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Improved bounds for Hadwiger's covering problem via thin-shell estimates

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Abstract. A central problem in discrete geometry, known as Hadwiger's covering problem, asks what the smallest natural number $N(n)$ is such that every convex body in $\mathbb{R}^n$ can be covered by a union of the interiors of at most $N(n)$ of its translates. Despite continuous efforts, the best general upper bound known for this number remains as it was more than sixty years ago, of the order of $\binom{2n}{n} n \ln n$.

In this note, we improve this bound by a subexponential factor. That is, we prove a bound of the order of $\binom{2n}{n} e^{-c\sqrt{n}}$ for some universal constant $c > 0$.

Our approach combines ideas from [3] by Artstein-Avidan and the second named author with tools from asymptotic geometric analysis. One of the key steps is proving a new lower bound for the maximum volume of the intersection of a convex body K with a translate of $-K$; in fact, we get the same lower bound for the volume of the intersection of K and $-K$ when they both have barycenter at the origin. To do so, we make use of measure concentration, and in particular of thin-shell estimates for isotropic log-concave measures.

Using the same ideas, we establish an exponentially better bound for $N(n)$ when restricting our attention to convex bodies that are ψ_2. By a slightly different approach, an exponential improvement is established also for classes of convex bodies with positive modulus of convexity.

Keywords. Covering number, convex body, measure of symmetry, isotropic, thin-shell, concentration

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1. Introduction

1.1. Hadwiger's covering problem

A long-standing problem in discrete geometry asks whether every convex body in $\mathbb{R}^n$ can be covered by a union of at most 2^n translates of its interior. It also asks whether 2^n translates are needed only for affine images of the n-cube.

This problem was posed by Hadwiger [30] for $n \geq 3$ but was already considered and settled for $n = 2$ a few years earlier by Levi [37]. An equivalent formulation, in which the interior of the convex body is replaced by smaller homothetic copies of it, was independently posed by Gohberg and Markus [27]. Other equivalent formulations of this problem were posed by Hadwiger [31] and Boltyanskiĭ [12] in terms of illuminating the boundary of a convex body by outer light sources. For a comprehensive survey of this problem and most of the progress made so far towards its solution see e.g. [8, 18, 44].

Putting things formally, a subset of $\mathbb{R}^n$ is called a *convex body* if it is a compact convex set with non-empty interior. The *covering number* of a set $A \subseteq \mathbb{R}^n$ by a set $B \subseteq \mathbb{R}^n$ is given by

$$N(A, B) = \min \left\{ N \in \mathbb{N} : \exists x_1, \ldots, x_N \in \mathbb{R}^n \text{ such that } A \subseteq \bigcup_{i=1}^{N} \{x_i + B\} \right\},$$

where $x + B = \{x + b : b \in B\}$. Denoting the interior of B by $\operatorname{int} B$ and letting $\lambda B = \{\lambda b : b \in B\}$ for $\lambda \in \mathbb{R}$, Hadwiger's conjecture states the following.

Conjecture. *Let $K \subseteq \mathbb{R}^n$ be a convex body. Then for some $0 < \lambda < 1$ one has $N(K, \lambda K) \leq 2^n$, or equivalently $N(K, \operatorname{int} K) \leq 2^n$. Moreover, equality holds only if K is an affine image of the n-cube.*

The currently best general upper bound known for $n \geq 3$ is $\binom{2n}{n}(n \ln n + n \ln\ln n + 5n)$, while the best bound for centrally-symmetric convex bodies (i.e. convex bodies K satisfying $K = -K$) is $2^n(n \ln n + n \ln\ln n + 5n)$. Both bounds are simple consequences of Rogers' estimates [48] for the asymptotic lower density of a covering of the whole space by translates of a general convex body, combined with the Rogers–Shephard inequality [49], as can be seen in [24] and [50]. For results in small dimensions, see [5, 9, 10, 14, 20, 33, 34, 46, 47]. We also mention in passing that Hadwiger's conjecture has been confirmed for certain classes of convex bodies such as constant width and fat spindle bodies (see [6, 51]), belt bodies (see [13, 15–17, 41]), bodies of Helly dimension 2 (see [14]), dual cyclic polytopes (see [7, 52]). We refer to the aforementioned surveys for a detailed account.

A fractional version of the illumination problem was considered by Naszódi [43], where the upper bounds of 2^n for the centrally-symmetric case, and $\binom{2n}{n}$ for the general case were obtained. The same bounds, as well as the extremity of the n-cube in the centrally-symmetric case, were established by Artstein-Avidan and the second named author [3] by considering fractional covering numbers of convex bodies. Moreover, together with an inequality linking integral covering numbers and fractional covering numbers (see Section 3 below), the aforementioned best known upper bounds for

Hadwiger's classical problem were recovered (technically, only the bound in the centrally-symmetric case was explicitly recovered, but the proof of the general bound is almost verbatim the same). These bounds were recovered once more in [39]. For additional recent results on Hadwiger's problem, see [38, 54], and references therein.

1.1.1. Main results. We combine ideas from [3] with a new result on the Kövner–Besicovitch measure of symmetry for convex bodies, which we discuss in Section 1.2. As a result, we obtain a new general upper bound for Hadwiger's problem:

Theorem 1.1. *There exist universal constants $c_1, c_2 > 0$ such that for all $n \geq 2$ and every convex body $K \subseteq \mathbb{R}^n$, one has*

$$N(K, \operatorname{int} K) \leq c_1 4^n e^{-c_2\sqrt{n}}.$$

For ψ_2 bodies (for definitions and more details see Section 2 below), we obtain the following exponential improvement:

Theorem 1.2. *Let $K \subseteq \mathbb{R}^n$ be a convex body with barycenter at the origin which is ψ_2 with constant $b_2 > 0$. Then*

$$N(K, \operatorname{int} K) \leq c_1 4^n e^{-c_2 b_2^{-2} n}.$$

1.2. The Kövner–Besicovitch measure of symmetry

Denote the family of all convex bodies in $\mathbb{R}^n$ by $\mathcal{K}_n$. Denote the Lebesgue volume of a measurable set $A \subseteq \mathbb{R}^n$ by $|A|$.

Let $K \subseteq \mathbb{R}^n$ be a convex body. Given a point $x \in \mathbb{R}^n$, let us call here the set

$$(K - x) \cap (x - K)$$

the *symmetric intersection* of K *at* x. As defined by Grünbaum [55], the following is a measure of symmetry for K, referred to as the *Kövner–Besicovitch measure of symmetry*:

$$\Delta_{KB}(K) = \max_{x \in \mathbb{R}^n} \frac{|(K - x) \cap (x - K)|}{|K|} = \max_{x \in \mathbb{R}^n} \frac{|K \cap (x - K)|}{|K|}.$$

To study this quantity, throughout this paper, we use the fact that the volume of the symmetric intersection of a convex body at a point x is the same as its convolution square at $2x$, i.e., we have the convolution relation

$$|(K - x) \cap (x - K)| = |K \cap (2x - K)| = (\mathbb{1}_K * \mathbb{1}_K)(2x),$$

where $\mathbb{1}_K$ is the indicator function of K. Combining this with the fact that the support of $\mathbb{1}_K * \mathbb{1}_K$ is $2K$, one easily sees by integration that

$$\min_{K \in \mathcal{K}_n} \Delta_{KB}(K) \geq 2^{-n}. \tag{1.1}$$

Denote by $\mathrm{b}(K)$ the barycenter of K. By fixing this as the point of reference, one may consider the volume ratio of the symmetric intersection of K at its barycenter as another measure of symmetry for K. A result of V. Milman and Pajor [42] tells us that

$$\frac{|(K - \mathrm{b}(K)) \cap (\mathrm{b}(K) - K)|}{|K|} \geq 2^{-n}. \tag{1.2}$$

The optimal lower bound, in both instances, is not known and conjectured to be attained by the simplex, which would imply a lower bound of the order of $(2/e)^n$ (see e.g. [55], [53] for more details).

1.2.1. A new lower bound. Our second goal in this note is to improve both (1.1) and (1.2). We consider two approaches, both of which involve using the property of a (properly normalised) log-concave measure to concentrate in a thin-shell, and in particular a quantitative form of it by Guédon and E. Milman [29]. More precisely, let X and Y be independent random vectors, uniformly distributed on a convex body $K \subseteq \mathbb{R}^n$. Our first approach is based on the comparison of the measure of a ball, whose boundary is between the two thin shells around which the distributions of X and $(X+Y)/2$ are concentrated, according to each of these measures; this leads to the improvement of (1.1).

The second approach, which allows us to bound the volume of the symmetric intersection of K at its barycenter and to improve (1.2), combines the above mentioned thin-shell estimates of Guédon and E. Milman with the notion of entropy. Given that there is not much reason to believe our bounds are optimal, we have chosen to present both approaches since either might have the potential to give further improvements.

To turn to details, we prove the following:

Theorem 1.3. *For some universal constant $c > 0$, we have*

$$\min_{K \in \mathcal{K}_n} \Delta_{KB}(K) \geq \min_{K \in \mathcal{K}_n : \mathrm{b}(K)=0} \frac{|K \cap (-K)|}{|K|} \geq \frac{\exp(cn^{1/2})}{2^n}.$$

Theorem 1.3 is a particular consequence of Propositions 2.2 and 5.3 below, which provide a lower bound for $\Delta_{KB}(K)$ and $|K \cap (-K)|/|K|$ by taking into account the ψ_α behavior of the convex body K (for definitions and more details see Section 2 below). In particular, for ψ_2 bodies, we have the following exponential improvement of (1.1) and (1.2).

Corollary 1.4 (of Propositions 2.2 and 5.3). *Let $K \subseteq \mathbb{R}^n$ be a convex body centered at the origin which is ψ_2 with constant $b_2 > 0$. Then*

$$\Delta_{KB}(K) \geq \frac{|K \cap (-K)|}{|K|} \geq \frac{\exp(cb_2^{-2}n)}{2^n}.$$

1.3. Positive modulus of convexity

The *modulus of convexity* of a centered convex body $K \subseteq \mathbb{R}^n$ is defined by

$$\delta_K(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|_K : \|x\|_K, \|y\|_K \leq 1,\ \|x-y\|_K \geq \varepsilon\right\},$$

where $\|x\|_K = \inf\{r > 0 : x \in rK\}$ is the gauge function of K. We say that K is *uniformly convex* if $\delta_K(\varepsilon) > 0$ for all $0 < \varepsilon < 2$. Note that in the finite-dimensional case, $K \subseteq \mathbb{R}^n$ is strictly convex (i.e. the boundary of K contains no line segments) if and only if it is uniformly convex.

Using a different concentration result of Arias-De-Reyna, Ball, and Villa [1], which was generalized by Gluskin and Milman [26], we extend Theorems 1.2 and 1.4 to the class of convex bodies whose modulus of convexity is positive for some $0 < \varepsilon < \sqrt{2}$. More precisely, for $0 < r < 1$ and $0 < \varepsilon < \sqrt{2}$, let $\mathcal{K}_{n,r,\varepsilon}$ be the class of centered convex bodies $K \subseteq \mathbb{R}^n$ for which $\delta_K(\varepsilon) \geq r$.

Theorem 1.5. *Let $0 < r < 1$, $0 < \varepsilon < \sqrt{2}$, and let $K \in \mathcal{K}_{n,r,\varepsilon}$. Then, for $\alpha := 1 - \exp(-(\sqrt{2} - \varepsilon)^2 n/4)$, we have*

$$\Delta_{KB}(K) \geq \alpha\, 2^{-n} \left(\frac{1}{1-r}\right)^n, \qquad \frac{|K \cap (-K)|}{|K|} \geq \frac{1}{e\sqrt{n}}\, 2^{-n} \left(\frac{1}{1-\alpha r}\right)^n.$$

Theorem 1.6. *Let $0 < r < 1$, $0 < \varepsilon < \sqrt{2}$, and let $K \in \mathcal{K}_{n,r,\varepsilon}$. Then*

$$N(K, \operatorname{int} K) \leq (1 - e^{-(\sqrt{2}-\varepsilon)^2 n/4})^{-1} (4(1-r))^n.$$

The paper is organized as follows. In Section 2 we prove the first part of Theorem 1.3 and of Corollary 1.4 (the bounds for the Kövner–Besicovitch measure of symmetry), and in Section 3 we apply these to Hadwiger's covering problem. Section 4 is devoted to the respective bounds in the case of uniformly convex bodies, i.e. the first part of Theorem 1.5 as well as Theorem 1.6. Finally, in Section 5 we complete the proofs of Theorems 1.3 and 1.5 and of Corollary 1.4 by showing via our second approach how to bound the volume of the symmetric intersection of K at its barycenter as well. A couple of concluding remarks are gathered at the end, including an application to a conjecture by Ehrhart in the geometry of numbers.

2. Bounding the convolution square

This section is devoted to the proof of Proposition 2.2 below. To that end, we need to recall some facts and results.

Denote the standard Euclidean inner product on $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$, and the corresponding Euclidean norm on $\mathbb{R}^n$ by $\|\cdot\|_2$. We shall also denote probability by $\mathbb{P}$ and expectation by $\mathbb{E}$.

Recall that a random vector in $\mathbb{R}^n$ is called *isotropic* if $\mathbb{E}X = 0$ (i.e., its barycenter is the origin) and $\mathbb{E}(X \otimes X) = \mathrm{Id}$ (i.e., its covariance matrix is the identity). We say that X is ψ_α with constant b_α if

$$(\mathbb{E}|\langle X, y\rangle|^p)^{1/p} \leq b_\alpha p^{1/\alpha} (\mathbb{E}|\langle X, y\rangle|^2)^{1/2} \qquad \forall p \geq 2,\ \forall y \in \mathbb{R}^n.$$

A function $f : \mathbb{R}^n \to [0, \infty)$ is called *log-concave* if $\ln f$ is concave on the support of f. It is well-known that any random vector X in $\mathbb{R}^n$ with a log-concave density is ψ_1 with $b_1 \leq C$, for some universal constant $C > 0$ (see e.g. [2, p. 115]).

We shall need the following thin-shell deviation estimate of Guédon and E. Milman:

Theorem 2.1 ([29, Theorem 1.1]). *Let X denote an isotropic random vector in $\mathbb{R}^n$ with log-concave density, which is in addition ψ_α ($\alpha \in [1, 2]$) with constant b_α. Then*

$$\mathbb{P}\left(\left|\|X\|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right) \leq C \exp\left(-c' b_\alpha^{-\alpha} \min(t^{2+\alpha}, t) n^{\alpha/2}\right) \qquad \forall t \geq 0,$$

where $c' > 0$ is some universal constant.

We remark that the dependence on n in Theorem 2.1 is optimal, while the dependence on t was recently improved by Lee and Vempala [35] in the ψ_1 case. However, in our approach t is going to be some fixed number which is bounded away from 0, thus optimizing over it cannot yield better bounds.

Proposition 2.2. *Suppose K is a convex body centered at the origin which is ψ_α with constant b_α. Then, for some universal constant $c > 0$,*

$$\Delta_{KB}(K) \geq \frac{\exp(c b_\alpha^{-\alpha} n^{\alpha/2})}{2^n}.$$

We remark that Theorem 1.3 is a particular consequence of Proposition 2.2, as all random vectors with log-concave densities are ψ_1 with the same universal constant.

Proof of Proposition 2.2. Let X and Y be independent random vectors, uniformly distributed on K. Since $\Delta_{KB}(K)$ is affine invariant, we may assume without loss of generality that K is *in isotropic position*: this means that $|K| = 1$, $\mathrm{b}(K) = 0$ as already assumed, and $\mathbb{E}(X \otimes X)$ is a multiple of the identity,

$$\mathbb{E}(X \otimes X) = L_K^2 \,\mathrm{Id}$$

where L_K is called the *isotropic constant* of K (note that this is another well-defined affine invariant of K). Equivalently, we ask that $|K| = 1$ and X/L_K is isotropic as defined above.

We are now looking for a lower bound for $\|f\|_\infty$ where $f = \mathbb{1}_K * \mathbb{1}_K$ is the density function for the random vector $X + Y$. Instead, we shall work with $(X + Y)/2$ so that both $(X + Y)/2$ and X have the same support. The probability density function of $(X + Y)/2$ is then $g(x) = f(2x)2^n$. There are many nice properties that $(X + Y)/2$ inherits from X. In particular, $(X + Y)/2$ has a centered log-concave density (the latter is a consequence of the Prékopa–Leindler inequality, see e.g. [2]). Moreover,

$$\begin{aligned}\mathbb{E}_{X,Y}\left(\frac{X+Y}{2}\right) \otimes \left(\frac{X+Y}{2}\right) &= \tfrac{1}{4}\mathbb{E}_{X,Y}(X \otimes X + X \otimes Y + Y \otimes X + Y \otimes Y) \\ &= \tfrac{1}{4}(L_K^2 I + 0 + 0 + L_K^2 I) = \tfrac{1}{2} L_K^2 I.\end{aligned}$$

Thus, $(X + Y)/2$ is isotropic up to scaling. Finally, $(X + Y)/2$ has more or less the same ψ_α behavior as X. Indeed, the above computations already show that

$$\left(\mathbb{E}\left|\left\langle \frac{X+Y}{2}, y \right\rangle\right|^2\right)^{1/2} = \frac{1}{\sqrt{2}} L_K \|y\|_2 = \frac{1}{\sqrt{2}} (\mathbb{E}|\langle X, y\rangle|^2)^{1/2}$$

for every $y \in \mathbb{R}^n$, hence a single application of Minkowski's inequality gives

$$\left(\mathbb{E}\left|\left\langle \frac{X+Y}{2}, y\right\rangle\right|^p\right)^{1/p} \le 2(\mathbb{E}|\langle X/2, y\rangle|^p)^{1/p} = (\mathbb{E}|\langle X, y\rangle|^p)^{1/p}$$
$$\le b_\alpha p^{1/\alpha}(\mathbb{E}|\langle X, y\rangle|^2)^{1/2} = \sqrt{2}\, b_\alpha p^{1/\alpha}\left(\mathbb{E}\left|\left\langle \frac{X+Y}{2}, y\right\rangle\right|^2\right)^{1/2},$$

assuming X is ψ_α with constant b_α. It is worth remarking however that, for our proof here, the fact that the distribution of $(X+Y)/2$ is ψ_1 suffices (and, as already mentioned, this is true for every log-concave distribution).

Observe now that for any $r > 0$ we have

$$\|g\|_\infty \ge \frac{\int_{rL_K\sqrt{n}\,B_2^n\cap K} g(x)\,dx}{\int_{rL_K\sqrt{n}\,B_2^n\cap K} 1\,dx} = \frac{\mathbb{P}\big(\|\frac{X+Y}{2}\|_2 \le rL_K\sqrt{n}\big)}{\mathbb{P}\big(\|X\|_2 \le rL_K\sqrt{n}\big)}.$$

Since $\mathbb{E}_{X,Y}\|\frac{X+Y}{2}\|_2^2 = \frac{1}{2}nL_K^2$ and $\mathbb{E}_X\|X\|_2^2 = nL_K^2$, we know that the distributions of X and $(X+Y)/2$ are concentrated within two different thin-shells. Thus, for $1/\sqrt{2} < r < 1$, we find that $\mathbb{P}_{X,Y}\big(\|\frac{X+Y}{2}\|_2 \le rL_K\sqrt{n}\big)$ is almost 1 since the set considered includes the "good" thin-shell of $(X+Y)/2$. On the other hand, $\mathbb{P}(\|X\|_2 \le rL_K\sqrt{n})$ is almost 0 since the set considered excludes the corresponding thin-shell of X. To quantify this, we apply Theorem 2.1: for any isotropic ψ_α log-concave vector Z the inequality in Theorem 2.1 is split into

$$\mathbb{P}(\|Z\|_2 \le (1-t)\sqrt{n}) \le C\exp\big(-c' b_\alpha^{-\alpha}\min(t^{2+\alpha}, t)\sqrt{n}\big) \qquad \forall t \in [0,1],$$
$$\mathbb{P}(\|Z\|_2 \ge (1+t)\sqrt{n}) \le C\exp(-c' b_\alpha^{-\alpha}\min(t^{2+\alpha}, t)\sqrt{n}) \qquad \forall t \ge 0.$$

Since we shall apply the first one with Z replaced by X/L_K and the second one with Z replaced by $\frac{X+Y}{2}\cdot\frac{\sqrt{2}}{L_K}$, we need $1-t = \frac{1+t}{\sqrt{2}}$ and hence $t = \frac{\sqrt{2}-1}{\sqrt{2}+1}$. We thus obtain

$$\mathbb{P}\left(\|X\|_2 \le \frac{2}{\sqrt{2}+1}L_K\sqrt{n}\right) \le \exp(-c' b_\alpha^{-\alpha} n^{\alpha/2}),$$

and

$$\mathbb{P}\left(\left\|\frac{X+Y}{2}\right\|_2 \le \frac{2}{\sqrt{2}+1}L_K\sqrt{n}\right) \ge 1 - \exp(-c' b_\alpha^{-\alpha} n^{\alpha/2}).$$

Therefore, we conclude that for some universal constant $c > 0$,

$$\|g\|_\infty \ge \exp(c b_\alpha^{-\alpha} n^{\alpha/2}),$$

and equivalently

$$\Delta_{KB}(K) = \frac{\|g\|_\infty}{2^n} \ge \frac{\exp(c b_\alpha^{-\alpha} n^{\alpha/2})}{2^n}. \qquad \blacksquare$$

3. A new bound for Hadwiger's covering problem

This section is devoted to the proof of Theorems 1.1 and 1.2. To that end, we need some preliminaries.

Let $\overline{N}(A,B) = \min\{N : \exists x_1,\dots,x_N \in A \text{ such that } A \subseteq \bigcup_{i=1}^N \{x_i + B\}\}$ be the covering number of A by translates of B that are centered in A. We shall need the following volume ratio bound.

Lemma 3.1. *Let $A, B \subseteq \mathbb{R}^n$ be convex bodies. Suppose B contains the origin in its interior. Then*

$$\overline{N}(A,B) \le 2^n \frac{\left|A + \frac{1}{2}(B \cap (-B))\right|}{|B \cap (-B)|}.$$

Proof. Recall that the *separation number* of A in B is defined as

$$M(A,B) = \max\{M : \exists x_1,\dots,x_M \in A \text{ such that } \forall i \neq j \ (x_i + B) \cap (x_j + B) = \emptyset\}.$$

It is an easy exercise (see e.g. [3]) to show that

$$M(A,B) \le \frac{|A+B|}{|B|}.$$

Next, note that for any convex body $T \subseteq \mathbb{R}^n$, one has $\overline{N}(A, T-T) \le M(A,T)$. Indeed, take a maximal T-separated set in A, that is, a set of points $x_1,\dots,x_M \in A$ such that for every point $x \in A$ one has $(x+T) \cap \bigcup_{i=1}^M \{x_i + T\} \neq \emptyset$. This means that $A \subseteq \bigcup_{i=1}^M \{x_i + T - T\}$ or, in other words, $\overline{N}(A, T-T) \le M(A,T)$. Since $\overline{N}(A,B) \le \overline{N}(A, B \cap (-B))$, it follows that $\overline{N}(A,B) \le M\big(A, \frac{1}{2}(B \cap (-B))\big)$, and hence

$$\overline{N}(A,B) \le 2^n \frac{\left|A + \frac{1}{2}(B \cap (-B))\right|}{|B \cap (-B)|}. \qquad \blacksquare$$

Next, we recall the notion of fractional covering numbers, as defined in [3]. Recall that $\mathbb{1}_A$ stands for the indicator function of a set $A \subseteq \mathbb{R}^n$. A sequence of pairs of points and weights, $S = \{(x_i, \omega_i) : x_i \in \mathbb{R}^n, \omega_i \in \mathbb{R}^+\}_{i=1}^N$, is said to be a *fractional covering* of a set $K \subseteq \mathbb{R}^n$ by a set $T \subseteq \mathbb{R}^n$ if for all $x \in K$ we have $\sum_{i=1}^N \omega_i \mathbb{1}_{x_i+T}(x) \ge 1$. The *total weight* of the covering is denoted by $\omega(S) = \sum_{i=1}^N \omega_i$. The *fractional covering number* of K by T is defined to be the infimal total weight over all fractional coverings of K by T and is denoted by $N_\omega(K,T)$.

We shall also need the following volume ratio bound from [3]:

Lemma 3.2 ([3, Proposition 2.9]). *Let $K, T \subseteq \mathbb{R}^n$ be convex bodies. Then*

$$N_\omega(K,T) \le \frac{|K-T|}{|T|}.$$

Finally, we shall need the following inequality that relates integral covering numbers and fractional covering numbers, and which was proved in [25], improving on a similar inequality in [3]. For any bounded Borel measurable sets, K, T_1 and T_2, one has

$$N(K, T_1 + T_2) \le N_\omega(K, T_1)\big(1 + \ln \overline{N}(K, T_2)\big). \tag{3.1}$$

To be more precise, (3.1) immediately follows from [25, Theorem 1.2], applied with $L = T_1 + T_2$ and $T = T_2$.

Proof of Theorem 1.1 . We can assume without loss of generality that $\mathrm{b}(K) = 0$. By Lemma 3.2, for $0 < \lambda < 1$ and any $x \in \mathbb{R}^n$ we have

$$\begin{aligned} N_\omega(K, \lambda K) \le N_\omega(K, \lambda(K \cap (x - K))) &\le \frac{|K - \lambda(K \cap (x - K))|}{|\lambda(K \cap (x - K))|} \\ &\le \left(\frac{1+\lambda}{\lambda}\right)^n \frac{|K|}{|K \cap (x - K)|}. \end{aligned}$$

By applying Theorem 1.3 with the point x which maximizes the above volume ratio, we get

$$N_\omega(K, \lambda K) \le \left(\frac{1+\lambda}{\lambda}\right)^n 2^n e^{-c\sqrt{n}}.$$

Using (3.1) with $T_1 = \alpha\lambda K, T_2 = (1 - \alpha)\lambda K$ for some $\alpha \in (0, 1)$, we obtain

$$N(K, \lambda K) \le \left(\frac{1+\alpha\lambda}{\alpha\lambda}\right)^n 2^n e^{-c\sqrt{n}}\big(1 + \ln \overline{N}(K, (1 - \alpha)\lambda K)\big).$$

Using Lemma 3.1 and taking the limit $\lambda \nearrow 1$, we get

$$\begin{aligned} N(K, \operatorname{int} K) &\le \left(\frac{1+\alpha}{\alpha}\right)^n 2^n e^{-c\sqrt{n}}\left(1 + \ln\left(2^n \frac{\left|K + \frac{1}{2}(1-\alpha)(K \cap (-K))\right|}{|(1-\alpha)(K \cap (-K))|}\right)\right) \\ &\le \left(\frac{1+\alpha}{\alpha}\right)^n 2^n e^{-c\sqrt{n}}\left(1 + \ln\left(\left(\frac{4}{1-\alpha}\right)^n \frac{|K|}{|K \cap (-K)|}\right)\right). \end{aligned}$$

Since K is centered at the origin, (1.2) (or its improvement in Theorem 1.3, which however cannot essentially affect the final estimate here) implies that

$$\begin{aligned} N(K, \operatorname{int} K) &\le \left(\frac{1+\alpha}{\alpha}\right)^n 2^n e^{-c\sqrt{n}}\left(1 + \ln\left(\left(\frac{4}{1-\alpha}\right)^n 2^n\right)\right) \\ &\le \left(\frac{1+\alpha}{\alpha}\right)^n 2^n e^{-c\sqrt{n}}\left(1 + n\ln\left(\frac{8}{1-\alpha}\right)\right). \end{aligned}$$

Plugging in $\alpha = 1 - 1/n$ shows that, for some universal constants $c_1, c_2 > 0$, we have

$$N(K, \operatorname{int} K) \le c_1 4^n e^{-c_2\sqrt{n}}. \qquad \blacksquare$$

The proof of Theorem 1.2 is the same as that of Theorem 1.1, except that one uses Corollary 1.4 instead of Theorem 1.3.

4. Positive modulus of convexity

Recall that the modulus of convexity of a centered convex body $K \subseteq \mathbb{R}^n$ is defined by

$$\delta_K(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|_K : \|x\|_K, \|y\|_K \leq 1,\ \|x-y\|_K \geq \varepsilon\right\}.$$

A result of Arias-De-Reyna, Ball, and Villa [1], which was generalized by Gluskin and Milman [26], tells us that if $K \subseteq \mathbb{R}^n$ is a convex body such that $0 \in \operatorname{int} K$ and $|K| = 1$ then for all $0 < \varepsilon' < 1$ one has

$$|\{(x,y) \in K \times K : \|x-y\|_K \leq \sqrt{2}\,(1-\varepsilon')\}| \leq e^{-\varepsilon'^2 n/2}. \tag{4.1}$$

We use this result to prove Theorem 1.5:

Proof of first part of Theorem 1.5. Without loss of generality, we assume that $|K| = 1$. Let X and Y be independent random vectors, uniformly distributed on K. Let $f(x) = |K \cap (x-K)|$ and recall that the density of $(X+Y)/2$ is $g(x) = 2^n f(2x)$.

Since, by assumption, $\delta_K(\varepsilon) \geq r$, the set $\theta = \{(x,y) \in K \times K : \|x-y\|_K \geq \varepsilon\}$ satisfies

$$\theta \subseteq \left\{(x,y) \in K \times K : \frac{x+y}{2} \in (1-r)K\right\}.$$

By (4.1), we have $|\theta| \geq 1 - e^{-(\sqrt{2}-\varepsilon)^2 n/4}$ and hence

$$\begin{aligned}\mathbb{P}\left(\frac{X+Y}{2} \in (1-r)K\right) &= \iint_{\{(x,y)\in K\times K : \frac{x+y}{2} \in (1-r)K\}} dx\,dy \\ &\geq \iint_{\theta} dx\,dy \geq 1 - e^{-(\sqrt{2}-\varepsilon)^2 n/4}.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\|g\|_\infty &\geq \frac{\int_{(1-r)K} g(x)\,dx}{\int_{(1-r)K} dx} = \frac{\mathbb{P}\left(\frac{X+Y}{2} \in (1-r)K\right)}{\mathbb{P}(X \in (1-r)K)} \\ &\geq \left(\frac{1}{1-r}\right)^n (1 - e^{-(\sqrt{2}-\varepsilon)^2 n/4}).\end{aligned}$$

∎

Repeating the proof of Theorem 1.1 but now using Theorem 1.5, one gets Theorem 1.6.

5. Bounding the convolution square at the barycenter

This section is devoted to the proof of Proposition 5.3 below (which will give the full proofs of Theorem 1.3 and Corollary 1.4) as well as completing that of Theorem 1.5 (the arguments will be very similar, just different applications of the same method). We recall that, for a random vector X in $\mathbb{R}^n$ with density f, we define its *entropy* as

$$\mathrm{Ent}[X] = -\int_{\mathbb{R}^n} f \ln f.$$

The conclusions of the following standard lemma are simple consequences of Jensen's inequality.

Lemma 5.1. *For any measurable function $h : \mathbb{R}^n \to [0, \infty)$ which is positive on the support of f we have*

$$\mathrm{Ent}[X] \le -\int_{\mathbb{R}^n} f \ln h + \ln\Big(\int_{\mathbb{R}^n} h\Big), \tag{5.1}$$

assuming all the quantities are finite. Moreover, if X has a log-concave density, then

$$\mathrm{Ent}[X] = \mathbb{E}[-\ln f(X)] \ge -\ln f(\mathbb{E}X). \tag{5.2}$$

Proof. To prove (5.1), we write

$$\mathrm{Ent}[X] + \int_{\mathbb{R}^n} f \ln h = \int_{\mathbb{R}^n} f \ln \frac{h}{f} \le \ln\Big(\int_{\mathbb{R}^n} h\Big),$$

with the inequality following by Jensen's inequality. As for (5.2), we note that if f is assumed log-concave, $-\ln f$ will be a convex function on $\mathbb{R}^n$, which allows one to apply Jensen's inequality again. ∎

Remark 5.2. We will apply Lemma 5.1 as follows. If $K \subset \mathbb{R}^n$ is a centered convex body, and X, Y are independent random vectors uniformly distributed on K, then the density f of X is given by $f(x) = \frac{1}{|K|}\mathbb{1}_K$, while the density g of $X + Y$ by $g(x) = \frac{1}{|K|^2}(\mathbb{1}_K * \mathbb{1}_K)(x) = \frac{1}{|K|^2}|K \cap (x - K)|$ (recall that $X + Y$ has a centered log-concave density, which is not hard to check using this identity). These show that $\mathrm{Ent}[X] = \ln |K|$, while, by (5.2),

$$-\ln\Big(\frac{|K \cap (-K)|}{|K|^2}\Big) = -\ln g(0) \le \mathrm{Ent}[X + Y].$$

Therefore,

$$-\ln\Big(\frac{|K \cap (-K)|}{|K|}\Big) = -\ln\Big(\frac{|K \cap (-K)|}{|K|^2}\Big) - \ln |K| \le \mathrm{Ent}[X + Y] - \mathrm{Ent}[X], \tag{5.3}$$

which we can combine with (5.1), applied for the vector $X + Y$, to obtain

$$-\ln\Big(\frac{|K \cap (-K)|}{|K|}\Big) \le \mathbb{E}[-\ln h(X + Y)] + \ln\Big(\int_{\mathbb{R}^n} h\Big) - \mathrm{Ent}[X] \tag{5.4}$$

for any integrable function $h : \mathbb{R}^n \to [0, \infty)$ which is positive on $2K$ (note that the first term on the right hand side depends only on values of h on $2K$, whereas the second term can only get smaller or stay the same when h is restricted to $2K$; in other words, replacing h with $h\mathbb{1}_{2K}$ might only improve the right hand side).

Observe that, by choosing h constant on $2K$ (and zero otherwise), one can recover (1.2). In the remainder of this section, we will choose different h in order to establish the improvements of (1.2) claimed earlier.

Proposition 5.3. *Suppose K is a convex body centered at the origin which is ψ_α with constant b_α. Then, for some universal constant $c > 0$,*

$$\frac{|K \cap (-K)|}{|K|} \geq \frac{\exp(c b_\alpha^{-\alpha} n^{\alpha/2})}{2^n}.$$

Proof. We begin by observing that both sides of (5.3) are invariant under invertible linear transformations of K, therefore we can assume without loss of generality that K is in isotropic position. We then apply (5.4) with $h(x) := \exp(-\lambda \|x\|_2^2)\mathbb{1}_{2K}$ for some constant λ to be specified later. The right hand side becomes

$$\begin{aligned}
&\mathbb{E}[\lambda \|X + Y\|_2^2] + \ln \int_{2K} \exp(-\lambda \|x\|_2^2)\, dx - \ln 1 \\
&\qquad\qquad = 2\mathbb{E}[\lambda \|X\|_2^2] + \ln \int_{2K} \exp(-\lambda \|x\|_2^2)\, dx \\
&\qquad\qquad = 2\lambda n L_K^2 + n \ln 2 + \ln \int_K \exp(-4\lambda \|x\|_2^2)\, dx. \qquad (5.5)
\end{aligned}$$

To estimate the last integral, we employ again the thin-shell estimates from Theorem 2.1, which imply that for $A_t := \{x \in K : \|x\|_2 \leq (1-t)\sqrt{n}\, L_K\}$, one has

$$|A_t| \leq C \exp(-c' b_\alpha^{-\alpha} t^{2+\alpha} n^{\alpha/2})$$

for all $t \in [0, 1]$. We can thus break the integral into two as follows:

$$\begin{aligned}
\int_K \exp(-4\lambda \|x\|_2^2)\, dx &= \int_{A_t} \exp(-4\lambda \|x\|_2^2)\, dx + \int_{K\setminus A_t} \exp(-4\lambda \|x\|_2^2)\, dx \\
&\leq C \exp(-c' b_\alpha^{-\alpha} t^{2+\alpha} n^{\alpha/2}) + \exp(-4\lambda (1-t)^2 n L_K^2).
\end{aligned}$$

We now set $t = 1 - 2/\sqrt{5}$ say, and then we choose our λ so that

$$c' b_\alpha^{-\alpha} t^{2+\alpha} n^{\alpha/2} = 4\lambda (1-t)^2 n L_K^2.$$

It follows that λ is of the order of $b_\alpha^{-\alpha} n^{\alpha/2-1} L_K^{-2}$. Combining these estimates with (5.4) and (5.5), we obtain

$$\begin{aligned}
-\ln\left(\frac{|K \cap (-K)|}{|K|}\right) &\leq 2\lambda n L_K^2 + n \ln 2 + \ln(C+1) - \tfrac{16}{5}\lambda n L_K^2 \\
&= n \ln 2 + \ln(C+1) - \tfrac{6}{5}\lambda n L_K^2 \\
&= n \ln 2 + \ln(C+1) - c'' b_\alpha^{-\alpha} n^{\alpha/2}
\end{aligned}$$

for some absolute constant c'' (which we can compute explicitly by the above relations). Exponentiating, we complete the proof. ∎

Proof of second part of Theorem 1.5. This time we only assume for simplicity that $|K| = 1$, and we apply (5.4) with $h(x) := \exp(-\lambda\|x\|_K)$ for some constant λ to be specified later. We immediately get

$$\begin{aligned} -\ln\left(\frac{|K \cap (-K)|}{|K|}\right) &\leq \mathbb{E}[\lambda\|X + Y\|_K] + \ln \int_{\mathbb{R}^n} \exp(-\lambda\|x\|_K)\, dx \\ &= \lambda\mathbb{E}[\|X + Y\|_K] + \ln(\lambda^{-n} n!|K|) \\ &= \lambda\mathbb{E}[\|X + Y\|_K] - n \ln \lambda + \ln(n!). \end{aligned}$$

Optimizing over λ yields

$$-\ln\left(\frac{|K \cap (-K)|}{|K|}\right) \leq n \ln \mathbb{E}[\|X + Y\|_K] + \ln \frac{n!e^n}{n^n}. \tag{5.6}$$

Given that $n! \leq en^{n+1/2}e^{-n}$, the last term is upper-bounded by $\ln(e\sqrt{n})$, so the final estimate will depend on how well we can bound $\mathbb{E}[\|X + Y\|_K]$. We will again use the concentration result of Arias-De-Reyna, Ball, and Villa. Note that by the triangle inequality $\|X + Y\|_K \leq 2$, and therefore, by the definition of the modulus of convexity, we have for any $\varepsilon \in (0, 2)$,

$$\begin{aligned} \mathbb{E}[\|X + Y\|_K] &= \mathbb{E}[\|X + Y\|_K \mathbb{1}_{\|X-Y\|_K \leq \varepsilon}] + \mathbb{E}[\|X + Y\|_K \mathbb{1}_{\|X-Y\|_K > \varepsilon}] \\ &\leq 2\mathbb{P}(\|X - Y\|_K \leq \varepsilon) + 2(1 - \delta_K(\varepsilon))\mathbb{P}(\|X - Y\|_K > \varepsilon) \\ &= 2[1 - \delta_K(\varepsilon)\mathbb{P}(\|X - Y\|_K > \varepsilon)]. \end{aligned}$$

Applying this now with some $\varepsilon \in (0, \sqrt{2})$ for which $\delta_K(\varepsilon) \geq r$, and recalling (4.1), we obtain

$$\begin{aligned} \mathbb{E}[\|X + Y\|_K] &\leq 2[1 - \delta_K(\varepsilon)(1 - \exp(-(\sqrt{2} - \varepsilon)^2 n/4))] \\ &\leq 2[1 - r(1 - \exp(-(\sqrt{2} - \varepsilon)^2 n/4))], \end{aligned}$$

which we can plug into (5.6) to complete the proof. ∎

6. Concluding remarks

We conclude this note with some remarks, questions and conjectures.

Conjecture 6.1. *There exists a universal constant $c > 0$ such that for every centered convex body $K \subseteq \mathbb{R}^n$ and some $0 < r < 1$ one has*

$$\frac{\mathbb{P}\left(\frac{X+Y}{2} \in rK\right)}{\mathbb{P}(X \in rK)} \geq (1 + c)^n,$$

where X and Y are independent random vectors, uniformly distributed on K.

We remark that the above conjecture implies an exponentially better upper bound for Hadwiger's covering problem. Moreover, the conjecture seems interesting in its own right and attempts in a way to quantify the intuition that the convolution of a uniform distribution with itself looks already more like a "bell curve" than like the flat distribution it originates from.

Another question that would capture this if answered in the affirmative is the following. Let X and Y be independent random vectors, uniformly distributed on a centred convex body K. Is it true that $\mathbb{E}\|X+Y\|_K \le 2-\Omega(n^{-\alpha})$ with $\alpha \in [0,1)$ independent of K? Or rather, given such an α, for which convex bodies in $\mathbb{R}^n$ does this bound hold? (Note that any such bound would improve on the trivial upper bound coming from the triangle inequality, which totally neglects independence: $\mathbb{E}\|X+Y\|_K \le 2\mathbb{E}\|X\|_K = 2\big(1-\frac{1}{n+1}\big)$.) In the previous section we proved that $\mathbb{E}\|X+Y\|_K$ is indeed upper-bounded by a constant smaller than 2 for convex bodies with a positive modulus of convexity. For the cube, however, it can be checked that $\mathbb{E}\|X+Y\|_K = 2\big(1-\frac{4^n}{(2n+1)\binom{2n}{n}}\big) \sim 2-\sqrt{\pi/n}$. Thus, we can also ask whether, in general, the bound $2-\Omega(n^{-1/2})$ is the worst case. If this is true, it would give another proof for our Main Theorems 1.1 and 1.3.

The quantity $\mathrm{Ent}[X+Y]-\mathrm{Ent}[X]$ which appears on the right hand side of (5.3) has been studied in the context of reverse entropy power inequalities for convex measures, a natural generalization of log-concave measures (see [11] and [40]). The upper bounds obtained there (when specialized to the log-concave case) as well as our improved bounds are perhaps far from optimal. To the best of our knowledge, a sharp upper bound is not known even in dimension 1. We believe the extremizer would be a one-sided exponential distribution.

Furthermore, in higher dimensions we can conjecture the following: for some universal constant $\varepsilon > 0$ and for every i.i.d. log-concave random vectors X and Y in $\mathbb{R}^n$, $\mathrm{Ent}[X+Y]-\mathrm{Ent}[X] \le n(\ln 2-\varepsilon)$. An even more ambitious guess here is that the extremizer should be the product one-sided exponential distribution (giving the upper bound $n\gamma$ with $\gamma = 0.57\ldots$ denoting the Euler–Mascheroni constant) and the simplex for uniform distributions on convex bodies.

Recall that our strategy from the proof of Proposition 5.3 for bounding $\mathrm{Ent}[X+Y]-\mathrm{Ent}[X]$ was to normalize X to be isotropic and choose a Gaussian function $h(x) = \exp(-\lambda\|x\|_2^2)$ in (5.3). Note however that any further improvements while working with this function h might be particularly hard: by relying on now classical volume concentration results as well as on reductions for the slicing problem from [19], it is possible to check that if this choice for h yields the bound $n(\ln 2-\varepsilon)$ for uniform random vectors, then this also implies logarithmic bounds for the slicing problem.

Theorem 1.3 has an immediate application in the geometry of numbers, and in particular to Ehrhart's conjecture from [21, 22]. This conjecture states that for every convex body K in $\mathbb{R}^n$ with barycenter at the origin and such that the only lattice point of $\mathbb{Z}^n$ in the interior of K is the origin, we have $|K| \le (n+1)^n/n!$ (with equality attained when K is the simplex $K = (n+1)\,\mathrm{conv}\{0,e_1,\ldots,e_n\}-(1,\ldots,1)$).

Ehrhart's conjecture has been confirmed for $n=2$ by Ehrhart, and in some special cases (see [4, 23, 45]), but it remains open in general for $n \ge 3$. The general bound

$|K| \le (n+1)^n(1-(1-1/n)^n)$ was established in [28]. The better bound $|K| \le 4^n$ is a direct consequence of a more general result, namely [36, Proposition 1.1], concerning a strengthening of Ehrhart's conjecture; see also [32] for a simpler derivation of the bound $|K| \le 4^n$, which we also follow below.

Both derivations of this bound made use of the Milman–Pajor inequality (1.2), so Theorem 1.3 allows us now to obtain the following improvement.

Proposition 6.2. *Let K be a convex body in $\mathbb{R}^n$ with $\mathrm{b}(K) = 0$ and $\mathrm{int}(K) \cap \mathbb{Z}^n = \{0\}$. Then $|K| \le 4^n e^{-c\sqrt{n}}$, where $c > 0$ is a universal constant.*

Proof. Since $K \cap (-K)$ is origin-symmetric and its interior contains no lattice point other than the origin, by Minkowski's theorem we obtain $|K \cap (-K)| \le 2^n$. Thus, Theorem 1.3 gives $2^n/|K| \ge 2^{-n}e^{c\sqrt{n}}$. ∎

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