J. Eur. Math. Soc. 24, 1593–1615 (2022) DOI 10.4171/JEMS/1114

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Stable Gabor phase retrieval for multivariate functions

Received March 1, 2019

Abstract. In recent work [P. Grohs and M. Rathmair, Stable Gabor phase retrieval and spectral clustering, Comm. Pure Appl. Math. (2018)] the instabilities of the Gabor phase retrieval problem, i.e., the problem of reconstructing a function f from its spectrogram $|\mathcal{G} f|$, where

$$\mathcal{G} f(x, y) = \int_{\mathbb{R}^d} f(t)e^{-\pi|t-x|^2}e^{-2\pi i t\cdot y}\,dt, \quad x, y \in \mathbb{R}^d,$$

have been completely classified in terms of the disconnectedness of the spectrogram. These findings, however, were crucially restricted to the one-dimensional case ($d = 1$) and therefore not relevant for many practical applications.

In the present paper we not only generalize the aforementioned results to the multivariate case but also significantly improve on them. Our new results have comprehensive implications in various applications such as ptychography, a highly popular method in coherent diffraction imaging.

Keywords. Phase retrieval, stability, Gabor transform, Cheeger constant, logarithmic derivative

1. Introduction

1.1. Motivation

Phase retrieval in its most general formulation is concerned with the reconstruction of a signal $f \in \mathcal{B}$ with $\mathcal{B}$ a Banach space from phaseless linear measurements

$$\mathcal{A} f := (|\phi_\omega(f)|)_{\omega\in\Omega}\,, \tag{1.1}$$

where $\Phi = (\phi_\omega)_{\omega\in\Omega} \subset \mathcal{B}'$, the dual of $\mathcal{B}$.

Problems of this kind appear in a vast number of physical applications, the most prominent example being coherent diffraction imaging [15, 18], where one seeks to

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Mathematics Subject Classification (2020): Primary 46E30; Secondary 32A15, 42B10, 30H20

recover a function from phaseless Fourier type measurements, so called diffraction patterns. Further applications include radar [14], astronomy [9], audio [19] and quantum mechanics [6] to mention only a few.

Usually the measurement vectors Φ are such that $f \mapsto (\phi_\omega(f))_{\omega\in\Omega}$ is nicely invertible, meaning that reconstructing f would not be a significant problem if the phases of the measurements were available. The removal of phase, however, not only involves the loss of a huge amount of information but also renders the problem nonlinear. It is therefore notoriously difficult to even decide whether a concrete phase retrieval problem is well posed, i.e., whether the measurements $\mathcal{A}f$ uniquely and stably determine the underlying signal f in the following sense:

Uniqueness: Is the mapping $f \mapsto \mathcal{A}f$ injective up to the identification $f \sim e^{i\alpha} f$ for $\alpha \in \mathbb{R}$?

Stability: What is the qualitative behaviour of the local stability constant, i.e., the smallest number $c(f)$ such that

$$d_{\mathcal{B}}(g, f) := \inf_{|a|=1} \|g - af\|_{\mathcal{B}} \le c(f) d'(\mathcal{A}g, \mathcal{A}f) \quad \forall g \in \mathcal{B}, \tag{1.2}$$

where d' denotes a suitable metric on the measurement space?

At this point we would like to draw attention to our very recent article together with Sarah Koppensteiner [11] where, among other things, our current understanding of uniqueness and stability for phase retrieval is summarized.

The present paper is concerned with the study of stability when the measurements arise from the so called Gabor transform. The Gabor transform is just the short time Fourier transform with Gaussian window.

Definition 1.1. The *Gabor transform* of $f \in L^2(\mathbb{R}^d)$ is defined by

$$\mathcal{G}f(x, y) = \int_{\mathbb{R}^d} f(t) e^{-\pi|t-x|^2} e^{-2\pi i t\cdot y} dt, \quad x, y \in \mathbb{R}^d.$$

By duality the definition can be extended to the dual of the space of Schwartz functions, i.e., the space of tempered distributions denoted by $\mathcal{S}'(\mathbb{R}^d)$.

Note that by choosing the measurement vectors to be time-frequency shifts of the Gaussian

$$\phi_\omega(t) = e^{-\pi|t-x|^2} e^{-2\pi i y\cdot t}, \quad \omega = (x, y) \in \Omega \subset \mathbb{R}^{2d},$$

the Gabor transform fits right into our setting, i.e., in that case $\mathcal{A}f$ as defined in (1.1) coincides with $|\mathcal{G}f|$.

The Gabor transform can be interpreted as localization of f at x followed by Fourier transform,

$$\mathcal{G}f(x, y) = \mathcal{F}\big(f e^{-\pi|\cdot-x|^2}\big)(y).$$

Thus, for a two-dimensional object, represented by $f \in L^2(\mathbb{R}^2)$, the magnitude of the Gabor transform $|\mathcal{G}f(x,\cdot)|$ describes the diffraction pattern of the localization of f at x.

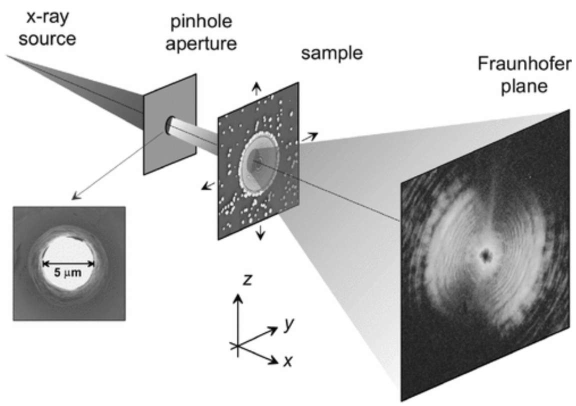

Fig. 1. Schematic setup of ptychographical experiment. Image taken from [16].

Hence the Gabor transform perfectly mimics the concept of *ptychography*, a popular and highly successful approach in coherent diffraction imaging based on the idea that multiple diffraction patterns of one and the same object are generated by illuminating different sections of the object separately in order to introduce redundancy (cf. Figure 1).

1.2. Related work

We will now briefly discuss results regarding stability properties of phase retrieval in infinite-dimensional spaces. All results into this direction are fairly recent.

First of all, inconveniently, phase retrieval in infinite dimensions is severely ill-posed as it can never be uniformly stable, in the sense that $c(f)$ in (1.2) can never be uniformly bounded, i.e., $\sup_{f\in\mathcal{B}} c(f) = +\infty$, under very general assumptions on $\mathcal{B}$, Φ and d' [3,5]. This means that there are functions f and $\tilde{f}$ such that the respective measurements $\mathcal{A}f$ and $\mathcal{A}\tilde{f}$ are arbitrarily close while f and $\tilde{f}$ are not similar at all. In that case f is informally referred to as an 'instability'.

Note that this behavior stands in stark contrast to the finite-dimensional situation where uniqueness readily implies global stability.

Furthermore, if the infinite-dimensional space $\mathcal{B}$ is approximated by an increasing sequence $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}$ of finite-dimensional subspaces with $\dim(\mathcal{B}_n) = n$ then the global stability constant of the restricted problem, i.e., the smallest number c_n such that

$$\inf_{|a|=1} \|g - af\|_{\mathcal{B}} \leq c_n d'(\mathcal{A}g, \mathcal{A}f) \qquad \forall f, g \in \mathcal{B}_n,$$

may degenerate exponentially in n [3,5].

For the concrete example of Gabor phase retrieval, explicit instabilities can be constructed by taking two functions f_1 and f_2 which have time-frequency support on two disjoint domains, meaning that $\mathcal{G}f_1$ and $\mathcal{G}f_2$ are essentially supported on disjoint domains.

In that case the spectrograms of $f_+ := f_1 + f_2$ and $f_- := f_1 - f_2$ approximately coincide since

$$|\mathcal{G}(f_1 \pm f_2)|^2 = |\mathcal{G} f_1|^2 \pm 2\Re\big(\mathcal{G} f_1 \overline{\mathcal{G} f_2}\big) + |\mathcal{G} f_2|^2 \approx |\mathcal{G} f_1|^2 + |\mathcal{G} f_2|^2.$$

For details see [4]. Qualitatively all instabilities obtained in this way are of the same type, namely their spectrograms essentially live on a domain which is a disconnected set in the time-frequency plane. A quantitative concept that precisely captures this kind of disconnectedness is provided by the so called Cheeger constant, which plays a prominent role in Riemannian geometry [7] and spectral graph theory [8].

Definition 1.2. Let $\Omega \subset \mathbb{R}^d$ be a domain and let w be a nonnegative, continuous function on Ω. Then the *Cheeger constant* of w is defined by

$$h(w, \Omega) := \inf_{C \in \mathcal{C}} \frac{\int_{\partial C \cap \Omega} w}{\min\{\int_C w, \int_{\Omega \setminus C} w\}},$$

where $\mathcal{C} := \{C \subset \Omega \text{ open} : \partial C \cap \Omega \text{ is smooth}\}$.

Remark 1.3. A small Cheeger constant $h(w, \Omega)$ indicates that the domain Ω can be partitioned into $C \subset \Omega$ and $\Omega \setminus C$ such that the weight w is small along the separating boundary and, at the same time, both C and $\Omega \setminus C$ approximately carry the same amount of L^1-energy with respect to w. In that sense w then consists of multiple components; we say that w is *of disconnected type*.

If on the other hand w is concentrated on a connected domain – the Gaussian being a prime example – a partition which accomplishes both objectives simultaneously does not exist. See Figure 2 where two concrete examples are considered.

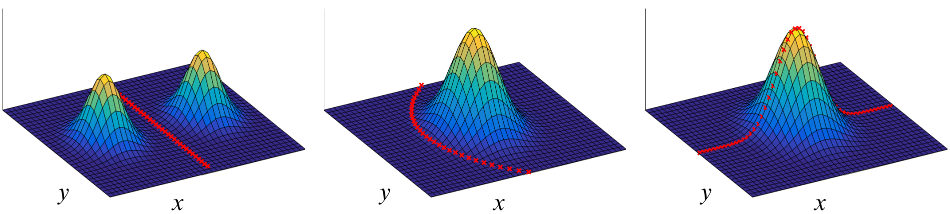

Fig. 2. Comparison of possible partitions of the domain in the disconnected (left) and connected case (center and right).

Based upon the work of one of the authors and his collaborators [2], the paramount discovery in our preceding article [12] is that the local stability constant $c(f)$ for phase retrieval from Gabor magnitudes of univariate functions can be essentially controlled by the reciprocal of the Cheeger constant of the spectrogram of f, i.e. by $h(|\mathcal{G} f|, \Omega)^{-1}$. This insight nicely complements the picture as it reveals that all instabilities are of disconnected type. However, the results are fundamentally restricted to the one-dimensional setting.

1.3. Contribution

The main contribution of this article is that we establish a connection between the Cheeger constant and stability of Gabor phase retrieval obtained in [12] for multivariate signals of arbitrary dimension. The function spaces best suited for our analysis are the so called *modulation spaces*.

Definition 1.4. For $p \geq 1$ the *modulation spaces* are defined by

$$\mathcal{M}^p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \mathcal{G}f \in L^p(\mathbb{R}^{2d})\}, \tag{1.3}$$

with induced norm $\|f\|_{\mathcal{M}^p(\mathbb{R}^d)} = \|\mathcal{G}f\|_{L^p(\mathbb{R}^{2d})}$.

With the modulation spaces at hand we can now state a special case of our main result, Theorem 4.4.

Theorem A. *Suppose that* $f \in \mathcal{M}^1(\mathbb{R}^d)$ *is such that* $|\mathcal{G}f|$ *has a global maximum at the origin and let* $q > 2d$. *Then for all* $g \in \mathcal{M}^1(\mathbb{R}^d)$,

$$\begin{aligned}\inf_{|a|=1} \|g - af\|_{\mathcal{M}^1(\mathbb{R}^d)} &\lesssim \left(1 + h(|\mathcal{G}f|, \mathbb{R}^{2d})^{-1}\right) \\ &\quad \cdot \left(\big\||\mathcal{G}g| - |\mathcal{G}f|\big\|_{W^{1,1}(\mathbb{R}^{2d})} + \|(1 + |\cdot|^{2d+2})(|\mathcal{G}g| - |\mathcal{G}f|)\|_{L^q(\mathbb{R}^{2d})}\right)\end{aligned} \tag{1.4}$$

where the implicit constant depends on d *and* q *only.*

Note however that Theorem 4.4 is way more general as it also covers the case where the phase of the Gabor transform on a domain $\Omega \subsetneq \mathbb{R}^{2d}$ is to be reconstructed given $|\mathcal{G}f(\omega)|, \omega \in \Omega$.

Our results have an immediate impact for substantial applications, one of them being ptychography – as briefly discussed in Section 1.1 – where the object of interest is represented by a function of more than one variable. Theorem A identifies precisely for which ptychographic measurements reconstruction is possible in a stable manner.

We would like to stress that the results in the present paper are not merely a straightforward generalization of our results from [12] to higher dimensions. The proof methods have undergone several modifications which not only makes for a slicker reading but also leads to notably improved results: Our earlier analysis only guaranteed estimates as in (1.4) where the implicit constant mildly depended on f. This dependence is now entirely removed, i.e., the stability constant can indeed be controlled in terms of the reciprocal of the Cheeger constant.

Our proof methods draw upon techniques from various fields of mathematics such as functional analysis, Riemannian geometry, complex analysis in several variables and potential theory. The second main emphasis is on the study of certain quantities, such as the logarithmic derivative, of entire functions satisfying specific growth restrictions. The results we derive in this direction play a vital role in our analysis of Gabor phase retrieval. These results do not only serve as an auxiliary intermediate step but are rather interesting in their own right, and therefore merit to be highlighted at this stage:

Theorem B. *Suppose that G is an entire function on $\mathbb{C}^d$ such that $\sup_{|z|\le r}|G(z)| \le |G(0)|e^{\alpha r^\beta}$ for all $r > 0$. Then*

$$\|G'/G\|_{L^1(B_r)} \lesssim r^{2d+\beta-1}, \qquad r > 0,$$

where the implicit constant depends on d, α and β but not on G.

Theorem B is a special case of Theorem 3.3.

1.4. Preliminaries and notation

In the present paper we will constantly identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$ via

$$(z_1,\dots,z_d) = (x_1+iy_1,\dots,x_d+iy_d) \longleftrightarrow (x_1,y_1,\dots,x_d,y_d);$$

accordingly, a domain Ω in $\mathbb{C}^d$ can be considered as a domain in $\mathbb{R}^{2d}$ and vice versa. We will denote balls of radius r centered at u by $B_r(u) := \{z : |z-u| < r\}$; if $u = 0$ we will just write B_r.

A complex valued function F on a domain $\Omega \subset \mathbb{C}^d$ is differentiable at $u \in \Omega$ if it is differentiable with respect to $x_1, y_1, \dots, x_d, y_d$. In that case we write

$$\nabla F(u) = \left(\frac{\partial}{\partial x_1}F(u), \frac{\partial}{\partial y_1}F(u), \dots, \frac{\partial}{\partial x_d}F(u), \frac{\partial}{\partial y_d}F(u)\right)^T.$$

We will also use the so called *Wirtinger derivatives* defined by $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ for $1 \le j \le d$. If F is complex differentiable at u we will occasionally use the notation

$$F'(u) = \left(\frac{\partial}{\partial z_1}F(u), \dots, \frac{\partial}{\partial z_d}F(u)\right)^T.$$

We denote the space of holomorphic functions on a domain $\Omega \subset \mathbb{C}^d$ by $\mathcal{O}(\Omega)$ and the space of meromorphic functions, i.e., functions that locally coincide with the quotient of two holomorphic functions, by $\mathcal{M}(\Omega)$.

For a measurable, nonnegative function w on Ω and $1 \le p < +\infty$ we denote the weighted Lebesgue space by $L^p(\Omega, w)$; it consists of all measurable functions F on Ω such that

$$\|F\|_{L^p(\Omega,w)} := \left(\int_\Omega |F|^p w\right)^{1/p} < +\infty. \tag{1.5}$$

In the unweighted case, i.e., if $w \equiv 1$, we will just write $L^p(\Omega)$ and $\|\cdot\|_{L^p(\Omega)}$ instead. Note that (1.5) also makes sense for vector valued functions F by understanding $|F|$ as the euclidean length.

The Sobolev norms are defined by

$$\|\cdot\|_{W^{1,p}(\Omega)} := \|\cdot\|_{L^p(\Omega)} + \|\nabla\cdot\|_{L^p(\Omega)}.$$

2. Stability and Cheeger constants

2.1. *A first stability result*

This section will unveil our key mechanism for deriving stability estimates for phase retrieval under the general assumption that the quotient of two measurements is meromorphic. This mechanism relies on the interplay between Poincaré inequalities, as defined next, and complex analysis.

Definition 2.1. Let $\Omega \subset \mathbb{C}^d$ be a domain equipped with a nonnegative, integrable weight w and let $1 \leq p < +\infty$. We say that Ω *supports a Poincaré inequality* if there exists a finite constant C such that

$$\inf_{c\in\mathbb{C}} \|F - c\|_{L^p(\Omega,w)} \leq C\|\nabla F\|_{L^p(\Omega,w)} \quad \text{for all } F \in \mathcal{M}(\Omega) \cap L^p(\Omega, w). \tag{2.1}$$

The smallest possible constant in (2.1) is called the *Poincaré constant* of Ω and denoted by $C_P(\Omega, w, p)$.

Remark 2.2. Note that in the defining inequality of Definition 2.1 functions are restricted to be meromorphic. This is certainly nonstandard but precisely the right concept for our purposes. Due to the famous *Lavrent'ev phenomenon* [20], which states that smooth functions need not necessarily be dense in weighted Sobolev spaces, a Poincaré inequality of the type defined above does not necessarily imply a Poincaré inequality in the usual sense.

The main result of this section provides an upper bound for the distance between two measurements whose quotient is assumed to be meromorphic in terms of an expression which only depends on the moduli of the two measurements.

Theorem 2.3. *Let $\Omega \subset \mathbb{C}^d$ be a domain and let $1 \leq p < +\infty$. Suppose that $F_1, F_2 \in L^p(\Omega)$ are such that their quotient F_2/F_1 is meromorphic. Then*

$$\begin{aligned}\inf_{|c|=1} \|F_2 - cF_1\| &\leq \big\||F_2| - |F_1|\big\|_{L^p(\Omega)} \\ &\quad + 2^{3/2} C_P(\Omega, |F_1|^p, p)\left(\big\|\nabla|F_1| - \nabla|F_2|\big\|_{L^p(\Omega)} + \left\|\frac{\nabla|F_1|}{|F_1|}(|F_1| - |F_2|)\right\|_{L^p(\Omega)}\right).\end{aligned} \tag{2.2}$$

Inequality (2.2) is already quite close to a stability estimate of the desired mould. If we neglect the logarithmic derivative $\nabla|F_1|/|F_1|$ for a moment, it states that – provided that $C_P(\Omega, |F_1|^p, p)$ is moderately small – the distance between two measurements is comparable to the distance of the respective moduli, as measured in the Sobolev norm.

The remainder of this section is devoted to proving Theorem 2.3. A key role in the proof will be played by the fact that for holomorphic functions F the local variation of $|F|$ coincides with the local variation of F up to a factor.

Lemma 2.4. *Suppose that F is holomorphic at a point $w \in \mathbb{C}^d$. Then*

$$\big|\nabla|F|(w)\big| = 2^{-1/2}|\nabla F(w)| = |F'(w)|.$$

Proof. Let us first assume that $d = 1$. We split $F = u + iv$ into its real and imaginary parts. At every point where F is differentiable,

$$\nabla|F| = \nabla(u^2+v^2)^{1/2} = \frac{1}{2}\frac{2u\nabla u + 2v\nabla v}{|F|} = \frac{u\nabla u + v\nabla v}{|F|}$$

and therefore

$$\big|\nabla|F|\big|^2 = \frac{u^2|\nabla u|^2 + v^2|\nabla v|^2 + 2uv(\nabla u\cdot\nabla v)}{u^2+v^2}.$$

Since F is assumed to be holomorphic at w, the Cauchy–Riemann equations hold at w. In particular, $|\nabla v(w)|^2 = |\nabla u(w)|^2$ and $\nabla u(w)\cdot\nabla v(w) = 0$. Thus, $|\nabla|F|(w)|^2 = |\nabla u(w)|^2$.

On the other hand,

$$|\nabla F(w)|^2 = |\nabla u(w) + i\nabla v(w)|^2 = |\nabla u(w)|^2 + |\nabla v(w)|^2 = 2|\nabla u(w)|^2, \tag{2.3}$$

where we have again used $|\nabla v(w)|^2 = |\nabla u(w)|^2$. Therefore the first identity $\big|\nabla|F|(w)\big| = 2^{-1/2}|\nabla F(w)|$ holds true.

For the second identity note that since F is holomorphic at w, we have $F'(w) = \frac{\partial}{\partial z}F(w) = \frac{\partial}{\partial x}F(w)$. Thus, by making use of the Cauchy–Riemann equations again and by (2.3), we have

$$|F'(w)|^2 = u_x^2(w) + v_x^2(w) = |\nabla u(w)|^2 = 2^{-1}|\nabla F(w)|^2$$

and therefore $|F'(w)| = 2^{-1/2}|\nabla F(w)|$.

The general case $d > 1$ follows from the univariate case: Note that for any $1 \le j \le d$ the mapping $\mathbb{C} \ni z \mapsto F(w_1,\ldots,w_{j-1},z,w_{j+1},\ldots,w_d)$ is holomorphic at w_j. Then by the first part,

$$\begin{aligned}|\nabla|F|(w)|^2 &= \sum_{j=1}^d\left[\left(\frac{\partial}{\partial x_j}|F|(w)\right)^2 + \left(\frac{\partial}{\partial y_j}|F|(w)\right)^2\right]\\ &= \sum_{j=1}^d 2^{-1}\left[\left(\frac{\partial}{\partial x_j}F(w)\right)^2 + \left(\frac{\partial}{\partial y_j}F(w)\right)^2\right] = 2^{-1}|\nabla F(w)|^2.\end{aligned}$$

Similarly we find that

$$\begin{aligned}|F'(w)|^2 &= \sum_{j=1}^d\left|\frac{\partial}{\partial z_j}F(w)\right|^2 = \sum_{j=1}^d 2^{-1}\left[\left(\frac{\partial}{\partial x_j}F(w)\right)^2 + \left(\frac{\partial}{\partial y_j}F(w)\right)^2\right]\\ &= 2^{-1}|\nabla F(w)|^2,\end{aligned}$$

which completes the proof. ∎

With Lemma 2.4 at hand we are ready to prove the main result of this section:

Proof of Theorem 2.3. In a first, preparatory step we show that the constraint $|c| = 1$ in the distance

$$\inf_{|c|=1} \|F_2 - cF_1\|_{L^p(\Omega)}$$

can effectively be dropped if the unsigned measurements are close (cf. inequality (2.7)). The distance term without the constraint on c will be controlled in the second step by making use of Poincaré's inequality as well as Lemma 2.4.

Step 1: Getting rid of the constraint. For $\epsilon > 0$ let $c_\epsilon \in \mathbb{C}$ be such that

$$\|F_2 - c_\epsilon F_1\|_{L^p(\Omega)} \leq \inf_{c\in\mathbb{C}} \|F_2 - cF_1\|_{L^p(\Omega)} + \epsilon. \tag{2.4}$$

Note that by continuity c_ϵ can always be chosen to be nonzero. By making use of the triangle inequality we estimate

$$\begin{aligned} \inf_{|c|=1} \|F_2 - cF_1\|_{L^p(\Omega)} &\leq \left\| F_2 - \frac{c_\epsilon}{|c_\epsilon|} F_1 \right\|_{L^p(\Omega)} \\ &\leq \|F_2 - c_\epsilon F_1\|_{L^p(\Omega)} + \left\| c_\epsilon F_1 - \frac{c_\epsilon}{|c_\epsilon|} F_1 \right\|_{L^p(\Omega)}. \end{aligned} \tag{2.5}$$

Furthermore the last term can be bounded by

$$\begin{aligned} \left\| c_\epsilon F_1 - \frac{c_\epsilon}{|c_\epsilon|} F_1 \right\|_{L^p(\Omega)} &= \big\| |c_\epsilon F_1| - |F_1| \big\|_{L^p(\Omega)} \\ &\leq \big\| |c_\epsilon F_1| - |F_2| \big\|_{L^p(\Omega)} + \big\| |F_2| - |F_1| \big\|_{L^p(\Omega)} \\ &\leq \|c_\epsilon F_1 - F_2\|_{L^p(\Omega)} + \big\| |F_2| - |F_1| \big\|_{L^p(\Omega)}. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) with (2.4) yields

$$\inf_{|c|=1} \|F_2 - cF_1\|_{L^p(\Omega)} \leq 2 \inf_{c\in\mathbb{C}} \|F_2 - cF_1\|_{L^p(\Omega)} + 2\epsilon + \big\| |F_2| - |F_1| \big\|_{L^p(\Omega)}.$$

Since $\epsilon > 0$ was arbitrary,

$$\inf_{|c|=1} \|F_2 - cF_1\|_{L^p(\Omega)} \leq 2 \inf_{c\in\mathbb{C}} \|F_2 - cF_1\|_{L^p(\Omega)} + \big\| |F_2| - |F_1| \big\|_{L^p(\Omega)}. \tag{2.7}$$

Step 2: Bound for the unconstrained distance. First we rewrite, for arbitrary $c \in \mathbb{C}$,

$$\|F_2 - cF_1\|_{L^p(\Omega)} = \|F_2/F_1 - c\|_{L^p(\Omega,|F_1|^p)}.$$

Note that by assumption the quotient F_2/F_1 is meromorphic, $\|F_2/F_1\|_{L^p(\Omega,|F_1|^p)} = \|F_2\|_{L^p(\Omega)} < +\infty$ and the weight $|F_1|^p$ is integrable due to the assumption that $F_1 \in L^p(\Omega)$. Therefore Poincaré's inequality can be applied and we obtain

$$\begin{aligned} \inf_{c\in\mathbb{C}} \|F_2 - cF_1\|_{L^p(\Omega)} &= \inf_{c\in\mathbb{C}} \|F_2/F_1 - c\|_{L^p(\Omega,|F_1|^p)} \\ &\leq C_P(\Omega, |F_1|^p, p) \left\| \nabla \frac{F_2}{F_1} \right\|_{L^p(\Omega,|F_1|^p)} \\ &= 2^{1/2} C_P(\Omega, |F_1|^p, p) \left\| \nabla \left| \frac{F_2}{F_1} \right| \right\|_{L^p(\Omega,|F_1|^p)} \end{aligned} \tag{2.8}$$

where the last equality follows from Lemma 2.4, since F_2/F_1 is holomorphic in Ω outside a set of measure zero. Using the fact that

$$\nabla\left|\frac{F_2}{F_1}\right| = |F_1|^{-2}\cdot(|F_1|\nabla|F_2| - |F_2|\nabla|F_1|)$$

almost everywhere in Ω we estimate

$$\begin{aligned}\left\|\nabla\left|\frac{F_2}{F_1}\right|\right\|_{L^p(\Omega,|F_1|^p)} &= \left\||F_1|^{-2}\cdot(|F_1|\nabla|F_2| - |F_2|\nabla|F_1|)\right\|_{L^p(\Omega,|F_1|^p)} \\ &\leq \left\|\frac{|F_1|\nabla|F_2| - |F_1|\nabla|F_1|}{|F_1|^2}\right\|_{L^p(\Omega,|F_1|^p)} + \left\|\frac{|F_1|\nabla|F_1| - |F_2|\nabla|F_1|}{|F_1|^2}\right\|_{L^p(\Omega,|F_1|^p)} \\ &= \left\|\nabla|F_1| - \nabla|F_2|\right\|_{L^p(\Omega)} + \left\|\frac{\nabla|F_1|}{|F_1|}(|F_1| - |F_2|)\right\|_{L^p(\Omega)}. \end{aligned} \tag{2.9}$$

By combining (2.7)–(2.9) we arrive at the desired bound in (2.2). ∎

2.2. *Cheeger's inequality*

Next we want to provide some insight into the Poincaré constant $C_P(\Omega,|F_1|^p,p)$, which by Theorem 2.3 is closely related to the question of local stability at F_1.

The following result is inspired by the work of Jeff Cheeger [7], where the smallest eigenvalue of the Laplacian on a Riemannian manifold is related to a geometric quantity which is similar to the Cheeger constant as introduced in Definition 1.2.

Theorem 2.5. *Let $1 \leq p \leq 2$, let $\Omega \subset \mathbb{R}^{2d}$ be a domain and let w be a nonnegative and continuous weight on Ω. Then*

$$C_P(\Omega,w,p) \leq 8\cdot h(w,\Omega)^{-1}.$$

Proof. A proof for the case $d = 1$ is carried out in [12, Appendix] and generalizes readily to the multivariate case. ∎

Theorem 2.5 immediately implies the following version of the stability result of Theorem 2.3.

Corollary 2.6. *Let $\Omega \subset \mathbb{C}^d$ be a domain and let $1 \leq p \leq 2$. Suppose that $F_1, F_2 \in L^p(\Omega)$ are such that F_2/F_1 is meromorphic and $|F_1|$ is continuous. Then*

$$\begin{aligned}\inf_{|c|=1}\|F_2 - cF_1\|_{L^p(\Omega)} &\leq \left\||F_2| - |F_1|\right\|_{L^p(\Omega)} \\ &\quad + 2^{9/2}\cdot h(|F_1|^p,\Omega)^{-1}\left(\left\|\nabla|F_1| - \nabla|F_2|\right\|_{L^p(\Omega)} + \left\|\frac{\nabla|F_1|}{|F_1|}(|F_1| - |F_2|)\right\|_{L^p(\Omega)}\right).\end{aligned}$$

3. On the growth of the logarithmic derivative of holomorphic functions

The estimates in Theorem 2.3 and Corollary 2.6 include the logarithmic derivative of the modulus of F_1, a term which is rather undesirable as we want to obtain a bound which depends on the difference of $|F_1|$ and $|F_2|$ only.

This section is devoted to the study of the logarithmic derivatives of entire functions that satisfy certain growth restrictions. More precisely, we consider the following class of entire functions on $\mathbb{C}^d$.

Definition 3.1. Let $\alpha, \beta > 0$. Then

$$\mathcal{O}_\alpha^\beta(\mathbb{C}^d) := \{G \in \mathcal{O}(\mathbb{C}^d) : M_G(r) \leq |G(0)| e^{\alpha r^\beta} \ \forall r > 0\},$$

where we set $M_G(r) := \max_{|z| \leq r} |G(z)|$.

Remark 3.2. Note that we require a pointwise inequality to hold, whereas in the definition of type and order a similar inequality only needs to hold in an asymptotic sense. Consequently, for an entire function G of type $\tau \leq a$ and order $\sigma \leq b$ in general we have $G \notin \mathcal{O}_\alpha^\beta(\mathbb{C}^d)$.

The quantity of our interest is the L^p-norm of $(\log G)' = G'/G$ on balls centered at the origin. Our results reveal that $\|(\log G)'\|_{L^p(B_r)}$ grows at most polynomially in r and provide explicit bounds that depend on α, β but are remarkably independent of $G \in \mathcal{O}_\alpha^\beta(\mathbb{C}^d)$.

The main theorem of this section reads as follows.

Theorem 3.3. *Let $1 \leq p < 1 + 1/(2d-1)$. There exists a constant $c > 0$ that only depends on d and p such that for all $G \in \mathcal{O}_\alpha^\beta(\mathbb{C}^d)$ with $G \neq 0$ and all $r > 0$,*

$$\|(\log G)'\|_{L^p(B_r)} \leq c\alpha 2^{2d+2\beta} r^{2d+\beta-1}. \tag{3.1}$$

The results of this section rely heavily on the formula of *Poisson–Jensen*. In the one-dimensional case the formula is well-known [1]. In higher dimensions a similar formula is established for subharmonic functions in potential theory [13]. Since $\log |G|$ is subharmonic for any holomorphic function G [17], the formula can be applied to $\log |G|$ and leads to the following result:

Theorem 3.4 (Poisson–Jensen). *Suppose $G : \mathbb{C}^d \to \mathbb{C}$ is entire. In case $d = 1$ let $z_1, z_2, \ldots$ denote the zeros of G repeated according to multiplicity. Then for $r > 0$ and $|z| < r$,*

$$\log |G(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |G(re^{i\theta})| \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \, d\theta - \sum_{k: |z_k| < r} \log \left| \frac{r^2 - \overline{z_k} z}{r(z - z_k)} \right|. \tag{3.2}$$

In case $d \geq 2$ there exists a Borel measure μ_G on $\mathbb{C}^d$ such that for any $r > 0$ and $|z| < r$,

$$\log|G(z)| = \frac{1}{S_{d-1} \cdot r} \int_{\partial B_r} \log|G(\xi)| \frac{r^2 - |z|^2}{|z-\xi|^{2d}} \, d\sigma(\xi) - \int_{B_r} \frac{1}{|z-\xi|^{2d-2}} - \left(\frac{r}{|\xi| \, \left|z - \xi r^2/|\xi|^2\right|} \right)^{2d-2} d\mu_G(\xi) \tag{3.3}$$

where S_{d-1} denotes the surface area of the unit sphere and σ denotes the surface measure on ∂B_r.

Since the formula of Poisson–Jensen takes different shapes depending on the dimension, in the following we will consider the cases $d = 1$ and $d \geq 2$ separately. Note however that qualitatively equations (3.2) and (3.3) are quite similar: First of all, both the integral in (3.2) and the first integral in (3.3) express a weighted average of $\log|G|$ over the surface of a ball. Secondly, the sum in (3.2) can be rewritten as

$$\sum_{k:\, |z_k| < r} \log \left| \frac{r^2 - \overline{z_k} z}{r(z - z_k)} \right| = \int_{B_r} \log \left| \frac{r^2 - \overline{\xi} z}{r(z-\xi)} \right| d\mu(\xi),$$

where $\mu := \sum_k \delta_{z_k}$, and therefore it is the integral of a function with a singularity at z with respect to a measure that is supported precisely on the zero set of G. The second integral in (3.3) can be interpreted similarly. More generally, the measure μ_G is related to the distribution of the zeros of G. As we will see next, the distribution of zeros can be controlled in terms of the growth of G.

Proposition 3.5. *Let $\alpha, \beta > 0$ and suppose $G \in \mathcal{O}_\alpha^\beta(\mathbb{C}^d)$ with $G \neq 0$. In case $d = 1$ let $z_1, z_2, \ldots$ denote the zeros of G repeated according to multiplicity. Then for all $r > 0$,*

$$\sharp\{k : |z_k| < r\} \leq \frac{2^\beta \alpha}{\log 2} r^\beta.$$

In case $d \geq 2$ let μ_G be defined by (3.3) and ν_G by $\nu_G(r) := \int_{B_r} |z|^{-2d+2} \, d\mu_G(z)$. Then for all $r > 0$,

$$\mu_G(B_r) \leq \alpha 2^{\beta+1} r^{2d-2+\beta} \quad \textit{and} \quad \nu_G(r) \leq \alpha 2^{\beta+1} r^\beta.$$

Proof. Note that $G \neq 0$ together with $G \in \mathcal{O}_\alpha^\beta(\mathbb{C}^d)$ implies that $G(0) \neq 0$. Since both equations (3.2) and (3.3) are invariant with respect to multiplication of G by a nonzero constant, we may assume that $|G(0)| = 1$.

We first prove the statement for $d = 1$. Since $|G(0)| = 1$, applying (3.2) for $z = 0$ yields

$$\sum_{k:\, |z_k| < r} \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log|G(re^{i\theta})| \, d\theta. \tag{3.4}$$

Since for $|z_k| < r/2$ we have $\log \frac{r}{|z_k|} \geq \log 2$, we can estimate

$$\begin{aligned}\sharp\{k : |z_k| < r/2\} &\leq \frac{1}{\log 2} \sum_{k:|z_k|<r} \log \frac{r}{|z_k|} \\ &= \frac{1}{\log 2} \frac{1}{2\pi} \int_0^{2\pi} \log |G(re^{i\theta})| \, d\theta \leq \frac{\alpha}{\log 2} r^\beta.\end{aligned}$$

Replacing of r by $2r$ concludes the proof of the first statement.

We proceed with the case $d \geq 2$. To simplify notation we set $\mu := \mu_G$ and $\nu := \nu_G$. Applying equation (3.3) for $z = 0$ gives

$$\int_{B_r} \left(\frac{1}{|\xi|^{2d-2}} - \frac{1}{r^{2d-2}} \right) d\mu(\xi) = \frac{1}{S_{d-1} \cdot r^{2d-1}} \int_{\partial B_r} \log |G(\xi)| \, d\sigma(\xi). \tag{3.5}$$

Since for $|\xi| < r/2$ we have

$$\frac{1}{|\xi|^{2d-2}} \leq 2 \cdot \left(\frac{1}{|\xi|^{2d-2}} - \frac{1}{r^{2d-2}} \right),$$

by using (3.5) we estimate

$$\nu(r/2) = \int_{B_{r/2}} |\xi|^{-2d+2} \, d\mu(\xi) \leq \frac{2}{S_{d-1} \cdot r^{2d-1}} \int_{\partial B_r} \log |G(\xi)| \, d\sigma(\xi) \leq 2\alpha r^\beta.$$

Replacing r by $2r$ yields

$$\nu(r) \leq 2^{\beta+1} \alpha r^\beta.$$

Since $\nu(r) \geq r^{-2d+2} \mu(B_r)$ the second claim follows immediately, i.e.,

$$\mu(B_r) \leq 2^{\beta+1} \alpha r^{2d-2+\beta}. \qquad \blacksquare$$

Before we go on to prove Theorem 3.3 let us provide some intuition for the case $d = 1$. In that case the zero set of any entire function $G \neq 0$ is discrete. Locally at a zero z_0 we can factorize

$$G(z) = (z - z_0)^m \cdot H(z),$$

where $m \in \mathbb{N}$ denotes the multiplicity of the zero z_0 and H is a locally nonvanishing, analytic function. Computing the logarithmic derivative of G gives

$$(\log G)'(z) = \frac{G'(z)}{G(z)} = \frac{m}{z - z_0} + (\log H)'(z),$$

thus the logarithmic derivative has a pole of order 1 at z_0. Proposition 3.5 allows us to control the number of zeros. If we choose p such that $z \mapsto |z|^{-1}$ is L^p-integrable we will be able to bound $\|(\log G)'\|_{L^p(B_r)}$.

For $d \geq 2$ the situation is more complicated as zeros are not discrete any more. Before we prove Theorem 3.3 we derive pointwise estimates, again by exploiting the representation formula of Poisson–Jensen.

Proposition 3.6. *Let $\alpha, \beta > 0$. Suppose $G \in \mathcal{O}_\alpha^\beta(\mathbb{C}^d)$ and $G \neq 0$. Then there exists a constant c that only depends on d such that for all $r > 0$ and $|z| < r/2$,*

$$|(\log G)'(z)| \leq c\left(\alpha 2^\beta r^{\beta-1} + \int_{B_r} \frac{d\mu(\xi)}{|z-\xi|^{2d-1}}\right), \tag{3.6}$$

where $\mu := \sum_k \delta_{z_k}$ – where z_k are the zeros of G repeated according to multiplicity – in case $d = 1$ and $\mu = \mu_G$ is defined by (3.3) in case $d \geq 2$.

Proof. The assumption that $G \in \mathcal{O}_\alpha^\beta(\mathbb{C}^d)$ is not the zero function implies that $G(0) \neq 0$. Since the logarithmic derivative is invariant with respect to multiplication by a nonzero constant, we may assume that $G(0) = 1$. Again the cases $d = 1$ and $d \geq 2$ are treated separately:

***d* = 1:** First we compute $\log G(z)$ for $|z| < r$ using (3.2):

$$\begin{aligned}\log G(z) &= 2\log|G(z)| - \log\overline{G(z)} \\ &= \frac{1}{\pi}\int_0^{2\pi} \log|G(re^{i\theta})| \cdot \frac{1}{2}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z} + \frac{re^{-i\theta}+\overline{z}}{re^{-i\theta}-\overline{z}}\right) d\theta \\ &\quad - \sum_{k:\,|z_k|<r}\left(\log\frac{r^2-\overline{z_k}z}{r(z-z_k)} + \log\frac{r^2-z_k\overline{z}}{r(\overline{z}-\overline{z_k})}\right) - \log\overline{G(z)}.\end{aligned} \tag{3.7}$$

Next we differentiate with respect to z. The antiholomorphic terms are annihilated by $\frac{\partial}{\partial z}$, i.e.,

$$\frac{\partial}{\partial z}\left(\frac{re^{-i\theta}+\overline{z}}{re^{-i\theta}-\overline{z}}\right) = 0, \qquad \frac{\partial}{\partial z}\left(\log\frac{r^2-z_k\overline{z}}{r(\overline{z}-\overline{z_k})}\right) = 0, \qquad \frac{\partial}{\partial z}\log\overline{G(z)} = 0.$$

Elementary computations show that

$$\frac{\partial}{\partial z}\left(\frac{1}{2}\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) = \frac{re^{i\theta}}{(re^{i\theta}-z)^2}, \qquad \frac{\partial}{\partial z}\left(\log\frac{r^2-\overline{z_k}z}{r(z-z_k)}\right) = \frac{|z_k|^2-r^2}{(z-z_k)(r^2-\overline{z_k}z)},$$

and thus

$$\begin{aligned}(\log G)'(z) &= \frac{1}{\pi}\int_0^{2\pi}\log|G(re^{i\theta})|\,\frac{re^{i\theta}}{(re^{i\theta}-z)^2}\,d\theta + \sum_{k:\,|z_k|<r}\frac{r^2-|z_k|^2}{(z-z_k)(r^2-\overline{z_k}z)} \\ &=: I(z) + II(z)\end{aligned} \tag{3.8}$$

We will now estimate $|(\log G)'(z)|$ for $|z| < r/2$. We treat I and II separately.

To estimate $|I(z)|$, note that applying Cauchy's integral formula to the function $z \mapsto z$ yields

$$\int_0^{2\pi}\frac{re^{i\theta}}{(z-re^{i\theta})^2}\,d\theta = 0 \quad \text{for all } z \in B_r.$$

Therefore

$$I(z) = \frac{1}{\pi} \int_0^{2\pi} \big(\log |G(re^{i\theta})| - \log |M_G(r)|\big) \frac{re^{i\theta}}{(z - re^{i\theta})^2} \, d\theta$$

and furthermore for $|z| < r/2$,

$$\begin{aligned} |I(z)| &\leq \frac{1}{\pi} \int_0^{2\pi} \big|\log |G(re^{i\theta})| - \log M_G(r)\big| \cdot \left| \frac{re^{i\theta}}{(z - re^{i\theta})^2} \right| d\theta \\ &\leq \frac{4}{\pi r} \int_0^{2\pi} \big(\log M_G(r) - \log |G(re^{i\theta})|\big) \, d\theta \end{aligned}$$

due to $|G(re^{i\theta})| \leq M_G(r)$. By having a look at (3.4) we observe that $\int_0^{2\pi} \log |G(re^{i\theta})| \, d\theta$ is nonnegative and therefore

$$|I(z)| \leq \frac{4\alpha}{\pi} r^{\beta - 1} \quad \text{for } |z| < r/2. \tag{3.9}$$

To estimate $|II(z)|$, note that for any $|z| < r/2$ and $|z_k| < r$ we have

$$\big|r^2 - |z_k|^2\big| \leq r^2, \quad |r^2 - \overline{z_k} z| \geq r^2 - |z_k| \, |z| \geq r^2/2;$$

making use of these estimates yields

$$|II(z)| \leq \sum_{k:\, |z_k| < r} \left| \frac{r^2 - |z_k|^2}{(z - z_k)(r^2 - \overline{z_k} z)} \right| \leq 2 \sum_{k:\, |z_k| < r} |z - z_k|^{-1}. \tag{3.10}$$

Combining (3.9) and (3.10) implies (3.6) for $d = 1$.

***$d \geq 2$*:** Let $\nu := \nu_G$ be defined as in Proposition 3.5. Then by utilizing (3.3) we know that

$$\begin{aligned} \log G(z) = &- \log \overline{G(z)} + \frac{2}{S_{d-1} r} \int_{\partial B_r} \log |G(\xi)| \cdot h(z, \xi) \, d\sigma(\xi) \\ &- 2 \int_{B_r} k(z, \xi) \, d\mu(\xi) \end{aligned} \tag{3.11}$$

for all $|z| < r$, where

$$h(z, \xi) = \frac{r^2 - |z|^2}{|z - \xi|^{2d}}, \quad k(z, \xi) = \frac{1}{|z - \xi|^{2d-2}} - \left(\frac{r}{|\xi| \cdot \left| z - \frac{\xi r^2}{|\xi|^2} \right|} . \right)^{2d-2}$$

We differentiate (3.11) with respect to the first component z_1 of z (differentation with respect to the other variables works in exactly the same way). Interchanging the order of integration and differentiation yields, since $\log \overline{G}$ is antiholomorphic with respect to z_1,

$$\begin{aligned} \frac{\partial}{\partial z_1} \log G(z) &= \frac{2}{S_{d-1} r} \int_{\partial B_r} \log |G(\xi)| \cdot \frac{\partial}{\partial z_1} h(z, \xi) \, d\sigma(\xi) - 2 \int_{B_r} \frac{\partial}{\partial z_1} k(z, \xi) \, d\mu(\xi) \\ &=: III(z) + IV(z). \end{aligned} \tag{3.12}$$

To compute the derivative of the kernel function h we write

$$h(z,\xi) = \frac{r^2 - z_1\overline{z_1} - |z'|^2}{\left((z_1-\xi_1)(\overline{z_1-\xi_1}) + |z'-\xi'|^2\right)^d},$$

where $z = (z_1, z')$ and $z' \in \mathbb{C}^{d-1}$ and similarly for ξ. Then

$$\begin{aligned}\frac{\partial}{\partial z_1}h(z,\xi) &= \frac{-\overline{z_1}|z-\xi|^{2d} - (r^2-|z|^2)d|z-\xi|^{2(d-1)}(\overline{z_1-\xi_1})}{|z-\xi|^{4d}} \\ &= |z-\xi|^{-2d-2}(-\overline{z_1}|z-\xi|^2 - d(r^2-|z|^2)(\overline{z_1-\xi_1})) \qquad (3.13)\end{aligned}$$

where we have used that $z_1 \mapsto \overline{z_1}$ is antiholomorphic. A similar computation yields

$$\frac{\partial}{\partial z_1}k(z,\xi) = (d-1)\cdot\left(\frac{r^{2d-2}}{|\xi|^{2d-2}}\cdot\frac{\overline{z_1-\hat{\xi}_1}}{|z-\hat{\xi}|^{2d}} - \frac{\overline{z_1-\xi_1}}{|z-\xi|^{2d}}\right) \qquad (3.14)$$

where we set $\hat{\xi} = \xi r^2/|\xi|^2$. Next we derive bounds for $|III(z)|$ and $|IV(z)|$ for $|z| < r/2$.

To handle $|III(z)|$, for $|\xi| = r$ we can now estimate

$$\left|\frac{\partial}{\partial z_1}h(z,\xi)\right| \lesssim_d r^{-2d+1},$$

where '$\lesssim_d$' means that the left hand side can be bounded by the right hand side times a constant that depends on d only. Thus

$$\begin{aligned}|III(z)| &\le \frac{2}{S_{d-1}\cdot r}\int_{\partial B_r}\left|\log|G(\xi)|\right|\cdot\left|\frac{\partial}{\partial z_1}h(z,\xi)\right| d\sigma(\xi) \\ &\lesssim_d r^{-2d}\int_{\partial B_r}\left|\log|G(\xi)|\right| d\sigma(\xi) \\ &\le r^{-2d}\left(\int_{\partial B_r}\left|\log M_G(r) - \log|G(\xi)|\right| d\sigma(\xi) + \int_{\partial B_r}|\log M_G(r)|\, d\sigma(\xi)\right) \\ &= r^{-2d}\left(\int_{\partial B_r}\left(\log M_G(r) - \log|G(\xi)|\right) d\sigma(\xi) + \int_{\partial B_r}|\log M_G(r)|\, d\sigma(\xi)\right)\end{aligned}$$

From (3.5) it follows that $\int_{\partial B_r}\log|G(\xi)|d\sigma(\xi)$ is nonnegative. Since G is holomorphic and $G(0) = 1$ we have $|\log M_G(r)| = \log M_G(r)$ for all r. Therefore we can further estimate

$$|III(z)| \lesssim_d r^{-2d}\int_{\partial B_r}\log M_G(r)\, d\sigma(\xi) \lesssim_d r^{-2d}r^{2d-1}\alpha r^\beta$$

and thus

$$|III(z)| \lesssim_d \alpha r^{\beta-1} \quad \text{for } |z| < r/2. \qquad (3.15)$$

To estimate $|IV(z)|$, first note that $|\hat{\xi}| > r$ whenever $|\xi| < r$. For $|z| < r/2$ we estimate

$$\left|\frac{\partial}{\partial z_1}k(z,\xi)\right| \lesssim_d \frac{r^{2d-2}}{|\xi|^{2d-2}}\cdot r^{-2d+1} + \frac{1}{|z-\xi|^{2d-1}}.$$

Therefore

$$
\begin{aligned}
|IV(z)| &\leq 2\int_{B_r}\left|\frac{\partial}{\partial z_1}k(z,\xi)\right| d\mu(\xi)\\
&\lesssim_d r^{-1}\int_{B_r}|\xi|^{-2d+2}\,d\mu(\xi)+\int_{B_r}\frac{d\mu(\xi)}{|z-\xi|^{2d-1}}\\
&= r^{-1}\cdot\nu(r)+\int_{B_r}\frac{d\mu(\xi)}{|z-\xi|^{2d-1}},
\end{aligned}
$$

where we set $\nu(r) := \int_{B_r}|\xi|^{-2d+2}d\mu(\xi)$. By Proposition 3.5,

$$
|IV(z)| \lesssim_d \alpha 2^\beta r^{\beta-1}+\int_{B_r}\frac{d\mu(\xi)}{|z-\xi|^{2d-1}} \quad \text{for } |z|<r/2. \tag{3.16}
$$

Combining (3.15) and (3.16) yields (3.6) for $d\geq 2$. ∎

We are set to prove Theorem 3.3.

Proof of Theorem 3.3. Let μ be defined as in Proposition 3.6. Then the pointwise estimate

$$
|(\log G)'(z)| \leq c\left(\alpha 2^\beta r^{\beta-1}+\int_{B_r}\frac{d\mu(\xi)}{|z-\xi|^{2d-1}}\right) \tag{3.17}
$$

holds for $z\in B_{r/2}$, where c only depends on d. We begin by bounding the norm of the second term on the right hand side; we may assume that $\mu(B_r)>0$, otherwise there is nothing to estimate. We have

$$
\begin{aligned}
\left\|z\mapsto\int_{B_r}\frac{d\mu(\xi)}{|z-\xi|^{2d-1}}\right\|_{L^p(B_{r/2})}^p &= \int_{B_{r/2}}\left(\int_{B_r}\frac{d\mu(\xi)}{|z-\xi|^{2d-1}}\right)^p dA(z)\\
&= \int_{B_{r/2}}\left(\int_{B_r}\frac{\mu(B_r)}{|z-\xi|^{2d-1}}\cdot\frac{d\mu(\xi)}{\mu(B_r)}\right)^p dA(z)\\
&\leq \mu(B_r)^{p-1}\int_{B_{r/2}}\int_{B_r}\frac{d\mu(\xi)}{|z-\xi|^{(2d-1)p}}\,dA(z)
\end{aligned}
$$

where we have used Jensen's inequality. By changing the order of integration we obtain

$$
\left\|z\mapsto\int_{B_r}\frac{d\mu(\xi)}{|z-\xi|^{2d-1}}\right\|_{L^p(B_{r/2})}^p \leq \mu(B_r)^{p-1}\int_{B_r}\int_{B_{r/2}}\frac{dA(z)}{|z-\xi|^{(2d-1)p}}\,d\mu(\xi).
$$

The inner integral can be bounded by

$$
\begin{aligned}
\int_{B_{r/2}}\frac{dA(z)}{|z-\xi|^{(2d-1)p}} &\leq \int_{B_{r/2}}\frac{dA(z)}{|z|^{(2d-1)p}}\\
&= \left\|z\mapsto|z|^{-2d+1}\right\|_{L^p(B_{r/2})}^p =: c_{d,p}^p.
\end{aligned}
$$

Note that $p < 1 + 1/(2d - 1)$ is precisely the condition for $c_{d,p}$ to be finite. Thus we arrive at

$$\left\| z \mapsto \int_{B_r} \frac{d\mu(\xi)}{|z - \xi|^{2d-1}} \right\|_{L^s(B_{r/2})}^p \leq c_{d,p}^p \mu(B_r)^p. \tag{3.18}$$

To estimate the first summand of the right hand side in (3.17) we compute the norm of the constant function

$$\|1\|_{L^p(B_{r/2})} = \mathrm{vol}(B_{r/2})^{1/p} = \left(\frac{\pi^d}{d!}(r/2)^{2d}\right)^{1/p}. \tag{3.19}$$

It follows from (3.17)–(3.19) and Proposition 3.5 that there exists a c' that only depends on d and p such that

$$\begin{aligned}\|(\log G)'\|_{L^p(B_{r/2})} &\leq c'\big(\alpha 2^\beta r^{\beta-1}\|1\|_{L^p(B_{r/2})} + \mu(B_r)\big) \\ &\leq c''\big(\alpha 2^\beta r^{\beta-1} \cdot (r/2)^{2d/p} + \alpha 2^{\beta+1} r^{2d-2+\beta}\big) \\ &\leq c'''\alpha 2^{\beta+1} r^{2d+\beta-1},\end{aligned}$$

where c'' and c''' again only depend on d and p. By replacing r by $2r$ we get the desired bound

$$\|(\log G)'\|_{L^p(B_r)} \leq c'''\alpha 2^{2d+2\beta} r^{2d+\beta-1}. \qquad \blacksquare$$

4. Stable Gabor phase retrieval

4.1. The main result

In the present section we will elaborate on how the results from the previous two sections enable us to derive stability estimates for the problem of phase retrieval from Gabor magnitudes. The Gabor transform $\mathcal{G}f$ enjoys the pleasant property of being an entire function (up to an exponential factor and a reflection) and therefore the tools we have developed thus far can be applied.

Lemma 4.1. *Let $\eta(z) = e^{\pi|z|^2/2 - \pi i x\cdot y}$. Then for any $f \in \mathcal{S}'(\mathbb{R}^d)$ the function $z \mapsto \mathcal{G}f(\bar{z})\eta(z)$ is entire.*

Proof. A proof for functions of polynomial growth can be found in [10]. ■

Due to Lemma 4.1 we can apply Corollary 2.6 to $F_1(z) = \mathcal{G}f(\bar{z})$ and $F_2(z) = \mathcal{G}g(\bar{z})$ to obtain the following result.

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^{2d}$ be a domain. Then for all $f, g \in \mathcal{M}^p(\mathbb{R}^d)$,*

$$\begin{aligned}\inf_{|c|=1} \|\mathcal{G}g - c\mathcal{G}f\|_{L^p(\Omega)} &\leq \big\||\mathcal{G}g| - |\mathcal{G}f|\big\|_{L^p(\Omega)} \\ &+ 2^{9/2} \cdot h(|\mathcal{G}f|^p, \Omega)^{-1}\left(\big\|\nabla|\mathcal{G}f| - \nabla|\mathcal{G}g|\big\|_{L^p(\Omega)} + \left\|\frac{\nabla|\mathcal{G}f|}{|\mathcal{G}f|}(|\mathcal{G}f| - |\mathcal{G}g|)\right\|_{L^p(\Omega)}\right).\end{aligned}$$

Except for the logarithmic derivative Corollary 4.2 already gives an estimate of the desired form. The following proposition aims at absorbing the logarithmic derivative in a polynomial weight which does not depend on f.

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^{2d}$ and let $1 \leq p < 2d/(2d-1)$ and $q > p/(1 - p\frac{2d-1}{2d})$. Suppose that $f \in \mathcal{M}^\infty(\mathbb{R}^d)$ is such that $|\mathcal{G} f|$ has a global maximum at z_0. Then there exists a constant c which only depends on d, p and q such that for all measurable functions H,*

$$\left\| \frac{\nabla|\mathcal{G} f|}{|\mathcal{G} f|} H \right\|_{L^p(\Omega)} \leq c \left\| (1 + |\cdot - z_0|)^{2d+2} H \right\|_{L^q(\Omega)}.$$

Proof. We can assume that $\Omega = \mathbb{R}^{2d}$ and that $z_0 = 0$ (otherwise translate and modulate f). The proof is split into two parts: First we derive uniform bounds of the norm of the logarithmic derivative on balls centered at the origin. In the second part the logarithmic derivative is absorbed in a polynomial weight by making use of Hölder's inequality and Part 1.

Part 1: By Lemma 4.1 we know that $G(z) := \mathcal{G} f(\bar{z})\eta(z)$ is entire. The gradient of the modulus of the Gabor transform can be computed in terms of G:

$$\nabla|\mathcal{G} f| = \nabla\big(|G| e^{-\frac{\pi}{2}|\cdot|^2}\big) = (\nabla|G|) e^{-\frac{\pi}{2}|\cdot|^2} + |G|\big(\nabla e^{-\frac{\pi}{2}|\cdot|^2}\big).$$

Since $|\nabla e^{-\frac{\pi}{2}|\cdot|^2}|(z) = \pi|z| \cdot e^{-\frac{\pi}{2}|z|^2}$ we obtain

$$\big|\nabla \log |\mathcal{G} f|(z)\big| \leq \big|\nabla \log |G|(z)\big| + \pi|z|. \tag{4.1}$$

Lemma 2.4 implies that the right hand side coincides with $2^{-1/2}|(\log G)'(z)| + \pi|z|$ almost everywhere.

The assumption that $|\mathcal{G} f|$ has a maximum at the origin implies that $G \in \mathcal{O}^2_{\pi/2}(\mathbb{C}^d)$ (see Definition 3.1). We can therefore apply Theorem 3.3 to obtain, for $r > 0$ and $1 \leq s < 1 + 1/(2d-1)$,

$$\big\|\nabla \log |\mathcal{G} f|\big\|_{L^s(B_r)} \leq 2^{-1/2}\|(\log G)'\|_{L^s(B_r)} + \pi\|z \mapsto z\|_{L^s(B_r)} \lesssim r^{2d+1}, \tag{4.2}$$

where the implicit constant depends on d and s only.

Part 2: We define s by the equation $1/p = 1/q + 1/s$. One can elementarily verify that the assumptions on p and q imply that $1 \leq s < 1 + 1/(2d-1)$. Thus the L^s-norm of the logarithmic derivative can be bounded as in Part 1 (see (4.2)).

Let $D_0 := B_1$ and $D_j := B_{2^j} \setminus B_{2^{j-1}}$ for $j \in \mathbb{N}$. Then

$$\|\nabla \log |\mathcal{G} f| \cdot H\|^p_{L^p(\mathbb{R}^{2d})} = \sum_{j \geq 0} \|\nabla \log |\mathcal{G} f| \cdot H\|^p_{L^p(D_j)}.$$

We apply now Hölder's inequality on every D_j to obtain

$$\|\nabla \log |\mathcal{G} f| \cdot H\|_{L^p(D_j)} \leq \big\|\nabla \log |\mathcal{G} f|\big\|_{L^s(D_j)} \|H\|_{L^q(D_j)} \lesssim 2^{j(2d+1)}\|H\|_{L^q(D_j)},$$

where we have used estimate (4.2) from Part 1. Let $r := q/p > 1$ and r' its Hölder conjugate, i.e., $1/r + 1/r' = 1$. By applying Hölder's inequality for sums we estimate further

$$\begin{aligned}\|\nabla \log|\mathcal{G}f|\cdot H\|_{L^p(\mathbb{R}^{2d})}^p &\lesssim \sum_{j\geq 0} 2^{j(2d+1)p}\|H\|_{L^q(D_j)}^p \\ &= \sum_{j\geq 0} 2^{-j/r'}\cdot 2^{j(2dp+p+1/r')}\|H\|_{L^q(D_j)}^p \\ &\leq \Big(\sum_{j\geq 0} 2^{-j}\Big)^{1/r'}\cdot\Big(\sum_{j\geq 0} 2^{j(2dp+p+1/r')r}\|H\|_{L^q(D_j)}^q\Big)^{p/q}.\end{aligned}$$

The first factor is a finite constant depending on r' and therefore ultimately on p and q only.

The second factor is estimated in the following way:

$$\begin{aligned}\sum_{j\geq 0} 2^{j(2dp+p+1/r')r}\|H\|_{L^q(D_j)}^q &\leq \sum_{j\geq 0}\int_{D_j} 2^{j(2d+2)pr}|H(z)|^q\,dA(z) \\ &= \sum_{j\geq 0}\int_{D_j} 2^{j(2d+2)q}|H(z)|^q\,dA(z) \\ &\lesssim \|(1+|\cdot|^{2d+2})H\|_{L^q(\mathbb{R}^{2d})}^q,\end{aligned}$$

where we have used $2^{j(2d+2)} \lesssim 1+|z|^{2d+2}$ for $z\in D_j$. Thus we get the desired estimate

$$\|\nabla \log|\mathcal{G}f|\cdot H\|_{L^p(\mathbb{R}^{2d})} \lesssim \|(1+|\cdot|^{2d+2})H\|_{L^q(\mathbb{R}^{2d})}. \qquad \blacksquare$$

The main stability result now follows directly from Proposition 4.3 together with Corollary 4.2.

Theorem 4.4. *Let $\Omega\subset\mathbb{R}^{2d}$ and let $1\leq p < 1+1/(2d-1)$ and $q > p/(1-p\frac{2d-1}{2d})$. Then for all $f\in\mathcal{M}^p(\mathbb{R}^{2d})$ whose spectrogram $|\mathcal{G}f|$ has a global maximum at z_0,*

$$\begin{aligned}&\inf_{|a|=1}\|\mathcal{G}g - a\mathcal{G}f\|_{L^p(\Omega)} \lesssim (1+h(|\mathcal{G}f|^p,\Omega)^{-1}) \\ &\cdot\Big(\big\||\mathcal{G}f|-|\mathcal{G}g|\big\|_{W^{1,p}(\Omega)} + \|(1+|\cdot - z_0|^{2d+2})(|\mathcal{G}f|-|\mathcal{G}g|)\|_{L^q(\Omega)}\Big) \quad \forall g\in\mathcal{M}^p(\mathbb{R}^d),\end{aligned}$$

where the implicit constant depends on d, p and q only.

4.2. Noise stability

In virtually any practical situation the measurements are corrupted by noise, i.e., one is faced with the problem of reconstructing f from $|\mathcal{G}f| + \gamma$ instead of $|\mathcal{G}f|$. As we will see next, the main theorem of the previous section also implies a stability result for reconstruction from noisy Gabor magnitudes.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^{2d}$ and let $1 \le p < 1 + 1/(2d-1)$ and $q > p/(1 - p\frac{2d-1}{2d})$. Suppose that $f \in \mathcal{M}^p(\mathbb{R}^{2d})$ is such that its spectrogram $|\mathcal{G} f|$ has a global maximum at z_0. Suppose that γ is a smooth function on Ω and suppose that $g \in \mathcal{M}^p(\mathbb{R}^d)$ is such that*

$$\big\| |\mathcal{G} f| + \gamma - |\mathcal{G} g| \big\|_{\mathcal{D}} \le \epsilon$$

where

$$\|F\|_{\mathcal{D}} := \|F\|_{W^{1,p}(\Omega)} + \|(1 + |\cdot - z_0|^{2d+2}) F\|_{L^q(\Omega)}.$$

Then

$$\inf_{|a|=1} \|\mathcal{G} g - a \mathcal{G} f\|_{L^p(\Omega)} \lesssim (1 + h(|\mathcal{G} f|^p, \Omega)^{-1})(\epsilon + \|\gamma\|_{\mathcal{D}}),$$

where the implicit constant depends on d, p and q only.

Proof. By Theorem 4.4 we have

$$\inf_{|a|=1} \|\mathcal{G} g - a \mathcal{G} f\|_{L^p(\Omega)} \lesssim (1 + h(|F_1|^p, \Omega)^{-1}) \big\| |\mathcal{G} g| - |\mathcal{G} f| \big\|_{\mathcal{D}}.$$

The statement then follows from the estimate

$$\big\| |\mathcal{G} g| - |\mathcal{G} f| \big\|_{\mathcal{D}} \le \big\| |\mathcal{G} g| - (|\mathcal{G} f| + \gamma) \big\|_{\mathcal{D}} + \|\gamma\|_{\mathcal{D}} \le \epsilon + \|\gamma\|_{\mathcal{D}}. \qquad \blacksquare$$

4.3. *Multicomponent stability*

In this section we discuss yet another consequence of the main stability result, Theorem 4.4, which tells us that instabilities for Gabor phase retrieval must be of disconnected type. In other words, reconstruction of the Gabor transform is stable on domains Ω where $|\mathcal{G} f|$ is connected.

We now want to pick up the multicomponent paradigm, which was introduced in earlier work by one of the authors and his collaborators [2]: Suppose that the phase retrieval problem is relaxed as we require no longer that $\mathcal{G} f$ be reconstructed up to a global phase factor but instead only demand that the phase factor is constant on each component but may take different values on different components. A *component* is here a subdomain Ω_i of Ω on which $|\mathcal{G} f|$ is connected, i.e., stable recovery on Ω_i is possible.

The multicomponent paradigm – i.e. to identify $F = \sum_{i=1}^k F_i$, where F_i is concentrated on Ω_i with $\sum_{i=1}^k a_i F_i$ whenever $|a_1|, \ldots, |a_k| = 1$ – is especially meaningful for applications in audio as a change of phase on individual components is usually imperceptible to the human ear.

The relaxation accomplishes that Gabor phase retrieval becomes stable. By applying Theorem 4.4 on every single component $\Omega_i \subset \Omega$ we obtain the following result.

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^{2d}$ and let $1 \le p < 1 + 1/(2d-1)$ and $q > p/\big(1 - p\frac{2d-1}{2d}\big)$. Suppose that $f \in \mathcal{M}^p(\mathbb{R}^d)$ is such that its spectrogram $|\mathcal{G} f|$ has a global maximum at z_0. Suppose that Ω is partitioned into subdomains $\Omega_1, \ldots, \Omega_k$, i.e., $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$ and $\bigcup_{i=1}^k \overline{\Omega_i} = \overline{\Omega}$. Then for all $g \in \mathcal{M}^p(\mathbb{R}^d)$,*

$$\inf_{|a_1|,\dots,|a_k|=1} \sum_{i=1}^{k} \|\mathcal{G}g - a_i \mathcal{G}f\|_{L^p(\Omega_i)}$$
$$\lesssim (1+h^*)\big(\big\||\mathcal{G}f| - |\mathcal{G}g|\big\|_{W^{1,p}(\Omega)} + \|(1+|\cdot - z_0|^{2d+2})(|\mathcal{G}f| - |\mathcal{G}g|)\|_{L^q(\Omega)}\big),$$

where $h^* := \max_{1\le i\le k} h(|\mathcal{G}f|^p, \Omega_i)^{-1}$ and where the implicit constant depends on d, p and q only.

Acknowledgments. PG would like to thank Wilhelm Schlag for inspiring discussions and for suggesting the problem of extending the results of [12] to the multivariate case.

Funding. This research has been supported by the Austrian Science Fund (FWF), grant P-30148. MR gratefully acknowledges the support by the Austrian Science Fund (FWF) through the START-Project Y963-N35.

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