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(with an appendix by Colin J. Bushnell and Guy Henniart)

Zeros of Rankin–Selberg L-functions at the edge of the critical strip

À la mémoire de Colin Bushnell, ami et co-auteur

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Abstract. Let π (respectively π_0) be a unitary cuspidal automorphic representation of GL_m (respectively GL_{m_0}) over $\mathbb{Q}$. We prove log-free zero density estimates for Rankin–Selberg L-functions of the form $L(s, \pi \times \pi_0)$, where π varies in a given family and π_0 is fixed. These estimates are unconditional in many cases of interest; they hold in full generality assuming an average form of the generalized Ramanujan conjecture. We consider applications of these estimates related to mass equidistribution for Hecke–Maaß forms, the rarity of Landau–Siegel zeros of Rankin–Selberg L-functions, the Chebotarev density theorem, and ℓ-torsion in class groups of number fields.

Keywords. Rankin–Selberg L-function, log-free zero density estimate, automorphic form

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1. Introduction and statement of the main results

The generalized Riemann hypothesis (GRH) for Dirichlet L-functions implies that if a and $q \geq 1$ are coprime integers, then there exists a prime[1] $p \ll (q \log q)^2$ such that $p \equiv a \pmod{q}$. Linnik [47] unconditionally proved that the least such prime is $O(q^A)$, where $A > 0$ is an absolute and effective constant; up to the quality of A, Linnik's result is commensurate with what GRH predicts. Linnik's proof developed powerful results for the distribution of zeros of Dirichlet L-functions near the point $s = 1$, including a log-free zero density estimate. In this paper, we prove a flexible log-free zero density estimate for families of L-functions and consider the arithmetic consequences of such an estimate in several different settings. We use this estimate to study mass equidistribution for Hecke–Maaß forms, the rarity of Landau–Siegel zeros for Rankin–Selberg L-functions, the Chebotarev density theorem, and ℓ-torsion in class groups of number fields.

In the spirit of Linnik's original result, Kowalski and Michel [39, Theorem 5] proved a log-free zero density estimate for general families of automorphic L-functions in the conductor aspect. To describe their result, let $\mathbb{A}_{\mathbb{Q}}$ be the ring of adeles over $\mathbb{Q}$, let $d \geq 1$ be a fixed integer, and let $\mathcal{A}(d)$ be the set of cuspidal automorphic representations of $\mathrm{GL}_d(\mathbb{A}_{\mathbb{Q}})$ with unitary central character. We implicitly assume that the central character of each $\pi \in \mathcal{A}(d)$ is trivial on the positive reals so that $\mathcal{A}(d)$ is discrete. For each $\pi \in \mathcal{A}(d)$, let

$$L(s,\pi) = \sum_{n\geq 1} \frac{a_\pi(n)}{n^s} = \prod_p \prod_{j=1}^{d} (1 - \alpha_{j,\pi}(p) p^{-s})^{-1}$$

be the standard L-function associated to π, where p runs through the primes. Consider a finite set $S(q)$ of distinct cuspidal automorphic representations $\pi \in \mathcal{A}(d)$ such that:

(1) There exists some $\delta > 0$ (depending at most on d) such that for each $\pi \in S(q)$, each $1 \leq j \leq d$, and each prime p, we have the bound $|\alpha_{j,\pi}(p)| \leq p^{1/4-\delta}$.

(2) There exists a constant $A > 0$ such that for all $\pi \in S(q)$, the conductor of π is $O(q^A)$.

(3) There exists a constant $M > 0$ such that $\#S(q) \ll q^M$.

(4) Each $\pi \in S(q)$ has the same component π_∞ at the infinite place of $\mathbb{Q}$.

[1] We write $f = O(g)$ or $f \ll g$ to mean that $|f| \leq c|g|$ for some absolute and effective constant $c > 0$. For a parameter ν, we write $f = O_\nu(g)$ to mean that c might depend on ν in an effective manner. We write $f \asymp g$ to mean that $f = O(g)$ and $g = O(f)$, and similarly for $f \asymp_\nu g$ and $f \sim_\nu g$.

Note that the generalized Ramanujan conjecture (GRC) predicts that $|\alpha_{j,\pi}(p)| \leq 1$ for all primes p; this has been verified in a small number of special cases. Define

$$N_\pi(\sigma, T) := \#\{\rho = \beta + i\gamma : \sigma \leq \beta,\ |\gamma| \leq T,\ L(\rho, \pi) = 0\}.$$

With these conventions and hypotheses, Kowalski and Michel prove that there exists a constant $c = c(A, \delta, M) > M$ and a constant $B > 0$ (depending on $S(q)$ but not q) such that

$$\sum_{\pi \in S(q)} N_\pi(\sigma, T) \ll T^B q^{c\frac{1-\sigma}{2\sigma-1}}, \qquad 3/4 < \sigma \leq 1, \quad T \geq 2. \tag{1.1}$$

If $\sigma \geq 1 - M/c$ and T is sufficiently small with respect to q, then (1.1) tells us that at most a vanishingly small proportion of low-lying zeros of the L-functions $L(s, \pi)$ with $\pi \in S(q)$ lie near $s = 1$. In many problems, such a result can serve as a powerful substitute for GRH. Until now, (1.1) appears to be the most flexible and robust zero density estimate for studying zeros of automorphic L-functions near $s = 1$.

For a pair of cuspidal automorphic representations $\pi \in \mathcal{A}(d)$ and $\pi_0 \in \mathcal{A}(d_0)$, consider the associated Rankin–Selberg L-function

$$L(s, \pi \times \pi_0) = \sum_{n \geq 1} \frac{a_{\pi \times \pi_0}(n)}{n^s} = \prod_p \prod_{j=1}^{d} \prod_{j_0=1}^{d_0} (1 - \alpha_{j,j_0,\pi\times\pi_0}(p) p^{-s})^{-1},$$

where

$$\begin{aligned} &\{\alpha_{j,j_0,\pi\times\pi_0}(p) : 1 \leq j \leq d,\ 1 \leq j_0 \leq d_0\} \\ &\qquad = \{\alpha_{j,\pi}(p)\alpha_{j_0,\pi_0}(p) : 1 \leq j \leq d,\ 1 \leq j_0 \leq d_0\} \end{aligned} \tag{1.2}$$

for all except finitely many primes p. In this paper, we establish log-free zero density estimates for families of Rankin–Selberg L-functions $L(s, \pi \times \pi_0)$, where π varies and π_0 is fixed. In order to make this precise, we let $\mathcal{A} = \bigcup_{d \geq 1} \mathcal{A}(d)$ and $\mathcal{F} \subseteq \mathcal{A}$. We define

$$\mathcal{F}_m := \{\pi \in \mathcal{F} : \pi \in \mathcal{F} \cap \mathcal{A}(d) \Rightarrow d \leq m\}, \qquad \mathcal{F}_m(Q) := \{\pi \in \mathcal{F}_m : C(\pi) \leq Q\}, \tag{1.3}$$

where $C(\pi)$ is the analytic conductor of π (see (3.3) for the definition). We require an average version of GRC.

Hypothesis 1.1. *Let $\pi \in \mathcal{A}(d)$. If $\varepsilon > 0$, then*

$$\prod_p \sum_{r=0}^{\infty} \frac{\max_{1 \leq j \leq d} |\alpha_{j,\pi}(p)|^{2r}}{p^{r(1+\varepsilon)}} \ll_{d,\varepsilon} C(\pi)^\varepsilon.$$

Remark 1.2. Indeed, if π satisfies GRC, then Hypothesis 1.1 follows with lots to spare. Brumley [7, Theorem 1 and Corollary 2] proved that each $\pi \in \mathcal{A}(d)$ satisfies Hypothesis 1.1 when $d \leq 4$ and gave sufficient conditions (strictly weaker than assuming GRC) under which π may satisfy Hypothesis 1.1 when $d \geq 5$.

Theorem 1.3. *Let $\pi_0 \in \mathcal{A}(m_0)$, and let $Q, T \geq 1$. Let $\mathcal{F}_m(Q)$ be as in (1.3), and suppose that π_0 and each $\pi \in \mathcal{F}_m(Q)$ satisfy Hypothesis 1.1. If $1/2 \leq \sigma \leq 1$, then*

$$\sum_{\pi \in \mathcal{F}_m(Q)} N_{\pi \times \pi_0}(\sigma, T) \ll_{m,m_0} (C(\pi_0)QT)^{10^8 (m_0 m)^4 (1-\sigma)}.$$

Remark 1.4. If $\mathbb{1} \in \mathcal{A}(1)$ is the trivial representation, whose L-function is the Riemann zeta function $\zeta(s)$, then $L(s, \pi \times \mathbb{1}) = L(s, \pi)$ and Theorem 1.3 immediately recovers (1.1) when $\pi_0 = \mathbb{1}$ (up to the quality of the coefficient of $1 - \sigma$) with the added benefit of improved dependence on T. Theorem 1.3 is new for all other choices of π_0, even if one assumes GRC in full.

Remark 1.5. We have made no attempt to optimize the exponent, but there is ample room for improvement (especially if one assumes GRC). Obtaining a strong numerical exponent was a key component of the work of Thorner and Zaman [64, Theorem 3.2].

Our proof of Theorem 1.3 in fact produces the upper bound

$$\sum_{\pi \in \mathcal{F}_m(Q)} N_{\pi \times \pi_0}(\sigma, T) \ll_{m,m_0} (C(\pi_0)QT\,\#\mathcal{F}_m(Q))^{2.05 \cdot 10^7 (m_0 m)^3 (1-\sigma)} \tag{1.4}$$

(see (6.3)). However, (1.4) only becomes meaningful when there exists a constant $c_m > 0$ (depending only on m) such that $\#\mathcal{F}_m(Q) \ll_{\mathcal{F},m} Q^{c_m}$. The situation is the same as in (1.1), which is why Kowalski and Michel assume the bound $\#S(q) \ll q^M$. A standard calculation for Dirichlet characters reveals that $\#\mathcal{F}_1(Q) \ll Q^2$, and the existence of some suitable $c_m > 0$ for $m \geq 2$ follows from work of Michel and Venkatesh [51, Section 2.6.5]. We expect that $\mathcal{F}_m(Q) \sim_{\mathcal{F},m} Q^{m+1}$ for all $m \geq 1$; Brumley and Milićević [8, Theorems 1.1 and 1.2] proved this claim (and much more) when $m = 2$. For $m \geq 3$, Brumley and Milićević prove the claim when each $\pi \in \mathcal{F}_m(Q)$ corresponds to a GL_m Hecke–Maaß newform. We prove an unconditional bound on $\#\mathcal{F}_m(Q)$ with an explicit exponent.

Theorem 1.6. *For all $\varepsilon > 0$, we have the bound $\#\mathcal{F}_m(Q) \ll_{\varepsilon,m} Q^{2m+\varepsilon}$.*

The truth of Theorem 1.6 follows immediately from Theorem A.1, which we prove in Appendix A. The bound in Theorem 1.6 together with (1.4) produces Theorem 1.3.

The estimate in Theorem 1.3 improves noticeably if there exists a primitive real Dirichlet character $\chi \pmod q$ such that $L(s, \chi)$ has a real zero close to $s = 1$.

Theorem 1.7. *Let $Q, T \geq 1$ and $1/2 \leq \sigma \leq 1$. Let $\chi \pmod q$ be a primitive real Dirichlet character such that $q \leq Q$ and $L(s, \chi)$ has a real zero $\beta_\chi \in (1/2, 1)$. Let $\pi_0 \in \mathcal{A}(m_0)$, and assume that $L(\beta_\chi, \pi_0 \times (\tilde{\pi}_0 \otimes \chi)) = 0$. Let*

$$\begin{aligned} &N^*_{\pi \times \pi_0}(\sigma, T) \\ &= \begin{cases} N_{\pi \times \pi_0}(\sigma, T) & \text{if } \pi \neq \tilde{\pi}_0 \otimes \chi, \\ \#\{\rho = \beta + i\gamma \neq \beta_\chi : \sigma \leq \beta,\ |\gamma| \leq T,\ L(\rho, \pi_0 \times (\tilde{\pi}_0 \otimes \chi)) = 0\} & \text{if } \pi = \tilde{\pi}_0 \otimes \chi. \end{cases} \end{aligned}$$

If π_0 and each $\pi \in \mathcal{F}_m(Q)$ satisfy Hypothesis 1.1, then

$$\sum_{\pi\in\mathcal{F}_m(Q)} N^*_{\pi\times\pi_0}(\sigma,T) \ll_{m,m_0} \min\{1,(1-\beta_\chi)\log(C(\pi_0)QT)\}\cdot(C(\pi_0)QT)^{10^8(m_0m)^4(1-\sigma)}.$$

Moreover, there exists an effectively computable constant $c_{m,m_0}>0$ such that if $(1-\beta_\chi)\log(C(\pi_0)Q)\le c_{m,m_0}$, then

$$\{\pi\in\mathcal{F}_m(Q)\colon L(\beta_\chi,\pi\times\pi_0)=0\}\cup\{\tilde{\pi}_0\otimes\chi\}=\{\tilde{\pi}_0\otimes\chi\}.$$

Remark 1.8. When $m_0=1$, the hypothesis that $L(\beta_\chi,\pi_0\times(\tilde{\pi}_0\otimes\chi))=0$ reduces to $L(\beta_\chi,\chi)=0$, which is true by definition. If $m_0\ge 2$, then $L(s,\pi_0\times(\tilde{\pi}_0\otimes\chi))/L(s,\chi)$ is expected to be entire for all primitive Dirichlet characters χ, which is far stronger than what Theorem 1.7 requires. This is known when $m_0=2$ by work of Gelbart and Jacquet [24] and when $m_0=3$ or 4 by work of Yang [70, Corollary 3]. For any $m_0\ge 2$, the conjectured automorphy of the adjoint square lift from GL_{m_0} to $\mathrm{GL}_{m_0^2-1}$ implies that $L(s,\pi_0\times(\tilde{\pi}_0\otimes\chi))/L(s,\chi)$ is entire for all primitive Dirichlet characters.

Remark 1.9. When $\pi_0=\mathbb{1}$ and $m=1$, in which case $\mathcal{F}_m(Q)$ is a set of primitive Dirichlet characters with conductor at most Q, Theorem 1.7 was proved in a less explicit form by Bombieri [5, §6]. Bombieri's ideas were extended to Hecke characters over a given number field by Weiss [69, Theorem 4.3].

Page's theorem [13, Chapter 14] tells us that there exists an absolute and effective constant $c_1>0$ such that for every $Q\ge 3$, there exists at most one modulus $q\in(Q/2,Q]$ and at most one primitive real character $\chi \pmod q$ such that $L(s,\chi)$ has a real zero β_χ with the property that $\beta_\chi>1-c_1/\log q$. Moreover, such a zero β_χ, which we call a *Landau–Siegel zero*, must be simple. If a primitive real character $\chi \pmod q$ with $q\in(Q/2,Q]$ has an associated Landau–Siegel zero β_χ, then Theorem 1.7 improves on Theorem 1.3. While it is well-known that Landau–Siegel zeros associated to real characters repel the zeros of Dirichlet L-functions from the line $\mathrm{Re}(s)=1$, Theorem 1.7 appears to be the first explicit instance in the literature where Landau–Siegel zeros associated to real characters repel zeros of high-degree L-functions. This adds to the growing literature on interesting consequences of the existence of Landau–Siegel zeros of Dirichlet L-functions [12, 14, 18–20, 22, 26].

Overview of the proof

Our proof of Theorem 1.3, which is noticeably different from that of (1.1), descends naturally from Gallagher's approach to log-free zero density estimates for Dirichlet L-functions [23]. Much like the classical approach to zero-free regions for L-functions, if $L(s,\pi\times\pi_0)$ has a zero ρ_0 such that $|\rho_0-(1+it)|\le\varepsilon$ for some small $\varepsilon>0$, then high derivatives of $-\frac{L'}{L}(s,\pi\times\pi_0)$ near $s=1+\varepsilon+it$ will be large; this is made quantitative via the lower bound for power sums due to Sós and Turán [59]. Moreover, one can show that if these high derivatives are large, then the mean value of a certain Dirichlet polynomial of the shape

$$P(t,\pi\times\pi_0)=\sum_{A<p<B}\frac{a_{\pi\times\pi_0}(p)\log p}{p^{1+it}}$$

must also be large when t is close to $\mathrm{Im}(\rho_0)$. A new "pre-sifted" large sieve inequality (Proposition 5.1) in the spirit of Duke and Kowalski [15, Theorem 4] shows that the mean value of $P(t,\pi\times\pi_0)$ cannot be too large for too many $\pi\in\mathcal{F}_m(Q)$ simultaneously; Theorem 1.3 follows from the interplay between the upper and lower bounds for the high derivatives. The coefficients of $P(t,\pi\times\pi_0)$ are supported on large unramified primes, in which case $a_{\pi\times\pi_0}(p)=a_\pi(p)a_{\pi_0}(p)$ by means of (1.2); this decisive identity facilitates the averaging over $\pi\in\mathcal{F}_m(Q)$ while π_0 is fixed. We prove Theorem 1.7 similarly by simultaneously considering the twists $L(s,\pi\times\pi_0)$ and $L(s,\pi\times(\pi_0\otimes\chi))$ and exploiting the fact that if χ is a real primitive Dirichlet character with a Landau–Siegel zero, then χ behaves like the Möbius function. This approach contrasts with the method of proof for (1.1), which uses mollification to detect zeros and a mean value theorem involving Selberg's pseudo-characters to show that the aggregate contribution from the zeros of each L-function is small. It is unclear to the authors how one would modify the proof of (1.1) to incorporate a twist by π_0 while maintaining a log-free estimate.

In [61, Corollary 2.6], Soundararajan and the second author establish the first unconditional log-free zero density estimate for each Rankin–Selberg L-function $L(s,\pi\times\pi_0)$ with an application to the weak subconvexity problem. The proof of [61, Corollary 2.6] relies on the same method of detecting zeros that we use here. Unfortunately, the means by which the proofs in [61] avoid appealing to a weak form of GRC (such as Hypothesis 1.1) appears to be incompatible with the process of averaging over $\pi\in\mathcal{F}_m(Q)$. In particular, Hypothesis 1.1 appears to be indispensable in the proof of Proposition 5.1 unless $\#\mathcal{F}_m(Q)=1$, which is precisely the case considered in [61].[2]

2. Arithmetic applications

2.1. Subconvexity and mass equidistribution

Let f be a Hecke–Maaß newform for the congruence subgroup $\Gamma_0(q_f)\subseteq\mathrm{SL}_2(\mathbb{Z})$ with Laplace eigenvalue λ_f and trivial central character. Define

$$\mathcal{G}(Q)=\{f\colon q_f\text{ squarefree},\ \lambda_f q_f\le Q\}. \tag{2.1}$$

Let f_0 denote a fixed Hecke–Maaß newform, and consider the L-functions $L(s,f\times f)$ and $L(s,f\times f\times f_0)$ as $f\in\mathcal{G}(Q)$ varies. Since q_f is squarefree, the conductor of $f\times f$ is q_f^2.

[2]Note added in proof: A year after the initial submission of this paper, the second and third authors unconditionally proved Proposition 5.1 in a stronger form using different ideas. This leads to a proof of Theorem 1.3 when $\pi_0=\mathbb{1}$ that requires no unproven hypotheses. See [66].

The generalized Lindelöf hypothesis (which follows from GRH) predicts that for all $\varepsilon > 0$ and all $f \in \mathscr{G}(Q)$, we have the bounds

$$\begin{aligned} L(1/2+it, f \times f) &\ll_\varepsilon ((|t|+1)^4 \lambda_f q_f^2)^\varepsilon, \\ L(1/2, f \times f \times f_0) &\ll_\varepsilon (\lambda_{f_0}^4 q_{f_0}^4 \lambda_f^2 q_f^4)^\varepsilon. \end{aligned}$$

The so-called convexity bounds

$$\begin{aligned} L(1/2+it, f \times f) &\ll ((|t|+1)^4 \lambda_f q_f^2)^{1/4}, \\ L(1/2, f \times f \times f_0) &\ll (\lambda_{f_0}^4 q_{f_0}^4 \lambda_f^2 q_f^4)^{1/4} \end{aligned} \tag{2.2}$$

follow from the work of Heath-Brown [27]. For fixed $\delta \in (0, 1/4)$, subconvexity bounds of the shape

$$\begin{aligned} L(1/2+it, f \times f) &\ll (|t|+1)(\lambda_f q_f^2)^{1/4-\delta}, \\ L(1/2, f \times f \times f_0) &\ll (\lambda_{f_0}^4 q_{f_0}^4 \lambda_f^2 q_f^4)^{1/4-\delta} \end{aligned} \tag{2.3}$$

are not yet known; obtaining bounds of these sorts is a very active area of research which has some spectacular partial results (see [33], for instance).

A standard calculation involving the classical large sieve [32, Theorem 7.13] and the approximate functional equation for Dirichlet L-functions shows that if Q is large, then for all except at most a density zero subset of the moduli $q \le Q$, we have the bound $L(1/2, \chi) \ll_\varepsilon q^\varepsilon$ for all primitive Dirichlet characters $\chi \pmod q$ (see [32, Theorem 7.34]). Similarly, a sufficiently strong large sieve for automorphic forms combined with the approximate functional equation will show that there exists a constant $\delta > 0$ such that (2.3) holds for almost all $f \in \mathscr{G}(Q)$. As of now, the best candidate for such a large sieve is that of Duke and Kowalski [15, Theorem 4], but this large sieve combined with the approximate functional equation is not strong enough to deduce (2.3) for almost all $f \in \mathscr{G}(Q)$ with any fixed $\delta > 0$, even under GRC. However, a straightforward application of Theorem 1.3 yields such an average result.

Theorem 2.1. *Let $\varepsilon > 0$, and let $\mathscr{G}(Q)$ be as in (2.1). For all except at most $O_{f_0}(Q^\varepsilon)$ of the Hecke–Maaß forms $f \in \mathscr{G}(Q)$, the bounds in (2.3) hold simultaneously with $\delta = 10^{-20}\varepsilon$.*

Remark 2.2. It follows from work of Brumley and Milićević [8] that $\#\mathscr{G}(Q) \asymp Q^2$. Thus the exceptional set in Theorem 2.1 is quite small.

Our interest in (2.3) is motivated by the quantum unique ergodicity conjecture. Lindenstrauss [46] and Soundararajan [60] proved that as f traverses the Hecke-Mass forms with $q_f = 1$ and $\lambda_f \to \infty$, the L^2 mass of f equidistributes in $\Gamma_0(1) \setminus \mathbb{H}$ with respect to the standard hyperbolic measure. This affirmatively resolved the quantum unique ergodicity conjecture of Rudnick and Sarnak [58] for the modular surface. More specifically, let

$$\mu_f(\phi) = \int_{\Gamma_0(q_f)\setminus\mathbb{H}} |f(z)|^2 \phi(z) \,\frac{dx\,dy}{y^2}, \qquad \mu(\phi) = \int_{\Gamma_0(1)\setminus\mathbb{H}} \phi(z)\,\frac{dx\,dy}{y^2}, \tag{2.4}$$

where ϕ is a bounded measurable function on $\Gamma_0(1) \setminus \mathbb{H}$. It is now known that as f traverses the Hecke–Maaß forms with $q_f = 1$ and eigenvalue $\lambda_f \to \infty$,

$$D_f(\phi) := \frac{\mu_f(\phi)}{\mu_f(1)} - \frac{\mu(\phi)}{\mu(1)} = o_\phi(1). \tag{2.5}$$

Unfortunately, the methods in [46, 60] do not yield any information about the rate of convergence in (2.5). See [30,53,55] for an unconditional proof of a version of (2.5) with an effective rate of convergence as f traverses the holomorphic cuspdial newforms of weight k_f and level q_f with $k_f q_f \to \infty$; this proof relies heavily on the fact that GRC is known for such newforms. For work in the direction of establishing (2.5) for Hecke–Maaß forms in q_f-aspect when q_f is large and prime, see [54].

We consider the problem of proving that for all except at most a density zero subset of $f \in \mathcal{G}(Q)$, one has (2.5) with a power-saving rate of convergence in the hybrid q_f and λ_f aspects. When f traverses the even Hecke–Maaß forms with $q_f = 1$, this follows from Zhao's computation of the quantum variance of the modular surface [72], which builds on work of Luo and Sarnak [49]. It is unclear to the authors whether one can adapt the proofs for the problem considered here.

It follows from work of Nelson [53] that for $f \in \mathcal{G}(Q)$ (given by (2.1)), subconvexity bounds of the form (2.3) imply the bound

$$D_f(\phi) \ll_\phi (\lambda_f q_f^2)^{-\delta+o(1)}. \tag{2.6}$$

(See [53, Remarks 1.4 and 1.7, and Section 4].) Thus the next result follows immediately from Theorem 2.1 and the remark that follows it.

Corollary 2.3. *Fix $\varepsilon > 0$, and let $\mathcal{G}(Q)$ be as in (2.1). For all except at most $O_\phi(Q^\varepsilon)$ of the Hecke–Maaß forms $f \in \mathcal{G}(Q)$, the bound (2.6) holds with $\delta = 10^{-21}\varepsilon$.*

Remark 2.4. By appealing to the extension of Watson's formula proved by Nelson, Pitale, and Saha [55] and the calculations in [15, p. 11], one can extend the definition of $\mathcal{G}(Q)$ to allow q_f to be *any* integer at the cost of allowing the exceptional set to be of size $O_\phi(Q^{1/2+\varepsilon})$ in Corollary 2.3. The proof is entirely analogous.

2.2. *Rarity of Landau–Siegel zeros*

Let $\mathcal{F}_m(Q)$ be as in (1.3), and let $\pi \in \mathcal{F}_m(Q) \cap \mathcal{A}(d)$. While GRH predicts that $L(s,\pi)$ has no zero in the region $\mathrm{Re}(s) > 1/2$, at present we know that $L(s,\pi)$ has at most one zero in the region

$$\mathrm{Re}(s) \geq 1 - \frac{c_2}{d^4 \log(C(\pi)(|\mathrm{Im}(s)| + 3))} \tag{2.7}$$

(see [32, Theorem 5.10]). If $L(s,\pi)$ has a zero in this region, then π is self-dual (so the Dirichlet coefficients of $L(s,\pi)$ are real), and the zero must be simple and real. We call such a zero a *Landau–Siegel zero*. Hoffstein and Ramakrishnan [29, Theorem A] proved that such Landau–Siegel zeros are quite rare. In particular, for some suitable effective constant $c(m) > 0$, there is at most one $\pi \in \mathcal{F}_m(Q)$ such that $L(s,\pi)$ has a real zero β

satisfying $\beta > 1 - c(m)/\log Q$. This generalizes Page's theorem for Dirichlet characters. Moreover, it is known by the work of Hoffstein and Ramakrishnan [29, Theorem A] and Banks [2] that if $m = 2$ or 3, then no $\pi \in \mathcal{F}_m(Q)$ has an L-function possessing a Landau–Siegel zero.

The situation for Rankin–Selberg L-functions is much more difficult. Currently, an unconditional zero-free region (with at most one exceptional zero) roughly of the shape (2.7) exists for $L(s, \pi \times \pi_0)$ when at least one of π and π_0 is self-dual [31, Theorem A.1]. In Lemma 4.5 below, we extend [29, Theorem A] to the context of Rankin–Selberg L-functions. Using Theorems 1.3 and 1.7 and Lemma 4.5, we show that with very few exceptions, the L-functions in the set $\{L(s, \pi \times \pi_0) \colon \pi \in \mathcal{F}_m(Q)\}$ have a much stronger zero-free region than what is provided by the usual approaches; moreover, there is no constraint on whether π or π_0 are self-dual.

Theorem 2.5. *Assume the above notation. Let $A \geq 1$, and let $\mathcal{S} = \mathcal{S}(A, Q, T, \mathcal{F}_m)$ be the set of all $\pi \in \mathcal{F}_m(Q)$ such that $L(s, \pi \times \pi_0)$ has a zero in the region*

$$s = \sigma + it, \quad |t| \leq T, \quad \sigma \geq 1 - \frac{A}{2 \cdot 10^8 (m_0 m)^4 \log(C(\pi_0) Q(T+2))}. \tag{2.8}$$

(i) *Under the notation and hypotheses of Theorem 1.3, $|\mathcal{S}| = O_{m,m_0}(e^A)$.*

(ii) *Under the notation and hypotheses of Theorem 1.7,*

$$|\mathcal{S} - \{\tilde{\pi}_0 \otimes \chi\}| = O_{m,m_0}(e^A \min\{1, (1-\beta_\chi)\log(C(\pi_0)QT)\}).$$

Remark 2.6. Suppose that $\pi \in \mathcal{F}_m(Q) - \{\tilde{\pi}_0\}$. By setting $T = Q$ and $A = \varepsilon \log(C(\pi_0)Q)$ for some small fixed $\varepsilon > 0$, it follows readily from Theorem 2.5(i) that apart from at most a few exceptional π in $\mathcal{F}_m(Q)$, one can obtain strong approximations for $L(1, \pi \times \pi_0)$ as a short Euler product. See [11, 25, 42] for further discussion and applications of such approximations.

Let $\chi \pmod q$ be a primitive real character such that $L(s, \chi)$ has a zero $\beta_\chi \in (1/2, 1)$, and define $\lambda_\chi := (1 - \beta_\chi)\log q$. Suppose that $q \leq C(\pi_0)QT \leq q^B$ for some $B \geq 1$, in which case $\lambda_\chi \asymp (1-\beta_\chi)\log(C(\pi_0)QT)$. Then Theorem 2.5(ii) implies $|\mathcal{S} - \{\tilde{\pi}_0 \otimes \chi\}| = O(Be^A \lambda_\chi)$. If A and B are constant and β_χ is a Landau–Siegel zero of $L(s, \chi)$, then $\lambda_\chi = o(1)$ as $q \to \infty$. Thus $|\mathcal{S} - \{\tilde{\pi}_0 \otimes \chi\}| = 0$ when q is sufficiently large. So under the hypotheses of Theorem 1.7 and the existence of a suitable sequence of primitive real characters $\chi_i \pmod{q_i}$ with $\lambda_{\chi_i} \to 0$ as $q_i \to \infty$, the only Rankin–Selberg L-functions in the set $\{L(s, \pi \times \pi_0) \colon \pi \in \mathcal{F}_m\}$ which have a zero in the region (2.8) are of the form $L(s, \pi_0 \times (\tilde{\pi}_0 \otimes \chi_i))$ (and in this case, Corollary 4.6 below shows that the Landau–Siegel zero associated to $\pi_0 \times (\tilde{\pi}_0 \otimes \chi)$ is precisely β_{χ_i}). In other words, Landau–Siegel zeros associated to quadratic characters repel all *other* Landau–Siegel zeros associated to Rankin–Selberg convolutions. This provides an interesting companion to another result of Hoffstein and Ramakrishnan [29, Theorem B], which roughly states that if all Rankin–Selberg L-functions factor into products of L-functions of cuspidal automorphic representations (as predicted by Langlands), then the only primitive L-functions over $\mathbb{Q}$ which could possibly admit a Landau–Siegel zero are those associated to primitive real Dirichlet characters.

2.3. The Chebotarev density theorem in families

Let K be a number field of degree $n = [K : \mathbb{Q}]$ with $D_K = |\mathrm{disc}(K/\mathbb{Q})|$ and Galois closure $\widetilde{K}$ over $\mathbb{Q}$. Let G be isomorphic to the Galois group of $\widetilde{K}/\mathbb{Q}$, and let C be a conjugacy class of $\mathrm{Gal}(\widetilde{K}/\mathbb{Q})$. Define

$$\pi_C(x, \widetilde{K}/\mathbb{Q}) := \#\left\{p \leq x : p \nmid D_{\widetilde{K}}, \left[\frac{\widetilde{K}/\mathbb{Q}}{p}\right] = C\right\},$$

where the Artin symbol $[\frac{\widetilde{K}/\mathbb{Q}}{p}]$ denotes the conjugacy class of Frobenius automorphisms attached to the prime ideals of $\widetilde{K}$ which lie over p. The Chebotarev density theorem states

$$\mathcal{E}_C(x, \widetilde{K}/\mathbb{Q}) := \left|\pi_C(x, \widetilde{K}/\mathbb{Q}) - \frac{|C|}{|G|}\pi(x)\right| = o\left(\frac{|C|}{|G|}\frac{x}{\log x}\right) \quad \text{as } x \to \infty,$$

where $\pi(x)$ is the number of rational primes up to x. It follows from the work of Lagarias and Odlyzko [41, Theorem 1.1] that GRH for the Dedekind zeta function $\zeta_{\widetilde{K}}(s)$ implies

$$\mathcal{E}_C(x, \widetilde{K}/\mathbb{Q}) \ll \frac{|C|}{|G|} x^{1/2} \log(D_{\widetilde{K}} x^{|G|}) \quad \text{for } x \geq (\log D_{\widetilde{K}})^2 (\log\log D_{\widetilde{K}})^4. \tag{2.9}$$

Unconditionally, refining a result of Lagarias and Odlyzko [41], it follows from work of Murty [52, Section 4] that

$$\begin{aligned}&\mathcal{E}_C(x, \widetilde{K}/\mathbb{Q})\\ &\ll \frac{|C|}{|G|}\left(\frac{x^{\beta_1}}{\log x} + \frac{x}{\exp(c_3(\log x)^{1/2}|G|^{-1/2})}\right) \quad \text{for } x \gg_{|G|} e^{c_4(\log D_{\widetilde{K}})^2/|G|},\end{aligned} \tag{2.10}$$

where β_1 is a potential Landau–Siegel zero of $\zeta_{\widetilde{K}}(s)$. Recent work of the authors [65] shows that for any $A > 1$, there exists $B = B(A) > 1$ such that

$$\mathcal{E}_C(x, \widetilde{K}/\mathbb{Q}) \ll_A \frac{|C|}{|G|}\left(\frac{x^{\beta_1}}{\log x} + \frac{x}{(\log x)^A}\right) \quad \text{for } x \gg_{|G|,A} D_{\widetilde{K}}^{B \log\log D_{\widetilde{K}}}. \tag{2.11}$$

For large x, (2.10) remains the strongest upper bound for $\mathcal{E}_C$ and it is nontrivial in the absence of a Landau–Siegel zero. On the other hand, (2.11) exhibits a weaker estimate but for much smaller values of x. Nonetheless, even when ignoring the Landau–Siegel zero, both (2.10) and (2.11) fall far short of exhibiting nontrivial bounds for values of x commensurate in size with (2.9). Even establishing such bounds for $x \geq D_{\widetilde{K}}^{o(1)}$ would be extremely desirable.

Substantial progress has recently been made by Pierce, Turnage-Butterbaugh, and Wood [56] when K varies in certain families. They show that the ranges of x in (2.10) and (2.11) can be significantly improved for most K. We briefly summarize their results. Let $G \in \{C_m, D_p, S_3, S_4, A_4\}$, where C_m is the cyclic group of order $m \geq 2$, S_m is the symmetric group acting on $m \geq 2$ elements, D_p is the dihedral group of order $2p$ with p

an odd prime, and A_4 is the alternating group acting on four elements. Let $\mathscr{F}(X) = \mathscr{F}(X; G, n, R_G)$ equal the set of number fields K given by

$$\{K\colon D_K \le X,\ [K:\mathbb{Q}] = n,\ \mathrm{Gal}(\widetilde{K}/\mathbb{Q}) \cong G,\ K \text{ satisfies } R_G\}, \tag{2.12}$$

where

$$R_G = \begin{cases} K \text{ is totally ramified} & \text{if } G = C_n, \\ K \text{ has square-free absolute discriminant} & \text{if } G = S_n, \\ \text{every prime } p \text{ that ramifies tamely in } K \text{ has its inertia group generated by an} & \\ \text{element in the conjugacy class of reflections} & \text{if } G = D_n \text{ for a prime } n \ge 3, \\ \text{every prime } p \text{ that ramifies tamely in } K \text{ has inertia group} & \\ \text{generated by an element in either } \{(1\,2\,3), (1\,3\,4), (1\,4\,2), (2\,4\,3)\} & \\ \text{or } \{(1\,3\,2), (1\,4\,3), (1\,2\,4), (2\,3\,4)\} & \text{if } n = 4 \text{ and } G = A_4. \end{cases} \tag{2.13}$$

As demonstrated in [56], there exists some constant $a = a(G,n) \in (0,1]$ such that, for all choices of G, n, and R_G under consideration, $\#\mathscr{F}(X) \gg_{G,n} X^a$.

With this setup in mind, let $A \ge 2$ and $\eta > 0$. Pierce, Turnage-Butterbaugh, and Wood [56, Theorem 1.4] proved that there exist effective constants $\alpha = \alpha(\eta, A, G, n) > 0$ and $\varepsilon = \varepsilon(G,n) > 0$ such that for all fields $K \in \mathscr{F}(X)$ with at most $O_{G,n}(X^{-\varepsilon}\#\mathscr{F}(X))$ exceptions, one has

$$\mathcal{E}_C(x, \widetilde{K}/\mathbb{Q}) \ll_A \begin{cases} \dfrac{|C|}{|G|}\dfrac{x}{(\log x)^A} & \text{if } e^{\alpha(\log\log D_{\widetilde{K}})^{5/3+\eta}} \le x \ll_{|G|} e^{c_4(\log D_{\widetilde{K}})^2/|G|}, \\ \dfrac{|C|}{|G|}\dfrac{x}{\exp\big(c_3\big(\frac{\log x}{|G|}\big)^{1/2}\big)} & \text{if } x \gg_{|G|} e^{c_4(\log D_{\widetilde{K}})^2/|G|}. \end{cases} \tag{2.14}$$

Notice that (2.14) eliminates the Landau–Siegel zero and, most importantly, goes beyond the range of x in (2.11). (We have only collected their unconditional results; see [56, Section 2] for a discussion regarding degree n S_n- and A_n-fields with $n \ge 5$.)

The proofs in [56] rely decisively on (1.1), and the T-dependence in (1.1) inhibits their proof from achieving a result that is more commensurate with what GRH predicts in (2.9). Using Theorems 1.3 and 8.5, we improve both the range of x and quality of the error term in (2.14). In particular, we obtain a range much closer to what GRH predicts with a power savings error term for small values of x.

Theorem 2.7. *Let G be one of the groups in (2.13). Let $C \subseteq G$ be a conjugacy class, and let $\mathscr{F}(X) = \mathscr{F}(X; G, n, R_G)$ be as in (2.12). There exist small positive constants $\eta = \eta(G,n)$ and $\varepsilon = \varepsilon(G,n)$ such that, for all fields $K \in \mathscr{F}(X)$ with at most $O_{G,n}(X^{-\varepsilon}\mathscr{F}(X))$*

exceptions,

$$\left|\pi_C(x,\widetilde{K}/\mathbb{Q})-\frac{|C|}{|G|}\pi(x)\right|$$
$$\ll \begin{cases} \dfrac{|C|}{|G|}x^{1-\eta} & \text{if } (\log D_{\widetilde{K}})^{2/\eta}\le x< D_{\widetilde{K}}^{1/(24\eta)},\\ \dfrac{|C|}{|G|}\dfrac{x}{\exp(c_3(\log x)^{1/2}|G|^{-1/2})} & \text{if } x\ge D_{\widetilde{K}}^{1/(24\eta)}.\end{cases}$$

Remark 2.8. For a more uniform version of the error term in Theorem 2.7, see (8.15).

2.4. *Landau–Siegel zeros and torsion in class groups*

Let us continue with the notation of Section 2.3. Let Cl_K denote the ideal class group of a number field K. It is widely believed that if ℓ is a positive integer, then the ℓ-torsion subgroup $\mathrm{Cl}_K[\ell]$ is of size $O_{\varepsilon,n,\ell}(D_K^{\varepsilon})$ for all $\varepsilon>0$. This bound is known to hold when $n=2$ and $\ell\ge 2$ (due to Gauss) and when ℓ is prime and K is an ℓ-extension of a given base field [38, Theorem 1.5]. The trivial bound is $O_{\varepsilon,\ell,n}(D_K^{1/2+\varepsilon})$, which follows from Minkowski's bound for the order of the entire group. Ellenberg and Venkatesh [16, Lemma 2.3 and Proposition 3.1] proved that if, for any $\varepsilon>0$, one has

$$\#\bigl\{p\le D_K^{\frac{1}{2\ell(n-1)}-\frac{\varepsilon}{4}}: p\nmid D_K \text{ and splits completely in } K\bigr\}\gg_{\varepsilon,n} D_K^{\frac{1}{2\ell(n-1)}-\frac{\varepsilon}{2}}, \tag{2.15}$$

then

$$|\mathrm{Cl}_K[\ell]|\ll_{\varepsilon,n,\ell} D_K^{\frac{1}{2}-\frac{1}{2\ell(n-1)}+\varepsilon}. \tag{2.16}$$

Since primes that split completely in $\widetilde{K}$ also split completely in K, the hypothesis (2.15) follows easily from (2.9), which is a consequence of GRH for $\zeta_{\widetilde{K}}(s)$. It is a straightforward consequence of (2.14) that for any positive integer ℓ, all except at most a density zero subset of the fields $K\in\mathscr{F}(X;G,n,R_G)$ satisfy (2.15), and hence (2.16), *unconditionally*. This provides the first nontrivial upper bounds for $|\mathrm{Cl}_K[\ell]|$, for all integers $\ell\ge 1$, applicable to infinite families of fields of arbitrarily large degree. This elegant application of (2.14) in [56] was achieved by exhibiting large zero-free regions for $\zeta_{\widetilde{K}}(s)$ for most fields K in a certain families. See also [68] for additional instances in which (2.16) can be improved pointwise.

We proceed in a complementary direction using Theorem 1.7. If the Dedekind zeta function of a quadratic subfield $\mathbb{Q}(\sqrt{d})$ has a Landau–Siegel zero, then Theorem 1.7 implies that certain number fields K, whose Galois closure does not contain $\mathbb{Q}(\sqrt{d})$ as a subfield, possess GRH-quality bounds on ℓ-torsion in their class groups.

Theorem 2.9. *Let $K/\mathbb{Q}$ be a number field of degree n with Galois closure $\widetilde{K}$ over $\mathbb{Q}$. Let $\ell\ge 1$ be a positive integer and $\varepsilon>0$ be arbitrary. Let χ be the real Dirichlet character modulo a fundamental discriminant d. Assume the following:*

(i) *$\zeta_{\widetilde{K}}(s)$ is the L-function of an automorphic representation of* $\mathrm{GL}_{[\widetilde{K}:\mathbb{Q}]}(\mathbb{A}_{\mathbb{Q}})$.

(ii) $\mathbb{Q}(\sqrt{d}) \cap \widetilde{K} = \mathbb{Q}$ *and* $\log D_K \asymp_{n,\varepsilon,\ell} \log |d|$.

(iii) *The Dirichlet L-function $L(s,\chi)$ has a real zero $\beta_\chi = 1 - \eta_\chi/\log d$ with η_χ sufficiently small, depending only on n, ε, and ℓ.*

Then

$$|\mathrm{Cl}_K[\ell]| \ll_{\varepsilon,n,\ell} D_K^{\frac{1}{2}-\frac{1}{2\ell(n-1)}+\varepsilon}.$$

Remark 2.10. In essence, the extra repulsion of zeros from the line $\mathrm{Re}(s) = 1$ induced by the presence of β_χ is barely sufficient to alleviate the need for the averaging process in [56]. This gives Theorem 2.9 the benefit of being a result for individual fields at the cost of assuming the existence of β_χ.

3. Properties of L-functions

We recall some standard facts about L-functions arising from cuspidal automorphic representations and their Rankin–Selberg convolutions. Much of the material we present here can be found in [6, Section 1]. We refer the reader there for a more detailed overview.

3.1. Standard L-functions

Let $d \geq 1$ be an integer, let $\mathbb{A}$ denote the ring of adeles over $\mathbb{Q}$, and let $\mathcal{A}(d)$ be the set of all cuspidal automorphic representations of $\mathrm{GL}_d(\mathbb{A})$. We consider each $\pi = \otimes_v \pi_v \in \mathcal{A}(d)$ to be normalized so that π has a unitary central character that is normalized to be trivial on the positive reals. The tensor product ranges over all places of $\mathbb{Q}$. At a nonarchimedean place corresponding with a prime p (resp. at the archimedean place), we write π_p (resp. π_∞) instead of π_v. We write $\widetilde{\pi} \in \mathcal{A}(d)$ for the representation which is contragredient to π.

Let $\pi = \otimes_v \pi_v \in \mathcal{A}(d)$, and let N_π denote the conductor of π. The standard L-function $L(s,\pi)$ associated to π is of the form

$$L(s,\pi) = \prod_p L(s,\pi_p) = \sum_{n\geq 1} \frac{a_\pi(n)}{n^s}.$$

The Euler product and Dirichlet series converge absolutely when $\mathrm{Re}(s) > 1$. For each p, the local factor $L(s,\pi_p)$ is given in the form

$$L(s,\pi_p) = \prod_{j=1}^{d} \left(1 - \frac{\alpha_{j,\pi}(p)}{p^s}\right)^{-1} = 1 + \sum_{j=1}^{\infty} \frac{a_\pi(p^j)}{p^{js}}$$

for suitable complex numbers $\alpha_{j,\pi}(p)$. With this convention, we have $\alpha_{j,\pi}(p) \neq 0$ for all j whenever $p \nmid N_\pi$, and it might be the case that $\alpha_{j,\pi}(p) = 0$ for some j when $p \mid N_\pi$. At the archimedean place of $\mathbb{Q}$, there are d complex Langlands parameters $\mu_\pi(j)$ from which we define

$$L(s,\pi_\infty) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s+\mu_\pi(j)}{2}\right).$$

By the work of Rudnick and Sarnak [58, Proposition A.1] and Blomer and Brumley [4, Corollary 1], we know that there exists a constant

$$\delta_d \in \left[0, \frac{1}{2} - \frac{1}{d^2+1}\right] \tag{3.1}$$

such that

$$|\alpha_{j,\pi}(p)| \leq p^{\delta_d} \quad \text{and} \quad \mathrm{Re}(\mu_\pi(j)) \geq -\delta_d \tag{3.2}$$

for all j and p. The generalized Selberg eigenvalue conjecture and GRC assert that $\delta_d = 0$ for all $d \geq 1$. For each p,

$$\{\alpha_{j,\tilde{\pi}}(p)\} = \{\overline{\alpha_{j,\pi}(p)}\}, \quad \{\mu_{\tilde{\pi}}(j)\} = \{\overline{\mu_\pi(j)}\}.$$

Let r_π denote the order of the pole of $L(s,\pi)$ at $s=1$ and κ_π be the residue of $L(s,\pi)$ at $s=1$. The completed L-function

$$\Lambda(s,\pi) = (s(s-1))^{r_\pi} N_\pi^{s/2} L(s,\pi) L(s,\pi_\infty)$$

is an entire function of order 1, and there exists a complex number $W(\pi)$ of modulus 1 such that for all $s \in \mathbb{C}$,

$$\Lambda(s,\pi) = W(\pi)\Lambda(1-s,\tilde{\pi}).$$

The trivial zeros of $L(s,\pi)$ are the poles of $s^{r_\pi} L(s,\pi_\infty)$. Since $\Lambda(s,\pi)$ is entire of order 1, it has an Hadamard factorization

$$\Lambda(s,\pi) = e^{a_\pi + b_\pi s} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where ρ runs through the nontrivial zeros of $L(s,\pi)$. These satisfy $0 < \mathrm{Re}(\rho) < 1$.

Finally, we define the analytic conductor of π to be

$$C(\pi,t) = N_\pi \prod_{j=1}^{d} (1 + |it + \mu_\pi(j)|), \quad C(\pi) = C(\pi,0). \tag{3.3}$$

3.2. Rankin–Selberg L-functions

Let $\pi = \otimes_v \pi_v \in \mathfrak{A}(d)$ and $\pi' = \otimes_v \pi'_v \in \mathfrak{A}(d')$. The Rankin–Selberg L-function $L(s,\pi\times\pi')$ associated to π and π' is of the form

$$L(s,\pi\times\pi') = \prod_p L(s,\pi_p\times\pi'_p) = \sum_{n\geq 1} \frac{a_{\pi\times\pi'}(n)}{n^s}.$$

The Euler product and Dirichlet series converge absolutely when $\mathrm{Re}(s) > 1$. For each p, the local factor $L(s,\pi_p\times\pi'_p)$ is given in the form

$$L(s,\pi_p\times\pi'_p) = \prod_{j=1}^{d}\prod_{j'=1}^{d'} (1 - \alpha_{j,j',\pi\times\pi'}(p) p^{-s})^{-1}$$

for suitable complex numbers $\alpha_{j,j',\pi\times\pi'}(p)$. With δ_d as in (3.1), we have the pointwise bound

$$|\alpha_{j,j',\pi\times\pi'}(p)| \leq p^{\delta_d+\delta_{d'}} \leq p^{1-\frac{1}{d'd}}. \tag{3.4}$$

If $p \nmid N_\pi N_{\pi'}$, then we have the equality of sets

$$\{\alpha_{j,j',\pi\times\pi'}(p)\} = \{\alpha_{j,\pi}(p)\alpha_{j',\pi'}(p)\}. \tag{3.5}$$

Rudnick and Sarnak [58, Appendix A.2] proved that $a_{\pi\times\tilde{\pi}}(n) \geq 0$ for all $n \geq 1$.

At the archimedean place of $\mathbb{Q}$, there are $d'd$ complex Langlands parameters $\mu_{\pi\times\pi'}(j,j')$ from which we define

$$L(s,\pi_\infty\times\pi'_\infty) = \pi^{-d'ds/2}\prod_{j=1}^{d}\prod_{j'=1}^{d'}\Gamma\Big(\frac{s+\mu_{\pi\times\pi'}(j,j')}{2}\Big).$$

These parameters satisfy

$$\{\mu_{\tilde{\pi}\times\tilde{\pi}'}(j,j')\} = \{\overline{\mu_{\pi\times\pi'}(j,j')}\}$$

and satisfy the pointwise bound

$$\mathrm{Re}(\mu_{\pi\times\pi'}(j,j')) \geq -\delta_d-\delta_{d'} \geq -1+(d'd)^{-1}. \tag{3.6}$$

Define

$$r_{\pi\times\pi'} := -\operatorname*{ord}_{s=1} L(s,\pi\times\pi'), \qquad \kappa_{\pi\times\pi'} := \operatorname*{Res}_{s=1} L(s,\pi\times\pi').$$

By our normalization for π and π', we know that $r_{\pi\times\pi'} = 1$ if and only if $\pi = \tilde{\pi}'$; otherwise, $r_{\pi\times\pi'} = 0$ and hence $\kappa_{\pi\times\pi'} = 0$. The function

$$\Lambda(s,\pi\times\pi') = (s(s-1))^{r_{\pi\times\pi'}} N_{\pi\times\pi'}^{s/2} L(s,\pi\times\pi')L(s,\pi_\infty\times\pi'_\infty)$$

is entire of order 1, and there exists a complex number $W(\pi\times\pi')$ of modulus 1 such that $\Lambda(s,\pi\times\pi')$ satisfies the functional equation

$$\Lambda(s,\pi\times\pi') = W(\pi\times\pi')\Lambda(1-s,\tilde{\pi}\times\tilde{\pi}').$$

The trivial zeros of $L(s,\pi\times\pi')$ are the poles of $s^{r_{\pi\times\pi'}}L(s,\pi_\infty\times\pi'_\infty)$. Since $\Lambda(s,\pi\times\pi')$ is entire of order 1, it has an Hadamard factorization

$$\Lambda(s,\pi\times\pi') = e^{a_{\pi\times\pi'}+b_{\pi\times\pi'}s}\prod_\rho\Big(1-\frac{s}{\rho}\Big)e^{s/\rho},$$

where ρ runs through the nontrivial zeros of $L(s,\pi\times\pi')$. These satisfy $0<\mathrm{Re}(\rho)<1$.

As with $L(s,\pi)$, we define the analytic conductor of $\pi\otimes\pi'$ to be

$$C(\pi\times\pi',t) = N_{\pi\times\pi'}\prod_{j=1}^{d}\prod_{j'=1}^{d'}(1+|it+\mu_{\pi\times\pi'}(j,j')|), \qquad C(\pi\times\pi') = C(\pi\times\pi',0).$$

It will be important to be able to decouple the dependencies of $C(\pi\times\pi',t)$ on π, π', and t. To this end, we have the combined work of Bushnell and Henniart [9, Theorem 1] and Brumley [31, Lemma A.2] which yields

$$C(\pi\times\pi',t) \ll C(\pi\times\pi')(1+|t|)^{d'd}, \qquad C(\pi\times\pi') \leq e^{O(d'd)}C(\pi)^{d'}C(\pi')^{d}. \tag{3.7}$$

4. Detecting zeros of L-functions

Let $\Lambda(n)$ be the von Mangoldt function, and define the numbers

$$\lambda_{\pi\times\pi'}(n) = \begin{cases} \sum_{j=1}^{d}\sum_{j'=1}^{d'} \alpha_{j,j',\pi\times\pi'}(p)^k & \text{if } n = p^k \text{ for a prime } p, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

so that

$$-\frac{L'}{L}(s,\pi\times\pi') = \sum_{n\geq 1}\frac{\lambda_{\pi\times\pi'}(n)\Lambda(n)}{n^s}.$$

It follows from the definition of $\lambda_{\pi\times\pi'}(n)$ that if $\gcd(n, N_\pi N_{\pi'}) = 1$, then

$$\lambda_{\pi\times\pi'}(n) = \lambda_\pi(n)\lambda_{\pi'}(n).$$

In particular, if $\gcd(n, N_\pi) = 1$, then $|\lambda_\pi(n)|^2 = \lambda_{\pi\times\tilde{\pi}}(n)$. During the proof of [58, Lemma A.1], it is shown that $\lambda_{\pi\times\tilde{\pi}}(n) \geq 0$ for all $n \geq 1$. Brumley [61, Appendix] proved that regardless of whether $\gcd(n, N_\pi N_{\pi'}) > 1$, we always have the inequality

$$|\lambda_{\pi\times\pi'}(n)| \leq \sqrt{\lambda_{\pi\times\tilde{\pi}}(n)\lambda_{\pi'\times\tilde{\pi}'}(n)} \leq \frac{\lambda_{\pi\times\tilde{\pi}}(n) + \lambda_{\pi'\times\tilde{\pi}'}(n)}{2}. \tag{4.2}$$

If χ is a primitive Dirichlet character modulo q, then $\chi\overline{\chi}$ is the trivial character modulo q, hence

$$|\lambda_{\pi\times(\pi'\otimes\chi)}(n)| \leq \sqrt{\lambda_{\pi\times\tilde{\pi}}(n)\lambda_{(\pi'\otimes\chi)\times(\tilde{\pi}'\otimes\overline{\chi})}(n)} \leq \frac{\lambda_{\pi\times\tilde{\pi}}(n) + \lambda_{\pi'\times\tilde{\pi}'}(n)}{2}. \tag{4.3}$$

The proof of Theorem 1.3 will use the following result on the detection of zeros near the line $\mathrm{Re}(s) = 1$.

Proposition 4.1. *Let $Q, T \geq 3$, and let $\tau \in \mathbb{R}$ satisfy $|\tau| \leq T$. Let $\pi \in \mathfrak{F}_m(Q)$ and $\pi_0 \in \mathcal{A}(m_0)$; suppose that both π and π_0 satisfy Hypothesis 1.1. Let $\chi \pmod q$ be a real primitive Dirichlet character. Let*

$$\frac{1}{\log(C(\pi_0)QT)} \leq \eta \leq \frac{1}{10^7(m_0 m)^2} \tag{4.4}$$

and

$$K \geq 8000(m_0 m)^3\eta\log(C(\pi_0)QT) + O_{m_0,m}(1) \tag{4.5}$$

with a sufficiently large implied constant. There exists an effectively computable constant $c_{m,m_0} \in (0, 1/4)$ such that the following is true.

Define

$$e_\chi := \begin{cases} 1 & \text{if } q > 1 \text{ and } L(s,\chi) \text{ has a real zero } \beta_\chi \geq 1 - c_{m,m_0}\eta, \\ 0 & \text{if } q = 1. \end{cases} \tag{4.6}$$

If $e_\chi = 1$, then assume that $L(\beta_\chi, \pi_0 \times (\tilde{\pi}_0 \otimes \chi)) = 0$. If $L(s, \pi\times\pi_0)$ has a zero ρ_0

(with $\rho_0 \neq \beta_\chi$ when $e_\chi = 1$) satisfying $|\rho_0 - (1 + i\tau)| \leq \eta$, then

$$1 \ll (200)^{4K}\Big[\eta^3 \int_{A_1}^{A_2}\Big|\sum_{A_1<p\leq u}\frac{\lambda_{\pi\times\pi_0}(p)(1+e_\chi\chi(p)p^{\beta_\chi-1})\log p}{p^{1+i\tau}}\Big|^2\frac{du}{u} + r_{\pi\times\pi_0}\mathbf{1}(\tau)\min\Big\{1, \frac{1-\beta_\chi}{\eta}\Big\}\Big],$$

where

$$A_1 := \exp(K/(300\eta)), \qquad A_2 := \exp(40K/\eta) \tag{4.7}$$

and

$$\mathbf{1}(\tau) := \begin{cases} 1 & \text{if } |\tau| < 200\eta, \\ 0 & \text{if } |\tau| \geq 200\eta, \end{cases} \tag{4.8}$$

and $\min\{1, (1-\beta_\chi)/\eta\}$ *is identified as* 1 *when* $e_\chi = 0$.

When $\pi, \pi_0 \in \mathcal{A}(1)$ and π_0 is trivial, Proposition 4.1 reduces to a result of Weiss [69, Proposition 4.2]; we follow Weiss's proof with the modifications which follow [44, 61] to allow for more general choices of π and π_0. Relative to the ideas in [44, 61, 69], there are three novelties here. First, we exploit the existence of an exceptional zero of a Dirichlet L-function in the zero-detection process for $L(s, \pi \times \pi_0)$, which generalizes [69, Proposition 4.2]. Second, we use Hypothesis 1.1 for both π and π_0 instead of assuming that at least one of π and π_0 satisfies GRC as in [44] so that, unlike the approach in [61], the Dirichlet polynomial can be supported on primes. Third, much like [61, Section 4], the proof here makes explicit some of the effective constants in [44, 69].

4.1. Preliminary estimates

Lemma 4.2. *Let $\pi \in \mathcal{F}_m(Q)$, let $\pi_0 \in \mathcal{A}(m_0)$, and let χ be a primitive Dirichlet character modulo q. If $\eta > 0$, then*

$$\sum_{n\geq 1}\frac{|\lambda_{\pi\times(\pi_0\otimes\chi)}(n)|\Lambda(n)}{n^{1+\eta}} \leq \frac{1}{\eta} + \frac{m_0 m}{2}\log(C(\pi_0)Q) + O((m_0 m)^2).$$

Proof. Suppose $\pi \in \mathcal{A}(d) \cap \mathcal{F}_m(Q)$. It follows from (4.3) and the discussion in [61] which follows Lemma 2.3 that

$$\sum_{n\geq 1}\frac{|\lambda_{\pi\times(\pi_0\otimes\chi)}(n)|\Lambda(n)}{n^{1+\eta}} \leq \frac{1}{\eta} + \frac{1}{4}\log C(\pi\times\tilde{\pi}) + \frac{1}{4}\log C(\pi_0\times\tilde{\pi}_0) + O((dm_0)^2),$$

The desired result follows from (3.7), the bound $C(\pi) \leq Q$, and the bound $d \leq m$. ∎

Lemma 4.3. *Let $\pi \in \mathcal{F}_m(Q)$, let $\pi_0 \in \mathcal{A}(m_0)$, and let χ be a primitive Dirichlet character modulo $q \leq Q$. Let $n(\eta; s)$ denote the number of zeros ρ of $L(s, \pi \times (\pi_0 \otimes \chi))$ with $|s - \rho| \leq \eta$. For all* $\mathrm{Re}(s) \geq 1$ *and all* $0 < \eta < 1/2$, *we have the bound*

$$n(\eta; s) \leq 20(mm_0)^2\eta\log(C(\pi_0)Q) + 5m_0 m\eta\log(|\mathrm{Im}(s)| + 2) + O((m_0 m)^2).$$

Proof. It suffices to prove the result for $n(\eta; 1+it)$ because $n(\eta; 1+it) \geq n(\eta; \sigma+it)$ for any $\sigma \geq 1$. For $\pi \in \mathcal{A}(d) \cap \mathcal{F}_m(Q)$, it follows from [61, Lemma 3.1] that for such s,

$$n(\eta; s) \leq 10dm_0\eta \log C(\pi \times (\pi_0 \otimes \chi)) + 5dm_0\eta \log(|\mathrm{Im}(s)|+2) + O((dm_0)^2).$$

The result now follows from (3.7). ■

Lemma 4.4. *If $\pi \in \mathcal{A}(d)$ satisfies Hypothesis 1.1, $y > C(\pi)$, and η is as in (4.4), then*

$$\sum_{\substack{n \in [y, y^{12000}] \\ n \text{ composite}}} \frac{\lambda_{\pi\times\tilde{\pi}}(n)\Lambda(n)}{n^{1+\eta}} \ll_d y^{-\frac{1}{2(d^2+1)}-\eta}(\log y)^3. \tag{4.9}$$

Proof. We first bound the contribution to the sum in (4.9) from the n which share a prime factor with N_π separately. Note that $O(\log y)$ primes divide N_π as $y > C(\pi) \geq N_\pi$. Thus by (3.4) and (4.1) applied to the ramified prime, we have

$$\begin{aligned}\sum_{\substack{n \in [y, y^{12000}] \\ n \text{ composite} \\ (n, N_\pi)>1}} \frac{\lambda_{\pi\times\tilde{\pi}}(n)\Lambda(n)}{n^{1+\eta}} &\ll d^2(\log y) \sum_{2 \leq r \leq 20000\log y} \sum_{\substack{y^{1/r} \leq p \leq y^{12000/r} \\ p \mid N_\pi}} p^{-r\frac{2}{d^2+1}-r\eta} \\ &\ll d^2(\log y)^2 \sum_{2\leq r\leq 20000\log y} y^{-\frac{2}{d^2+1}-\eta} \ll d^2 y^{-\frac{2}{d^2+1}-\eta}(\log y)^3.\end{aligned}$$

If $p \nmid N_\pi$, then $\lambda_{\pi\times\tilde{\pi}}(p^r) = |\lambda_\pi(p^r)|^2$. From (4.1), we see that

$$|\lambda_\pi(p^r)|^2 \leq d^2 \max_{1\leq j\leq d} |\alpha_{j,\pi}(p)|^{2r}.$$

Define

$$\beta_p = p^{-1} \max_{1\leq j\leq d} |\alpha_{j,\pi}(p)|^2.$$

Note that $\beta_p \leq p^{-2/(d^2+1)}$ by (3.2). Thus the contribution to the sum in (4.9) arising from the integers n which are coprime to N_π is

$$\begin{aligned}&\ll d^2(\log y)\sum_{r=2}^{\infty} \sum_{y^{1/r}\leq p\leq y^{12000/r}} \beta_p^r p^{-r\eta} \\ &\ll d^2(\log y) \sum_{2\leq R\leq 20000\log y} \sum_{y^{1/R}<p\leq y^{12000/R}} \sum_{r=R}^{\infty} \beta_p^r p^{-r\eta}.\end{aligned}$$

Subject to Hypothesis 1.1, we will prove that

$$S_R := \sum_{y^{1/R}<p\leq y^{12000/R}} \sum_{r=R}^{\infty} \beta_p^r p^{-r\eta} \ll_d y^{-\frac{1}{2(d^2+1)}-\eta}(\log y) \tag{4.10}$$

uniformly for all $2 \leq R \leq 20000\log y$, which suffices to prove the lemma.

The inner sum is geometric, so

$$S_R = \sum_{y^{1/R} < p \le y^{12000/R}} \frac{(\beta_p p^{-\eta})^R}{1 - \beta_p p^{-\eta}} \le y^{-\frac{1}{d^2+1} - \eta} \sum_{y^{1/R} < p \le y^{12000/R}} \frac{\beta_p}{1 - \beta_p p^{-\eta}}.$$

We decompose the sum according to whether p is greater than 2^{d^2} (in which case $1 - \beta_p p^{-\eta} \ge 1/2$) or not. The contribution from the latter range to the sum is $O_d(1)$, so we have

$$\begin{aligned} S_R &\le y^{-\frac{1}{d^2+1} - \eta}\Big(2 \sum_{y^{1/R} < p \le y^{12000/R}} \beta_p + O_d(1)\Big) \\ &\le y^{-\frac{1}{d^2+1} - \eta}\Big(2 y^{\frac{1}{2(d^2+1)}} \sum_{y^{1/R} < p \le y^{12000/R}} \beta_p p^{-\frac{1}{12000(d^2+1)}} + O_d(1)\Big). \end{aligned} \tag{4.11}$$

Note that $x \le 2\log(x+1)$ for all $0 \le x \le 5/2$. Thus, since $0 \le \beta_p p^{-\frac{1}{12000(d^2+1)}} \le 1$ for all p, the final sum in (4.11) is bounded by

$$\begin{aligned} 2 \sum_{y^{1/R} < p \le y^{12000/R}} \log\big(1 + \beta_p p^{-\frac{1}{12000(d^2+1)}}\big) &\le 2 \sum_p \log\Big(\sum_{r=0}^{\infty} \beta_p^r p^{-\frac{r}{12000(d^2+1)}}\Big) \\ &= 2\log\bigg(\prod_p \sum_{r=0}^{\infty} \frac{\max_{1 \le j \le d} |\alpha_{j,\pi}(p)|^{2r}}{p^{r(1+\frac{1}{12000(d^2+1)})}}\bigg). \end{aligned}$$

The above display is $\ll_d \log y$ by Hypothesis 1.1, which yields (4.10). ∎

We record an analogue of [29, Theorem A] for Rankin–Selberg L-functions.

Lemma 4.5. *Let $\pi_0 \in \mathfrak{A}(m_0)$, and let $\mathcal{H}_{m_1}(R) \subseteq \bigcup_{d \le m_1} \mathfrak{A}(d)$ be a set of cuspidal automorphic representations of dimension at most m_1 having analytic conductor at most R. There exists an effectively computable constant $c_{m_1,m_0} \in (0, 1/4)$ such that for at most one $\pi \in \mathcal{H}_{m_1}(R)$, $L(s, \pi \times \pi_0)$ has a real zero in the interval*

$$1 - \frac{c_{m_1,m_0}}{\log(C(\pi_0)R)} \le s < 1. \tag{4.12}$$

Proof. Suppose to the contrary that there exist distinct $\pi, \pi' \in \mathcal{H}_{m_1}(R)$ such that $L(s, \pi \times \pi_0)$ and $L(s, \pi' \otimes \pi_0)$ have a real zero in the region (4.12). Consider the isobaric (noncuspidal) representation $\Pi = \pi_0 \boxplus \tilde{\pi} \boxplus \tilde{\pi}'$ and the L-function

$$\begin{aligned} L(s, \Pi \times \tilde{\Pi}) = {} & L(s, \pi \times \tilde{\pi}) L(s, \pi' \times \tilde{\pi}') L(s, \pi_0 \times \tilde{\pi}_0) L(s, \pi \times \pi_0) L(s, \pi' \times \pi_0) \\ & \cdot L(s, \tilde{\pi} \times \pi') L(s, \tilde{\pi} \times \tilde{\pi}_0) L(s, \tilde{\pi}' \times \tilde{\pi}_0) L(s, \pi \times \tilde{\pi}'). \end{aligned}$$

By [29, Lemma a], the Dirichlet coefficients of $-\frac{L'}{L}(s, \Pi \times \tilde{\Pi})$ are real and nonnegative. Proceeding as in [32, Lemma 5.9], we find that if the order of the pole at $s = 1$ of the $L(s, \Pi \times \tilde{\Pi})$ is r, then $L(s, \Pi \times \tilde{\Pi})$ has at most r real zeros in the region (4.12) for some suitable $c_{m_1,m_0} > 0$. (We have implicitly used (3.7) to bound the analytic conductors for the factors of $L(s, \Pi \times \widetilde{\Pi})$.)

If $\pi, \pi' \neq \widetilde{\pi}_0$, then $r = 3$. If $L(s, \pi \times \pi_0)$ has a real zero in the interval (4.12), then by the functional equation and complex conjugation, so does $L(s, \widetilde{\pi} \times \widetilde{\pi}_0)$. This applies also to $L(s, \pi' \times \pi_0)$ and $L(s, \widetilde{\pi}' \times \widetilde{\pi}_0)$. Thus $L(s, \Pi \times \widetilde{\Pi})$ has at least four zeros in (4.12), a contradiction. If either $\pi = \widetilde{\pi}_0$ or $\pi' = \widetilde{\pi}_0$, then by a similar argument, we find that $r = 5$ while $L(s, \Pi \times \widetilde{\Pi})$ has at least six real zeros in (4.12). Again, we reach a contradiction. Since this handles all permissible possibilities, the proof is complete. ■

Corollary 4.6. *Let* $Q \geq 3$, $\pi_0 \in \mathcal{A}(m_0)$, *and* $\chi \pmod q$ *be a primitive quadratic character such that* $q \leq Q$ *and* $L(s, \chi)$ *has a real zero* $\beta_\chi \in (1/2, 1)$. *Suppose that* $L(\beta_\chi, \pi_0 \times (\widetilde{\pi}_0 \otimes \chi)) = 0$. *There exists an effectively computable constant* c_{m,m_0} *in* $(0, 1/4)$ *(the same as in Proposition 4.1) such that if* $(1 - \beta_\chi) \log(C(\pi_0)Q) \leq c_{m,m_0}$ *and* $\pi \in \mathcal{F}_m(Q) \cup \{\widetilde{\pi}_0 \otimes \chi\}$, *then* $L(\beta_\chi, \pi \times \pi_0) = 0$ *if and only if* $\pi = \widetilde{\pi}_0 \otimes \chi$, *in which case* β_χ *is a simple zero. In other words, if* $\pi \in \mathcal{F}_m(Q)$ *and* $(1 - \beta_\chi) \log(C(\pi_0)Q) \leq c_{m,m_0}$, *then*

$$\operatorname*{ord}_{s=\beta_\chi} L(s, \pi \times \pi_0) = r_{\pi \times (\pi_0 \otimes \chi)}.$$

Proof. By Lemma 4.5 with $m_1 = \max\{m, m_0\}$, $R = \max\{Q, C(\pi_0)\}$, and $\mathcal{H}_{m_1}(R) = \mathcal{F}_m(Q) \cup \{\widetilde{\pi}_0 \otimes \chi\}$, there exists an effectively computable constant $c_{m,m_0} \in (0, 1/4)$ such that at most one $\pi \in \mathcal{F}_m(Q) \cup \{\widetilde{\pi}_0 \otimes \chi\}$ has the property that $L(s, \pi \times \pi_0)$ has a zero in the region

$$1 - \frac{c_{m,m_0}}{\log(C(\pi_0)Q)} \leq s < 1. \tag{4.13}$$

If $(1 - \beta_\chi) \log(C(\pi_0)Q) \leq c_{m,m_0}$, then β_χ lies in the interval (4.13). Since we have assumed that $L(\beta_1, \pi_0 \times (\widetilde{\pi}_0 \otimes \chi)) = 0$, it remains to show that β_χ is a simple zero of $L(s, \pi_0 \times (\widetilde{\pi}_0 \otimes \chi))$.

To prove this, we modify our approach in Lemma 4.5 with a different choice of Π, namely $\Pi = \pi \boxplus \pi \otimes \chi \boxplus \pi \otimes \chi$. We have the identity

$$L(s, \Pi \times \widetilde{\Pi}) = L(s, \pi_0 \times \widetilde{\pi}_0)^5 L(s, \pi_0 \times (\widetilde{\pi}_0 \otimes \chi))^4,$$

which uses the hypothesis that χ is quadratic. By [29, Lemma a], the Dirichlet coefficients of $-\frac{L'}{L}(s)$ are real and nonnegative. Proceeding as in [32, Lemma 5.9], we find that $L(s)$ has at most five real zeros in the interval (4.13) since $L(s)$ has a pole of order 5 at $s = 1$. However, since $L(s, \pi_0 \times (\widetilde{\pi}_0 \otimes \chi))$ occurs as a factor of $L(s)$ with multiplicity 4, we achieve a contradiction unless β_χ is simple. ■

4.2. A lower bound for high derivatives

Let $\pi \in \mathcal{A}(d) \cap \mathcal{F}_m(Q)$ and $\pi_0 \in \mathcal{A}(m_0)$, and let $\chi \pmod q$ be a primitive real Dirichlet character with $q \leq Q$. Suppose that $L(s, \pi \times \pi_0)$ has a zero ρ_0 such that $|\rho_0 - (1 + i\tau)| \leq \eta$, where $\tau \in \mathbb{R}$, $|\tau| \leq T$, and

$$\frac{1}{\log(C(\pi_0)QT)} \leq \eta \leq \frac{1}{10^7 (m_0 m)^2}. \tag{4.14}$$

Let $s = 1 + \eta + i\tau$. Let $\chi \pmod{q}$ be a primitive real Dirichlet character with $q \le Q$, let e_χ be as in (4.6), and define

$$F(z) := \frac{L'}{L}(z, \pi \times \pi_0) + e_\chi \frac{L'}{L}(z + 1 - \beta_\chi, \pi \times (\pi_0 \otimes \chi)), \qquad G_k(z) := \frac{(-1)^k}{k!} F^{(k)}(z).$$

We proceed as in [61, (4.2)] or [69, Lemma 4.2] (see also [32, Proposition 5.7]) and find that $G_k(s)$ equals

$$\sum_{\substack{L(\rho,\pi\times\pi_0)=0\\ |s-\rho|\le 200\eta}} \frac{1}{(s-\rho)^{k+1}} + \sum_{\substack{L(\rho',\pi\times(\pi_0\otimes\chi))=0\\ |s-\rho'|\le 200\eta}} \frac{e_\chi}{(s+1-\beta_\chi-\rho')^{k+1}}$$
$$-\frac{r_{\pi\times\pi_0}}{(s-1)^{k+1}} - \frac{r_{\pi\times(\pi_0\otimes\chi)}e_\chi}{(s-\beta_\chi)^{k+1}} + O\Big(\frac{(m_0m)^2\eta(\log(C(\pi\times\pi_0))+\log(C(\pi\times(\pi_0\otimes\chi))))}{(200\eta)^k}\Big).$$

We isolate the contribution from β_χ, if it exists, using Corollary 4.6 along with the facts that $L(s, \pi \times (\pi_0 \otimes \chi)) = L(s, (\pi \otimes \chi) \times \pi_0)$ and $r_{(\pi\otimes\chi)\times(\pi_0\otimes\chi)} = r_{\pi\times\pi_0}$. Consequently,

$$\begin{aligned} G_k(s) &= \sum_{\substack{\rho\ne\beta_\chi\\ L(\rho,\pi\times\pi_0)=0\\ |s-\rho|\le 200\eta}} \frac{1}{(s-\rho)^{k+1}} + \sum_{\substack{\rho'\ne\beta_\chi\\ L(\rho',\pi\times(\pi_0\otimes\chi))=0\\ |s-\rho|\le 200\eta}} \frac{e_\chi}{(s+1-\beta_\chi-\rho')^{k+1}} \\ &\quad + \frac{e_\chi \operatorname{ord}_{s=\beta_\chi} L(s,\pi\times\pi_0)}{(s-\beta_\chi)^{k+1}} + \frac{e_\chi \operatorname{ord}_{s=\beta_\chi} L(s,(\pi\otimes\chi)\times\pi_0)}{(s+1-2\beta_\chi)^{k+1}} - \frac{r_{\pi\times\pi_0}}{(s-1)^{k+1}} \\ &\quad - \frac{r_{\pi\times(\pi_0\otimes\chi)}e_\chi}{(s-\beta_\chi)^{k+1}} + O\Big(\frac{(m_0m)^3\eta\log(C(\pi_0)QT)}{(200\eta)^k}\Big) \\ &= \sum_{\substack{\rho\ne\beta_\chi\\ L(\rho,\pi\times\pi_0)=0\\ |s-\rho|\le 200\eta}} \frac{1}{(s-\rho)^{k+1}} + \sum_{\substack{\rho'\ne\beta_\chi\\ L(\rho',\pi\times(\pi_0\otimes\chi))=0\\ |s-\rho|\le 200\eta}} \frac{e_\chi}{(s+1-\beta_\chi-\rho')^{k+1}} \\ &\quad + r_{\pi\times\pi_0}\Big(\frac{e_\chi}{(s+1-2\beta_\chi)^{k+1}} - \frac{1}{(s-1)^{k+1}}\Big) \\ &\quad + O\Big(\frac{(m_0m)^3\eta\log(C(\pi_0)QT)}{(200\eta)^k}\Big). \end{aligned} \tag{4.15}$$

We have used (3.7) to bound the analytic conductors in the O-term.

Lemma 4.3 implies that our two sums over zeros have, in total, at most K terms for any

$$K \ge 8000(m_0m)^3\eta\log(C(\pi_0)QT) + O((m_0m)^2). \tag{4.16}$$

Just like [44,61,69], we rely on the following diophantine result due to Sós and Turán [59].

Lemma 4.7. *Let $z_1, \dots, z_\nu \in \mathbb{C}$. If $K \ge \nu$, then there exists an integer $k \in [K, 2K]$ such that $|z_1^k + \cdots + z_m^k| \ge \big(\frac{1}{50}|z_1|\big)^k$.*

If $L(z,\pi\times\pi_0)$ has a zero ρ_0 (not equal to β_χ if $e_\chi=1$) satisfying $|\rho_0-(1+i\tau)|\le\eta$, then by Lemma 4.7, we can bound the zero contribution to (4.15) from below as follows:

$$\begin{aligned}\Big|\sum_{\substack{\rho\neq\beta_\chi\\|s-\rho|\le 200\eta}}\frac{1}{(s-\rho)^{k+1}}+\sum_{\substack{\rho'\neq\beta_\chi\\|s-\rho'|\le 200\eta}}\frac{e_\chi}{(s+1-\beta_\chi-\rho')^{k+1}}\Big| &\ge\Big(\frac{1}{50|s-\rho_0|}\Big)^{k+1}\\ &\ge\frac{1}{(100\eta)^{k+1}}.\end{aligned}$$

Therefore, if the implied constant in (4.16) is sufficiently large, then for some $k\in[K,2K]$,

$$\begin{aligned}\eta^{k+1}\Big|G_k(s)-r_{\pi\times\pi_0}\Big(\frac{e_\chi}{(s+1-2\beta_\chi)^{k+1}}-\frac{1}{(s-1)^{k+1}}\Big)\Big|&\\ \ge\frac{1}{(100)^{k+1}}-O\Big(\frac{(m_0m)^3\eta^2\log(C(\pi_0)QT)}{(200)^k}\Big)&\ge\frac{3}{4(100)^{k+1}}.\end{aligned}\tag{4.17}$$

It follows from a calculation identical to [69, pp. 80–81] that

$$\eta^k\Big|\frac{e_\chi}{(s+1-2\beta_\chi)^{k+1}}-\frac{1}{(s-1)^{k+1}}\Big|\le\frac{1}{4(100)^{k+1}}+\mathbf{1}(\tau)\min\{1,2^{k+4}((1-\beta_\chi)/\eta)^{1/2}\},$$

where $\mathbf{1}(\tau)$ is given by (4.8) and we identify $\min\{1,2^{k+4}((1-\beta_\chi)/\eta)^{1/2}\}$ as 1 when $e_\chi=0$. In summary, we conclude that

$$\eta^{k+1}|G_k(s)|+r_{\pi\times\pi_0}\mathbf{1}(\tau)\min\{1,2^{k+4}((1-\beta_\chi)/\eta)^{1/2}\}\ge\frac{1}{2(100)^{k+1}}.\tag{4.18}$$

4.3. An upper bound for high derivatives

We proceed to bound $|G_k(s)|$ from above. Since $\eta>0$, we can use the absolute convergence of the Dirichlet series which defines $F(s)$ to directly compute

$$\eta^{k+1}|G_k(s)|=\eta\Big|\sum_{n\ge1}\frac{(\lambda_{\pi\times\pi_0}(n)+e_\chi\lambda_{\pi\times(\pi_0\otimes\chi)}(n)n^{\beta_\chi-1})\Lambda(n)}{n^{1+i\tau}}j_k(\eta\log n)\Big|,\tag{4.19}$$

where $j_k(u):=(k!)^{-1}u^ke^{-u}$. Let A_1 and A_2 be as in (4.7). Suppressing summands, we write the right hand side of (4.19) as

$$\eta\sum_{n\ge1}=\eta\Big(\sum_{n\notin[A_1,A_2]}+\sum_{\substack{n\in[A_1,A_2]\\ n\text{ composite}}}+\sum_{p\in[A_1,A_2]}\Big).\tag{4.20}$$

First, we bound the contribution from $n\notin[A_1,A_2]$. Since $k!\ge(k/e)^k$, we find from a small numerical calculation [61, proof of Lemma 4.3] that

$$j_k(\eta\log n)\le(110)^{-k}n^{-\eta/2}\qquad\text{if } n\notin[A_1,A_2].\tag{4.21}$$

By (4.21),

$$\Big|\eta\sum_{n\notin[A_1,A_2]}\Big|\ll\eta(110)^{-k}\sum_{n\geq 1}\frac{(|\lambda_{\pi\times\pi_0}(n)|+|e_\chi\lambda_{\pi\times(\pi_0\otimes\chi)}(n)|)\Lambda(n)}{n^{1+\eta/2}}.$$

By Lemma 4.2, the above display is $\ll\eta(110)^{-k}(\eta^{-1}+(m_0m)^2\log(C(\pi_0)QT))$. Using (4.16), we see that the contribution from $n\notin[A_1,A_2]$ is $O_{m_0,m}(k(110)^{-k})$.

Second, we bound the contribution from the composite $n\in[A_1,A_2]$. As $(\log u)^k\leq k!u$ for all $k\geq 1$ and $u\geq 1$, we find that

$$j_k(\eta\log n)=\frac{(\eta\log n)^k}{k!n^\eta}=\frac{1}{n^\eta}(110)^{-k}\frac{(\log n^{110\eta})^k}{k!}\leq\frac{1}{n^\eta}(110)^{-k}n^{110\eta}.$$

This estimate and (4.3) imply that

$$\begin{aligned}\Big|\eta\sum_{\substack{n\in[A_1,A_2]\\ n\text{ composite}}}\Big|&\leq\eta(110)^{-k}\sum_{\substack{n\in[A_1,A_1^{12000}]\\ n\text{ composite}}}\frac{(|\lambda_{\pi\times\pi_0}(n)|+|e_\chi\lambda_{\pi\times(\pi_0\otimes\chi)}(n)|)\Lambda(n)}{n^{1+\eta}}n^{110\eta}\\&\leq\eta(110)^{-k}\sum_{\substack{n\in[A_1,A_1^{12000}]\\ n\text{ composite}}}\frac{(\lambda_{\pi\times\tilde{\pi}}(n)+\lambda_{\pi_0\times\tilde{\pi}_0}(n))\Lambda(n)}{n^{1+\eta}}n^{110\eta}\\&\leq\eta(110)^{-k}A_1^{1320000\eta}\sum_{\substack{n\in[A_1,A_1^{12000}]\\ n\text{ composite}}}\frac{(\lambda_{\pi\times\tilde{\pi}}(n)+\lambda_{\pi_0\times\tilde{\pi}_0}(n))\Lambda(n)}{n^{1+\eta}}.\end{aligned}$$

By Lemma 4.4 and (4.14), the above display is

$$\ll_{d,m_0}\eta(110)^{-k}A_1^{1320000\eta-\frac{1}{2((dm_0)^2+1)}-\eta}(\log A_1)^2\ll_{d,m_0}\eta(110)^{-k}\ll_{m,m_0}k(110)^{-k}.$$

Finally, we estimate the contribution from the primes $p\in[A_1,A_2]$. Summation by parts gives us the identity

$$\begin{aligned}\eta\sum_{p\in[A_1,A_2]}&=j_k(\eta\log A_2)\eta\sum_{p\in[A_1,A_2]}\frac{(\lambda_{\pi\times\pi_0}(p)+e_\chi\lambda_{\pi\times(\pi_0\otimes\chi)}(p)p^{\beta_\chi-1})\Lambda(p)}{p^{1+i\tau}}\\&-\eta^2\int_{A_1}^{A_2}j_k'(\eta\log u)\sum_{p\in[A_1,u]}\frac{(\lambda_{\pi\times\pi_0}(p)+e_\chi\lambda_{\pi\times(\pi_0\otimes\chi)}(p)p^{\beta_\chi-1})\Lambda(p)}{p^{1+i\tau}}\frac{du}{u}.\end{aligned}\tag{4.22}$$

Much like the above calculations, we use Lemma 4.2 to deduce that the sum over $p\in[A_1,A_2]$ in (4.22) is

$$\ll\eta(110)^{-k}A_2^{-\eta/2}\sum_{n<A_2}\frac{(|\lambda_{\pi\times\pi_0}(n)|+|e_\chi\lambda_{\pi\times(\pi_0\otimes\chi)}(n)|)\Lambda(n)}{n}\ll\frac{k}{(110)^k}.$$

Since

$$\left|\frac{d}{du} j_k(u)\right| = |j_{k-1}(u) - j_k(u)| \leq j_{k-1}(u) + j_k(u) \leq 1,$$

we find that

$$\Big|\eta \sum_{p\in[A_1,A_2]}\Big| \leq \eta^2 \int_{A_1}^{A_2} \Big|\sum_{p\in[A_1,u]} \frac{(\lambda_{\pi\times\pi_0}(p) + e_\chi \lambda_{\pi\times(\pi_0\otimes\chi)}(p)p^{\beta_\chi - 1})\Lambda(p)}{p^{1+i\tau}}\Big| \frac{du}{u} + O\Big(\frac{k}{(110)^k}\Big).$$

If K satisfies (4.16), then the condition $p \in [A_1, A_2]$ implies that $p \nmid N_\pi N_{\pi_0} q$. Therefore, by (3.5) and (4.1),

$$(\lambda_{\pi\times\pi_0}(p) + e_\chi \lambda_{\pi\times(\pi_0\otimes\chi)}(p)p^{\beta_\chi-1})\Lambda(p) = \lambda_{\pi\times\pi_0}(p)(1 + e_\chi\chi(p)p^{\beta_\chi-1})\log p.$$

We collect our estimates for the three sums in (4.20) to find that for all $k \in [K, 2K]$ with K satisfying (4.5),

$$\eta^{k+1}|G_k(s)| \leq \eta^2 \int_{A_1}^{A_2} \Big|\sum_{A_1<p\leq u} \frac{\lambda_{\pi\times\pi_0}(p)\log p}{p^{1+i\tau}}(1 + e_\chi\chi(p)p^{\beta_\chi-1})\Big| \frac{du}{u} + O_{m_0,m}\Big(\frac{k}{(110)^k}\Big). \tag{4.23}$$

4.4. Proof of Proposition 4.1

We enlarge K according to (4.5), which we are free to do. If $k \in [K, 2K]$ and the implied constant in (4.5) is sufficiently large, then $O_{m,m_0}(k(110)^{-k}) \leq \frac{1}{4}(100)^{-k-1}$. Therefore, it follows from (4.18) and (4.23) that if $L(s, \pi\times\pi_0)$ has a zero ρ_0 (not equal to β_χ when $e_\chi = 1$) which satisfies $|\rho_0 - (1 + i\tau)| \leq \eta$, then with K satisfying (4.5), we have the bound

$$1 \leq 4(100)^{2K+1}\eta^2 \int_{A_1}^{A_2} \Big|\sum_{A_1<p\leq u} \frac{\lambda_{\pi\times\pi_0}(p)\log p}{p^{1+i\tau}}(1 + e_\chi\chi(p)p^{\beta_\chi-1})\Big| \frac{du}{u} + 4r_{\pi\times\pi_0}(100)^{2K+1}\mathbf{1}(\tau)\min\{1, 16\cdot 2^{2K}((1-\beta_\chi)/\eta)^{1/2}\}.$$

We square both sides and apply the Cauchy–Schwarz inequality to obtain the bound

$$1 \ll (100)^{4K}\eta^4\Big(\int_{A_1}^{A_2}\frac{du}{u}\Big)\Big(\int_{A_1}^{A_2}\Big|\sum_{A_1<p\leq u} \frac{\lambda_{\pi\times\pi_0}(p)\log p}{p^{1+i\tau}}(1 + e_\chi\chi(p)p^{\beta_\chi-1})\Big|^2 \frac{du}{u}\Big) + r_{\pi\times\pi_0}(100)^{4K}\mathbf{1}(\tau)\min\{1, 2^{4K}\eta^{-1}(1-\beta_\chi)\}.$$

Since $\int_{A_1}^{A_2} u^{-1}du \ll K/\eta$, Proposition 4.1 follows.

5. A new large sieve inequality

To apply Proposition 4.1 in our proof of Theorems 1.3 and 1.7, we must show that as $\pi \in \mathcal{F}_m(Q)$ varies, the integral in Proposition 4.1 is small on average. To prove this, we modify the large sieve for Dirichlet coefficients of automorphic representations due to Duke and Kowalski [15, Theorem 4]. As observed by Brumley [7], one can adjust their proof to show that if each $\pi \in \mathcal{F}_m(Q)$ satisfies Hypothesis 1.1 and $Q, x \geq 2$, then

$$\sum_{\pi \in \mathcal{F}_m(Q)} \Big|\sum_{n \leq x} a_\pi(n) b(n)\Big|^2 \ll_{\varepsilon,m} (Qx)^\varepsilon (x + Q^{\frac{m}{m^2+1}} x^{1-\frac{1}{m^2+1}} \#\mathcal{F}_m(Q)) \sum_{n \leq x} |b(n)|^2, \tag{5.1}$$

where $b(n)$ is any complex-valued function supported on the integers. We require two modifications to (5.1). First, we need to take sums over n in intervals of length x/T, where T is arbitrarily large. Second, we need a variant of (5.1) which applies with more sensitivity to sequences $b(n)$ supported on the primes.

We establish a "pre-sifted" large sieve inequality over short intervals for families of automorphic representations which satisfy Hypothesis 1.1. We anticipate that this will be useful in contexts beyond this paper. Since much stronger results are available when $m = 1$ (see [23, Theorem 4]), we assume that $m \geq 2$ throughout this section.

Proposition 5.1. *Let $b(n)$ be a complex-valued function supported on the integers, and suppose that each $\pi \in \mathcal{F}_m(Q)$ (see (1.3)) satisfies Hypothesis 1.1. If $Q \geq 3$, $T \geq 1$, $x > 0$, and $z \gg_m Q^{6m}$ with a sufficiently large implied constant, then for every $\varepsilon > 0$,*

$$\sum_{\pi \in \mathcal{F}_m(Q)} \Big| \sum_{\substack{x < p \leq xe^{1/T} \\ p > z}} a_\pi(p) b(p)\Big|^2 \ll_{\varepsilon,m} \left(\frac{x}{T \log z} + Q^{\frac{3}{2m}} T^{\frac{3}{4}} x^{1-\frac{1}{m^2}+\frac{1}{m^4}} z^{2+\varepsilon} \#\mathcal{F}_m(Q)\right) \sum_{\substack{x < p \leq xe^{1/T} \\ p > z}} |b(p)|^2.$$

5.1. *The naïve Rankin–Selberg L-function*

Let $\pi \in \mathcal{A}(d)$ and $\pi' \in \mathcal{A}(d')$. For each prime $p \nmid N_\pi N_{\pi'}$, define

$$L^{RS}(s, \pi_p \times \pi'_p) = 1 + \sum_{j=1}^{\infty} \frac{a_\pi(p^j) a_{\pi'}(p^j)}{p^{js}}. \tag{5.2}$$

We call the Dirichlet series

$$L^{RS}(s, \pi \times \pi') := \sum_{\substack{n \geq 1 \\ (n, N_\pi N_{\pi'})=1}} \frac{a_\pi(n) a_{\pi'}(n)}{n^s} = \prod_{p \nmid N_\pi N_{\pi'}} L^{RS}(s, \pi_p \times \pi'_p) \tag{5.3}$$

the *naïve Rankin–Selberg L-function*. We access the Dirichlet coefficients of $L^{RS}(s, \pi \times \pi')$ by relating $L^{RS}(s, \pi \times \pi')$ to $L(s, \pi \times \pi')$. The next result is due to

Brumley (see [7, proof of Corollary 3]), which improves upon [15, Proposition 2] by the insertion of the second order average estimate of Hypothesis 1.1.

Lemma 5.2 (Brumley). *Suppose that $\pi, \pi' \in \mathcal{F}_m(Q)$ satisfy Hypothesis 1.1. For each prime p, define $H(s, \pi_p \times \pi'_p)$ by the equality*

$$L^{RS}(s, \pi_p \times \pi'_p) = L(s, \pi_p \times \pi'_p) H(s, \pi_p \times \pi'_p).$$

For all $\varepsilon > 0$, the Euler product

$$H(s, \pi \times \pi') := \prod_{p \nmid N_\pi N_{\pi'}} H(s, \pi_p \times \pi'_p)$$

converges absolutely for $\mathrm{Re}(s) > 1 - (m^2 + 1)^{-1}$. *This yields the factorization*

$$L^{RS}(s, \pi \times \pi') = L(s, \pi \times \pi') H(s, \pi \times \pi') \prod_{p \mid N_\pi N_{\pi'}} L(s, \pi_p \times \pi'_p)^{-1}$$

in the region $\mathrm{Re}(s) > 1 - (m^2 + 1)^{-1}$. *Furthermore, $H(s, \pi \times \pi') \ll_{\varepsilon,m} Q^\varepsilon$ in this region.*

5.2. *Preliminary estimates*

Let $\pi, \pi' \in \mathcal{F}_m(Q)$ satisfy Hypothesis 1.1. Define

$$g_d^{RS}(s, \pi \times \tilde{\pi}') := \prod_{p \mid d} (1 - L^{RS}(s, \pi_p \times \tilde{\pi}'_p)^{-1}), \tag{5.4}$$

and let $d \geq 1$ be a square-free integer such that[3] $(d, N_\pi N_{\pi'}) = 1$. Consider the Dirichlet series

$$L_d^{RS}(s, \pi \times \tilde{\pi}') := \sum_{\substack{n \geq 1 \\ d \mid n \\ (n, N_\pi N_{\pi'}) = 1}} \frac{a_\pi(n) \overline{a_{\pi'}(n)}}{n^s} = L^{RS}(s, \pi \times \tilde{\pi}') g_d^{RS}(s, \pi \times \tilde{\pi}').$$

A bound for $g_d^{RS}(s, \pi \times \tilde{\pi}')$ follows readily from (3.4).

Lemma 5.3. *Let $d \geq 1$ be square-free and $\pi, \pi' \in \mathcal{F}_m(Q)$ satisfy Hypothesis 1.1. In the region $\sigma > 1 - (m^2 + 1)^{-1}$, we have*

$$g_d^{RS}(s, \pi \times \tilde{\pi}') \ll_{\varepsilon,m} d^\varepsilon.$$

If $d \geq 2$, then $0 \leq g_d^{RS}(1, \pi \times \tilde{\pi}) < 1$.

Proof. The fact that $0 \leq g_d^{RS}(1, \pi \times \tilde{\pi}) < 1$ for $d \geq 2$ follows immediately from (5.2). The bound (3.4) yields $|L^{RS}(s, \pi_p \times \tilde{\pi}'_p)^{-1}| \ll_m 1$ for $\mathrm{Re}(s) > 1 - (m^2 + 1)^{-1}$. The

[3] This is an abuse of notation since we have already used d for the dimension of π, but this abuse is limited in its appearance and does not compromise the exposition.

lemma now follows from the well-known bound $\omega(n) \ll (\log\log n)^{-1} \log n$, where $\omega(n)$ is the number of distinct prime factors of n. ∎

We require some uniform estimates for $L_d^{RS}(s, \pi \times \widetilde{\pi}')$.

Lemma 5.4. *Let $s = \sigma + it$, let $\pi, \pi' \in \mathcal{F}_m(Q)$, and let $\varepsilon > 0$. For any squarefree integer $d \geq 1$ coprime to $N_\pi N_{\pi'}$ and any $\sigma \geq 1 - \frac{1}{m^2} + \frac{1}{m^4}$, we have the uniform bound*

$$|(\sigma - 1)^{r(\pi\times\widetilde{\pi}')} L_d^{RS}(s, \pi \times \widetilde{\pi}')| \ll_{\varepsilon,m} d^\varepsilon Q^{\frac{3}{2m}} (1 + |t|)^{3/4}.$$

Proof. First, we establish the bound

$$|(\sigma - 1)^{r_{\pi\times\widetilde{\pi}'}} L(s, \pi \times \widetilde{\pi}')| \ll_{\varepsilon_0,m} (Q^{2m}(1 + |t|)^{m^2})^{\max\{\frac{1}{2}(1-\sigma),0\}+\varepsilon_0}, \qquad 1/2 \leq \sigma \leq 3, \tag{5.5}$$

for every $\varepsilon_0 > 0$. By the work of Li [45, Theorem 2], we know that there exists a constant $c_m > 0$ (depending at most on m) such that

$$(\sigma - 1)^{r_{\pi\times\widetilde{\pi}'}} |L(\sigma, \pi \times \widetilde{\pi}')| \ll \exp\left(c_m \frac{\log C(\pi \times \widetilde{\pi}')}{\log\log C(\pi \times \widetilde{\pi}')}\right) \ll_{\varepsilon_0,m} C(\pi \times \widetilde{\pi}')^{\varepsilon_0}, \quad 1 \leq \sigma \leq 3. \tag{5.6}$$

By replacing $\widetilde{\pi}'$ with $\widetilde{\pi}' \otimes |\det|^{it}$ in the proof of (5.6) (which does not change the proof substantially), we obtain

$$|(\sigma - 1)^{r_{\pi\times\widetilde{\pi}'}} L(\sigma + it, \pi \times \widetilde{\pi}')| \ll_{\varepsilon_0,m} C(\pi \times \widetilde{\pi}', t)^{\varepsilon_0}, \qquad 1 \leq \sigma \leq 3. \tag{5.7}$$

The refined convexity bound for L-functions proved by Heath-Brown [27] yields

$$|L(1/2 + it, \pi \times \widetilde{\pi}')| \ll_m |L(3/2 + it, \pi \times \widetilde{\pi}')|^2 C(\pi \times \widetilde{\pi}', t)^{1/4}. \tag{5.8}$$

Hence, by (5.7),

$$|L(1/2 + it, \pi \times \pi')| \ll_{\varepsilon_0,m} C(\pi \times \pi', t)^{1/4+\varepsilon_0}. \tag{5.9}$$

Thus (5.5) follows from (3.7), (5.7), (5.9), and the Phragmén–Lindelöf principle.

We see from (3.4) and the bound $\omega(n) \ll (\log\log n)^{-1} \log n$ that for every $\varepsilon_0 > 0$, one has the bound

$$\prod_{p | N_\pi N_{\pi'}} |L(s, \pi_p \times \widetilde{\pi}_p')^{-1}| \ll_{\varepsilon_1,m} Q^{\varepsilon_0}, \qquad \mathrm{Re}(s) > 1 - (m^2 + 1)^{-1}.$$

Therefore, by (3.7) and (5.5), we have

$$|(\sigma - 1)^{r_{\pi\times\widetilde{\pi}'}} L^{RS}(s, \pi \times \widetilde{\pi}')| \ll_{\eta,m} (Q^{2m}(1 + |t|)^{m^2})^{\max\{\frac{1}{2}(1-\sigma),0\}+2\varepsilon_0}, \qquad 1 - (m^2 + 1)^{-1} < \sigma \leq 3.$$

Once we choose

$$\sigma \geq 1 - \frac{1}{m^2} + \frac{1}{m^4} > 1 - (m^2+1)^{-1} \quad \text{and} \quad \varepsilon_0 = (8m^2)^{-1} + (4m^4)^{-1},$$

the lemma follows from the above estimate, Lemma 5.2, and Lemma 5.3. ∎

Fix a smooth function ϕ whose support is a compact subset of $(-2,2)$. Let

$$\widehat{\phi}(s) = \int_{-\infty}^{\infty} \phi(y) e^{sy}\, dy.$$

Thus $\widehat{\phi}(s)$ is entire, and integrating by parts several times yields the bound

$$\widehat{\phi}(s) \ll_{\phi,k} \frac{e^{2|\mathrm{Re}(s)|}}{|s|^k} \tag{5.10}$$

for any integer $k \geq 0$. Let $T \geq 1$; by Fourier inversion, for any $c > 0$, one has the identity

$$\phi(T \log x) = \frac{1}{2\pi i T} \int_{c-i\infty}^{c+i\infty} \widehat{\phi}(s/T) x^{-s} ds.$$

Lemma 5.5. *Let $\pi, \pi' \in \mathcal{F}_m(Q)$ with $m \geq 2$. Let $x > 0$, $T \geq 1$, and $d \geq 1$ be a square-free integer which is coprime to $N_\pi N_{\pi'}$. Define*

$$R(\pi,\pi') = \kappa_{\pi\times\widetilde{\pi}'} H(1, \pi\times\widetilde{\pi}') \prod_{p|N_\pi N_{\pi'}} L^{RS}(1, \pi_p \times \widetilde{\pi}'_p)^{-1}. \tag{5.11}$$

(1) *If ϕ is as above, then*

$$\Bigg| \sum_{\substack{n\geq 1 \\ d|n \\ (n,N_\pi N_{\pi'})=1}} a_\pi(n)\overline{a_{\pi'}(n)}\phi\Big(T\log\frac{n}{x}\Big) - R(\pi,\pi') x \frac{\widehat{\phi}(1/T)}{T} g_d^{RS}(1,\pi\times\widetilde{\pi}')\Bigg| \ll_{\varepsilon,m,\phi} d^\varepsilon Q^{\frac{3}{2m}} T^{3/4} x^{1-\frac{1}{m^2}+\frac{1}{m^4}}.$$

(2) *If*

$$z \gg_m Q^{6m}$$

with a sufficiently large implied constant, then

$$\sum_{\substack{n\leq z \\ (n,N_\pi)=1}} \frac{|a_\pi(n)|^2}{n} \geq \frac{R(\pi,\pi)}{20} \log z + \frac{1}{2}.$$

(3) $R(\pi,\pi') > 0$ *when* $\pi = \pi'$ *and* $R(\pi,\pi') = 0$ *otherwise.*

Proof. For (1), the quantity we want to estimate equals, by Lemma 5.2,

$$\frac{1}{2\pi i T} \int_{1-\frac{1}{m^2}+\frac{1}{m^4}-i\infty}^{1-\frac{1}{m^2}+\frac{1}{m^4}+i\infty} L_d^{RS}(s,\pi\times\widetilde{\pi}')\widehat{\phi}(s/T) x^s ds.$$

By Lemma 5.4 and (5.10), the above integral is

$$\begin{aligned}
&\ll_{\varepsilon,m} \frac{x^{1-\frac{1}{m^2}+\frac{1}{m^4}} d^{\varepsilon} Q^{\frac{3}{2m}}}{T} \int_{-\infty}^{\infty} \left|\widehat{\phi}\left(\frac{1-\frac{1}{m^2}+\frac{1}{m^4}+it}{T}\right)\right| (1+|t|)^{3/4} dt \\
&\ll_{\varepsilon,m,\phi} \frac{x^{1-\frac{1}{m^2}+\frac{1}{m^4}} d^{\varepsilon} Q^{\frac{3}{2m}}}{T} \int_{-\infty}^{\infty} \min\left\{1, \frac{T^2}{(|t|+2)^2}\right\} (1+|t|)^{3/4}\, dt \\
&\ll_{\varepsilon,m,\phi} d^{\varepsilon} Q^{\frac{3}{2m}} T^{3/4} x^{1-\frac{1}{m^2}+\frac{1}{m^4}}.
\end{aligned}$$

We proceed to (2). Let

$$\phi(t) = \begin{cases} \exp(16 + t^{-1}(t+1/2)^{-1}) & \text{if } t \in (-1/2, 0), \\ 0 & \text{otherwise,} \end{cases}$$

which is a smooth pointwise lower bound for the indicator function of the interval $[-1/2, 0]$. Observe that if $z \geq 4$, then by Lemmas 5.2 and 5.4, and (5.10),

$$\begin{aligned}
&\sum_{\substack{n\geq 1 \\ (n,N_\pi)=1}} \frac{|a_\pi(n)|^2}{n} \phi\left(\log\frac{n}{z}\right) - R(\pi,\pi)\widehat{\phi}(0) \\
&= \frac{1}{2\pi i} \int_{-\frac{1}{m^2}+\frac{1}{m^4}-i\infty}^{-\frac{1}{m^2}+\frac{1}{m^4}+i\infty} L_d^{RS}(s+1, \pi\times\widetilde{\pi})\widehat{\phi}(s) z^s\, ds \ll_m Q^{\frac{3}{2m}} z^{-\frac{1}{m^2}+\frac{1}{m^4}}.
\end{aligned}$$

The intervals $[2^{-j}e^{-1/2}z, 2^{-j}z]$ and $[2^{-j-1}e^{-1/2}z, 2^{-j-1}z]$ are disjoint for all integers $0 \leq j \leq \frac{\log z}{\log 4}$, so

$$\begin{aligned}
\sum_{\substack{n\leq z \\ (n,N_\pi)=1}} \frac{|a_\pi(n)|^2}{n} &\geq 1 + \sum_{j \leq \frac{\log z}{\log 4}} \sum_{\substack{n\geq 1 \\ (n,N_\pi)=1}} \frac{|a_\pi(n)|^2}{n} \phi\left(\log\frac{n}{z/2^j}\right) \\
&= 1 + \left\lfloor \frac{\log z}{\log 4} \right\rfloor R(\pi,\pi)\widehat{\phi}(0) + O_m(Q^{\frac{3}{2m}} z^{(-\frac{1}{m^2}+\frac{1}{m^4})/2}).
\end{aligned}$$

Since $\widehat{\phi}(0) \geq 1/10$, the result follows once $z \gg_m Q^{6m}$.

For (3), note that $\kappa_{\pi\times\widetilde{\pi}'} > 0$ if and only if $\pi = \pi'$, and $\kappa_{\pi\times\widetilde{\pi}'} = 0$ otherwise. From (5.2) and the fact that $\overline{a_\pi(n)} = a_{\widetilde{\pi}}(n)$, we have $\prod_{p|N_\pi} L^{RS}(1, \pi_p \times \widetilde{\pi}_p)^{-1} > 0$. It remains to show that $H(1, \pi\times\widetilde{\pi}) > 0$. To see this, note that the estimate over composite n in Lemma 4.4 implies that as $x \to \infty$,

$$\begin{aligned}
\log x \sim \sum_{n\in[x,2x]} \frac{\lambda_{\pi\times\widetilde{\pi}}(n)\Lambda(n)}{n} &= \sum_{p\in[x,2x]} \frac{a_{\pi\times\widetilde{\pi}}(p)\log p}{p} + \sum_{\substack{n\in[x,2x] \\ n \text{ composite}}} \frac{\lambda_{\pi\times\widetilde{\pi}}(n)\Lambda(n)}{n} \\
&\sim \sum_{p\in[x,2x]} \frac{a_{\pi\times\widetilde{\pi}}(p)\log p}{p}.
\end{aligned}$$

By partial summation, the sum $\sum_p a_{\pi\times\widetilde{\pi}}(p)/p$ diverges, hence $\sum_p |a_\pi(p)|^2/p$ diverges. Consequently, $\sum_n |a_\pi(n)|^2/n$ diverges, in which case Lemma 5.2 implies that $L^{RS}(s, \pi \times \widetilde{\pi})$ has a positive residue at $s = 1$. Since the same is true for $L(s, \pi \times \widetilde{\pi}) \prod_{p\nmid N_\pi} L(s, \pi_p \times \widetilde{\pi}_p)^{-1}$, the holomorphy of $H(s, \pi \times \widetilde{\pi})$ on the line $\mathrm{Re}(s) = 1$ implies that $H(1, \pi \times \widetilde{\pi}) > 0$. ■

5.3. *Proof of Proposition* 5.1

We begin by constructing Selberg sieve weights for each $\pi \in \mathcal{F}_m(Q)$. Define

$$g_\pi(d) := g_d(1, \pi \times \widetilde{\pi}), \qquad P_\pi(z) := \prod_{\substack{p<z,\, p\nmid N_\pi \\ g_\pi(p)\neq 0}} p, \qquad \mathcal{D}_\pi(z) := \{d\colon d \le z,\ d|P_\pi(z)\},$$

where $g_d(1, \pi \times \widetilde{\pi})$ is given by (5.4). Let $\rho_\pi(d)$ be a real-valued function satisfying

$$\rho_\pi(1) = 1, \qquad \rho_\pi(d) = 0 \text{ unless } d \in \mathcal{D}_\pi(z), \qquad |\rho_\pi(d)| \le 1 \text{ for all } d. \tag{5.12}$$

Our requirements (5.12) for $\rho_\pi(d)$ imply that if the least prime dividing n is greater than z, then the condition $d \mid n$ implies that either $d = 1$ or $\rho_\pi(d) = 0$.

For a given integer q, let $\mathbf{1}_q(n)$ be the indicator function of the integers n such that $(n, q) = 1$. Consider the linear operator A defined by the mapping

$$(b(n))_{n\in(x,xe^{1/T}]} \mapsto \Big(\sum_{n\in(x,xe^{1/T}]} a_\pi(n)\mathbf{1}_{N_\pi}(n)\Big[\sum_{d|(n,P_\pi(z))} \rho_\pi(d)\Big]b(n)\Big)_{\pi\in\mathcal{F}_m(Q)}.$$

It suffices to consider $b(n)$ normalized so that $\sum_{n\in(x,xe^{1/T}]} |b(n)|^2 = 1$. By duality, the square of the operator norm of A equals the square of the operator norm of the adjoint operator A^*, namely

$$C(\mathcal{F}_m, Q, T, x, z) := \sup_{\|\beta\|_2=1} \sum_{n\in(x,xe^{1/T}]} \Big|\sum_{\pi\in\mathcal{F}_m(Q)} a_\pi(n)\mathbf{1}_{N_\pi}(n)\Big[\sum_{d|(n,P_\pi(z))} \rho_\pi(d)\Big]\beta(\pi)\Big|^2, \tag{5.13}$$

Here, the supremum ranges over the functions $\beta\colon \mathcal{F}_m(Q) \to \mathbb{C}$ such that

$$\|\beta\|_2^2 = \sum_{\pi\in\mathcal{F}_m(Q)} |\beta(\pi)|^2 = 1.$$

Since $\sum_{d|(p,P_\pi(z))} \rho_\pi(d) = 1$ for all p such that $p > z$, Proposition 5.1 follows from

$$C(\mathcal{F}_m, Q, T, x, z) \ll_m \frac{x}{T\log z} + Q^{\frac{3}{2m}} T^{3/4} x^{1-\frac{1}{m^2}+\frac{1}{m^4}} z^{2+\varepsilon}\, \#\mathcal{F}_m(Q), \qquad z \gg_m Q^{6m}. \tag{5.14}$$

We proceed to prove (5.14).

Fix a compactly supported, infinitely differentiable function ϕ such that $\phi(t) \ge 1$ for $t \in [0, 1]$ and $\phi(t) \ge 0$ otherwise. Then $\phi\big(T \log \frac{n}{x}\big)$ is a nonnegative upper bound for the indicator function of the interval $(x, xe^{1/T}]$. Thus (5.13) is bounded above by the supremum over all β with $\|\beta\|_2 = 1$ of

$$\sum_{n\geq 1}\Big|\sum_{\pi\in\mathcal{F}_m(Q)} a_\pi(n)\mathbf{1}_{N_\pi}(n)\Big[\sum_{d\mid(n,P_\pi(z))}\rho_\pi(d)\Big]\beta(\pi)\Big|^2\phi\Big(T\log\frac{n}{x}\Big). \tag{5.15}$$

We expand the square and swap the order of summation so that (5.15) equals

$$\begin{aligned}
&\sum_{\pi,\pi'\in\mathcal{F}_m(Q)}\beta(\pi)\overline{\beta(\pi')}\Big[\sum_{(n,N_\pi N_{\pi'})=1} a_\pi(n)a_{\widetilde{\pi}'}(n)\Big[\sum_{d\mid(n,P_\pi(z))}\rho_\pi(d)\Big]\\
&\qquad\qquad\qquad\times\Big[\sum_{d\mid(n,P_{\pi'}(z))}\rho_{\pi'}(d)\Big]\phi\Big(T\log\frac{n}{x}\Big)\Big]\\
&=\sum_{\pi,\pi'\in\mathcal{F}_m(Q)}\beta(\pi)\overline{\beta(\pi')}\\
&\qquad\times\sum_{\substack{d\in\mathcal{D}_\pi(z)\\ d'\in\mathcal{D}_{\pi'}(z)}}\rho_\pi(d)\rho_{\pi'}(d')\Big[\sum_{\substack{[d,d']\mid n\\ (n,N_\pi N_{\pi'})=1}} a_\pi(n)a_{\widetilde{\pi}'}(n)\phi\Big(T\log\frac{n}{x}\Big)\Big].
\end{aligned}\tag{5.16}$$

We use Lemma 5.5 and (5.12) to conclude that (5.16) equals

$$\begin{aligned}
&x\frac{\widehat{\phi}(1/T)}{T}\sum_{\pi\in\mathcal{F}_m(Q)}|\beta(\pi)|^2R(\pi,\pi)\sum_{d,d'\in\mathcal{D}_\pi(z)}\rho_\pi(d)\rho_\pi(d')g_\pi([d,d'])\\
&+O_{m,\varepsilon}\Big(Q^{\frac{3}{2m}}T^{3/4}x^{1-\frac{1}{m^2}+\frac{1}{m^4}}\sum_{\pi,\pi'\in\mathcal{F}_m(Q)}|\beta(\pi)\beta(\pi')|\sum_{\substack{d\in\mathcal{D}_\pi(z)\\ d'\in\mathcal{D}_{\pi'}(z)}}|\rho_\pi(d)\rho_{\pi'}(d')|[d,d']^{\varepsilon/2}\Big)\\
&=x\frac{\widehat{\phi}(1/T)}{T}\sum_{\pi\in\mathcal{F}_m(Q)}|\beta(\pi)|^2R(\pi,\pi)\sum_{d,d'\in\mathcal{D}_\pi(z)}\rho_\pi(d)\rho_\pi(d')g_\pi([d,d'])\\
&\quad+O_{m,\varepsilon}\Big(Q^{\frac{3}{2m}}T^{3/4}x^{1-\frac{1}{m^2}+\frac{1}{m^4}}z^{2+\varepsilon}\sum_{\pi,\pi'\in\mathcal{F}_m(Q)}|\beta(\pi)\beta(\pi')|\Big).
\end{aligned}\tag{5.17}$$

By proceeding as in the formulation of the Selberg sieve in [21, Theorem 7.1], we find that for each $\pi\in\mathcal{F}_m(Q)$, there exists a choice of $\rho_\pi(d)$ satisfying (5.12) such that

$$\begin{aligned}
\sum_{d,d'\in\mathcal{D}_\pi(z)}\rho_\pi(d)\rho_\pi(d')g([d_1,d_2]) &= \sum_{\substack{d\leq z^2\\ d\mid P_\pi(z)}}\prod_{p\mid d}\frac{g_\pi(p)}{1-g_\pi(p)}\\
&\leq\Big(\sum_{\substack{n\leq z\\ (n,N_\pi)=1}}\frac{|a_\pi(n)|^2}{n}\Big)^{-1}.
\end{aligned}$$

Hence (5.17) is

$$\leq x\frac{\widehat{\phi}(1/T)}{T}\sum_{\pi\in\mathcal{F}_m(Q)}|\beta(\pi)|^2\frac{R(\pi,\pi)}{\sum_{n\leq z,\,(n,N_\pi)=1}\frac{|a_\pi(n)|^2}{n}}$$
$$+\,O_{m,\varepsilon}\Big(Q^{\frac{3}{2m}}T^{3/4}x^{1-\frac{1}{m^2}+\frac{1}{m^4}}z^{2+\varepsilon}\sum_{\pi,\pi'\in\mathcal{F}_m(Q)}|\beta(\pi)\beta(\pi')|\Big). \tag{5.18}$$

Since $\|\beta\|_2 = 1$, the inequality of arithmetic and geometric means implies that (5.18) equals

$$x\frac{\widehat{\phi}(1/T)}{T}\sum_{\pi\in\mathcal{F}_m(Q)}|\beta(\pi)|^2\frac{R(\pi,\pi)}{\sum_{n\leq z,\,(n,N_\pi)=1}\frac{|a_\pi(n)|^2}{n}}$$
$$+\,O_{m,\varepsilon}(Q^{\frac{3}{2m}}T^{3/4}x^{1-\frac{1}{m^2}+\frac{1}{m^4}}z^{2+\varepsilon}\,\#\mathcal{F}_m(Q)). \tag{5.19}$$

By Lemma 5.5, the fact that $R(\pi,\pi) > 0$, and the upper bound $\widehat{\phi}(1/T) \ll 1$ from (5.10), we find that if $z \gg_m Q^{6m}$ with a sufficiently large implied constant, then

$$\frac{R(\pi,\pi)}{\sum_{n\leq z,\,(n,N_\pi)=1}\frac{|a_\pi(n)|^2}{n}} \leq \frac{R(\pi,\pi)}{R(\pi,\pi)\frac{\log z}{20}+\frac{1}{2}} \ll \frac{1}{\log z}.$$

This establishes the bound (5.14), which concludes the proof of Proposition 5.1.

5.4. *Mean values of Dirichlet polynomials*

Using Proposition 5.1, we bound the mean value of the Dirichlet polynomial appearing as the integrand in Proposition 4.1.

Proposition 5.6. *Suppose that each $\mathcal{F}_m(Q)$ satisfies Hypothesis 1.1, and let $\pi_0 \in \mathcal{A}(m_0)$. Let $Q \geq 3$, $T \geq 1$, and $y \geq c_m(C(\pi_0)QT\,\#\mathcal{F}_m(Q))^{32(m_0m)^3}$, where $c_m > 0$ is a sufficiently large constant depending at most on m. For any $u \in [y, y^{12000}]$,*

$$\sum_{\pi\in\mathcal{F}_m(Q)}\int_{-T}^{T}\Big|\sum_{y<p\leq u}\frac{\lambda_{\pi\times\pi_0}(p)\log p}{p^{1+it}}(1+e_\chi\chi(p)p^{\beta_\chi-1})\Big|^2dt$$
$$\ll_m \sum_{y<p\leq u}\frac{\lambda_{\pi_0\times\tilde{\pi}_0}(p)(1+e_\chi\chi(p)p^{\beta_\chi-1})^2\log p}{p}.$$

Proof. A result of Gallagher [23, Theorem 1] states that for any sequence of complex numbers a_n and any $T \geq 1$, we have

$$\int_{-T}^{T}\Big|\sum_{n\geq 1}a_n n^{-it}\Big|^2dt \ll T^2\int_0^\infty\Big|\sum_{x<n\leq xe^{1/T}}a_n\Big|^2\frac{dx}{x}.$$

Assume $z \geq c_m Q^{6m}$ with c_m sufficiently large. If $b(n)$ is as in Proposition 5.1, then the above result with $a_n = b(n)a_\pi(n)$ yields the bound

$$\sum_{\pi \in \mathcal{F}_m(Q)} \int_{-T}^{T} \Big|\sum_{p>z} a_\pi(p)b(p)p^{-it}\Big|^2 dt \ll T^2 \int_0^\infty \sum_{\pi \in \mathcal{F}_m(Q)} \Big| \sum_{\substack{x<p\leq xe^{1/T} \\ p>z}} a_\pi(p)b(p)\Big|^2 \frac{dx}{x}.$$

We apply Proposition 5.1 and bound the right hand side of the above display by

$$\ll_{\varepsilon,m} T^2 \int_0^\infty \Big(\frac{x}{T\log z} + Q^{\frac{3}{2m}} T^{3/4} x^{1-\frac{1}{m^2}+\frac{1}{m^4}} z^{2+\varepsilon} \#\mathcal{F}_m(Q)\Big) \sum_{\substack{x<p\leq xe^{1/T} \\ p>z}} |b(p)|^2 \frac{dx}{x}$$

$$\ll_{\varepsilon,m} \sum_{p>z} |b(p)|^2 p \Big(\frac{1}{\log z} + p^{-\frac{1}{m^2}+\frac{1}{m^4}} Q^{\frac{2}{3m}} T^{\frac{7}{4}} z^{2+\varepsilon} \#\mathcal{F}_m(Q)\Big)$$

$$\ll_m \frac{1}{\log z} \sum_{p>z} |b(p)|^2 p (1 + p^{-\frac{1}{m^2}+\frac{1}{m^4}} Q^{\frac{2}{3m}} T^{\frac{7}{4}} z^3 \#\mathcal{F}_m(Q)).$$

Choose y such that $y \geq c_m(C(\pi_0)QT\#\mathcal{F}_m(Q))^{32(m_0m)^3}$ and $z = y^{1/(5m^2)}$, and choose $b(p)$ to be supported on the primes $p > y$. Then the above display is

$$\ll_m (1 + Q^{\frac{3}{2m}} T^{7/4} y^{-\frac{1}{m^2}+\frac{1}{m^4}} z^3 \#\mathcal{F}_m(Q)) \frac{1}{\log z} \sum_{p>y} |b(p)|^2 p.$$

By our assumptions on y and z, we have $z \geq c_m Q^{6m}$. It follows that

$$\sum_{\pi \in \mathcal{F}_m(Q)} \int_{-T}^{T} \Big|\sum_{p>y} b(p)a_\pi(p)p^{-it}\Big|^2 dt \ll_m \frac{1}{\log y} \sum_{p>y} |b(p)|^2 p. \tag{5.20}$$

Now, select

$$b(p) = \begin{cases} a_{\pi_0}(p)(1 + e_\chi \chi(p)p^{\beta_\chi - 1})\frac{\log p}{p} & \text{if } y < p \leq u, \\ 0 & \text{otherwise.} \end{cases}$$

Since $y > C(\pi)C(\pi_0)$ for any $\pi \in \mathcal{F}_m(Q)$, we see by (3.5) that

$$a_\pi(p)a_{\pi_0}(p) = \lambda_{\pi\times\pi_0}(p), \qquad |a_{\pi_0}(p)|^2 = \lambda_{\pi_0\times\tilde{\pi}_0}(p)$$

for every $p > y$. Therefore, we may conclude from (5.20) that if $u \in [y, y^{12000}]$, then

$$\begin{aligned} \sum_{\pi \in \mathcal{F}_m(Q)} \int_{-T}^{T} \Big|\sum_{y<p\leq u} \frac{\lambda_{\pi\times\pi_0}(p)\log p}{p^{1+it}}(1 + e_\chi\chi(p)p^{\beta_\chi-1})\Big|^2 dt \\ \ll_m \frac{1}{\log y} \sum_{y<p\leq u} \frac{\lambda_{\pi_0\times\tilde{\pi}_0}(p)(1+e_\chi\chi(p)p^{\beta_\chi-1})^2(\log p)^2}{p} \\ \ll_m \sum_{y<p\leq u} \frac{\lambda_{\pi_0\times\tilde{\pi}_0}(p)(1+e_\chi\chi(p)p^{\beta_\chi-1})^2 \log p}{p}, \end{aligned}$$

as desired. ∎

6. Proofs of Theorems 1.3 and 1.7 and the rarity of Landau–Siegel zeros

We now begin the proofs of Theorems 1.3 and 1.7, both of which use Propositions 4.1 and 5.6. Theorem 2.5 will follow as a straightforward consequence of Theorems 1.3 and 1.7. The proofs of Theorems 1.3 and 1.7 run parallel for the most part and deviate only at the very end.

Let η satisfy (4.4), and let $\tau \in \mathbb{R}$ satisfy $|\tau| \leq T$. In order to simultaneously satisfy Propositions 4.1 and 5.6, we choose

$$K = 240000(m_0 m)^3 \eta \log(C(\pi_0) qQT \, \#\mathcal{F}_m(Q)) + O_{m_0,m}(1), \tag{6.1}$$

where the implied constant is sufficiently large. Lemma 4.3 implies that there are $\ll (m_0 m)^2 \log(C(\pi_0)QT)$ zeros of $L(s, \pi \times \pi_0)$ satisfying $|\rho - (1 + i\tau)| \leq \eta$. Thus if $L(s, \pi \times \pi_0)$ has a zero ρ_0 (not equal to β_χ if $e_\chi = 1$) such that $|\rho_0 - (1 + i\tau)| \leq \eta$, then by Proposition 4.1,

$$\begin{aligned}
&\frac{\#\{\rho = \beta + i\gamma \colon \beta \geq 1 - \eta/2,\ |\gamma - \tau| \leq \eta/2\}}{(m_0 m)^2 \eta \log(C(\pi_0)QT)} \\
&\ll (200)^{4K} \Big[\eta^3 \int_{A_1}^{A_2} \Big| \sum_{A_1 < p \leq u} \frac{\lambda_{\pi \times \pi_0}(p) \log p}{p^{1+i\tau}} (1 + e_\chi \chi(p) p^{\beta_\chi - 1}) \Big|^2 \frac{du}{u} \\
&\qquad + r_{\pi \times \pi_0} \mathbf{1}(\tau) \min\Big\{1, \frac{1 - \beta_\chi}{\eta}\Big\} \Big],
\end{aligned}$$

where A_1 and A_2 are as in (4.7). (The zero β_χ is not counted on the left hand side when $e_\chi = 1$.) We integrate both sides over $|\tau| \leq T$ and use the bound

$$(m_0 m)^2 \eta \log(C(\pi_0)QT) \ll K$$

to conclude that $N_{\pi \times \pi_0}(1 - \eta/2, T)$ (excluding β_χ when $e_\chi = 1$) is

$$\begin{aligned}
&\ll (201)^{4K} \Big[\eta^2 \int_{-T}^{T} \int_{A_1}^{A_2} \Big| \sum_{A_1 \leq p \leq u} \frac{\lambda_{\pi \times \pi_0}(p) \log p}{p^{1+i\tau}} (1 + e_\chi \chi(p) p^{\beta_\chi - 1}) \Big|^2 \frac{du \, d\tau}{u} \\
&\qquad + r_{\pi \times \pi_0} \min\Big\{1, \frac{1 - \beta_\chi}{\eta}\Big\} \Big].
\end{aligned}$$

We now sum over $\pi \in \mathcal{F}_m(Q)$. Since $r_{\pi \times \pi_0} = 1$ for at most one $\pi \in \mathcal{F}_m(Q)$, it follows from Proposition 5.6 that

$$\begin{aligned}
&\sum_{\pi \in \mathcal{F}_m(Q)} N_{\pi \times \pi_0}(1 - \eta/2, T) \\
&\ll_m (201)^{4K} \\
&\times \Big[\eta^2 \int_{A_1}^{A_2} \sum_{A_1 < p \leq u} \frac{\lambda_{\pi_0 \times \tilde{\pi}_0}(p)(1 + e_\chi \chi(p) p^{\beta_\chi - 1})^2 \log p}{p} \frac{du}{u} + \min\Big\{1, \frac{1 - \beta_\chi}{\eta}\Big\} \Big]
\end{aligned}$$

$$\begin{aligned}
&\ll_m (201)^{4K}\\
&\quad\times\Big[\eta^2\log\frac{A_2}{A_1}\sum_{A_1<p\le A_2}\frac{\lambda_{\pi_0\times\tilde\pi_0}(p)(1+e_\chi\chi(p)p^{\beta_\chi-1})^2\log p}{p}+\min\Big\{1,\frac{1-\beta_\chi}{\eta}\Big\}\Big]\\
&\ll_{m,m_0} (202)^{4K}\Big[\eta\sum_{A_1<p\le A_2}\frac{\lambda_{\pi_0\times\tilde\pi_0}(p)(1+e_\chi\chi(p)p^{\beta_\chi-1})^2\log p}{p}+\min\Big\{1,\frac{1-\beta_\chi}{\eta}\Big\}\Big]
\end{aligned}\tag{6.2}$$

where we identify $\min\{1,(1-\beta_\chi)/\eta\}$ with 1 when $e_\chi=0$ and omit β_χ from the count when $e_\chi=1$.

Proof of Theorem 1.3. Let $q=1$, so $1+e_\chi\chi(p)p^{\beta_\chi-1}=1$ and $\min\{1,(1-\beta_\chi)/\eta\}=1$. It follows from Lemma 4.2 (with $\eta=1/\log A_2$) that

$$\sum_{A_1<p\le A_2}\frac{\lambda_{\pi_0\times\tilde\pi_0}(p)\log p}{p}\ll_{m_0}\frac{K}{\eta}.$$

It follows that (6.2) is $\ll_{m,m_0}(203)^{4K}$. Unraveling our choice of K in (6.1) and choosing $\sigma=1-\eta/2$, we find that

$$\sum_{\pi\in\mathcal{F}_m(Q)}N_{\pi\times\pi_0}(\sigma,T)\ll_{m_0,m}(C(\pi_0)QT\,\#\mathcal{F}_m(Q))^{2.05\cdot10^7(m_0m)^3(1-\sigma)}\tag{6.3}$$

when $1-\frac{1}{2\cdot10^7(m_0m)^2}\le\sigma\le1-\frac{1}{2\log(C(\pi_0)QT)}$. If $\sigma>1-\frac{1}{2\log(C(\pi_0)QT)}$, then we still achieve (6.3) by the bound

$$\sum_{\pi\in\mathcal{F}_m(Q)}N_{\pi\times\pi_0}(\sigma,T)\le\sum_{\pi\in\mathcal{F}_m(Q)}N_{\pi\times\pi_0}\Big(1-\frac{1}{2\log(C(\pi_0)QT)},T\Big)$$

and Theorem 1.6. If $\sigma\le1-\frac{1}{2\cdot10^7(m_0m)^2}$, then our result is trivial since $N_{\pi\times\pi_0}(1/2,T)\ll_{m_0,m}T\log(C(\pi_0)QT)$ for each $\pi\in\mathcal{F}_m(Q)$ (see [32, Theorem 5.8]). Once we invoke Theorem 1.6 to bound $\#\mathcal{F}_m(Q)$, we have proved Theorem 1.3 in all cases. ∎

Proof of Theorem 1.7. Let $\chi\ (\mathrm{mod}\ q)$ be a primitive quadratic nontrivial character modulo $q\le Q$ such that $L(s,\chi)$ has a real zero $\beta_\chi\in(1/2,1)$. Since Theorem 1.7 follows from Theorem 1.3 when $(1-\beta_\chi)\log(C(\pi_0)QT)>c_{m,m_0}$, we may assume without loss of generality that $(1-\beta_\chi)\log(C(\pi_0)QT)\le c_{m,m_0}$ (with c_{m,m_0} as in Corollary 4.6), in which case $e_\chi=1$ and $\min\{1,(1-\beta_\chi)/\eta\}=(1-\beta_\chi)/\eta$. We will prove that

$$\sum_{A_1<p\le A_2}\frac{\lambda_{\pi_0\times\tilde\pi_0}(p)(1+\chi(p)p^{\beta_\chi-1})^2\log p}{p}\ll\frac{K^2}{\eta}\cdot\frac{1-\beta_\chi}{\eta}.\tag{6.4}$$

Once we insert (6.4) into (6.2), the proof proceeds just as for Theorem 1.3. (The verification that our estimate is trivial for $\sigma\le1-\frac{1}{2\cdot10^7(m_0m)^2}$ uses the bound $1-\beta_\chi\gg Q^{-1/2}$.) Our proof of (6.4) is a modification of an idea due to Bombieri [5, §6].

To prove (6.4), we begin with an application of Taylor's theorem:

$$(1+\chi(p)p^{\beta_\chi-1})^2 \ll 1+\chi(p)+(1-\beta_\chi)\log p.$$

Therefore, by Lemma 4.2, (6.4) is

$$\begin{aligned}
&\ll \sum_{A_1<p\leq A_2}\frac{\lambda_{\pi_0\times\tilde{\pi}_0}(p)(1+\chi(p))\log p}{p}+(1-\beta_\chi)\sum_{A_1<p\leq A_2}\frac{\lambda_{\pi_0\times\tilde{\pi}_0}(p)(\log p)^2}{p}\\
&\ll_{m,m_0}\frac{K}{\eta}\Big(\sum_{A_1<p\leq A_2}\frac{\lambda_{\pi_0\times\tilde{\pi}_0}(p)(1+\chi(p))}{p}+(1-\beta_\chi)\sum_{A_1<p\leq A_2}\frac{\lambda_{\pi_0\times\tilde{\pi}_0}(p)\log p}{p}\Big)\\
&\ll_{m,m_0}\frac{K}{\eta}\Big(\sum_{A_1<p\leq A_2}\frac{\lambda_{\pi_0\times\tilde{\pi}_0}(p)(1+\chi(p))}{p}+\frac{K}{\eta}(1-\beta_\chi)\Big).
\end{aligned}$$

Thus the bound (6.4) follows from the bound

$$\sum_{A_1<p\leq A_2}\frac{\lambda_{\pi_0\times\tilde{\pi}_0}(p)(1+\chi(p))}{p}\ll_{m,m_0}\frac{K}{\eta}(1-\beta_\chi), \tag{6.5}$$

which we will now prove.

Recall that $\mathbb{1}$ is the trivial representation in $\mathcal{A}(1)$, and set $\Pi := \pi_0\times(\tilde{\pi}_0\otimes(\mathbb{1}\boxplus\chi))$. Note that $L(s,\Pi)=L(s,\pi_0\times\tilde{\pi}_0)L(s,\pi_0\times(\tilde{\pi}_0\otimes\chi))$, and for all $p\in(A_1,A_2]$, we have $a_\Pi(p)=\lambda_{\pi_0\times\tilde{\pi}_0}(p)(1+\chi(p))$. It is important that $L(s,\Pi)$ has a pole of order 1 at $s=1$, and

$$\kappa_\Pi := \operatorname*{Res}_{s=1} L(s,\Pi)\gg_{m_0}\frac{1}{(C(\pi_0)q)^{11m_0^2}} \tag{6.6}$$

per (3.7) and [43, Theorem A.1]. We restate our goal in (6.5) as

$$\sum_{A_1<p\leq A_2}\frac{a_\Pi(p)}{p}\ll_{m,m_0}\frac{K}{\eta}(1-\beta_\chi). \tag{6.7}$$

The function $a_\Pi(n)$ is multiplicative, and it is also nonnegative.[4] Since every integer $n\in[1,A_1]$ is coprime to every prime $p\in(A_1,A_2]$, it follows that

$$\Big(\sum_{n\leq A_1}\frac{a_\Pi(n)}{n}\Big)\Big(\sum_{A_1<p\leq A_2}\frac{a_\Pi(p)}{p}\Big)=\sum_{\substack{A_1<p\leq A_2\\ n\leq A_1}}\frac{a_\Pi(np)}{np}\leq\sum_{A_1\leq n\leq A_1A_2}\frac{a_\Pi(n)}{n}.$$

Let κ_Π denote the residue at $s=1$ for $L(s,\Pi)$. The bound (6.7), and hence (6.4) and Theorem 1.7, follows from the two bounds

$$\sum_{A_1\leq n\leq A_1A_2}\frac{a_\Pi(n)}{n}\ll_{m,m_0}\kappa_\Pi\frac{K}{\eta},\qquad \sum_{n\leq A_1}\frac{a_\Pi(n)}{n}\gg_{m,m_0}\frac{\kappa_\Pi}{1-\beta_\chi}. \tag{6.8}$$

[4]This follows from [1, Proposition 6.9] and [29, Lemma a].

The first bound in (6.8) follows from a straightforward contour integral estimate:

$$\begin{aligned}\sum_{A_1\leq n\leq A_1A_2}\frac{a_\Pi(n)}{n}&\ll\sum_{n=1}^{\infty}\frac{a_\Pi(n)}{n}(e^{-\frac{n}{A_1A_2}}-e^{-\frac{2n}{A_1}})\\&=\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}L(s+1,\Pi)((A_1A_2)^s-(A_1/2)^s)\Gamma(s)\,ds\\&=\kappa_\Pi\log(2A_2)+\int_{-1/2-i\infty}^{-1/2+i\infty}L(s+1,\Pi)((A_1A_2)^s-(A_1/2)^s)\Gamma(s)\,ds.\end{aligned}$$

By (3.7), (5.9), (6.1), and (4.7), we have the convexity bound

$$|L(1/2+it,\Pi)|A_1^{-1/2}\ll_{m,m_0}(qC(\pi_0))^{-15m_0^3}(3+|t|)^{m_0^2/2}. \tag{6.9}$$

The integral is then $\ll_{m,m_0}(qC(\pi_0))^{-15m_0^3}$, which is majorized by $\kappa_\Pi\log(2A_2)$ because of (6.6). The first bound in (6.8) now follows since $\log(2A_2)\ll_{m,m_0}K/\eta$.

Our proof of the second bound in (6.8) uses the fact that if $1\leq n\leq A_1$ and $\beta_\chi\in(1/2,1)$, then $A_1/n\geq(A_1/n)^{\beta_\chi}$:

$$\begin{aligned}\sum_{n\leq A_1}\frac{a_\Pi(n)}{n}\geq\sum_{n\leq A_1}\frac{a_\Pi(n)}{n^{\beta_\chi}}A_1^{\beta_\chi-1}&\geq\sum_{n\leq A_1}\frac{a_\Pi(n)}{n^{\beta_\chi}}A_1^{\beta_\chi-1}\Big(1-\frac{n}{A_1}\Big)^{3m_0^2}\\&=\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}L(s+\beta_\chi,\Pi)\frac{A_1^{s+\beta_\chi-1}}{s(s+1)\cdots(s+3m_0^2)}\,ds.\end{aligned}$$

We push the contour to the line $\mathrm{Re}(s)=1/2-\beta_\chi$, picking up residues at $s=1-\beta_\chi$ and $s=0$:

$$\begin{aligned}&\frac{\kappa_\Pi}{(1-\beta_\chi)\prod_{j=1}^{3m_0^2}(1-\beta_\chi+j)}+\frac{A_1^{\beta_\chi-1}}{(3m_0^2)!}L(\beta_\chi,\Pi)\\&\qquad+\int_{1/2-\beta_\chi-i\infty}^{1/2-\beta_\chi+i\infty}L(s+\beta_\chi,\Pi)\frac{A_1^{s+\beta_\chi-1}}{s(s+1)\cdots(s+3m_0^2)}.\end{aligned}$$

Our hypothesis that $L(\beta_\chi,\pi_0\times(\tilde{\pi}_0\otimes\chi))=0$ implies that $L(\beta_\chi,\Pi)=0$. We invoke (6.9) to bound the integral. Since $\beta_\chi\in(1/2,1)$, it follows that

$$\sum_{n\leq A_1}\frac{a_\Pi(n)}{n}\gg_{m,m_0}\frac{\kappa_\Pi}{1-\beta_\chi}+O_{m,m_0}((qC(\pi_0))^{-15m_0^3}).$$

Since the contribution from κ_Π dominates the second term by (6.6), the second bound in (6.8) follows. ■

Proof of Theorem 2.5. Part (1) (resp. (2)) follows from Theorem 1.3 (resp. Theorem 1.7 and Corollary 4.6) by choosing $\sigma=1-\frac{A}{10^8(m_0m)^4\log(C(\pi_0)QT)}$. ■

7. Subconvexity and mass equidistribution

Proof of Theorem 2.1. Recall the notation and setup of Section 2.1, especially the definition of $\mathscr{G}(Q)$ in (2.1). To each $f \in \mathscr{G}(Q)$, there corresponds a cuspidal automorphic representation $\pi_f \in \mathcal{A}(2)$ with trivial central character. Let $\mathcal{F}$ denote the set of all such π_f, and define $\mathcal{F}_2(Q)$ according to (1.3). Since $L(s,f) = L(s,\pi_f)$, it suffices for us to work with $\mathcal{F}_2(Q)$ instead of $\mathscr{G}(Q)$. We denote by $\pi_0 \in \mathcal{A}(2)$ the representation corresponding to f_0.

Given $\pi \in \mathcal{F}_2(Q)$, let $\mathrm{Ad}^2\pi \in \mathcal{A}(3)$ denote the adjoint square lift of π; then $C(\mathrm{Ad}^2\pi) \asymp \lambda_f q_f^2 \le Q^2$. If $\pi \in \mathcal{F}_2(Q)$ and $\pi_0 \in \mathcal{A}(2)$, then it follows from the uniform bound $|\alpha_{j,\pi}(p)|, |\alpha_{j,\pi_0}(p)| \le p^{7/64}$ that both $L(3/2+it, \mathrm{Ad}^2\pi)$ and $L(3/2, \mathrm{Ad}^2\pi\times\pi_0)$ are defined by absolutely convergent sums which are bounded independently of π and π_0. (The bound $|\alpha_{j,\pi}(p)| \le p^{7/64}$ was proved by Kim and Sarnak [36, Appendix] when p is unramified; the ramified case was handled by Blomer and Brumley [3].) Theorem 1.1 of [61], together with (3.7), now implies that for any $0 \le \delta < 1/2$, we have the bounds

$$\begin{aligned} &\log|L(1/2, \mathrm{Ad}^2\pi\times\pi_0)| \\ &\qquad\le \Big(\frac{1}{4}-\frac{\delta}{10^9}\Big)\log(C(\mathrm{Ad}^2\pi\times\pi_0)) + \frac{\delta}{10^7}N_{\mathrm{Ad}^2\pi\times\pi_0}(1-\delta,6) + O(1) \\ &\qquad\le \Big(\frac{1}{4}-\frac{\delta}{10^9}\Big)\log(\lambda_f^2 q_f^4 \lambda_{f_0}^3 q_{f_0}^3) + \frac{\delta}{10^7}N_{\mathrm{Ad}^2\pi\times\pi_0}(1-\delta,6) + O(1) \end{aligned} \tag{7.1}$$

and

$$\log|L(1/2,\mathrm{Ad}^2\pi)| \le \Big(\frac{1}{4}-\frac{\delta}{10^9}\Big)\log(C(\mathrm{Ad}^2\pi)) + \frac{\delta}{10^7}N_{\mathrm{Ad}^2\pi}(1-\delta,6) + O(1).$$

The effect of replacing $1/2$ with $1/2+it$ is that we add it to the Langlands parameters $\mu_{\mathrm{Sym}^2 f}(j)$. After an application of (3.7), the net effect on the above bound is

$$\begin{aligned} &\log|L(1/2+it,\mathrm{Ad}^2\pi)| \\ &\quad\le \Big(\frac{1}{4}-\frac{\delta}{10^9}\Big)\log(\lambda_f q_f^2(6+|t|)^3) + \frac{\delta}{10^7}N_{\mathrm{Ad}^2\pi}(1-\delta,|t|+6) + O(1). \end{aligned} \tag{7.2}$$

Note that the bound

$$|L(1/2+it,\mathrm{Ad}^2\pi)| \ll (|t|+1)(\lambda_f q_f^2)^{1/4-\delta}$$

follows immediately from (2.2) when $|t|+1 \ge \lambda_f^{4\delta/3} q_f^{8\delta/3}$. Also, the convexity bound

$$L(1/2,\pi_0) \ll \lambda_{f_0}^{1/4} q_{f_0}^{1/4}$$

follows from work of Heath-Brown [27]. Therefore, whenever we can prove the bound

$$L(1/2,\mathrm{Ad}^2\pi\times\pi_0) \ll (\lambda_{f_0}^3 q_{f_0}^3 \lambda_f^2 q_f^4)^{\frac{1}{4}-\frac{\varepsilon/259200000000}{10^9}},$$

it follows that

$$L(1/2, f \times f \times f_0) = L(1/2, \pi_0) L(1/2, \mathrm{Ad}^2 \pi \times \pi_0) \ll (\lambda_{f_0}^4 q_{f_0}^4 \lambda_f^2 q_f^4)^{\frac{1}{4} - \frac{\varepsilon}{10^{21}}}.$$

Consequently, by (7.1) and (7.2), we find for $\varepsilon \in (0, 1)$ that the size of the exceptional set in Theorem 2.1 is

$$\begin{aligned} &\leq \sum_{\pi \in \mathcal{F}_2(Q)} N_{\mathrm{Ad}^2 \pi \otimes \pi_0}\Big(1 - \frac{\varepsilon}{259200000000}, 6\Big) \\ &\quad + \sum_{\pi \in \mathcal{F}_2(Q)} N_{\mathrm{Ad}^2 \pi}\Big(1 - \frac{\varepsilon}{259200000000}, Q + 6\Big). \end{aligned}$$

By the definition of $\mathcal{G}(Q)$, each $\pi \in \mathcal{F}_2(Q)$ has squarefree conductor and trivial central character; it then follows from work of Ramakrishnan [57, Theorem 4.2 and Corollary 4.3] that if $\pi, \pi' \in \mathcal{F}_2(Q)$ and $\mathrm{Ad}^2 \pi = \mathrm{Ad}^2 \pi'$, then $\pi' = \pi$. Therefore, if we let $\mathcal{G}_3(Q^2)$ be the image of $\mathcal{F}_2(Q)$ in $\mathcal{A}(3)$ under the adjoint square lift, then the map $\mathrm{Ad}^2\colon \mathcal{F}_2(Q) \to \mathcal{G}_3(Q^2)$ is bijective. The size of the exceptional set in Theorem 2.1 is now

$$\leq \sum_{\pi \in \mathcal{G}_3(Q^2)} N_{\pi \otimes \pi_0}\Big(1 - \frac{\varepsilon}{259200000000}, 6\Big) + \sum_{\pi \in \mathcal{G}_3(Q^2)} N_{\pi}\Big(1 - \frac{\varepsilon}{259200000000}, Q + 6\Big).$$

By Theorem 1.3, this is

$$\ll (Q^2)^{10^8 \cdot (3 \cdot 2)^4 \cdot \frac{\varepsilon}{259200000000}} + (Q^2 \cdot Q)^{10^8 \cdot (3 \cdot 1)^4 \cdot \frac{\varepsilon}{259200000000}} \ll Q^{\varepsilon},$$

as desired. ■

8. The Chebotarev density theorem in families

The goal of this section is to prove Theorem 2.7. Let $L/\mathbb{Q}$ be a Galois extension of number fields. We begin by establishing a flexible variant of the Chebotarev density theorem. Given any zero-free region for the Dedekind zeta function $\zeta_L(s)$, we would like to compute an asymptotic expression for $\pi_C(x, L/\mathbb{Q})$ with an error term depending on the zero-free region in an explicit form.

Proposition 8.1. *Let $L/\mathbb{Q}$ be a Galois extension of number fields with Galois group G. Let $\Delta\colon [3, \infty) \to (0, \infty)$ be a function such that $\zeta_L(s)/\zeta_{\mathbb{Q}}(s) \neq 0$ in the region* $\mathrm{Re}(s) > 1 - \Delta(|\mathrm{Im}(s)| + 3)$. *Define*

$$\eta(x) = \inf_{t \geq 3} [\Delta(t) \log x + \log t]. \tag{8.1}$$

Let C be a conjugacy class of G, and suppose there exists an abelian subgroup H of G such that $H \cap C$ is nonempty and $\zeta_{L^H}(s)/\zeta_{\mathbb{Q}}(s)$ is entire, where L^H is the subfield of L fixed by H. For $x \geq (\log D_L)^4$,

$$\Big|\pi_C(x, L/\mathbb{Q}) - \frac{|C|}{|G|}\pi(x)\Big| \ll \frac{|C|}{|G|} \frac{x e^{-\frac{1}{8}\eta(x)}}{\log x} \log D_L + \frac{|C|}{|G|} \frac{x^{3/4}}{\log x}.$$

Remark 8.2. The existence of this abelian subgroup H is a mild condition for our purposes. In the special case $C = \{1\}$, one can take $H = \{1\}$ and this follows unconditionally from the Aramata–Brauer theorem as $L^H = L$ is Galois over $\mathbb{Q}$. For an arbitrary conjugacy class C, one can take $H = \langle g \rangle$ to be the cyclic subgroup generated by some element $g \in C$, in which case this assumption follows easily from the strong Artin conjecture for $\zeta_L(s)$ over $\mathbb{Q}$. The strong Artin conjecture is known for all examples under consideration in Theorem 2.7.

Remark 8.3. An analogous result holds for any Galois extension L/F with $\pi(x)$ replaced by the number of prime ideals of F up to x and $\zeta_{\mathbb{Q}}(s)$ replaced by $\zeta_F(s)$. We restrict to $F = \mathbb{Q}$ for simplicity and with Theorem 2.7 in mind.

Proof of Proposition 8.1. For the proof, we will borrow heavily from results recorded in [65] and will therefore remain consistent with the notation therein. Let $g \in H \cap C$ be arbitrary and set $C_H = \{g\}$. Let $K = L^H$ be the fixed field of L by H. Select $f(\cdot) = f(\cdot\,; x, \ell, \varepsilon)$ in [65, Lemma 2.2] with

$$\varepsilon = \min\{1/8, 8e^{-\eta(x)/4}\} + x^{-1/4}, \qquad \ell = 2. \tag{8.2}$$

Note that $0 \le f(t) \le 1$ for all $t \in \mathbb{R}$, $f(t)$ is supported in $\left[\frac{1}{2} - \frac{\varepsilon}{\log x}, 1 + \frac{\varepsilon}{\log x}\right]$ and $f(t) = 1$ for $t \in [1/2, 1]$. Its Laplace transform $F(z) = \int_0^\infty f(t)e^{-zt}\,dt$ is entire and satisfies many properties recorded in [65, Lemma 2.2]. Consider the weighted prime sum $\widetilde{\psi}_{C_H}(x, f) = \widetilde{\psi}_{C_H}(x, L/L^H; f)$ given by [65, (2.13)], or equivalently

$$\widetilde{\psi}_{C_H}(x; f) = \frac{|C_H|}{|H|} \sum_{\chi \in \widehat{H}} \overline{\chi}(C_H) \frac{\log x}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \chi, L/L^H) F(-s\log x)\, ds,$$

where χ runs over all the (Hecke) characters of the dual group $\widehat{H}$. By [65, Lemma 4.3], the bound $\varepsilon \ge x^{-1/4}$ from (8.2), and the bounds $n_L \ll \log D_L \le x^{1/4}$, it follows that

$$\frac{|H|}{|C_H|} \frac{\widetilde{\psi}_{C_H}(x; f)}{\log x} = F(-\log x) - \sum_{\chi \in \widehat{H}} \overline{\chi}(C_H) \sum_{\rho_\chi} F(-\rho_\chi \log x) + O\Big(\frac{x^{1/2}}{\log x}\Big), \tag{8.3}$$

where ρ_χ runs over all nontrivial zeros of the Hecke L-functions $L(s, \chi, L/L^H)$. Note

$$\zeta_L(s) = \zeta_K(s) \prod_{\substack{\chi \in \widehat{H} \\ \chi \ne 1}} L(s, \chi, L/L^H)$$

and, by assumption, $\zeta_K(s)/\zeta_{\mathbb{Q}}(s)$ is entire. Therefore, in (8.3), the zeros of $\zeta_{\mathbb{Q}}(s)$ only contribute to the zeros of the Hecke L-function associated to the trivial character $\chi = 1$. From these observations, it follows that

$$F(-\log x) - \sum_{\chi \in \widehat{H}} \overline{\chi}(C_H) \sum_{\rho_\chi} F(-\rho_\chi \log x) = S(x) + O\Big(\sum_{\substack{\rho \\ \frac{\zeta_L}{\zeta_{\mathbb{Q}}}(\rho)=0}} |F(-\rho\log x)|\Big), \tag{8.4}$$

where

$$S(x) = F(-\log x) - \sum_{\substack{\rho \\ \zeta_{\mathbb{Q}}(\rho)=0}} F(-\rho \log x).$$

By standard arguments using Mellin inversion, one can verify that

$$\begin{aligned}(\log x)S(x) &= \sum_{n\geq 1} \Lambda(n) f\left(\frac{\log n}{\log x}\right) + (\log x)F(0) \\ &\quad - \frac{\log x}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} -\frac{\zeta'_{\mathbb{Q}}}{\zeta_{\mathbb{Q}}}(s)F(-s\log x)\,ds. \end{aligned} \tag{8.5}$$

By [65, Lemma 2.2(iv)], $|F(0)| \ll 1$. From the properties of f described immediately following (8.2) and the prime number theorem,

$$\sum_{n\geq 1} \Lambda(n) f\left(\frac{\log n}{\log x}\right) = \sum_{n\leq x} \Lambda(n) + O(\varepsilon x + x^{1/2}).$$

For $\mathrm{Re}(s) = -1/2$, we have that

$$-\frac{\zeta'_{\mathbb{Q}}}{\zeta_{\mathbb{Q}}}(s) \ll \log(|\mathrm{Im}(s)| + 3), \qquad (\log x)|F(-s\log x)| \ll \varepsilon^{-2}x^{-1/4} \ll x^{1/4},$$

which follow from [65, Lemmas 2.2(vi) and 2.6] and (8.2). Combining all of these observations with (8.5) and noting $\varepsilon \ll e^{-\eta(x)/4} + x^{-1/4}$ by (8.2), it follows that

$$(\log x)S(x) = \sum_{n\leq x} \Lambda(n) + O(xe^{-\eta(x)/4} + x^{3/4}). \tag{8.6}$$

All that remains is to consider the error term in (8.4). By [65, Lemma 4.4] and the assumption $\log D_L \leq x^{1/4}$, the zeros ρ with $|\rho| \leq 1/4$ have negligible contribution; namely,

$$\sum_{\substack{\rho \\ \frac{\zeta_L}{\zeta_{\mathbb{Q}}}(\rho)=0}} |F(-\rho\log x)| = \sum_{\substack{|\rho|\geq 1/4 \\ \frac{\zeta_L}{\zeta_{\mathbb{Q}}}(\rho)=0}} |F(-\rho\log x)| + O(x^{1/2}) \qquad \text{for } x \geq 3.$$

Write $\rho = \beta + i\gamma$ for each nontrivial zero ρ. By (8.1), one can see that $\frac{x^{-(1-\beta)}}{|\gamma|+3} \leq e^{-\eta(x)}$. Thus, [65, Lemma 2.2(iv)] and (8.2) imply that, for $|\rho| \geq 1/4$,

$$(\log x)|F(-\rho\log x)| \ll \frac{x^{\beta}}{|\gamma|+3} \cdot \frac{\varepsilon^{-2}}{(|\gamma|+3)^2} \ll xe^{-\eta(x)} \cdot \frac{e^{\eta(x)/2}}{(|\gamma|+3)^3}.$$

Summing over all such zeros, it follows that

$$\sum_{\substack{|\rho|\geq 1/4 \\ \frac{\zeta_L}{\zeta_{\mathbb{Q}}}(\rho)=0}} |F(-\rho\log x)| \ll \frac{xe^{-\eta(x)/2}}{\log x} \sum_{\frac{\zeta_L}{\zeta_{\mathbb{Q}}}(\rho)=0} \frac{1}{(|\gamma|+3)^3}.$$

Applying a standard estimate for the zeros of the Dedekind zeta function [65, Lemma 2.5] and Minkowski's bound $n_L \ll \log D_L$, we see that the above expression is

$$\ll \frac{xe^{-\eta(x)/2}}{\log x} \sum_{T=1}^{\infty} \sum_{\substack{\frac{\zeta_L}{\zeta_{\mathbb{Q}}}(\rho)=0 \\ T-1\le|\gamma|<T}} \frac{\log D_L + n_L \log(T+3)}{T^3} \ll \frac{xe^{-\eta(x)/2}\log D_L}{\log x}. \tag{8.7}$$

Substituting (8.7), (8.6), and (8.4) into (8.3), we conclude that

$$\frac{|H|}{|C_H|}\widetilde{\psi}_{C_H}(x;f) = \sum_{n\le x}\Lambda(n) + O(xe^{-\eta(x)/4}\log D_L + x^{3/4}) \quad \text{for } x \ge (\log D_L)^4.$$

Via [65, Lemma 2.3], we may replace $\widetilde{\psi}_{C_H}(x;f)$ by the usual prime counting function $\psi_{C_H}(x)$ given by [65, (2.1)] at the cost of $O(\varepsilon x + x^{1/2})$. From (8.2), this cost is absorbed into the existing error term in the above expression. By partial summation (see [65, Lemma 2.1 and (5.3)]), it therefore follows that

$$\frac{|H|}{|C_H|}\pi_{C_H}(x) = \pi(x) + O\Big((\log D_L)\frac{x}{\log x}\sup_{\sqrt{x}\le y\le x} e^{-\eta(y)/4} + \frac{x^{3/4}}{\log x} + \log D_L\Big).$$

By (8.1), one can verify that $\eta(y)$ is an increasing function of y and also $\eta(x^{1/2}) \ge \frac{1}{2}\eta(x)$. With these observations and the assumption $\log D_L \le x^{1/4}$, we conclude that

$$\pi_{C_H}(x) = \frac{|C_H|}{|H|}\pi(x) + O\Big(\frac{|C_H|}{|H|}\frac{x}{\log x}e^{-\eta(x)/8}\log D_L + \frac{|C_H|}{|H|}\frac{x^{3/4}}{\log x}\Big).$$

Proposition 8.1 now follows by an application of [65, Lemma 5.2] from class field theory. To absorb the arising secondary error term, we again use $n_L \ll \log D_L \le x^{1/4}$. ∎

First, we record a classical zero-free region for $\zeta_L(s)$ [40, Lemma 2.3].

Lemma 8.4. *The Dedekind zeta function $\zeta_L(s)$ has at most one zero in the region*

$$\mathrm{Re}(s) > 1 - \frac{c_5}{\log D_L + n_L \log(|\mathrm{Im}(s)|+3)}.$$

If the exceptional zero exists, then it must be both real and simple.

Assuming a strong zero-free region for the Dedekind zeta function, we arrive at a natural form of the Chebotarev density theorem.

Theorem 8.5. *Let $L/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and $L \ne \mathbb{Q}$. Let C be a conjugacy class of G satisfying the hypotheses of Proposition 8.1. Let $0 < \delta \le 1/2$ and $T \ge (\log D_L)^{24}$ be arbitrary. Assume $\zeta_L(s)/\zeta_{\mathbb{Q}}(s)$ has no zeros in the region*

$$\mathrm{Re}(s) > 1-\delta, \qquad |\mathrm{Im}(s)| \le T. \tag{8.8}$$

For $x \geq (\log D_L)^{16/\delta}$,

$$\Big|\pi_C(x, L/\mathbb{Q}) - \frac{|C|}{|G|}\pi(x)\Big| \ll \frac{|C|}{|G|}\frac{x}{\log x}\big(x^{-\delta/8} + T^{-1/24}e^{-\frac{1}{24}\sqrt{c_5(\log x)/n_L}} + T^{-1/24}e^{-\frac{1}{24}\frac{c_5\log x}{\log D_L}}\big).$$

Proof. By Proposition 8.1 and Lemma 8.4, it remains to compute $\eta(x)$ for

$$\Delta(t) = \begin{cases} \delta, & 3 \leq t \leq T, \\ c_5(\log D_L + n_L \log t)^{-1}, & t > T. \end{cases}$$

Define $\eta(x) = \min\{\eta_1(x), \eta_2(x)\}$, where

$$\eta_1(x) = \inf_{3 \leq t \leq T}(\delta \log x + \log t) \quad \text{and} \quad \eta_2(x) = \inf_{t \geq T}\Big(\frac{c_5 \log x}{\log D_L + n_L \log t} + \log t\Big).$$

If $\eta(x) = \eta_1(x)$, then $\eta(x) \geq \delta \log x$. Otherwise, we may assume $\eta(x) = \eta_2(x)$. Arguing as in [65, Lemma 4.6], the expression $\frac{c_5 \log x}{\log D_L + n_L u} + u$ is positive for $u \geq 0$ and is globally minimized in this interval at $u = \max\{0, u_0\}$ where $u_0 = (c_5 \log x)^{1/2}/n_L^{1/2} - (\log D_L)/n_L$. Therefore,

$$\eta(x) = \eta_2(x) \geq \min\Big\{\frac{c_5 \log x}{\log D_L}, \sqrt{\frac{c_5 \log x}{n_L}}\Big\}.$$

Since one always has the lower bound $\eta_2(x) \geq \log T \geq 24 \log\log D_L$, we see in all cases that

$$\begin{aligned} e^{-\eta(x)/8} &\leq e^{-\eta_1(x)/8} + e^{-\eta_2(x)/8} \leq x^{-\delta/8} + e^{-\eta_2(x)/24}T^{-1/24}(\log D_L)^{-1} \\ &\leq (\log D_L)^{-1}\big(x^{-\delta/16} + T^{-1/24}e^{-\frac{c_5 \log x}{24 \log D_L}} + T^{-1/24}e^{-\frac{1}{24}\sqrt{c_5(\log x)/n_L}}\big) \end{aligned}$$

because $x \geq (\log D_L)^{16/\delta}$. This estimate, along with Proposition 8.1 and Lemma 8.4, yields the result. ∎

We conclude this section with the proof of Theorem 2.7.

Proof of Theorem 2.7. Let $\mathscr{F}(X) = \mathscr{F}(X; G, n, R_G)$ be given by (2.12). Let $K \in \mathscr{F}(X)$ and recall $\widetilde{K}/\mathbb{Q}$ is the Galois closure of K over $\mathbb{Q}$. For $\mathrm{Re}(s) > 1$,

$$\zeta_{\widetilde{K}}(s) = \zeta_{\mathbb{Q}}(s)\prod_{\rho \neq 1} L(s, \rho, \widetilde{K}/\mathbb{Q})^{\dim \rho}, \tag{8.9}$$

where ρ runs over the *nontrivial* irreducible Artin representations of G. In all cases under consideration, the strong Artin conjecture is known to hold for all nontrivial Artin representations ρ of G. That is, $L(s, \rho, \widetilde{K}/\mathbb{Q}) = L(s, \pi)$ for some cuspidal automorphic representation $\pi = \pi_\rho$ of $\mathrm{GL}_d(\mathbb{A}_{\mathbb{Q}})$ with d equal to the degree of ρ. Observe that d is bounded by m, where $m = m(G)$ is the maximum degree of the irreducible representations of G. The map

$$\rho \mapsto \pi_\rho \tag{8.10}$$

has image $\mathscr{A}(X) = \mathscr{A}(X; G, n, R_G)$, the set of automorphic representations π obtained this way from $\mathscr{F}(X)$.

Let $M(X) = M(X; G, n, R_G)$ be the maximum size of the fibres of the map in (8.10). As shown in [56],

$$M(X) \leq \max_{F \neq \mathbb{Q}} \#\{K \in \mathscr{F}(X) \colon \mathbb{Q} \subsetneq F \subseteq \widetilde{K}\}, \tag{8.11}$$

where the maximum runs over all number fields $F \neq \mathbb{Q}$. Since our notation differs from theirs, we explain (8.11) for the sake of clarity. Fix some $\pi \in \mathscr{A}(X)$. By a result of Klüners and Nicolae [37, Theorem 5] refined by Pierce, Turnage-Butterbaugh and Wood [56, Lemma 7.4], it follows that[5] $L(s, \rho_1, \widetilde{K}_1/\mathbb{Q}) = L(s, \rho_2, \widetilde{K}_2/\mathbb{Q}) = L(s, \pi)$ if and only if $\widetilde{K}_1^{\ker(\rho_1)} = \widetilde{K}_2^{\ker(\rho_2)} = F$ for some number field F. Note that $F \neq \mathbb{Q}$ since the representations ρ_1, ρ_2 are nontrivial. Hence, the size of the fibre above $\pi \in \mathscr{A}(X)$ in (8.10) equals $\#\{K \in \mathscr{F}(X) \colon \mathbb{Q} \subsetneq F \subseteq \widetilde{K}\}$ for some number field $F \neq \mathbb{Q}$, implicitly depending on π. This implies (8.11).

In light of (8.11), it follows from [56, Proposition 7.9] and [56, Theorem 7.1] that there exists a sufficiently small $\varepsilon = \varepsilon(n, G) > 0$ such that

$$M(X) \ll_{n,G,\varepsilon} X^{-2\varepsilon} \#\mathscr{F}(X). \tag{8.12}$$

This result is one of the key innovations of [56].

Now, we verify the assumptions of Theorem 1.3 with $\pi_0 \in \mathcal{A}(1)$ taken to be the trivial representation. Take $m = m(G)$ to be the maximum degree of the irreducible representations of G, $Q = X^{|G|/2}$, and $\mathcal{F}_m(Q) = \mathscr{A}(X)$. By (8.9) and (8.10), each $\pi \in \mathcal{F}_m(Q)$ satisfies

$$\deg(\pi) \leq m \quad \text{and} \quad C(\pi) \leq D_{\widetilde{K}} \quad \text{for some } K \in \mathscr{F}(X).$$

Since $D_{\widetilde{K}} \leq D_K^{|G|/2} \leq X^{|G|/2} = Q$ for any $K \in \mathscr{F}(X)$, we indeed have $C(\pi) \leq Q$ for every $\pi \in \mathcal{F}_m(Q)$. Moreover, $\pi \in \mathcal{F}_m(Q)$ satisfies GRC (and hence Hypothesis 1.1) since it corresponds to an Artin representation via (8.10). Thus, by Theorem 1.3, it follows that

$$\sum_{\pi \in \mathscr{A}(X)} N_\pi(1-\delta, T) \ll_{n,G} (X^{|G|/2} T)^{10^8 m^4 \delta} \tag{8.13}$$

uniformly for $T \geq 1$ and $0 < \delta < 1/2$. For $\varepsilon \in (0,1)$ arbitrary, select

$$T = Q(\log Q)^{24}, \qquad \delta = \frac{\varepsilon}{10^9 |G| m^4}.$$

[5] Here we crucially use the fact that the base field is $\mathbb{Q}$.

Thus, by (8.13) and our definition of $Q = X^{|G|/2}$, for all except at most $O_{n,G,\varepsilon}(X^{\varepsilon})$ automorphic representations $\pi \in \mathscr{A}(X)$, the L-function $L(s,\pi)$ is zero-free in the region

$$\mathrm{Re}(s) > 1-\delta, \qquad |\mathrm{Im}(s)| \le Q(\log Q)^{24}. \tag{8.14}$$

Each exceptional π corresponds to at most $M(X)$ exceptional fields $K \in \mathscr{F}(X)$. Throwing out all of these exceptional fields, it follows by (8.12) that $\zeta_{\widetilde{K}}(s)/\zeta_{\mathbb{Q}}(s)$ is zero-free in the region (8.14) for all $K \in \mathscr{F}(X)$ with at most $O_{n,G,\varepsilon}(X^{-\varepsilon}\,\#\mathscr{F}(X))$ exceptions.

Now, let $K \in \mathscr{F}(X)$ be a nonexceptional field. By Theorem 8.5, we have

$$\left|\pi_C(x,\widetilde{K}/\mathbb{Q}) - \frac{|C|}{|G|}\pi(x)\right| \ll \frac{|C|}{|G|}\frac{x}{\log x}E(x) \qquad \text{for } x \ge (\log D_{\widetilde{K}})^{16/\delta}, \tag{8.15}$$

where

$$\begin{aligned} E(x) = x^{-\delta/8} &+ D_{\widetilde{K}}^{-1/24}\exp\left[-\frac{1}{24}\left(\frac{c_5\log x}{|G|}\right)^{1/2}\right] \\ &+ D_{\widetilde{K}}^{-1/24}\exp\left[-\frac{1}{24}\frac{c_5\log x}{\log D_{\widetilde{K}}}\right]. \end{aligned}$$

Note we used $D_{\widetilde{K}} \le Q$ to express $E(x)$ in terms of $D_{\widetilde{K}}$ instead of Q. Choose $\eta = \delta/8$. For $(\log D_{\widetilde{K}})^{2/\eta} \le x \le D_{\widetilde{K}}^{1/(24\eta)}$, one can directly verify that $E(x) \ll x^{-\delta/8} = x^{-\eta}$. If $(24\eta)^{-1}\log D_{\widetilde{K}} \le \log x \le c_5^{-1}|G|(\log D_{\widetilde{K}})^2$ then one can verify that

$$E(x) \ll D_{\widetilde{K}}^{-1/24} \ll e^{-\frac{1}{24}\sqrt{c_5(\log x)/|G|}}.$$

Finally, if $\log x \ge c_5^{-1}|G|(\log D_{\widetilde{K}})^2$ then one can verify that

$$E(x) \ll e^{-\frac{1}{24}\sqrt{c_5(\log x)/|G|}} + e^{-\frac{c_5\log x}{24\log D_{\widetilde{K}}}} \ll e^{-\frac{1}{24}\sqrt{c_5(\log x)/|G|}}.$$

This completes the proof of Theorem 2.7. ∎

9. Landau–Siegel zeros and torsion in class groups

This section is dedicated to the proof of Theorem 2.9. The first ingredient is a lemma due to Ellenberg–Venkatesh [16, Lemma 2.3]. It establishes a connection between the existence of small split primes and bounds for the class group.

Lemma 9.1 (Ellenberg–Venkatesh). *Let $K/\mathbb{Q}$ be a number field of degree n and let $\ell \ge 1$ be a positive integer. Set $0 < \delta < \frac{1}{2\ell(n-1)}$ and suppose there exist M rational primes $p \le D_K^{\delta}$ which are unramified and split completely in K. For any $\varepsilon > 0$,*

$$|\mathrm{Cl}_K[\ell]| \ll_{\varepsilon,\ell,n} D_K^{1/2+\varepsilon}M^{-1}.$$

To make use of Lemma 9.1, we require a proposition relating low-lying zero-free regions to the existence of small primes with a given splitting behaviour.

Proposition 9.2. *Let $L/\mathbb{Q}$ be a Galois extension of number fields and let $0 < \varepsilon < \delta/2$ be arbitrary. Suppose $\zeta_L(s)$ has no zeros in the region*

$$\mathrm{Re}(s) > 1 - \frac{H_{\delta,\varepsilon}}{\log D_L}, \qquad |\mathrm{Im}(s)| \leq 1, \tag{9.1}$$

where $H_{\delta,\varepsilon} \geq 1$ is sufficiently large. Then, for any conjugacy class $C \subseteq G$,

$$\pi_C(D_L^\delta, L/\mathbb{Q}) \geq \frac{\varepsilon}{8\delta}\frac{|C|}{|G|}D_L^{\delta-\varepsilon} + O_{\delta,\varepsilon}\left(\frac{|C|}{|G|}D_L^{\delta-\varepsilon}(\log D_L)^{-3}\right).$$

Proof. This essentially follows from the arguments found in [71]. We will outline the proof here and borrow heavily from [71], so we will remain as consistent as possible with the notation therein. In particular, set $\mathscr{L} = \log D_L$. Select f as in [71, Lemma 2.6] with $\ell = 2$, $B = \delta$, and $A = \varepsilon/4$. Then:

- $0 \leq f(t) \leq A^{-1} = 4\varepsilon^{-1}$ for all $t \in \mathbb{R}$.
- The support of f is contained in $[B - 2\ell A, B] = [\delta - \varepsilon, \delta]$.
- The Laplace transform $F(z) = \int_0^\infty f(t)e^{-zt}\,dt$ is entire and given by
$$F(z) = e^{-(B-2\ell A)z}\left(\frac{1-e^{-Az}}{Az}\right)^{2\ell} = e^{-(\delta-\varepsilon)z}\left(\frac{1-e^{-\varepsilon z/4}}{\varepsilon z/4}\right)^4.$$
- For $s = \sigma + it \in \mathbb{R}$ with $\sigma < 1$ and $t \in \mathbb{R}$, we have
$$|F((1-s)\mathscr{L})| \ll_\varepsilon e^{-(\delta-\varepsilon)(1-\sigma)\mathscr{L}} \min\{1, |(1-s)\mathscr{L}|^{-4}\}.$$

  Furthermore, $F(0) = 1$.

We will use these properties frequently and often without mention. Define

$$S = \sum_{\substack{p \text{ prime} \\ p \nmid D_L}} \frac{\log p}{p} f\left(\frac{\log p}{\mathscr{L}}\right)\mathbf{1}_C(p),$$

where, for primes p unramified in L, $\mathbf{1}_C(p) = 1$ if $\left[\frac{L/\mathbb{Q}}{p}\right] = C$ and 0 otherwise. By the properties of f, one can verify that

$$S \leq \frac{\delta\mathscr{L}}{e^{(\delta-\varepsilon)\mathscr{L}}} \cdot 4\varepsilon^{-1} \cdot \sum_{\substack{p \leq D_L^\delta \\ p \nmid D_L}} \mathbf{1}_C(p) \leq (4\delta\varepsilon^{-1}D_L^{-\delta+\varepsilon}\log D_L) \cdot \pi_C(D_L^\delta, L/\mathbb{Q}). \tag{9.2}$$

Now, from the proof of [71, Lemma 4.1], we have

$$\mathscr{L}^{-1}S = \sum_\psi \overline{\psi}(C)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \psi, L/\mathbb{Q})F((1-s)\mathscr{L})\,ds$$
$$+ O_{\delta,\varepsilon}(\mathscr{L}^2 e^{-\delta\mathscr{L}/4}),$$

where ψ runs over the irreducible Artin characters of $\mathrm{Gal}(L/\mathbb{Q})$. Using standard class field theory arguments (see [71, Section 4.2]), one can shift the contour as in [71, Lemma 4.2] with $T_\star = 1$. This yields

$$\frac{|G|}{|C|}\mathscr{L}^{-1}S = 1 + O_{\delta,\varepsilon}\Big(\sum_{|\mathrm{Im}(\rho)|\le 1} |F((1-\rho)\mathscr{L})| + \mathscr{L}^{-3}\Big), \tag{9.3}$$

where ρ runs over all nontrivial zeros of $\zeta_L(s)$ satisfying $|\mathrm{Im}(\rho)| \le 1$. We apply [71, Lemma 4.3] (with $J = 1, T_1 = 1$, and $R_1 = H_{\delta,\varepsilon}$ in their notation) to deduce that

$$\sum_{|\mathrm{Im}(\rho)|\le 1} |F((1-\rho)\mathscr{L})| = \sum_{\substack{|\mathrm{Im}(\rho)|\le 1\\ \mathrm{Re}(\rho)>1-\frac{H_{\delta,\varepsilon}}{\log D_L}}} |F((1-\rho)\mathscr{L})| + O_{\delta,\varepsilon}(e^{-\delta H_{\delta,\varepsilon}/2}).$$

By assumption (9.1), the remaining sum over zeros is empty. Combining these estimates with (9.3) implies that

$$\begin{aligned} S &= \frac{|C|}{|G|}(\log D_L)\big(1 + O_{\delta,\varepsilon}(e^{-\delta H_{\delta,\varepsilon}/2} + (\log D_L)^{-3})\big) \\ &\ge \frac{1}{2}\frac{|C|}{|G|}(\log D_L)\big(1 + O_{\delta,\varepsilon}((\log D_L)^{-3})\big), \end{aligned}$$

since $H_{\delta,\varepsilon}$ is sufficiently large. Substituting this lower bound into (9.2) yields the result. ∎

Let K be a number field, let $n = [K:\mathbb{Q}]$, and let $\widetilde{K}$ be the Galois closure of K over $\mathbb{Q}$. Our application of Proposition 9.2 assumes that $\zeta_{\widetilde{K}}(s)$ is the L-function associated to a (noncuspidal) automorphic representation Π of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$, where $m = [\widetilde{K}:\mathbb{Q}] \le n!$. By a result of Langlands along with strong multiplicity one, there exists an integer $1 \le r \le m$, integers $1 \le m_j \le m$ and $1 \le d_j \le m$ such that $\sum_{j=1}^r m_j = m$ and $\pi_j \in \mathcal{A}(m_j)$ occurs as a cuspidal constituent of Π with multiplicity d_j. Consequently,

$$\zeta_{\widetilde{K}}(s) = \prod_{j=1}^{r} L(s,\pi_j)^{d_j}. \tag{9.4}$$

This gives a factorization of $\zeta_{\widetilde{K}}(s)$ into irreducible L-functions (see [50, Remark 1.1]).

Lemma 9.3. *Let $K/\mathbb{Q}$ be a number field of degree n with Galois closure $\widetilde{K}$ over $\mathbb{Q}$. Let $\ell \ge 1$ be a positive integer, and let $\varepsilon > 0$ be arbitrary. Let χ be a real primitive Dirichlet character modulo a fundamental discriminant d. Assume the following:*

(i) *$\zeta_{\widetilde{K}}(s)$ is the L-function of an isobaric automorphic representation Π of the group $\mathrm{GL}_{[\widetilde{K}:\mathbb{Q}]}(\mathbb{A}_\mathbb{Q})$.*

(ii) $\mathbb{Q}(\sqrt{d}) \cap \widetilde{K} = \mathbb{Q}$.

(iii) *The Dirichlet L-function $L(s,\chi)$ has a real zero $\beta_\chi = 1 - \eta_\chi/\log|d|$ with η_χ sufficiently small, depending on n.*

Then $\chi \ne \pi_j$ for all $1 \le j \le r$ in (9.4).

Proof. Suppose to the contrary that $\chi = \pi_1$ in (9.4), in which case $|d| \leq D_{\widetilde{K}} \leq D_K^{n!/2}$. By [41, Section 8], there exists an effectively computable constant $c_n > 0$ such that $\zeta_{\widetilde{K}}(s)$ has at most one real simple zero in the interval

$$s \geq 1 - \frac{c_n}{\log D_K}. \tag{9.5}$$

Similarly, if η_χ is sufficiently small, then by results in [13, Chapter 14], β_χ is a real simple zero of $L(s,\chi)$. Since χ is a cuspidal constituent of Π by hypothesis, it follows that $\zeta_{\widetilde{K}}(s)/L(s,\chi)$ is holomorphic on $\mathbb{C} - \{1\}$. In particular, if $\eta_\chi < c_n$ (which can be guaranteed by item (iii)), then the sole real simple zero of $\zeta_{\widetilde{K}}(s)$ in the interval (9.5) is β_χ.

A result of Stark [62, Theorem 3] implies that $\widetilde{K}$ contains a quadratic subfield M such that β_χ is a real simple zero of $\zeta_M(s)$. Define χ' (mod d') to be the primitive quadratic character whose Dirichlet L-function equals $\zeta_M(s)/\zeta(s)$. Since M is a subfield of $\widetilde{K}$, we have $|d'| \leq D_{\widetilde{K}} \leq D_K^{n!/2}$.

Note that $L(s,\chi)$ and $L(s,\chi')$ both have β_χ as a real simple zero, and β_χ lies in the interval (9.5). Since $|d|, |d'| \leq D_K^{n!/2}$, a theorem of Page [13, Chapter 14] implies that $\chi = \chi'$ once c_n (hence η_χ) is made sufficiently small in terms of n (which is permissible by (iii)). Thus $M = \mathbb{Q}(\sqrt{d})$, so $\mathbb{Q}(\sqrt{d})$ is a subfield of $\widetilde{K}$. This contradicts item (ii) in the statement of the lemma, so χ cannot appear as a cuspidal constituent of Π, as desired. ■

Proof of Theorem 2.9. Recall that K is a number field of degree n whose Galois closure over $\mathbb{Q}$ is $\widetilde{K}$, and let $m = [\widetilde{K} : \mathbb{Q}] \leq n!$. By assumption, there exists a (noncuspidal) automorphic representation $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ whose L-function $\zeta_{\widetilde{K}}(s) = L(s,\Pi)$. Thus we may assume the existence of a factorization of the form (9.4). Clearly, $L(s,\Pi)$ satisfies GRC. Our hypotheses and Lemma 9.3 ensure that $\pi_j \neq \chi$ for all $1 \leq j \leq r$ in (9.4). Let

$$Q = \max\{D_{\widetilde{K}}, 2d\}, \qquad 0 < \varepsilon < \frac{1}{4\ell(n-1)}, \qquad \delta = \Big(\frac{1}{2\ell(n-1)} - \varepsilon\Big)\frac{\log D_K}{\log D_{\widetilde{K}}},$$

and let $H_{\delta,\varepsilon} \geq 1$ be sufficiently large. From the estimate $D_K^{|G|/n} \leq D_{\widetilde{K}} \leq D_K^{|G|/2}$, one can see that $\delta < 1$ and δ is bounded away from zero *uniformly* in terms of n, ℓ, and ε. Thus, when a quantity depends on δ (such as $H_{\delta,\varepsilon}$), we may replace this dependence with n, ℓ, and ε. In particular, we may treat δ as independent of D_K and $D_{\widetilde{K}}$.

Recall that $\beta_\chi = 1 - \eta_\chi/\log q$. We apply Theorem 1.7 with π_0 trivial, $\mathcal{F}_m(Q) = \{\pi_j : 1 \leq j \leq r\}$, $T = 1$, and $\sigma = 1 - H_{\delta,\varepsilon}/\log D_{\widetilde{K}}$. Since $\widetilde{\pi}_0 \otimes \chi = \chi \notin \mathcal{F}_m(Q)$, Theorem 1.7 implies that

$$N_\pi(1 - H_{\delta,\varepsilon}/\log D_{\widetilde{K}}, 1) \ll_n ((1-\beta_\chi)\log Q)Q^{10^8 m^4 H_{\delta,\varepsilon}/\log D_{\widetilde{K}}} \ll_{n,\ell,\varepsilon} \eta_\chi,$$

where we have used the assumed bounds $\log Q \asymp_{n,\varepsilon,\ell} \log D_{\widetilde{K}} \asymp_{n,\varepsilon,\ell} \log d$ and $m \leq n!$. As η_χ is sufficiently small depending only on n, ℓ, ε, it follows that $\zeta_{\widetilde{K}}(s)$ has no zeros in the region $\mathrm{Re}(s) > 1 - H_{\delta,\varepsilon}/\log D_{\widetilde{K}}$ and $|\mathrm{Im}(s)| \leq 1$. Thus, by Proposition 9.2, there are $\gg_{\varepsilon,n,\ell} D_{\widetilde{K}}^{\delta-\varepsilon} = D_K^{\frac{1}{2\ell(n-1)}-2\varepsilon}$ rational primes $p \leq D_{\widetilde{K}}^{\delta} = D_K^{\frac{1}{2\ell(n-1)}-\varepsilon}$ which split completely in $\widetilde{K}$, provided D_K is sufficiently large depending on ε, n, and ℓ. The result now follows from Lemma 9.1 after rescaling ε appropriately. ∎

Appendix A. Explicit upper bound on the universal family for GL_n

Let F be a number field of degree d over $\mathbb{Q}$ and discriminant D and let $n \geq 1$ be an integer. Let $\mathscr{A}_{\mathrm{cusp}}$ denote the set of unitary cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_F)$, with normalized central character, ordered by analytic conductor $C(\pi)$. We recall that $C(\pi) = N_\pi K_\pi$, where $N_\pi = \mathrm{Norm}(\mathfrak{q}_{\pi_f}) \in \mathbb{N}$ is the arithmetic conductor (the norm of the Jacquet–Piatetski-Shapiro–Shalika conductor of π_f), and K_π the archimedean conductor, as in (3.3).

For $Q \geq 1$, let

$$\mathscr{F}(Q) = \{\pi \in \mathscr{A}_{\mathrm{cusp}} \colon C(\pi) \leq Q\}.$$

We present an argument, due to Venkatesh [67] and based on results in [6], to deduce a polynomial upper bound on the cardinality $|\mathscr{F}(Q)|$. We can in fact make this polynomial bound explicit, using subsequent refinements of *loc. cit.*, as in the following

Theorem A.1. *We have, for all fixed $\varepsilon > 0$, $|\mathscr{F}(Q)| \ll_{d,n,\varepsilon} (DQ)^\varepsilon D^{n^2} Q^{2n}$.*

Remark A.2. As we consider d and n as being fixed, we shall henceforth systematically suppress the dependence of implied constants on n and d in the notation.

Remark A.3. The expected value of the exponent of Q in Theorem A.1 is $n+1$, and indeed this was shown (with an asymptotic) in [8], with one caveat: for $n \geq 3$ the authors restrict to the subfamily of $\mathscr{F}(Q)$ consisting of Maass forms. This restriction is fortunate, in a way, since it provides an occasion for this appendix, which has sat for a long time in a drawer (or inbox) and whose methods are quite different. While Theorem A.1 says nothing about existence, and the upper bound is not sharp, we believe that the proof itself is of sufficient interest to merit circulation.

Remark A.4. The results of [8] make no claim of uniformity in the number field F. The upper bound in Theorem A.1 is, however, uniform in D, making this perhaps the most novel aspect of the result.

Remark A.5. In our definition of the analytic conductor, we have not included a factor of the discriminant, as some authors do (including Iwaniec and Sarnak [33]). For them, the analytic conductor of π would be $C_{\mathrm{IS}}(\pi) = D^n N_\pi K_\pi$, and the analytic conductor of the Rankin–Selberg L-function $L(s, \pi \times \pi')$ would be of the form $C_{\mathrm{IS}}(\pi \times \pi'; s) = D^{n^2} N_{\pi\times\pi'} K_{\pi\times\pi'}(s)$ rather than our $N_{\pi\times\pi'} K_{\pi\times\pi'}(s)$; see §A.2.4. Note that one can deduce Theorem A.1 for either definition from the other, by a simple scaling argument. One reason for our convention is that the Bushnell–Henniart bounds [9] for the Rankin–Selberg conductor on $\mathrm{GL}_m \times \mathrm{GL}_n$ would yield (if blindly applied to the definition of

Iwaniec–Sarnak) the lossy $D^{mn} \leq D^{2mn}$ in the discriminant aspect. This could be corrected by writing $C_{\mathrm{IS}}(\pi \times \pi') \ll D^{-mn} C_{\mathrm{IS}}(\pi)^n C_{\mathrm{IS}}(\pi')^m$, but we have preferred to avoid this.

The proof of Theorem A.1 combines two ingredients: Rankin–Selberg theory and sphere packing bounds in large dimensions. It is natural to ask what effect assuming standard conjectures on these L-functions would have on the quality of the resulting bound. For example, a similar argument to the one we present here was used in [17] to count ℓ-adic sheaves of bounded complexity. In that article, Deligne's proof of the Riemann hypothesis over finite fields is used to show that certain trace functions form a quasi-orthogonal system with small enough angular separation to deduce a polynomial upper bound. We show that the exponent $2n$ can be improved to $n+1$ under standard conjectures, demonstrating the strength of the method of proof.

Theorem A.6. *Assume the generalized Ramanujan conjecture and the generalized Riemann hypothesis for Rankin–Selberg L-functions. Then, for all fixed $\varepsilon > 0$,*

$$|\mathscr{F}(Q)| \ll_\varepsilon (DQ)^\varepsilon D^{n^2/2} Q^{n+1}.$$

Remark A.7. Note that, by the results in [8], the exponent of Q in Theorem A.6 is sharp, up to the ε. Moreover, the D dependence here and that of the *main term* of the asymptotic given in [8] are in agreement.

Remark A.8. The method of proof of Theorems A.1 and A.6 is sensitive to any loss of information incurred in the application of the Bushnell–Henniart bounds [9]. Recall that the main result in *loc. cit.* provides upper bounds for the Rankin–Selberg Artin exponent $\mathrm{Ar}(\pi_v \times \widetilde{\pi}'_v)$ at finite places v in terms the standard Artin exponents $\mathrm{Ar}(\pi_v)$ and $\mathrm{Ar}(\widetilde{\pi}'_v)$, and the integers n, n', where π_v and π'_v are smooth irreducible representations of $\mathrm{GL}_n(F_v)$ and $\mathrm{GL}_{n'}(F_v)$, respectively.

While the bounds in *loc. cit.* are sharp in general, we apply them under additional hypotheses on π_v and π'_v. Namely, in the course of the proof, we assume that

(1) the dimensions $n = n'$ are the same,

(2) the Artin exponents $a = \mathrm{Ar}(\pi) = \mathrm{Ar}(\pi')$ are the same,

(3) the central characters are the same, say equal to χ.

Under (1) and (2) above, Theorem 1 in [9] establishes the sharp bound $\mathrm{Ar}(\pi_v \times \widetilde{\pi}'_v) \leq (2n-1)a$. In Theorem B.1 of Appendix B, Bushnell and Henniart show that, under the additional assumption of (3), this bound can be improved to $\mathrm{Ar}(\pi_v \times \widetilde{\pi}'_v) \leq (2n-2)a$.

This improved bound is an ingredient in the explicit exponents given in Theorems A.1 and A.6. Without this improvement, the unconditional bound in Theorem A.1 would have an additional factor of Q, and the conditional bound in Theorem A.6 would have an additional factor of $Q^{1/2}$.

Remark A.9. The method of proof of Theorems A.1 and A.6 requires fixing certain representation-theoretic data, of combinatorial nature. This data encodes the dimensional blocks of the inducing supercuspidal representations in the Bernstein–Zelevinsky classification, as well as the partition of these blocks according to the underlying twist equivalence classes. See §A.2 for more details. After bounding the size of the subfamily associated with such data, one then sums over the finite number of such choices.

This decomposition allows one to prove, in principle, refined bounds for the cardinality of these subfamilies, since the Bushnell–Henniart bounds [9] can often be improved under such assumptions. For example, if the combinatorial data that one takes is “trivial”, in the sense that it corresponds to π_v and π'_v supercuspidal on GL_n, then (keeping the assumptions (2) and (3) of the previous remark) one can use the bound $\mathrm{Ar}(\pi \times \widetilde{\pi}') \leq na$ of [10, Corollary C], which is, in general, far better than the general bound of $(2n-2)a$ cited above. In this way one can show that, under Ramanujan and Riemann as in Theorem A.6, the subfamily of $\mathscr{F}(Q)$ consisting of π which

(1) are supercuspidal at all the places at which they ramify,

(2) have archimedean component lying in some fixed compact of the unitary dual,

has cardinality $O(Q^{n/2+2})$ (ignoring the discriminant dependence). This bound is surprisingly strong, and no trace formula was used to derive it. We have not found this type of interplay between conductor dropping phenomenon and improved bounds on dimension counts of automorphic forms elsewhere in the literature.

A.1. Idea of proof

We present here the basic argument to prove Theorem A.1. We shall later need to modify the presentation to obtain the best possible exponent.

Let $\mathfrak{q}$ be an integral ideal of $\mathcal{O}_F$. Let χ be a character of $\mathbb{A}_f^\times$, where $\mathbb{A}_f$ is the ring of finite adeles of F. Let

$$\mathscr{A}_{\mathrm{cusp}}(\mathfrak{q},\chi) = \{\pi \in \mathscr{A}_{\mathrm{cusp}} : \mathfrak{q}_{\pi_f} = \mathfrak{q},\ \chi_{\pi_f} = \chi\},$$
$$\mathscr{F}_{\mathfrak{q},\chi}(Q) = \mathscr{F}(Q) \cap \mathscr{A}_{\mathrm{cusp}}(\mathfrak{q},\chi).$$

Here, $\mathfrak{q}_{\pi_f}$ is the conductor of π_f and χ_{π_f} is the central character of π_f. If $\mathscr{A}_{\mathrm{cusp}}(\mathfrak{q},\chi)$ is nonempty then the conductor $\mathfrak{d}$ of χ necessarily divides $\mathfrak{q}$. Then

$$|\mathscr{F}(Q)| = \sum_{\mathrm{Norm}(\mathfrak{q}) \leq Q} \sum_{\mathfrak{d} | \mathfrak{q}} \sum_{\substack{\chi \\ \mathrm{cond}\ \mathfrak{d}}} |\mathscr{F}_{\mathfrak{q},\chi}(Q)|. \tag{A.1}$$

The argument we sketch below provides a bound on $|\mathscr{F}_{\mathfrak{q},\chi}(Q)|$ of the form

$$O_\varepsilon((D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n+n^2})^{1+\varepsilon}).$$

Executing the triple sum over all $(\mathfrak{q}, \mathfrak{d}, \chi)$, this would produce a bound of $O_\varepsilon((D^{n^2}Q^{2n+n^2})^{1+\varepsilon})$. We will later show (see §A.3) how to remove the n^2 to establish Theorem A.1, as well as the sharp conditional bounds in Theorem A.6.

A.1.1. Mapping $\mathscr{A}_{\mathrm{cusp}}(\mathfrak{q},\chi)$ to a Hermitian space. We begin by describing a way to map $\mathscr{A}_{\mathrm{cusp}}(\mathfrak{q},\chi)$ to a Hermitian space, whose inner product can be understood in terms of Rankin–Selberg L-functions. The reader is encouraged to read ahead to the next subsection describing the Dirichlet coefficients of these L-functions for a motivation of the following constructions.

Recall that a *partition* $\mu=(\mu_i)$ is a sequence of nonincreasing nonnegative integers $\mu_1\geq\mu_2\geq\cdots$ with only finitely many nonzero entries. Write $\mathcal{P}$ for the set of all partitions. The *length* of $\mu\in\mathcal{P}$, denoted $\ell(\mu)$, is the number of its nonzero entries. Write

$$\mathcal{P}_\ell=\{\mu\colon\mu_1\geq\cdots\geq\mu_\ell\geq 0\}$$

for the partitions of length at most ℓ. Finally, for $\mu=(\mu_i)\in\mathcal{P}$, write $|\mu|=\sum_i\mu_i$. For an integer r, let $\mathcal{P}_\ell(r)=\{\mu\in\mathcal{P}_\ell\colon|\mu|=r\}$; this is an empty set when r is negative.

Let S be a finite set of finite places. Let I^S denote the set of integral ideals of $\mathcal{O}_F$ supported outside of S. When S is empty we abbreviate this to I for the set of all integral ideals. Given an $\mathfrak{n}=\prod_{\mathfrak{p}}\mathfrak{p}^{r_{\mathfrak{p}}}\in I$ we write $\mathscr{P}_{n-1}(\mathfrak{n})$ for the set of sequences $\mathfrak{u}=(\mu_{\mathfrak{p}})_{\mathfrak{p}}$ of partitions such that $\mu_{\mathfrak{p}}\in\mathcal{P}_{n-1}(r_{\mathfrak{p}})$. A *$\mathcal{P}_{n-1}$-decorated prime-to-$S$ ideal* is a pair $(\mathfrak{n},\mathfrak{u})$, where $\mathfrak{n}\in I^S$ and $\mathfrak{u}\in\mathscr{P}_{n-1}(\mathfrak{n})$. Let $\mathscr{I}^S$ denote the set of $\mathcal{P}_{n-1}$-decorated prime-to-S ideals. We have a map $\mathscr{I}^S\to I^S$, $(\mathfrak{n},\mathfrak{u})\mapsto\mathfrak{n}$, where we forget the decoration and take the underlying ideal. Observe that several $(\mathfrak{n},\mathfrak{u})$ can have the same underlying ideal $\mathfrak{n}$. We shall sometimes write $\tilde{\mathfrak{n}}$ for a $\mathcal{P}_{n-1}$-decorated ideal with underlying ideal $\mathfrak{n}$.

For a parameter $X>1$, let $\mathscr{I}^S(X)=\{\tilde{\mathfrak{n}}\in\mathscr{I}^S\colon\mathrm{Norm}(\mathfrak{n})\leq X\}$; this is the set of pairs $(\mathfrak{n},\mathfrak{u})$ with $\mathrm{Norm}(\mathfrak{n})\leq X$ and $\mathfrak{u}\in\mathscr{P}_{n-1}(\mathfrak{n})$. Let $V^S(X)$ be the vector space of complex valued functions on $\mathscr{I}^S(X)$. Endow $V^S(X)$ with the standard scalar product

$$\langle f,g\rangle=\sum_{\tilde{\mathfrak{n}}\in\mathscr{I}^S(X)}f(\tilde{\mathfrak{n}})\overline{g(\tilde{\mathfrak{n}})}.$$

For an integral ideal $\mathfrak{q}$ of $\mathcal{O}_F$, with support S, we shall map $\mathscr{A}_{\mathrm{cusp}}(\mathfrak{q},\chi)$ to $V^S(X)$ in the following way. For a partition $\mu\in\mathcal{P}_{n-1}$ let s_μ denote the associated Schur function in n variables. If μ is the zero partition, then s_μ is identically 1. For $(\mathfrak{n},\mathfrak{u})\in\mathscr{I}^S$ set

$$a_\pi(\mathfrak{n},\mathfrak{u})=\prod_{\mathfrak{p}\notin S}s_{\mu_{\mathfrak{p}}}(A_\pi(\mathfrak{p})),\quad\text{where}\quad A_\pi(\mathfrak{p})=(\alpha_{1,\pi}(\mathfrak{p}),\ldots,\alpha_{n,\pi}(\mathfrak{p})).\tag{A.2}$$

We note that if $n=2$ and π_f has trivial central character, then the decoration $\mathfrak{u}=(\mu_{\mathfrak{p}})_{\mathfrak{p}}$ is necessarily $\mu_{\mathfrak{p}}=(r_{\mathfrak{p}},0,\ldots)$ and the (A.2) just recovers the Hecke eigenvalue of π at $\mathfrak{n}$. In fact, more generally, when $n\geq 2$ and π_f has trivial central character, if we take $\mathfrak{u}=(\mu_{\mathfrak{p}})_{\mathfrak{p}}$ to satisfy $\mu_{\mathfrak{p}}=(r_{\mathfrak{p}},0,\ldots)$, then we once again recover the Hecke eigenvalue at $\mathfrak{n}=\prod_{\mathfrak{p}}\mathfrak{p}^{r_{\mathfrak{p}}}$.

Let $f\colon\mathbb{R}\to\mathbb{R}$ be a nonnegative smooth function supported in $[1/2,1]$ and having Lebesgue integral 1. Write

$$F_X^S(\mathfrak{n})=\sum_{(\mathfrak{m},S)=1}f(\mathrm{Norm}(\mathfrak{n}\mathfrak{m}^n)/X).$$

For every $\pi \in \mathscr{A}_{\mathrm{cusp}}(\mathfrak{q},\chi)$ we define a vector $\mathbf{v}_\pi^S \in V^S(X)$ by the rule

$$\mathbf{v}_\pi^S : (\mathfrak{n},\mathfrak{y}) \mapsto \sqrt{F_X^S(\mathfrak{n})}\, a_\pi(\mathfrak{n},\mathfrak{y}).$$

Note that for $\mathrm{Norm}(\mathfrak{n}) > X$ we have $F_X^S(\mathfrak{n}) = 0$; in this way the function $\widetilde{\mathfrak{n}} \mapsto \mathbf{v}_\pi^S(\widetilde{\mathfrak{n}})$ can indeed be viewed as an element of $V^S(X)$.

A.1.2. Relation to Rankin–Selberg L-functions. We now recall the description of the Rankin–Selberg Dirichlet coefficients. This will clarify the choice of map $\pi \mapsto \mathbf{v}_\pi^S$ and the inner product we put on $V^S(X)$. Let $\pi,\pi' \in \mathscr{A}_{\mathrm{cusp}}(\mathfrak{q},\chi)$. The prime-to-$S$ part of the Rankin–Selberg L-function is defined, for $\mathrm{Re}(s) > 1$, by the Euler product

$$L^S(s,\pi\times\widetilde{\pi}') = \prod_{\mathfrak{p}\notin S}\prod_{j=1}^{n}\prod_{j'=1}^{n}\left(1-\alpha_{j,\pi}(\mathfrak{p})\overline{\alpha_{j',\pi'}(\mathfrak{p})}\mathrm{Norm}(\mathfrak{p})^{-s}\right)^{-1}.$$

We write $a_{\pi\times\widetilde{\pi}'}(\mathfrak{n})$, for $(\mathfrak{n},S)=1$, for the Dirichlet coefficients of $L^S(s,\pi\times\widetilde{\pi}')$, so that

$$L^S(s,\pi\times\widetilde{\pi}') = \sum_{(\mathfrak{n},S)=1} a_{\pi\times\widetilde{\pi}'}(\mathfrak{n})\mathrm{Norm}(\mathfrak{n})^{-s}.$$

Cauchy's identity shows that

$$a_{\pi\times\widetilde{\pi}'}(\mathfrak{p}^r) = \sum_{\mu\in\mathscr{P}_n(r)} s_\mu(A_\pi(\mathfrak{p}))s_\mu(A_{\widetilde{\pi}'}(\mathfrak{p})).$$

Following the exposition in [7, §2], for a partition $\mu = (\mu_1,\ldots,\mu_{n-1},k,0,\ldots) \in \mathscr{P}_n$ we let $\widehat{\mu} = (\mu_1-k,\ldots,\mu_{n-1}-k,0,\ldots) \in \mathscr{P}_{n-1}$. Then $s_\mu(A_\pi(\mathfrak{p})) = \chi^k(\varpi_\mathfrak{p})s_{\widehat{\mu}}(A_\pi(\mathfrak{p}))$. Now, for any pair (μ,k), where $\mu\in\mathscr{P}_{n-1}$ and $k\geq 0$, there is a unique $\lambda\in\mathscr{P}_n$ such that $|\lambda| = |\mu| + kn$ and $\widehat{\lambda} = \mu$ (add k to each of the first n entries of μ). Applying this we get

$$a_{\pi\times\widetilde{\pi}'}(\mathfrak{p}^r) = \sum_{k\geq 0}\sum_{\mu\in\mathscr{P}_{n-1}(r-nk)} s_\mu(A_\pi(\mathfrak{p}))s_\mu(A_{\widetilde{\pi}'}(\mathfrak{p})). \tag{A.3}$$

The sum on k is finite, going up to the integer part of r/n. Note that, in the above expression, we have used the fact that $\chi_{\pi_f} = \chi_{\pi'_f} = \chi$; this explains why we have decomposed according to central character in (A.1). Thus, for $(\mathfrak{n},S)=1$, we have

$$\begin{aligned} a_{\pi\times\widetilde{\pi}'}(\mathfrak{n}) &= \prod_{\mathfrak{p}^{r_\mathfrak{p}}\|\mathfrak{n}}\sum_{k_\mathfrak{p}\geq 0}\sum_{\mu_\mathfrak{p}\in\mathscr{P}_{n-1}(r_\mathfrak{p}-nk_\mathfrak{p})} s_{\mu_\mathfrak{p}}(A_\pi(\mathfrak{p}))s_{\mu_\mathfrak{p}}(A_{\widetilde{\pi}'}(\mathfrak{p})) \\ &= \sum_{\substack{(\mathfrak{m},S)=1\\ \mathfrak{m}^n|\mathfrak{n}}}\sum_{\mathfrak{y}\in\mathscr{P}_{n-1}(\mathfrak{n}/\mathfrak{m}^n)} a_\pi(\mathfrak{n}/\mathfrak{m}^n,\mathfrak{y})\overline{a_{\pi'}(\mathfrak{n}/\mathfrak{m}^n,\mathfrak{y})}. \end{aligned} \tag{A.4}$$

We now consider the smooth sum of coefficients

$$S(X) = \sum_{(\mathfrak{a},S)=1} a_{\pi\times\widetilde{\pi}'}(\mathfrak{a})f(\mathrm{Norm}(\mathfrak{a})/X).$$

We have

$$
\begin{aligned}
S(X) &= \sum_{(\mathfrak{a},S)=1} f(\mathrm{Norm}(\mathfrak{a})/X) \sum_{\substack{(\mathfrak{m},S)=1\\ \mathfrak{m}^n|\mathfrak{a}}} \sum_{\mu\in\mathscr{P}_{n-1}(\mathfrak{a}/\mathfrak{m}^n)} a_\pi(\mathfrak{a}/\mathfrak{m}^n,\mu)\overline{a_{\pi'}(\mathfrak{a}/\mathfrak{m}^n,\mu)}\\
&= \sum_{(\mathfrak{n},S)=1} \sum_{(\mathfrak{m},S)=1} f(\mathrm{Norm}(\mathfrak{n}\mathfrak{m}^n)/X) \sum_{\mu\in\mathscr{P}_{n-1}(\mathfrak{n})} a_\pi(\mathfrak{n},\mu)\overline{a_{\pi'}(\mathfrak{n},\mu)}\\
&= \sum_{(\mathfrak{n},\mu)\in\mathscr{I}^S} F_X^S(\mathfrak{n}) a_\pi(\mathfrak{n},\mu)\overline{a_{\pi'}(\mathfrak{n},\mu)} = \sum_{\tilde{\mathfrak{n}}\in\mathscr{I}^S(X)} \mathbf{v}_\pi^S(\tilde{\mathfrak{n}})\overline{\mathbf{v}_\pi^S(\tilde{\mathfrak{n}})}.
\end{aligned}
$$

We recognize this as $\langle \mathbf{v}_\pi^S, \mathbf{v}_{\pi'}^S\rangle$. On the other hand, if we let

$$\hat{f}(s) = \int_0^\infty f(x)x^s \frac{dx}{x}$$

be the Mellin transform of f, then by the Mellin inversion formula one has

$$S(X) = \frac{1}{2\pi i}\int_{(2)} L^S(s,\pi\times\tilde{\pi}')\hat{f}(s)X^s\,ds.$$

This allows us to read off the orthogonality properties of $\mathbf{v}_\pi^S$ and $\mathbf{v}_{\pi'}^S$ in terms of the analytic information of $L^S(s,\pi\times\tilde{\pi}')$.

A.1.3. Strategy of proof. Let

$$\mathbf{u}_\pi^S = \frac{\mathbf{v}_\pi^S}{\langle \mathbf{v}_\pi^S, \mathbf{v}_\pi^S\rangle^{1/2}}$$

be the projection of the vector $\mathbf{v}_\pi^S$ to the unit sphere in $V^S(X)$. The idea behind the proof of Theorem A.1 is to show that, for X large relative to Q,

(1) the map $\mathscr{F}_{\mathfrak{q},\chi}(Q) \to V^S$ given by $\pi \mapsto \mathbf{v}_\pi^S$ is injective;

(2) when $\pi,\pi' \in \mathscr{F}_{\mathfrak{q},\chi}(Q)$ are distinct, the vectors $\mathbf{u}_\pi^S$ and $\mathbf{u}_{\pi'}^S$ are quasi-orthogonal;

(3) there cannot be too many such quasi-orthogonal vectors.

Moreover, each of these steps will be seen to be quantifiable, polynomially in Q.

There is only one problem with this approach: we have thrown out the information at ramified primes. While this allows for a simpler presentation, the price to pay is a weaker bound in Theorem A.1. Indeed one obtains in this way the exponent $2n + n^2 + \varepsilon$ in the parameter Q, with or without assuming the Ramanujan conjecture and the Riemann hypothesis. See Remark A.16 for more details on the source of this loss by a power of n^2.

To obtain the unconditional bound of Theorem A.1 (as well as the conditional bound of Theorem A.6, which is sharp up to ε), we shall need to take into account the information at ramified primes. To adapt the above argument along these lines, one must explicate the Rankin–Selberg coefficients at ramified primes, which has been done by the first author in [61, Appendix]. In particular, we shall see in §A.2 that the “combinatorial distance to supercuspidal" of $\pi_S = \otimes_{\mathfrak{p}\in S} \pi_{\mathfrak{p}}$ governs the shape of the ramified Rankin–Selberg coefficients. Then, in §A.3, we further decompose $\mathscr{F}_{\mathfrak{q},\chi}(Q)$ according to this data. After an appropriate enrichening of the space $V^S(X)$ to take into account this information, we then execute the above three steps.

A.2. *Rankin–Selberg theory*

We now recall some of the basic local and global properties of the Rankin–Selberg L-function that we shall need in the proof of Theorem A.1.

A.2.1. Induction data. Let v be a finite place of F associated with a prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$. Let q_v be the cardinality of the residue field. Let π_v be an irreducible unitary generic representation of $\mathrm{GL}_n(F_v)$.

Recall that by the Bernstein–Zelevinsky description of the admissible dual, we may associate with π_v (see [61, §A.1]) the following *combinatorial* data:

(C1) a standard Levi subgroup $M \simeq \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ of GL_n;

(C2) a partition $\underline{J} = [J_1, \ldots, J_A]$ of the set $\{1, \ldots, r\}$;

(C3) an integer vector $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$, where $d_j \mid n_j$, such that $m_j = n_j/d_j$ is constant (say equal to m_a) along $j \in J_a$;

(C4) an integer vector $\mathbf{e} = (e_1, \ldots, e_A) \in \mathbb{N}^A$, where each e_a divides n;

the following *analytic* data:

(A1) real numbers $\sigma_1 \geq \cdots \geq \sigma_r$;

(A2) real numbers $t_1, \ldots, t_r$;

encoded in the complex numbers

$$s_j = \sigma_j + it_j \quad \text{and} \quad z_j = q_v^{-s_j - n_j/2};$$

as well as the following *arithmetic* data:

(SC) a set $\{\varrho_1, \ldots, \varrho_A\}$ of pairwise twist-inequivalent unitary supercuspidal representations ϱ_a of $\mathrm{GL}_{m_a}(F_v)$ having torsion number e_a.

The representation is π_v arises through induction in stages from the above data, as recalled in [61, §A.1].

A.2.2. Rankin–Selberg local factors. The local Rankin–Selberg L-factor can be expressed using the above combinatorial and analytic data. (The epsilon factor, on the other hand, encodes the arithmetic information contained in the choice of supercuspidal representations on each block. We do not define the epsilon factors here, but they are used implicitly in Appendix B.) We let $\mathrm{Comb}_v = \{(M, \underline{J}, \mathbf{d}, \mathbf{e})\}$ denote the collection of combinatorial data C1, C2, C3, C4. Let π_v and π_v' both have the same combinatorial type $(M, \underline{J}, \mathbf{d}, \mathbf{e})$. Let z_j, z_j' denote their respective analytic data.

By [61, §A.2, Example 6] we have

$$L(s,\pi_v\times\tilde{\pi}'_v)=\prod_{a=1}^{A}\prod_{j,k\in J_a}\prod_{\nu=1}^{\min(n_j,n_k)}\left(1-(q_v^\nu z_j\overline{z'_k})^{e_a}q_v^{-e_a s}\right)^{-1}$$
$$=\prod_{a=1}^{A}\prod_{\nu=1}^{n}\prod_{j,k\in J_a^\nu}\left(1-(q_v^\nu z_j\overline{z'_k})^{e_a}q_v^{-e_a s}\right)^{-1}, \tag{A.5}$$

where $J_a^\nu=\{j\in J_a: n_j\geq\nu\}$. We expand the expression (A.5) into the local Dirichlet series, which we again denote by $a_{\pi\times\tilde{\pi}'}(\mathfrak{p}^r)$. We shall now describe these in terms of the analytic data z_j, similarly to the unramified setting of §A.1.

We now furthermore assume that the central characters of π_v and π'_v coincide. We fix a and ν in (A.5) and expand the product over j and k. We obtain

$$\prod_{j,k\in J_a^\nu}\left(1-(q_v^\nu z_j\overline{z'_k})^{e_a}X^{e_a}\right)^{-1}=\sum_{r\geq 0}a_{\pi\times\tilde{\pi}'}(\mathfrak{p}^{e_a r};\nu,a)X^{e_a r}.$$

Cauchy's identity will once again allow us to describe the coefficients $a_{\pi\times\tilde{\pi}'}(\mathfrak{p}^{e_a r};\nu,a)$ as a combinatorial expression in terms of the local roots. With this in mind, we let $A_\pi(\mathfrak{p};a,\nu)$ denote the set of parameters $q_v^{\nu/2}z_j$, for $j\in J_a^\nu$, completed to a size n multiset by adding $n-|J_a^\nu|$ remaining zeros. For an integer $e\geq 1$ we write $A_\pi^e(\mathfrak{p};a,\nu)$ for the set of e-th powers of the parameters in $A_\pi(\mathfrak{p};a,\nu)$. We may then evaluate the Schur functions in n variables on $A_\pi^e(\mathfrak{p},a,\nu)$. Reasoning as in (A.3), we find

$$a_{\pi\times\tilde{\pi}'}(\mathfrak{p}^{e_a r};\nu,a)=\sum_{k\geq 0}\sum_{\mu\in\mathcal{P}_{n-1}(r-nk)}s_\mu(A_\pi^{e_a}(\mathfrak{p};a,\nu))s_\mu(A_{\tilde{\pi}'}^{e_a}(\mathfrak{p};a,\nu)). \tag{A.6}$$

Multiplying out ν and a in (A.5), we deduce that $a_{\pi\times\tilde{\pi}'}(\mathfrak{p}^r)$ is the complete homogeneous polynomial of degree r in the coefficients $a_{\pi\times\tilde{\pi}'}(\mathfrak{p}^{e_a f};\nu,a)$.

Remark A.10. The combinatorial data $M=T$, $\underline{J}=\{1,\dots,n\}$ (so that $A=1$), $\mathbf{d}=(1,\dots,1)$, and $\mathbf{e}=1$ corresponds to representations π_v which are, up to a character twist, unramified. In this case, the coefficient $a_{\pi\times\tilde{\pi}'}(\mathfrak{p}^r;\nu,1)$ is zero for all $\nu>1$, since all $n_j=1$. Thus $a_{\pi\times\tilde{\pi}'}(\mathfrak{p}^r)=a_{\pi\times\tilde{\pi}'}(\mathfrak{p}^r;1,1)$. Note that when π_v is unramified, $A_\pi(\mathfrak{p};1,1)$ is the set of the Satake parameters $A_\pi(\mathfrak{p})$, and (A.6) recovers (A.3).

Remark A.11. In [61, (A.6)], it is shown that when π_v and π'_v are irreducible unitary generic representations of $\mathrm{GL}_n(F_v)$ and $\mathrm{GL}_m(F_v)$, respectively, then

$$L(s,\pi_v\times\pi'_v)=\prod_{(a,b)\in\Delta}\prod_{j\in J_a}\prod_{k\in K_b}\prod_{\nu=1}^{\min(n_j,n'_k)}\left(1-(q_v^\nu z_j z'_k)^{e_{\ell(a,b)}}q_v^{-e_{\ell(a,b)}s}\right)^{-1}. \tag{A.7}$$

The expression (A.5) is a special case of this, when both π_v and π'_v have the same combinatorial type. See *loc. cit.* for relevant notation.

The local roots $q_v^\nu z_j z_k'$ in (A.7) satisfy $|q_v^\nu z_j z_k'| = q_v^{\nu-\sigma_j-\sigma_k'-n_j/2-n_k'/2}$. Under the Ramanujan conjecture, we have $\sigma_j = \sigma_k' = 0$, so that

$$|q_v^\nu z_j z_k'| = q_v^{\nu-n_j/2-n_k'/2} \le 1. \tag{A.8}$$

Unconditionally, the Jacquet–Shalika bounds [34] show that $0 \le |\sigma_j|, |\sigma_k'| < 1/2$, so that

$$|q_v^\nu z_j z_k'| < q_v^{\nu+1-n_j/2-n_k'/2} \le q_v. \tag{A.9}$$

Rudnick–Sarnak [58, Appendix] improved this to $q_v^{1-\delta}$, where $\delta = 1/(n^2+1) + 1/(m^2+1)$.

A.2.3. General formula for Dirichlet coefficients. We put together the descriptions of the prime-to-S coefficients in (A.2) with the ramified coefficients in (A.6).

We continue to write v for a finite place with associated prime ideal $\mathfrak{p}$. Recall the set Comb_v from §A.2.2, whose elements index the combinatorial data $\mathscr{C}_v = (M_v, \underline{J}_v, \mathbf{d}_v, \mathbf{e}_v)$ described in §A.2.1. Via the expansion (A.5), $\mathscr{C}_v$ gives rise to a set

$$\{A_\pi^{e_v}(\mathfrak{p}; a_v, \nu_v) : 1 \le a_v \le A_v, 1 \le \nu_v \le n\},$$

encoding the analytic data. We shall write $\mathrm{Index}(\mathscr{C}_v)$ for the indexing set of pairs (a_v, ν_v).

Now let S once again denote the prime support of the ideal $\mathfrak{q}$ and put $\mathrm{Comb}_S = \prod_{v\in S} \mathrm{Comb}_v$. Furthermore, for $\mathscr{C} \in \mathrm{Comb}_S$ we let $\mathrm{Index}(\mathscr{C}) = \prod_{v\in S} \mathrm{Index}(\mathscr{C}_v)$. For any $\mathscr{C} \in \mathrm{Comb}_S$, we let $\mathscr{A}_{\mathrm{cusp}}(\mathfrak{q}, \chi, \mathscr{C})$ denote the set of $\pi \in \mathscr{A}_{\mathrm{cusp}}(\mathfrak{q}, \chi)$ such that π_S has combinatorial data $\mathscr{C}$. Let $\pi, \pi' \in \mathscr{A}_{\mathrm{cusp}}(\mathfrak{q}, \chi, \mathscr{C})$. Recalling the notation in §A.1.1, let $\mathfrak{n}$ be an integral ideal and $\mathfrak{u} \in \mathscr{P}_{n-1}(\mathfrak{n})$. Let $(a, \nu) \in \mathrm{Index}(\mathscr{C})$. Generalizing (A.2), we write

$$a_\pi(\mathfrak{n}, \mathfrak{u}; a, \nu) = \prod_{\mathfrak{p}\notin S} s_{\mu_\mathfrak{p}}(A_\pi(\mathfrak{p})) \prod_{\mathfrak{p}\in S} s_{\mu_\mathfrak{p}}(A_\pi^{e_{a_\mathfrak{p}}}(\mathfrak{p}; a_\mathfrak{p}, \nu_\mathfrak{p})).$$

Then the Dirichlet coefficients of $L(s, \pi \times \widetilde{\pi}')$, denoted $a_{\pi\times\widetilde{\pi}'}(\mathfrak{n})$, can be written as

$$a_{\pi\times\widetilde{\pi}'}(\mathfrak{n}) = \sum_{\mathfrak{m}^n|\mathfrak{n}} \sum_{\mathfrak{u}\in\mathscr{P}_{n-1}(\mathfrak{n}/\mathfrak{m}^n)} \sum_{(a,\nu)\in\mathscr{C}} a_\pi(\mathfrak{n}/\mathfrak{m}^n, \mathfrak{u}; a, \nu)\overline{a_{\pi'}(\mathfrak{n}/\mathfrak{m}^n, \mathfrak{u}; a, \nu)}, \tag{A.10}$$

extending (A.4) to all ideal $\mathfrak{n} \in I$.

A.2.4. Global Rankin–Selberg estimates. We now recall a few basic analytic properties of the Rankin–Selberg L-function $L(s, \pi \times \widetilde{\pi}')$ associated with a pair $(\pi, \pi') \in \mathscr{F}_{\mathfrak{q},\chi}(Q) \times \mathscr{F}_{\mathfrak{q},\chi}(Q)$.

The convexity bound of Li [45] (see also [7] for the cases $n = 3, 4$) states that

$$\frac{s-1}{s-2} L(s, \pi \times \widetilde{\pi}') \ll (D^{n^2} C(\pi \times \widetilde{\pi}', s))^{(1-\sigma)/2} \quad (\mathrm{Re}(s) \le 1). \tag{A.11}$$

We have the factorization $C(\pi \times \widetilde{\pi}', s) = N_{\pi\times\widetilde{\pi}'} K_{\pi\times\widetilde{\pi}'}(s)$. For π_f and π'_f of conductor $\mathfrak{q}$, whose central characters are equal up to an unramified twist, Theorem B.1 of Appendix B implies that

$$N_{\pi\times\widetilde{\pi}'} \leq \mathrm{Norm}(\mathfrak{q})^{2n-2}. \tag{A.12}$$

Moreover, the bounds [43, Lemma A.2] imply

$$K_{\pi\times\widetilde{\pi}'}(s) \ll (1+|s|)^{dn^2}(K_\pi K_{\widetilde{\pi}'})^n. \tag{A.13}$$

We deduce that, for $\pi, \pi' \in \mathscr{F}_{\mathfrak{q},\chi}(Q)$, we have

$$\frac{s-1}{s-2}L(s, \pi\times\widetilde{\pi}') \ll (D^{n^2}(1+|s|)^{dn^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{(1-\sigma)/2} \qquad (\mathrm{Re}(s) \leq 1). \tag{A.14}$$

The function $L(s, \pi \times \widetilde{\pi}')$ is regular at $s = 1$ if and only if $\pi' \neq \pi$. In the case where $\pi' = \pi$, we have a lower bound of polynomial type on the residue at $s = 1$. Indeed, [6, Theorem 3] establishes the existence of an $A > 0$ such that

$$\operatorname*{Res}_{s=1} L(s, \pi \times \widetilde{\pi}) \gg (D^n Q)^{-A}. \tag{A.15}$$

Remark A.12. In [43] it is shown that $\operatorname*{Res}_{s=1} L(s, \pi \times \widetilde{\pi}) \gg (D^{4n^2} C(\Pi \times \widetilde{\Pi}))^{-\frac{7}{8}+\frac{5}{8n}-\varepsilon}$, where $\Pi = \pi \boxplus \pi$. From the upper bound (A.12), one may obtain an explicit admissible value of A. This exponent will not play a role in Theorem A.1.

A.3. Refining the setup in §A.1

We put $\mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q) = \mathscr{F}(Q) \cap \mathscr{A}_{\mathrm{cusp}}(\mathfrak{q}, \chi, \mathscr{C})$. Then

$$|\mathscr{F}_{\mathfrak{q},\chi}(Q)| = \sum_{\mathscr{C}\in\mathrm{Comb}_S} |\mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q)|. \tag{A.16}$$

We shall prove that

$$|\mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q)| \ll_\varepsilon (D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{1+\varepsilon}, \tag{A.17}$$

uniformly in $\mathfrak{q}$. Note that for every v we have $|\mathscr{C}_v| = O_n(1)$. Thus the number of terms in (A.16) is $|\mathscr{C}| = O(|S|^{O_n(1)}) = O(\log^{O_n(1)} Q)$. Inserting this into (A.16) and (A.1) will then prove Theorem A.1.

Recall the set $\mathscr{I}^S$ of $\mathcal{P}_{n-1}$-decorated prime-to-S ideals from §A.1.1. We shall now enrich $\mathscr{I}^S$ at the places in S to account for the combinatorial information $\mathscr{C} \in \mathrm{Comb}_S$. We shall define a $(\mathcal{P}_{n-1}, \mathscr{C})$*-decorated ideal* to be a triple $(\mathfrak{n}, \mathfrak{p}, (a, \nu))$, where $\mathfrak{n} \in I$ is an integral ideal, $\mathfrak{p} \in \mathscr{P}_{n-1}(\mathfrak{n})$, and $(a, \nu) \in \mathrm{Index}(\mathscr{C})$. We shall generally write this as $(\mathfrak{n}, \mathfrak{p}; a, \nu)$. The set of such triples will be denoted $\mathscr{I}_S$. We have a map $\mathscr{I}_S \to I$, $(\mathfrak{n}, \mathfrak{p}; a, \nu) \mapsto \mathfrak{n}$, where we forget the decorations and take the underlying ideal $\mathfrak{n}$. We sometimes write $\widetilde{\mathfrak{n}}$ for an element in $\mathscr{I}_S$ with underlying ideal $\mathfrak{n}$. Let $\mathscr{I}_S(X)$ denote the set of $\widetilde{\mathfrak{n}} \in \mathscr{I}_S$ with $\mathrm{Norm}(\mathfrak{n}) \leq X$.

Let $V_S(X)$ be the vector space of complex valued functions on $\mathscr{I}_S(X)$. Endow $V_S(X)$ with the standard scalar product

$$\langle a, b\rangle = \sum_{\tilde{\mathfrak{n}} \in \mathscr{I}_S(X)} f(\tilde{\mathfrak{n}})\overline{g(\tilde{\mathfrak{n}})}.$$

We shall map $\mathscr{A}_{\mathrm{cusp}}(\mathfrak{q}, \chi, \mathscr{C})$ to $V_S(X)$ by sending $\pi \in \mathscr{A}_{\mathrm{cusp}}(\mathfrak{q}, \chi, \mathscr{C})$ to the vector $\mathbf{v}_\pi \in V_S(X)$ given by the formula

$$\mathbf{v}_\pi(\tilde{\mathfrak{n}}) = \sqrt{F_X(\mathfrak{n})}\, a_\pi(\tilde{\mathfrak{n}}),$$

where $f : \mathbb{R} \to \mathbb{R}$ is as in §A.1 and

$$F_X(\mathfrak{n}) = \sum_{\mathfrak{m}} f(\mathrm{Norm}(\mathfrak{n}\mathfrak{m}^n)/X).$$

The above enrichment allows us to identify the inner product $\langle \mathbf{v}_\pi, \mathbf{v}_{\pi'}\rangle$ in terms of the *full* finite part Rankin–Selberg L-function. Indeed, by (A.10) and Mellin inversion we have

$$\langle \mathbf{v}_\pi, \mathbf{v}_{\pi'}\rangle = \sum_{\tilde{\mathfrak{n}} \in \mathscr{I}_S} F_X(\mathfrak{n}) a_\pi(\tilde{\mathfrak{n}})\overline{a_{\pi'}(\tilde{\mathfrak{n}})} = \frac{1}{2\pi i}\int_{(2)} L(s, \pi \times \tilde{\pi}')\hat{f}(s) X^s\, ds. \tag{A.18}$$

The above formula is the culmination of the combinatorial explication of the Rankin–Selberg L-functions in §A.1–A.3. It is the basis of the following section.

A.4. *Executing steps* (1) *and* (2)

We now execute the first two steps of the proof outline in §A.1, using the facts we collected from Rankin–Selberg theory in §A.2.4.

A.4.1. First step. We begin by establishing the following result.

Proposition A.13. *Let $\varepsilon > 0$ and*

$$X \gg (D^{n^2} \mathrm{Norm}(\mathfrak{q})^{-2} Q^{2n})^{1+\varepsilon}.$$

Then the map $\mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q) \to V_S(X)$ given by $\pi \mapsto \mathbf{v}_\pi$ is injective.

Proof. Indeed, [6, Theorem 7] shows the existence of a $B > 0$ such that when $X \gg (D^{n/2}Q)^B$ any pair $(\pi, \pi') \in \mathscr{F}_{\mathfrak{q}}(Q) \times \mathscr{F}_{\mathfrak{q}}(Q)$ satisfying $a_\pi(\tilde{\mathfrak{n}}) = a_{\pi'}(\tilde{\mathfrak{n}})$ for $\tilde{\mathfrak{n}} \in \mathscr{I}_S$ lies along the diagonal $\pi = \pi'$. It is shown in [48] that an admissible value for the exponent B is $2n + \varepsilon$, for any $\varepsilon > 0$. (In *loc. cit.* the discriminant dependence is actually $D^{2n^2+\varepsilon}$ rather than $D^{n^2+\varepsilon}$ as we have written. This is due to an inefficient application of the Bushnell–Henniart bounds in the discriminant aspect. See Remark A.5.) In fact, their result can be refined, under the assumption that π_f and π'_f have the same (finite) conductor $\mathfrak{q}$ and central character χ. Indeed, in this case, the bounds of Theorem B.1 of Appendix B save $\mathrm{Norm}(\mathfrak{q})^2$ off of this. ∎

A.4.2. Second step. As in §A.1, we let

$$\mathbf{u}_\pi = \frac{\mathbf{v}_\pi}{\langle \mathbf{v}_\pi, \mathbf{v}_\pi\rangle^{1/2}}$$

be the projection of the vector $\mathbf{v}_\pi$ to the unit sphere in V. We now proceed to show that the vectors $\mathbf{u}_\pi$ and $\mathbf{u}_{\pi'}$ (for $\pi \neq \pi'$) are quasi-orthogonal, in a quantifiable sense.

Proposition A.14. *Let $(\pi, \pi') \in \mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q) \times \mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q)$. For $\varepsilon > 0$ let*

$$X \gg (D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{1/2+\varepsilon}.$$

If $\pi' \neq \pi$ then $\langle \mathbf{u}_\pi, \mathbf{u}_{\pi'}\rangle \ll_{\varepsilon,r} (D^n Q)^{-r}$ for all $r > 0$.

Proof. We shall show that there is $C > 0$ such that for $X \gg (D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{1/2+\varepsilon}$ any pair $(\pi, \pi') \in \mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q) \times \mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q)$ satisfies

$$\begin{cases} \langle \mathbf{v}_\pi, \mathbf{v}_\pi\rangle \gg (D^n Q)^{-C} & \text{if } \pi = \pi', \\ \langle \mathbf{v}_\pi, \mathbf{v}_{\pi'}\rangle \ll_{\varepsilon,r} (D^n Q)^{-r} & \text{if } \pi \neq \pi'. \end{cases} \tag{A.19}$$

(Using Remark A.12, we can find an explicit value of C but this value is irrelevant for the proof of this proposition.) These two estimates imply the result.

Recall the identity (A.18). By hypothesis $\hat{f}(1) = 1$, and since f is of compact support, $\hat{f}(s)$ is entire. Using (A.14), we shift the contour to $(-r)$ for $r > 0$ to obtain

$$\langle \mathbf{v}_\pi, \mathbf{v}_{\pi'}\rangle = \operatorname*{Res}_{s=1} L(s, \pi \times \tilde{\pi}')X + O_r((D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{(1+r)/2}X^{-r}).$$

If $\pi \neq \pi'$ then the residual term vanishes, and hence

$$\langle \mathbf{v}_\pi, \mathbf{v}_{\pi'}\rangle \ll_r (D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{(1+r)/2}X^{-r}.$$

If $\pi = \pi'$ we recall the lower bound (A.15). This produces

$$\langle \mathbf{v}_\pi, \mathbf{v}_\pi\rangle \gg (D^n Q)^{-A}X + O_r((D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{(1+r)/2}X^{-r}).$$

Letting $X \gg (D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{1/2+\varepsilon}$, we take r sufficiently large (relative to n and ε) to arrive at the two estimates in (A.19). ∎

Remark A.15. We note that we could avoid quoting the convexity bound (A.11) of [45] by dualizing the L-function, as was done, for example, in [48]. This does not, however, lead to an improvement in the resulting bounds.

Remark A.16. The analog of Proposition A.14, when stated with $\langle \mathbf{v}_\pi^S, \mathbf{v}_{\pi'}^S\rangle$, would incur a loss of n^2 in the power of Q. Indeed, with the setup of §A.1 one needs to bound

$$L^S(s, \pi \times \tilde{\pi}') = L(s, \pi \times \tilde{\pi}')L_S(s, \pi \times \tilde{\pi}')^{-1}$$

for $\mathrm{Re}(s) = \sigma \to -\infty$. From (A.5), each local correction factor $L_v(s, \pi \times \widetilde{\pi}')^{-1}$ is the product of at most n^2 local factors of the form $1 - \alpha_{\pi\times\pi'}(\mathfrak{p}; v, j, k)^e q_v^{-es}$. By Remark A.11, and in particular the bound (A.9) on the Rankin–Selberg local roots, we deduce that $L_v(s, \pi \times \widetilde{\pi}')^{-1} \ll (1 + q_v^{1-\sigma})^{n^2}$. Thus, for $\mathrm{Re}(s) = \sigma < 0$, we have $L_S(s, \pi \times \widetilde{\pi}')^{-1} \ll (\prod_{v|\mathfrak{q}} q_v)^{n^2(1-\sigma)}$, which accounts for the weakened exponent. Moreover, the same loss by Q^{n^2} would arise in the proof of Proposition A.13, were we only to assume that $\pi_\mathfrak{p} \simeq \pi'_\mathfrak{p}$ for $\mathfrak{p} \nmid \mathfrak{q}$.

Note that in the critical strip the correction factor $L_S(s, \pi \times \widetilde{\pi}')^{-1}$ is uniformly bounded under the Ramanujan conjecture (see (A.8)). Nevertheless, a contour shift to anywhere within the critical strip leads to insufficient correlation bounds relative to known sphere packing bounds.

A.5. *Executing step* (3)

We finally come to the fact that a large-dimensional sphere can only contain so many quasi-orthogonal vectors.

Let N denote the cardinality of the set $\mathscr{I}_S(X)$; this is the same as the dimension of $V_S(X)$. Denote by K the cardinality of $\mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q)$. We shall show that $K \le N$ when $X = (D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{1+\varepsilon}$. Since $N \asymp X = (D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{1+\varepsilon}$, this will complete the proof of (A.17), and hence of Theorem A.1.

By our choice of X, we may apply both Propositions A.13 and A.14, so that $\mathscr{F}_{\mathfrak{q},\chi,\mathscr{C}}(Q)$ can be viewed as a finite system of unitary quasi-orthogonal vectors in $V_S(X)$. The following abstract result establishes the desired bound. To apply it to our situation, we identify $V_S(X) = \mathbb{C}^N = \mathbb{R}^M$, where $M = 2N$.

Proposition A.17. *Let $M \ge 2$ and put $V = \mathbb{R}^M$. Let $\mathbf{u}_1, \dots, \mathbf{u}_K \in V$ be unitary vectors such that $|\langle \mathbf{u}_i, \mathbf{u}_j\rangle| < M^{-1}$ for $i \ne j$. Then $K \le M$.*

Before passing to the proof of Proposition A.17, we make several remarks.

Remark A.18. The conclusion of the proposition is sharp, since one can certainly put M orthonormal vectors (and no more) on the unit sphere in $\mathbb{R}^M$. The idea of the proof of Proposition A.17 is that, in high dimensions, a $1/M$ error off of strict orthogonality is imperceptible. (In fact, a $\frac{1}{2}M^{-1/2}$ error is provably imperceptible: see Remark A.19.)

Note that the quasi-orthogonality relations established in Proposition A.14 for the family $\{\mathbf{u}_\pi : \pi \in \mathscr{F}(Q)\}$ are much stronger (rapid decay) that the required bounds for Proposition A.17. However, it is of no advantage to have $O_r(M^{-r})$ correlation decay, instead of the required rate of $O(1/M)$, since in any case, strictly vanishing off-diagonal correlations (an orthonormal basis) still produce $K = M$.

Remark A.19. Let $M \ge 2$ and $\theta \in [0, \pi)$. Denote by $A(M, \theta)$ the maximum cardinality of a subset $\{\mathbf{u}_1, \dots, \mathbf{u}_K\}$ of S^{M-1} with $\max_{i\ne j}\langle \mathbf{u}_i, \mathbf{u}_j\rangle \le \cos\theta$. Such a subset is called a *spherical code*.

If $\theta > \pi/2$, then an elementary argument shows that $A(M,\theta)$ is bounded by an expression depending only on θ. Indeed, as remarked in [28, §3.2], we have

$$0 \leq \langle \mathbf{u}_1 + \cdots + \mathbf{u}_K, \mathbf{u}_1 + \cdots + \mathbf{u}_K \rangle \leq K + K(K-1)\cos\theta. \tag{A.20}$$

Thus, if $\cos\theta$ is strictly negative, this provides a bound for $A(M,\theta)$ which depends only on θ. In particular, if $\theta > \pi/2$ is *fixed*, then $A(M,\theta)$ is bounded uniformly in M.

If $\theta > \pi/2$ is now allowed to depend on M, then (A.20) still yields an upper bound on $A(M,\theta)$. For example, if $\cos\theta = -M^{-\alpha}$ for some $\alpha \geq 0$, we obtain $A(M,\theta) \leq M^{\alpha}$. As α varies through the interval $[0,1]$, this provides an interpolation of the uniformly bounded range (where $\theta > 1/2$ is fixed) and the range treated by Proposition A.17.

On the other hand, when $\theta < \pi/2$ is fixed, then $A(M,\theta)$ grows exponentially in M. The work of Kabatyanskiĭ–Levenshteĭn [35] provides upper bounds in this regime. It is known, however, that for $\theta = \pi/2 - c/\sqrt{M}$, one still retains a polynomial upper bound. See [17, Theorem 2.1] and [63]. Indeed, Lemma 2 of *loc. cit* shows that one retains a *linear bound* as long as $\cos\theta \leq \frac{1}{2}M^{-1/2}$. This last result would in fact be sufficient for our purposes.

Proof of Proposition A.17. An elementary exercise establishes the result for $M = 2$. Suppose the result is true in dimension $M-1$. We claim this implies the result in dimension M.

Let W be the orthogonal complement to $\mathbf{u}_K$ in $V = \mathbb{R}^M$. For every $1 \leq i \leq K-1$ let $\mathbf{w}_i$ be the projection of the vector $\mathbf{u}_i$ to W. We define $\lambda_i \in \mathbb{R}$ by the equality $\mathbf{w}_i = \mathbf{u}_i - \lambda_i \mathbf{u}_K$; then $\lambda_i = \langle \mathbf{u}_i, \mathbf{u}_K \rangle$. For $1 \leq i, j \leq K-1$ we have $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \langle \mathbf{u}_i, \mathbf{u}_j \rangle + \lambda_i \lambda_j$. If $i = j$ we obtain $\|\mathbf{w}_i\|^2 = 1 + \lambda_i^2$. By hypothesis, $|\lambda_i| < 1/M$, which implies that $\|\mathbf{w}_i\|^2 > 1 - M^{-2} = M^{-2}(M^2-1)$. Moreover, if $i \neq j$ we have $|\langle \mathbf{u}_i, \mathbf{u}_j \rangle| < 1/M$; thus

$$|\langle \mathbf{w}_i, \mathbf{w}_j \rangle| < 1/M + 1/M^2 = M^{-2}(M+1).$$

Now, consider the $K-1$ unitary vectors $\mathbf{u}'_j = \mathbf{w}_j/\|\mathbf{w}_j\|$ in the $(M-1)$-dimensional subspace W. For $1 \leq i \neq j \leq K-1$ we have

$$|\langle \mathbf{u}'_i, \mathbf{u}'_j \rangle| = \|\mathbf{w}_i\|^{-1}\|\mathbf{w}_j\|^{-1}|\langle \mathbf{w}_i, \mathbf{w}_j \rangle| < \frac{M^2}{M^2-1} \cdot \frac{M+1}{M^2} = \frac{1}{M-1}.$$

From our recurrence hypothesis, we deduce that $K-1 \leq M-1$, as claimed. ∎

Remark A.20. The above induction argument works under the more general hypothesis that $\max_{i\neq j} |\langle \mathbf{u}_i, \mathbf{u}_j \rangle| < f(M)$, for any function f satisfying $\frac{f(M)}{1-f(M)} \leq f(M-1)$. But if $f(M) = M^{-\alpha}$, this inequality reads $(1-1/M)^{\alpha} \leq 1 - M^{-\alpha}$. The left hand side is approximated by $1 - \alpha/M$, and one sees that one can do no better than $\alpha = 1$.

A.6. Proof of Theorem A.6

We now address the question of improving the upper bound on $|\mathscr{F}(Q)|$ in Theorem A.1, under the Riemann hypothesis for Rankin–Selberg L-functions as well as the Ramanujan conjecture at finite places for members of $\mathscr{A}_{\mathrm{cusp}}$.

It is easy to see that the exponent of $2n$ in Theorem A.1 can be improved to $n+1$ under these assumptions, and that the discriminant dependence is as described there. This is due to the fact that, under Riemann and Ramanujan, the map $\pi \mapsto \mathbf{v}_\pi$ is injective as soon as $X \gg \log^2 Q$ (see, for example, [32, Proposition 5.22]). This replaces step (1) in the proof of Theorem A.1. On the other hand, the proof of Proposition A.14 is insensitive to the Riemann hypothesis and the Ramanujan conjecture, despite the fact that the residue of the $L(s, \pi \times \widetilde{\pi})$ is bounded below by $1/\log Q$ under these assumptions (see [32, Theorem 5.19]). In any case, with Theorem A.13 improved, we may take $X = (D^{n^2}\mathrm{Norm}(\mathfrak{q})^{-2}Q^{2n})^{1/2+\varepsilon}$ in executing step (3). Indeed, the exponent of Q required for the value of X in step (3) is the maximum of the exponents coming from Propositions A.13 and A.14. Inserting this into (A.16) and (A.1) will then prove Theorem A.6.

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Appendix B. A bound for the Artin exponent of a pair

Let F be a locally compact nonarchimedean field, and n, m two positive integers. Let π be a smooth irreducible representation of $\mathrm{GL}_n(F)$, with central character ω_π and Artin conductor $\mathrm{Ar}(\pi) = a$, and let ρ be a smooth irreducible representation of $\mathrm{GL}_m(F)$, with central character ω_ρ and Artin conductor $\mathrm{Ar}(\rho) = b$.

In [9] and [10, Theorem C], we proved that the pair (π, ρ) satisfies

$$\mathrm{Ar}(\pi \times \rho) \leq ma + nb - \min(a, b). \tag{B.1}$$

That bound cannot be improved in general but here, prompted by a query of F. Brumley, we improve (B.1) under an additional hypothesis.

Theorem B.1. *Assume that $\omega_\pi\omega_\rho$ is unramified. Then*

$$\mathrm{Ar}(\pi \times \rho) \leq ma + nb - 2\min(a, b). \tag{B.2}$$

When $n = m$ and $a = b$, this gives $\mathrm{Ar}(\pi \times \rho) \leq (2n-2)a$, as used in the main text. Note also that when $n = m = 1$ the hypothesis implies $a = b$ and $\mathrm{Ar}(\pi \times \rho) = 0$, which is fortunate since the right hand side of (B.2) is also 0!

Thanks to the Langlands correspondence, we may express the theorem in terms of Weil–Deligne representations, and we indeed use that language in the proofs. We fix a separable algebraic closure F^{sep} of F and let W_F be the Weil group of F^{sep} over F. We write σ, τ for the Weil–Deligne representations corresponding to π, ρ: they are directs sums of indecomposable Weil–Deligne representations. The theorem above is then equivalent to

Theorem B.2. *Assume that* $\det\sigma \det\tau$ *is unramified. Then*

$$\mathrm{Ar}(\sigma \otimes \tau) \leq ma + nb - 2\min(a, b). \tag{B.3}$$

Remark B.3. Assume that σ is the direct sum of characters of W_F, all trivial but one, which then has to be $\det\sigma$. Take for τ the contragredient $\widetilde{\sigma}$ of σ. Then $\mathrm{Ar}(\sigma \otimes \widetilde{\sigma}) = (2n-2)a$, so one cannot improve (B.3) or (B.2) in general, even assuming that $\tau = \widetilde{\sigma}$.

We now proceed to the proof, relying on the results and techniques of [10].

B.1. A basic point is a stronger inequality than (B.1) when σ and τ are indecomposable.

Lemma B.4. *Assume that σ and τ are indecomposable. Then*

$$\mathrm{Ar}(\sigma \otimes \tau)/(nm) \leq \max(a/n, b/m), \quad \text{with equality if } a/n \neq b/m.$$

Proof. The case of inequality is [10, Proposition 6.3]. The case of equality can be deduced from [10, Proposition 5.5]. ∎

Lemma B.5. *Assume that σ is indecomposable. Then* $\mathrm{Ar}(\det \sigma) \leq a/n$.

Proof. By [10, Fact 2.1] and the notation there, we have $\sigma = \mathrm{St}_r(\sigma')$, for some positive integer r and some irreducible representation σ' of W_F.

If σ' is an unramified character of W_F then $r = n$ and $a = n - 1$, whereas $\det \sigma$ is unramified, so $\mathrm{Ar}(\det \sigma) = 0 \leq a/n$. If σ' is not an unramified character, then $a = r\,\mathrm{Ar}(\sigma')$ and $\det \sigma = (\det \sigma)^r$, so it is enough to treat the case where $\sigma = \sigma'$ is irreducible (and not unramified). But then $a - n$ is the Swan exponent of σ, so, using [10, Fact 2.3],

$$\frac{a}{n} - 1 = \inf\{\varepsilon > 0 \colon \sigma(W_F^{\varepsilon}) = 1\}.$$

Since $\det \sigma$ is certainly trivial on the ramification subgroup W_F^{ε} if σ is, we see that the Swan exponent of $\det \sigma$ is at most $a/n - 1$, so $\mathrm{Ar}(\det \sigma) \leq a/n$. ∎

Let us define the list of slopes of σ. When indecomposable, σ has a list of n slopes, all equal to a/n. In general the list of slopes of σ is obtained by gathering the lists of slopes of its indecomposable summands, in increasing order. We write $(a_1, \dots, a_n)$ for the list of slopes of σ, and $(b_1, \dots, b_m)$ for the list of slopes of τ; in particular, $a = a_1 + \cdots + a_n$ and $b = b_1 + \cdots + b_m$.

Applying Lemmas B.4 and B.5 to the indecomposable summands of σ and τ we get

Corollary B.6. *The following holds:*

(i) $\mathrm{Ar}(\det \sigma) \leq a_n$.

(ii) $\mathrm{Ar}(\sigma \otimes \tau) \leq n\,\mathrm{Ar}(\tau)$ *if* $a_n \leq b_1$, *with equality if* $a_n < b_1$.

B.2. In this subsection, we assume $n = 1$. As the case $n = m = 1$ is done, we also assume $m > 1$.

We first deal with the case $b_{m-1} < b_m$. Then we can write $\tau = \tau' \oplus \eta$ for a character η of W_F with $\mathrm{Ar}(\eta) = b_m$. By Corollary B.6(i), $\mathrm{Ar}(\det \tau') \leq b_{m-1}$ and since $\det \tau = (\det \tau')\eta$ we get $\mathrm{Ar}(\det \tau) = b_m$. But $\sigma = \det \sigma$ and $\det \sigma \det \tau$ is unramified, so we have $a = b_m$.

By Corollary B.6(ii), we have $\mathrm{Ar}(\sigma \otimes \tau') = (m-1)a$ since $a = b_m > b_{m-1}$. We also have $\mathrm{Ar}(\sigma \otimes \eta) = \mathrm{Ar}((\det \sigma)\eta) = \mathrm{Ar}((\det \tau)^{-1}\eta)$, so $\mathrm{Ar}(\sigma \otimes \eta) = \mathrm{Ar}(\det \tau') \leq b_{m-1}$. Adding, we get $\mathrm{Ar}(\sigma \otimes \tau) \leq (m-1)a + b_{m-1}$.

On the other hand, $b \geq b_m = a$ hence $\min(a, b) = a$ and

$$ma + b - 2\min(a, b) = (m-2)a + b \geq (m-1)a + b_{m-1}$$

because $b \geq b_{m-1} + b_m = a + b_{m-1}$. We have proved (B.3) when $b_{m-1} < b_m$.

We now assume that $b_{m-1} = b_m$. By Corollary B.6(i), $\mathrm{Ar}(\det \tau) \le b_m$ and reasoning as above, we now get $a \le b_m$ from Corollary B.6(ii). Write $\tau = \tau' \oplus \eta$, where η is a Weil–Deligne representation with dimension $d \ge 2$ and all slopes equal to b_m. Let $b' = \mathrm{Ar}(\tau')$, so $b = b' + db_m$. We have $\mathrm{Ar}(\sigma \otimes \tau') \le (m-d)a + b' - \min(a, b')$: this follows from (B.1) if $\tau' \ne 0$, and $m = d$, $b' = 0$ if $\tau' = 0$.

On the other hand, $\mathrm{Ar}(\sigma \otimes \eta) \le db_m$ by Corollary B.6(ii), since $a \le b_m$. Adding, we obtain

$$\mathrm{Ar}(\sigma \otimes \tau) \le (m-d)a + b' + db_m - \min(a, b'),$$

so the result follows, provided $da + \min(a, b') \ge \min(a, b' + db_m)$, which is clear since $d \ge 2$. This again proves (B.3).

B.3. From now on we assume $n, m > 1$.

We first deal with the situation where $a_{n-1} < a_n$ and $b_{m-1} < b_m$. Accordingly, we write $\sigma = \sigma' \oplus \chi$ for a character χ of W_F with $\mathrm{Ar}(\chi) = a_n$, and $\tau = \tau' \oplus \eta$, for a character η of W_F with $\mathrm{Ar}(\eta) = b_m$. We put $a' = \mathrm{Ar}(\sigma')$, $b' = \mathrm{Ar}(\tau')$, so $a = a' + a_n$ and $b = b' + b_m$. Reasoning as above, we get $\mathrm{Ar}(\det \sigma') \le a_{n-1}$, $\mathrm{Ar}(\det \tau') \le b_{m-1}$, $a_n = b_m$, and $\mathrm{Ar}(\chi\eta) \le \max(a_{n-1}, b_{m-1})$.

On the other hand, by (B.1) we have

$$\mathrm{Ar}(\sigma' \otimes \tau') \le (m-1)a' + (n-1)b' - \min(a', b'),$$

and by Corollary B.6(ii) again, $\mathrm{Ar}(\sigma' \otimes \eta) = (n-1)b_m = (n-1)a_n$ and $\mathrm{Ar}(\chi \otimes \tau') = (n-1)a_n$. Adding, we get

$$\mathrm{Ar}(\sigma \otimes \tau) \le (m-1)a + (n-1)b - \min(a', b') + \max(a_{n-1}, b_{m-1}).$$

The result then follows provided that $a + b + \min(a', b') \ge 2\min(a, b) + \max(a_{n-1}, b_{m-1})$, or equivalently

$$a' + b' + \min(a', b') \ge 2\min(a', b') + \max(a_{n-1}, b_{m-1}). \tag{B.4}$$

But $a' + b' = \min(a', b') + \max(a', b')$ and $\max(a', b') \ge \max(a_{n-1}, b_{m-1})$ because $a' \ge a_{n-1}$ and $b' \ge b_{m-1}$, establishing (B.4).

B.4. We turn to the case where $a_{n-1} < a_n$ but $b_{m-1} = b_m$. Write $\sigma = \sigma' \oplus \chi$ for a character χ of W_F with $\mathrm{Ar}(\chi) = a_n$, and $\tau = \tau' \oplus \eta$ for a Weil–Deligne representation η with dimension $d \ge 2$ and all slopes equal to b_m. Put $a' = \mathrm{Ar}(\sigma')$ and $b' = \mathrm{Ar}(\tau')$, so $a = a' + a_n$ and $b = b' + db_m$.

As in the second case of §B.2, we get $a_n \le b_m$ and $\mathrm{Ar}(\chi \otimes \eta) \le db_m$ by Corollary B.6(ii).

By (B.1) (or because $\tau' = 0$) we have

$$\mathrm{Ar}(\sigma' \otimes \tau') \le (m-d)a' + (n-1)b' - \min(a', b').$$

Because $a_n \le b_m$ we have $a_{n-1} < b_m$ so, by Corollary B.6(ii), $\mathrm{Ar}(\sigma' \otimes \eta) = (n-1)db_m$. Applying Lemma B.4 to $\chi \otimes \tau_j$ where τ_j is an indecomposable summand of τ', we obtain

$$\mathrm{Ar}(\chi \otimes \tau') \le \sum_{i=1}^{m-d} \max(a_n, b_i).$$

Adding gives

$$\mathrm{Ar}(\sigma \otimes \tau) \le (m-d)a' + ndb_m + (n-1)b' - \min(a', b') + \sum_{i=1}^{m-d} \max(a_n, b_i). \tag{B.5}$$

We claim that the right hand side of (B.5) is at most $m(a' + a_n) + n(b' + db_m) - 2\min(a' + a_n, b' + db_m)$, or equivalently that

$$2\min(a' + a_n, b' + db_m) + \sum_{i=1}^{m-d} \max(a_n, b_i) \le ma_n + da' + b' + \min(a', b'). \tag{B.6}$$

Indeed, since $\sum_{i=1}^{m-d} \max(a_n, b_i) \le (m-d)a_n + b'$ and $d \ge 2$, we have

$$\begin{aligned} 2\min(a' + a_n, b' + db_m) + \sum_{i=1}^{m-d} \max(a_n, b_i) &\le 2(a' + a_n) + (m-d)a_n + b' \\ &\le ma_n + da' + b', \end{aligned}$$

establishing (B.6). By symmetry, the case where $a_{n-1} = a_n$ but $b_{m-1} < b_m$ also holds.

B.5. The final case is when $a_{n-1} = a_n$ and $b_{m-1} = b_m$. Here the hypothesis that $\det\sigma \det\tau$ is unramified plays no role. By symmetry we may and do assume $a_n \le b_m$.

We write $\sigma = \sigma' \oplus \chi$ for a Weil–Deligne representation χ with dimension $e \ge 2$ and all slopes equal to a_n, and $\tau = \tau' \oplus \eta$ as in §B.4. We put $a' = \mathrm{Ar}(\sigma')$, $b' = \mathrm{Ar}(\tau')$, so that $a = a' + ea_n$, $b = b' + db_m$. By (B.1) (or because σ' or τ' is 0) we have $\mathrm{Ar}(\sigma' \otimes \tau') \le (m-d)a' + (n-e)b' - \min(a', b')$. Since $a_n \le b_m$ by hypothesis, from Corollary B.6 we get $\mathrm{Ar}(\sigma' \otimes \eta) \le d(n-e)b_m$ and $\mathrm{Ar}(\chi \otimes \eta) \le db_m$. As in §B.4 we get, by Lemma B.4,

$$\mathrm{Ar}(\chi \otimes \tau') \le \sum_{i=1}^{m-d} e \max(a_n, b_i).$$

Adding, this gives

$$\mathrm{Ar}(\sigma \otimes \tau) \le (m-d)a' + (n-e)b' + ndb_m - \min(a', b') + e\sum_{i=1}^{m-d} \max(a_n, b_i). \tag{B.7}$$

We claim that the right hand side of (B.7) is at most $ma' + mea_n + nb' + ndb_m - \min(a' + ea_n, b' + db_m)$. This is equivalent to the inequality

$$da' + mea_n + eb' + \min(a', b') \\ \geq \min(a' + ea_n, b' + db_m) + e \sum_{i=1}^{m-d} \max(a_n, b_i). \quad \text{(B.8)}$$

Indeed, using $\sum_{i=1}^{m-d} \max(a_n, b_i) \leq (m-d)a_n + b'$ and $d \geq 2$, as in §B.4, we deduce (B.8).

B.6. With an entirely similar reasoning, but replacing Artin exponents Ar with Swan exponents Sw, we get the following result, improving [10, Theorem CS] in a special case.

Theorem B.7. *Let σ and τ be semisimple representations of W_F. Assume that* $\mathrm{Sw}(\det\sigma\det\tau) = 0$. *Then*

$$\mathrm{Sw}(\sigma \otimes \tau) \leq (\dim\tau)\mathrm{Sw}(\sigma) + (\dim\sigma)\mathrm{Sw}(\tau) - 2\min(\mathrm{Sw}(\sigma), \mathrm{Sw}(\tau)).$$

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