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Subconvexity for GL(3) × GL(2) L-functions in t-aspect

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Abstract. Let π be a Hecke–Maass cusp form for $\mathrm{SL}(3,\mathbb{Z})$ and f be a holomorphic (or Maass) Hecke cusp form for $\mathrm{SL}(2,\mathbb{Z})$. In this paper we prove the subconvex bound

$$L(1/2+it,\pi\times f)\ll_{\pi,f,\varepsilon}(1+|t|)^{3/2-1/51+\varepsilon}.$$

Keywords. Subconvexity, Rankin–Selberg L-functions, GL(3) Maass forms

1. Introduction

For π a Hecke–Maass cusp form for $\mathrm{SL}(3,\mathbb{Z})$, and f a holomorphic Hecke cusp form for $\mathrm{SL}(2,\mathbb{Z})$ the associated Rankin–Selberg L-series is given by

$$L(s,\pi\times f)=\sum\sum_{n,r=1}^{\infty}\frac{\lambda_\pi(n,r)\lambda_f(n)}{(nr^2)^s}$$

in the half-plane $\sigma>1$. (Here λ_π and λ_f are the normalized Fourier coefficients of the forms.) This series extends to an entire function and satisfies a functional equation of Riemann type,

$$\gamma(s,\pi)L(s,\pi\times f)=i^k\gamma(1-s,\bar{\pi})L(1-s,\bar{\pi}\times f),$$

with a gamma factor of degree 6,

$$\gamma(s,\pi)=\pi^{-3s}\Gamma\left(\frac{s+\frac{k-1}{2}-\alpha_1}{2}\right)\Gamma\left(\frac{s+\frac{k-1}{2}-\alpha_2}{2}\right)\Gamma\left(\frac{s+\frac{k-1}{2}-\alpha_3}{2}\right)$$
$$\times\Gamma\left(\frac{s+\frac{k+1}{2}-\alpha_1}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}-\alpha_2}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}-\alpha_3}{2}\right)$$

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where k is the weight of the holomorphic form f, and α_i are the Langlands parameters of the form π. (This functional equation can be worked out using the template given in [4, p. 315].) This class of Rankin–Selberg L-functions plays a crucial role in arithmetic quantum chaos as their central critical values appear in Watson's formula for the period integrals arising from Weyl's equidistribution criterion (see [10, Lectures 4 and 5] and also the survey [19]). Hence it is important to understand the size of these functions inside the critical strip. As a first step, in this paper we study the growth of this function on the central line $s = 1/2 + it$ for any given pair of forms (π, f). A standard consequence of the functional equation is the easy convexity bound

$$L(1/2 + it, \pi \times f) \ll_{\pi,f,\varepsilon} (1 + |t|)^{3/2+\varepsilon},$$

for any $\varepsilon > 0$, where the implied constant depends on the forms and ε. The Lindelöf hypothesis predicts that such a bound holds with any positive exponent in place of $3/2+\varepsilon$. But even breaking the convexity barrier is hard and has remained open so far. The purpose of this paper is to prove the following subconvex bound.

Theorem 1. *Let π be a Hecke–Maass cusp form for* $\mathrm{SL}(3, \mathbb{Z})$, *and f a holomorphic Hecke cusp form for* $\mathrm{SL}(2, \mathbb{Z})$. *Then*

$$L(1/2 + it, \pi \times f) \ll_{\pi,f,\varepsilon} (1 + |t|)^{3/2-1/51+\varepsilon}.$$

Let us mention that we have not tried to obtain the best possible exponent, as our goal here is to describe a method that works in the current scenario which was beyond existing technology. Subconvex bounds in the t-aspect are known for L-functions of degree up to 3 over the field of rationals. The pioneering work of Weyl [20] yields the famous bound for the Riemann zeta function

$$\zeta(1/2 + it) \ll t^{1/6+\varepsilon},$$

an analogue of which in the case of degree 2 was established by Good [4],

$$L(1/2 + it, f) \ll t^{1/3+\varepsilon}.$$

Though a bound of the same strength is not yet known for degree 3 L-functions, a subconvex bound in this case was established in [17]:

$$L(1/2 + it, \pi) \ll t^{3/4-1/16+\varepsilon}.$$

Similar bounds are also known for the Rankin–Selberg L-function $L(s, f \times g)$ for two GL(2) forms f and g (see e.g. [11]).[1] The t-aspect subconvexity for genuine GL(4) L-functions remains an important open problem. Our method of proving Theorem 1 follows the template given in [17] and is based on the separation of oscillation technique (as

[1] Of course, Michel–Venkatesh [11] does far more than just t-aspect, but the author is not aware of any work which specifically addresses the t-aspect subconvexity problem for such L-functions.

introduced in [15]). The key reason for a similar argument to be effective here – a much more complicated case of degree 6 L-function – is the curious fact that the character sum

$$\sum_{a \bmod q}^{\star} S(\bar{a}, n; q) e\left(\frac{\bar{a}m}{q}\right) \tag{1}$$

essentially boils down to the additive character $qe(-\bar{m}n/q)$. In other words, the GL(3), GL(2) Voronoi summations together transform the Ramanujan sums

$$\sum_{a \bmod q}^{\star} e\left(\frac{a(n-m)}{q}\right),$$

arising in the delta method, to additive characters with respect to the GL(3) variable (see Section 2.3). Hence we save more by applying the Poisson summation after Cauchy's inequality. This is the vital structural input in this paper, and is very specific to Rankin–Selberg convolutions of the type GL(n) × GL($n-1$). This is the main input of this paper which goes far beyond [17], and opens the door for tackling the subconvexity problem for higher rank groups. (One can compare this with the more obvious, yet vital, observation that the GL(2) Voronoi summation transforms the Kloosterman sum into Ramanujan sum. This has been crucial in reducing the subconvexity problem to shifted convolution sum problem in several GL(2) scenarios; see e.g. [2, Section 6].)

The same feature helps us to prove a subconvex bound for these L-functions in the GL(2) spectral aspect (thereby extending the main result of Li [9] to non-self-dual setting). Also the same technique coupled with the 'transfer of mass' trick introduced in [18] settles the subconvexity problem for twists of the above L-function by Dirichlet characters. These will appear in follow up papers. Let us also note that our argument works for Maass forms f, after mild alterations. In fact the argument can be extended to Rankin–Selberg convolutions of general GL(3) and general GL(2) automorphic forms over $\mathbb{Q}$. The cuspidality condition on π and f can also be removed. Thus, replacing π or f or both by suitable Eisenstein series one gets subconvex bounds in t-aspect for GL(1), GL(2) and GL(3) L-functions at one stroke. Needless to say, the exponents turn out to be much worse than what are already available in the literature.

The main technical heart of [17] was the analysis of integral transforms. In this paper we give a simpler analysis of these integrals. This is very much desired as the technique of [17] leads to the Weyl bound in the case of GL(2) and GL(1) L-functions (see [1]), and now perhaps with this simplification one can go further.

2. The set-up

Let $\lambda_\pi(n,m)$ denote the normalised Fourier coefficients of the form π (see [3, Chapter 6]) and let $\lambda_f(n)$ denote the normalised Fourier coefficients of the form f (see [6]). Suppose $t > 2$. Then taking a smooth dyadic subdivision of the approximate functional equation (see [6, Theorem 5.3, Proposition 5.4] and [7, Section 3]) we get

$$L(1/2+it,\pi\times f)\ll t^{\varepsilon}\sup_{N\leq t^{3+\varepsilon}}\frac{|S(N)|}{N^{1/2}}+t^{-2018} \tag{2}$$

where $N=2^{\alpha/2}$ with $\alpha=[-1,\infty)\cap\mathbb{Z}$, and $S(N)$ is a sum of type

$$S(N):=\sum\sum_{n,r=1}^{\infty}\lambda_{\pi}(n,r)\lambda_f(n)(nr^2)^{-it}V\left(\frac{nr^2}{N}\right)$$

for some smooth function V supported in $[1,2]$ and satisfying $V^{(j)}(x)\ll_j 1$. Note that the error term $O(t^{-2018})$ takes into account the negligible contribution of the tail, i.e. those terms with nr^2 larger than the square root of the conductor of the L-function.

Remark 1 (Notation). In this paper the notation $\alpha\ll A$ will mean that for any $\varepsilon>0$, there is a constant c such that $|\alpha|\leq cAt^{\varepsilon}$. The dependence of the constant on π, f and ε, when occurring, will be ignored.

Using the Ramanujan bounds on average (see [13])

$$\sum\sum_{n_1^2n_2\leq x}|\lambda_{\pi}(n_1,n_2)|^2\ll x^{1+\varepsilon}\quad\text{and}\quad\sum_{n\leq x}|\lambda_f(n)|^2\ll x^{1+\varepsilon}, \tag{3}$$

we get $S(N)\ll N$, and consequently (using the above convention)

$$L(1/2+it,\pi\times f)\ll\sup_{t^{3-\theta}<N\leq t^{3+\varepsilon}}\frac{|S(N)|}{N^{1/2}}+t^{(3-\theta)/2}.$$

We set

$$S_r(M):=\sum_{n=1}^{\infty}\lambda_{\pi}(n,r)\lambda_f(n)n^{-it}V\left(\frac{n}{M}\right).$$

Then

$$\frac{S(N)}{N^{1/2}}\ll\sum_{r\ll N^{1/2}}\frac{|S_r(N/r^2)|}{N^{1/2}}\ll\sum_{r\leq t^{\theta}}\frac{|S_r(N/r^2)|}{N^{1/2}}+\sum_{r>t^{\theta}}\frac{|S_r(N/r^2)|}{N^{1/2}}.$$

Employing (3) we get

$$\begin{aligned}\sum_{r>t^{\theta}}\frac{|S_r(N/r^2)|}{N^{1/2}}&\ll\frac{1}{N^{1/2}}\sum_{r>t^{\theta}}\sum_{n\ll N/r^2}|\lambda_{\pi}(n,r)|\,|\lambda_f(n)|\\&\ll\frac{1}{N^{1/2}}\Big[\sum_{r>t^{\theta}}\sum_{n\ll N/r^2}|\lambda_f(n)|^2\Big]^{1/2}\Big[\sum\sum_{nr^2\ll N}|\lambda_{\pi}(n,r)|^2\Big]^{1/2}\\&\ll\frac{N^{1/2}}{t^{\theta/2}}.\end{aligned}$$

Also for $t^{3-\theta} < N \leq t^{3+\varepsilon}$ we find that

$$\sum_{r\leq t^\theta} \frac{|S_r(N/r^2)|}{N^{1/2}} \ll \sum_{r\leq t^\theta} \frac{1}{r} \sup_{\frac{t^{3-\theta}}{r^2}\leq M\leq\frac{t^{3+\varepsilon}}{r^2}} \frac{|S_r(M)|}{M^{1/2}} \ll \sup_{r\leq t^\theta} \sup_{\frac{t^{3-\theta}}{r^2}\leq M\leq\frac{t^{3+\varepsilon}}{r^2}} \frac{|S_r(M)|}{M^{1/2}}.$$

Plugging these bounds in (2) we conclude that

$$L(1/2+it,\pi\times f) \ll \sup_{r\leq t^\theta} \sup_{\frac{t^{3-\theta}}{r^2}\leq N\leq\frac{t^{3+\varepsilon}}{r^2}} \frac{|S_r(N)|}{N^{1/2}} + t^{(3-\theta)/2}. \tag{4}$$

Hence to establish subconvexity we need to show cancellation in the sum $S_r(N)$ for N roughly of size t^3 and r small. We can and shall further normalize V, for convenience, so that $\int V(y)\,\mathrm{d}y = 1$.

2.1. The delta method

There are three oscillatory factors contributing to the sum $S_r(N)$. Our method is based on separating these oscillations using the circle method. In the present situation we will use a version of the delta method of Duke, Friedlander and Iwaniec.[2] More specifically, we will use the expansion (20.157) given in Chapter 20 of [6]. Let $\delta : \mathbb{Z} \to \{0, 1\}$ be defined by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We seek a Fourier expansion which matches with δ in the range $[-2M, 2M]$. For this we pick $Q = 2M^{1/2}$. Then we have

$$\delta(n) = \frac{1}{Q} \sum_{1\leq q\leq Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} g(q,x) e\left(\frac{nx}{qQ}\right) \mathrm{d}x \tag{5}$$

for $n \in \mathbb{Z} \cap [-2M, 2M]$ (and $e(z) = e^{2\pi i z}$). The $\star$ on the sum indicates that the sum over a is restricted by the condition $(a, q) = 1$. The function g is the only part in the formula which is not explicitly given. We only need the following three properties (see [6, (20.158) and (20.159)], and [5, Lemma 15])

$$\begin{aligned} &g(q,x) = 1 + O\left(\frac{Q}{q}\left(\frac{q}{Q} + |x|\right)^A\right), \qquad g(q,x) \ll |x|^{-A} \text{ for any } A > 1, \\ &x^j \frac{\partial^j}{\partial x^j} g(q,x) \ll \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|}\right\} \end{aligned} \tag{6}$$

[2] The choice is guided by the curious observation about the character sum (1). This phenomenon is absent in Kloosterman's version of circle method, however it is present in the GL(2) δ-method in a different guise.

for $j \geq 1$. In particular the second property implies that the effective range of the integral in (5) is $[-M^\varepsilon, M^\varepsilon]$. Also it follows that if $q \ll Q^{1-\varepsilon}$ and $x \ll Q^{-\varepsilon}$, then $g(q,x)$ can be replaced by 1 at a cost of a negligible error term. In the complementary range we have $x^j g^{(j)}(q,x) \ll Q^\varepsilon$. Finally, by Parseval and Cauchy we get

$$\int (|g(q,x)| + |g(q,x)|^2)\,\mathrm{d}x \ll Q^\varepsilon,$$

i.e. $g(q,x)$ has average size 1 in the L^1 and L^2 sense.

2.2. *Separation of oscillation*

We apply (5) directly to $S_r(N)$ as a device to separate the oscillations of $\lambda(n,r)$ and $\lambda_f(n)n^{-it}$. This by itself does not suffice, and as in [16] and [17] we need a 'conductor lowering mechanism'. For this purpose we introduce an extra integral, namely

$$S_r(N) = \frac{1}{K}\int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum\sum_{\substack{n,m=1\\ n=m}}^{\infty} \lambda_\pi(n,r)\lambda_f(m)m^{-it}\left(\frac{n}{m}\right)^{iv} V\left(\frac{n}{N}\right)U\left(\frac{m}{N}\right)\mathrm{d}v,$$

where $t^\varepsilon < K < t^{1-\varepsilon}$ is a parameter which will be chosen optimally later, and U is a smooth function supported in $[1/2, 5/2]$, with $U(x) = 1$ for $x \in [1,2]$ and $U^{(j)} \ll_j 1$. For $n, m \asymp N$, the integral

$$\frac{1}{K}\int_{\mathbb{R}} V\left(\frac{v}{K}\right)\left(\frac{n}{m}\right)^{iv}\mathrm{d}v$$

is negligibly small (i.e. $O_A(t^{-A})$ for any $A > 0$) if $|n-m| \gg Nt^\varepsilon/K$. Hence we can apply (5) with

$$Q = t^\varepsilon \left(\frac{N}{K}\right)^{1/2} \tag{7}$$

and we find that up to a negligible error term, $S_r(N)$ is given by

$$\begin{aligned}\frac{1}{QK}\int_{\mathbb{R}} W(x) \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{1\leq q\leq Q} \frac{g(q,x)}{q} \sideset{}{^\star}\sum_{a \bmod q} \\ \sum_{n=1}^{\infty} \lambda_\pi(n,r) e\left(\frac{an}{q}\right) e\left(\frac{nx}{qQ}\right) n^{iv} V\left(\frac{n}{N}\right) \\ \times \sum_{m=1}^{\infty} \lambda_f(m) m^{-i(t+v)} e\left(-\frac{am}{q}\right) e\left(-\frac{mx}{qQ}\right) U\left(\frac{m}{N}\right)\mathrm{d}v\,\mathrm{d}x,\end{aligned} \tag{8}$$

where W is a smooth bump function with support $[-t^\varepsilon, t^\varepsilon]$.

2.3. *Sketch of proof*

We end this section with a brief sketch of the proof. For simplicity let us focus on the generic case, i.e. $N = t^3$, $r = 1$ and $q \sim Q = t^{3/2}/K^{1/2}$, so that the main object of study is given by

$$\int_{v\sim K} \sum_{q\sim Q} \sideset{}{^\star}\sum_{a \bmod q} \sum_{n\sim N} \lambda_\pi(n,1) e\left(\frac{an}{q}\right) n^{iv} \sum_{m\sim N} \lambda_f(m) e\left(-\frac{am}{q}\right) m^{-i(t+v)}\, \mathrm{d}v.$$

Our aim is to save N plus a 'little more'. First we apply the Voronoi summation formulae to both the m and n sums. In the GL(2) (resp. GL(3)) Voronoi we save $(NK)^{1/2}/t$ (resp. $N^{1/4}/K^{3/4}$) and the dual length becomes $m^\star \sim t^2/K$ (resp. $n^\star \sim K^{3/2}N^{1/2}$). Also we save $\sqrt{Q}$ in the a sum and $\sqrt{K}$ in the v integral. Hence in total we have saved N/t, and it remains to save t plus a little extra in a sum of the form

$$\sum_{q\sim Q} \sum_{n\sim K^{3/2}N^{1/2}} \lambda_\pi(1,n) \sum_{m\sim t^2/K} \lambda_f(m) \mathfrak{C}\mathfrak{I}$$

where $\mathfrak{I}$ is an integral transform which oscillates like n^{iK} with respect to n, and the character sum is given by

$$\mathfrak{C} = \sideset{}{^\star}\sum_{a \bmod q} S(\bar{a}, n; q) e\left(\frac{\bar{a}m}{q}\right) \rightsquigarrow q e\left(-\frac{\bar{m}n}{q}\right).$$

Here $\rightsquigarrow$ means that the left hand side essentially reduces to the right hand side. Next applying the Cauchy inequality we arrive at

$$\sum_{n\sim K^{3/2}N^{1/2}} \left| \sum_{q\sim Q} \sum_{m\sim t^2/K} \lambda_f(m) e\left(-\frac{\bar{m}n}{q}\right) \mathfrak{I} \right|^2 \tag{9}$$

where we seek to save t^2 plus extra. Opening the absolute value square we apply the Poisson summation formula on the sum over n. We save enough in the zero frequency (diagonal contribution) if $t^2 Q/K > t^2$, i.e. if $K < t$. On the other hand, we save enough in the non-zero frequencies if $K^{3/2}N^{1/2}/K^{1/2} > t^2$, which boils down to $K > t^{1/2}$.

Remark 2. Notice that since the character sum boils down to an additive character we are saving more than usual. In the general case, one would have a character sum modulo q in place of the additive character inside the absolute value in (9), and hence after applying Poisson in the next step one would obtain a more complicated character sum. Hence in the general case, we would only hope to save square root of the modulus in the character sum. Consequently, in the non-zero frequencies we would have saved $K^{3/2}N^{1/2}/(QK^{1/2})$ (in place of $K^{3/2}N^{1/2}/K^{1/2}$), which would be larger than t^2 only if we had $K > t^{4/3}$. This would contradict the upper bound $K < t$.

3. Voronoi summation formulae

3.1. GL(2) *Voronoi*

Consider the sum over m in (8). The Voronoi summation formula (see [6, Section 4.5]) transforms this sum into

$$\frac{N^{1-i(t+v)}}{q}\sum_{m=1}^{\infty}\lambda_f(m)e\left(\frac{\bar{a}m}{q}\right)\int_0^{\infty}U(y)y^{-i(t+v)}e\left(-\frac{Nxy}{qQ}\right)J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right)\mathrm{d}y$$

where k is the weight of the form f. Extracting the oscillation of the J-Bessel function (see [7, Section 4])

$$J_r(2\pi x)=e(x)W_r(x)+e(-x)\bar{W}_r(x),$$

with

$$x^j\frac{\mathrm{d}^j}{\mathrm{d}x^j}W_r(x)\ll_j 1/\sqrt{x},$$

we see that the above sum is essentially given by a sum of two terms of the form

$$\frac{N^{3/4-i(t+v)}}{q^{1/2}}\sum_{m=1}^{\infty}\frac{\lambda_f(m)}{m^{1/4}}e\left(\frac{\bar{a}m}{q}\right)\int_0^{\infty}U(y)y^{-i(t+v)}e\left(-\frac{Nxy}{qQ}\pm\frac{2\sqrt{mNy}}{q}\right)\mathrm{d}y. \tag{10}$$

(Notice the slight abuse of notation: the weight function U in (10) is different from the one in the previous expression.) By repeated integration by parts it follows that the integral is negligibly small if

$$m\gg t^{\varepsilon}\left(\frac{t^2q^2}{N}+K\right)=:M_0.$$

In the complementary range the size of the integral is given by the second derivative bound. However we need a more precise analysis of the integral based on the stationary phase expansion, which will be taken up later, e.g. in Lemma 5. At this point we note that if $Nx/(qQ\ll t^{1-\varepsilon})$ then $m\asymp(qt)^2/N$, otherwise the integral is negligibly small.

3.2. GL(3) *Voronoi*

Next we apply the GL(3) Voronoi summation (of Miller–Schmid [12]) to the sum over n in (8). A similar sum occurred in [17]. The only difference is that there we had $r=1$, while here r is allowed to take small values $r\ll t^{\theta}$. This only introduces certain cosmetic complications. Let $\{\alpha_i : i=1,2,3\}$ be the Langlands parameters for $\bar{\pi}$. Let g be a compactly supported smooth function on $(0,\infty)$. We define, for $\ell=0,1$,

$$\gamma_\ell(s):=\frac{\pi^{-3s-3/2}}{2}\prod_{i=1}^{3}\frac{\Gamma\left(\frac{1+s+\alpha_i+\ell}{2}\right)}{\Gamma\left(\frac{-s-\alpha_i+\ell}{2}\right)},$$

set $\gamma_\pm(s) = \gamma_0(s) \mp i\gamma_1(s)$ and let

$$G_\pm(y) = \frac{1}{2\pi i}\int_{(\sigma)} y^{-s}\gamma_\pm(s)\tilde{g}(-s)\,\mathrm{d}s,$$

where $\sigma > -1 + \max\{-\mathrm{Re}(\alpha_1), -\mathrm{Re}(\alpha_2), -\mathrm{Re}(\alpha_3)\}$ and $\tilde{g}$ is the Mellin transform of g. The GL(3) Voronoi summation formula (see [9] and [17]) is given by

$$\sum_{n=1}^{\infty}\lambda_\pi(n,r)e\left(\frac{an}{q}\right)g(n)$$
$$= q\sum_{\pm}\sum_{n_1|qr}\sum_{n_2=1}^{\infty}\frac{\lambda_\pi(n_1,n_2)}{n_1n_2}S(r\bar{a},\pm n_2;qr/n_1)G_\pm\left(\frac{n_1^2n_2}{q^3r}\right). \tag{11}$$

In the present case we have $g(n) = e(nx/(qQ))n^{iv}V(n/N)$. Next we need to extract the oscillation of the integral transform as in the case of GL(2) above. To this end we employ Lemma 6.1 of [8], which gives an explicit expansion of the integral transform together with a bound for the error term,

$$G_\pm(y) = y\int_0^\infty g(z)\sum_{j=1}^{K}\frac{c_{j,\pm}e(3(zy)^{1/3}) + d_{j,\pm}e(-3(zy)^{1/3})}{(zy)^{j/3}}\,\mathrm{d}z + O((yN)^{(5-K)/3}),$$

where $c_{j,\pm}$, $d_{j,\pm}$ are constants depending on the Langlands parameters of the form π. (The proof is based on standard properties of the Mellin transform and exploits well-known relations between the gamma function and the Bessel function. In particular, in the proof, the error term is estimated trivially by shifting the contour integral and using the trivial bound $|\tilde{g}(-s)| \ll N^{-\sigma}$ in terms of the support of g.) This expansion can be used in our context for the range

$$\frac{n_1^2n_2N}{q^3r} \gg t^\varepsilon, \tag{12}$$

where we are able to replace the expression in (11) essentially by

$$\frac{N^{2/3+iv}}{qr^{2/3}}\sum_{\pm}\sum_{n_1|qr}n_1^{1/3}\sum_{n_2=1}^{\infty}\frac{\lambda_\pi(n_1,n_2)}{n_2^{1/3}}S(r\bar{a},\pm n_2;qr/n_1)$$
$$\times\int_0^\infty V(z)z^{iv}e\left(\frac{Nxz}{qQ} \pm \frac{3(Nn_1^2n_2z)^{1/3}}{qr^{1/3}}\right)\mathrm{d}z. \tag{13}$$

Next by repeated integration by parts we see that the integral is negligibly small if

$$n_1^2n_2 \gg t^\varepsilon\left(\frac{(qK)^3r}{N} + K^{3/2}N^{1/2}rx^3\right) =: N_0.$$

We now substitute (10) in place of the third line and (13) in place of the second line of (8), to get the object of focus.

3.3. Error term

Consider the range complementary to (12). In this case we move the contour in the definition of $G_\pm$ to the left up to $\sigma = -5/2$ passing through the poles given by

$$\frac{1+s+\alpha_i+\ell}{2} = 0, \quad \text{i.e.} \quad s = -1-\ell-\alpha_i.$$

The second derivative bound yields

$$\tilde{g}(-s) = \int_0^\infty e\left(\frac{xz}{qQ}\right) z^{iv-s} V\left(\frac{z}{N}\right) \frac{\mathrm{d}z}{z} \ll N^{-\sigma} \sqrt{\frac{qQ}{Nx}}.$$

To the gamma factor we use Stirling, which ensures absolute convergence of the integral over the contour $\sigma = -5/2$. Consequently, we get the bound

$$G_\pm\left(\frac{n_1^2 n_2}{q^3 r}\right) \ll \left(\frac{n_1^2 n_2 N}{q^3 r}\right)^{5/2} \sqrt{\frac{qQ}{Nx}} + \sum_{\ell=0,1} \sum_{i=1}^{3} \left(\frac{n_1^2 n_2 N}{q^3 r}\right)^{1+\ell+\mathrm{Re}(\alpha_i)} \sqrt{\frac{qQ}{Nx}}.$$

The first term accounts for the integral over $\sigma = -5/2$ and the second term accounts for the contribution of the poles. Now since $1+\ell+\mathrm{Re}(\alpha_i) > 1/2$, and since we are in the range complementary to (12), i.e. $n_1^2 n_2 N/(q^3 r) \ll t^\varepsilon$, it follows that

$$G_\pm\left(\frac{n_1^2 n_2}{q^3 r}\right) \ll \left(\frac{n_1^2 n_2 Q}{q^2 r x}\right)^{1/2}.$$

Plugging this bound in (11), and using the Weil bound for Kloosterman sums, and the Ramanujan bound on average (3), we find that the expression in (11) is bounded by

$$\frac{q^2\sqrt{Qr}}{\sqrt{Nx}}.$$

(We can ignore the $\sqrt{x}$ in the denominator as the integral over x in (5) balances this up.) Comparing with the left hand side of (11) we see that in this case GL(3) Voronoi saved $N^{3/2}/(q^2\sqrt{Qr})$. Returning to our analysis in Section 3.1, and using the second derivative bound for the oscillating integral in (10) we see that the GL(2) Voronoi saved $\sqrt{N/M_0}$. As we will see later we will have square root cancellation in the complete sum over a mod q. So in the present case we are able to save $N^2/(q^{3/2}\sqrt{M_0 Qr})$ over the trivial bound in (8), and consequently the total contribution of the terms in the range complementary to (12) to (8) is bounded by

$$Q^2\sqrt{r}\left(\frac{tQ}{\sqrt{N}} + \sqrt{K}\right) \ll \sqrt{r}\left(\frac{tN}{K^{3/2}} + \frac{N}{\sqrt{K}}\right). \tag{14}$$

4. Reduction of integrals

4.1. Simplifying the integrals

We have transformed the sum in (8) into a new object with four integrals

$$\int_{\mathbb{R}} W(x)g(q,x)\int_{\mathbb{R}} V\left(\frac{v}{K}\right)\int_0^\infty U(y)y^{-i(t+v)}\int_0^\infty V(z)z^{iv} \times e\left(\frac{Nx(z-y)}{qQ} \pm \frac{2\sqrt{mNy}}{q} \pm \frac{3(Nn_1^2n_2z)^{1/3}}{qr^{1/3}}\right) \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}v\, \mathrm{d}x, \tag{15}$$

which we need to simplify. Consider the integral over x which is given by

$$\int_{\mathbb{R}} W(x)g(q,x)e\left(\frac{Nx(z-y)}{qQ}\right) \mathrm{d}x.$$

First consider the case where $q \ll Q^{1-\varepsilon}$. Break the integral over x into two parts using a smooth partition of unity. In the first part we have

$$\int_{\mathbb{R}} U(x)g(q,x)e\left(\frac{Nx(z-y)}{qQ}\right) \mathrm{d}x,$$

where U is supported in $[-Q^{-\varepsilon}, Q^{-\varepsilon}]$, and satisfies $U^{(j)} \ll_j Q^{\varepsilon j}$. In this range we can replace $g(q,x)$ by 1 up to a negligible error term (see (6)). Then by repeated integration by parts we see that the integral is negligibly small unless $|z-y| \ll t^\varepsilon q/(QK)$. In the remaining part we have

$$\int_{\mathbb{R}} V(x)g(q,x)e\left(\frac{Nx(z-y)}{qQ}\right) \mathrm{d}x,$$

with V supported in $[-Q^\varepsilon, -Q^{-\varepsilon}/2] \cup [Q^{-\varepsilon}/2, Q^\varepsilon]$ and satisfying $V^{(j)} \ll_j Q^{\varepsilon j}$. Now using (6) and by repeated integration by parts we see that the integral is negligibly small unless $|z-y| \ll t^\varepsilon q/(QK)$. It remains to consider the case where $q \gg Q^{1-\varepsilon}$. Here the derivatives of $g(q,x)$ can be large for x near 0. However, in this case we can use the v integral to conclude that $|z-y| \ll t^\varepsilon/K \ll t^\varepsilon q/(QK)$. Note that the x integral (or the v integral) is used only to get the above restriction on z, where we effectively save QK/q (roughly K) in the length of z, which is equivalent to 'square root' saving in both x (or v) and z integral. (Of course we cannot expect to save in both x and v integrals.) Next writing $z = y + u$ with $|u| \ll t^\varepsilon q/(QK)$ we arrive at the y integral

$$I(m, n_1^2n_2, q) := \int_0^\infty U(y)y^{-it}e\left(\pm\frac{2\sqrt{mNy}}{q} \pm \frac{3(Nn_1^2n_2(y+u))^{1/3}}{qr^{1/3}}\right) \mathrm{d}y. \tag{16}$$

Observe that $(z/y)^{iv} = (1+u/y)^{iv} = e^{iv\log(1+u/y)}$, which is not oscillating as a function of y, and hence can be absorbed (as elsewhere in this paper) in the weight function U. Thus we have reduced the four-fold integral in (15) to

$$\frac{q}{QK} \times K \times w(q) \times \int_0^\infty U(y)y^{-it}e\left(\pm\frac{2\sqrt{mNy}}{q} \pm \frac{3(Nn_1^2n_2(y+u))^{1/3}}{qr^{1/3}}\right) \mathrm{d}y,$$

with the understanding that one needs to take supremum over $|u| \ll C/(QK)$ at the end when the sum over q is restricted to the dyadic block $C \leq q < 2C$ (i.e. $q \sim C$). One will notice that our analysis below is uniform with respect to u in the given range. Note that the factor $q/(QK)$ (resp. K) in front of the integral reflects the length of the u integral (resp. v integral). Also the weight $w(q) \ll 1$ comes from the integral

$$\int W(x)g(q,x)e\left(\frac{nux}{qQ}\right)\mathrm{d}x.$$

4.2. Size of the integral $I(\dots)$

Suppose $K = t^{1-\eta}$ for some $\eta > 0$. Then we claim that essentially $I(\dots) \ll t^{-1/2}$. We will prove that the bound holds in the L^2-sense.

Lemma 1. *Let*

$$L = \int W(w)|I(m, N_0 w^3, q)|^2\,\mathrm{d}w$$

where W is a bump function. Then

$$L \ll \min\{1/t, qr^{1/3}/(NN_0)^{1/3}\}.$$

Proof. To prove this assertion we make the change of variable $z = y^{1/2}$, so that the phase function in (16) reduces to

$$P = -\frac{t}{\pi}\log z \pm \frac{2\sqrt{mN}\,z}{q} \pm \frac{3(NN_0(z^2+u))^{1/3}w}{qr^{1/3}}.$$

Then

$$P'' = \frac{t}{\pi z^2} \mp \frac{2(NN_0)^{1/3}w}{3qr^{1/3}z^{4/3}} + \text{smaller order terms.}$$

By the second derivative bound one gets

$$I(\dots) \ll \min\{1/t^{1/2}, q^{1/2}r^{1/6}/(NN_0)^{1/6}\},$$

except possibly when $3(NN_0)^{1/3}w/(qr^{1/3}) \asymp t$. In this special case opening the absolute value square we arrive at

$$L \ll \iint U(y_1)U(y_2)\left|\int W(w)e\left(\frac{3w(NN_0)^{1/3}}{qr^{1/3}}((y_1+u)^{1/3}-(y_2+u)^{1/3})\right)\mathrm{d}w\right|\mathrm{d}y_1\,\mathrm{d}y_2$$
$$\ll \iint_{|y_1-y_2|\ll 1/t} U(y_1)U(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 + t^{-2018} \ll 1/t \asymp qr^{1/3}/(NN_0)^{1/3}.$$

The lemma follows. ∎

5. Cauchy and Poisson

5.1. Cauchy inequality

Substituting (10) and (13) to (8), and using (7), we find that the expression in (8) essentially reduces to

$$\frac{N^{5/12}}{r^{2/3}} \sum_{1\leq q\leq Q} \frac{w(q)}{q^{3/2}} \sideset{}{^\star}\sum_{a \bmod q} \sum_{\pm} \sum_{n_1|qr} n_1^{1/3} \sum_{n_2\ll N_0/n_1^2} \frac{\lambda_\pi(n_1,n_2)}{n_2^{1/3}} S(r\bar{a},\pm n_2;qr/n_1) \times \sum_{m\ll M_0} \frac{\lambda_f(m)}{m^{1/4}} e\left(\frac{\bar{a}m}{q}\right) I(m,n_1^2n_2,q).$$

The weights $w(q)\ll 1$ will not be of any concern as we will see below. Splitting the sum over q into dyadic blocks $q\sim C$, and writing $q=q_1q_2$ with $q_1 \mid (n_1r)^\infty$, $(n_1r,q_2)=1$, we see that the contribution of the C-block to the above sum is dominated by

$$\frac{N^{5/12}}{r^{2/3}C^{3/2}} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \sum_{n_2\ll N_0/n_1^2} \frac{|\lambda_\pi(n_1,n_2)|}{n_2^{1/3}} \times \left| \sum_{q_2\sim C/q_1} w(q_1q_2) \sum_{m\ll M_0} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(\dots) I(m,n_1^2n_2,q)\right|, \tag{17}$$

where the character sum $\mathcal{C}(\dots)$ is given by

$$\sideset{}{^\star}\sum_{a \bmod q} S(r\bar{a},\pm n_2;qr/n_1) e\left(\frac{\bar{a}m}{q}\right) = \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1\\ n_1\alpha\equiv -m \bmod d}} e(\pm\frac{\bar{\alpha}n_2}{qr/n_1}).$$

To analyse the sum in (17) further we break the sum over m into dyadic blocks. Then applying Cauchy's inequality and using the Ramanujan bound on average we see that the expression in (17) is dominated by

$$\sup_{M_1\ll M_0} \frac{N^{5/12}}{r^{2/3}C^{3/2}} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \Omega^{1/2} \tag{18}$$

where

$$\Theta = \sum_{n_2\ll N_0/n_1^2} \frac{|\lambda_\pi(n_1,n_2)|^2}{n_2^{2/3}}, \tag{19}$$

$$\Omega = \sum_{n_2\ll N_0/n_1^2} \left| \sum_{q_2\sim C/q_1} w(q_1q_2) \sum_{m\sim M_1} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(\dots) I(m,n_1^2n_2,q)\right|^2, \tag{20}$$

with

$$M_1 \ll M_0 = t^\varepsilon\left(K+\frac{C^2t^2}{N}\right), \qquad N_0 = t^\varepsilon\left(\frac{(CK)^3r}{N}+K^{3/2}N^{1/2}rx^3\right). \tag{21}$$

5.2. Poisson summation

Smoothing out the outer sum in (20) with an appropriate bump function and opening the absolute value square we get

$$\Omega \ll \sum\sum_{q_2,q_2'\sim C/q_1} w(q_1q_2)\overline{w(q_1q_2')} \sum\sum_{m,m'\sim M_1} \frac{\lambda_f(m)\lambda_f(m')}{(mm')^{1/4}}$$
$$\times \sum_{n_2\in\mathbb{Z}} \mathcal{C}(\dots)\overline{\mathcal{C}(\dots)}\, I(m,n_1^2n_2,q)\overline{I(m',n_1^2n_2,q')}W\left(\frac{n_2n_1^2}{N_0}\right)$$

where $q'=q_1q_2'$. Next we apply the Poisson summation formula (see [6, (4.25)]) to the sum over n_2 with modulus $q_1q_2q_2'r/n_1$. After a standard change of variables in the Fourier transform, and a simple evaluation of the character sum, we arrive at

$$\Omega \ll \frac{N_0}{n_1^2M_1^{1/2}} \sum\sum_{q_2,q_2'\sim C/q_1} \sum\sum_{m,m'\sim M_1} \sum_{n_2\in\mathbb{Z}} |\mathfrak{C}|\,|\mathfrak{I}|, \tag{22}$$

where

$$\mathfrak{C} = \sum\sum_{\substack{d|q\\ d'|q'}} dd'\mu\left(\frac{q}{d}\right)\mu\left(\frac{q'}{d'}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1\\ n_1\alpha\equiv -m \bmod d}} \sideset{}{^\star}\sum_{\substack{\alpha' \bmod q'r/n_1\\ n_1\alpha'\equiv -m' \bmod d'\\ \bar\alpha q_2'-\bar\alpha' q_2\equiv n_2 \bmod q_2q_2'q_1r/n_1}} 1,$$

$$\mathfrak{I} = \int W(w)I(m,N_0w,q)\overline{I(m',N_0w,q')}e\left(-\frac{N_0n_2w}{n_1q_2q_2'q_1r}\right)\mathrm{d}w.$$

(Note that since we are considering the case of f holomorphic, we can simple use the Deligne bound for the Fourier coefficients $\lambda_f(m)$. For Maass forms one needs to apply the Ramanujan bound on average after a suitable application of the Cauchy inequality at the end.) By repeated integration by parts we see that the integral is negligibly small if

$$|n_2| \gg t^{\varepsilon}\frac{CN^{1/3}r^{2/3}n_1}{q_1N_0^{2/3}} =: N_2. \tag{23}$$

In the complementary range Lemma 1 yields a bound for the integral $\mathfrak{I}$.

5.3. The zero frequency

The zero frequency $n_2=0$ has to be treated differently. Let Ω_0 denote the contribution of the zero frequency to Ω, and let Σ_0 be its contribution to (18).

Lemma 2. *We have*

$$\Omega_0 \ll \frac{N_0M_1^{1/2}C^2r}{n_1^2q_1}\min\left\{\frac{1}{t},\frac{Cr^{1/3}}{(NN_0)^{1/3}}\right\}(C+M_1), \qquad \Sigma_0 \ll r^{1/2}N^{1/2}t^{3/2-\eta/2},$$

where $K=t^{1-\eta}$.

Proof. In the case $n_2 = 0$ it follows from the congruence conditions in the definition of $\mathfrak{C}$ that $q_2 = q_2'$ and $\alpha = \alpha'$. So the character sum is bounded as

$$\mathfrak{C} \ll \sum\sum_{d,d'|q} dd' \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ n_1\alpha\equiv -m \bmod d \\ n_1\alpha\equiv -m' \bmod d'}} 1 \ll \sum\sum_{\substack{d,d'|q \\ (d,d')|(m-m')}} dd' \frac{qr}{[d,d']},$$

and hence we get (using Lemma 1)

$$\begin{aligned}\Omega_0 &\ll \frac{N_0}{n_1^2 M_1^{1/2} Y} \sum_{q_2\sim C/q_1} qr \sum\sum_{d,d'|q} (d,d') \sum\sum_{\substack{m,m'\sim M_1 \\ (d,d')|m-m'}} 1 \\ &\ll \frac{N_0}{n_1^2 M_1^{1/2} Y} \sum_{q_2\sim C/q_1} qr \sum\sum_{d,d'|q} (M_1(d,d') + M_1^2),\end{aligned}$$

where $Y = \max\{t, (NN_0)^{1/3}/(Cr^{1/3})\}$. Trivially handling the remaining sums we get the first part of the lemma.

This bound when substituted in place of Ω in (18) yields

$$\sup_{M_1\ll M_0} \frac{N^{5/12}}{r^{2/3}C^{3/2}} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \frac{(N_0 r)^{1/2} M_1^{1/4} C}{n_1(q_1Y)^{1/2}} (\sqrt{C} + \sqrt{M_1}). \tag{24}$$

Substituting the expression for M_0 in place of M_1, using the trivial bound $N_0 \ll K^{3/2}N^{1/2}r$, and replacing the range for n_1 by the longer range $n_1 \ll Cr$, one arrives at

$$\frac{N^{2/3}K^{3/4}r^{1/3}}{Y^{1/2}} \left(1 + \frac{K^{1/2}}{C^{1/2}} + \frac{C^{1/2}t}{N^{1/2}}\right)\left(K^{1/4} + \frac{(Ct)^{1/2}}{N^{1/4}}\right) \sum_{n_1\ll Cr} \frac{(r,n_1)^{1/2}}{n_1^{7/6}} \Theta^{1/2}.$$

To the last sum we apply Cauchy to get

$$\sum_{n_1\ll Cr} \frac{(r,n_1)^{1/2}}{n_1^{7/6}} \Theta^{1/2} \ll \left[\sum_{n_1\ll Cr} \frac{(r,n_1)}{n_1}\right]^{1/2} \left[\sum\sum_{n_1^2 n_2\ll N_0} \frac{|\lambda_\pi(n_1,n_2)|^2}{(n_1^2n_2)^{2/3}}\right]^{1/2} \ll N_0^{1/6} \tag{25}$$

where in the last sum we applied (3) and partial summation. It follows that the expression in (24) is bounded by

$$\frac{N^{3/4}Kr^{1/2}}{Y^{1/2}} \left(1 + \frac{K^{1/2}}{C^{1/2}} + \frac{C^{1/2}t}{N^{1/2}}\right)\left(K^{1/4} + \frac{(Ct)^{1/2}}{N^{1/4}}\right). \tag{26}$$

Here if we substitute $\sqrt{N/K}$ (resp. t) in place of C (resp. Y) and use the fact that $K = t^{1-\eta}$, then we get $O(r^{1/2}N^{1/2}t^{3/2-\eta/2})$ as the final bound on (18). This takes care of all the terms in (26) except the single term which has $C^{1/2}$ in the denominator. For this term we substitute $(NN_0)^{1/3}/(Cr^{1/3})$ in place of Y. Hence, this particular term of (26) is bounded by

$$\frac{N^{3/4}Kr^{1/2}}{((NN_0)^{1/3}/Cr^{1/3})^{1/2}}\frac{K^{3/4}}{C^{1/2}} \ll \frac{N^{7/12}K^{7/4}r^{2/3}}{N_0^{1/6}} \ll \frac{K^{3/2}(Nr)^{1/2}}{|x|^{1/2}}.$$

(In the last inequality we have used (21).) The integral over x takes care of the $x^{1/2}$ in the denominator, and we see that the total contribution of this term to (18) is dominated by $O(r^{1/2}N^{1/2}t^{3/2-3\eta/2})$. The lemma follows. ∎

6. Analysis of non-zero frequencies

6.1. The character sum

Our next lemma gives a bound for $\mathfrak{C}$.

Lemma 3. *We have*

$$\mathfrak{C} \ll \frac{q_1^2 r(n_1,m')}{n_1} \sum\sum_{\substack{d_2|(q_2,q_2'n_1+mn_2)\\ d_2'|(q_2',q_2n_1+m'n_2)}} d_2d_2'.$$

Proof. The 'character sum' $\mathfrak{C}$ can be dominated by a product of two sums, $\mathfrak{C} \ll \mathfrak{C}_1\mathfrak{C}_2$, where

$$\mathfrak{C}_1 = \sum\sum_{d_1,d_1'|q_1} d_1d_1' \mathop{\sum\nolimits^{\star}}_{\substack{\alpha \bmod q_1r/n_1\\ n_1\alpha\equiv -m \bmod d_1}} \mathop{\sum\nolimits^{\star}}_{\substack{\alpha' \bmod q_1r/n_1\\ n_1\alpha'\equiv -m' \bmod d_1'\\ \bar\alpha q_2' - \bar\alpha' q_2 \equiv n_2 \bmod q_1r/n_1}} 1,$$

$$\mathfrak{C}_2 = \sum\sum_{\substack{d_2|q_2\\ d_2'|q_2'}} d_2d_2' \mathop{\sum\nolimits^{\star}}_{\substack{\alpha \bmod q_2\\ n_1\alpha\equiv -m \bmod d_2}} \mathop{\sum\nolimits^{\star}}_{\substack{\alpha' \bmod q_2'\\ n_1\alpha'\equiv -m' \bmod d_2'\\ \bar\alpha q_2' - \bar\alpha' q_2 \equiv n_2 \bmod q_2q_2'}} 1.$$

In the second sum, since $(n_1, q_2q_2') = 1$, we get $\alpha \equiv -m\bar n_1 \bmod d_2$ and $\alpha' \equiv -m'\bar n_1 \bmod d_2'$. Then using the congruence modulo q_2q_2' we are able to conclude that

$$\mathfrak{C}_2 \ll \sum\sum_{\substack{d_2|(q_2,q_2'n_1+mn_2)\\ d_2'|(q_2',q_2n_1+m'n_2)}} d_2d_2'.$$

In the first sum $\mathfrak{C}_1$ the congruence condition determines α uniquely in terms of α', and hence

$$\mathfrak{C}_1 \ll \sum\sum_{d_1,d_1'|q_1} d_1d_1' \mathop{\sum\nolimits^{\star}}_{\substack{\alpha' \bmod q_1r/n_1\\ n_1\alpha'\equiv -m' \bmod d_1'}} 1 \ll \sum\sum_{d_1,d_1'|q_1} d_1d_1' \frac{q_1r(m',n_1)}{n_1d_1'} \ll \frac{q_1^2r(m',n_1)}{n_1}.$$

This completes the proof of the lemma. ∎

We now substitute these bounds in (22). Writing q_2d_2 in place of q_2 and $q_2'd_2'$ in place of q_2' we find that the contribution of the non-zero frequencies to Ω is

$$\Omega_{\neq 0} \ll \frac{N_0 q_1^2 r}{n_1^3 M_1^{1/2}} \sum\sum_{d_2,d_2'} d_2 d_2' \sum\sum_{\substack{q_2\sim C/(q_1d_2)\\ q_2'\sim C/(q_1d_2')}} \sum\sum_{\substack{m,m'\sim M_1\\ q_2'd_2'n_1+mn_2\equiv 0 \bmod d_2\\ q_2d_2n_1+m'n_2\equiv 0 \bmod d_2'}} \sum_{n_2\in\mathbb{Z}-\{0\}} (m',n_1)|\mathfrak{I}|. \tag{27}$$

We denote by $\Sigma_{\neq 0}$ the term we get by substituting this for Ω in (18).

6.2. *The case of small modulus*

In this section we will consider the case where $q\sim C\ll t^{1+\varepsilon}$. Recall that $\mathfrak{I}\ll 1/t$ and $n_2\neq 0$. Also note that while writing the bounds we will be using the convention from Remark 1.

Lemma 4. *The contribution of $q\sim C\ll t^{1+\varepsilon}$ and $n_2\neq 0$ to (18) is bounded by*

$$\Sigma_{\neq 0,\, small} \ll r^{1/2} t^{3-\eta/2}.$$

Proof. We use the congruences to count the number of (m,m') in (27). We have

$$\sum_{\substack{m'\sim M_1\\ q_2d_2n_1+m'n_2\equiv 0 \bmod d_2'}} (m',n_1) \ll \sum_{\delta|n_1} \delta \sum_{\substack{m'\sim M_1/\delta\\ q_2d_2\bar{\delta}n_1+m'n_2\equiv 0 \bmod d_2'}} 1 \ll (d_2',n_2)\left(n_1+\frac{M_1}{d_2'}\right).$$

Recall that $(d_2',n_1)=1$. Counting the number of m in a similar fashion we see that the number of pairs (m,m') is dominated by

$$O\big((d_2,q_2'd_2'n_1)(d_2',n_2)(1+M_1/d_2)(n_1+M_1/d_2')\big).$$

It follows that the contribution of this to $\Omega_{\neq 0}$ is dominated by

$$\frac{N_0 q_1^2 r}{n_1^3 M_1^{1/2} t} \sum\sum_{d_2,d_2'} d_2 d_2' \sum\sum_{\substack{q_2\sim C/(q_1d_2)\\ q_2'\sim C/(q_1d_2')}} \sum_{1\leq n_2\ll N_2} (d_2,q_2'd_2'n_1)(d_2',n_2)\left(1+\frac{M_1}{d_2}\right)\left(n_1+\frac{M_1}{d_2'}\right).$$

Summing over n_2 and q_2 we arrive at

$$\frac{N_0 q_1 r C N_2}{n_1^3 M_1^{1/2} t} \sum\sum_{d_2,d_2'} d_2' \sum_{q_2'\sim C/(q_1d_2')} (d_2,q_2'd_2'n_1)\left(1+\frac{M_1}{d_2}\right)\left(n_1+\frac{M_1}{d_2'}\right).$$

Next summing over d_2 we get

$$\frac{N_0 q_1 r C N_2}{n_1^3 M_1^{1/2} t} \sum_{d_2'} d_2' \sum_{q_2'\sim C/(q_1d_2')} \left(\frac{C}{q_1}+M_1\right)\left(n_1+\frac{M_1}{d_2'}\right).$$

Handling the remaining sums we get

$$\frac{r}{n_1^3}\left(\frac{N_0N_2C^4n_1}{M_1^{1/2}tq_1^2}+\frac{N_0N_2C^3M_1^{1/2}n_1}{tq_1}+\frac{N_0N_2C^2M_1^{3/2}}{t}\right). \tag{28}$$

Using (21) and (23) we get

$$N_0N_2 \ll \frac{rn_1}{q_1}(NK)^{1/2}C. \tag{29}$$

Let us first consider the third term of the expression (28). Substituting this bound for Ω in (18) we get

$$\sup_{M_1\ll M_0}\frac{N^{5/12}}{r^{2/3}C^{3/2}}\sum_{\pm}\sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty}\frac{r}{n_1q_1^{1/2}}\frac{(NK)^{1/4}C^{3/2}M_1^{3/4}}{t^{1/2}},$$

which reduces to (using (25) and the trivial bound $M_0 \ll t^4/N$, as $C \ll t$)

$$\begin{aligned}\frac{K^{1/4}t^{5/2}r^{1/3}}{N^{1/12}}\sum_{\frac{n_1}{(n_1,r)}\ll C}\frac{(n_1,r)^{1/2}}{n_1^{7/6}}\Theta^{1/2} &\ll \frac{K^{1/4}t^{5/2}r^{1/3}N_0^{1/6}}{N^{1/12}}\\ &\ll K^{1/2}t^{5/2}r^{1/2} \ll r^{1/2}t^{3-\eta/2}.\end{aligned}$$

The contribution of the second term in (28) to (18) is given by

$$\sup_{M_1\ll M_0}\frac{N^{5/12}}{r^{2/3}C^{3/2}}\sum_{\pm}\sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty}\frac{r}{q_1n_1^{1/2}}\frac{(NK)^{1/4}C^2M_1^{1/4}}{t^{1/2}},$$

and again using the bounds $M_1 \ll M_0 \ll t^4/N$, $C \ll t$, we arrive at

$$N^{5/12}K^{1/4}tr^{1/3}\sum_{\frac{n_1}{(n_1,r)}\ll C}\frac{(n_1,r)}{n_1^{7/6}}\Theta^{1/2} \ll N^{5/12}K^{1/4}tr^{5/6}\sum_{\frac{n_1}{(n_1,r)}\ll C}\frac{(n_1,r)^{1/2}}{n_1^{7/6}}\Theta^{1/2}.$$

This is smaller than the bound obtained for the third term as $N \ll t^{3+\varepsilon}/r^2$. Next consider the contribution of the first term of (28). We will get a satisfactory bound for this term in two cases. As the first case suppose $M_1 \gg Cn_1^{1/4}/q_1$. Then the first term of (28) is dominated by

$$\frac{r}{n_1^3}\frac{N_0N_2C^4n_1}{M_1^{1/2}tq_1^2} \ll \frac{r^2}{n_1^{7/8}q_1^{5/2}}\frac{(NK)^{1/2}C^{9/2}}{t},$$

which when substituted for Ω in (18) yields

$$N^{2/3}(Kt)^{1/4}r^{1/3}\sum_{n_1\ll Cr} n_1^{1/3-7/16}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty}\frac{1}{q_1^{5/4}} \ll N^{2/3}(Kt)^{1/4}r^{1/3}\sum_{n_1\ll Cr}\frac{(n_1,r)^{5/4}}{n_1^{7/6+3/16}}\Theta^{1/2}.$$

This is dominated by

$$N^{2/3}(Kt)^{1/4}r^{1/3+9/16}\sum_{n_1\ll Cr}\frac{(n_1,r)^{1/2}}{n_1^{7/6}}\Theta^{1/2} \ll N^{3/4}K^{1/2}t^{1/4}r^{1/2}r^{9/16} \ll t^{3-\eta/2}.$$

In the second case we suppose $M_1 \asymp (tC)^2/N$. Then

$$\frac{r}{n_1^3}\,\frac{N_0N_2C^4n_1}{M_1^{1/2}tq_1^2} \ll \frac{r^2}{n_1q_1^3}\,\frac{NK^{1/2}C^4}{t^2},$$

which when substituted for Ω in (18) yields

$$\frac{N^{11/12}K^{1/4}r^{1/3}}{t^{1/2}}\sum_{n_1\ll Cr} n_1^{-1/6}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty}\frac{1}{q_1^{3/2}} \ll \frac{N^{11/12}K^{1/4}r^{1/3}}{t^{1/2}}\sum_{n_1\ll Cr}\frac{(n_1,r)^{3/2}}{n_1^{5/3}}\Theta^{1/2}.$$

This is dominated by

$$\frac{N^{11/12}K^{1/4}r^{1/3}r^{1/2}}{t^{1/2}}\sum_{n_1\ll Cr}\frac{(n_1,r)^{1/2}}{n_1^{7/6}}\Theta^{1/2} \ll \frac{NK^{1/2}r^{1/2}r^{1/2}}{t^{1/2}} \ll t^{3-\eta/2}.$$

It remains to analyze the range where $M_1 \ll Cn_1^{1/4}/q_1$ and M_1 is not of size $(tC)^2/N$. Then according to the comment at the end of Section 3.1 one has $Nx/(CQ) \gg t^{1-\varepsilon}$, and hence $N_0 \asymp N^2rx^3/Q^3 \gg (Ct)^3r/N$. In this case we adopt a different strategy for counting. (Let $d_2 \sim D \ll D' \sim d_2'$.) We have

$$q_2d_2n_1 + m'n_2 \ll Cn_1/q_1 + M_1N_2 \ll Cn_1/q_1 + Nn_1^{5/4}/(q_1^2t^2).$$

Using the congruence relations in (27) we write

$$q_2d_2n_1 + m'n_2 = -d_2'h \quad \text{with} \quad h \ll Cn_1/(q_1D') + Nn_1^{5/4}/(q_1^2t^2D') =: H.$$

With this we transform (27) to

$$\frac{N_0 q_1^2 r}{n_1^3 M_1^{1/2}} \sum\sum_{d_2,d_2'} d_2 d_2' \sum\sum_{\substack{h\ll H\\ q_2'\sim C/(q_1 d_2')}} \sum\sum_{\substack{m,m'\sim M_1\\ q_2'd_2'n_1+mn_2\equiv 0 \bmod d_2\\ hd_2'+m'n_2\equiv 0 \bmod d_2}} \sum_{n_2\in\mathbb{Z}-\{0\}} (m',n_1)|\mathfrak{I}|. \tag{30}$$

The first congruence gives the number of m, which is $O((n_2,d_2)(1+M_1/D))$. Using the second congruence we either count the number of d_2', which turns out to be $O((d_2,h)D'/D)$ for $h\neq 0$, or count d_2, which turns out to be $O(t^{\varepsilon})$ for $h=0$. It follows that (30) is dominated by

$$\frac{N_0 q_1^2 r}{n_1^3 M_1^{1/2} t} \sum_{d_2\sim D} D'^2 \sum\sum_{\substack{h\ll H\\ q_2'\sim C/(q_1 D')}} \sum_{m'\sim M_1} \sum_{0<n_2\ll N_2} (n_2,d_2)(h,d_2)(m',n_1)\left(1+\frac{M_1}{D}\right).$$

(Observe that this also takes care of the $h=0$ case.) First summing over n_2, and then over m' and d_2, we arrive at

$$\frac{N_0 q_1^2 r}{n_1^3 M_1^{1/2} t} M_1 N_2 D D'^2 \sum\sum_{\substack{h\ll H\\ q_2'\sim C/(q_1 D')}} \left(1+\frac{M_1}{D}\right),$$

which is dominated by

$$\frac{N_0 r}{n_1^3 M_1^{1/2} t} M_1 N_2 C\left(Cn_1+\frac{Nn_1^{3/2}}{q_1 t^2}\right)(D+M_1) \ll \frac{N_0 N_2 C^{5/2} r}{n_1^{3/2} q_1^{3/2}} \ll \frac{(NK)^{1/2} C^{7/2} r^2}{q_1^{5/2} n_1^{1/2}}. \tag{31}$$

To get the middle inequality we substitute $D\ll C/q_1$, $M_1\ll Cn_1^{1/4}/q_1$ and $C\ll t^{1+\varepsilon}$, and use the fact that $Nr^2\ll t^3$, $q_1\geq n_1/r$. (Also observe that $H\gg n_1\gg 1$.) When the above bound is substituted in place of Ω in (18) we get

$$\frac{N^{5/12}}{r^{2/3}C^{3/2}} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1 r)^\infty} \frac{(NK)^{1/4}C^{7/4} r}{q_1^{5/4} n_1^{1/4}},$$

which is dominated by

$$N^{2/3}(Kt)^{1/4} r^{1/3+3/4} \sum_{n_1\ll Cr} \frac{(n_1,r)^{1/2}}{n_1^{7/6}} \Theta^{1/2} \ll t^{3-\eta/2}.$$

The lemma follows. ∎

6.3. *The generic case*

It now remains to tackle the case where $C \gg t^{1+\varepsilon}$ and $n_2 \neq 0$.

Lemma 5. *The contribution of* $q \sim C \gg t^{1+\varepsilon}$ *and* $n_2 \neq 0$ *to* (18) *is bounded by*

$$\Sigma_{\neq 0,\, generic} \ll N^{1/2} t^{3/2-1/6+3\eta/4}.$$

Proof. In this case we need a better bound for $\mathfrak{I}$. To this end we seek to apply the stationary phase analysis to the integral $I(\dots)$ in (16), namely

$$\int_0^\infty U(y) e\left(-\frac{t}{2\pi}\log y \pm A\sqrt{y} \pm B(y+u)^{1/3}\right) \mathrm{d}y,$$

where $A = 2\sqrt{mN}/q$ and $B = 3(Nn_1^2 n_2)^{1/3}/(qr^{1/3})$. Since $C \gg t^{1+\varepsilon}$, from (10) we deduce that we have A with a plus sign and that $A \asymp t$. From (13) we conclude that $B \ll t^{1-\eta/2}$. (Otherwise the integrals in (10) and (13) are negligibly small.) Hence the stationary point, which is a solution of

$$-\frac{t}{2\pi y} + \frac{A}{2\sqrt{y}} \pm \frac{B}{3(y+u)^{2/3}} = 0,$$

can be found using Newton's method. We can write the stationary point as $y_0 + y_1 + y_2 + \cdots$ with $y_i \ll (B/t)^i$. Explicit calculation yields

$$y_0 = \left(\frac{t}{\pi A}\right)^2, \qquad y_1 = \mp\frac{4\pi B}{3t}\left(\frac{t}{\pi A}\right)^{8/3},$$

and in general $y_k = f_k(A/t)(B/t)^k$ for some function $f_k \ll 1$. It follows that $I(m, n_1^2 n_2, q)$ is essentially given by

$$\frac{1}{t^{1/2}} y_0^{-it} e\left(Bg_1(A) + B^2 g_2(A) + O\left(\frac{B^3}{t^2}\right)\right)$$

where $g_1(A) = \mp t^{2/3}/(3(\pi A)^{2/3}) \ll 1$ and $g_2(A) \ll 1/t$. Also note that we have $B \asymp (NN_0)^{1/3}/(qr^{1/3})$. It follows that the integral $\mathfrak{I}$ is given by

$$\frac{1}{t}\int W(y) e\left((Bg_1(A) - B'g_1(A')) + (B^2 g_2(A) - B'^2 g_2(A')) + O\left(\frac{NN_0}{C^3 r t^2}\right)\right) \times e\left(-\frac{N_0 n_2 y}{n_1 q_2 q_2' q_1 r}\right) \mathrm{d}y$$

where

$$B = \frac{3(NN_0 y)^{1/3}}{qr^{1/3}}, \qquad B' = \frac{3(NN_0 y)^{1/3}}{q' r^{1/3}}.$$

Since $n_2 \neq 0$ we get

$$\gamma := \frac{N_0 n_2}{n_1 q_2 q_2' q_1 r} \gg \frac{N_0}{(n_1, r) C^2 r} \gg t^{\varepsilon} \frac{N N_0}{C^3 r t^2} \tag{32}$$

because $C \gg t^{1+\varepsilon}$ and $N r^2 \ll t^3$. Making the change of variable $y = z^3$ we write the integral as

$$\frac{1}{t} \int V(z) e(f(z))\, \mathrm{d}z$$

where

$$f(z) = \alpha z + \beta z^2 - \gamma z^3 + H(z) \quad \text{with} \quad z^j \frac{\partial^j}{\partial z^j} H(z) \ll \frac{N N_0}{C^3 r t^2}.$$

(Note that the leading term involving B and B' is quadratic in z.) Let

$$g(z) = f''(z) = 2\beta - 6\gamma z + H''(z).$$

Then because of (32) we have $g'(z) \asymp \gamma$, which implies that g is monotonic. Also by the mean value theorem the length of the interval where $g(z) \ll \gamma^{2/3}$ turns out to be $O(\gamma^{-1/3})$. Over this interval we use the trivial bound for the integral. In the complementary range, since $g(z) = f''(z) \gg \gamma^{2/3}$, we use the second derivative bound. It follows that

$$\mathfrak{I} \ll \frac{1}{t \gamma^{1/3}} \ll \frac{1}{t} \left(\frac{n_1 q_2 q_2' q_1 r}{N_0 n_2} \right)^{1/3} \ll \frac{C r^{1/3} t^{2/3}}{t (N N_0)^{1/3}}.$$

In our bounds for Ω (see (28) and (31)), we had the factor $N_0 N_2$ which boils down to $C(N N_0)^{1/3} r^{2/3}/(n_1 q_1)$ by substituting the value of N_2. Now when we incorporate the new bound for the integral, this factor is replaced by $C^2 t^{2/3} r/(n_1 q_1)$. In other words, the bound in (29) is replaced by

$$\frac{r}{n_1 q_1} (C t^{2/3}) C \ll \frac{r}{n_1 q_1} (N K)^{1/2} C \ \times t^{\eta - 1/3}$$

(as $K = t^{1-\eta}$ and $C \ll (N/K)^{1/2}$). Hence we save $t^{1/3-\eta}$ in our estimate for Ω. Taking this into account and substituting $C \ll (N/K)^{1/2}$ (in place of $C \ll t$), and accordingly $M_0 \ll t^2/K$, in place of the corresponding bounds in the proof of Lemma 4, we get Lemma 5. ∎

6.4. *Conclusion*

We now put together the bounds from Lemmas 2, 4, and 5 and the bound for the error term in (14) to get

$$\frac{S_r(N)}{N^{1/2} t^{3/2}} \ll t^{-1/2 + 3\eta/2} + r^{1/2} \frac{t^{3/2 - \eta/2}}{N^{1/2}} + t^{-1/6 + 3\eta/4},$$

where $t^{3-\theta}/r^2 < N < t^3/r^2$. It follows that

$$\frac{S_r(N)}{N^{1/2}t^{3/2}} \ll t^{-1/2+3\eta/2} + t^{2\theta-\eta/2} + t^{-1/6+3\eta/4}$$

for $r \ll t^\theta$. Hence (for subconvexity) we need at least $1/3 > \eta > 4\theta$, and consequently the last term dominates the first. Equating the last two terms we find that the optimal choice for η is given by $\eta = 8\theta/5 + 2/15$. Plugging this in (4) we get

$$L(1/2 + it, \pi \times f) \ll t^{3/2+6\theta/5-1/15} + t^{3/2-\theta/2},$$

and with the optimal choice $\theta = 2/51$ we obtain the bound given in Theorem 1.

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