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Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators

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Abstract. We introduce a notion of β-almost periodicity and prove quantitative lower spectral/quantum dynamical bounds for general bounded β-almost periodic potentials. Applications include the first sharp arithmetic spectral criterion for the entire family of supercritical analytic quasiperiodic Schrödinger operators and arithmetic spectral/quantum dynamical criteria for families with zero Lyapunov exponents, with applications to Sturmian potentials and the critical almost Mathieu operator. In particular, we disprove a 1994 conjecture of Wilkinson–Austin.

Keywords. Quasiperiodic Schrödinger operator, spectral dimension, almost Mathieu operator

1. Introduction

Singular continuous spectral measures of Schrödinger operators, usually defined by what they are not, are still not very well understood. The aim of direct spectral theory is to obtain properties of spectral measures/spectra and associated quantum dynamics based on the properties of the potential. In the context of 1D operators this is most often done via the study of solutions/transfer matrices/dynamics of transfer-matrix cocycles. Indeed, there are many beautiful results linking the latter to either dimensional properties of spectral measures (going back to [40]) or directly to quantum dynamics (e.g. [24, 50]). There is also a long thread of results relating dimensional properties of spectral measures to quantum dynamics (e.g. [8, 53, 60] and references therein) as well as results connecting spectral/dynamical properties to some further aspects (e.g. [12, 51]). Many of those have been used to obtain dimensional/quantum dynamical results (sometimes sharp) for several concrete families (e.g. [21]). However, there were no results directly linking easily formulated properties of the potential to dimensional/quantum dynamical results, other

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than for specific families or a few that ensure either the mere singularity or continuity of spectral measures (and their immediate consequences). In particular, we do not know of any quantitative results of this type.

In this paper we prove the first such result. It is abstract, and thus applicable in various contexts. In particular, it leads to a number of powerful conclusions, both for the general class of analytic quasiperiodic operators, e.g. Theorem 2, the first arithmetic if-and-only-if criterion for this family beyond the Kotani theory, and for the specific popular models, e.g. Theorem 4, disproving a 25-year old conjecture of Wilkinson–Austin.

To start with the most general setting, consider the Schrödinger operator on $\ell^2(\mathbb{Z})$ given by

$$(Hu)_n = u_{n+1} + u_{n-1} + V(n)u_n. \tag{1.1}$$

For $\beta > 0$, we say a real sequence $\{V(n)\}_{n\in\mathbb{Z}}$ *has β-repetitions* if there is a sequence of positive integers $q_n \to \infty$ such that

$$\max_{1\le j\le q_n} |V(j) - V(j \pm q_n)| \le e^{-\beta q_n}. \tag{1.2}$$

We will say that $\{V(n)\}_{n\in\mathbb{Z}}$ *has ∞-repetitions* if (1.2) holds for any $\beta > 0$. For $\beta < \infty$, we will say that $\{V(n)\}_{n\in\mathbb{Z}}$ is *β-almost periodic* if, for some $\epsilon > 0$, $V(\cdot + kq_n)$ satisfies (1.2) for any $|k| \le e^{\epsilon\beta q_n}/q_n$, i.e.,

$$\max_{1\le j\le q_n,\, |k|\le e^{\epsilon\beta q_n}/q_n} |V(j + kq_n) - V(j + (k \pm 1)q_n)| \le e^{-\beta q_n} \tag{1.3}$$

for any n. We will say that $\{V(n)\}_{n\in\mathbb{Z}}$ is *∞-almost periodic* if it is β-almost periodic for any $\beta < \infty$. We note that β-almost periodicity and even ∞-almost periodicity does not imply almost periodicity in the usual sense. In particular, it is easily seen that there is an explicit set of generic skew shift potentials that satisfy this condition.

We will prove

Theorem 1. *Let H be given by* (1.1) *and V be bounded and β-almost periodic. Then, for an explicit $C = C(\epsilon, V) > 0$, for any*

$$\gamma < 1 - C/\beta \tag{1.4}$$

the spectral measure is γ-spectral continuous.

For the definition of spectral continuity (a property that also implies packing continuity and thus lower bounds on quantum dynamics) see Section 1.1. We formulate a more precise version (specifying the dependence of C on ϵ, V) in Theorem 7.

Our result can simultaneously be viewed as a quantitative version of two well-known statements:

- *Periodicity implies absolute continuity.* Indeed, we prove that a quantitative weakening (β-almost periodicity) implies quantitative continuity of the (fractal) spectral measure.
- *Gordon condition* (*an infinite sequence of single/double almost repetition*) *implies continuity of the spectral measure.* Indeed, we prove that a quantitative strengthening (multiple almost repetitions) implies quantitative continuity of the spectral measure.

Potentials with ∞-repetitions are known in the literature as Gordon potentials.[1] This property has been used fruitfully in the spectral theory in various situations; see reviews [16, 18] and references therein. In many cases those potentials were automatically β-almost periodic or even ∞-almost periodic, so satisfied almost repetitions over sufficiently many periods. However, even in such cases, what all those papers used was the strength of the approximation over two (almost) periods based on the Gordon Lemma type arguments. See, e.g., [38] where a sharp abstract version is presented. The only exception is Last [53] who implicitly used ∞-almost periodicity to construct an example of potentials with spectral measure of zero Hausdorff dimension and almost ballistic quantum dynamics (later slightly improved in [25] to an example with pure point spectrum and almost ballistic quantum dynamics). Our main technical accomplishment here is that we find a new algebraic argument and develop a technology that allows one to obtain *quantitative* corollaries from the fact that the approximation stays strong over many periods, thus exploring this feature analytically and effectively for the first time. One important technical step is quantitative bounds on ellipticity of almost-period-length transfer matrices for spectrally almost every energy (Theorem 10). This means that spectrally almost every energy eventually falls into *shrunk* spectral bands of the periodic approximants (vs *enlarged* ones, in the previous treatments, see Remark 2.1), with moreover a quantitative control.

Lower bounds on spectral dimension lead to lower bounds on packing dimension, thus also for the packing/upper box counting dimensions of the spectrum as a set and for the upper rate of quantum dynamics. Therefore, we obtain corresponding non-trivial results for all the above quantities.

It is clear that our general result only goes in one direction, as even absolute continuity of the spectral measures does not imply β-almost periodicity for $\beta > 0$, as analytic potentials with Diophantine frequencies (so no β-almost periodicity for positive β) have absolutely continuous spectrum for small couplings.

However, in the important context of analytic quasiperiodic operators this leads to a sharp if-and-only-if result.

Let $H = H_{\theta,\alpha,V}$ be a Schrödinger operator on $\ell^2(\mathbb{Z})$ given by

$$(Hu)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n, \quad n \in \mathbb{Z},\ \theta \in \mathbb{T}, \tag{1.5}$$

where V is the potential, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is the *frequency* and $\theta \in \mathbb{T}$ is the *phase*. Let $\mu = \mu_{\theta,\alpha}$ be the spectral measure associated with vectors $\delta_0, \delta_1 \in l^2(\mathbb{Z})$ in the usual sense.

Given $\alpha \in (0,1)$, let p_n/q_n be the continued fraction approximants to α. Define

$$\beta(\alpha) := \limsup_n \frac{\log q_{n+1}}{q_n} \in [0, \infty]. \tag{1.6}$$

Let $S := \{E \in \sigma(H) : L(E) > 0\}$, where $\sigma(H)$ is the spectrum of H and $L(E)$ is the Lyapunov exponent, be the set of supercritical energies (or, equivalently, the set of E

[1]While ∞-repetitions are usually used in the definition of Gordon potentials, typically β-repetitions for sufficiently large β are enough for the applications.

such that the corresponding transfer-matrix cocycle is non-uniformly hyperbolic). The set S depends on α and V but not on θ.

Our main application is

Theorem 2. *For any analytic V and any θ, the spectral measure μ restricted to S is of full spectral dimension if and only if $\beta(\alpha) = \infty$.*

Full spectral dimensionality is defined through the boundary behavior of the Borel transform of the spectral measure (see details in Section 1.1). It implies a range of properties, in particular, maximal packing dimension and quasiballistic quantum dynamics. Thus our criterion links in a sharp way a purely analytic property of the spectral measure to arithmetic properties of the frequency. The result is local (so works for any subset of the supercritical set, see Theorem 5 for more details) and quantitative (so we obtain separately quantitative spectral singularity and spectral continuity statements for every finite value of β, see Theorems 6 and 7).

The study of one-dimensional one-frequency quasiperiodic operators with general analytic potentials has seen remarkable advances in the last two decades, focusing mainly on two regimes: (almost) reducibility (which does not intersect with S) and hyperbolicity (i.e. S), dubbed, correspondingly, sub- and supercritical in [1]. The results in the regime of positive Lyapunov exponents can be divided into two classes:

- Those that hold for all frequencies (e.g. [11, 24, 41, 45–47]).
- Those that have arithmetic (small denominator type) obstructions preventing their holding for all frequencies, thus requiring a Diophantine type condition (e.g. [10, 20, 31]).[2]

Results of the first kind often (but not always [11, 69]) do not require analyticity and hold in higher generality. Results of the second kind describe phenomena where there is a transition in the arithmetics of the frequency. Thus an extremely interesting question is to determine where this transition happens and to understand the neighborhood of the transition. However, even though some improvements on the frequency range of some results above have been obtained (e.g. [72]), most existing proofs often require a removal of a non-arithmetically-defined measure zero set of frequencies, thus cannot be expected to work up to the transition.

There have been remarkable recent advances in obtaining complete arithmetic criteria in the presence of transitions [7, 42–44] or non-transitions [3] for explicit popular Hamiltonians: the almost Mathieu operator and the Maryland model, but there have been no such results that work for large families of potentials. For the reducibility regime, where many recent advances are described in [71], there are also both dynamical and spectral phenomena that do have arithmetic obstructions, but there have been no results

[2] Not all results can currently be classified this way, the most notable example being the Cantor structure of the spectrum [32], currently proved for a non-arithmetically-defined full measure set of frequencies, while the statement has no known arithmetic obstructions. Theoretically there may also be results such as [3] which formally should belong to the first group but the proof requires an argument that highly depends on the arithmetics, so they must be in the second group, in spirit. In some sense [11] is a result of this type.

yet on if-and-only-if arithmetic criteria in the presence of transitions, for large families. Theorem 2, holding for general analytic potentials, is the first theorem of this kind.

A natural way to distinguish between different singular continuous spectral measures is by their Hausdorff dimension. However, Hausdorff dimension is a poor tool for characterizing the singular continuous spectral measures arising in the regime of positive Lyapunov exponents, as it is always zero (for a.e. phase for any ergodic case [65], and for every phase for one frequency analytic potentials [41][3]). Similarly, the lower transport exponent is always zero for piecewise Lipschitz potentials [24, 47]. Thus those two quantities do not even distinguish between pure point and singular continuous situations. In contrast, our quantitative version of Theorem 2, contained in Theorems 6 and 7, shows that spectral dimension is a good tool to finely distinguish between different kinds of singular continuous spectra appearing in the supercritical regime for analytic potentials.

The continuity part of Theorem 2 is robust and only requires some regularity of V. Besides the above-mentioned criterion, Theorem 1 allows us to obtain new results for other popular models, such as the critical almost Mathieu operator, Sturmian potentials, and others.

Indeed, our lower bounds are effective for $\beta > C \sup_{E\in\sigma(H)} L(E)$ where $L(E)$ is the Lyapunov exponent (see Theorem 7), thus the range of β is increased for smaller Lyapunov exponents, and in particular we obtain full spectral dimensionality (and therefore quasiballistic motion) as long as $\beta(\alpha) > 0$, when Lyapunov exponents are zero on the spectrum. This applies, in particular, to Sturmian potentials and the critical almost Mathieu operator.

As an example, setting $S_0 = \{E : L(E) = 0\}$ we have

Theorem 3. *For Lipschitz V, the quantum dynamics is quasiballistic*

(1) *for any $\beta(\alpha) > 0$ if $S_0 \neq \emptyset$,*

(2) *for $\beta(\alpha) = \infty$ otherwise.*

A similar statement also holds for full spectral dimensionality or packing/box counting dimension 1. The Lipschitz condition can be relaxed to piecewise Lipschitz (or even Hölder), leading to part (1) also holding for Sturmian potentials. This in turn leads to first explicit examples of operators whose integrated density of state has different Hausdorff and packing dimensions, within both the critical almost Mathieu and Sturmian families.

The fact that quantum motion can be quasiballistic for highly Liouville frequencies was first realized by Last [53] who proved that the almost Mathieu operator with an appropriate Liouville frequency (constructed step by step) is quasiballistic. The quasiballistic property is a G_δ in any regular (à la Simon's Wonderland theorem [63]) space [15, 28], thus this was known for (unspecified) topologically generic frequencies. Here we show a precise arithmetic condition on α depending on whether or not Lyapunov exponent vanishes somewhere on the spectrum. Thus, in the regime of positive Lyapunov exponents, the quantum motion is very interesting, with dynamics almost bounded along some

[3]The result of [41] is formulated for a trigonometric polynomial v, but it extends to the analytic case – and more – by the method of [47].

scales [47] (this property is sometimes called quasilocalization) and almost ballistic along others. For finite values of $\beta(\alpha)$ in this regime our result also yields power-law quantum dynamics along certain scales while bounded along others.

In particular, for the almost Mathieu operator, a combination of Theorems 1, 2, and 3 leads to optimal results on the quasiballistic behavior in terms of the exponent $\beta(\alpha)$ (Corollary 3).

Almost Mathieu operators, which are operators (1.5) with $V(x) = 2\lambda \cos 2\pi x$, go back to the work of Peierls [59] and are known (and well-studied) in physics as Harper's or Azbel–Hofstadter model. They were dubbed almost Mathieu and were popularized in math through "Simon's problems" [62, 64], which guided the development of the theory of quasiperiodic operators. One of the most important remaining open questions is the dimension of the spectrum of the critical ($\lambda = 1$) almost Mathieu operator (e.g. [66]). This question has received a lot of attention in the physics literature since the early 80's. A conjecture, usually attributed to Thouless (e.g. [70]), and supported by significant numerics (e.g. [27,67,68]), was that the fractal dimension is equal to $1/2$. Wilkinson and Austin [70] gave numerical and heuristic evidence that the box counting dimension is less than $1/2$ for the golden mean, and goes to zero as $n \to \infty$ for α with continued fraction expansion of the form $[n, n, n, \ldots]$, leading to the conjecture that the box counting dimension is less than $1/2$ for *all* α. There have been a number of interesting rigorous results on the Hausdorff dimension of the spectrum of the critical almost Mathieu operator [6, 35, 36, 54], all showing zero or small dimension for various α. Recently, it was proved that Hausdorff dimension is always bounded above by $1/2$ [39], which, however, does not imply a similar bound for box counting dimension. Let $\dim_{\mathrm{B}}(A)$ denote the box counting dimension of a Borel set A, and let Σ denote the spectrum of the critical almost Mathieu operator. An immediate corollary of the proof of Theorem 3 is

Theorem 4. *If* $\beta(\alpha) > 0$, *then* $\dim_{\mathrm{B}}(\Sigma) = 1$.

This disproves the Wilkinson–Austin conjecture through an explicit family of examples. More details are given in Section 1.3.

Finally, we mention that the concepts and methods introduced in this paper were already extended to singular Jacobi matrices in [34], leading also to sharp results, both for the general analytic case and for the extended Harper's model.

1.1. Main application

Fractal properties of Borel measures on $\mathbb{R}$ are linked to the boundary behavior of their Borel transforms [25]. Let

$$M(E + i\varepsilon) = \int \frac{\mathrm{d}\mu(E')}{E' - (E + i\varepsilon)} \tag{1.7}$$

be the Borel transform of the measure μ. Fix $0 < \gamma < 1$. If for μ-a.e. E,

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| < \infty, \tag{1.8}$$

we say that μ is (upper) γ-*spectral continuous*. Note that spectral continuity (and singularity) captures the lim inf power law behavior of $M(E+i\varepsilon)$, while the corresponding lim sup behavior is linked to Hausdorff dimension [25]. Define the (upper) *spectral dimension* of μ to be

$$s(\mu) = \sup\{\gamma \in (0,1) : \mu \text{ is } \gamma\text{-spectral continuous}\}. \tag{1.9}$$

For a Borel subset $S \subset \mathbb{R}$, let μ_S be the restriction of μ on S. A reformulation of Theorem 2 is

Theorem 5. *Suppose V is real analytic and $L(E) > 0$ for every E in some Borel set $S \subset \mathbb{R}$. Then for any $\theta \in \mathbb{T}$, $s(\mu_S) = 1$ if and only if $\beta(\alpha) = \infty$.*

Remark 1.1. If for μ-a.e. E,

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E+i\varepsilon)| = \infty, \tag{1.10}$$

we say that μ is (upper) γ-*spectral singular*. We can also consider

$$\tilde{s}(\mu) = \inf\{\gamma \in (0,1) : \mu \text{ is } \gamma\text{-spectral singular}\}. \tag{1.11}$$

Obviously, $s(\mu) \leq \tilde{s}(\mu)$. Actually, we will prove an upper bound for $\tilde{s}(\mu)$. Therefore, the conclusion in Theorem 5 also holds for $\tilde{s}(\mu)$.

1.2. Spectral singularity, continuity and proof of Theorem 5

We first study the γ-spectral singularity of μ. We are going to show that under the assumption of Theorem 5 we have

Theorem 6. *Assume $L(E) > a > 0$ for $E \in S$. There exists $c = c(a) > 0$ such that for any α, θ, if*

$$\gamma > \frac{1}{1 + \frac{c}{\beta(\alpha)}}, \tag{1.12}$$

then μ_S is γ-spectral singular.

Obviously, Theorem 6 implies that if $\beta < \infty$, then

$$s(\mu_S) \leq \tilde{s}(\mu_S) \leq \frac{1}{1 + c/\beta} < 1. \tag{1.13}$$

The analyticity of the potential and the positivity of the Lyapunov exponent are only needed for spectral singularity. We now formulate a more precise version of the general spectral continuity result, Theorem 1.

For $S \subset \sigma(H)$ assume there are constants $\Lambda > 0$ and $n_0 \in \mathbb{N}$ such that for any $k \in \mathbb{Z}$, $E \in S$ and $n \geq n_0$,

$$\left\| \begin{pmatrix} E - V(k+n) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V(k+1) & -1 \\ 1 & 0 \end{pmatrix} \right\| \leq e^{\Lambda n}. \tag{1.14}$$

Clearly, such a Λ always exists for bounded V, with $n_0 = 1$.

As before we denote by μ_S the spectral measure of H restricted to a Borel set $S \subset \sigma(H)$.

Theorem 7. *Let H be given by (1.1) and suppose V satisfies (1.14) and is β-almost periodic with $\epsilon > 0$. Then, for a $C(\epsilon) = C_0(1 + 1/\epsilon)$, with Λ satisfying (1.14), if*

$$\beta > C(\epsilon)\frac{\Lambda}{1-\gamma} \tag{1.15}$$

then μ_S is γ-spectral continuous. Here C_0 is a universal constant. Consequently,

$$\tilde{s}(\mu_S) \geq s(\mu_S) \geq 1 - C(\epsilon)\frac{\Lambda}{\beta}. \tag{1.16}$$

Proof of Theorem 5. Under the assumption of Theorem 5, if $\beta < \infty$, Theorem 6 provides the upper bound (1.13) for the spectral dimension.

We will now get the lower bound using Theorem 7. Let $V_\theta(n) := V(\theta + n\alpha)$. By boundedness of V and compactness of the spectrum, there is a constant $\Lambda_V < \infty$ such that (1.14) holds uniformly for $E \in \sigma(H_\theta), \theta \in \mathbb{T}$. In order to apply Theorem 7, it is enough to show that for any $\beta < \beta(\alpha)$, $V(\theta + j\alpha)$ has β-repetitions for any $\theta \in \mathbb{T}, j \in \mathbb{Z}$. Indeed, by (1.6), there is a subsequence q_{n_k} such that

$$(\log q_{n_k+1})/q_{n_k} > \beta.$$

Since V is analytic, for any θ, j and $1 \leq n \leq q_{n_k}$ we have

$$|V(\theta + j\alpha + n\alpha) - V(\theta + j\alpha + n\alpha \pm q_{n_k}\alpha)| \leq C\|q_{n_k}\alpha\| \leq C\frac{1}{q_{n_k+1}} \leq Ce^{-\beta q_{n_k}}.$$

Thus if $\beta(\alpha) = \infty$, then $\tilde{s}(\mu_S) = s(\mu_S) = 1$. ∎

Property (1.14) naturally holds in a *sharp* way in the context of ergodic potentials with uniquely ergodic underlying dynamics. Assume the potential $V = V_\theta$ is generated by some homeomorphism T of a compact metric space Ω and a function $f : \Omega \to \mathbb{R}$ by

$$V_\theta(n) = f(T^n\theta), \qquad \theta \in \Omega, n \in \mathbb{Z}. \tag{1.17}$$

Assume (Ω, T) is uniquely ergodic with an ergodic measure ν. It is known that the spectral type of H_θ is ν-almost surely independent of θ (e.g. [13]). In general, however, the spectral type (locally) does depend on θ (see [49]). If f is continuous then, by uniform upper semicontinuity (e.g. [26]),

$$\limsup_n \sup_\theta \frac{1}{n}\log\left\|\begin{pmatrix} E - V_\theta(n) & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} E - V_\theta(1) & -1 \\ 1 & 0 \end{pmatrix}\right\| \leq L(E), \qquad \forall E. \tag{1.18}$$

This was recently extended in [47] to almost continuous f. Following [47], we will say a function f is *almost continuous* if it is bounded and its set of discontinuities has a closure of ν-measure zero. By [47, Corollary 3.2], if f is bounded and almost continuous then

(1.18) also holds for every E. Moreover, if the Lyapunov exponent $L(E)$ is continuous on some compact set S, then, by compactness and subadditivity, the lim sup in (1.18) will also be uniform in $E \in S$. Since by upper semicontinuity $L(E)$ is continuous on the set where it is zero, as a consequence of Theorem 7 we obtain

Corollary 1. *Assume the function f in (1.17) is bounded and almost continuous and $L(E) = 0$ on some Borel subset S of $\sigma(H_\theta)$. If $V_\theta(n)$ is β-almost periodic for some $\beta, \epsilon > 0$, then $s(\mu_S^\theta) = 1$.*

Proof. For any $0 < \gamma < 1$, set $\Lambda' = \beta(1-\gamma)/(2C)$ where $C = C(\epsilon)$ is given in Theorem 7.[4] Since $L(E) = 0$ on S, by the arguments above there is $n_0 = n_0(\Lambda')$ independent of θ and E such that

$$\left\| \begin{pmatrix} E - V_\theta(n) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V_\theta(1) & -1 \\ 1 & 0 \end{pmatrix} \right\| \le e^{\Lambda' n}, \qquad n \ge n_0,\ E \in S,\ \theta \in \Omega.$$

Obviously, $\beta > C\Lambda'/(1-\gamma)$, so Theorem 7 is applicable and (1.16) holds. Therefore, $s(\mu_{S,\theta}) \ge 1 - C\Lambda'/\beta > \gamma$. ■

Let $S_0 = \{E : L(E) = 0\}$ and $S_+ = \{E : L(E) > 0\}$.

As an immediate consequence we obtain

Theorem 8. *If $V_\theta(n)$ is given by (1.17) with uniquely ergodic (Ω, T) and almost continuous f, then for every θ we have*

(1) $s(\mu_{S_0}) = 1$ *as long as V is β-almost periodic with $\beta > 0$,*

(2) $s(\mu_{S_+}) = 1$ *as long as V is β-almost periodic with $\beta = \infty$.*

Remark 1.2. (1) $\beta > 0$ is not a necessary condition in general for $s(\mu_{S_0}) = 1$, for $s(\mu^{\mathrm{ac}}) = 1$ even if V is not β-almost periodic for any β, and the support of the absolutely continuous spectrum is contained in (and may coincide with) S_0. It is a very interesting question to specify a quantitative almost periodicity condition for $s(\mu_{S_0}^{\mathrm{sing}}) = 1$, in particular, find an arithmetic criterion for analytic one-frequency potentials for $s(\mu_{S_{\mathrm{cr}}}) = 1$ where $S_{\mathrm{cr}} \subset S_0$ is the set of critical energies in the sense of Avila's global theory.

(2) According to Theorem 5, $\beta = \infty$ is also necessary if f is analytic and T is an irrational rotation of the circle (β will depend on T). In case f has lower regularity, it is an interesting question to determine an optimal condition on β.

1.3. Corollaries for the AMO, Sturmian potentials, and transport exponents

If we replace the lim inf by lim sup in the definition of upper spectral dimension, we will define correspondingly the lower spectral dimension which will coincide with the Hausdorff dimension $\dim_{\mathrm{H}}(\mu)$ of a measure μ.

Also one can consider the packing dimension of μ, denoted by $\dim_{\mathrm{P}}(\mu)$. The packing dimension can be defined in a similar way to (1.9) through the γ-dimensional lower

[4] If $\beta = \infty$ take any finite β instead.

derivative $\underline{D}^{\gamma}\mu(E)$. It can be easily shown that $\underline{D}^{\gamma}\mu(E) \leq \liminf_{\varepsilon\downarrow 0} \varepsilon^{1-\gamma}|M(E+i\varepsilon)|$. Thus the relation between packing dimension and upper spectral dimension is $\dim_{\mathrm{P}}(\mu) \geq \tilde{s}(\mu)$.[5] Therefore, the lower bound we get in Theorem 7 also holds for the packing dimension.

Lower bounds on spectral dimension also have immediate applications to lower bounds on quantum dynamics. Denote by δ_j the vector $\delta_j(n) = \chi_j(n)$. For $p > 0$, define

$$\langle |X|_{\delta_0}^{p}\rangle(T) = \frac{2}{T}\int_0^{\infty} e^{-2t/T}\sum_n |n|^p |\langle e^{-itH}\delta_0, \delta_n\rangle|^2. \tag{1.19}$$

The growth rate of $\langle |X|_{\delta_0}^{p}\rangle(T)$ characterizes how fast $e^{-itH}\delta_0$ spreads out. In order to get power law bounds for $\langle |X|_{\delta_0}^{p}\rangle(T)$, it is natural to define the following upper and lower dynamical exponents:

$$\beta_{\delta_0}^{+}(p) = \limsup_{T\to\infty} \frac{\log\langle |X|_{\delta_0}^{p}\rangle(T)}{p\log T}, \quad \beta_{\delta_0}^{-}(p) = \liminf_{T\to\infty} \frac{\log\langle |X|_{\delta_0}^{p}\rangle(T)}{p\log T}. \tag{1.20}$$

The dynamics is called *ballistic* if $\beta_{\delta_0}^{-}(p) = 1$ for all $p > 0$, and *quasiballistic* if $\beta_{\delta_0}^{+}(p) = 1$ for all $p > 0$. We will also say that the dynamics is *quasilocalized* if $\beta_{\delta_0}^{-}(p) = 0$ for all $p > 0$.

In [33], it is shown that the upper and lower transport exponents of a discrete Schrödinger operator (1.1) can be bounded from below by the packing and Hausdorff dimension of its spectral measure respectively. Therefore, by [33] we have $\beta_{\delta_0}^{+}(p) \geq s(\mu)$ for all p. As a direct consequence of Theorem 7 we have

Corollary 2. *If $V(n)$ is bounded and ∞-almost periodic, the upper dynamical exponent $\beta_{\delta_0}^{+}(p)$ of the operator (1.5) is 1 for any $p > 0$, and the associated dynamics is quasi-ballistic.*

This has nice immediate consequences. In particular, consider the *almost Mathieu operator*

$$(H_{\lambda,\theta,\alpha}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos 2\pi(\theta + n\alpha)\, u_n, \qquad \lambda > 0. \tag{1.21}$$

As a consequence of the formula for the Lyapunov exponent and Theorem 5, one has

Corollary 3. *The almost Mathieu operator (1.21) is quasiballistic[6] for any (and all) $\theta \in \mathbb{T}$ and*

(1) *for $\lambda < 1$ for all α;*

(2) *for $\lambda = 1$ as long as $\beta(\alpha) > 0$;*

(3) *for $\lambda > 1$ as long as $\beta(\alpha) = \infty$.*

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[5] Unlike for Hausdorff dimension, the relation for packing dimension only goes in one direction, in general, contrary to what is claimed in [14].

[6] And has spectral dimension 1 and packing dimension 1 of the spectral measure.

Statement (1) is a corollary of absolute continuity [2, 51] and is listed here for completeness only. Statements (2) and (3) are direct corollaries of Theorem 8.

For $\lambda > 1$, the Hausdorff dimension of the spectral measure of the almost Mathieu operator is equal to zero [41] and $\beta^-(p) = 0$ for all $p > 0$ [24]. Thus almost Mathieu operators with $\lambda > 1$ and $\beta(\alpha) = \infty$ provide a family of *explicit* examples of operators that are simultaneously quasilocalized and quasiballistic and whose spectral measures satisfy

$$0 = \dim_{\mathrm{H}}(\mu) < \dim_{\mathrm{P}}(\mu) = 1.$$

The same holds of course for cos replaced with any almost continuous f as long as the Lyapunov exponent is positive everywhere on the spectrum, in particular for $f = \lambda g$ where g is either bi-Lipschitz (as in [37]) or analytic, and $\lambda > \lambda(g)$ is sufficiently large.

Let $\mathrm{d}N$ be the density states measure of the almost Mathieu operator and Σ be the spectrum. It is well-known that in the critical case, $\lambda = 1$, Σ has Lebesgue measure zero [5, 52]. It is then interesting to consider the fractal dimension of the spectrum (as a set). Since $\mathrm{d}N = \mathbb{E}(\mathrm{d}\mu_\theta)$ and $\mathrm{supp}_{\mathrm{top}}(\mathrm{d}N) = \Sigma$, by the discussion above we have

Corollary 4. *For the critical almost Mathieu operator, $\lambda = 1$, and $\beta(\alpha) > 0$ we have* $\dim_{\mathrm{P}}(\mathrm{d}N) = \dim_{\mathrm{P}}(\Sigma) = 1$.

Last and Shamis [54] (see also [61]) proved that for a dense G_δ set of α (which therefore has a generic intersection with the set $\{\alpha : \beta(\alpha) > 0\}$), the Hausdorff dimension of the spectrum is equal to zero. This was recently extended to the entire set of α with $\beta(\alpha) > 0$ [6]. Thus critical almost Mathieu operators with topologically generic frequencies (namely, those with $\beta(\alpha) > 0$) and any θ provide an explicit family of operators that all have spectra satisfying

$$0 = \dim_{\mathrm{H}}(\Sigma) < \dim_{\mathrm{P}}(\Sigma) = \dim_{\mathrm{B}}(\Sigma) = 1.$$

Another well-known family is *Sturmian Hamiltonians* given by

$$(Hu)_n = u_{n+1} + u_{n-1} + \lambda\chi_{[1-\alpha,1)}(n\alpha + \theta \bmod 1)u_n, \tag{1.22}$$

where $\lambda > 0$ and $\alpha = \mathbb{R}\backslash\mathbb{Q}$. If $\alpha = \frac{\sqrt{5}-1}{2}$, it is called the *Fibonacci Hamiltonian*. The spectral properties of the Fibonacci Hamiltonian have been thoroughly studied in a series of papers in the past three decades (see [17, 19] for more references). Recently, Damanik, Gorodetski and Yessen [21] proved that for every $\lambda > 0$, the density-of-states measure $\mathrm{d}N_\lambda$ is *exact-dimensional* (the Hausdorff and upper box counting dimension are the same) and $\dim_{\mathrm{H}}(\mathrm{d}N_\lambda) < \dim_{\mathrm{H}}(\Sigma_\lambda)$.

Our results show that the exact dimensionality properties of the Sturmian Hamiltonians strongly rely on the arithmetic properties of α. It was shown in [9] that if α is irrational, the Lyapunov exponent of a Sturmian operator restricted to the spectrum is zero. Also the spectrum $\Sigma_{\lambda,\alpha}$ of the Sturmian Hamiltonian is always a Cantor set with Lebesgue measure zero. Moreover, for Sturmian potentials results similar to those for the critical almost Mathieu operator in Corollary 4 also hold. Let μ_θ be the spectral measure of the Sturmian operator (1.22) and let $\mathrm{d}N_{\lambda,\alpha}$ be the density-of-states measure and $\Sigma_{\lambda,\alpha}$

be the spectrum. We say that a phase θ is *α-Diophantine* if there exist $\gamma < \infty$ and $\tau > 1$ such that $\|\theta + m\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \gamma/(|m|+1)^\tau$ for all $m \in \mathbb{Z}$. Clearly, this is a full measure condition. We have

Theorem 9. *For the Sturmian operator $H_{\theta,\lambda,\alpha}$ with $\beta(\alpha) > 0$ and $\lambda > 0$, if θ is α-Diophantine, the spectral dimension of μ_θ is* 1.

As a consequence, if $\beta(\alpha) > 0$ and $\lambda > 0$, then the packing dimensions of $\mathrm{d}N_{\lambda,\alpha}$ and of $\Sigma_{\lambda,\alpha}$ are both equal to 1.

Previously, Liu, Qu and Wen [57, 58] studied the Hausdorff and upper box counting dimension of $\Sigma_{\lambda,\alpha}$ of Sturmian operators. For large couplings, they gave a criterion on $\alpha \in (0,1)$ for the Hausdorff dimension of the spectrum to be 1. Combining Theorem 9 with their results, we have

Corollary 5. *Let $\Sigma_{\lambda,\alpha}$ be the spectrum of the Sturmian Hamiltonian with $\lambda > 20$. There are explicit α such that for a.e. θ,*

$$\dim_{\mathrm{H}}(\mu^\theta_{\lambda,\alpha}) < s(\mu^\theta_{\lambda,\alpha}) = \dim_{\mathrm{P}}(\mu^\theta_{\lambda,\alpha}) = 1, \tag{1.23}$$

$$\dim_{\mathrm{H}}(\mathrm{d}N_{\lambda,\alpha}) < s(\mathrm{d}N_{\lambda,\alpha}) = \dim_{\mathrm{P}}(\mathrm{d}N_{\lambda,\alpha}) = 1. \tag{1.24}$$

The proof for the Sturmian case is given in Section 4.

The rest of this paper is organized as follows. After giving the preliminaries in Section 1.4 we proceed to the proof of the general continuity statement in Section 2. First we quickly reduce Theorem 7 to Lemma 2.1 where we also specify the constant C_0 appearing in Theorem 7. We note that we do not aim to optimize the constants here and many of our arguments have room for improvement. Lemma 2.1 is further reduced to the estimate on the traces of the transfer matrices over eventual almost periods, Theorem 10, through its corollaries, Lemmas 2.2 and 2.3. Theorem 10 is the key element and the most technical part of the proof. It is of interest in its own right as it can be viewed as the quantitative version of the fact that period-length transfer matrices of periodic operators are elliptic: it provides quantitative bounds on the traces of transfer matrices over almost periods based on quantitative almost periodicity for spectrally a.e. energy. In Section 2.2 we separate this statement into hyperbolic and almost parabolic parts, in Lemmas 2.4 and 2.5. In Section 2.3 we use the extended Schnol Theorem to study the hyperbolic case and in Section 2.4 we combine estimates on level sets of polynomials, power-law subordinacy bounds, and an elementary but very useful algebraic representation of matrix powers (Lemma 2.9) to study the almost parabolic case. Lemmas 2.2 and 2.3 are proved in Section 2.5, completing the continuity part. In Section 3 we focus on analytic quasiperiodic potentials and prove Theorem 6. The proof is based on a lemma about density of localized blocks (Lemma 3.3). Finally, we discuss Sturmian potentials in Section 4, proving Theorem 9 and then providing explicit examples for Corollary 5.

1.4. Preliminaries

1.4.1. m-function and subordinacy theory. We now briefly introduce the power-law extension of the Gilbert–Pearson subordinacy theory [29, 30], developed in [40]. We will also list the necessary related facts on the Weyl–Titchmarsh m-function. More details can be found, e.g., in [13].

Let H be as in (1.5) and $z = E + i\varepsilon \in \mathbb{C}$. Consider the equation

$$Hu = zu \tag{1.25}$$

with the family of normalized phase boundary conditions:

$$u_0^{\varphi} \cos\varphi + u_1^{\varphi} \sin\varphi = 0, \quad -\pi/2 < \varphi \le \pi/2, \ |u_0^{\varphi}|^2 + |u_1^{\varphi}|^2 = 1. \tag{1.26}$$

Let $\mathbb{Z}^+ = \{1, 2, \ldots\}$ and $\mathbb{Z}^- = \{\ldots, -2, -1, 0\}$. Denote by $u^{\varphi} = \{u_j^{\varphi}\}_{j\ge 0}$ the right half-line solution on $\mathbb{Z}^+$ of (1.25) with boundary condition (1.26) and by $u^{\varphi,-} = \{u_j^{\varphi,-}\}_{j\le 0}$ the left half-line solution on $\mathbb{Z}^-$ of the same equation. Also denote by v^{φ} and $v^{\varphi,-}$ the right and left half-line solutions of (1.25) with boundary conditions orthogonal to u^{φ} and $u^{\varphi,-}$. That is, $v^{\varphi} = u^{\varphi+\pi/2}, v^{\varphi,-} = u^{\varphi+\pi/2,-}, \varphi \le 0$, and $v^{\varphi} = u^{\varphi-\pi/2}, v^{\varphi,-} = u^{\varphi-\pi/2,-}, \varphi > 0$. For any function $u : \mathbb{Z}^+ \to \mathbb{C}$ we denote by $\|u\|_l$ the norm of u over a lattice interval of length l, that is,

$$\|u\|_l = \Big(\sum_{n=1}^{[l]} |u(n)|^2 + (l - [l])|u([l] + 1)|^2\Big)^{1/2}, \tag{1.27}$$

where [ ] is integer part. Similarly, for $u : \mathbb{Z}^- \to \mathbb{C}$, we define

$$\|u\|_l = \Big(\sum_{n=1}^{[l]-1} |u(-n)|^2 + (l - [l])|u(-[l])|^2\Big)^{1/2}. \tag{1.28}$$

Now given any $\varepsilon > 0$, we define the length $l = l(\varphi, \varepsilon, E)$ by requiring the equality

$$\|u^{\varphi}\|_{l(\varphi,\varepsilon,E)} \|v^{\varphi}\|_{l(\varphi,\varepsilon,E)} = \frac{1}{2\varepsilon}. \tag{1.29}$$

We also define $l^-(\varphi)$ by using $u^{\varphi,-}, v^{\varphi,-}$ through the same equation. By the constancy of the Wronskian and orthogonality of the boundary conditions, we have

$$u^{\varphi}(n+1)v^{\varphi}(n) - v^{\varphi}(n+1)u^{\varphi}(n) = u^{\varphi}(1)v^{\varphi}(0) - v^{\varphi}(1)u^{\varphi}(0) = 1,$$

which implies by Cauchy–Schwarz that

$$\|u^{\varphi}\|_l \cdot \|v^{\varphi}\|_l \ge \tfrac{1}{2}([l] - 1). \tag{1.30}$$

Denote by $m_{\varphi} : \mathbb{C}^+ \to \mathbb{C}^+$ and $m_{\varphi}^- : \mathbb{C}^+ \to \mathbb{C}^+$ the right and left Weyl–Titchmarsh m-functions associated with the boundary condition (1.26). Let $m = m_0$ and $m^- = m_0^-$ be the half line m-functions corresponding to the Dirichlet boundary conditions. The following key inequality [40] relates $m_{\varphi}(E + i\varepsilon)$ to the solutions u^{φ} and v^{φ} given by (1.25), (1.26).

Lemma 1.1 (J-L inequality, [40, Theorem 1.1]). *For $E \in \mathbb{R}$ and $\varepsilon > 0$, and for any $\varphi \in (-\pi/2, \pi/2]$,*

$$\frac{5 - \sqrt{24}}{|m_{\varphi}(E + i\varepsilon)|} < \frac{\|u^{\varphi}\|_{l(\varphi,\varepsilon)}}{\|v^{\varphi}\|_{l(\varphi,\varepsilon)}} < \frac{5 + \sqrt{24}}{|m_{\varphi}(E + i\varepsilon)|}. \tag{1.31}$$

We need to study the whole-line m-function which is given by the Borel transform of the spectral measure μ of the operator H (see e.g. [13]). The following relation between the whole-line m-function M and the half-line m-function m_φ was first shown in [22] as a corollary of the maximal modulus principle. One can also find in [4] a different proof based on a direct computation in the hyperbolic plane.

Proposition 1 ([22, Corollary 21]). *For $E \in \mathbb{R}$ and $\varepsilon > 0$,*

$$|M(E+i\varepsilon)| \le \sup_{\varphi} |m_\varphi(E+i\varepsilon)|. \tag{1.32}$$

This proposition implies that in order to obtain an upper bound for the whole-line m-function, namely, the continuity of the whole-line spectrum, it is enough to obtain a uniform upper bound of the half-line m-function for any boundary condition.

On the other hand, consider the unitary operator $U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ defined by $(U\psi)_n = \psi_{-n+1}$ for $n \in \mathbb{Z}$. For any operator H on $l^2(\mathbb{Z})$, we define an operator $\widetilde{H}$ on $l^2(\mathbb{Z})$ by $\widetilde{H} = UHU^{-1}$. Denote by $\widetilde{m}, \widetilde{m}_\varphi, \widetilde{u}^\varphi$ and $\widetilde{l}(\varphi)$ the parameters m, m_φ, u^φ and $l(\varphi)$ of the operator $\widetilde{H}$. We will need the following well-known facts (see e.g. [41, Section 3]). For any $\varphi \in (-\pi/2, \pi/2]$ we have

$$M(z) = \frac{m_\varphi(z)\widetilde{m}_{\pi/2-\varphi} - 1}{m_\varphi(z) + \widetilde{m}_{\pi/2-\varphi}} \tag{1.33}$$

and

$$\widetilde{l}(\pi/2-\varphi) = l^-(\varphi), \qquad \|u\|_l = \|Uu\|_l. \tag{1.34}$$

Similar to [41, Lemma 5], a direct consequence of (1.33) is the following result.

Lemma 1.2. *For any $0 < \gamma < 1$, suppose that there exists a $\varphi \in (-\pi/2, \pi/2]$ such that for μ-a.e. E in some Borel set S, we have $\liminf_{\varepsilon\to 0} \varepsilon^{1-\gamma}|m_\varphi(E+i\varepsilon)| = \infty$ and $\liminf_{\varepsilon\to 0} \varepsilon^{1-\gamma}|\widetilde{m}_{\pi/2-\varphi}(E+i\varepsilon)| = \infty$. Then $\liminf_{\varepsilon\to 0} \varepsilon^{1-\gamma}|M(E+i\varepsilon)| = \infty$ for μ-a.e. E in S, so the restriction $\mu(S \cap \cdot)$ is γ-spectral singular.*

1.4.2. Transfer matrices and Lyapunov exponents. Although Theorem 7 does not involve any further conditions on the potential, it will be convenient in what follows to use dynamical notations. Let $\Omega = \mathbb{R}^{\mathbb{Z}}$ and $T : \Omega \to \Omega$ be given by $(T\theta)(n) = \theta(n+1)$. Let $f(\theta) := \theta(0)$. Then any potential V can be written as in (1.17), $V_\theta(n) := \theta(n) = f(T^n\theta)$. Thus for a fixed $\{V_n\}_{n\in\mathbb{Z}} = \theta \in \Omega$, we will rewrite the potential V as $V_\theta(n) = f(T^n\theta)$ as in (1.17). For our general theorem we do not introduce any topology, etc.; this is being done purely for the notational convenience. Denote the n-step transfer matrix by $A_n(\theta, E)$:

$$A_n(\theta, E) = A(T^n\theta, E)A(T^{n-1}\theta, E)\cdots A(T\theta, E), \qquad n > 0, \tag{1.35}$$

and

$$A_0 = \mathrm{Id}, \qquad A_n(\theta, E) = A_{-n}^{-1}(T^n\theta, E), \qquad n < 0,$$

where

$$A(\theta, E) = \begin{pmatrix} E - f(\theta) & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.36}$$

The connection to Schrödinger operators is clear since a solution of $Hu = Eu$ can be reformulated as

$$A_n(\theta, E)\begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}, \qquad n \in \mathbb{Z}. \tag{1.37}$$

In other words, the spectral properties of Schrödinger operators H are closely related to the dynamics of the family of skew products $(T, A(\theta, E))$ over $\Omega \times \mathbb{R}^2$. We will often suppress either θ or E or both from the notations if the corresponding parameters are fixed through the argument.

If V is actually dynamically defined by (1.17) with a certain underlying ergodic base dynamics (Ω, T, ν) then, by the general properties of subadditive ergodic cocycles, we can define the Lyapunov exponent

$$L(E, T) = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \|A_n(\theta, E)\| \, d\nu = \inf_{n>0} \frac{1}{n} \int_\Omega \log \|A_n(\theta, E)\| \, d\nu. \tag{1.38}$$

2. Spectral continuity

2.1. Proof of Theorem 7

Throughout this section we assume (1.14) is satisfied uniformly for $E \in S$ and $\theta \in \Omega$ (see Section 1.4.2). Assume V is β-almost periodic for some $\epsilon > 0$. The proof of Theorem 7 is based on the following estimates on the growth of the l-norm of half-line solutions. Let u^φ, v^φ be given as in (1.25)–(1.27).

Lemma 2.1. *For $0 < \gamma < 1$, assume $\beta > 100(1 + 1/\epsilon)\Lambda/(1 - \gamma)$. For μ-a.e. E, there is a sequence of positive numbers $\eta_k \to 0$ such that for any φ,*

$$\frac{1}{16}(L_k)^\gamma \le \|v^\varphi\|_{L_k}^2 \le (L_k)^{2-\gamma} \tag{2.1}$$

where $L_k = l(\varphi, \eta_k, E)$ is as in (1.29).

Proof of Theorem 7. Fix $0 < \gamma < 1$. Set $C(\epsilon) := 100(1 + 1/\epsilon)$. Lemma 2.1 can be applied to any $\beta > C(\epsilon)\frac{\Lambda}{1-\gamma}$. According to (2.1) and the J-L inequality (1.31), for μ-a.e. E and any φ,

$$\begin{aligned} \eta_k^{1-\gamma} |m_\varphi(E + i\eta_k)| &\le \frac{1}{(2\|u^\varphi\|_{L_k}\|v^\varphi\|_{L_k})^{1-\gamma}} \cdot (5 + \sqrt{24}) \frac{\|v^\varphi\|_{L_k}}{\|u^\varphi\|_{L_k}} \\ &\le C_\gamma \cdot \frac{(L_k^{(2-\gamma)/2})^\gamma}{\left(\frac{1}{4}L_k^{\gamma/2}\right)^{2-\gamma}} = C < \infty. \end{aligned}$$

Since η_k is independent of φ, for a fixed E and η_k, we can take the supremum with respect to φ. By (1.32) in Proposition 1, for μ-a.e. E we have

$$\eta_k^{1-\gamma}|M(E+i\eta_k)|<C,$$

i.e.,

$$\liminf_{\varepsilon\downarrow 0}\varepsilon^{1-\gamma}|M(E+i\varepsilon)|<\infty, \qquad \mu\text{-a.e. } E,$$

which proves the γ-spectral continuity of Theorem 7. The lower bound (1.16) comes from the definition of spectral dimensionality. ■

Lemma 2.1 follows from the following estimates on the trace of the transfer matrix. Let q_k be the sequence given in (1.3).

Theorem 10. *If*

$$\beta>(37+11/\epsilon)\Lambda, \tag{2.2}$$

then for μ-a.e. E, there is $K(E)$ such that

$$|\text{Trace } A_{q_k}(E)|<2-e^{-10\Lambda q_k}, \qquad k\geq K(E). \tag{2.3}$$

This theorem is the key estimate for spectral continuity. It can be viewed as a quantitative version of the classical fact that period-length transfer matrices of periodic operators are elliptic on the spectrum. Indeed, we prove that β-almost periodicity implies quantitative bounds on ellipticity. The proof will be given in the following two subsections. As a direct consequence of Theorem 10, we have the following estimates on the norm of the transfer matrices. They show that if the trace of the transfer matrix over an almost period is strictly less than 2, then there is a sublinearly bounded subsequence. We will use this result to prove Lemma 2.1 first. The proofs of Lemmas 2.2 and 2.3 will be postponed to Section 2.5. Let $K(E)$ be as in Theorem 10.

Lemma 2.2. *For any $\xi>0$ set $N_k=[e^{\xi q_k}]$ and suppose that, in addition to the conditions of Theorem 10,*

$$\beta>15\Lambda+(2+1/\epsilon)\xi. \tag{2.4}$$

Then for μ-a.e. E,

$$\sum_{n=1}^{N_k q_k}\|A_n(E)\|^2\leq e^{(\xi+15\Lambda)q_k}, \qquad k\geq K(E) \tag{2.5}$$

Lemma 2.3. *For $0<\gamma<1$, assume that in addition to the conditions of Lemma 2.2,*

$$\xi\geq\frac{17\Lambda}{1-\gamma}. \tag{2.6}$$

Then

$$\sum_{n=1}^{N_k q_k}\|A_n(E)\|^2\leq (N_k q_k)^{2-\gamma}, \qquad k\geq K(E). \tag{2.7}$$

Proof of Lemma 2.1. It is enough to prove the right-hand inequality of (2.1), which is an upper bound of $\|v^\varphi\|_{L_k}$ for any φ. Then by the relation $u^\varphi = v^{\varphi+\pi/2}$, one obtains the same upper bound for $\|u^\varphi\|_{L_k}$. Note that $\|u^\varphi\|_{L_k}\|v^\varphi\|_{L_k} \geq L_k/4$ by (1.30), provided $L_k \geq 2$. Therefore,

$$\|v^\varphi\|_{L_k} \geq \frac{1}{4}L_k\frac{1}{\|u^\varphi\|_{L_k}} \geq \frac{1}{4}L_k^{\gamma/2},$$

which is the left-hand inequality of (2.1).

For any $0 < \gamma < 1$, set $\beta_0 = 100(1 + 1/\epsilon)\frac{\Lambda}{1-\gamma}$, $\xi = \frac{17\Lambda}{1-\gamma}$. Then (2.2), (2.4) and (2.6) are satisfied for all $\beta > \beta_0$. Therefore, (2.7) holds the with above choice of parameters. Let $l_k = [e^{\xi q_k}]q_k$. Rewrite (2.7) as $\sum_{n=1}^{l_k}\|A_n(E)\|^2 < l_k^{2-\gamma}$. Thus $\|v^\varphi\|_{l_k}^2 \leq 4l_k^{2-\gamma}$ for any φ. By (1.30), we have

$$\frac{1}{4}l_k \leq \|u^\varphi\|_{l_k}\|v^\varphi\|_{l_k} \leq 4l_k^{2-\gamma}. \tag{2.8}$$

Set

$$\varepsilon_k(\varphi) := \frac{1}{2\|u^\varphi\|_{l_k}\|v^\varphi\|_{l_k}}. \tag{2.9}$$

Then

$$\eta_k = \inf_\varphi \varepsilon_k(\varphi) \geq \frac{1}{8l_k^{2-\gamma}} > 0 \tag{2.10}$$

is well-defined. Set $L_k(\varphi) := l(\varphi, \eta_k, E)$. Then the length scale satisfies

$$\eta_k = \frac{1}{2\|u^\varphi\|_{L_k(\varphi)}\|v^\varphi\|_{L_k(\varphi)}}. \tag{2.11}$$

By (2.10),

$$L_k(\varphi) \leq 4\|u^\varphi\|_{L_k}\|v^\varphi\|_{L_k} = \frac{2}{\eta_k} \leq 16l_k^{2-\gamma}.$$

Since $\eta_k \leq \varepsilon_k(\varphi)$ and $\|u^\varphi\|_l\|v^\varphi\|_l$ is increasing in l, for any φ we obtain

$$l_k \leq L_k(\varphi) \leq 16l_k^{2-\gamma}. \tag{2.12}$$

By the definition of l_k, for large k,

$$e^{(\xi-\frac{\Lambda}{200(1-\gamma)})q_k}q_k \leq L_k(\varphi) \leq e^{((2-\gamma)\xi+\Lambda/200)q_k}q_k. \tag{2.13}$$

Write $L_k(\varphi) = [L_k(\varphi)] + \widetilde{L}_k(\varphi)$ and

$$[L_k(\varphi)] = (N_k(\varphi) - 1)q_k + r_k(\varphi), \qquad N_k(\varphi) \in \mathbb{N},\ 0 \leq r_k(\varphi) < q_k. \tag{2.14}$$

Define

$$\xi_k(\varphi) = \frac{\log N_k(\varphi)}{q_k}. \tag{2.15}$$

We have

$$[L_k(\varphi)] = (e^{\xi_k(\varphi)q_k} - 1)q_k + r_k(\varphi), \qquad e^{\xi_k(\varphi)q_k} \in \mathbb{N},\ 0 \leq r_k(\varphi) < q_k. \tag{2.16}$$

For large q_k, it is easy to check that

$$e^{(\xi_k(\varphi)-\Lambda/200)q_k} q_k \leq (e^{\xi_k(\varphi)q_k} - 1)q_k \leq L_k(\varphi) \leq e^{\xi_k(\varphi)q_k} q_k.$$

Using (2.13), for any φ we have

$$\xi - \Lambda/200 \leq \xi_k(\varphi) \leq (2-\gamma)\xi + \Lambda/100 \leq 2\xi + \Lambda/100. \tag{2.17}$$

Together with the choice of β and ξ, we have

$$\beta > \beta_0 > 15\Lambda + (2 + 1/\epsilon)\xi_k(\varphi).$$

Now we can again apply Lemma 2.2 with parameters β, $\xi_k(\varphi)$ and the length scale $N_k(\varphi) = e^{\xi_k(\varphi)q_k}$ to get

$$\sum_{n=1}^{N_k(\varphi)q_k} \|A_n(E)\|^2 \leq e^{(\xi_k(\varphi)+15\Lambda)q_k}. \tag{2.18}$$

Notice $L_k(\varphi) \geq e^{(\xi_k(\varphi)-\Lambda/200)q_k}$ implies that

$$\frac{1}{(L_k)^{2-\gamma}} \sum_{n=1}^{N_k(\varphi)q_k} \|A_n(E)\|^2 \leq e^{(-(1-\gamma)\xi_k+16\Lambda)q_k}.$$

By the left-hand inequality of (2.17), we obtain

$$(1-\gamma)(\xi_k(\varphi) + \Lambda/200) > (1-\gamma)\xi = 17\Lambda,$$

which implies

$$(1-\gamma)\xi_k(\varphi) > 17\Lambda - (1-\gamma)\Lambda/200 > 16.5\Lambda,$$

and

$$\frac{1}{(L_k)^{2-\gamma}} \sum_{n=1}^{N_k q_k} \|A_n(E)\|^2 \leq e^{-\Lambda q_k/2} \leq 1. \tag{2.19}$$

Finally, by Lemma 2.3 we have

$$\begin{aligned} \|v^\varphi\|_{L_k}^2 &\leq \sum_{n=1}^{[L_k]+1} |v_n^\varphi|^2 \leq \sum_{n=1}^{N_k(\varphi)q_k} (|v_n^\varphi|^2 + |v_{n+1}^\varphi|^2) \\ &\leq \sum_{n=1}^{N_k(\varphi)q_k} \|A_n(E)\|^2 \leq (L_k)^{2-\gamma}. \end{aligned}$$

■

2.2. *Proof of Theorem* 10

The proof of Theorem 10 will be divided into two cases. We will first exclude the energies where the trace is much greater than 2 infinitely many times using the extended Schnol Theorem (Lemma 2.6). Then we will estimate the measure of energies where the trace is close to 2 through subordinacy theory. The conclusion consists of the following two lemmas. Again let q_k be the sequence given by (1.3) with certain $\beta, \epsilon > 0$.

Lemma 2.4. *For any $\tau > 0$, if*

$$\beta > (3 + 1/\epsilon)\tau + (7 + 1/\epsilon)\Lambda, \tag{2.20}$$

then for spectrally a.e. E, there is $K_1(E)$ such that

$$|\text{Trace } A_{q_k}(E)| < 2 + e^{-\tau q_k}, \qquad k \geq K_1(E). \tag{2.21}$$

Lemma 2.5. *If*

$$\beta > (25 + 1/\epsilon)\Lambda, \tag{2.22}$$

then for spectrally a.e. E, there is $K_2(E)$ such that

$$|\text{Trace } A_{q_k}(E) \pm 2| > e^{-10\Lambda q_k}, \qquad k \geq K_2(E). \tag{2.23}$$

Proof of Theorem 10. The theorem follows immediately by combining Lemma 2.4 with $\tau = 10\Lambda$ and Lemma 2.5. ∎

Remark 2.1. It may be interesting to compare Theorem 10 with the technique Last used in his proof of zero Hausdorff dimensionality of the spectral measures of supercritical Liouville almost Mathieu operators [53]. An important step there was using Schnol's Theorem to show that eventually spectrally almost every energy is in the union of the spectral bands of the periodic approximants *enlarged* by a factor of q_k^2. Here we show that spectrally almost every energy is in the union of the *shrunk* spectral bands of the periodic approximants, a much more delicate statement, technically, hence with more powerful consequences.

2.3. *The hyperbolic case: Proof of Lemma* 2.4

We are going to show that if q is an 'approximate' period as in (1.3) with certain $\beta, \epsilon > 0$ and satisfies

$$|\text{Trace } A_q(E)| \geq 2 + e^{-\tau q} \tag{2.24}$$

then the trace of the transfer matrix at the scale $e^{\tau q/2}$ will be very large and any generalized eigenfunction of $Hu = Eu$ will be bounded from below at the scale $e^{\tau q/2}$. If this happens for infinitely many q, then any generalized eigenfunction will have at least $1/2$ power law growth (in index) along some fixed subsequence. By the extended Schnol Theorem, such an E must belong to a set of spectral measure zero.

Claim 1. *Fix any $\tau > 0$. There exist $x_q^i \in \mathbb{Z}$, $i = 1, \dots, 4, q \in \mathbb{N}$, (depending only on τ, Λ) such that $|x_q^i| \to \infty$ as $q \to \infty$ and the following is true: if for $q \to \infty$, $|\text{Trace } A_q(E)| \geq 2 + e^{-\tau q}$ and*

$$\max_{1 \leq j \leq q,\, |k| \leq e^{\epsilon \beta q}/q} |V(j + kq) - V(j + (k \pm 1)q)| \leq e^{-\beta q}, \quad \epsilon > 0, \tag{2.25}$$

with

$$\beta > (3 + 1/\epsilon)\tau + (7 + 1/\epsilon)\Lambda, \tag{2.26}$$

then for any $|u_0|^2 + |u_1|^2 = 1$, we have $\max_{i=1,\dots,4} |u^E_{x_q^i}| > e^q/16$, where u_n^E is a solution with boundary values (u_0, u_1).

Lemma 2.6 (Extended Schnol Theorem). *Fix any $y > 1/2$. For any sequence $|x_k| \to \infty$ (where the sequence is independent of E), for spectrally a.e. E, there is a generalized eigenvector u^E of $Hu = Eu$ such that*

$$|u^E_{x_k}| < C(1 + |k|)^y.$$

We can now prove Lemma 2.4.

Proof of Lemma 2.4. Let q_k be as in (1.3). Let E be such that there is a subsequence $q_{k_j} \to \infty$ satisfying $|\text{Trace } A_{q_{k_j}}(E, \alpha)| \geq 2 + e^{-\tau q_{k_j}}$. For simplicity, we still denote the subsequence by q_k. By Claim 1 the set $\bigcup_{i=1}^4 \{x^i_{q_k}\}$ has the property that for any generalized eigenfunction u^E, one has

$$\max_{i=1,\dots,4} |u^E_{x^i_{q_k}}| > e^{q_k}/16.$$

Let $\tilde{x}_k$ be the one in $\bigcup_{i=1}^4 \{x^i_{q_k}\}$ where the maximum of $|u^E_{x^i_{q_k}}|$ is attained. Then $|u^E_{\tilde{x}^i_k}| > e^{q_k}/16 \geq |k|$. According to Lemma 2.6, the set of such E must have spectral measure zero. ■

Claim 1 is based on the following results. First, we need to estimate the norm of the conjugation matrix for any hyperbolic $SL(2, \mathbb{R})$ matrix with respect to the distance between its trace and 2:

Lemma 2.7. *Suppose $G \in SL(2, \mathbb{R})$ with $2 < |\text{Trace } G| \leq 6$. The invertible matrix B such that*

$$G = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1} \tag{2.27}$$

where $\rho^{\pm 1}$ are the two conjugate real eigenvalues of G with $|\det B| = 1$ satisfies

$$\|B\| = \|B^{-1}\| < \frac{\sqrt{\|G\|}}{\sqrt{|\text{Trace } G| - 2}}. \tag{2.28}$$

If $|\text{Trace } G| > 6$, then $\|B\| \leq \frac{2\sqrt{\|G\|}}{\sqrt{|\text{Trace } G| - 2}}$.

The proof is based on a direct computation of conjugate matrices. For the sake of completeness, we present it in Appendix A.1.

Fix $\tau > 0$ and apply Lemma 2.7 to A_q satisfying

$$|\text{Trace } A_q| > 2 + e^{-\tau q}. \tag{2.29}$$

Using the bound $\|A_q\| \le e^{\Lambda q}$ in (1.14) and $2 \le e^{\Lambda q/200}$ for q large, we then have

Claim 2. *For large q,*

$$A_q = B\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1} \tag{2.30}$$

where $\rho^{\pm 1}$ are the two conjugate real eigenvalues of A_q with $|\rho| > 1$ and B satisfies $|\det B| = 1$ and

$$\|B\| = \|B^{-1}\| < e^{(\tau/2+\Lambda/2+\Lambda/200)q}. \tag{2.31}$$

Second, we need to use the almost periodicity (2.25) of the potential to obtain approximation statements for the transfer matrices. Set

$$N = [e^{(\tau/2+\Lambda/100)q}]. \tag{2.32}$$

Under the assumption (2.25) and (2.26) on V as in Claim 1, for q large enough (depending on Λ and n_0) we have

Claim 3. *Under the conditions of Claim 1,*

$$\|A_{Nq} - A_q^N\| \le 2e^{-\Lambda q}|\rho|^N \le 2e^{-\Lambda q}|\text{Trace } A_q^N|, \tag{2.33}$$

$$\|[A_{Nq}]^{-1} - A_{-Nq}\| \le 4e^{-\Lambda q}|\rho|^N \le 4e^{-\Lambda q}|\text{Trace } A_q^N|. \tag{2.34}$$

Proof of Claim 1. Decomposing A_q as in (2.30), we obtain $|\rho| > 1 + e^{-\tau q/2}$. Obviously, $|\text{Trace } A_q^N| \ge |\rho|^N$. By (2.32), $N > 2e^{\tau q/2}q$, thus

$$|\text{Trace } A_q^N| \ge (1 + e^{-\tau q/2})^{2e^{\tau q/2}q} \ge e^q.$$

Assume q is so large that $4e^{-\Lambda q} \le 1/10$. By (2.33), we have

$$|\text{Trace } A_{Nq} - \text{Trace } A_q^N| \le 2\|A_{Nq} - A_q^N\| \le 4e^{-\Lambda q}|\text{Trace } A_q^N|.$$

Therefore,

$$|\text{Trace } A_{Nq}| > (1 - 4e^{-\Lambda q})|\text{Trace } A_q^N| \ge \tfrac{9}{10}e^q. \tag{2.35}$$

Now consider the solution u of $Hu = Eu$ with normalized initial value

$$X = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}, \qquad \|X\| = 1.$$

Then by (1.37),

$$A_{Nq} \cdot X = \begin{pmatrix} u_{Nq+1} \\ u_{Nq} \end{pmatrix}, \qquad A_{-Nq} \cdot X = \begin{pmatrix} u_{-Nq+1} \\ u_{-Nq} \end{pmatrix}. \tag{2.36}$$

(2.34) implies that

$$\begin{aligned}\|[A_{Nq}]^{-1}\cdot X\| &\le \|A_{-Nq}\|\cdot\|X\| + 4e^{-\Lambda q}|\text{Trace } A_q^N|\cdot\|X\|\\ &\le \|A_{-Nq}\|\cdot\|X\| + \frac{1}{10}|\text{Trace } A_q^N|\cdot\|X\|.\end{aligned}$$

Thus combining the Cayley–Hamilton theorem with (2.35), we have

$$\begin{aligned}\tfrac{9}{10}|\text{Trace } A_q^N|\cdot\|X\| &\le \|\text{Trace } A_{Nq}X\| = \|A_{Nq}\cdot X + [A_{Nq}]^{-1}\cdot X\|\\ &\le \|A_{Nq}\cdot X\| + \|A_{-Nq}\cdot X\| + \tfrac{1}{10}|\text{Trace } A_q^N|\cdot\|X\|.\end{aligned}$$

Then

$$\|A_{Nq}\cdot X\| + \|A_{-Nq}\cdot X\| \ge \tfrac{8}{10}|\text{Trace } A_q^N|\cdot\|X\| \ge \tfrac{1}{2}|\text{Trace } A_q^N|,$$

which is equivalent to

$$\max\left\{\left\|\begin{pmatrix}u_{Nq+1}\\ u_{Nq}\end{pmatrix}\right\|, \left\|\begin{pmatrix}u_{-Nq+1}\\ u_{-Nq}\end{pmatrix}\right\|\right\} \ge \tfrac{1}{4}|\text{Trace } A_q^N|.$$

Therefore,

$$\max\{|u_{Nq+1}|, |u_{Nq}|, |u_{-Nq+1}|, |u_{-Nq}|\} \ge \tfrac{1}{16}e^{q}.$$

Let $x_q^i = (-1)^i Nq + 1 - [i/3], i = 1, \ldots, 4$. Then for every q and one of $i = 1, \ldots, 4$, $|u_{x_q^i}| > \frac{1}{16}e^q$. ■

It now remains to prove (2.33) and (2.34) in Claim 3. Set

$$\Delta_i = A_q(T^{(i-1)q}\theta, E) - A_q(\theta, E), \quad i = -N+1, \ldots, N. \tag{2.37}$$

Claim 4. *Suppose* (1.14) *holds for* $n \ge n_0$ *and is uniform in* $E \in S$. *Fix* $E \in S$ *and* $\theta \in \Omega$. *If* V_θ *satisfies* (2.25) *with* $\epsilon > 0$, *then there is a constant* C_{n_0} *(depending only on* n_0 *and an upper bound of* $\|V\|_\infty$*) such that*

$$\|\Delta_i(\theta, E)\| \le |i-1|qC_{n_0}e^{(\Lambda-\beta)q}, \quad |i| = 1, \ldots, [e^{\epsilon\beta q}/q]. \tag{2.38}$$

Proof. The proof is quite standard. Suppose $1 \le i \le [e^{\epsilon\beta q}/q]$. Then for $|k| < i$, $|k|q < e^{\epsilon\beta q}$. Since $V_{T^{kq}\theta}(n) = V_\theta(n+kq)$, (2.25) implies that for $|k| < i$ the following holds:

$$|V_{T^{kq}\theta}(j) - V_{T^{(k+1)q}\theta}(j)| \le e^{-\beta q}, \quad 1 \le j \le q,$$

which implies

$$\|A(T^{kq+j}\theta) - A(T^{(k+1)q+j}\theta)\| \le e^{-\beta q}, \quad 1 \le j \le q,\ |k| < i.$$

By a standard telescoping argument, for any $\theta' = T^{kq}\theta, |k| < i$,

$$\|A_q(T^q\theta') - A_q(\theta')\| \le qC_V^{n_0}e^{(\Lambda-\beta)q} = qC_{n_0}e^{(\Lambda-\beta)q}$$

where C_V is such that $\|A(\theta', E)\| \le C_V$ for all θ', E. In the above estimate, if $n > n_0$, we use the bound (1.14). When $n \le n_0$, we use the trivial bound $\|A_n\| \le C_V^{n_0}$. We have

$$A_{iq}(\theta, E) = A_q(T^{(i-1)q}\theta) \cdots A_q(T^q\theta)A_q(\theta).$$

Therefore, for $1 \le i \le [e^{\epsilon\beta q}/q]$,

$$\|\Delta_i\| \le \sum_{k=1}^{i-1} \|A_q(T^{kq}\theta) - A_q(T^{(k-1)q}\theta))\| \le (i-1)qC_{n_0}e^{(\Lambda-\beta)q}.$$

Since (2.25) is symmetric with respect to $T \to T^{-1}$, (2.38) for $i \le 0$ follows by taking $T' = T^{-1}$. ∎

Proof of Claim 3. Write, for any i,

$$A_q^i = B^{-1}\begin{pmatrix} \rho^i & 0 \\ 0 & \rho^{-i} \end{pmatrix} B,$$

so $\|A_q^i\| \le \|B\|^2|\rho|^i$. Set $G(\theta) = \frac{1}{\rho}A_q(\theta)$ and $G_j = G(T^{(j-1)q}\theta)$. By (2.31), we have $\|G^i\| \le \|B\|^2 \le e^{(\tau+\Lambda+\Lambda/100)q}$. Under the assumption (2.26), we have $\tau/2 + \Lambda/100 < \epsilon\beta$, so by (2.32), $Nq < e^{\epsilon\beta q}$. Then Claim 4 implies, for $j = -N, \dots, N$ and large q, that

$$\|G_j - G\| = \frac{1}{\rho}\|\Delta_j\| \le NqC_{n_0}e^{(\Lambda-\beta)q} \le e^{(-\beta+\tau/2+\Lambda+\Lambda/50)q}.$$

Now we want to apply Lemma A.1 in Appendix A.3 to these G_j, with $M = e^{(\tau+\Lambda+\Lambda/100)q}$ and $\delta = e^{(-\beta+\tau/2+\Lambda+\Lambda/50)q}$. Direct computation gives

$$NM^2\delta < e^{(-\beta+3\tau+3\Lambda+\Lambda/20)q}.$$

By (2.26) we have $\beta - (3\tau + 3\Lambda + \Lambda/20) > \Lambda$. Therefore, for q large enough (depending on n_0), we have $NM\delta < NM^2\delta < e^{-\Lambda q}$. Then Lemma A.1 implies that

$$\left\| \coprod_{j=1}^N G_j - G^N \right\| \le 2NM^2\delta \le 2e^{-\Lambda q}, \qquad \left\| \coprod_{j=1}^N G_{-N+j} - G^N \right\| \le 2NM^2\delta \le 2e^{-\Lambda q}.$$

Therefore,

$$\|A_{Nq} - A_q^N\| = |\rho|^N \cdot \left\| \coprod_{j=1}^N G_j - G^N \right\| \le 2e^{-\Lambda q}|\text{Trace } A_q^N|, \tag{2.39}$$

establishing (2.33) , and

$$\|A_{Nq}(T^{-Nq}\theta) - A_q^N(\theta)\| = |\rho|^N \cdot \left\| \coprod_{j=1}^N G_{-N+j} - G^N \right\| \le 2e^{-\Lambda q}|\text{Trace } A_q^N|. \tag{2.40}$$

For any two matrices $F, G \in \mathrm{SL}(2,\mathbb{R})$, direct computation of the entries shows that $\|F+G\| = \|F^{-1}+G^{-1}\|$.

Using this simple fact, we find that

$$\begin{aligned}\|A_{-Nq}(\theta) - [A_q^{-1}(\theta)]^N\| &= \|[A_{Nq}]^{-1}(T^{-Nq}\theta) - [A_q^N(\theta)]^{-1}\| \\ &= \|[A_{Nq}](T^{-Nq}\theta) - [A_q^N(\theta)]\|.\end{aligned}$$

This implies

$$\|A_{-Nq}(\theta) - [A_q^{-1}(\theta)]^N\| \le 2e^{-\Lambda q}|\mathrm{Trace}\, A_q^N|.$$

Also,

$$\|[A_{Nq}]^{-1} - [A_q^{-1}]^N\| = \|[A_{Nq}] - [A_q]^N\|. \tag{2.41}$$

Therefore, by (2.39), we obtain (2.34). ■

Lemma 2.6 is proved in the same way as the standard Schnol Theorem, but the statement in this form, while very useful, does not seem to be in the literature (we learned it from S. Molchanov, see Acknowledgments). For the sake of completeness, we include a short proof in the Appendix.

2.4. Energies with trace close to 2: Proof of Lemma 2.5

Throughout this section, we will assume again that all q are large enough and satisfy (1.3) with certain $\beta, \epsilon > 0$, i.e.,

$$\max_{1\le j\le q,\, |k|\le e^{\epsilon\beta q}/q} |V(j+kq) - V(j+(k\pm 1)q)| \le e^{-\beta q}. \tag{2.42}$$

We are going to show that spectrally almost surely, there are only finitely many q such that Trace A_q is close to 2.

In fact, we are going to prove the following quantitative estimate on the measure of energies where the trace of the associated transfer matrix is close to 2.

Lemma 2.8. *Let Λ be as in* (1.14) *on some set $S \subset \sigma(H)$. Let*

$$S_q = \{E : 0 < |\mathrm{Trace}\, A_q \pm 2| < e^{-10\Lambda q}\}. \tag{2.43}$$

Assume (2.42) *holds and*

$$\beta > (25 + 11/\epsilon)\Lambda. \tag{2.44}$$

Then

$$\mu(S_q) < 4qe^{-\Lambda q/15} < e^{-\Lambda q/20} \tag{2.45}$$

where $\mu = \mu_S$ is the spectral measure restricted to S.

Once we have Lemma 2.8, the Borel–Cantelli lemma immediately implies Lemma 2.5. So the main task is to prove (2.45).

In order to estimate the spectral measure of S_q, first we recall the following results on the structure of S_q. Let $\mathcal{P}_n(\mathbb{R})$ denote the polynomials over $\mathbb{R}$ of exact degree n. Let $\mathcal{P}_{n;n}(\mathbb{R})$ be the elements in $\mathcal{P}_n(\mathbb{R})$ with n *distinct* real zeros.

Proposition 2 ([45, Theorem 6.1]). *Let $p \in \mathcal{P}_{n;n}(\mathbb{R})$ with $y_1 < \cdots < y_{n-1}$ the local extrema of p. Let*

$$\zeta(p) := \min_{1 \le j \le n-1} |p(y_j)| \tag{2.46}$$

and $0 \le a < b$. Then

$$|p^{-1}(a,b)| \le 2\,\mathrm{diam}(z(p-a)) \max\left\{\frac{b-a}{\zeta(p)+a}, \left(\frac{b-a}{\zeta(p)+a}\right)^{1/2}\right\} \tag{2.47}$$

where $z(p)$ is the zero set of p and $|\cdot| = \mathrm{Leb}(\cdot)$ denotes the Lebesgue measure.

Fix any $\tau > 0$. We apply Proposition 2 to the polynomial $\mathrm{Trace}\, A_q(E) \in \mathcal{P}_{q;q}(\mathbb{R})$, with $a = 2$ and $b = 2 + e^{-\tau q}$. Clearly, $\mathrm{diam}(z(\mathrm{Trace}\, A_q - 2))$ is bounded from above by some constant that only depends on $\|V\|_\infty$. We also have $|\zeta(\mathrm{Trace}\, A_q)| \ge 2$. Since $b - a < 1$, we have $|(\mathrm{Trace}\, A_q)^{-1}(a,b)| \le C_V\sqrt{b-a} = C_V e^{-\tau q/2}$ where C_V is some constant that only depends on $\|V\|_\infty$. Since $(\mathrm{Trace}\, A_q)^{-1}(a,b)$ contains at most q bands, setting $S_q = \{E : 2 < \mathrm{Trace}\, A_q < 2 + e^{-\tau q}\}$, we have

$$S_q = \bigcup_{j=1}^{q} I_j, \quad |I_j| \le |S_q| \le C_V e^{-\tau q/2}. \tag{2.48}$$

The same analysis works for $(a,b) = (2 - e^{-\tau q}, 2), (-2 - e^{-\tau q}, -2), (-2, -2 + e^{-\tau q})$. Thus the structure (2.48) also holds for the other three cases.

Denote

$$\varepsilon_q^j = |I_j| < e^{(-\tau/2 + \Lambda/200)q}. \tag{2.49}$$

If $I_j \cap \Sigma \neq \emptyset$, pick $E_j \in I_j \cap \Sigma$ where $\Sigma = \sigma(H)$ is the spectrum. Set $\widetilde{I}_j = (E_j - \varepsilon_q^j, E_j + \varepsilon_q^j)$. Then $I_j \subset \widetilde{I}_j$, so it is enough to estimate the spectral measure of $\bigcup \widetilde{I}_j$.

Set $N_q = [e^{(\tau/2 - \Lambda/200)q}]$. For any $\varepsilon_q > 0$, define $l_q = l(\varphi, \varepsilon_q, E), u^\varphi, v^\varphi$ as in (1.29). Write $l_q = [l_q] + l_q - [l_q]$, and $[l_q] = K_q \cdot q + r_q$, where $0 \le r_q = [l_q] \bmod q < q$ and $0 \le l_q - [l_q] < 1$. We need the following power law estimate, which is key to the proof of Lemma 2.8.

Claim 5. *Suppose $E \in S_q \cap \Sigma$ and $0 < \varepsilon_q < e^{(-\tau/2 + \Lambda/200)q}$. Suppose (2.42) holds. Assume that $\beta > (2 + 1/\epsilon)\tau + (5 + 1/\epsilon)\Lambda$ and $\tau \ge 10\Lambda$. Then for every initial condition φ,*

$$\|u^\varphi\|_{l_q}^2 \ge e^{\Lambda q/10}. \tag{2.50}$$

Combining (2.50) with subordinacy theory, we are ready to estimate the m-function and the spectral measure.

Proof of Lemma 2.8. Take $\tau = 10\Lambda$. Then $\beta > (25 + 11/\epsilon)\Lambda$ satisfies the requirement in Claim 5. Let $E_j \in I_j \cap \Sigma \subset S_q \cap \Sigma$. For any φ, let $u^{\varphi,E_j}, v^{\varphi,E_j}$ be the right half-line solution associated with the energy E_j. According to (2.49), Claim 5 can be applied to all u^{φ,E_j}.

For any φ, we have

$$\|u^{\varphi,E_j}\|^2_{l_q(j)} \geq e^{\Lambda q/10}, \quad j = 1, \ldots, q,$$

where $l_q(j) = l(\varphi, E_j, \varepsilon_q^j)$.

Then by the J-L inequality (1.31) and the definition of $l_q(j)$, we have

$$\varepsilon_q^j |m_\varphi(E_j + i\varepsilon_q^j)| < \frac{5+\sqrt{24}}{2\|u^{\varphi,E_j}\|_{l_q}\|v^{\varphi,E_j}\|_{l_q}} \cdot \frac{\|v^{\varphi,E_j}\|_{l_q}}{\|u^{\varphi,E_j}\|_{l_q}} < \frac{5+\sqrt{24}}{2} \cdot e^{-\Lambda q/10}$$

Notice that the interval I_j is independent of the boundary condition φ, and so is ε_q^j. Therefore, we can take the supremum with respect to φ on both sides of the above inequality. By Proposition 1, we have

$$\varepsilon_q^j |M(E_j + i\varepsilon_q^j)| \leq \frac{5+\sqrt{24}}{2} e^{-\Lambda q/10}.$$

On the other hand, by the definition of $M(z)$ in (1.7), we have

$$\mathrm{Im}\, M(E + i\varepsilon) \geq \frac{1}{2\varepsilon}\mu(E - \varepsilon, E + \varepsilon), \quad E \in \mathbb{R},\ \varepsilon > 0.$$

Therefore,

$$\mu(E_j - \varepsilon_q^j, E_j + \varepsilon_q^j) \leq 2\varepsilon_q^j |M(E_j + i\varepsilon_q^j)| \leq (5 + \sqrt{24})e^{-\Lambda q/10},$$

which implies

$$\mu(I_j) \leq \mu(\widetilde{I}_j) \leq e^{-\Lambda q/15}.$$

Since in (2.48) there are four cases for S_q and each of them satisfies the previous estimates, the spectral measure of S_q will be bounded by $4qe^{-\Lambda q/15} \leq e^{-\Lambda q/20}$. ■

The proof of Claim 5 relies on the following estimates on transfer matrices. The first one is a formula for the power of a general $\mathrm{SL}(2, \mathbb{R})$ matrix. It is elementary but turned out particularly useful and will be an important part of our quantitative estimates in both hyperbolic and nearly parabolic cases. As we did not find it in the literature, we provide a proof of it, as well as of the next lemma, in the Appendix.

Lemma 2.9. *Suppose $A \in \mathrm{SL}(2, \mathbb{R})$ has eigenvalues $\rho^{\pm 1}$. For any $k \in \mathbb{N}$, if* Trace $A \neq 2$, *then*

$$A^k = \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left(A - \frac{\mathrm{Trace}\, A}{2} \cdot I\right) + \frac{\rho^k + \rho^{-k}}{2} \cdot I. \tag{2.51}$$

Otherwise, $A^k = k(A - I) + I$.

The key to the estimates in the nearly parabolic case is the following simple

Lemma 2.10. *There are universal constants $1 < C_1 < \infty$ and $c_1 > 1/3$ such that for all $E \in S_q$ and $1 \leq k \leq N_q$, we have*

$$c_1 < \frac{\rho^k + \rho^{-k}}{2} < C_1, \quad c_1 k < \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} < C_1 k \tag{2.52}$$

Second, since $A_q(\theta)$ is almost periodic (with an exponential error), the iteration of $A_q(\theta)$ along the orbit will be close to its power. The argument is similar to what we used in the proof of (2.33) in the previous part.

Claim 6. *Fix $\theta \in \Omega$, $E \in S_q \cap \Sigma$ and $\tau > 0$. Suppose (2.42) holds with $\beta > (2 + 1/\epsilon)\tau + (5 + 1/\epsilon)\Lambda$. Then for any $1 \le k \le N_q$, we have*

$$\|A_{kq} - A_q^k\| \le 2e^{-\Lambda q}. \tag{2.53}$$

Proof. Set $\Delta_j = A_q(T^{j-1}\theta) - A_q(\theta)$. By Claim 4, $\|\Delta_j\| \le jqCe^{(-\beta+\Lambda)q}$ for $j < [e^{\epsilon\beta q}/q]$. Recall that $N_q = [e^{(\tau/2-\Lambda/200)q}]$. The condition on β guarantees $N_q < [e^{\epsilon\beta q}/q]$, therefore we have $\|\Delta_j\| \le e^{(-\beta+\tau/2+\Lambda+\Lambda/100)q}$ for all $j = 1, \dots, N_q$. We need to check the other requirements of Lemma A.1. According to Lemmas 2.9 and 2.10,

$$\|A_q^j\| < C_1 j \left\| A_q - \frac{\text{Trace}\, A_q}{2} \cdot I \right\| + C_1 < 3C_1 N_q \|A_q\| < e^{(\tau/2+\Lambda+\Lambda/100)q}.$$

Now apply Lemma A.1 to the sequence $A_q(\theta), \dots, A_q(T^{j-1}\theta), \dots, A_q(T^{k-1}\theta)$ with $M = e^{(\tau/2+\Lambda+\Lambda/100)q}$ and $\delta = e^{(-\beta+\tau/2+\Lambda+\Lambda/100)q}$. We have $N_q M^2 \delta < e^{(-\beta+2\tau+3\Lambda+\Lambda/40)q}$. Since $\beta > (2 + 1/\epsilon)\tau + (5 + 1/\epsilon)\Lambda > 2\tau + 5\Lambda$ we have $\beta - (2\tau + 3\Lambda + \Lambda/40) > \Lambda$. Therefore, for q large enough, $N_q M\delta < N_q M^2 \delta < e^{-\Lambda q}$. Consequently, by Lemma A.1, we have $\|A_{kq} - A_q^k\| = \| \coprod_{j=1}^k A_q(T^{j-1}\theta) - A_q^k(\theta)\| \le 2e^{-\Lambda q}$. ∎

Now we are ready to finish the proof of the most technical part.

Proof of Claim 5. We first show the following lower bound for $K_q = [[\ell_q]/q]$:

$$K_q > e^{\Lambda q/6} > 18C_1 e^{\Lambda q/8}. \tag{2.54}$$

Actually, if $K_q \ge N_q$, then (2.54) is automatically satisfied since $\tau \ge 10\Lambda$.

Now assume $K_q < N_q$. For any $n \le [l_q] + 1$, write $n = kq + r$, where $0 \le k \le K_q$, $0 \le r \le q$. Set $X_\varphi = \binom{\cos\varphi}{-\sin\varphi}$. According to (2.51), (2.52) we have, for any φ and $1 \le k \le K_q < N_q$,

$$\|A_q^k \cdot X_\varphi\| < C_1 k \left\| A_q - \frac{\text{Trace}\, A_q}{2} \cdot I \right\| + C_1 < C_1 k(\|A_q\| + 3/2) + C_1$$

and by (2.53),

$$\|A_{kq} \cdot X_\varphi\| \le \|A_q^k \cdot X_\varphi\| + \|(A_{kq} - A_q^k) \cdot X_\varphi\| \le C_1 k(\|A_q\| + 3/2) + C_1 + 1.$$

For $n_0 < r \le q$, and for any $\theta' \in \Omega$, $\|A_r(\theta')\| \le e^{\Lambda q}$. For $1 \le r \le n_0$, we bound $\|A_r(\theta')\|$ by C^{n_0} as in the proof of Claim 4. Therefore, $\|A_r(\theta')\| \le e^{\Lambda q}$ for all $1 \le r \le q$ with q large. Thus,

$$\begin{aligned}\|A_{kq+r}(\theta) \cdot X_\varphi\| &\le \|A_r(T^{kq}\theta)\| \cdot \|A_{kq}(\theta) \cdot X_\varphi\| \le e^{\Lambda q}\big(C_1 k(\|A_q\| + 3/2) + C_1 + 1\big) \\ &\le e^{\Lambda q}\big(C_1 k(e^{\Lambda q} + 3/2) + C_1 + 1\big) \le k e^{(2\Lambda+\Lambda/200)q}.\end{aligned}$$

Recalling that $\binom{u^{\varphi}_{n+1}}{u^{\varphi}_n} = A_n \cdot X_{\varphi}$, direct computation shows

$$\begin{aligned}\|u^{\varphi}\|^2_{l_q} &\leq \sum_{n=1}^{[l_q]+1} \|A_n \cdot X_{\varphi}\|^2 \leq \sum_{r=1}^{q} \|A_r \cdot X_{\varphi}\|^2 + \sum_{k=1}^{K_q} \sum_{r=1}^{q} \|A_{kq+r} X_{\varphi}\|^2 \\ &\leq q e^{2\Lambda q} + \sum_{k=1}^{K_q} \sum_{r=1}^{q} k^2 e^{(4\Lambda+\Lambda/100)q} \\ &\leq q e^{2\Lambda q} + K_q^3 q e^{(4\Lambda+\Lambda/100)q} \leq K_q^3 e^{(4\Lambda+\Lambda/20)q}.\end{aligned}$$

Since φ is arbitrary and $\binom{v^{\varphi}_{n+1}}{v^{\varphi}_n} = A_n \cdot X_{\varphi+\pi/2}$, $\|v^{\varphi}\|^2_{l_q}$ has the same upper bound. Therefore, $\|u^{\varphi}\|_{l_q} \|v^{\varphi}\|_{l_q} \leq K_q^3 e^{(4\Lambda+\Lambda/20)q}$. On the other hand, since $\varepsilon_q < e^{(-\tau/2+\Lambda/200)q}$, we have

$$\|u^{\varphi}\|_{l_q} \|v^{\varphi}\|_{l_q} = \frac{1}{2\varepsilon_q} \geq e^{(\tau/2-\Lambda/100)q}. \tag{2.55}$$

With $\tau \geq 10\Lambda$, this implies that $K_q^3 \geq e^{q\Lambda/2}$. Therefore,

$$K_q > e^{\Lambda q/6} > 18 C_1 e^{\Lambda q/8} \tag{2.56}$$

as claimed.

In order to get the lower bound on $\|u^{\varphi}\|^2_{l_q}$ in (2.50), we need to consider two cases.

Case I. For φ such that

$$\left\| \left(A_q - \frac{\text{Trace } A_q}{2} \cdot I \right) \cdot X_{\varphi} \right\| \geq e^{-\Lambda q/8}, \tag{2.57}$$

using (2.51) of Lemma 2.9, for any $1 \leq k \leq 18 C_1 e^{\Lambda q/8} \leq N_q$ we have

$$\begin{aligned}\|A_q^k \cdot X_{\varphi}\| &= \left\| \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left(A_q - \frac{\text{Trace } A_q}{2} \cdot I \right) X_{\varphi} + \frac{\rho^k + \rho^{-k}}{2} \cdot X_{\varphi} \right\| \\ &\geq \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left\| \left(A_q - \frac{\text{Trace } A_q}{2} \cdot I \right) X_{\varphi} \right\| - \frac{\rho^k + \rho^{-k}}{2} \cdot \|X_{\varphi}\| \\ &\geq \tfrac{1}{3} k e^{-\Lambda q/8} - C_1\end{aligned}$$

where in the last inequality we use (2.52) of Lemma 2.10. Due to (2.53), we then have

$$\|A_{kq} \cdot X_{\varphi}\| \geq \|A_q^k \cdot X_{\varphi}\| - \|(A_{kq} - A_q^k) \cdot X_{\varphi}\| \geq \tfrac{1}{3} k e^{-\Lambda q/8} - 2C_1.$$

Therefore, for $9 C_1 e^{\Lambda q/8} \leq k \leq 18 C_1 e^{\Lambda q/8}$, we have

$$\|A_{kq} \cdot X_{\varphi}\| \geq C_1 > 1. \tag{2.58}$$

By (2.56) and (2.58) we obtain

$$\|u^{\varphi}\|_{l_q}^2 \geq \frac{1}{2}\sum_{n=1}^{[l_q]-1} \|A_n \cdot X_{\varphi}\|^2 \geq \frac{1}{2}\sum_{k=[9C_1e^{\Lambda q/8}]+1}^{[18C_1e^{\Lambda q/8}]} \|A_{kq} \cdot X_{\varphi}\|^2 \geq e^{\Lambda q/10}.$$

Case II. For φ such that

$$\left\|\left(A_q - \frac{\text{Trace } A_q}{2} I\right) \cdot X_{\varphi}\right\| < e^{-\Lambda q/8}, \tag{2.59}$$

by (2.51), for any $1 \leq k \leq N_q$ we get

$$\begin{aligned}\|A_q^k \cdot X_{\varphi}\| &= \left\|\frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left(A_q - \frac{\text{Trace } A_q}{2} I\right)X_{\varphi} + \frac{\rho^k + \rho^{-k}}{2} \cdot X_{\varphi}\right\| \\ &\geq \frac{\rho^k + \rho^{-k}}{2} \cdot \|X_{\varphi}\| - \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left\|\left(A_q - \frac{\text{Trace } A_q}{2} I\right)X_{\varphi}\right\| \\ &\geq 1/3 - C_1 k e^{-\Lambda q/8}\end{aligned}$$

where in the last step we use (2.52) of Lemma 2.10 again. Combining this with (2.53), we have

$$\begin{aligned}\|A_{kq} \cdot X_{\varphi}\| &\geq \|A_q^k \cdot X_{\varphi}\| - \|(A_{kq} - A_q^k) \cdot X_{\varphi}\| \geq 1/3 - C_1 k e^{-\Lambda q/8} - 2e^{-\Lambda q} \\ &\geq 1/4 - C_1 k e^{-\Lambda q/8}\end{aligned}$$

provided $2e^{-\Lambda q} \leq 1/12$. Then for $1 \leq k \leq \frac{1}{8C_1}e^{\Lambda q/8} \leq K_q \leq N_q$, we obtain $\|A_{kq} \cdot X_{\varphi}\| \geq 1/8$. This implies

$$\|u^{\varphi}\|_{l_q}^2 \geq \frac{1}{2}\sum_{k=1}^{[\frac{1}{8C_1}e^{\Lambda q/8}]} \|A_{kq} \cdot X_{\varphi}\|^2 \geq e^{\Lambda q/10}.$$

2.5. *Proofs of Lemmas 2.2 and 2.3*

Proof of Lemma 2.2. Assume that

$$|\text{Trace } A_q| < 2 - e^{-\tau q} < 2. \tag{2.60}$$

Then in the expression in Lemma 2.9, $\rho = e^{i\psi}$ with $\psi \in (-\pi, \pi)$. For any j, we have

$$A_q^j = \frac{\sin j\psi}{\sin \psi} \cdot \left(A_q - \frac{\text{Trace } A_q}{2} \cdot I\right) + \frac{\cos j\psi}{2} \cdot I, \qquad \psi \in (-\pi, \pi). \tag{2.61}$$

Then $|2\cos\psi| = |\text{Trace } A_q| < 2 - e^{-\tau q}$ implies $|\sin\psi| > \sqrt{1 - \left(1 - \frac{1}{2}e^{-\tau q}\right)^2} > Ce^{-\tau q/2}$. Therefore,

$$\|A_q^j\| \leq Ce^{\tau q/2}(\|A_q\| + 1) + 1$$

(here q is large enough so that $\|A_q\| \le e^{\Lambda q}$). Now, by Theorem 10 for μ-a.e. E and $k > K(E)$ we have (2.60) with $\tau = 10\Lambda$, so we obtain

$$\|A_q^j\| \le e^{(6\Lambda+\Lambda/100)q}.$$

Now let $N = [e^{\xi q}]$. By the same argument as used for the proof of (2.33) and (2.53) (based on Lemma A.1), if $\beta > 15\Lambda + (2+1/\epsilon)\xi$, then for any $j \le N$,

$$\|A_q^j - A_{jq}\| < e^{(-\beta+13\Lambda+2\xi+\Lambda/20)q} < e^{-\Lambda q}.$$

As a consequence, we have $\|A_{jq}\| \le e^{(6\Lambda+\Lambda/50)q}$, and $\|A_{jq+r}\| \le e^{(7\Lambda+\Lambda/50)q}$ for all $0 \le r \le q$ and $0 \le j \le N$. Therefore,

$$\sum_{n=1}^{Nq} \|A_n(E)\|^2 \le \sum_{k=0}^{N}\sum_{r=1}^{q} \|A_{kq+r}(\theta, E)\|^2 \le Nqe^{(14\Lambda+\Lambda/25)q} \le e^{(\xi+15\Lambda)q}. \qquad \blacksquare$$

Proof of Lemma 2.3. By the choice of N, $Nq > e^{(\xi-\Lambda/200)q}$ for q large, thus for any $\gamma < 1$,

$$\frac{1}{(Nq)^{2-\gamma}} \sum_{n=1}^{Nq} \|A_n(E)\|^2 \le e^{(\xi+15\Lambda)q} e^{-(2-\gamma)(\xi-\Lambda/200)q}$$
$$\le e^{(-(1-\gamma)\xi+16\Lambda)q}.$$

If $\xi \ge \frac{17\Lambda}{1-\gamma}$, then $(1-\gamma)\xi - 16\Lambda \ge \frac{1}{2}\Lambda$. Therefore,

$$\frac{1}{(Nq)^{2-\gamma}} \sum_{n=1}^{Nq} \|A_n(E)\|^2 \le e^{-\frac{1}{2}\Lambda q} \le 1. \qquad \blacksquare$$

3. Spectral singularity

3.1. Power law estimates and proof of Theorem 6

Throughout this section, our potential will be given by $V_\theta(n) = V(\theta + n\alpha), n \in \mathbb{Z}$, where $V(\theta)$ is a real analytic function defined on the torus with analytic extension to the strip $\{z : |\mathrm{Im}\, z| < \rho\}$.

According to Lemma 1.2, it is enough to find a φ such that both m_φ and $\widetilde{m}_{\pi/2-\varphi}$ are γ-spectral singular. The main technical tool to estimate the m-function is subordinacy theory, Lemma 1.1. We also need one more general statement about the existence of generalized eigenfunctions with sublinear growth of their l-norm (see [55]). That is, for μ_θ-a.e. E, there exists $\varphi \in (-\pi/2, \pi/2]$ such that u^φ and $u^{\varphi,-}$ both obey

$$\limsup_{l\to\infty} \frac{\|u\|_l}{l^{1/2}\log l} < \infty. \tag{3.1}$$

This inequality provides us an upper bound for the l-norm of the solution. To apply subordinacy theory, we also need a lower bound for the l-norm. It will be derived from the following lower bounds on the norm of transfer matrices. Denote

$$\widetilde{A}_n(\theta, E, \alpha) = A_n(\theta - \alpha, E, -\alpha). \tag{3.2}$$

Note that $\widetilde{A}_n$ is the n-step propagator for $u^{\varphi,-}$, $v^{\varphi,-}$. We have

Lemma 3.1. *Fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $\beta = \beta(\alpha) < \infty$. Assume that $L(E) \geq a > 0$ and $E \in S$. There is $c = c(a, \rho) > 0$ such that for $l > l(E, \beta, \rho)$, and any $\theta \in \mathbb{T}$,*

$$\sum_{k=1}^{l} \|A_k(\theta, E, \alpha)\|^2 \geq l^{1+2c/\beta}, \tag{3.3}$$

$$\sum_{k=1}^{l} \|\widetilde{A}_k(\theta, E, \alpha)\|^2 \geq l^{1+2c/\beta}. \tag{3.4}$$

Proof of Theorem 6. For any φ, we have

$$\|u^{\varphi}\|_l^2 + \|v^{\varphi}\|_l^2 \geq \frac{1}{2}\sum_{k=1}^{l} \|A_k(\theta)\|^2, \tag{3.5}$$

$$\|u^{\varphi,-}\|_l^2 + \|v^{\varphi,-}\|_l^2 \geq \frac{1}{2}\sum_{k=1}^{l} \|\widetilde{A}_k(\theta)\|^2. \tag{3.6}$$

Therefore, a direct consequence of (3.3) is the power law estimate for the left-hand side of (3.5), i.e., $\|u^{\varphi}\|_l^2 + \|v^{\varphi}\|_l^2 \geq l^{1+2c/\beta}$ for l large.

On the other hand, according to (3.1), for μ_θ-a.e. E, there exist $\varphi(E)$ and $C = C(E) < \infty$ such that for large l,

$$\|u^{\varphi}\|_l \leq C l^{1/2} \log l, \qquad \|u^{\varphi,-}\|_l \leq C l^{1/2} \log l. \tag{3.7}$$

Let us consider the right half-line estimates for u^{φ}, m_{φ} first. From (3.5) and (3.7), we have

$$\|v^{\varphi}\|_l^2 \geq l^{1+2c/\beta} - C l (\log l)^2 \geq \tfrac{1}{4} l^{1+2c/\beta}$$

and then

$$\|v^{\varphi}\|_l \geq \tfrac{1}{2} l^{1/2+c/\beta} \tag{3.8}$$

provided $\beta < \infty$ and $l > l(\beta, E, \rho)$.

Applying subordinacy theory (1.31) to (3.7) and (3.8) one has, for any $\gamma \in (0, 1)$ and any $\varepsilon > 0$,

$$\begin{aligned}\varepsilon^{1-\gamma}|m_{\varphi}(E + i\varepsilon)| &\geq \frac{1}{(2\|u^{\varphi}\|_{l(\varepsilon)}\|v^{\varphi}\|_{l(\varepsilon)})^{1-\gamma}} \cdot (5 - \sqrt{24})\frac{\|v^{\varphi}\|_{l(\varepsilon)}}{\|u^{\varphi}\|_{l(\varepsilon)}} \\ &\geq c_\gamma \frac{\|v^{\varphi}\|_l^{\gamma}}{\|u^{\varphi}\|_l^{2-\gamma}} \geq c_\gamma l^{(1+c/\beta)\gamma - 1} \log^{-2} l\end{aligned}$$

where $c_\gamma > 0$ may denote different constants that only depend on γ. Set $\gamma_0 = \gamma_0(\beta) = \frac{1}{1+c/\beta} < 1$, since $\beta < \infty$. For any $\gamma > \gamma_0$, we have

$$\varepsilon^{1-\gamma}|m_\varphi(E + i\varepsilon)| \geq c_\gamma l^{\gamma/\gamma_0 - 1} \log^{-2} l \to \infty$$

as $\varepsilon \to 0$.

Using (3.6) and (3.7), the same argument works for $u^{\varphi,-}$, $v^{\varphi,-}$ and m_φ^-. Therefore, for spectrally a.e. E, $\liminf_{\varepsilon\to 0} \varepsilon^{1-\gamma}|m_\varphi(E+i\varepsilon)| = \infty$ and $\liminf_{\varepsilon\to 0} \varepsilon^{1-\gamma}|m_\varphi^-(E+i\varepsilon)| = \infty$. According to Lemma 1.2, μ is γ-spectral singular for any $\gamma > \gamma_0$. The conclusion for the spectral dimension follows from the definition directly. ■

The proof of Lemma 3.1 depends on the following lemmas about the localization density of the half-line solution. The key observation is that in the regime of positive Lyapunov exponents we can guarantee transfer matrix growth at scale q_n somewhere within any interval of length q_n, giving a contribution to (3.3).

Lemma 3.2. *Assume that $L(E) \geq a > 0$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. There are $c_2 = c_2(a,\rho) > 0$ and a positive integer $d = d(\rho)$ such that for $E \in S$ and $n > n(E,\rho)$, there exists an interval Δ_n such that*

$$\mathrm{Leb}(\Delta_n) \geq \frac{c_2}{4dn} \tag{3.9}$$

and for any $\theta \in \Delta_n$, we have[7]

$$\|A_n(\theta, E, \alpha)\|_{\mathrm{HS}} > e^{nL(E)/16}. \tag{3.10}$$

In the following, we will use $\|\cdot\|$ for the HS norm $\|\cdot\|_{\mathrm{HS}}$. Now let c_2 and d be as in Lemma 3.2. Denote

$$k_n = \left[\frac{c_2 q_n}{4d}\right] - 1 \tag{3.11}$$

where as before q_n are the denominators of the continued fraction approximants to α. Based on Lemma 3.2, one can show that

Lemma 3.3. *Fix $E \in S$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Let k_n be as in (3.11). Suppose q_n is large enough so that (3.9) holds for Δ_{k_n}. Then for any θ, and any $N \in \mathbb{N}$, there is $j_N(\theta) \in [2Nq_n, 2(N+1)q_n)$ such that*

$$\|A_{j_N}(\theta, E, \alpha)\| > e^{k_n L(E)/32}. \tag{3.12}$$

We first use Lemmas 3.2 and 3.3 to finish the proof of Lemma 3.1. The proofs of these two lemmas are postponed to the next section.

Proof of Lemma 3.1. For l sufficiently large, there is q_n such that $l \in [2q_n, 2q_{n+1})$. Write l as

$$l = 2Nq_n + r,$$

[7] We denote by $\|\cdot\|_{\mathrm{HS}}$ the Hilbert–Schmidt norm of an $\mathrm{SL}(2,\mathbb{R})$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\|A\|_{\mathrm{HS}} = \sqrt{a^2+b^2+c^2+d^2}$.

where $0 \leq r < 2q_n$ and $1 \leq N < q_{n+1}/q_n$. Suppose q_n is large enough so that (3.9) holds for Δ_{k_n}. Then Lemma 3.3 is applicable. Fix θ. Consider $A_n(\theta, E, \alpha)$ first. Let $j_n(\theta) \in [2nq_n, 2(n+1)q_n)$, $n = 0, 1, \ldots, N$, be as in (3.12). Direct computation shows that

$$\sum_{k=1}^{l} \|A_k(\theta)\|^2 \geq \|A_{j_0}(\theta)\|^2 + \|A_{j_1}(\theta)\|^2 + \cdots + \|A_{j_{N-1}}(\theta)\|^2 \geq Ne^{k_n L(E)/32}.$$

We have $l = 2Nq_n + r < 4Nq_n$, i.e., $N > l/(4q_n)$. (3.11) implies $c_2 q_n/(5d) < k_n < c_2 q_n/(4d)$ for q_n large, so we have

$$\sum_{k=1}^{l} \|A_k(\theta)\|^2 \geq \frac{l}{4q_n} e^{k_n L(E)/32} \geq \frac{l}{4q_n} e^{16cq_n}$$

where $c = c(c_2, d, a)$, where a is the lower bound for L from Lemma 3.2. Then for sufficiently large l, we have

$$\sum_{k=1}^{l} \|A_k(\theta)\|^2 > le^{8cq_n}.$$

We also assume l is large enough (meaning q_n is large enough), so that $\frac{\log q_{n+1}}{q_n} < 2\beta$, i.e., $e^{q_n} > q_{n+1}^{1/(2\beta)}$. Then

$$\sum_{k=1}^{l} \|A_k(\theta)\|^2 \geq lq_{n+1}^{4c/\beta} \geq l \cdot (l/2)^{4c/\beta} \geq l^{1+2c/\beta}.$$

For the same θ, repeat the above procedure for $A_n(\theta - \alpha, -\alpha, E)$. Notice that $\widetilde{A}_n(\theta, E, \alpha) = A_n(\theta - \alpha, E, -\alpha)$. Therefore, we have a sequence of positive integers $\tilde{j}_N(\theta - \alpha) \in [2Nq_n, 2(N+1)q_n)$ for any $N \in \mathbb{N}$ such that

$$\|\widetilde{A}_{\tilde{j}_N}(\theta, E, \alpha)\| > e^{k_n L(E)/32}. \tag{3.13}$$

The rest of the computations are exactly the same as for $A_n(\theta, E, \alpha)$. Notice that the constants c_2 and d in Lemma 3.2 are independent of the choice of α or $-\alpha$ and θ. So k_n and c will be the same for A_n and $\widetilde{A}_n$. ∎

3.2. Proofs of the density lemmas

Proof of Lemma 3.2. Denote

$$f_n(\theta) = \|A_n(\theta, E, \alpha)\|_{\mathrm{HS}}^2. \tag{3.14}$$

Obviously, $f_n(\theta)$ is a real analytic function with analytic extension to the strip $\{z : |\mathrm{Im}\, z| < \rho\}$. For bounded S we have

$$\|f_n\|_\rho := \sup_{|\mathrm{Im}\, z| < \rho} |f_n(z)| < e^{C_1 n}, \qquad E \in S, \tag{3.15}$$

where $C_1 = C_1(S, \|V\|_\rho)$ can be taken uniform for all $E \in S$. Expand $f_n(\theta)$ into its Fourier series on $\mathbb{T}$ as

$$f_n(\theta) = \sum_{k\in\mathbb{Z}} b_n(k) e^{2\pi i k\theta} \tag{3.16}$$

where $b_n(k)$ is the k-th Fourier coefficient of $f_n(\theta)$ which satisfies

$$|b_n(k)| < \|f_n\|_\rho e^{-2\pi\rho|k|}, \quad \forall k \in \mathbb{Z}. \tag{3.17}$$

We split $f_n(\theta)$ into two parts, for some positive integer d which will be specified later:

$$f_n(\theta) = g_n(\theta) + R_n(\theta), \quad g_n(\theta) = \sum_{|k|\le dn} b_n(k) e^{2\pi i k\theta}, \quad R_n(\theta) = \sum_{|k|>dn} b_n(k) e^{2\pi i k\theta}.$$

For any $\theta \in \mathbb{T}$,

$$|R_n(\theta)| \le \sum_{|k|>dn} |b_n(k)| \le \sum_{|k|>dn} \|f_n\|_\rho e^{-2\pi\rho|k|} \le \frac{2}{1-e^{-2\pi\rho}} e^{C_1 n} e^{-2\pi\rho d n}.$$

Now pick

$$d = \left[\frac{C_1}{2\pi\rho}\right] + 2. \tag{3.18}$$

With this choice of d, we have $2\pi\rho d > C_1 + 1$, so for any $\theta \in \mathbb{T}$,

$$|R_n(\theta)| \le \frac{2}{1-e^{-2\pi\rho}} e^{-n} < 1, \quad n > n_0(\rho). \tag{3.19}$$

Now we assume that the Lyapunov exponent $L(E)$ of $A(\theta, E)$ is positive. Denote

$$\Theta_n^1 = \{\theta : f_n(\theta) > e^{nL(E)/8}\},$$
$$\Theta_n^2 = \{\theta : g_n(\theta) > e^{nL(E)/4}\},$$
$$\Theta_n^3 = \{\theta : f_n(\theta) > e^{nL(E)/2}\}.$$

According to (3.19), we see that if $f_n(\theta) > e^{nL(E)/2}$, then

$$g_n(\theta) > f_n(\theta) - |R_n(\theta)| > e^{nL(E)/2} - 1 > e^{nL(E)/4}, \quad n > n(E),$$

and if $g_n(\theta) > e^{nL(E)/4}$, then

$$f_n(\theta) > g_n(\theta) - |R_n(\theta)| > e^{nL(E)/8}, \quad n > n(E).$$

Therefore, for large n,

$$\Theta_n^3 \subseteq \Theta_n^2 \subseteq \Theta_n^1. \tag{3.20}$$

On the other hand,

$$\begin{aligned} 2nL(E) &\le 2\int_{\mathbb{T}} \log \|A_n(\theta)\|_{\mathrm{HS}} \, \mathrm{d}\theta = \int_{\mathbb{T}} \log f_n(\theta) \, \mathrm{d}\theta \\ &\le \mathrm{Leb}(\Theta_n^3) \log \|f_n\|_\rho + (1 - \mathrm{Leb}(\Theta_n^3)) \log e^{nL(E)/2} \\ &\le \mathrm{Leb}(\Theta_n^3) \cdot C_1 n + (1 - \mathrm{Leb}(\Theta_n^3)) \cdot nL(E)/2, \end{aligned}$$

which implies $\mathrm{Leb}(\Theta_n^3) \geq \frac{3L(E)}{2C_1 - L(E)}$. Since $L(E) \geq a > 0$ for $E \in S$, we have

$$\mathrm{Leb}(\Theta_n^3) \geq \frac{3a}{2C_1 - a} =: c_2(a, \rho) > 0. \tag{3.21}$$

Thus

$$\mathrm{Leb}(\Theta_n^2) \geq c_2(a, \rho) > 0. \tag{3.22}$$

Since $g_n(\theta)$ is a trigonometric polynomial of degree $2dn$, the set Θ_n^2 consists of no more than $4dn$ intervals. Therefore, there exists a segment $\Delta_n \subset \Theta_n^2$ with $\mathrm{Leb}(\Delta_n) > \frac{c_2}{4dn}$. Obviously, Δ_n is also contained in Θ_n^1, i.e., for any $\theta \in \Delta_n$,

$$\|A_n(\theta)\|_{\mathrm{HS}}^2 > e^{nL(E)/8}$$

and

$$\mathrm{Leb}(\Delta_n) > \frac{c_2}{4dn}, \qquad n > n(E, \rho),$$

where d only depends on ρ and is independent of n. ■

The following standard lemma is proved e.g. in [41].

Lemma 3.4 ([41, Lemma 9]). *Let $\Delta \subset [0,1]$ be an arbitrary interval. If $|\Delta| > 1/q_n$ then for any θ there exists a j in $\{0, 1, \ldots, q_n + q_{n-1} - 1\}$ such that $\theta + j\alpha \in \Delta$.*

Proof of Lemma 3.3. The case $N = 0$ is already covered by Lemma 3.4. The proof for $N > 0$ follows the same strategy. Notice that (3.11) implies $|\Delta_{k_n}| > \frac{c_2}{4dk_n} > \frac{1}{q_n}$ for large q_n. Applying Lemma 3.4 to $\theta + 2Nq_n$, we find that there exists a j in $\{0, 1, \ldots, q_n + q_{n-1} - 1\}$ such that $\theta + 2Nq_n\alpha + j\alpha \in \Delta_{k_n}$, i.e.,

$$\|A_{k_n}(\theta + 2Nq_n\alpha + j\alpha)\| > e^{k_n L(E)/16}.$$

Since

$$A_{2Nq_n+j+k_n}(\theta) = A_{k_n}(\theta + 2Nq_n\alpha + j\alpha) A_{2Nq_n+j}(\theta)$$

and A_i is unimodular, we see that either

$$\|A_{2Nq_n+j}(\theta)\| \geq e^{k_n L(E)/32} \quad \text{or} \quad \|A_{2Nq_n+j+k_n}(\theta)\| \geq e^{k_n L(E)/32}.$$

Let j_N be $2Nq_n + j$ or $2Nq_n + j + k_n$, so that j_N satisfies (3.12). Clearly,

$$2Nq_n \leq 2Nq_n + j < 2Nq_n + j + k_n < 2Nq_n + 2q_n.$$

Therefore, $j_N \in [2Nq_n, 2(N+1)q_n)$. ■

4. Sturmian Hamiltonian

Liu, Qu and Wen [57, 58] studied the Hausdorff and upper box counting dimension of $\Sigma_{\lambda,\alpha}$ with general irrational frequencies. For any irrational $\alpha \in (0,1)$ with continued fraction expansion $[0; a_1, a_2, \ldots]$, define

$$K_*(\alpha) = \liminf_{k\to\infty} \Big(\prod_{i=1}^{k} a_i\Big)^{1/k} \quad \text{and} \quad K^*(\alpha) = \limsup_{k\to\infty} \Big(\prod_{i=1}^{k} a_i\Big)^{1/k}. \tag{4.1}$$

Then [58, Theorem 1], [57, Theorem 1.1]) for large coupling constant λ, $\dim_{\mathrm{H}}(\Sigma_{\alpha,\lambda}) = 1$ iff $K_*(\alpha) = \infty$ and $\overline{\dim}_{\mathrm{B}}(\Sigma_{\alpha,\lambda}) = 1$ iff $K^*(\alpha) = \infty$.

The usual way to study Sturmian Hamiltonians is to decompose Sturmian potentials into canonical words, which obey recursive relations. Here we present an alternative approach to study spectral dimension properties of Sturmian Hamiltonians based on the techniques we developed in Theorem 7.

We will first prove Theorem 9. Set

$$V_\theta(n) = \lambda \chi_{[1-\alpha,1)}(\theta + n\alpha \bmod 1). \tag{4.2}$$

It is well-known that for Sturmian H_θ, the restriction of the Lyapunov exponent on the spectrum is zero (see [23, Theorem 1]). By the discussion after (1.18) (see [47]) or else, specifically for Sturmian potentials, by [56], for arbitrarily small $\Lambda > 0$ and $n \geq n_0(\Lambda)$, $\|A_n(\theta, E)\| \leq e^{\Lambda n}$ uniformly in θ and $E \in \sigma(H_\theta)$. Here we will apply Corollary 1 directly.

Let q_k be the subsequence of denominators of the continued fraction approximants of α such that $\|q_k\alpha\| < e^{-\beta q_k/2}$. In order to apply Corollary 1, it is enough to verify that $V_\theta(n)$ given by (4.2) is $\beta(\alpha)$-almost periodic for all α-Diophantine $\theta \in \mathbb{T}$. Fix $\tau > 1$. If θ is α-Diophantine there is $\gamma > 0$ such that $\|\theta + m\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \gamma/(|m|+1)^\tau$ for any $m \in \mathbb{Z}$. Then for $|m| \leq q$,

$$\mathrm{dist}(\theta + m\alpha, \{\mathbb{Z}, 1-\alpha+\mathbb{Z}\}) \geq \min_{|m|\leq q+1} \|\theta + m\alpha\|_{\mathbb{R}/\mathbb{Z}}.$$

Therefore,

$$\min_{|m|\leq q} \mathrm{dist}(\theta + m\alpha, \{\mathbb{Z}, 1-\alpha+\mathbb{Z}\}) \geq \min_{|m|\leq q+1} \frac{\gamma}{(|m|+1)^\tau} \geq \frac{\gamma}{(q+2)^\tau}.$$

Let $N = [e^{\beta q/4}]$, where $q > q_0(\gamma, \beta)$ is an element of the subsequence q_k chosen above. Then for $|j| \leq N$, and any $|m| \leq q$, we have

$$\|jq\alpha\| \leq |j| \cdot \|q\alpha\| \leq e^{-\beta q/4} \leq \frac{\gamma}{10(q+2)^\tau} \leq \frac{1}{10}\mathrm{dist}(\theta + m\alpha, \{\mathbb{Z}, 1-\alpha+\mathbb{Z}\}).$$

Therefore, for any $|m| \leq q$ and $|j| \leq N$, $\theta + m\alpha \bmod 1$ and $\theta + m\alpha + jq\alpha \bmod 1$ belong to the same of the two open intervals $(0, 1-\alpha)$, $(1-\alpha, 1)$, which implies that

$$\chi_{[1-\alpha,1)}(\theta + m\alpha \bmod 1) = \chi_{[1-\alpha,1)}(\theta + m\alpha + jq\alpha \bmod 1), \qquad |m| \leq q,\ |j| \leq N$$

Therefore, for $0 \leq m \leq q$,

$$V_\theta(m) = V_\theta(m+q) = \cdots = V_\theta(m+Nq),$$

which immediately implies $\beta(\alpha)$-almost periodicity for the sequence q_k with $\epsilon = 1/4$.

Since the set of α-Diophantine θ has full Lebesgue measure, the conclusion for the density of states follows directly from $\mathrm{d}N = \mathbb{E}(\mathrm{d}\mu_\theta)$.

Next we will construct α to prove Corollary 5. We will define inductively the continued fraction coefficients $a_n, n \geq 1$, so $\alpha = [a_1, \ldots, a_n, \ldots]$. Fix $\beta > 0$. Start with some n_0 large. For $1 \leq i \leq n_0$, set $a_i = 1$. Set $[a_1, \ldots, a_n] = p_n/q_n$. Now, for $k = 1, 2, \ldots$ define $n_k = q_{n_0} + q_{n_1} + \cdots + q_{n_{k-1}}$ and

$$a_n = \begin{cases} e^{\beta q_{n_k}}, & n = n_k + 1, \\ 1, & n_k + 2 \leq n \leq n_{k+1}, \end{cases} \qquad \text{for } k = 0, 1, \ldots.$$

It is easy to check that

$$\beta + \frac{\log q_{n_k}}{q_{n_k}} = \frac{\log a_{n_k+1} q_{n_k}}{q_{n_k}} < \frac{\log q_{n_k+1}}{q_{n_k}} < \beta + \frac{\log 2q_{n_k}}{q_{n_k}},$$

so

$$\frac{\log q_{n_k+1}}{q_{n_k}} \to \beta,$$

and

$$\begin{aligned}(a_1 a_2 \cdots a_{n_k})^{1/n_k} &= (a_{n_0+1} a_{n_1+1} \cdots a_{n_{k-1}+1})^{1/n_k} \\ &= (e^{\beta q_{n_0}} e^{\beta q_{n_1}} \cdots e^{\beta q_{n_{k-1}}})^{1/(q_{n_0}+q_{n_1}+\cdots+q_{n_{k-1}})} = e^{\beta} < \infty.\end{aligned}$$

Therefore, α constructed in the above way satisfies $\beta(\alpha) > 0$ while $K_*(\alpha) < \infty$. Then Corollary 5 follows from [58] and Theorem 9. ∎

On the other hand, if we take $\alpha = [0; 1, 2, 3, \ldots]$, then $K_*(\alpha) = \infty$ while $\beta(\alpha) = 0$. By [58], for the Sturmian Hamiltonian with frequencies α such that $K_*(\alpha) = \infty$, we have $\dim_{\mathrm{H}}(\Sigma_{\alpha,\lambda}) = \dim_{\mathrm{P}}(\Sigma_{\alpha,\lambda}) = 1$.

Appendix A

A.1. Proof of Lemma 2.7

Suppose that u, v are the two normalized eigenvectors of G such that

$$Gu = \rho u, \quad Gv = \rho^{-1} v, \quad \|u\| = \|v\| = 1.$$

Denote the angle between u and v by θ. Without loss of generality we assume further that $|\theta| < \pi/2$. Set $\widetilde{B} = (u, v)$, $B = \widetilde{B}/\sqrt{|\det \widetilde{B}|}$. Obviously, $\|\widetilde{B}\| \leq 1$, $|\det B| = 1$, and $\det \widetilde{B} = \|u\| \cdot \|v\| \cdot \sin\theta$. Therefore,

$$\|B\| \leq \frac{1}{\sqrt{|\sin\theta|}}.$$

On the other hand, $G(u - v) = \rho u - \rho^{-1} v$, which implies that

$$\rho - \rho^{-1} = \rho\|u\| - \rho^{-1}\|v\| \leq \|\rho u - \rho^{-1} v\| = \|G(u - v)\| \leq \|G\| \cdot \|u - v\|.$$

By the law of cosines, $\|u-v\| = 2\sin\frac{\theta}{2}$. Then

$$2\sin\frac{\theta}{2} \geq \frac{\rho-\rho^{-1}}{\|G\|} = \frac{\sqrt{(|\text{Trace}\, G|+2)(|\text{Trace}\, G|-2)}}{\|G\|}$$

$|\text{Trace}\, G| \leq 6$ implies that $|\text{Trace}\, G| + 2 \geq 2(|\text{Trace}\, G|-2)$, so $2\sin\frac{\theta}{2} \geq \frac{\sqrt{2}(|\text{Trace}\, G|-2)}{\|G\|}$. Therefore,

$$\sin\theta \geq 2\sin\frac{\theta}{2}\cdot\frac{1}{\sqrt{2}} \geq \frac{|\text{Trace}\, G|-2}{\|G\|}, \quad \text{so} \quad \|B\| \leq \frac{\sqrt{\|G\|}}{\sqrt{|\text{Trace}\, G|-2}}.$$

It is also easy to see that if $|\text{Trace}\, G| > 6$, then $\|B\| \leq \frac{2\sqrt{\|G\|}}{\sqrt{|\text{Trace}\, G|-2}}$. ■

A.2. Proofs of Lemmas 2.9 and 2.10

Proof of Lemma 2.9. Suppose $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \text{SL}(2,\mathbb{R})$ has eigenvalues ρ and ρ^{-1}.

Case I: Trace $A \neq 2$. Obviously, $\rho \neq 1$ and

$$A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} = B\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1}\end{pmatrix}B^{-1} \tag{A.1}$$

where B is the conjugation matrix. Suppose $\rho \neq d$. We can pick the conjugation matrix as

$$B = \begin{pmatrix} 1 & \frac{b}{\rho^{-1}-a} \\ \frac{c}{\rho-d} & 1\end{pmatrix}, \qquad B^{-1} = \frac{\rho-d}{\rho-\rho^{-1}}\begin{pmatrix} 1 & -\frac{b}{\rho^{-1}-a} \\ -\frac{c}{\rho-d} & 1\end{pmatrix}. \tag{A.2}$$

If $\rho = d$, it is easy to see that $bc = 0$. Without loss of generality, we assume $c = 0, b \neq 0$. We can pick the conjugation matrix as

$$B = \begin{pmatrix} 1 & 1 \\ \frac{d-d^{-1}}{b} & 0\end{pmatrix}, \qquad B^{-1} = \frac{b}{d^{-1}-d}\begin{pmatrix} 0 & -1 \\ -\frac{d-d^{-1}}{b} & 1\end{pmatrix}. \tag{A.3}$$

Direct computation using (A.1)–(A.3) shows that for any $k \in \mathbb{N}$,

$$A^k = \frac{\rho^k-\rho^{-k}}{\rho-\rho^{-1}}\cdot\left(A - \frac{a+d}{2}\cdot I\right) + \frac{\rho^k+\rho^{-k}}{2}\cdot I. \tag{A.4}$$

Case II: Trace $A = 2$. Also follows by a (simpler) direct computation, considering separately $a = 1$ and $a \neq 1$.

Proof of Lemma 2.10. Now assume $E \in S_q$ and $1 \leq k \leq N_q$. Apply (A.4) to $A_q(E)$. First, suppose $2 < \text{Trace}\, A_q(E) < 2 + e^{-\tau q}$. Then

$$1 < \rho = \frac{\text{Trace}\, A_q(E) + \sqrt{(\text{Trace}\, A_q(E))^2-4}}{2} < \frac{2+e^{-\tau q} + \sqrt{(2+e^{-\tau q})^2-4}}{2}$$
$$< 1 + e^{(-\tau/2+\Lambda/200)q}.$$

There is a universal constant C such that for any $1 \le k \le N_q < e^{(\tau/2-\Lambda/200)q}$,

$$1 < \rho^k < (1 + e^{(-\tau/2+\Lambda/200)q})^{N_q} < C.$$

Therefore,

$$1 < \frac{\rho^k + \rho^{-k}}{2} < C. \tag{A.5}$$

On the other hand,

$$\frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} = \sum_{i=1}^{k} \rho^{k-2i+1},$$

and therefore

$$k \le \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} < C_1 k. \tag{A.6}$$

Now assume $2 - e^{-\tau q} < \text{Trace}\, A_q(E) < 2$. Then $\rho = e^{i\psi}$ and (A.4) can be expressed as

$$A_q^k = \frac{\sin k\psi}{\sin \psi} \cdot \left(A_q - \frac{\text{Trace}\, A_q}{2} \cdot I \right) + \frac{\cos k\psi}{2} \cdot I. \tag{A.7}$$

We have $1 - \frac{1}{2}e^{-\tau q} < \cos\psi < 1$. Then $|\sin\psi| < e^{-\tau q/2}$, and $|\psi| < \frac{\pi}{2}|\sin\psi| < 2e^{-\tau q/2}$. As in the hyperbolic case, we set $N_q = [e^{(\tau/2-\Lambda/200)q}]$. For $k \le N_q$,

$$|k\psi| < 2e^{-\Lambda q/200}.$$

Then for q large enough, we have $\frac{2}{\pi}|k\psi| \le |\sin k\psi| \le |k\psi| \le \sqrt{3}/2$. Therefore, $\frac{2}{\pi}k \le \left|\frac{\sin k\psi}{\sin\psi}\right| < \frac{\pi}{2}k$ and $1 \ge \cos k\psi > 1/2$.

Exactly the same argument works for the cases $\{E : -2 < \text{Trace}\, A_q < -2 + e^{-\tau q}\}$ and $\{E : -2 - e^{-\tau q} < \text{Trace}\, A_q < -2\}$. ∎

A.3. Some estimates on matrix products

Lemma A.1. *Suppose G is a 2×2 matrix satisfying*

$$\|G^j\| \le M < \infty \quad \textit{for all } 0 < j \le N, \tag{A.8}$$

where $M \ge 1$ only depends on N. Let $G_j = G + \Delta_j$, $j = 1, \ldots, N$, be a sequence of 2×2 matrices and let

$$\delta = \max_{1 \le j \le N} \|\Delta_j\|. \tag{A.9}$$

If

$$NM\delta < 1/2, \tag{A.10}$$

then for any $n \le N$,

$$\left\| \coprod_{j=1}^{n} G_j - G^n \right\| \le 2NM^2\delta. \tag{A.11}$$

Proof. Denote

$$D = \max_{1\le k_1,k_2\le N} \Big\| \coprod_{j=k_1}^{k_2} G_j \Big\|.$$

Then a simple perturbation argument, as e.g. in [48], shows that $D \le M(\delta DN + 1)$. Thus $D \le \frac{M}{1-M\delta N}$. Direct computation shows that for any $1 \le n \le N$,

$$\coprod_{j=1}^{n} G_j - G^n = \sum_{k=0}^{n-1} \Big(\coprod_{j=k+2}^{n} G_j \Big) \Delta_{k+1} G^k.$$

Therefore,

$$\Big\| \coprod_{j=1}^{n} G_j - G^n \Big\| \le ND\delta M \le \frac{M^2 \delta N}{1 - M\delta N}.$$

Clearly, if $M\delta N < 1/2$, then $\| \coprod_{j=1}^{n} G_j - G^n \| \le 2NM^2\delta$. ∎

A.4. Extended Schnol Theorem (Lemma 2.6)

Let $y > 1/2$ and x_k be any sequence such that $|x_k| \to \infty$ as $k \to \infty$. For a Borel set $B \subset \mathbb{R}$, denote

$$\mu_{n,m}(B) = \langle \delta_n, \chi_B(H)\delta_m \rangle, \quad \rho(B) = \sum_n a_n(\mu_{n,n}(B) + \mu_{n+1,n+1}(B)),$$

where

$$a_n = \begin{cases} c(1 + |k|)^{-2y}, & n = x_k, \\ c(1 + |n|)^{-2y}, & \text{else} \end{cases}$$

with $c > 0$ chosen so that $\sum_n a_n = 1/2$. Then ρ is a Borel probability measure with $\rho(B) = 0$ if and only if $\mu(B) = 0$, i.e., ρ and μ are mutually absolutely continuous. By the Cauchy–Schwarz inequality,

$$|\mu_{n,m}(B)| \le \mu_{n,n}(B)^{1/2} \mu_{m,m}(B)^{1/2}.$$

so $\mu_{n,m}$ is absolutely continuous with respect to ρ. By the Radon–Nikodym Theorem, there exists a measurable density

$$F_{n,m}(E) = \Big[\frac{\mathrm{d}\mu_{n,m}}{\mathrm{d}\rho} \Big](E), \quad \rho\text{-a.e. } E,$$

with

$$\mu_{n,m}(B) = \int \chi_B(E) F_{n,m}(E)\, \mathrm{d}\rho(E).$$

Then for every bounded measurable function f, we have

$$\langle \delta_n, f(H)\delta_m \rangle = \int f(E) F_{n,m}(E)\, \mathrm{d}\rho(E).$$

In particular, if g is compactly supported and bounded, we may set $f(E) = Eg(E)$ and have

$$\int g(E)(EF_{n,m}(E))\,\mathrm{d}\rho(E) = \langle \delta_n, Hg(H)\delta_m\rangle = \langle \delta_{n+1} + \delta_{n-1} + V_n\delta_n, g(H)\delta_m\rangle$$
$$= \int g(E)F_{n+1,m}(E)\,\mathrm{d}\rho(E) + \int g(E)F_{n-1,m}(E)\,\mathrm{d}\rho(E) + \int g(E)V_nF_{n,m}(E)\,\mathrm{d}\rho(E)$$
$$= \int g(E)[F_{n+1,m}(E) + F_{n-1,m}(E) + V_nF_{n,m}(E)]\,\mathrm{d}\rho(E).$$

For any fixed $m \in \mathbb{Z}$, let $u^E(n) = F_{n,m}(E)$. Thus for any g we have

$$\int g(E)((H - E)u^E)(n)\,\mathrm{d}\rho(E) = 0,$$

i.e., $\{u^E(n)\}_{n\in\mathbb{Z}}$ is a generalized eigenfunction of $Hu = Eu$ for ρ-a.e. E.

On the other hand, let

$$B_n = \{E : F_{n,n} \geq 1/a_n\}.$$

Then

$$\rho(B_n) = \sum_k a_k\mu_{k,k}(B_n) \geq a_n\mu_{n,n}(B_n) = a_n \int_{B_n} F_{n,n}(E)\,\mathrm{d}\rho(E),$$

while

$$\int_{B_n} F_{n,n}(E)\,\mathrm{d}\rho(E) \geq \frac{1}{a_n}\rho(B_n).$$

Therefore,

$$\int_{B_n} (a_nF_{n,n}(E) - 1)\,\mathrm{d}\rho(E) \leq 0.$$

Hence, $\rho(B_n) = 0$, i.e., for ρ-a.e. E, $F_{n,n}(E) \leq \frac{1}{a_n}$, thus

$$|F_{n,m}| \leq a_n^{-1/2}a_m^{-1/2}.$$

Fix $m = 0$, and let $u^E(n) = F_{n,0}$. Then according to the previous proof, for ρ-a.e. E, u^E is a generalized eigenfunction of $Hu = Eu$ and obeys the estimate

$$|u^E(n)| \leq a_0^{-1/2}a_n^{-1/2}.$$

By the choice of a_n, we have

$$|u^E(x_k)| \leq (1 + |k|)^y. \qquad \blacksquare$$

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