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A polynomial excluded-minor approximation of treedepth

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Abstract. Treedepth is a minor-monotone graph invariant in the family of "width measures" that includes treewidth and pathwidth. The characterization and approximation of these invariants in terms of excluded minors has been a topic of interest in the study of sparse graphs. A celebrated result of Chekuri and Chuzhoy (2014) shows that treewidth is polynomially approximated by the largest $k \times k$ grid minor in a graph. In this paper, we give an analogous polynomial approximation of treedepth via three distinct obstructions: grids, balanced binary trees, and paths. Namely, we show that there is a constant c such that every graph with treedepth $\Omega(k^c)$ has at least one of the following minors (each of treedepth at least k):

- a $k \times k$ grid,
- a complete binary tree of height k, or
- a path of order 2^k.

Moreover, given a graph G we can, in randomized polynomial time, find an embedding of one of these minors or conclude that treedepth of G is at most $O(k^c)$. This result has applications in various settings where bounded treedepth plays a role. In particular, we describe one application in finite model theory, an improved homomorphism preservation theorem over finite structures [Rossman, 2017], which was the original motivation for our investigation of treedepth.

Keywords. Treedepth, excluded minor

1. Introduction

The *treedepth* of a graph G is defined as the minimum height of a rooted forest F with the same set of vertices such that any two adjacent vertices in G have an ancestor-descendant relationship in F. This well-studied graph invariant arises in many settings and has several equivalent characterizations: it appears in the literature as vertex ranking number [35], ordered chromatic number [21] and minimum elimination tree height [29], before being systematically studied as treedepth by Ossona de Mendes and Nešetřil [26]. Bounded treedepth graphs play an important role in areas such as the theory of sparse graph classes [27, 28], parameterized complexity theory [12, 17, 30], and finite model theory [33].

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Treedepth (**td** for short) belongs to a family of decomposition-based "width measures" that includes treewidth (**tw**) and pathwidth (**pw**). (See Section 3 for definitions.) Roughly speaking, whereas **tw** and **pw** measure how far a graph is from being a tree or path, **td** measures how far a graph is from being a star (a connected graph with at most one vertex of degree > 1); in particular, graphs with $\mathbf{tw} = 1$, $\mathbf{pw} = 2$ and $\mathbf{td} = 2$ are precisely unions of trees, paths and stars. These three invariants are related by inequalities

$$\mathbf{tw}(G) + 1 \leq \mathbf{pw}(G) + 1 \leq \mathbf{td}(G) \leq (\mathbf{tw}(G) + 1) \cdot \log |V(G)|. \tag{1}$$

Treedepth is also tied to the length (number of vertices) of the longest path in G, denoted $\mathbf{lp}(G)$:

$$\log(\mathbf{lp}(G) + 1) \leq \mathbf{td}(G) \leq \mathbf{lp}(G) \tag{2}$$

where $\log(\cdot)$ is the base-2 logarithm. (See [27, Chapter 6] for proofs of (1) and (2).)

Invariants **td**, **tw**, **pw** and **lp** are all minor-monotone, that is, non-decreasing under the graph minor relation. Recall that a graph H is a *minor* of G, denoted $H \preceq G$, if H can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. A graph invariant $f : \{\text{graphs}\} \to \mathbb{N}$ is *minor-monotone* if $f(H) \leq f(G)$ for all $H \preceq G$. Equivalently, f is minor-monotone if the class $\{G : f(G) \leq k\}$ is minor-closed for all $k \in \mathbb{N}$, where a class $\mathcal{C}$ is *minor-closed* if $G \in \mathcal{C} \Rightarrow H \in \mathcal{C}$ for all $H \preceq G$. By the Robertson–Seymour Graph Minor Theorem [31], every minor-closed class $\mathcal{C}$ is characterized by a finite set $\mathcal{F}$ of *obstructions* (also known as excluded or forbidden minors) with the property that

$$G \in \mathcal{C} \iff (\forall F \in \mathcal{F})(F \npreceq G)$$

for all graphs G. Moreover, the obstruction set $\mathcal{F}$ is unique up to isomorphism of its elements subject to *minimality* (i.e., $F \npreceq F'$ for all distinct $F, F' \in \mathcal{F}$). A minor-monotone graph invariant f is thus characterized by the sequence $(\mathcal{F}_1, \mathcal{F}_2, \dots)$ of finite minimal obstruction sets $\mathcal{F}_k$ for the classes $\{G : f(G) \leq k\}$.

Characterizing the minimal obstruction sets $\mathcal{F}_k$ for minor-monotone invariants is a longstanding question in graph theory (see [1, 9]). Minimal obstructions for treedepth specifically have been studied by multiple sets of authors [4, 5, 16, 18]. A complete classification of minimal obstructions for treedepth $\leq k$ remains elusive even for small values of k (less than 5). The number of minimal obstructions is a doubly exponential function of k [16], while the minimal obstructions themselves may have exponentially many vertices [11]. The situation is similar for treewidth and many other minor-monotone invariants such as genus. This severely limits the usefulness of minimal obstruction sets in applications such as parameterized algorithms on bounded treedepth graphs.

On the other hand, there are many applications where a reasonable *approximation* of invariants like treedepth or treewidth serves a good enough purpose. This raises the question whether a given minor-monotone invariant admits a polynomial approximation in terms of a "nice" (uniformly described and easily recognizable) sequence of *non-minimal* obstruction sets. A breakthrough result of Chekuri and Chuzhoy [10] gives precisely such an approximation of treewidth (resolving a longstanding conjecture in graph minor theory).

Theorem 1.1 (Polynomial Grid Minor Theorem for Treewidth [10]). *There is an absolute constant c such that every graph with treewidth $\Omega(k^c)$ has a $k \times k$ grid minor.*

Since the $k \times k$ grid has treewidth k, Theorem 1.1 establishes that the treewidth of a graph is polynomially approximated by the size of its largest grid minor. (Prior to [10], treewidth was known to be exponential in the size of the largest grid minor.) In this paper, we give an analogous polynomial excluded-minor approximation of treedepth in terms of three distinct obstructions: grids, complete binary trees, and paths.

Theorem 1.2 (Polynomial Grid/Tree/Path Minor Theorem for Treedepth). *There is an absolute constant c such that every graph with treedepth $\Omega(k^c)$ has one or more of the following minors:*

- *a $k \times k$ grid,*
- *a complete binary tree of height k, or*
- *a path of order 2^k.*

Since each of the above graphs has treedepth at least k, the largest such obstruction gives a polynomial approximation of $\mathbf{td}(G)$. Moreover, all three obstructions in Theorem 1.2 are necessary for a polynomial approximation of treedepth. (The longest path alone gives an exponential approximation of treedepth by (2).) Theorem 1.2 follows from combining Theorem 1.1 with the following result, which is the technical main theorem of this paper.

Theorem 1.3 (Main Theorem). *Every graph with treedepth $\Omega(k^5 \log^2 k)$ satisfies one or more of the following conditions:*

- *G has treewidth at least k,*
- *G contains a subdivision of a complete binary tree of height k, or*
- *G contains a path of order 2^k.*

Due to the constructive nature of the proofs, we also obtain algorithmic versions of Theorems 1.2 and 1.3 (Corollaries 7.1 and 7.3).

Outline of the paper. Section 2 contains a discussion of related work. Section 3 states the basic definitions of width measures **tw**, **pw** and **td**, as well as graph minors and minor-monotonicity of graph invariants. Section 4 gives some simple lemmas on tree decompositions. Section 5 gives additional lemmas on rooted trees, including a proof of Theorem 1.3 in the case where G is a tree. Section 6 gives the full proof of Theorem 1.3. Section 7 describes algorithmic versions of our main results. Finally, Section 8 discusses applications of Theorem 1.3 in circuit complexity and finite model theory.

2. Related work and applications

The results of this paper have interesting applications in complexity theory and logic, as well as the theory of sparse graphs. Theorem 1.3 was used by Kush and Rossman [24] to lower bound the AC^0 circuit complexity of the subgraph isomorphism problem in terms of the treedepth of pattern graph. This result has a further application in finite model theory: a polynomial-rank homomorphism preservation theorem over finite structures [33]. These results were in fact the original motivation for the study of excluded-minor approximations of treedepth initiated in this paper.

Another application is found in recent work of Kun, O'Brien, Pilipczuk and Sullivan [23] on *linear colorings*, defined as functions $\alpha : V(G) \to \mathbb{Z}$ such that every path in G contains a vertex with a unique color. (This notion generalizes *centered colorings*, which arise in the theory of bounded expansion graph classes [26].) Our excluded-minor approximation of treedepth is used in [23] to show that any linear coloring of a graph G requires $\Omega(\mathbf{td}(G)^\varepsilon)$ colors for an absolute constant $\varepsilon > 0$.

2.1. *Improved bound in Theorem 1.3*

Following the initial conference publication of this paper [22], the bound of Theorem 1.3 was improved from $\Omega(k^5 \log^2 k)$ to $\Omega(k^3)$ by Czerwiński, Nadara and Pilipczuk [14]. It remains an open problem to further improve the bound of Theorem 1.3 to $\Omega(k^c)$ for any $c < 3$; examples in [14] show that the optimal constant c is at least 2. The key lemma from their paper qualitatively improves the results in Section 5 of this paper on excluded minors of trees.

Lemma 2.1 ([14]). *Every tree of treedepth k contains a subcubic* (*maximum degree* 3) *subtree of treedepth $\Omega(k)$.*

Czerwiński, Nadara and Pilipczuk also combine Lemma 2.1 with the machinery in Section 6 of this paper to show the following algorithmic result.

Theorem 2.2 ([14]). *Given a graph G, one can in polynomial time compute a rooted tree of height*

$$O(\mathbf{td}(G) \cdot \mathbf{tw}(G) \log^{3/2} \mathbf{tw}(G))$$

whose closure contains G.

2.2. *Excluded-minor approximation of pathwidth*

In the conference version [22], we conjectured that treewidth and complete binary tree minors (i.e., the first two cases in Theorem 1.3) give a polynomial approximation of pathwidth:

Conjecture 2.3. *There is an absolute constant c such that every graph with pathwidth $\Omega(k^c)$ has treewidth at least k, or contains a subdivision of a complete binary tree of height k.*

We are pleased to report that this conjecture is now a theorem of Groenland, Joret, Nadara and Walczak [19], who moreover obtain the optimal constant $c = 2$. Their result, together with Theorems 1.1 and 1.2, completes a satisfying three-way excluded-minor approximation of the parameters **tw**, **pw** and **td**.

Corollary 2.4. *Writing G_k for the $k \times k$ grid, B_k for the complete binary tree of height k, and P_k for the path of order k, there is an absolute constant $\varepsilon > 0$ such that*

$$\begin{aligned}\text{excludes } G_k \text{ as a minor} &\Rightarrow \mathbf{tw} \geq k \Rightarrow \text{excludes } G_{k^\varepsilon} \text{ as a minor},\\ \text{excludes } G_k,\ B_k \text{ as minors} &\Rightarrow \mathbf{pw} \geq k \Rightarrow \text{excludes } G_{k^\varepsilon},\ B_{k^\varepsilon} \text{ as minors},\\ \text{excludes } G_k,\ B_k,\ P_{2^k} \text{ as minors} &\Rightarrow \mathbf{td} \geq k \Rightarrow \text{excludes } G_{k^\varepsilon},\ B_{k^\varepsilon},\ P_{2^{k^\varepsilon}} \text{ as minors}.\end{aligned}$$

The optimal constant ε in Corollary 2.4 can be shown to be at least $1/11 - o(1)$, using the bounds in [14, 19] along with the best known constant $c = 9 + o(1)$ in Theorem 1.1 due to Chuzhoy and Tan [13].

3. Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$; for $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$; and let $\log(\cdot)$ be the base-2 logarithm.

All graphs in this paper are finite simple graphs. Formally, a *graph* is a pair $G = (V(G), E(G))$ where $E(G) \subseteq \binom{V(G)}{2}$. A *tree* is a connected acyclic graph. A *rooted tree* is a tree with a designated root. A tree is *subcubic* if it has maximum degree at most 3; for rooted trees, we also require that the root has degree at most 2 (i.e., a rooted tree is *subcubic* if every node has at most two children). Examples of subcubic (rooted) trees include paths and binary trees.

Definition 3.1 (Tree decompositions, treewidth, pathwidth).

- A *tree decomposition* of a graph G is a pair $(T, \mathcal{W})$ where T is a tree and $\mathcal{W} = \{W_t\}_{t \in V(T)}$ is a family of sets $W_t \subseteq V(G)$ such that
  - $\bigcup_{t \in V(T)} W_t = V(G)$, and every edge of G has both ends in some W_t,
  - if $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.
- The *width* of a tree decomposition $(T, \mathcal{W})$ is defined as $\max_{t \in V(T)} |W_t| - 1$.
- The *treewidth* of G, denoted $\mathbf{tw}(G)$, is the minimum width of a tree decomposition for G.
- The *pathwidth* of G, denoted $\mathbf{pw}(G)$, is the minimum width of a tree decomposition $(T, \mathcal{W})$ for G such that T is a path.

Definition 3.2 (Rooted trees). A *rooted tree* is a tree T with a designated root vertex. The *height* of T is the maximum number of vertices on a root-to-leaf path. We use the following notation:

- $\vec{E}(T)$ is the set of ordered pairs xy such that x is a child of y in T. (We write xy instead of (x, y) and think of this pair as a directed edge.)
- $<_T$ is the partial order on $V(T)$ defined by $x <_T y$ iff x is a proper descendent of y; we write $x \leq_T y$ iff $x <_T y$ or $x = y$; for $W \subseteq V(T)$, we write $W \leq_T x$ iff $w \leq_T x$ for all $w \in W$.
- The *closure* of T, denoted $\mathrm{Clos}(T)$, is the graph with vertex set $V(T)$ and edge set $\{\{x, y\} : x <_T y \text{ or } y <_T x\}$. (In other words, two vertices are joined by an edge in $\mathrm{Clos}(T)$ iff they lie on a common branch in T.)

Definition 3.3 (Treedepth). The *treedepth* of a connected graph G, denoted $\mathbf{td}(G)$, is the minimum height of a rooted tree T such that $G \subseteq \mathrm{Clos}(T)$. The *treedepth* of a disconnected graph is the maximum treedepth of its connected components.

For general graphs G, treedepth is the minimum height of a rooted forest F such that $G \subseteq \mathrm{Clos}(F)$ (where $\mathrm{Clos}(F)$ is defined similarly to $\mathrm{Clos}(T)$).[1]

Definition 3.4 (Graph minors and minor-monotonicity).

- A graph F is a *minor* of G, denoted $F \preceq G$, if F is isomorphic to a graph that can be obtained from G by a sequence of edge deletions and edge contractions.
- A graph invariant $f : \{\text{graphs}\} \to \mathbb{N}$ is *minor-monotone* if $f(F) \leq f(G)$ for all graphs $F \preceq G$.

Width measures **tw**, **pw** and **td** are easily shown to be minor-monotone, as is the invariant **lp** (the order of the longest path).

4. Lemmas on tree decompositions

Our first lemma bounds the treedepth of a graph G in terms of the width of one of its tree decomposition $(T, \mathcal{W})$ and the treedepth of T. This lemma may be considered folklore; it is implicit in proofs of the inequality $\mathbf{td}(G) \leq (\mathbf{tw}(G) + 1) \log |V(G)|$ [7, 27]. We could not find a proof in the literature, so we include one for completeness.

Lemma 4.1. *If* $(T, \mathcal{W})$ *is a width-*w *tree decomposition of a graph* G*, then*

$$\mathbf{td}(G) \leq (w + 1) \cdot \mathbf{td}(T).$$

Proof. Suppose $(T, \mathcal{W})$ is a width-w tree decomposition of the graph G. We will construct a rooted tree R of height at most $(w + 1) \cdot \mathbf{td}(T)$ such that $G \subseteq \mathrm{Clos}(R)$. (The construction is illustrated in Figure 1. The tree decomposition $(T, \mathcal{W})$ in that example happens to be a path.)

By definition of treedepth, there exists a rooted tree S such that $T \subseteq \mathrm{Clos}(S)$ and $\mathbf{td}(T) = \mathrm{height}(S)$. Without loss of generality, we may assume that $V(S) = V(T)$ (by deleting any vertices of $V(S) \setminus V(T)$).

Recall that $\mathcal{W}$ is a family $\{W_t\}_{t \in V(T)}$ where $W_t \subseteq V(G)$. For each $t \in V(T)$, define the set $U_t \subseteq W_t$ by $U_t := W_t \setminus \bigcup_{u : t <_S u} W_u$. Let $\mathcal{U} := \{U_t\}_{t \in V(T)}$ and note that $\mathcal{U}$ forms a partition of $V(G)$ (where some of the sets U_t may be empty).

For each $t \in V(T)$, fix an arbitrary linear order $<_t$ on U_t. Define a partial order $<^\star$ on $V(G)$ by

$$x <^\star y \overset{\text{def}}{\iff} \Big(\bigvee_{t \in V(T)} x, y \in U_t \text{ and } x <_t y \Big) \text{ or } \Big(\bigvee_{t,u \in V(T) : t <_S u} x \in U_t \text{ and } y \in U_u \Big).$$

[1] Elsewhere in the literature, rooted forests F satisfying $G \subseteq \mathrm{Clos}(F)$ are called *treedepth decompositions* of G. We avoid this terminology in this paper, to avoid confusion with the more common notion of tree decompositions.

That is, we have $x <^\star y$ iff either x, y belong to the same set U_t and $x <_t y$, or x, y belong to distinct U_t, U_u respectively where $t <_S u$.

It is easy to see that $<^\star$ is equivalent to $<_R$ for a unique rooted R with $V(R) = V(G)$. (This follows from the observation that $<^\star$ is a partial order on $V(G)$; it has a unique maximal element (namely, the $<_t$-maximal element of U_t $(= W_t)$ where $t = \text{root}(S)$); and for every $x \in V(G)$, the set $\{y : x <^\star y\}$ is totally ordered by $<^\star$.) Note that

$$\mathbf{td}(G) \le \text{height}(R) \le \max_{t \in V(T)} |W_t| \cdot \text{height}(S) = (w+1) \cdot \mathbf{td}(T).$$

To complete the proof, it remains to establish that $G \subseteq \text{Clos}(R)$. Consider an edge $\{x, y\} \in E(G)$. By definition of $(T, \mathcal{W})$ being a tree decomposition of G, the set $\{t \in V(T) : \{x, y\} \subseteq W_t\}$ is non-empty; let p be any $<_S$-maximal element in this set. Consider the set $\{u \in V(T) : p \le_S u \text{ and } \{x, y\} \cap W_t \ne \emptyset\}$; let q be the unique $<_S$-maximal element in this set. There are now two cases to consider:

- Assume $p = q$. Then $x, y \in U_p$. We can assume that $x <_p y$. Then we have $x <_R y$ and hence $\{x, y\} \in E(\text{Clos}(R))$.
- Assume $p \ne q$. Then $|\{x, y\} \cap W_q| = 1$. We can assume that $\{x, y\} \cap W_q = \{y\}$. Then we have $x \in U_p$ and $y \in U_q$ and $p <_S q$. It follows that $x <_R y$ and hence $\{x, y\} \in E(\text{Clos}(R))$.

Since $\{x, y\} \in E(\text{Clos}(R))$ in both cases, we conclude that $G \subseteq \text{Clos}(R)$. ∎

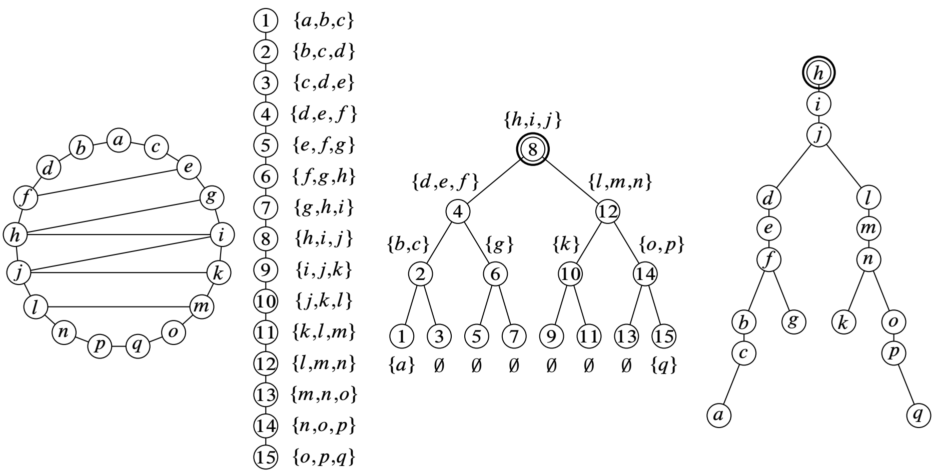

Fig. 1. From left to right: G, $(T, \mathcal{W})$, $(S, \mathcal{U})$, R.

We next introduce a normal form for tree decompositions of connected graphs, which witnesses tight upper bounds for both treewidth and treedepth (as shown in Lemmas 4.5 and 4.6).

Definition 4.2 (Greedy rooted tree decomposition).

- A *greedy rooted tree decomposition* of a connected graph G is a rooted tree T with the following properties:
  (i) $V(T) = V(G)$,
  (ii) $G \subseteq \mathrm{Clos}(T)$,
  (iii) for every child-parent pair $xy \in \vec{E}(T)$, there exists $w \leq_T x$ with $\{w, y\} \in E(G)$.

  (Given properties (i) and (ii), property (iii) is equivalent to the following: for every $x \in V(T)$, the induced subgraph of G on $\{w : w \leq_T x\}$ is connected.)
- For each $x \in V(G)$, we define the set $\mathrm{Bag}_{T,G}(x) \subseteq V(G)$ by

$$\mathrm{Bag}_{T,G}(x) \\ := \{x\} \cup \big\{y : \text{there exists } w \text{ such that } w \leq_T x <_T y \text{ and } \{w, y\} \in E(G)\big\}.$$

- The *width* of T with respect to G is defined by $\max_{x \in V(G)} |\mathrm{Bag}_{T,G}(x)| - 1$.

Remark 4.3. Our notion of greedy rooted tree decompositions is defined only for connected graphs for simplicity. However, Definition 4.2 extends naturally to general graphs by considering rooted forests instead of rooted trees.

The same notion appears at least twice in the literature: in [12] under the name *good treedepth decomposition* and in [8] under the name *reduced separation forest*. An even "greedier" class of tree decompositions appears in [17] under the name *minimal rooted trees*. Every minimal rooted tree for a connected graph G (in the sense of [17]) is a greedy rooted tree decomposition of G (in our sense), but not conversely. (The notion of minimal rooted trees would not work for our purposes, as Lemma 4.6 is false with respect to this more restrictive class of tree decompositions.)

The following three lemmas establish the key properties of greedy rooted tree decompositions. (These properties are also noted in [8, 12].) The first lemma establishes that greedy rooted tree decompositions are, in fact, tree decompositions in the sense of Definition 3.1.

Lemma 4.4. *If T is a greedy rooted tree decomposition of a connected graph G, then T together with $\{\mathrm{Bag}_{T,G}(x)\}_{x \in V(G)}$ is a tree decomposition of G.*

Proof. Straightforward from definitions. ∎

The next two lemmas show that height-optimal (resp. width-optimal) greedy rooted tree decomposition witness the treedepth (resp. treewidth) of connected graphs.

Lemma 4.5. *Every connected graph G has a greedy rooted tree decomposition of height* $\mathbf{td}(G)$.

Proof. By definition of treedepth, there exists a rooted tree T of height $\mathbf{td}(G)$ such that $G \subseteq \mathrm{Clos}(T)$. We may assume that $V(T) = V(G)$ (by contracting an edge of T incident with each vertex of $V(T) \setminus V(G)$). Thus, T satisfies conditions (i) and (ii) of Definition 4.2. If T satisfies condition (iii), then we are done. So we assume that T violates condition (iii).

Consider any child-parent pair $xy \in \vec{E}(T)$ witnessing the violation of condition (iii), that is, y is the parent of x in T and there is no edge in G between y and any element of $\{w : w \leq_T x\}$. Note that y cannot be the root of T (since it would then follow from $G \subseteq \mathrm{Clos}(T)$ that G is disconnected). Let z be the parent of y in T. Let T' be the rooted tree obtained from T by removing the edge $\{x, y\}$ and adding the edge $\{x, z\}$. Note the following:

- T' satisfies conditions (i) and (ii) (that is, $V(T') = V(G)$ and $G \subseteq \mathrm{Clos}(T')$).
- $\mathrm{height}(T') \leq \mathrm{height}(T)$.
- $\mathrm{width}(T', G) \leq \mathrm{width}(T, G)$.
- We have $\phi(T') < \phi(T)$ where $\phi : \{\text{rooted trees}\} \to \mathbb{N}$ is the potential function $\phi(S) := \sum_{v \in V(S)} \mathrm{depth}_S(v)$ where $\mathrm{depth}_S(v)$ is the distance between v and the root of S. This is clear, since $V(T') = V(T)$ and

$$\mathrm{depth}_{T'}(v) = \begin{cases} \mathrm{depth}_T(v) - 1 & \text{if } v \leq_T x, \\ \mathrm{depth}_T(v) & \text{otherwise.} \end{cases}$$

It follows that finitely many operations $T \mapsto T'$ transform T into a greedy rooted tree decomposition of G of at most the same height and width. In particular, the height is at most $\mathbf{td}(T)$, which proves the lemma. ■

Lemma 4.6. *Every connected graph G has a greedy rooted tree decomposition of width* $\mathbf{tw}(G)$.

Proof. By definition of treewidth, there exists a tree decomposition $(T, \mathcal{W})$ of G of width $\mathbf{tw}(G)$. We may assume that W_t is nonempty for all $t \in V(T)$. We now make T into a rooted tree by arbitrarily fixing a choice of $\mathrm{root}(T) \in V(T)$. Without increasing width, if $|W_{\mathrm{root}(T)}| = \{v_1, \ldots, v_k\}$ where $k \geq 2$, then replace $\mathrm{root}(T)$ by a path on fresh vertices $t_1, \ldots, t_k$ where $W_{t_i} = \{v_1, \ldots, v_i\}$; if $|W_s \setminus W_t| = k \geq 2$ for some $st \in \vec{E}(T)$, then replace the edge $\{s, t\}$ in T by a path of length $k - 1$ with appropriate sets W_u at the newly created vertices u. Then the tree decomposition $(T, \mathcal{W})$ is such that

- $|W_{\mathrm{root}(T)}| = 1$,
- $|W_s \setminus W_t| = 1$ for all every child-parent pair $st \in \vec{E}(T)$.

We may now identify $V(T)$ with $V(G)$ by identifying $\mathrm{root}(T)$ with the unique element of $W_{\mathrm{root}(T)}$ and identifying each non-root t with the unique element of $W_t \setminus W_u$ where u is the parent of t.

Thus identified, the rooted tree T now satisfies conditions (i) and (ii) of Definition 4.2, that is, $V(T) = V(G)$ and $G \subseteq \mathrm{Clos}(T)$. Moreover, we have $\mathrm{width}(T, G) \leq \mathrm{width}(T, \mathcal{W})$. Finally, we repeat the same operation $T \mapsto T'$ as in the proof of Lemma 4.5 until T satisfies condition (iii) with respect to G. Since this operation does not increase width, we obtain a greedy rooted tree decomposition of G of width at most $\mathbf{tw}(G)$, which proves the lemma. ■

5. Lemmas on rooted trees

In this section, we show that every tree with treedepth k, as well as every subcubic tree with 2^k vertices, contains a path of length $2^{\Omega(\sqrt{k})}$ or a subdivision of a complete binary tree of height $\Omega(\sqrt{k})$ (Lemmas 5.11 and 5.13).

Definition 5.1 (Rooted trees P_k and B_h).

- For $k \geq 1$, let P_k denote the path of order k rooted at one of its endpoints.
- For $h \geq 1$, let B_h denote the rooted complete binary tree of height h (with $2^h - 1$ vertices).

Note that P_1 and B_1 are both the rooted trees of size 1 (i.e., isolated roots).

We next introduce useful notation for describing the structure of rooted trees.

Definition 5.2 (Rooted tree-building operations $*$ and $\langle\rangle$).

- For rooted trees S and T, let $S * T$ denote the rooted tree formed by taking the disjoint union of S and T and identifying the two roots. (For example, $P_2 * \cdots * P_2$ is a star rooted at its central vertex.) This operation is associative and commutative with identity element P_1. For a sequence of rooted trees $T_1, \ldots, T_m$ ($m \in \mathbb{N}$), we adopt the convention that $T_1 * \cdots * T_m = P_1$ if $m = 0$.
- For a rooted tree T, let $\langle T \rangle$ denote the rooted tree obtained from T by creating a new root ρ and placing an edge between ρ and the old root of T.
- For a sequence of rooted trees $T_1, \ldots, T_m$ ($m \geq 1$), let

$$\langle T_1, \ldots, T_m \rangle := \langle T_1 * \langle T_2 * \ldots \langle T_{m-1} * \langle T_m \rangle \rangle \ldots \rangle \rangle.$$

  That is, $\langle T_1, \ldots, T_m \rangle$ is the rooted tree obtained by identifying the root of T_i with the ith vertex from the root on the rooted path P_{m+1}.

These operations on rooted trees are illustrated in Figure 2 below.

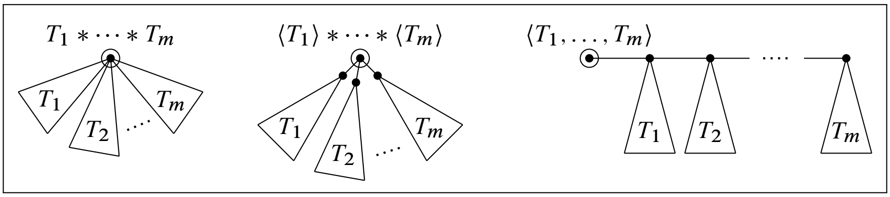

Fig. 2

Note that $P_1 = B_1 = \langle\rangle$ and for $k, h \geq 2$,

$$P_k = \langle P_{k-1} \rangle = \langle \underbrace{P_1, \ldots, P_1}_{k-1 \text{ times}} \rangle \quad \text{and} \quad B_h = \langle B_{h-1} \rangle * \langle B_{h-1} \rangle.$$

To prove the main lemmas of this section, we analyze the topological minor relation on rooted trees.

Definition 5.3 (Topological minors). For rooted trees S and T, we say that S is a *topological minor* of T (denoted $S \preceq_{\text{top}} T$) if some subdivision of S is isomorphic to a rooted subtree of T.

Note that $S \preceq_{\text{top}} T$ implies $\deg(\text{root}(S)) \le \deg(\text{root}(T))$; and $S \preceq_{\text{top}} T$ also implies $S \preceq T$ (that is, S is a minor of T as undirected graphs). However, the converse does not hold: for example, if the root of S has degree ≥ 2, then $S \not\preceq_{\text{top}} \langle S \rangle$ (whereas $T \preceq \langle T \rangle$ holds for all trees T). Another observation we will use: $\langle S \rangle \preceq_{\text{top}} T$ implies $\langle S \rangle \preceq_{\text{top}} \langle T \rangle$ (by further subdividing the edge from the root of $\langle S \rangle$).

Lemma 5.4. *Every rooted tree T has a decomposition of the form $\langle T_1 \rangle * \cdots * \langle T_l \rangle$ for rooted trees $T_1, \dots, T_l$ unique up to ordering. Further, for all rooted trees S, we have*

$$\langle S \rangle \preceq_{\text{top}} T \iff \exists i \in [l],\ \langle S \rangle \preceq_{\text{top}} \langle T_i \rangle.$$

Proof. Straightforward from definitions. Here l is the degree of $\text{root}(T)$ and $T_1, \dots, T_l$ are the subtrees rooted at the children of $\text{root}(T)$ (see Figure 2). Note that $l = 0$ in this decomposition if, and only if, T is an isolated root. ∎

The next lemmas characterize the structure of rooted trees T that omit binary trees $\langle B_h \rangle$ as topological minors. We first consider rooted trees $\langle T \rangle$ where the root has degree 1.

Lemma 5.5. *If T is a rooted tree such that $\langle B_h \rangle \not\preceq_{\text{top}} \langle T \rangle$, then there exist $m \ge 1$ and rooted trees $S_1, \dots, S_m$ such that $T = S_1 * \langle S_2, \dots, S_m \rangle$ and $\langle B_{h-1} \rangle \not\preceq_{\text{top}} S_i$ for all $i \in [m]$.*

Proof. Assume $\langle B_h \rangle \not\preceq_{\text{top}} T$ and note that this implies $h \ge 2$ (since $B_1 \preceq_{\text{top}} T$). We argue by induction on $|V(T)|$. In the base case $T = P_1$, the conclusion of the lemma is satisfied with $m := 1$ and $S_1 := T$.

For the induction step, assume $|V(T)| \ge 2$ and let $T = \langle T_1 \rangle * \cdots * \langle T_l \rangle$ be the decomposition given by Lemma 5.4. We claim that there is at most one $i \in [l]$ such that $\langle B_{h-1} \rangle \preceq_{\text{top}} \langle T_i \rangle$; otherwise, we would have $B_h = \langle B_{h-1} \rangle * \langle B_{h-1} \rangle \preceq_{\text{top}} T$, from which it follows that $\langle B_h \rangle \preceq_{\text{top}} \langle T \rangle$ (contradicting our assumption).

We now consider two cases depending whether there are zero or one indices $i \in [l]$ such that $\langle B_{h-1} \rangle \preceq_{\text{top}} \langle T_i \rangle$. If $\langle B_{h-1} \rangle \not\preceq_{\text{top}} \langle T_i \rangle$ for all $i \in [l]$, then we have $\langle B_{h-1} \rangle \not\preceq_{\text{top}} T$ by Lemma 5.4. In this case, the conclusion of the lemma is satisfied with $m := 1$ and $S_1 := T$.

Finally, consider the case that there exists a unique $i \in [l]$ such that $\langle B_{h-1} \rangle \preceq_{\text{top}} \langle T_i \rangle$; without loss of generality, assume $i = l$. Let $S_1 := \langle T_1 \rangle * \cdots * \langle T_{l-1} \rangle$ and $T' := T_l$. We have $\langle B_{h-1} \rangle \not\preceq_{\text{top}} S_1$ by Lemma 5.4. Now note that $|V(T)| \ge 1 + |V(T')|$ (since $\langle T' \rangle$ is a proper subtree of T) and $\langle B_h \rangle \not\preceq_{\text{top}} \langle T' \rangle$ (since $\langle B_h \rangle \not\preceq_{\text{top}} \langle T \rangle$). By the induction hypothesis applied to T', there exist $S_2, \dots, S_m$ ($m \ge 2$) such that $T' = S_2 * \langle S_3, \dots, S_m \rangle$ and $\langle B_{h-1} \rangle \not\preceq_{\text{top}} S_i$ for all $i \in \{2, \dots, m\}$. We are now done, since $T = S_1 * \langle T' \rangle = S_1 * \langle S_2, \dots, S_m \rangle$. ∎

Lemma 5.6. *If T is a rooted tree such that $\langle B_h \rangle \not\preceq_{\text{top}} T$, then there exist $m \geq 0$ and $l_1, \dots, l_m \geq 1$ and rooted trees $S_{i,j}$ $(i \in [m],\ j \in [l_i])$ such that*

$$T = \langle S_{1,1}, \dots, S_{1,l_1} \rangle * \cdots * \langle S_{m,1}, \dots, S_{m,l_m} \rangle$$

and $\langle B_{h-1} \rangle \not\preceq_{\text{top}} S_{i,j}$ for all $i \in [m]$ and $j \in [l_i]$.

Proof. Assume $\langle B_h \rangle \not\preceq_{\text{top}} T$. Let $T = \langle T_1 \rangle * \cdots * \langle T_m \rangle$ be the decomposition given by Lemma 5.4, where $\langle B_h \rangle \not\preceq_{\text{top}} \langle T_i \rangle$ for all $i \in [m]$. By Lemma 5.5, there exist rooted trees $S_{i,1}, \dots, S_{i,l_i}$ such that $T_i = S_{i,1} * \langle S_{i,2}, \dots, S_{i,l_i} \rangle$ and $\langle B_{h-1} \rangle \not\preceq_{\text{top}} S_{i,j}$ for all $j \in [l_i]$. Finally, we have $\langle T_i \rangle = \langle S_{i,1}, \dots, S_{i,l_i} \rangle$, which gives the desired formula for T. ■

5.1. Treedepth bounds

The next lemmas give bounds on the treedepth of (unrooted) trees T. These lemmas play a key role in the proof of Theorem 1.3 in Section 6.

Lemma 5.7 ([26, 27]). *For all $k, h \geq 1$, we have*

$$\mathbf{td}(P_k) = \lceil \log(k+1) \rceil \quad \textit{and} \quad \mathbf{td}(B_h) = h. \qquad ■$$

Note that the embedding $P_{15} \subseteq \mathrm{Clos}(B_4)$, which witnesses the bound $\mathbf{td}(P_{15}) \leq 4$, is depicted in Figure 1.

Lemma 5.8. *For all $m \geq 0$ and rooted trees $T_1, \dots, T_m$,*

$$\mathbf{td}(T_1 * \cdots * T_m) \leq \max\{\mathbf{td}(T_1), \dots, \mathbf{td}(T_m)\} + 1.$$

Proof. Let $\mathbf{td}_{\text{rooted}}(T)$ denote the minimum height of a rooted tree T' such that $\mathrm{root}(T') = \mathrm{root}(T)$ and $E(T) \subseteq E(\mathrm{clos}(T'))$. It is easy to see that $\mathbf{td}(T) \leq \mathbf{td}_{\text{rooted}}(T)$ and $\mathbf{td}_{\text{rooted}}(T_1 * \cdots * T_m) = \max\{\mathbf{td}(T_1), \dots, \mathbf{td}(T_m)\} + 1$. ■

Lemma 5.9. *For all $m \geq 0$ and rooted trees $T_1, \dots, T_m$,*

$$\mathbf{td}(\langle T_1, \dots, T_m \rangle) \leq \lceil \log(m+2) \rceil + \max\{\mathbf{td}(T_1), \dots, \mathbf{td}(T_m)\}.$$

Proof. For each $i \in [m]$, fix a rooted tree T_i' of height $\mathbf{td}(T_i)$ with $E(T_i) \subseteq E(\mathrm{clos}(T_i'))$. Invoking Lemma 5.7, let T_0' be a rooted tree of height $\lceil \log(m+2) \rceil$ such that $E(P_{m+1}) \subseteq E(\mathrm{clos}(T_0'))$. Label the vertices of P_{m+1} as $v_0, \dots, v_m$ with v_0 being the root. Let T' be the rooted tree, with root v_0, obtained from the disjoint union of $T_0', \dots, T_m'$ by identifying the vertices v_i and $\mathrm{root}(T_i')$ for each $i \in [m]$. Note that $E(\langle T_1, \dots, T_m \rangle) \subseteq E(\mathrm{clos}(T'))$ and

$$\mathrm{height}(T') \leq \mathrm{height}(T_0') + \max_{i \in [m]} \mathrm{height}(T_i') = \lceil \log(m+2) \rceil + \max_{i \in [m]} \mathbf{td}(T_i). \qquad ■$$

Lemma 5.10. *For every rooted tree T and $h \geq 0$ and $k \geq 1$, if $\langle B_h \rangle \not\preceq_{\text{top}} T$ and $P_k \not\preceq_{\text{top}} T$, then*

$$\mathbf{td}(T) \leq h \cdot (\lceil \log(k+1) \rceil + 1).$$

Proof. The lemma is proved by induction on h. The base case $h = 0$ is vacuous, since $\langle B_0 \rangle = P_1$ is a rooted minor of every rooted tree. For the induction step, let $h \geq 1$ and assume $\langle B_h \rangle \not\preceq_{\text{top}} T$ and $P_k \not\preceq_{\text{top}} T$. By Lemma 5.6, there exist $m \geq 0$ and $l_1, \ldots, l_m \geq 1$ and rooted trees $S_{i,j}$ ($i \in [m]$, $j \in [l_i]$) such that

$$T = \langle S_{1,1}, \ldots, S_{1,l_1} \rangle * \cdots * \langle S_{m,1}, \ldots, S_{m,l_m} \rangle$$

and $\langle B_{h-1} \rangle \not\preceq_{\text{top}} S_{i,j}$ for all $i \in [m]$ and $j \in [l_i]$. We also clearly have $l_i < k$ and $P_k \not\preceq_{\text{top}} S_{i,j}$ for all $i \in [m]$ and $j \in [l_i]$. By the induction hypothesis,

$$\mathbf{td}(S_{i,j}) \leq (h-1) \cdot \lceil \log(k+1) \rceil.$$

By Lemma 5.9, we have

$$\begin{aligned} \mathbf{td}(\langle S_{i,1}, \ldots, S_{i,l_i} \rangle) &\leq \lceil \log(l_i + 2) \rceil + \max \{\mathbf{td}(S_{i,1}), \ldots, \mathbf{td}(S_{i,l_i})\} \\ &\leq \lceil \log(k+1) \rceil + (h-1) \cdot (\lceil \log(k+1) \rceil + 1) \\ &= h \cdot (\lceil \log(k+1) \rceil + 1) - 1. \end{aligned}$$

Finally, by Lemma 5.8,

$$\begin{aligned} \mathbf{td}(T) &\leq \max \{\mathbf{td}(\langle S_{1,1}, \ldots, S_{1,l_1} \rangle), \ldots, \mathbf{td}(\langle S_{m,1}, \ldots, S_{m,l_m} \rangle)\} + 1 \\ &\leq h \cdot (\lceil \log(k+1) \rceil + 1). \end{aligned}$$

∎

Lemma 5.11. *Every rooted tree with treedepth $\geq d$ contains a subcubic rooted subtree (i.e., every vertex has at most two children) of order $\geq 2^{\sqrt{d}-2}$.*

Proof. We prove the contrapositive. Suppose T is a rooted tree that does not contain a subcubic rooted subtree of order $\geq 2^{\sqrt{d}-2}$. In particular, T does not have $\langle B_h \rangle$ or P_k as a rooted minor where $h = \lceil \sqrt{d} - 2 \rceil$ and $k = 2^h$. By Lemma 5.10, it follows that

$$\mathbf{td}(T) \leq h \cdot (\lceil \log(k+1) \rceil + 1) \leq (\sqrt{d} - 1)(\lceil \log(2^{\sqrt{d}-1} + 1) \rceil + 1) < d.$$

∎

5.2. *Bounded-degree graphs that omit P_k and B_h minors*

The final two lemmas of this section bound the size of bounded-degree graphs that omit P_k and B_h minors.

Lemma 5.12. *Let $h, k, c \geq 1$ and suppose T is a rooted tree such that $\langle B_h \rangle \not\preceq_{\text{top}} T$ and $P_k \not\preceq_{\text{top}} T$ and every vertex of T has at most c children. Then $|V(T)| \leq (ck)^{h-1}$.*

Proof. The lemma is proved by induction on h. In the base case $h = 1$, the assumption $\langle B_1 \rangle \not\preceq_{\text{top}} T$ implies that T is an isolated root (since $\langle B_1 \rangle = P_2$). Therefore $|V(T)| = 1 = (ck)^{h-1}$.

For the induction step, suppose that $h \geq 2$. Again by Lemma 5.6, there exist $m \geq 0$ and $l_1, \ldots, l_m \geq 1$ and rooted trees $S_{i,j}$ ($i \in [m]$, $j \in [l_m]$) with the property that $T = \langle S_{1,1}, \ldots, S_{1,l_1} \rangle * \cdots * \langle S_{m,1}, \ldots, S_{m,l_m} \rangle$ and $\langle B_{h-1} \rangle \not\preceq_{\text{top}} S_{i,j}$ for all $i \in [m]$ and $j \in [l_m]$.

Note that $m \le c$ and $l_i \le k-1$ and $P_{k-1} \not\le_{\text{top}} S_{i,j}$ for all $i \in [m]$ and $j \in [l_i]$. By the induction hypothesis, we have $|V(S_{i,j})| \le (c(k-1))^{h-2}$ for all i and j. Therefore,

$$\begin{aligned}|V(T)| &= 1 + \sum_{i=1}^{m}\sum_{j=1}^{l_i} |V(S_{i,j})| \\ &\le 1 + c(k-1)(c(k-1))^{h-2} = 1 + (c(k-1))^{h-1} \le (ck)^{h-1}. \qquad \blacksquare\end{aligned}$$

Lemma 5.13. *Let $h, c \ge 1$ and suppose G is a connected graph with maximum degree $\le c+1$ such that $B_h \not\le G$ and $P_{c^h} \not\le G$. Then $|V(G)| \le c^{h^2}$.*

Proof. Let F be any spanning tree of G rooted at any of its leaves. Since G has maximum degree $c+1$, every node of F has at most c children. The assumption that $P_{c^h} \not\le G$ and $B_h \not\le G$ implies that $P_{c^h} \not\le_{\text{top}} T$ and $\langle B_h \rangle \not\le_{\text{top}} F$. Therefore, by Lemma 5.12,

$$|V(G)| = |V(F)| \le (c^{h+1})^{h-1} \le c^{h^2}. \qquad \blacksquare$$

6. Proof of Theorem 1.3

We now prove our main result, Theorem 1.3. The following is an equivalent rephrasing:

Theorem 6.1. *Every graph G with treewidth $< k$ contains a path of order 2^h or a subdivision of B_h where $h = \Omega((\mathbf{td}(G)/k)^{1/4}/\sqrt{\log k})$.*

Theorem 1.3 follows immediately: If G is a graph such that $\mathbf{tw}(G) < k$ and $\mathbf{td}(G) \ge Ck^5 \log^2 k$ (for a sufficiently large constant C), then by Theorem 6.1, G contains a path of order 2^k or a subdivision of B_k.

Proof of Theorem 6.1. It suffices to prove the conclusion for connected graphs. Let G be any connected graph with treewidth $< k$.

We will construct four trees $T \supseteq S$ and $F \supseteq Q$ where

- T is a greedy rooted tree decomposition of G of width $\mathbf{tw}(G)$,
- S is a subcubic rooted subtree of T of size $2^{\Omega(\sqrt{\mathbf{td}(G)/k})}$,
- F is a spanning tree of G, and
- Q is a subtree of F such that $V(Q) \supseteq V(S)$ and maximum-degree$(Q) \le k+1$.

It follows by Lemma 5.13 that Q contains a path of order $(\mathbf{tw}(G)+1)^h$ $(\ge 2^h)$ or a subdivision of B_h where h is the largest integer such that $|V(Q)| > (\mathbf{tw}(G)+1)^{h^2}$ $(= 2^{h^2 \log(\mathbf{tw}(G)+1)})$. On the other hand, $|V(Q)|$ is at least $2^{\Omega(\sqrt{\mathbf{td}(G)/k})}$ (because $V(Q) \supseteq V(S)$). Therefore, $h = \Omega((\mathbf{td}(G)/k)^{1/4}/\sqrt{\log k})$. Since $Q \subseteq G$, we conclude that G contains a path of order 2^h or a subdivision of B_h.

We proceed with the construction of trees T, S, F, Q in three steps. Various definitions will be stated as items (a), (b), (c), etc.

Step 1: The greedy rooted tree decomposition T and rooted subtree $S \subseteq T$

(a) By Lemma 4.6, G has a greedy rooted tree decomposition T of width $\mathbf{tw}(G)$. Fix any such T. Note $V(T) = V(G)$ (by Definition 4.2(i)) and $\mathbf{td}(T) \geq \mathbf{td}(G)/(\mathbf{tw}(G)+1) \geq \mathbf{td}(G)/k$ (by Lemma 4.1).

(b) By Lemma 5.11, T has a subcubic rooted subtree S with at least $2^{\sqrt{\mathbf{td}(T)}-2}$ ($\geq 2^{\sqrt{\mathbf{td}(G)/k}-2}$) vertices. Fix any such S.

(c) Let $V = V(T)$ ($= V(G)$) and $V' = V \setminus \{\mathrm{root}(T)\}$, and let $U = V(S)$ and $U' = V(S) \setminus \{\mathrm{root}(S)\}$. (Note that $U \subseteq V$ and $U' \subseteq V'$, since $\mathrm{root}(S) = \mathrm{root}(T)$ by definition of rooted subtree.)

Step 2: The spanning tree $F \subseteq G$

(d) For each $x \in V'$, let $\widehat{x}$ denote the parent of x in T.

(e) By Definition 4.2(iii), for each $x \in V'$, there exists a vertex $\widecheck{x} \in V'$ such that $\widecheck{x} \leq_T x$ and $\{\widecheck{x}, \widehat{x}\} \in E(G)$. Fix a choice of $\widecheck{x}$ for each x.

(f) Let F be the spanning subgraph of G with edge set $E(F) = \{\{\widehat{x}, \widecheck{x}\} : x \in V'\}$.

Claim 1. *For all $x \in V$, the induced subgraph of F on $\{v : v \leq_T x\}$ is a tree. In particular, F itself is a spanning tree of G.*

► Straightforward induction starting from the leaves of T. ◄Claim 1

Claim 2. *For all $x \in V'$, if $(p_0, p_1, \ldots, p_\ell)$ is the path in F from $p_0 = \widehat{x}$ to $p_\ell = x$, then $p_1 = \widecheck{x}$ and $p_1, \ldots, p_\ell \leq_T x$.*

► Since F induces a tree on the set $\{v : v \leq_T x\}$, the path in F from $\widecheck{x}$ to x stays within this set. Since F includes the edge $\{\widehat{x}, \widecheck{x}\}$, the claim follows. ◄Claim 2

Step 3: The subtree $Q \subseteq F$

(g) For each $x \in U$, let Q_x be the minimum subtree of F that includes all vertices in the set $\{u : u \leq_S x\}$.

(h) Let Q ($= Q_{\mathrm{root}(S)}$) be the minimum subtree of F such that $U \subseteq V(Q)$.

(i) For each $x \in U'$,
   - let $x^\star$ be first vertex on the path in F from $\widehat{x}$ to x such that $x^\star \in V(Q_x)$ (this is well-defined since $x \in V(Q_x)$),
   - let P_x be the path in F between $\widehat{x}$ and $x^\star$.

   (Note that $x^\star \leq_T x$ by Claim 2, but not necessarily $x^\star \leq_S x$.)

Claim 3. *The tree Q is the union of edge-disjoint paths: $E(Q) = \bigsqcup_{x \in U'} E(P_x)$. Moreover, for all distinct $x \neq y \in U'$, the paths P_x and P_y have no interior vertices in common (i.e., the sets $V(P_x) \setminus \{\widehat{x}, x^\star\}$ and $V(P_y) \setminus \{\widehat{y}, y^\star\}$ are disjoint).*

► Straightforward from definitions. ◄Claim 3

Claim 4. *For all $q \in V(Q)$, we have* $\deg_Q(q) \le |\{x \in U' : q = x^\star\}| + 2$.

► Consider any $q \in V(Q)$. By Claim 3, we have $E(Q) = \bigsqcup_{x \in U'} E(P_x)$ and hence

$$\begin{aligned}\deg_Q(q) = \sum_{x \in U'} \deg_{P_x}(q) &= 2\cdot|\{x \in U' : \deg_{P_x}(q) = 2\}| + |\{x \in U' : \deg_{P_x}(q) = 1\}| \\ &= 2\cdot|\{x \in U' : \deg_{P_x}(q) = 2\}| \\ &\quad + |\{x \in U' : q = \widehat{x}\}| + |\{x \in U' : q = x^\star\}|.\end{aligned}$$

We now consider two cases depending whether or not $q \in U$.

- Suppose $q \in U$. Then $\{x \in U' : \deg_{P_x}(q) = 2\}$ is empty by definition of P_x. Also, $|\{x \in U' : q = \widehat{x}\}| \le 2$ since S is a subcubic rooted tree (i.e., q is the parent of at most two vertices of S). Therefore, $\deg_Q(q) \le |\{x \in U' : q = x^\star\}| + 2$ as required.
- Suppose $q \notin U$. Then $\{x \in U' : q = \widehat{x}\}$ is empty, since $\widehat{x} \in U$ for all $x \in U'$ (this follows from S being a rooted subtree of T). Also, $|\{x \in U' : \deg_{P_x}(q) = 2\}| \le 1$ since no vertex q is an interior vertex of P_x for more than one $x \in U'$ by Claim 3. Therefore, $\deg_Q(q) \le |\{x \in U' : q = x^\star\}| + 2$ as required. ◄Claim 4

Claim 5. *For all $q \in V(Q)$, we have* $|\{x \in U' : q = x^\star\}| \le |\mathrm{Bag}_{T,G}(q)| - 1 \le \mathbf{tw}(G)$.

► Recall the tree decomposition of G induced by T (Definition 4.2). The bags of this tree decomposition are given by

$$\mathrm{Bag}_{T,G}(x) = \{x\} \cup \big\{y : \text{there exists } w \text{ such that } w \le_T x <_T y \text{ and } \{w,y\} \in E(G)\big\}.$$

For each $x \in U'$, we have $x^\star \le_T x <_T \widehat{x}$. As only one endpoint of P_x is $\le_T x$, it follows that P_x contains an edge $\{\widehat{w}, \breve{w}\}$ for some $w \in V'$ such that $\breve{w} \le_T x^\star$ and $\widehat{w} \not\le_T x^\star$. It follows that $\breve{w} \le_T x^\star \le_T w <_T \widehat{w}$. Since $\{\widehat{w}, \breve{w}\} \in E(P_x) \subseteq E(G)$, we have $\widehat{w} \in \mathrm{Bag}_{T,G}(x^\star) \setminus \{x^\star\}$.

Fix a choice of $w_x \in V'$ for each $x \in U'$ (satisfying $\{\widehat{w}_x, \breve{w}_x\} \in E(P_x)$ and $\breve{w}_x \le_T x^\star \le_T w_x <_T \breve{w}_x$). Now consider any $q \in V(Q)$. Suppose $x \ne y \in U'$ are such that $q = x^\star = y^\star$. We claim that $\widehat{w}_x \ne \widehat{w}_y$. To see why, assume for contradiction that $\widehat{w}_x = \widehat{w}_y$. Then w_x and w_y are siblings. Since $q \le_T w_x$ and $q \le_T w_y$, it follows that $w_x = w_y$. But this means that $\{\widehat{w}_x, \breve{w}_x\}$ is an edge of both P_x and P_y, contradicting the edge-disjointness of these paths by Claim 3.

This argument shows that $x \mapsto \widehat{w}_x$ is a one-to-one map from $\{x \in U' : q = x^\star\}$ to $\mathrm{Bag}_{T,G}(q) \setminus \{q\}$. Therefore, $|\{x \in U' : q = x^\star\}| \le |\mathrm{Bag}_{T,G}(q)| - 1$. Finally, we have $|\mathrm{Bag}_{T,G}(q)| - 1 \le \mathbf{tw}(G)$ by our choice of T, which satisfies $\mathrm{width}(T,G) = \max_{x \in V} |\mathrm{Bag}_{T,G}(q)| - 1 = \mathbf{tw}(G)$. ◄Claim 5

Claims 4 and 5 together imply that Q has maximum degree at most $\mathbf{tw}(G) + 2$ ($\le k+1$). We also have $|V(Q)| \ge 2^{\sqrt{\mathbf{td}(G)/k} - 2}$, since $V(Q)$ contains $W = V(S)$. By Lemma 5.13, we are able to conclude that Q (and hence G) contains a path of length 2^h or a subdivision of B_h where $h = \Omega((\mathbf{td}(G)/k)^{1/4}/\sqrt{\log k})$. ■

7. Algorithmic results

In this section, we describe the algorithmic versions of our main results. From the constructive nature of the proof of Theorem 6.1, we have the following

Corollary 7.1 (Algorithmic version of Theorem 6.1). *There is a polynomial-time algorithm which, given a graph G and a width $< k$ tree decomposition of G, outputs a minor embedding of P_{2^h} or B_h where $h = \Omega((\mathbf{td}(G)/k)^{1/4}/\sqrt{\log k})$.*

Results of Bodlaender et al. [6, 7] give a polynomial-time algorithm which, given a graph G, outputs a tree decomposition of G of width $O(\mathbf{tw}(G)^2)$. (This is actually a combination of two polynomial-time approximation algorithms for treewidth: an $O(\log n)$-approximation for arbitrary n-vertex graphs and 5-approximation when $\mathbf{tw}(G) \le \log n$ [7].) Combining this algorithm with Corollary 7.2, we get

Corollary 7.2. *There is a polynomial-time algorithm which, given a graph G, outputs a minor embedding of P_{2^h} or B_h where $h = \Omega(\mathbf{td}(G)^{1/4}/\sqrt{\mathbf{tw}(G)\log(\mathbf{tw}(G)+1)})$.*

To obtain the algorithmic version of Theorem 1.2, we combine Corollary 7.2 with the randomized polynomial-time algorithm of Chekuri and Chuzhoy [10] which, given a graph G, outputs a minor embedding of the $k \times k$ grid where $k = \mathbf{tw}(G)^{\Omega(1)}$.

Corollary 7.3 (Algorithmic version of Theorem 1.2). *There is an absolute constant $c > 0$ and a randomized polynomial-time algorithm which, given a graph G, outputs a minor embedding of one of the following graphs where $k = \Omega(\mathbf{td}(G)^c)$:*

- *a $k \times k$ grid,*
- *a complete binary tree of height k, or*
- *a path of order 2^k.*

The algorithm of Corollary 7.3 finds a $k \times k$ grid minor via the Chekuri–Chuzhoy algorithm and a P_{2^h} or B_h minor via Corollary 7.2. It outputs the grid minor if $k > h$ and the P_{2^h} or B_h minor otherwise.

8. Applications

In this section we briefly describe applications of our results in complexity theory and logic that were the original motivation for our investigation of excluded-minor approximations of treedepth. For details, the reader is referred to Kush and Rossman [24] and Rossman [33] where these results first appeared.

8.1. The AC^0 *formula size of subgraph isomorphism*

For a fixed graph G, the COLORED G-SUBGRAPH ISOMORPHISM problem, denoted SUB(G) for short, is the following: *Given an n-vertex graph H and a vertex-coloring $V(H) \to V(G)$, does H contain a properly colored subgraph isomorphic to G (i.e., a subgraph that maps isomorphically to G under the coloring map)?* Straightforward (folklore) upper bounds show that this problem is solvable by Boolean formulas of size $O(n^{\mathbf{td}(G)})$, as well by Boolean circuits of size $O(n^{\mathbf{tw}(G)+1})$ and non-deterministic branching programs of size $O(n^{\mathbf{pw}(G)+1})$. (Similar upper bounds for the harder *uncolored* version of G-SUBGRAPH ISOMORPHISM problem are known by the "color-coding" technique of [3].) Moreover, these upper bounds can be implemented in constant depth (for a constant depending on G) by so-called AC^0 *formulas* and *circuits* with unbounded fan-in AND and OR gates in addition to NOT gates. A nearly matching lower bound on the constant-depth AC^0 formula complexity of SUB(G) was recently proved in [24].

Theorem 8.1 ([24]). *There is an absolute constant $\varepsilon > 0$ such that for all graphs G, constant-depth* AC^0 *formulas solving* SUB(G) *require size* $n^{\Omega(\mathbf{td}(G)^{\varepsilon})}$.

The proof of Theorem 8.1 crucially relies on Theorem 1.3, as we will explain. It was observed by Li, Razborov and Rossman [25] that the complexity of SUB(G) is a minor-monotone function of G.

Lemma 8.2 ([25]). *If F is a minor of G, then* SUB(F) *reduces to* SUB(G) *via a monotone projection* (*a mapping of variables of* SUB(F) *to variables of* SUB(G) *or constant*). *Consequently, if* SUB(F) *is not solvable by* AC^0 *formulas of a given size and depth, then neither is* SUB(G).

The core of Theorem 8.1 consists of three lower bounds, corresponding to the three cases of Theorem 6.1:

- Constant-depth AC^0 formulas solving SUB(G) require size $n^{\Omega(\mathbf{tw}(G)/\log \mathbf{tw}(G))}$ for all graphs G [25]. (In fact, this lower bound is proved more generally for AC^0 circuits.)
- Constant-depth AC^0 formulas solving SUB(P_{2^k}) require size $n^{\Omega(k)}$ for all k [34].
- Constant-depth AC^0 formulas solving SUB(B_k) require size $n^{\Omega(k)}$ for all k [24].

Theorem 8.1 follows immediately from these three lower bounds, together with Lemma 8.2 and Theorem 1.3. Quantitatively, a lower bound $n^{\Omega((\mathbf{td}(G)/\log \mathbf{td}(G))^{1/3})}$ in Theorem 8.1 can be shown using the improved $\Omega(k^3)$ bound in Theorem 1.3 due to Czerwiński, Nadara and Pilipczuk [14].

8.2. *Polynomial-rank homomorphism preservation theorem on finite structures*

Theorem 8.1 has a further corollary in finite model theory (the study of logical definability on finite structures). Before stating this result, we first recall some basic definitions of first-order logic. Fix an arbitrary *relational vocabulary*, that is, a set of relation symbols each associated with a positive integer "arity". (Without loss of generality, it suffices to consider the vocabulary with just a single binary relation symbol; by standard arguments, everything we will say extends to general vocabularies that may also include constant and function symbols.)

A *structure* $\mathcal{A}$ consists of a set A, called the *universe* of $\mathcal{A}$, together with an interpretation $R^{\mathcal{A}} \subseteq A^r$ for each r-ary relation symbol R. A *homomorphism* from structure $\mathcal{A}$ to structure $\mathcal{B}$ is a function $f : A \to B$ with $(a_1, \ldots, a_r) \in R^{\mathcal{A}} \Rightarrow (f(a_1), \ldots, f(a_r)) \in R^{\mathcal{B}}$ for each r-ary relation symbol R and r-tuple $(a_1, \ldots, a_r) \in A^r$.

First-order formulas are logical expressions built up from *atomic formulas* $x = y$ and $Rx_1 \dots x_r$ and *negated atomic formulas* $x \neq y$ and $\neg Rx_1 \dots x_r$ via *connectives* $\varphi \wedge \psi$ and $\varphi \vee \psi$ and *quantifiers* $\forall x\ \varphi$ and $\exists x\ \varphi$, where variables x, y, z, etc. range over the universe of a structure. (Note that under this definition, negations occur only at the level of atomic formulas, while quantifiers are not necessarily arranged at the front of a formula.) *Quantifier-rank* is defined as the maximum nesting depth of quantifiers. For example, the formula $\exists x\ ((\forall y\ Rxy) \vee (\exists z\ \neg Rzx))$ has quantifier-rank 2.

Let φ be a first-order formula without free variables. Notation $\mathcal{A} \models \varphi$ indicates that $\mathcal{A}$ satisfies φ. We say that φ is *preserved under homomorphisms* [*on finite structures*] if $\mathcal{A} \models \varphi \Rightarrow \mathcal{B} \models \varphi$ for all [finite] structures $\mathcal{A}$ and $\mathcal{B}$ such that there exists a homomorphism $\mathcal{A} \to \mathcal{B}$. Finally, we say that φ is:

- *existential* if it contains no universal quantifiers (i.e., no occurrences of $\forall$);
- *positive* if it contains no negated atomic formulas (i.e., no occurrences of $\neq$ or $\neg$);
- *existential-positive* if it is both existential and positive.

We are ready to state the corollary in finite model theory which was the original motivation for the results of this paper.

Theorem 8.3 (Polynomial-Rank Homomorphism Preservation Theorem on Finite Structures [33]). *Let φ be a first-order formula of quantifier-rank k that is preserved under homomorphisms on finite structures. Then φ is logically equivalent on finite structures to an existential-positive formula ψ of quantifier-rank $k^{O(1)}$ (at most $O(k^{1/\varepsilon})$ where $\varepsilon \geq 1/3 - o(1)$ is the constant of Theorem* 8.1).

If we remove "on finite structures" from this statement, we get the original Homomorphism Preservation Theorem from classical model theory (where the quantifier-rank of ψ can be shown to be at most k). This is a close relative of the Łoś–Tarski (resp. Lyndon) Preservation Theorems, which state that first-order formulas preserved under injective (resp. surjective) homomorphisms are logically equivalent to existential (resp. positive) formulas. All three preservation theorems were originally proved using the compactness property of first-order logic, which fails when restricted to finite structures. As one might expect, the Łoś–Tarski and Lyndon Preservation Theorems also fail on finite structures [2, 20]. In contrast, the Homomorphism Preservation Theorem was shown in [32] to survive on finite structures, however with a non-elementary bound on the quantifier-rank of ψ (eventually greater than any finite tower of exponentials, 2^k, 2^{2^k}, etc.). The polynomial bound $k^{O(1)}$ in Theorem 8.3 was obtained in [33] as a consequence of Theorem 8.1, as we explain next following a few key definitions.

Definition 8.4 (Gaifman graph, hom-preserved classes, model checking problem).

- The *Gaifman graph* of a structure $\mathcal{A}$, denoted $G(\mathcal{A})$, is the simple graph with vertex set A and edges between all pairs of vertices that appear together in a tuple of a relation of $\mathcal{A}$.
- Notation $\mathcal{A} \to \mathcal{B}$ (resp. $\mathcal{A} \hookrightarrow \mathcal{B}$) indicates the existence of a homomorphism (resp. injective homomorphism) from $\mathcal{A}$ to $\mathcal{B}$.

- A class of finite structures $\mathscr{C}$ is *hom-closed* if ($\mathcal{A} \in \mathscr{C}$ and $\mathcal{A} \to \mathcal{B}$) $\Rightarrow \mathcal{B} \in \mathscr{C}$ for all finite structures $\mathcal{A}$ and $\mathcal{B}$.
- For such a class $\mathscr{C}$, we say that $\mathcal{A} \in \mathscr{C}$ is a *minimal element* if $\mathcal{A} \hookrightarrow \mathcal{B}$ for all $\mathcal{B} \in \mathscr{C}$ such that $\mathcal{B} \to \mathcal{A}$.
- The MODEL CHECKING problem for $\mathscr{C}$, denoted $\mathrm{MC}(\mathscr{C})$ for short, is: *Given a finite structure $\mathcal{A}$, determine whether or not $\mathcal{A} \in \mathscr{C}$.*

The following lemma relates first-order definability and existential-positive definability of hom-closed classes with the AC^0 formula complexity of the associated model checking problem. Parts (1), (2), (3) all have elementary proofs; see references [15,32,33] for details.

Lemma 8.5. *Let $\mathscr{C}$ be a class of finite structures.*

(1) *If $\mathscr{C}$ is definable on finite structures by a first-order formula of quantifier-rank k, then $\mathrm{MC}(\mathscr{C})$ is solvable on structures of size n by AC^0 formulas of size $O(n^k)$ and depth $O(k)$* [15].

(2) *$\mathscr{C}$ is definable on finite structures by an existential-positive formula of quantifier-rank k if, and only if, $\mathscr{C}$ is hom-closed and $\mathbf{td}(G(\mathcal{A})) \le k$ for every minimal element $\mathcal{A} \in \mathscr{C}$* [32].

(3) *If $\mathscr{C}$ is hom-closed, then for every minimal element $A \in \mathscr{C}$, there is a reduction from $\mathrm{SUB}(G(\mathcal{A}))$ to $\mathrm{MC}(\mathscr{C})$ via linear-size AC^0 formulas* [33].

Theorem 8.3 is a direct consequence of Theorem 8.1 and the three parts of Lemma 8.5. To see why, suppose φ is a first-order formula of quantifier-rank k that is preserved under homomorphisms on finite structures. Let $\mathscr{C}$ be the hom-closed class of finite models of φ, and consider any minimal element $\mathcal{A} \in \mathscr{C}$. By Theorem 8.1, $\mathrm{SUB}(G(\mathcal{A}))$ requires constant-depth AC^0 formula size $n^{\Omega(\mathbf{td}(G(\mathcal{A}))^{\varepsilon})}$. Therefore, by Lemma 8.5(3), we get the same lower bound for $\mathrm{MC}(\mathscr{C})$. On the other hand, by Lemma 8.5(1), $\mathrm{MC}(\mathscr{C})$ has constant-depth AC^0 formulas of size $n^{O(k)}$. It follows that $\mathbf{td}(G(\mathcal{A})) = O(k^{1/\varepsilon})$. Since $\mathcal{A}$ is an arbitrary minimal element of $\mathscr{C}$, the "if" direction of Lemma 8.5(2) implies that $\mathscr{C}$ is definable on finite structures by an existential-positive formula ψ of quantifier-rank $O(k^{1/\varepsilon})$. Finally, φ and ψ are logically equivalence on finite structures, since they both define the class $\mathscr{C}$.

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