $\varepsilon > 0$, We apply the (ARP) in (1) to these parameters to find the corresponding n. We claim that $\mathbf{X}_n = (\ell_\infty^{n+k}, \pi_n^{(k)})$ works. Let $c : \mathrm{Emb}(\mathbf{X}, \mathbf{X}_n) \to r$, and let $\widehat{c} : \mathrm{Emb}(\mathbf{X}_d, \mathbf{X}_n) \to r$ be defined by $\widehat{c}(\gamma) := c(\gamma \circ i)$, where $i \in \mathrm{Emb}(\mathbf{X}, \mathbf{X}_d)$ is defined by $i(x) := (x, s(x))$. Let $\gamma \in \mathrm{Emb}(\mathbf{X}_m, \mathbf{X}_n)$ and $\widehat{r} < r$ be such that $\gamma \circ \mathrm{Emb}(\mathbf{X}_d, \mathbf{X}_m) \subseteq (\widehat{c}^{-1}(\widehat{r}))_\varepsilon$. Let $\gamma_0 := \gamma \circ j \in \mathrm{Emb}(\mathbf{Y}, \mathbf{X}_n)$ where $j \in \mathrm{Emb}(\mathbf{Y}, \mathbf{X}_m)$ is $j(y) := (y, t(y))$. Then $\gamma_0 \circ \mathrm{Emb}(\mathbf{X}, \mathbf{Y}) \subseteq (c^{-1}(\widehat{r}))_\varepsilon$, because given $\eta \in \mathrm{Emb}(\mathbf{X}, \mathbf{Y})$, if we define $\eta_0 \in \mathrm{Emb}(\mathbf{X}_d, \mathbf{X}_m)$ by $\eta_0(x, y) = (\eta(x), y)$, then we have $\eta_0 \circ i = j \circ \eta$, hence $\gamma_0 \circ \eta = \gamma \circ j \circ \eta = (\gamma \circ \eta_0) \circ i$, and consequently $c(\gamma_0 \circ \eta) = \widehat{c}(\gamma \circ \eta_0)$.

(3) follows trivially from (2); and (4) follows from (3) and from

Claim 3.6.1. *For every finite-dimensional R-function system $\mathbf{X}$ and every δ there is some $\mathbf{Y} \in \mathcal{B}_R$ such that* $\mathrm{Emb}_\delta(\mathbf{X}, \mathbf{Y}) \neq \emptyset$.

Proof of Claim. Since a function space is a unital closed subspace of some function system, it follows for example from the existence of partitions of unity that for every finite-dimensional function system X and every δ there is some n and some unital δ-embedding $\gamma : X \to \ell_\infty^n$. If in addition $s : X \to R$ is a unital contraction, then there must be some k and some unital contraction $t : X \to R_k$ such that $\|s - t\| \le \delta$. Since ℓ_∞^k is an injective function system, we can find a unital contraction $u : \ell_\infty^n \to \ell_\infty^k$ such that $\|u \circ \gamma - t\| \le \delta$, and consequently $\gamma \in \mathrm{Emb}_{2\delta}((X, s), (\ell_\infty^n, u))$. ∎

This ends the proof of Theorem 3.5.

Proof of Lemma 3.6. The proof of the dual form of this statement is that of Lemma 2.17 with the natural modifications that we sketch: Fix d, m, r and ε. First, fix a finite ε-dense subset $\mathcal{D}$ of $\mathrm{Ball}^+(\ell_1^d)$ containing $\{u_j\}_{j<d}$ such that for every non-zero $x \in \mathrm{Ball}^+(\ell_1^d)$ there is $y \in \mathcal{D}$ such that $\|y - x\|_1 \le \varepsilon$ and $\|y\|_1 < \|x\|_1$. Let $\mathrm{emb}(d, m)$ be the collection of 1-1 mappings from d into m, and for each such mapping f, let $h_f : \ell_1^d \to \ell_1^m$ be the positive isometry sending u_j to $u_{f(j)}$. Observe that for each positive quotient mapping $\sigma \in \mathrm{Quo}^+(\ell_1^m, \ell_1^d)$ there is some $f \in \mathrm{emb}(d, m)$ such that $\sigma \circ h_f = \mathrm{Id}_{\ell_1^d}$. Let $\Delta := \mathcal{D} \times \mathrm{emb}(d, m)$. We linearly order $\mathcal{D}$ by $\prec$ in such a way that if $\|x\|_1 < \|y\|_1$, then $x \prec y$, we order $\mathrm{emb}(d, m)$ arbitrarily, and then we consider Δ lexicographically ordered. Then $n := \mathbf{GR}(|\mathcal{D}|, |\Delta|, r)$ works. Given $c : \mathrm{Quo}^+(\ell_1^n, \ell_1^d) \to r$ one defines $\widehat{c} : \mathrm{Epi}(n, \mathcal{D}) \to r$ by $\widehat{c}(\sigma) := c(\Phi(\sigma))$ where $\Phi(\sigma)(u_j) := \sigma(j)$ for $j < n$. Let $\varrho \in \mathrm{Epi}(n, \Delta)$ and $\widehat{r} < r$ be such that $\mathrm{Epi}(\Delta, \mathcal{D}) \circ \varrho$ is monochromatic with color $\widehat{r}$. Let $e \in \mathrm{Quo}^+(\ell_1^n, \ell_1^m)$ be linearly defined by $e(u_j) := h_f(v)$, where $(v, f) = \varrho(j)$. Then it can be shown as in the proof of Lemma 2.17 that for every $T \in \mathrm{Quo}^+(\ell_1^m, \ell_1^d)$ there is some $\sigma \in \mathrm{Epi}(\Delta, \mathcal{D})$ such that $\|\Phi(\sigma \circ \varrho) - T \circ e\| \le \varepsilon$, and consequently $\mathrm{Quo}^+(\ell_1^m, \ell_1^d) \circ e \subseteq (c^{-1}(j))_\varepsilon$. ■

Theorem 3.5 and the (KPT) correspondence in Theorem 3.4 yield

Proof of Theorem 3.3. We rephrase (1) of Theorem 3.5 geometrically. We identify the n-dimensional standard simplex Δ_n with the state space $S(\ell_\infty^{n+1}) \subset \ell_1^{n+1}$. Let $\mathrm{Epi}(\Delta_n, \Delta_d)$ be the space of surjective continuous affine maps from Δ_n to Δ_d endowed with the metric $d(\phi, \psi) := \sup_{p \in \Delta_n} \|\phi(p) - \psi(p)\|_{\ell_1^d}$. We also let $\mathrm{Epi}_0(\Delta_n, \Delta_d)$ be the subspace of $\phi \in \mathrm{Epi}(\Delta_n, \Delta_d)$ such that $\phi(u_n) = u_d$. One can (isometrically) identify $\mathrm{Epi}(\Delta_n, \Delta_d)$ isometrically with the space of trace-preserving quotients from ℓ_1^n onto ℓ_1^d, and the space $\mathrm{Epi}_0(\Delta_n, \Delta_d)$ with $\mathrm{Quo}((\ell_1^n, \pi_n^{(1)}), (\ell_1^d, \pi_n^{(1)}))$. The following statement is therefore equivalent to Theorem 3.5 for $k = 1$ and isometric embeddings.

Corollary 3.7. *For any $d, m, r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any r-coloring of the space $\mathrm{Epi}_0(\Delta_n, \Delta_d)$ there exists $\gamma \in \mathrm{Epi}_0(\Delta_n, \Delta_m)$ such that $\mathrm{Epi}_0(\Delta_m, \Delta_d) \circ \gamma$ is ε-monochromatic.* ■

3.3. *Closed faces of the Poulsen simplex*

Theorem 3.3 can be restated geometrically in terms of a property of the Poulsen simplex. The Poulsen simplex $\mathbb{P}$ has the following universality and homogeneity property for faces: any metrizable Choquet simplex is affinely homeomorphic to a closed proper face of $\mathbb{P}$, and any affine homeomorphism between closed proper faces of $\mathbb{P}$ extends to an affine homeomorphism of $\mathbb{P}$ [36, Theorems 2.3 and 2.5]. This can be seen as a consequence of the following geometric version of Proposition 3.2.

Proposition 3.8. *Let F be a metrizable Choquet simplex, and let $s : A(\mathbb{P}) \to A(F)$ be a unital quotient. The following assertions are equivalent:*

(1) *s belongs to the $\mathrm{Aut}(\mathbb{P})$-orbit of the generic $A(F)$-state $\Omega_{A(F)}$.*

(2) *There is a closed proper face $\bar{F}$ of $\mathbb{P}$ affinely homeomorphic to F such that s is the restriction map $A(\mathbb{P}) \to A(\bar{F})$, $f \mapsto f\restriction_{\bar{F}}$.*

In particular, the $\mathrm{Aut}(\mathbb{P})$*-orbit of the generic state $\Omega_{\mathbb{R}} : A(\mathbb{P}) \to \mathbb{R}$ is the set of extreme points of $\mathbb{P}$.*

Hence, Theorem 3.3 can be reformulated as follows.

Theorem 3.9. *Suppose that F is a closed proper face of the Poulsen simplex $\mathbb{P}$. Then the pointwise stabilizer of F with respect to the canonical action* $\mathrm{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$ *is extremely amenable.* ∎

3.4. The universal minimal flows of $\mathbb{P}$

Using Theorem 3.9 we can compute the universal minimal flow of the affine homeomorphism group $\mathrm{Aut}(\mathbb{P})$ of the Poulsen simplex.

Theorem 3.10. *The universal minimal flow of* $\mathrm{Aut}(\mathbb{P})$ *is the canonical action* $\mathrm{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$.

Proof. The action $\mathrm{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$ is minimal by a result of Glasner [19]. This can be seen directly using the homogeneity property of $A(\mathbb{P})$ and the fact that for any $\varepsilon > 0$ and $d \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for any state s on ℓ_∞^d and t on ℓ_∞^m there exists a unital linear isometry $\phi : \ell_\infty^d \to \ell_\infty^m$ such that $\|t \circ \phi - s\| < \varepsilon$. Consider the generic state $\Omega_{\mathbb{R}}$ on $A(\mathbb{P})$. We know from Proposition 3.8 that the state $\Omega_{\mathbb{R}}$ is an extreme point of $\mathbb{P}$, whose $\mathrm{Aut}(\mathbb{P})$-orbit is dense. The stabilizer $\mathrm{Aut}(\boldsymbol{A(\mathbb{P})}_{\mathbb{R}})$ of $\Omega_{\mathbb{R}}$ is extremely amenable by Theorem 3.3. The canonical $\mathrm{Aut}(\mathbb{P})$-equivariant map from the quotient $\mathrm{Aut}(\mathbb{P})$-space $\mathrm{Aut}(\mathbb{P}) /\!\!/ \mathrm{Aut}(\boldsymbol{A(\mathbb{P})}_{\mathbb{R}})$ to $\mathbb{P}$ is a uniform equivalence. This follows from the homogeneity property of $\boldsymbol{A(\mathbb{P})}_{\mathbb{R}}$ as the Fraïssé limit of the class of finite-dimensional function systems with a distinguished state; see also [37, §5.4]. This allows one to conclude via a standard argument—see [44, Theorem 1.2]—that the action $\mathrm{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$ is the universal minimal compact $\mathrm{Aut}(\mathbb{P})$-space. ∎

The universal minimal flows for the non-commutative versions of the Poulsen simplex have been computed in [5, 7]. It has recently been proved in [10] that the situation in Theorem 3.10 is typical. Whenever G is a Polish group whose universal compact G-space $M(G)$ is metrizable, there exists a closed extremely amenable subgroup H of G such that the completion of the homogeneous quotient G-space G/H is G-equivariantly homeomorphic to $M(G)$.

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