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Global existence of entropy-weak solutions to the compressible Navier–Stokes equations with non-linear density dependent viscosities

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Abstract. In this paper, we considerably extend the results on global existence of entropy-weak solutions to the compressible Navier–Stokes system with density dependent viscosities obtained, independently (using different strategies) by Vasseur–Yu [Invent. Math. 206 (2016) and arXiv:1501.06803 (2015)] and by Li–Xin [arXiv:1504.06826 (2015)]. More precisely, we are able to consider a physical symmetric viscous stress tensor $\sigma = 2\mu(\rho)\,\mathbb{D}(u) + (\lambda(\rho)\operatorname{div} u - P(\rho))\,\mathrm{Id}$ where $\mathbb{D}(u) = [\nabla u + \nabla^T u]/2$ with shear and bulk viscosities (respectively $\mu(\rho)$ and $\lambda(\rho)$) satisfying the BD relation $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$ and a pressure law $P(\rho) = a\rho^\gamma$ (with $a > 0$ a given constant) for any adiabatic constant $\gamma > 1$. The non-linear shear viscosity $\mu(\rho)$ satisfies some lower and upper bounds for low and high densities (our result includes the case $\mu(\rho) = \mu\rho^\alpha$ with $2/3 < \alpha < 4$ and $\mu > 0$ constant). This provides an answer to a longstanding question on compressible Navier–Stokes equations with density dependent viscosities, mentioned for instance by F. Rousset [Bourbaki 69ème année, 2016–2017, exp. 1135].

Keywords. Global weak solutions, compressible Navier–Stokes equations, vacuum, degenerate viscosities

1. Introduction

When a fluid is governed by the barotropic compressible Navier–Stokes equations, the existence of global weak solutions, in the sense of J. Leray [35], in space dimension greater than 2 remained for a long time without answer, because of the weak control of the divergence of the velocity field which may provide the possibility for the density to vanish (vacuum state) even if initially this is not the case.

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There exists a huge literature on this question in the case of constant shear viscosity μ and constant bulk viscosity λ. Many authors, including Hoff [27], Jiang–Zhang [29], Kazhikhov–Shelukhin [33], Serre [47], Vaĭgant–Kazhikhov [48] (to cite but a few), obtained partial answers. The first rigorous approach to this problem in its generality is due to P.-L. Lions [38] when the pressure law in terms of the density is given by $P(\rho) = a\rho^\gamma$ where a and γ are strictly positive constants. He presented in 1998 a complete theory for $P(\rho) = a\rho^\gamma$ with $\gamma \geq 3d/(d+2)$ (where d is the space dimension) allowing one to obtain global existence of weak solutions à la Leray in dimensions $d = 2$ and 3 and for general initial data belonging to the energy space. His result was then extended in 2001 to the case $P(\rho) = a\rho^\gamma$ with $\gamma > d/2$ by Feireisl–Novotný–Petzeltová [24] introducing an appropriate method of truncation. Note also the 2014 paper on compressible Navier–Stokes equations with constant viscosities by Plotnikov–Weigant [45] in dimension 2 for the linear pressure law, that is, $\gamma = 1$. In 2002, Feireisl [23] also proved it is possible to consider a pressure law $P(\rho)$ non-monotone on a compact set $[0, \rho_*]$ (with ρ_* constant) and monotone elsewhere. This was relaxed in 2018 by Bresch–Jabin [16] allowing one to consider real non-monotone pressure laws. They also proved that it is possible to consider some constant anisotropic viscosities. The Lions theory has also been extended recently by Vasseur–Wen–Yu [49] to pressure laws depending on two phases (see also Maltese & al. [39], Novotný [43] and Novotný–Pokorný [44]). The method introduced by Bresch–Jabin [16] has also been recently developed in the bifluid framework by Bresch–Mucha–Zatorska [17].

When the shear and bulk viscosities (respectively μ and λ) are assumed to depend on the density ρ, the mathematical framework is completely different. It was discussed, mathematically, initially in a paper by Bernardi–Pironneau [6] related to viscous shallow-water equations and by P.-L. Lions [38]. The main ingredient in the constant case, which is the compactness in space of the effective flux $F = (2\mu + \lambda)\operatorname{div} u - P(\rho)$, is no longer true for density dependent viscosities. In space dimension greater than 1, a new mathematical framework was initiated with a series of papers by Bresch–Desjardins [8–11] (started in 2003 with Lin [12] in the context of Navier–Stokes–Korteweg with linear shear viscosity case) who identified a piece of information related to the gradient of a function of the density if the viscosities satisfy what is called the Bresch–Desjardins constraint. This information is usually called the BD entropy, allowing the introduction of the concept of entropy-weak solutions. Using such extra information, they obtained the global existence of entropy-weak solutions in the presence of appropriate drag terms or singular pressure close to vacuum. Concerning the case one-dimensional in space, or the spherical case, many important results have been obtained for instance by Burtea–Haspot [18], Ducomet–Nečasová–Vasseur [22], Constantin–Drivas–Nguyen–Pasqualotto [20], Guo–Jiu–Xin [25], Haspot [26], Jiang–Xin–Zhang [28], Jiang–Zhang [29], Kanel' [32], Li–Li–Xin [36], Mellet–Vasseur [41], Shelukhin [47] without drag terms. Stability and construction of approximate solutions in space dimension 2 or 3 have been investigated during more than fifteen years with a first important stability result without drag terms or singular pressure by Mellet–Vasseur [40]. Several important works, for instance by Bresch–Desjardins [8–11] and Bresch–Desjardins–Lin [12], Bresch–Desjardins–Zatorska [13], Li–Xin [37], Mellet–Vasseur [40], Mucha–Pokorný–Zatorska

[42], Vasseur–Yu [50, 51], and Zatorska [52], have also been written trying to find a way to construct approximate solutions. Recently a real breakthrough has been made in two papers by Li–Xin [37] and Vasseur–Yu [50]: Using two different ways, they got the global existence of entropy-weak solutions for the compressible system when $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$. Note that Li–Xin [37] also consider more general viscosities satisfying the BD relation but with a non-symmetric stress diffusion ($\sigma = \mu(\rho)\nabla u + (\lambda(\rho)\operatorname{div} u - P(\rho))\,\mathrm{Id}$) and more restrictive conditions on the shear viscosity $\mu(\rho)$ and bulk viscosity $\lambda(\rho)$ and on the pressure law $P(\rho)$ compared to the present paper.

The objective of this paper is to extend the results on existence of global entropy-weak solutions obtained independently (using different strategies) by Vasseur–Yu [50] and Lin–Xin [37] in order to answer a longstanding mathematical question on compressible Navier–Stokes equations with density dependent viscosities, mentioned for instance by Rousset [46]. More precisely, extending and coupling carefully the two-velocities framework by Bresch–Desjardins–Zatorska [13] with the generalization of the quantum Böhm identity found by Bresch–Couderc–Noble–Vila [7] (proving a generalization of the dissipation inequality used by Jüngel [30] for the Quantum Navier–Stokes system and established by Jüngel–Matthes [31]) and with the renormalized solutions introduced by Lacroix-Violet and Vasseur [34], we get global existence of entropy-weak solutions to the following Navier–Stokes equations:

$$\begin{aligned} &\rho_t + \operatorname{div}(\rho u) = 0,\\ &(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - 2\operatorname{div}\Big(\sqrt{\mu(\rho)}\,\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{S}_\mu)\,\mathrm{Id}\Big) = 0, \end{aligned} \tag{1.1}$$

where

$$\sqrt{\mu(\rho)}\,\mathbb{S}_\mu = \mu(\rho)\mathbb{D}(u)$$

with data

$$\rho|_{t=0} = \rho_0(x) \geq 0, \qquad \rho u|_{t=0} = m_0(x) = \rho_0 u_0, \tag{1.2}$$

and where $P(\rho) = a\rho^\gamma$ denotes the pressure with constants $a > 0$ and $\gamma > 1$, ρ is the density of the fluid, u stands for the velocity of the fluid, and $\mathbb{D}u = [\nabla u + \nabla^T u]/2$ is the strain tensor. As usual, we consider

$$\begin{aligned} &u_0 = \frac{m_0}{\rho_0} \text{ when } \rho_0 \neq 0 \text{ and } u_0 = 0 \text{ elsewhere},\\ &\frac{|m_0|^2}{\rho_0} = 0 \text{ a.e. on } \{x \in \Omega : \rho_0(x) = 0\}. \end{aligned}$$

Remark 1.1. We record the following identity:

$$2\operatorname{div}\Big(\sqrt{\mu(\rho)}\,\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{S}_\mu)\,\mathrm{Id}\Big) = 2\operatorname{div}(\mu(\rho)\mathbb{D}u) + \nabla(\lambda(\rho)\operatorname{div} u).$$

The viscosity coefficients $\mu = \mu(\rho)$ and $\lambda = \lambda(\rho)$ satisfy the Bresch–Desjardins relation introduced in [9],

$$\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho)). \tag{1.3}$$

The relation between the stress tensor $\mathbb{S}_\mu$ and the triple $(\mu(\rho)/\sqrt{\rho}, \sqrt{\rho}\, u, \sqrt{\rho}\, v)$ where $v = 2\nabla s(\rho)$ with $s'(\rho) = \mu'(\rho)/\rho$ will be proved in the following way: The matrix valued $\mathbb{S}_\mu$ is the symmetric part of the matrix valued function $\mathbb{T}_\mu$, namely

$$\mathbb{S}_\mu = \frac{\mathbb{T}_\mu + {}^t\mathbb{T}_\mu}{2}, \tag{1.4}$$

where $\mathbb{T}_\mu$ is defined through

$$\sqrt{\mu(\rho)}\, \mathbb{T}_\mu = \nabla\Big(\sqrt{\rho}\, u \frac{\mu(\rho)}{\sqrt{\rho}}\Big) - \sqrt{\rho}\, u \otimes \sqrt{\rho}\, \nabla s(\rho) \tag{1.5}$$

with

$$s'(\rho) = \mu'(\rho)/\rho, \tag{1.6}$$

and

$$\frac{\lambda(\rho)}{2\mu(\rho)} \operatorname{Tr}(\sqrt{\mu(\rho)}\, \mathbb{T}_\mu) \operatorname{Id} = \Big[\operatorname{div}\Big(\frac{\lambda(\rho)}{\mu(\rho)} \sqrt{\rho}\, u\, \frac{\mu(\rho)}{\sqrt{\rho}}\Big) - \sqrt{\rho}\, u \cdot \sqrt{\rho}\, \nabla s(\rho) \frac{\rho\mu''(\rho)}{\mu'(\rho)}\Big] \operatorname{Id}. \tag{1.7}$$

Remark 1.2. Compared to the case $\mu(\rho) = \rho$, the definition of $\mathbb{T}_\mu$ is given through the two compatible identities (1.5) and (1.7).

For the sake of simplicity, we will consider the case of periodic boundary conditions in three dimensions in space, namely $\Omega = \mathbb{T}^3$. In the whole paper, we assume

$$\mu \in C^0(\mathbb{R}^+; \mathbb{R}^+) \cap C^2(\mathbb{R}^+_*; \mathbb{R}), \tag{1.8}$$

where $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^+_* = (0, \infty)$. We also assume that there exist positive numbers α_1, α_2 such that

$$\begin{aligned} &2/3 < \alpha_1 \le \alpha_2 < 4, \\ &0 < \frac{1}{\alpha_2}\rho\mu'(\rho) \le \mu(\rho) \le \frac{1}{\alpha_1}\rho\mu'(\rho) \quad \text{for any } \rho > 0, \end{aligned} \tag{1.9}$$

and there exists a constant $C > 0$ such that

$$\left|\frac{\rho\mu''(\rho)}{\mu'(\rho)}\right| \le C < \infty. \tag{1.10}$$

Note that if $\mu(\rho)$ and $\lambda(\rho)$ satisfy (1.3) and (1.9), then

$$\lambda(\rho) + 2\mu(\rho)/3 \ge 0,$$

and thanks to (1.9),

$$\mu(0) = \lambda(0) = 0.$$

Remark 1.3. Note that the hypotheses (1.9)–(1.10) allow the shear viscosity to be of the form $\mu(\rho) = \mu\rho^\alpha$ with $\mu > 0$ a constant where $2/3 < \alpha < 4$ and the bulk viscosity satisfying the BD relation $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$.

Remark 1.4. Note that the restriction $2/3 < \alpha_1 \le \alpha_2$ comes from the hypothesis that there exists $\varepsilon > 0$ such that $2\mu(\rho) + 3\lambda(\rho) \ge \varepsilon\mu(\rho)$, which gives $2\mu(\rho) + 3\lambda(\rho) > 0$ (far from vacuum), the usual physical restriction between the shear and bulk viscosities. Meanwhile, for technical reasons in the proof of Lemma 2.1, we need to restrict $\alpha_2 < 4$ in hypothesis (1.9). More precisely, we get $\|\nabla\nabla Z(\rho)\|_{L^2((0,T)\times\Omega)}$ and $\|\nabla Z_1(\rho)\|_{L^4((0,T)\times\Omega)}$ controlled if the two constants in front of them in Lemma 2.1 are positive.

It is important to remark that in the recent paper [1], the authors have indicated how Lemma 2.1 may be used for the full range $2/3 < \alpha < \infty$ when $\mu(\rho) = \rho^\alpha$ and $\lambda(\rho) = 2(\alpha - 1)\rho^\alpha$. It is enough to be able to compare $\|\nabla\nabla Z(\rho)\|_{L^2((0,T)\times\Omega)}$ to $\|\nabla Z_1(\rho)\|_{L^4((0,T)\times\Omega)}$ to relax the assumptions. This is based on the following inequality: For any $d \ge 1$ and any positive function θ in $H^2(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} |\nabla\theta^{1/2}|^4\,dx \le \frac{9}{16}\int_{\mathbb{T}^d} (\Delta\theta)^2\,dx.$$

Remark 1.5. In [50] and [37] the case $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$ is considered, and in [37] more general cases have been considered but with a non-symmetric viscous term in the three-dimensional in space case, namely $-\operatorname{div}(\mu(\rho)\nabla u) - \nabla(\lambda(\rho)\operatorname{div} u)$. In [37] the viscosities $\mu(\rho)$ and $\lambda(\rho)$ satisfy (1.3) with $\mu(\rho) = \mu\rho^\alpha$ where $\alpha \in [3/4, 2)$ and with the following assumption on γ for the pressure $p(\rho) = a\rho^\gamma$:

$$\gamma \in \begin{cases} (1, 6\alpha - 3) & \text{if } \alpha \in [3/4, 1], \\ [2\alpha - 1, 3\alpha - 1] & \text{if } \alpha \in (1, 2). \end{cases}$$

Definitions

Following [34] (based on [50]), we will show the existence of renormalized solutions in u. Then, we will show that this renormalized solution is a weak solution. The renormalization provides weak stability of the advection terms $\rho u \otimes u$ and $\rho u \otimes v$. Let us first define a renormalized solution:

Definition 1.1. Consider $\mu > 0$, $3\lambda + 2\mu > 0$, $r_0 \ge 0$, $r_1 \ge 0$, $r_2 \ge 0$, $\delta \ge 0$ and $r \ge 0$. We say that $(\sqrt{\rho}, \sqrt{\rho}u)$ is a *renormalized weak solution* in u of the compressible Navier–Stokes equations (with an extra capillarity term, with drag terms, with a supplementary pressure if respectively $r \ne 0$, $(r_0, r_1, r_2) \ne 0$ and $\delta \ne 0$) if it satisfies (1.23)–(1.26) below, and for any $\varphi \in W^{2,\infty}(\mathbb{R}^3)$, there exist measures $R_\varphi, \overline{R}^1_\varphi, \overline{R}^2_\varphi \in \mathcal{M}(\mathbb{R}^+ \times \Omega)$ with

$$\|R_\varphi\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|\overline{R}^1_\varphi\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|\overline{R}^2_\varphi\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} \le C\|\nabla\nabla\varphi\|_{L^\infty(\mathbb{R}^3)},$$

where the constant C depends only on the solution $(\sqrt{\rho}, \sqrt{\rho}\, u)$, and for any function $\psi \in C_c^\infty(\mathbb{R}^+ \times \Omega)$,

$$\int_0^T \int_\Omega (\rho \psi_t + \sqrt{\rho}\, \sqrt{\rho}\, u \cdot \nabla \psi)\, dx\, dt = 0,$$

$$\begin{aligned}
&\int_0^T \int_\Omega \big(\rho \varphi(u) \psi_t + \rho \varphi(u) \otimes u : \nabla \psi\big)\; dx\, dt \\
&\quad - \int_0^T \int_\Omega \bigg(2\Big(\sqrt{\mu(\rho)}\, \mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)} \operatorname{Tr}(\sqrt{\mu(\rho)}\, \mathbb{S}_\mu) \operatorname{Id}\Big) \varphi'(u)\bigg) \cdot \nabla \psi \, dx\, dt \\
&\quad - r \int_0^T \int_\Omega \bigg(2\Big(\sqrt{\mu(\rho)}\, \mathbb{S}_r + \frac{\lambda(\rho)}{2\mu(\rho)} \operatorname{Tr}(\sqrt{\mu(\rho)}\, \mathbb{S}_r) \operatorname{Id}\Big) \varphi'(u)\bigg) \cdot \nabla \psi \, dx\, dt \\
&\quad + \int_0^T \int_\Omega F(\rho, u) \varphi'(u) \psi \, dx\, dt = \langle R_\varphi, \psi \rangle,
\end{aligned}$$

$$\int_0^T \int_\Omega \bigg(\mu(\rho)\psi_t + \frac{\mu(\rho)}{\sqrt{\rho}} \sqrt{\rho}\, u \cdot \nabla \psi\bigg)\, dx\, dt - \int_0^T \int_\Omega \frac{\lambda(\rho)}{2\mu(\rho)} \operatorname{Tr}(\sqrt{\mu(\rho)}\, \mathbb{T}_\mu) \psi \, dx\, dt = 0,$$

where $\mathbb{S}_\mu$ is given in (1.4) and $\mathbb{T}_\mu$ is given in (1.7). The matrix $\mathbb{S}_r$ is compatible in the sense of (1.19)–(1.21) below.

The vector valued function F is given by

$$\begin{aligned}
F(\rho, u) = &\sqrt{\frac{P'(\rho)\rho}{\mu'(\rho)}}\, \nabla \int_0^\rho \sqrt{\frac{P'(s)\mu'(s)}{s}}\, ds \\
&+ \delta \sqrt{\frac{P_\delta'(\rho)\rho}{\mu'(\rho)}}\, \nabla \int_0^\rho \sqrt{\frac{P_\delta'(s)\mu'(s)}{s}}\, ds - r_0 u - r_1 \rho |u| u - \frac{r_2}{\mu'(\rho)} \rho |u|^2 u.
\end{aligned} \tag{1.11}$$

For every i, j, k between 1 and d we have

$$\sqrt{\mu(\rho)}\, \varphi_i'(u) [\mathbb{T}_\mu]_{jk} = \partial_j (\mu(\rho) \varphi_i'(u) u_k) - \sqrt{\rho}\, u_k \varphi_i'(u) \sqrt{\rho}\, \partial_j s(\rho) + \overline{R}_\varphi^1, \tag{1.12}$$

$$r \varphi_i'(u) [\nabla(\sqrt{\mu(\rho)}\, \nabla Z(\rho))]_{jk} = r \partial_j (\sqrt{\mu(\rho)}\, \varphi_i'(u) \partial_k Z(\rho)) + \overline{R}_\varphi^2; \tag{1.13}$$

moreover

$$\|\overline{R}_\varphi^1\|_{\mathcal{M}(\mathbb{R}^+ \times \Omega)} + \|\overline{R}_\varphi^2\|_{\mathcal{M}(\mathbb{R}^+ \times \Omega)} + \|R_\varphi\|_{\mathcal{M}(\mathbb{R}^+ \times \Omega)} \le C \|\nabla \nabla \varphi\|_{L^\infty},$$

and for any $\overline{\psi} \in C_c^\infty(\Omega)$,

$$\begin{aligned}
&\lim_{t \to 0} \int_\Omega \rho(t, x) \overline{\psi}(x)\, dx = \int_\Omega \rho_0(x) \overline{\psi}(x)\, dx, \\
&\lim_{t \to 0} \int_\Omega \rho(t, x) u(t, x) \overline{\psi}(x)\, dx = \int_\Omega m_0(x) \overline{\psi}(x)\, dx, \\
&\lim_{t \to 0} \int_\Omega \mu(\rho)(t, x) \overline{\psi}(x)\, dx = \int_\Omega \mu(\rho_0)(x) \overline{\psi}(x)\, dx.
\end{aligned}$$

Remark 1.6. The notion of renormalized solutions was introduced by R. DiPerna and P.-L. Lions [21], and it was adapted to the study of the compressible Navier–Stokes equations by P.-L. Lions [38]. In Lions' framework, this notion allows one to handle the issue of low regularity of the density. However, in our paper, we have more uniform bounds on the density and less regularity of the velocity. With our definition of renormalized solution in velocity, this allows us to get the weak stability of the solution sequence even if we are not able to have extra control on $\rho|u|^2$. It allows us to get rid of the Mellet–Vasseur type inequality for passing to the limits and allows us to establish the existence result for any $\gamma > 1$.

We define a global weak solution of the approximate system or the compressible Navier–Stokes equation (when $r = r_0 = r_1 = r_2 = \delta = 0$) as follows:

Definition 1.2. Let $\mathbb{S}_\mu$ the symmetric part of $\mathbb{T}_\mu$ in $L^2((0,T)\times\Omega)$ satisfying (1.4)–(1.7) and $\mathbb{S}_r$ the capillary quantity in $L^2((0,T)\times\Omega)$ given by (1.19)–(1.21). Set $P(\rho) = a\rho^\gamma$ and $P_\delta(\rho) = \delta\rho^{10}$. We say that (ρ, u) is a *weak solution* to (1.17)–(1.20) if it satisfies the *a priori* estimates (1.23)–(1.26) and for any $\psi \in \mathcal{C}_c^\infty((0,T)\times\Omega)$,

$$
\begin{aligned}
&\int_0^T\int_\Omega (\rho\partial_t\psi + \rho u\cdot\nabla\psi)\,dx\,dt = 0,\\
&\int_0^T\int_\Omega (\rho u\partial_t\psi + \rho u\otimes u : \nabla\psi)\,dx\,dt\\
&\qquad -\int_0^T\int_\Omega 2\Big(\sqrt{\mu(\rho)}\,\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{S}_\mu)\operatorname{Id}\Big)\cdot\nabla\psi\,dx\,dt\\
&\qquad - r\int_0^T\int_\Omega 2\Big(\sqrt{\mu(\rho)}\,\mathbb{S}_r + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{S}_r)\operatorname{Id}\Big)\cdot\nabla\psi\,dx\,dt\\
&\qquad + \int_0^T\int_\Omega F(\rho,u)\,\psi\,dx\,dt = 0,\\
&\int_0^\infty\int_\Omega \Big(\mu(\rho)\psi_t + \frac{\mu(\rho)}{\sqrt{\rho}}\sqrt{\rho}u\cdot\nabla\psi\Big)\,dx\,dt\\
&\qquad\qquad - \int_0^T\int_\Omega \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{T}_\mu)\psi\,dx\,dt = 0,
\end{aligned}
\tag{1.14}
$$

with F given through (1.11), and for any $\overline{\psi} \in \mathcal{C}_c^\infty(\Omega)$,

$$
\begin{aligned}
\lim_{t\to 0}\int_\Omega \rho(t,x)\overline{\psi}(x)\,dx &= \int_\Omega \rho_0(x)\overline{\psi}(x)\,dx,\\
\lim_{t\to 0}\int_\Omega \rho(t,x)u(t,x)\overline{\psi}(x)\,dx &= \int_\Omega m_0(x)\overline{\psi}(x)\,dx,\\
\lim_{t\to 0}\int_\Omega \mu(\rho)(t,x)\overline{\psi}(x)\,dx &= \int_\Omega \mu(\rho_0)(x)\overline{\psi}(x)\,dx.
\end{aligned}
$$

Remark 1.7. As mentioned in [15], the equation on $\mu(\rho)$ is important: By taking $\psi = \operatorname{div}\varphi$ for all $\varphi \in \mathcal{C}_0^\infty$, we can write the equation satisfied by $\nabla\mu(\rho)$, namely

$$\begin{aligned}\partial_t \nabla\mu(\rho) + \operatorname{div}(\nabla\mu(\rho)\otimes u) &= \operatorname{div}(\nabla\mu(\rho)\otimes u) - \nabla \operatorname{div}(\mu(\rho)u) \\ &\quad - \nabla\left(\frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{T}_\mu)\right) \\ &= -\operatorname{div}(\sqrt{\mu(\rho)}\,{}^t\mathbb{T}_\mu) - \nabla\left(\frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{T}_\mu)\right).\end{aligned} \tag{1.15}$$

This will justify in some sense the two-velocities formulation introduced in [13] with the extra velocity linked to $\nabla\mu(\rho)$.

The main result of our paper reads as follows:

Theorem 1.1. *Let $\mu(\rho)$ satisfy (1.8)–(1.10) and let μ and λ satisfy (1.3). Assume that the initial data satisfy*

$$\begin{aligned}\int_\Omega \left(\frac{1}{2}\rho_0|u_0 + 2\kappa\nabla s(\rho_0)|^2 + \kappa(1-\kappa)\rho_0\frac{|2\nabla s(\rho_0)|^2}{2}\right)dx \\ + \int_\Omega\left(a\frac{\rho_0^\gamma}{\gamma-1} + \mu(\rho_0)\right)dx \le C < \infty\end{aligned} \tag{1.16}$$

with $k \in (0,1)$ given. Let $0 < T < \infty$ be given. Then, for any $\gamma > 1$, there exists a renormalized solution to (1.1)–(1.2) as defined in Definition 1.1 with r, r_0, r_1, r_2 and δ all zero. Moreover, this renormalized solution with initial data satisfying (1.16) is a weak solution to (1.1)–(1.2) in the sense of Definition 1.2.

Our result may be considered as an improvement of [37] for two reasons: First, it takes into account a physical symmetric viscous tensor, and secondly, it extends the range of the coefficients α and γ. The method is based on the consideration of an approximate system with an extra pressure quantity, appropriate non-linear drag terms and appropriate capillarity terms. This generalizes the Quantum Navier–Stokes system with quadratic drag terms considered in [50,51]. First we prove that weak solutions of the approximate system are renormalized solutions of the system, in the sense of [34]. Then we pass to the limit with respect to r_2, r_1, r_0, r, δ to get renormalized solutions of the compressible Navier–Stokes system. The final step is the proof that a renormalized solution of the compressible Navier–Stokes system is a global weak solution of the compressible Navier–Stokes system. Note that thanks to the technique of renormalized solution introduced in [34], it is not necessary to derive the Mellet–Vasseur type inequality in this paper; this allows us to cover the whole range $\gamma > 1$.

First step. Motivated by the work of [34], the first step is to establish the existence of a global κ-entropy weak solution (in the sense of Theorem 1.2 below) to the following approximation:

$$
\begin{aligned}
&\rho_t + \operatorname{div}(\rho u) = 0,\\
&(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \nabla P_\delta(\rho)\\
&\qquad\qquad - 2\operatorname{div}\Big(\sqrt{\mu(\rho)}\,\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{S}_\mu)\operatorname{Id}\Big)\\
&\qquad\qquad - 2r\operatorname{div}\Big(\sqrt{\mu(\rho)}\,\mathbb{S}_r + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{S}_r)\operatorname{Id}\Big)\\
&\qquad\qquad + r_0 u + r_1\rho|u|u + r_2\frac{\rho}{\mu'(\rho)}|u|^2 u = 0
\end{aligned}
\tag{1.17}
$$

where the barotropic pressure law and the extra pressure term are respectively

$$
P(\rho) = a\rho^\gamma, \qquad P_\delta(\rho) = \delta\rho^{10} \quad \text{with } \delta > 0. \tag{1.18}
$$

The matrix $\mathbb{S}_\mu$ is defined in (1.4) and $\mathbb{T}_\mu$ is given in (1.5)–(1.7). The matrix $\mathbb{S}_r$ is compatible in the following sense:

$$
r\sqrt{\mu(\rho)}\,\mathbb{S}_r = 2r\big[2\sqrt{\mu(\rho)}\,\nabla\nabla Z(\rho) - \nabla(\sqrt{\mu(\rho)}\,\nabla Z(\rho))\big], \tag{1.19}
$$

where

$$
\begin{aligned}
Z(\rho) &= \int_0^\rho [\mu(s)^{1/2}\mu'(s)]/s\,ds,\\
k(\rho) &= \int_0^\rho [\lambda(s)\mu'(s)]/\mu(s)^{3/2}\,ds
\end{aligned}
\tag{1.20}
$$

and

$$
\begin{aligned}
r\frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{S}_r)\operatorname{Id} &= r\Big(\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} + \frac{1}{2}k(\rho)\Big)\Delta Z(\rho)\operatorname{Id}\\
&\quad - \frac{r}{2}\operatorname{div}[k(\rho)\nabla Z(\rho)]\operatorname{Id}.
\end{aligned}
\tag{1.21}
$$

Remark 1.8. Note that the previous system is a generalization of the quantum viscous Navier–Stokes system considered by Lacroix-Violet and Vasseur [34] (see also the interesting papers by Antonelli–Spirito [2, 3] and by Carles–Carrapatoso–Hillairet [19]). Indeed, if we consider $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$, we can write

$$
\sqrt{\mu(\rho)}\,\mathbb{S}_r = 4\sqrt{\rho}\,[\nabla\nabla\sqrt{\rho} - 4(\nabla\rho^{1/4}\otimes\nabla\rho^{1/4})],
$$

using $Z(\rho) = 2\sqrt{\rho}$. The Navier–Stokes equations for quantum fluids were also considered by A. Jüngel [30].

As a first step generalizing [50], we prove the following result.

Theorem 1.2. *Let $\mu(\rho)$ satisfy (1.8)–(1.10) and let $\lambda(\rho)$ be given by (1.3). If $r_0 > 0$, then also assume that $\inf_{s\in[0,\infty)} \mu'(s) = \epsilon_1 > 0$. Assume that r_1 is small enough compared to δ, r_2 is small enough compared to r and the initial values satisfy*

$$\int_\Omega \rho_0\Big(\frac{|u_0 + 2\kappa\nabla s(\rho_0)|^2}{2} + (\kappa(1-\kappa)+r)\frac{|2\nabla s(\rho_0)|^2}{2}\Big)\,dx + \int_\Omega\Big(a\frac{\rho_0^\gamma}{\gamma-1} + \mu(\rho_0) + \delta\frac{\rho_0^{10}}{9} + \frac{r_0}{\varepsilon_1}|(\ln\rho_0)_-|\Big)\,dx < \infty, \tag{1.22}$$

for a fixed $\kappa\in(0,1)$. Then there exists a κ-entropy weak solution $(\rho, u, \mathbb{T}_\mu, \mathbb{S}_r)$ to (1.17)–(1.21) satisfying the initial conditions (1.2), in the sense that $(\rho, u, \mathbb{T}_\mu, \mathbb{S}_r)$ satisfies the mass and momentum equations in a weak form, and satisfies the compatibility formula in the sense of Definition 1.2. In addition, it satisfies the following estimates:

$$\begin{aligned}
&\|\sqrt{\rho}\,(u+2\kappa\nabla s(\rho))\|^2_{L^\infty(0,T;L^2(\Omega))} \le C, \quad a\|\rho\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))} \le C,\\
&\|\mathbb{T}_\mu\|^2_{L^2(0,T;L^2(\Omega))} \le C, \quad (\kappa(1-\kappa)+r)\|\sqrt{\rho}\,\nabla s(\rho)\|^2_{L^\infty(0,T;L^2(\Omega))} \le C,\\
&\kappa\|\sqrt{\mu'(\rho)\rho^{\gamma-2}}\,\nabla\rho\|^2_{L^2(0,T;L^2(\Omega))} \le C,
\end{aligned} \tag{1.23}$$

and

$$\begin{aligned}
&\delta\|\rho\|^{10}_{L^\infty(0,T;L^{10}(\Omega))} \le C, && \delta\Big\|\sqrt{\mu'(\rho)\rho^8}\,\nabla\rho\Big\|^2_{L^2(0,T;L^2(\Omega))} \le C,\\
&r_2\Big\|\Big(\frac{\rho}{\mu'(\rho)}\Big)^{1/4}u\Big\|^4_{L^4(0,T;L^4(\Omega))} \le C, && r_1\|\rho^{1/3}|u|\|^3_{L^3(0,T;L^3(\Omega))} \le C,\\
&r_0\|u\|^2_{L^2(0,T;L^2(\Omega))} \le C, && r\|\mathbb{S}_r\|^2_{L^2(0,T;L^2(\Omega))} \le C,
\end{aligned} \tag{1.24}$$

where C does not depend on the parameters κ, δ, r, r_0, r_1, r_2. Note that the bounds (1.23) provide the following control on the velocity field:

$$\|\sqrt{\rho}\,u\|^2_{L^\infty(0,T;L^2(\Omega))} \le C.$$

Moreover letting

$$Z(\rho) = \int_0^\rho \frac{\sqrt{\mu(s)}\,\mu'(s)}{s}\,ds \quad \text{and} \quad Z_1(\rho) = \int_0^\rho \frac{\mu'(s)}{\mu(s)^{1/4}s^{1/2}}\,ds,$$

we have the extra control

$$r\Big[\int_0^T\int_\Omega |\nabla^2 Z(\rho)|^2\,dx\,dt + \int_0^T\int_\Omega |\nabla Z_1(\rho)|^4\,dx\,dt\Big] \le C, \tag{1.25}$$

and

$$\begin{aligned}
&\|\mu(\rho)\|_{L^\infty(0,T;W^{1,1}(\Omega))} + \|\mu(\rho)u\|_{L^\infty(0,T;L^{3/2}(\Omega))\cap L^2(0,T;W^{1,1}(\Omega))} \le C,\\
&\|\partial_t\mu(\rho)\|_{L^\infty(0,T;W^{-1,1}(\Omega))} \le C,\\
&\|Z(\rho)\|_{L^\infty(0,T;L^{1+}(\Omega))} + \|Z_1(\rho)\|_{L^\infty(0,T;L^{1+}(\Omega))} \le C,
\end{aligned} \tag{1.26}$$

where $C > 0$ is a constant which depends only on the initial data.

Sketch of proof of Theorem 1.2. To show Theorem 1.2, we need to build a smooth solution to an approximation associated to (1.17). Here, we adapt the ideas developed in [13] to construct this approximation. More precisely, we consider an augmented version of the system which will be more appropriate to construct approximate solutions. Let us explain the idea.

First step: the augmented system. Define a new velocity field generalizing the one introduced in the BD entropy estimate:

$$w = u + 2\kappa \nabla s(\rho)$$

and a drift velocity $v = 2\nabla s(\rho)$ with $s(\rho)$ defined in (1.6).

Assuming to have a smooth solution of (1.17) with damping terms, (ρ, w, v) satisfies the following system of equations:

$$\rho_t + \operatorname{div}(\rho w) - 2\kappa \Delta \mu(\rho) = 0$$

and

$$\begin{aligned}
&(\rho w)_t + \operatorname{div}(\rho u \otimes w) - 2(1-\kappa)\operatorname{div}(\mu(\rho)\mathbb{D}w) - 2\kappa \operatorname{div}(\mu(\rho)\mathbf{A}(w)) \\
&\quad - (1-\kappa)\nabla(\lambda(\rho)\operatorname{div}(w - \kappa v)) + \nabla\rho^\gamma + \delta\nabla\rho^{10} + 4(1-\kappa)\kappa \operatorname{div}(\mu(\rho)\nabla^2 s(\rho)) \\
&\quad = -r_0(w - 2\kappa\nabla s(\rho)) - r_1\rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho)) \\
&\qquad - r_2\frac{\rho}{\mu'(\rho)}|w - 2\kappa\nabla s(\rho)|^2(w - 2\kappa\nabla s(\rho)) + r\rho\nabla\Big(\sqrt{K(\rho)}\,\Delta\Big(\int_0^\rho \sqrt{K(s)}\,ds\Big)\Big),
\end{aligned}$$

where $\mathbf{A}(w) = (\nabla w - {}^t\nabla w)/2$, and

$$(\rho v)_t + \operatorname{div}(\rho u \otimes v) - 2\kappa \operatorname{div}(\mu(\rho)\nabla v) + 2\operatorname{div}(\mu(\rho)\nabla^t w) + \nabla(\lambda(\rho)\operatorname{div}(w - \kappa v)) = 0,$$

where

$$v = 2\nabla s(\rho), \qquad w = u + \kappa v, \qquad K(\rho) = 4(\mu'(\rho))^2/\rho.$$

This is the augmented version for which we will show that there exist global weak solutions, adding a hyperdiffusivity $\varepsilon_2[\Delta^{2s} w - \operatorname{div}((1 + |\nabla w|^2)\nabla w)]$ to the equation satisfied by w, and passing to the limit as ε_2 goes to zero.

Important remark. Note that recently Bresch–Couderc–Noble–Vila [7] showed the following relation:

$$\begin{aligned}
&\rho\nabla\Big(\sqrt{K(\rho)}\,\Delta\Big(\int_0^\rho \sqrt{K(s)}\,ds\Big)\Big) \\
&\qquad = \operatorname{div}(F(\rho)\nabla^2\psi(\rho)) + \nabla\big((F'(\rho)\rho - F(\rho))\Delta\psi(\rho)\big)
\end{aligned}$$

with $F'(\rho) = \sqrt{K(\rho)\rho}$ and $\sqrt{\rho}\,\psi'(\rho) = \sqrt{K(\rho)}$. Thus choosing

$$F(\rho) = 2\mu(\rho) \quad \text{and therefore} \quad F'(\rho)\rho - F(\rho) = \lambda(\rho),$$

this gives $\psi(\rho) = 2s(\rho)$ and thus

$$\rho\nabla\Big(\sqrt{K(\rho)}\,\Delta\Big(\int_0^\rho \sqrt{K(s)}\,ds\Big)\Big) = 2\,\mathrm{div}\big(\mu(\rho)\nabla^2(2s(\rho))\big) + \nabla\big(\lambda(\rho)\Delta(2s(\rho))\big). \tag{1.27}$$

This identity will play a crucial role in the proof, because it defines the appropriate capillarity term to consider in the approximate system to be compatible with the stress tensor. This form is compatible with the various multipliers which are used to get the κ-entropy estimates and to control regularity of the density. Other identities will be used to define a weak solution for the Navier–Stokes–Korteweg system and to pass to the limit in it, namely

$$\begin{aligned}2\mu(\rho)\nabla^2(2s(\rho)) + \lambda(\rho)\Delta(2s(\rho)) &= 4\big[2\sqrt{\mu(\rho)}\,\nabla\nabla Z(\rho) - \nabla(\sqrt{\mu(\rho)}\,\nabla Z(\rho))\big]\\ &\quad + \Big(\frac{2\lambda(\rho)}{\sqrt{\mu(\rho)}} + k(\rho)\Big)\Delta Z(\rho)\,\mathrm{Id} - \mathrm{div}[k(\rho)\nabla Z(\rho)]\,\mathrm{Id},\end{aligned} \tag{1.28}$$

where

$$Z(\rho) = \int_0^\rho [\mu(s)^{1/2}\mu'(s)]/s\,ds \quad \text{and} \quad k(\rho) = \int_0^\rho \frac{\lambda(s)\mu'(s)}{\mu(s)^{3/2}}\,ds.$$

Note that the case considered in [34, 50, 51] is related to $\mu(\rho) = \rho$ and $K(\rho) = 4/\rho$ which corresponds to the Quantum Navier–Stokes system. Note that two very interesting papers by Antonelli–Spirito [4, 5] consider Navier–Stokes–Korteweg systems without such relation between the shear viscosity and the capillary coefficient.

Remark 1.9. The additional pressure $\delta\rho^{10}$ is used in (2.19) thanks to $3\alpha_2 - 2 \le 10$. It could be possible to take $\rho^{3\alpha_2-2}$ but we have chosen ρ^{10} for simplicity.

Second step and main result concerning the compressible Navier–Stokes system. To prove global existence of weak solutions of the compressible Navier–Stokes equations, we follow the strategy introduced in [34, 50]. To do so, first we approximate the viscosity μ by a viscosity μ_{ε_1} such that $\inf_{s\in[0,\infty)} \mu'_{\varepsilon_1}(s) \ge \varepsilon_1 > 0$. Then we use Theorem 1.2 to construct a κ-entropy weak solution to the approximate system (1.17). We then show that it is a renormalized solution of (1.17) in the sense of [34]. More precisely we prove the following theorem:

Theorem 1.3. *Let $\mu(\rho)$ satisfy (1.8)–(1.10), and let $\lambda(\rho)$ be given by (1.3). If $r_0 > 0$, then assume also that $\inf_{s\in[0,\infty)} \mu'(s) = \epsilon_1 > 0$. Assume that r_1 is small enough compared to δ and r_2 is small enough compared to r (as in Theorem 1.2) and the initial values satisfy*

$$\begin{aligned}C_{\mathrm{in}} := &\int_\Omega \Big(\rho_0\Big(\frac{|u_0 + 2\kappa\nabla s(\rho_0)|^2}{2} + (\kappa(1-\kappa) + r)\frac{|2\nabla s(\rho_0)|^2}{2}\Big)\Big)\,dx\\ &+ \int_\Omega \Big(a\frac{\rho_0^\gamma}{\gamma-1} + \mu(\rho_0) + \delta\frac{\rho^{10}}{9} + \frac{r_0}{\varepsilon_1}|(\ln\rho_0)_-|\Big)\,dx < \infty.\end{aligned} \tag{1.29}$$

Then the κ-entropy weak solution is a renormalized solution of (1.17) in the sense of Definition 1.1.

We then pass to the limit with respect to the parameters r, r_0, r_1, r_2 and δ to obtain a renormalized weak solution of the compressible Navier–Stokes equations and prove our main theorem.

2. The first level of the approximation procedure

The goal of this section is to construct a sequence of approximate solutions satisfying the compactness structure to prove Theorem 1.2, that is, the existence of weak solutions of the approximation system with capillarity and drag terms. Here we present the first level of the approximation procedure.

1. The continuity equation is

$$\rho_t + \operatorname{div}(\rho[w]_{\varepsilon_3}) = 2\kappa \operatorname{div}([\mu'(\rho)]_{\varepsilon_4} \nabla\rho) \tag{2.1}$$

with modified initial data

$$\rho(0,x) = \rho_0 \in C^{2+\nu}(\bar{\Omega}), \quad 0 < \underline{\rho} \le \rho_0(x) \le \bar{\rho}.$$

Here $[f(t,x)]_{\varepsilon_3}$ denotes the standard regularization by mollification with respect to space, and $[f(t,x)]_{\varepsilon_4}$ is in time. This is a parabolic equation if $\inf_{[0,\infty)} \mu'(s) > 0$. Thus, we can apply the standard theory of parabolic equations to solve it when w is smooth enough. In fact, the same equation was solved in [13].

2. The momentum equation with drag terms is replaced by its Faedo–Galerkin approximation with the additional regularizing term $\varepsilon_2[\Delta^{2s} w - \operatorname{div}((1+|\nabla w|^2)\nabla w)]$ where $s \ge 2$,

$$\begin{aligned}
&\int_\Omega \rho w \cdot \psi - \int_0^t \int_\Omega \Big(\rho\Big([w]_{\varepsilon_3} - 2\kappa \frac{[\mu'(\rho)]_{\varepsilon_4}}{\rho}\nabla\rho\Big) \otimes w\Big) : \nabla\psi \\
&\quad + 2(1-\kappa)\int_0^t \int_\Omega \mu(\rho)\mathbb{D}w : \nabla\psi + 2\kappa \int_0^t \int_\Omega \mu(\rho)\mathbf{A}(w) : \nabla\psi \\
&\quad + (1-\kappa)\int_0^t \int_\Omega \lambda(\rho)\operatorname{div} w \operatorname{div}\psi - 2\kappa(1-\kappa)\int_0^t \int_\Omega \mu(\rho)\nabla v : \nabla\psi \\
&\quad - \kappa(1-\kappa)\int_0^t \int_\Omega \lambda(\rho)\operatorname{div} v \operatorname{div}\psi - \int_0^t \int_\Omega \rho^\gamma \operatorname{div}\psi - \delta \int_0^t \int_\Omega \rho^{10}\operatorname{div}\psi \\
&\quad + \varepsilon_2 \int_0^t \int_\Omega \big(\Delta^s w \cdot \Delta^s \psi + (1+|\nabla w|^2)\nabla w : \nabla\psi\big) \\
&= -\int_0^t \int_\Omega r_0(w - 2\kappa\nabla s(\rho)) \cdot \psi - r_1 \int_0^t \int_\Omega \rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho)) \cdot \psi \\
&\quad - r_2 \int_0^t \int_\Omega \frac{\rho}{\mu'(\rho)}|w - 2\kappa\nabla s(\rho)|^2(w - 2\kappa\nabla s(\rho)) \cdot \psi \\
&\quad - r\int_0^t \int_\Omega \sqrt{K(\rho)}\,\Delta\Big(\int_0^\rho \sqrt{K(s)}\,ds\Big)\operatorname{div}(\rho\psi) + \int_\Omega \rho_0 w_0 \cdot \psi,
\end{aligned} \tag{2.2}$$

satisfied for any $t > 0$ and any test function $\psi \in C([0,T]; X_n)$, where $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$, $s'(\rho) = \mu'(\rho)/\rho$, $X_n = \text{span}\,\{e_i\}_{i=1}^n$ and $\{e_i\}_{i=1}^\infty$ is an orthonormal basis in $W^{1,2}(\Omega)$ with $e_i \in C^\infty(\Omega)$ for any integers $i > 0$.

3. The Faedo–Galerkin approximation for the equation on the drift velocity v reads

$$\begin{aligned}\int_\Omega \rho v\cdot\phi - \int_0^t\int_\Omega \Big(\rho\Big([w]_{\varepsilon_3} - 2\kappa\frac{[\mu'(\rho)]_{\varepsilon_4}}{\rho}\nabla\rho\Big)\otimes v\Big) : \nabla\phi \\ + 2\kappa\int_0^t\int_\Omega \mu(\rho)\nabla v : \nabla\phi + \kappa\int_0^t\int_\Omega \lambda(\rho)\,\text{div}\, v\,\text{div}\,\phi \\ - \int_0^t\int_\Omega \lambda(\rho)\,\text{div}\, w\,\text{div}\,\phi + 2\int_0^t\int_\Omega \mu(\rho)\nabla^T w : \nabla\phi = \int_\Omega \rho_0 v_0\cdot\phi,\end{aligned} \tag{2.3}$$

satisfied for any $t > 0$ and any test function $\phi \in C([0,T]; Y_n)$, where $Y_n = \text{span}\,\{b_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^\infty$ is an orthonormal basis in $W^{1,2}(\Omega)$ with $b_i \in C^\infty(\Omega)$ for any integers $i > 0$.

The above full approximation is similar to the ones in the papers [13]–[14] which are two parts dedicated to augmented systems similar to the one we consider. We can repeat the same argument as in [13] to obtain the local existence of solutions to the Galerkin approximation. For the sake of completeness, in Section 2.1 we show how to obtain the local solution to the approximation system.

2.1. Local existence

For any fixed $w \in C([0,T]; X_n)$, we can solve the continuity equation (2.1) with its initial data. In fact, it is a quasi-linear parabolic equation with smooth coefficients, and the classical parabolic theory yields the following result [13, 24].

Theorem 2.1. *Let $\nu \in (0,1)$ and suppose that the initial condition is*

$$\rho(0,x) = \rho_0 \in C^{2+\nu}(\bar\Omega), \qquad 0 < \underline{\rho} \le \rho_0(x) \le \bar\rho.$$

Then (2.1) has a unique classical solution $\rho = \rho(w)$ that belongs to

$$V_{[0,T]} = \{\rho \in C([0,T]; C^{2+\nu}(\Omega)) \cap C^1([0,T]\times\Omega) : \rho_t \in C^{\nu/2}([0,T]; C(\Omega))\}$$

and satisfies

$$0 < \underline{\rho} \le \rho(t,x) \le \bar\rho < \infty. \tag{2.4}$$

Moreover, $w \mapsto \rho(w)$ maps bounded sets in $C([0,T]; X_n)$ into bounded sets in $V_{[0,T]}$ and is continuous with values in $C([0,T]; C^{2+\nu'}(\Omega))$, $0 < \nu' < \nu < 1$.

With Theorem 2.1 at hand, we are ready to proceed to the Galerkin approximation. In particular, we are going to show that there is a unique solution to (2.2) and (2.3) on a short time interval by using a fixed point argument. More precisely, there exists $T_n > 0$ such that

$$(w, v) \in C([0,T_n]; X_n) \times C([0,T_n]; Y_n)$$

satisfying (2.2) and (2.3). To this end, we denote

$$\begin{aligned}(w(t), v(t)) \\ &= \Big(\mathcal{M}_{\rho(t)}\Big[P_{X_n}(\rho w)^0 + \int_0^t \mathcal{K}(w)(s)\,ds\Big], \mathcal{N}_{\rho(t)}\Big[P_{Y_n}(\rho v)^0 + \int_0^t \mathcal{L}(v)(s)\,ds\Big]\Big) \\ &= \mathcal{T}(w, v)(t),\end{aligned}$$

where $\rho = \rho(w)$ is a solution to the continuity equations when w is given in X_n. The space X_n^* is identified with X_n; we use $\langle e, \phi\rangle_{(X_n,X_n)}$ to express the action of a functional from X_n^* on an element from X_n, and similarly for $\langle b, \phi\rangle_{(Y_n,Y_n)}$. Thus, we have

$$\mathcal{M}_{\rho(t)} : X_n \to X_n, \quad \int_\Omega \rho\mathcal{M}_{\rho(t)}(e)\cdot\phi = \langle e, \psi\rangle_{(X_n,X_n)}, \quad e, \psi \in X_n,$$
$$\mathcal{N}_{\rho(t)} : Y_n \to Y_n, \quad \int_\Omega \rho\mathcal{N}_{\rho(t)}(b)\cdot\phi = \langle b, \phi\rangle_{(Y_n,Y_n)}, \quad b, \psi \in Y_n,$$

where P_{X_n}, P_{Y_n} are the projections of $L^2(\Omega)$ onto X_n and Y_n respectively. The operator $\mathcal{K} : X_n \to X_n$ is defined as follows:

$$\begin{aligned}\langle \mathcal{K}(w), \psi\rangle_{(X_n,X_n)} &= \int_\Omega \Big(\rho\Big([w]_{\varepsilon_3} + 2\kappa\frac{[\mu'(\rho)]_{\varepsilon_4}}{\rho}\nabla\rho\Big)\otimes w\Big) : \nabla\psi \\ &- 2(1-\kappa)\int_\Omega \mu(\rho)\mathbb{D}w : \nabla\psi - 2\kappa\int_\Omega \mu(\rho)\mathbf{A}(w) : \nabla\psi \\ &- (1-\kappa)\int_\Omega \lambda(\rho)\,\mathrm{div}\, w\,\mathrm{div}\,\psi + 2\kappa(1-\kappa)\int_\Omega \mu(\rho)\nabla v : \nabla\psi \\ &+ \kappa(1-\kappa)\int_\Omega \lambda(\rho)\,\mathrm{div}\, v\,\mathrm{div}\,\psi + \int_\Omega \rho^\gamma\,\mathrm{div}\,\psi + \delta\int_\Omega \rho^{10}\,\mathrm{div}\,\psi \\ &- \varepsilon_2\int_\Omega \big(\Delta^s w\cdot\Delta^s\psi - (1+|\nabla w|^2)\nabla w : \nabla\psi\big) - \int_\Omega r_0(w - 2\kappa\nabla s(\rho))\cdot\psi \\ &- r_1\int_\Omega \rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho))\cdot\psi \\ &- r_2\int_\Omega \frac{\rho}{\mu'(\rho)}|w - 2\kappa\nabla s(\rho)|^2(w - 2\kappa\nabla s(\rho))\cdot\psi \\ &- r\int_\Omega \sqrt{K(\rho)}\,\Delta\Big(\int_0^\rho \sqrt{K(s)}\;ds\Big)\,\mathrm{div}(\rho\psi);\end{aligned}$$

and the operator $\mathcal{L} : Y_n \to Y_n$ is defined by

$$\begin{aligned}\langle \mathcal{L}(v), \phi\rangle = &\int_\Omega \Big(\rho\Big([w]_{\varepsilon_3} - 2\kappa\frac{[\mu'(\rho)]_{\varepsilon_4}}{\rho}\nabla\rho\Big)\otimes v\Big) : \nabla\phi \\ &- 2\kappa\int_\Omega \mu(\rho)\nabla v : \nabla\phi - \kappa\int_\Omega \lambda(\rho)\,\mathrm{div}\, v\,\mathrm{div}\,\phi \\ &- \int_\Omega \lambda(\rho)\,\mathrm{div}\, w\,\mathrm{div}\,\phi - 2\int_\Omega \mu(\rho)\,{}^t\nabla w : \nabla\phi.\end{aligned}$$

Thanks to (2.4), we derive

$$\|\mathcal{M}_{\rho(t)}\|_{L(X_n,X_n)} \le 1/\underline{\rho}, \qquad \|\mathcal{N}_{\rho(t)}\|_{L(Y_n,Y_n)} \le 1/\underline{\rho},$$

and

$$\|\mathcal{M}_{\rho^1(t)} - \mathcal{M}_{\rho^2(t)}\|_{L(X_n,X_n)} + \|\mathcal{N}_{\rho^1(t)} - \mathcal{N}_{\rho^2(t)}\|_{L(Y_n,Y_n)} \le c\|\rho^1 - \rho^2\|_{L^1(\Omega)}, \quad (2.5)$$

where c only depends on n and $\underline{\rho}$. Observe that by the equivalence of norms in the finite-dimensional spaces X_n and Y_n, this yields

$$\|\mathcal{K}(w)\|_{X_n} + \|\mathcal{L}(v)\|_{Y_n} \le c(\underline{\rho}, \overline{\rho}, \|\nabla\rho\|_{L^2}, \|w\|_{X_n}, \|v\|_{Y_n}). \quad (2.6)$$

We denote

$$\mathcal{B}_{M,\tau} = \{(w,v) \in C([0,\tau]; X_n) \times C([0,\tau]; Y_n) : \|w\|_{C([0,\tau];X_n)} + \|v\|_{C([0,\tau];Y_n)} \le M\},$$

where $M > 0$ is a given constant; this is a ball in $C([0,\tau]; X_n) \times C([0,\tau]; Y_n)$. By Theorem 2.1 and estimates (2.5), (2.6), one can show that the mapping $\mathcal{T}(w,v)$ is a continuous mapping of the ball $\mathcal{B}_{M,\tau}$ into itself, and it is a contraction for small time $T_n > 0$. Thus, the fixed point argument gives us a unique solution to (2.2) and (2.3).

Thus, we have obtained a local solution (ρ_n, w_n, v_n) on $[0, T_n]$, where $T_n \le T$. In order to extend the local solution to a global one, uniform bounds are necessary so that the corresponding procedure can be iterated.

2.2. *The energy estimate if the solution is regular enough*

For any fixed $n > 0$, choosing test functions $\psi = w$, $\phi = v$ in (2.2) and (2.3), we find that (ρ, w, v) satisfies the κ-entropy equality

$$\begin{aligned}
&\int_\Omega \left(\rho\left(\frac{|w|^2}{2} + (1-\kappa)\kappa\frac{|v|^2}{2}\right) + \frac{\rho^\gamma}{\gamma-1} + \delta\frac{\rho^{10}}{9}\right) + 2(1-\kappa)\int_0^t\int_\Omega \mu(\rho)|\mathbb{D}w - \kappa\nabla v|^2 \\
&\quad + (1-\kappa)\int_0^t\int_\Omega \lambda(\rho)(\operatorname{div} w - \kappa \operatorname{div} v)^2 + 2\kappa\int_0^t\int_\Omega \frac{\mu'(\rho)p'(\rho)}{\rho}|\nabla\rho|^2 \\
&\quad + 2\kappa\int_0^t\int_\Omega \mu(\rho)|Aw|^2 + \varepsilon_2\int_0^t\int_\Omega \left(|\Delta^s w|^2 + (1+|\nabla w|^2)|\nabla w|^2\right) \\
&\quad + r\int_0^t\int_\Omega \sqrt{K(\rho)}\,\Delta\left(\int_0^\rho \sqrt{K(s)}\,ds\right)\operatorname{div}(\rho w) + 20\kappa\delta\int_0^t\int_\Omega \mu'(\rho)\rho^8|\nabla\rho|^2 \\
&\quad + r_0\int_0^t\int_\Omega (w - 2\kappa\nabla s(\rho))\cdot w + r_1\int_0^t\int_\Omega \rho|w-2\kappa\nabla s(\rho)|(w-2\kappa\nabla s(\rho))\cdot w \\
&\quad + r_2\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|w-2\kappa\nabla s(\rho)|^2(w-2\kappa\nabla s(\rho))\cdot w \\
&= \int_\Omega \left(\rho_0\left(\frac{|w_0|^2}{2} + (1-\kappa)\kappa\frac{|v_0|^2}{2}\right) + \frac{\rho_0^\gamma}{\gamma-1} + \delta\frac{\rho_0^{10}}{9}\right) - \int_0^T\int_\Omega \rho^\gamma \operatorname{div}([w]_{\varepsilon_3} - w) \\
&\quad - \delta\int_0^T\int_\Omega \rho^{10}\operatorname{div}([w]_{\varepsilon_3} - w),
\end{aligned} \quad (2.7)$$

where $s' = \mu'(\rho)/\rho$ and $p(\rho) = \rho^\gamma$. Compared to the calculations made in [13], we have to take care of the capillary term and then to take care of the drag terms showing that they can be controlled using $\inf_{s\in[0,\infty)} \mu'(s) \geq \varepsilon_1 > 0$ for the linear drag, using the extra pressure term $\delta\rho^{10}\, ds$ for the quadratic drag term and using the capillary term $r\rho\nabla(\sqrt{K(\rho)}\,\Delta(\int_0^\rho \sqrt{K(s)}\,ds)$ for the cubic drag term. To do so, let us provide some properties of the capillary term and rewrite the terms coming from the drag quantities.

2.2.1. Some properties of the capillary term. Using the mass equation, the capillary term in the entropy estimates reads

$$r\int_\Omega \sqrt{K(\rho)}\,\Delta\Big(\int_0^\rho \sqrt{K(s)}\,ds\Big)\,\mathrm{div}(\rho w) = \frac{r}{2}\frac{d}{dt}\int_\Omega \Big|\nabla\int_0^\rho \sqrt{K(s)}\,ds\Big|^2 + 2\kappa r\int_\Omega \sqrt{K(\rho)}\,\Delta\Big(\int_0^\rho \sqrt{K(s)}\,ds\Big)\Delta\mu(\rho) = I_1 + I_2. \tag{2.8}$$

In fact, we write I_1 as

$$\frac{r}{2}\frac{d}{dt}\int_\Omega \Big|\nabla\int_0^\rho \sqrt{K(s)}\,ds\Big|^2 = \frac{r}{2}\frac{d}{dt}\int_\Omega \rho|\nabla s(\rho)|^2.$$

By (1.27), we have

$$\begin{aligned} I_2 &= 2\kappa r\int_\Omega \sqrt{K(\rho)}\,\Delta\Big(\int_0^\rho \sqrt{K(s)}\,ds\Big)\Delta\mu(\rho)\\ &= -2\kappa r\int_\Omega \rho\nabla\Big(\sqrt{K(\rho)}\,\Delta\Big(\int_0^\rho \sqrt{K(s)}\,ds\Big)\Big)\cdot\nabla s(\rho)\\ &= 2\kappa r\int_\Omega \big(2\mu(\rho)|2\nabla^2 s(\rho)|^2 + \lambda(\rho)|2\Delta s(\rho)|^2\big). \end{aligned}\tag{2.9}$$

Control of norms using I_2. Let us first recall that since

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)) > -2\mu(\rho)/3,$$

there exists $\eta > 0$ such that

$$2\int_0^T\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2 + \int_0^T\int_\Omega \lambda(\rho)|\Delta s(\rho)|^2 \geq \eta\Big[2\int_0^T\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2 + \frac{1}{3}\int_0^T\int_\Omega \mu(\rho)|\Delta s(\rho)|^2\Big].$$

As the second term on the right-hand side is positive, a lower bound on the quantity

$$\int_0^T\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2 \tag{2.10}$$

will provide the same lower bound on I_2.

Let us now specify the norms which are controlled by (2.10). To do so, we need to rely on the following lemma on the density. In this lemma, we prove a more general entropy dissipation inequality than the one introduced by Jüngel [30] and more general than those by Jüngel–Matthes [31].

Lemma 2.1. *Let $\mu'(\rho)\rho < k\mu(\rho)$ for $2/3 < k < 4$ and*

$$s(\rho) = \int_0^\rho \frac{\mu'(s)}{s}\, ds, \qquad Z(\rho) = \int_0^\rho \frac{\sqrt{\mu(s)}}{s}\mu'(s)\, ds, \qquad Z_1(\rho) = \int_0^\rho \frac{\mu'(s)}{\mu(s)^{1/4}s^{1/2}}\, ds.$$

(i) *Assume $\rho > 0$ and $\rho \in L^2(0,T;H^2(\Omega))$. Then there exists $\varepsilon(k) > 0$ such that*

$$\int_0^T \int_\Omega |\nabla^2 Z(\rho)|^2 + \varepsilon(k) \int_0^T \int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4 \le \frac{C}{\varepsilon(k)} \int_0^T \int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2,$$

where C is a universal positive constant.

(ii) *Consider a sequence of smooth densities $\rho_n > 0$ such that $Z(\rho_n)$ and $Z_1(\rho_n)$ converge strongly in $L^1((0,T)\times\Omega)$ respectively to $Z(\rho)$ and $Z_1(\rho)$ and $\sqrt{\mu(\rho_n)}\,\nabla^2 s(\rho_n)$ is uniformly bounded in $L^2((0,T)\times\Omega)$. Then*

$$\int_0^T \int_\Omega |\nabla^2 Z(\rho)|^2 + \varepsilon(k) \int_0^T \int_\Omega |\nabla Z_1(\rho)|^4 \le C < \infty.$$

Remark 2.1. The case of $Z = 2\sqrt{\rho}$ was proved in [30], which is critical to deriving a uniform bound on the approximate velocity in $L^2(0,T;L^2(\Omega))$ in [50, 51]. The above lemma will play a similar role in this paper.

Proof of Lemma 2.1. Let us first prove (i). Noting that $Z'(\rho) = \frac{\sqrt{\mu(\rho)}}{\rho}\mu'(\rho)$, we get

$$\begin{aligned}
\sqrt{\mu(\rho)}\,\nabla^2 s(\rho) &= \sqrt{\mu(\rho)}\,\nabla\left(\frac{\nabla\mu(\rho)}{\rho}\right)\\
&= \sqrt{\mu(\rho)}\,\nabla\left(\frac{1}{\sqrt{\mu(\rho)}}\nabla Z(\rho)\right)\\
&= \nabla^2 Z(\rho) - \frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}} \otimes \nabla\sqrt{\mu(\rho)}\\
&= \nabla^2 Z(\rho) - \frac{\rho\nabla Z(\rho)\otimes\nabla Z(\rho)}{2\mu(\rho)^{3/2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2 &= \int_\Omega |\nabla^2 Z(\rho)|^2 + \frac{1}{4}\int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4\\
&\quad - \int_\Omega \frac{\rho}{\mu(\rho)^{3/2}}\nabla^2 Z(\rho) : (\nabla Z(\rho)\otimes\nabla Z(\rho)). \qquad (2.11)
\end{aligned}$$

By integration by parts, the cross product term reads

$$
\begin{aligned}
&-\int_\Omega \frac{\rho}{\mu(\rho)^{3/2}}\nabla^2 Z(\rho) : (\nabla Z(\rho)\otimes\nabla Z(\rho))\\
&\qquad = -\int_\Omega \frac{\rho\sqrt{\mu(\rho)}}{\mu(\rho)}\nabla^2 Z(\rho) : \left(\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}}\otimes\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}}\right)\\
&\qquad = \int_\Omega \frac{\rho}{\mu(\rho)}\sqrt{\mu(\rho)}\,\nabla Z(\rho)\cdot \operatorname{div}\left(\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}}\otimes\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}}\right)\\
&\qquad\quad + \int_\Omega \nabla\left(\frac{\rho}{\sqrt{\mu(\rho)}}\right)\otimes\nabla Z(\rho) : \frac{\nabla Z(\rho)\otimes\nabla Z(\rho)}{\mu(\rho)}\\
&\qquad = I_1 + I_2.
\end{aligned}
\tag{2.12}
$$

We are able to control I_1 directly:

$$
\begin{aligned}
|I_1| &\le \varepsilon\int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4 + \frac{C}{\varepsilon}\int_\Omega \mu(\rho)\left|\nabla\left(\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}}\right)\right|^2\\
&\le \varepsilon\int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4 + \frac{C}{\varepsilon}\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2,
\end{aligned}
\tag{2.13}
$$

where C is a universal positive constant. We also calculate

$$
\begin{aligned}
I_2 &= \int_\Omega \nabla\left(\frac{\rho}{\sqrt{\mu(\rho)}}\right)\otimes\nabla Z(\rho) : \frac{\nabla Z(\rho)\otimes\nabla Z(\rho)}{\mu(\rho)}\\
&= \int_\Omega \frac{\nabla\rho\otimes\nabla Z(\rho)}{\mu(\rho)^{3/2}} : (\nabla Z(\rho)\otimes\nabla Z(\rho))\\
&\quad - \int_\Omega \frac{\rho}{\mu(\rho)^2}\nabla\sqrt{\mu(\rho)}\otimes\nabla Z(\rho) : (\nabla Z(\rho)\otimes\nabla Z(\rho))\\
&= \int_\Omega \frac{\rho}{\mu(\rho)^2\mu(\rho)'}|\nabla Z(\rho)|^4 - \frac{1}{2}\int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4.
\end{aligned}
\tag{2.14}
$$

Relying on (2.11)–(2.14), we have

$$
\begin{aligned}
\int_\Omega |\nabla^2 Z(\rho)|^2 + \int_\Omega \frac{\rho}{\mu(\rho)^2\mu'(\rho)}|\nabla Z(\rho)|^4 - \left(\frac{1}{4}+\varepsilon\right)\int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4\\
\le \frac{C}{\varepsilon}\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2.
\end{aligned}
$$

Since $k_1\mu'(s)s \le \mu(s)$, we have

$$
\frac{s}{\mu(s)^2\mu'(s)} - \left(\frac{1}{4}+\varepsilon\right)\frac{s^2}{\mu(s)^3} \ge \left(k_1 - \frac{1}{4} - \varepsilon\right)\frac{s^2}{\mu(s)^3} > \varepsilon\frac{s^2}{\mu(s)^3},
$$

where we choose $k_1 > 1/4$. This implies

$$\int_\Omega |\nabla^2 Z(\rho)|^2 + \varepsilon \int_\Omega \frac{\rho^2}{\mu(\rho)^3} |\nabla Z(\rho)|^4 \le \frac{C}{\varepsilon} \int_\Omega \mu(\rho) |\nabla^2 s(\rho)|^2.$$

This ends the proof of (i). Concerning (ii), it suffices to pass to the limit in the inequality proved previously using the lower semicontinuity on the left-hand side. ■

2.2.2. Drag terms control. We have to discuss three kinds of drag terms: linear, quadratic and cubic.

(a) *Linear drag term.* As in previous works [8, 51, 52], we need to choose a linear drag with constant coefficients,

$$r_0 \int_0^t \int_\Omega (w - 2\kappa \nabla s(\rho)) \cdot w = r_0 \int_0^t \int_\Omega |w - 2\kappa \nabla s(\rho)|^2 + r_0 \int_0^t \int_\Omega (w - 2\kappa \nabla s(\rho)) \cdot (2\kappa \nabla s(\rho)). \tag{2.15}$$

The second term on the right side of (2.15) reads

$$\begin{aligned} r_0 \int_0^t \int_\Omega (w - 2\kappa \nabla s(\rho)) \cdot (2\kappa \nabla s(\rho)) &= r_0 \int_0^t \int_\Omega \rho (w - 2\kappa \nabla s(\rho)) \cdot \frac{2\kappa \nabla s(\rho)}{\rho} \\ &= r_0 \int_0^t \int_\Omega \rho (w - 2\kappa \nabla s(\rho)) \cdot 2\kappa \nabla g(\rho) \\ &= 2\kappa r_0 \int_0^t \int_\Omega \rho_t g(\rho), \end{aligned}$$

where $g'(\rho) = \frac{s'(\rho)}{\rho} = \frac{\mu'(\rho)}{\rho^2}$ and $g(\rho) = \int_1^\rho \frac{\mu'(r)}{r^2}\, dr$. Letting

$$G(\rho) = \int_1^\rho \int_1^r \frac{\mu'(\zeta)}{\zeta^2}\, d\zeta\, dr,$$

we get

$$r_0 \int_\Omega \rho_t g(\rho) = r_0 \frac{\partial}{\partial_t} \int_\Omega G(\rho),$$

which implies

$$r_0 \int_0^t \int_\Omega \rho_t g(\rho) = r_0 \int_\Omega G(\rho).$$

Meanwhile, since $\lim_{\zeta \to 0} \mu'(\zeta) = \varepsilon_1 > 0$, for any small $\epsilon > 0$ and any $|\zeta| < \epsilon$, we have $\mu'(\zeta) \ge \varepsilon_1/2$. Thus,

$$G(\rho) \ge \frac{\varepsilon_1}{2} \int_1^\rho \left(1 - \frac{1}{r}\right) dr = \frac{\varepsilon_1}{2} (\rho - 1 - \ln \rho) \ge -\frac{\varepsilon_1}{4} (\ln \rho)_-$$

for any $\rho \leq \epsilon$. Similarly, we can show that

$$G(\rho) \leq 4\varepsilon_1(\ln \rho)_+$$

for any $\rho \leq \epsilon$. For a given $\epsilon_0 > 0$, if $\rho \geq \epsilon_0$, then

$$0 \leq G(\rho) \leq C \int_1^\rho \int_1^r \mu'(\zeta)\, d\zeta\, dr \leq C\mu(\rho)\rho.$$

(b) *Quadratic drag term.* We use the same argument as in [13] to handle this term. The quadratic drag term gives

$$\begin{aligned} r_1 \int_0^t \int_\Omega \rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho)) \cdot w \\ &= r_1 \int_0^t \int_\Omega \rho|w - 2\kappa\nabla s(\rho)|^3 \\ &\quad + r_1 \int_0^t \int_\Omega \rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho)) \cdot (2\kappa\nabla s(\rho)). \end{aligned} \tag{2.16}$$

The second drag term on the right–hand side can be controlled as follows:

$$\begin{aligned} r_1 \left| \int_0^t \int_\Omega \rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho)) \cdot (2\kappa\nabla s(\rho)) \right| \\ &\leq 2\kappa r_1 \int_0^t \int_\Omega \mu(\rho)|u|\,|\mathbb{D}u| \\ &\leq \kappa \int_0^t \int_\Omega \mu(\rho)|\mathbb{D}u|^2 + \kappa r_1^2 \int_0^t \int_\Omega \mu(\rho)|u|^2, \end{aligned} \tag{2.17}$$

and

$$\|\sqrt{\mu(\rho)}\,|u|\|_{L^2(0,T;L^2(\Omega))} \leq C\,\|\rho^{1/3}|u|\|_{L^3(0,T;L^3(\Omega))} \left\| \frac{\sqrt{\mu(\rho)}}{\rho^{1/3}} \right\|_{L^6(0,T;L^6(\Omega))}.$$

Note that

$$\begin{aligned} \int_0^t \int_\Omega \frac{\mu(\rho)^3}{\rho^2} &= \int_0^t \int_{0\leq\rho\leq 1} \frac{\mu(\rho)^3}{\rho^2} + \int_0^t \int_{\rho\geq 1} \frac{\mu(\rho)^3}{\rho^2} \\ &\leq C \int_0^t \int_{0\leq\rho\leq 1} \mu(\rho)(\mu'(\rho))^2 + \int_0^t \int_{\rho\geq 1} \frac{\mu(\rho)^3}{\rho^2} \\ &\leq C + \int_0^t \int_{\rho\geq 1} \frac{\mu(\rho)^3}{\rho^2}. \end{aligned} \tag{2.18}$$

From (1.9), for any $\rho \geq 1$, we have

$$c'\rho^{\alpha_1} \leq \mu(\rho) \leq c\rho^{\alpha_2},$$

where $2/3 < \alpha_1 \leq \alpha_2 < 4$. This yields

$$\int_0^t \int_{\rho\geq 1} \frac{\mu(\rho)^3}{\rho^2} \leq c \int_0^t \int_{\rho\geq 1} \rho^{3\alpha_2-2} \leq c \int_0^t \int_\Omega \rho^{10} \tag{2.19}$$

for any time $t > 0$.

(c) *Cubic drag term.* The non-linear cubic drag term gives

$$\begin{aligned} r_2 \int_0^t \int_\Omega \frac{\rho}{\mu'(\rho)} |w - 2\kappa\nabla s(\rho)|^2 (w - 2\kappa\nabla s(\rho)) \cdot w \\ = r_2 \int_0^t \int_\Omega \frac{\rho}{\mu'(\rho)} |w - 2\kappa\nabla s(\rho)|^4 \\ + r_2 \int_0^t \int_\Omega \frac{\rho}{\mu'(\rho)} |w - 2\kappa\nabla s(\rho)|^2 (w - 2\kappa\nabla s(\rho)) \cdot (2\kappa\nabla s(\rho)). \end{aligned} \tag{2.20}$$

The novelty now is to show that we control the second drag term of the right-hand side using the Korteweg-type information on the left-hand side,

$$\begin{aligned} r_2 \int_0^t \int_\Omega \frac{\rho}{\mu'(\rho)} |w - 2\kappa\nabla s(\rho)|^2 (w - 2\kappa\nabla s(\rho)) \cdot (2\kappa\nabla s(\rho)) \\ \leq r_2 \Big(\frac{3}{4} \int_0^t \int_\Omega \frac{\rho}{\mu'(\rho)} |w - 2\kappa\nabla s(\rho)|^4 + \frac{(2\kappa)^4}{4} \int_0^t \int_\Omega \frac{\rho}{\mu'(\rho)} |\nabla s(\rho)|^4 \Big). \end{aligned} \tag{2.21}$$

Note that the first term on the right-hand side may be absorbed by the first term in (2.20). Let us now prove that if r_2 small enough, the second term on the right-hand side may be absorbed by the term coming from the capillary quantity in the energy. From Lemma 2.1, we have

$$\int_0^t \int_\Omega \frac{\rho^2}{\mu(\rho)^3} |\nabla Z(\rho)|^4 = \int_0^t \int_\Omega \frac{1}{\mu(\rho)\rho^2} |\nabla \mu(\rho)|^4.$$

It remains to check that

$$\int_0^t \int_\Omega \frac{\rho}{\mu'(\rho)} |\nabla s(\rho)|^4 = \int_0^t \int_\Omega \frac{1}{\mu'(\rho)\rho^3} |\nabla \mu(\rho)|^4 \leq C \int_0^t \int_\Omega \frac{1}{\mu(\rho)\rho^2} |\nabla \mu(\rho)|^4.$$

This concludes the proof provided r_2 is small enough compared to r.

2.2.3. The κ-entropy estimate. Using the previous calculations, assuming r_2 small enough compared to r, and denoting

$$\begin{aligned} E[\rho, u + 2\kappa\nabla s(\rho), \nabla s(\rho)] \\ = \int_\Omega \Big(\rho \Big(\frac{|u + 2\kappa\nabla s(\rho)|^2}{2} + (1-\kappa)\kappa \frac{|\nabla s(\rho)|^2}{2} \Big) + \frac{\rho^\gamma}{\gamma - 1} + \frac{\delta\rho^{10}}{9} + G(\rho) \Big), \end{aligned}$$

we get the following κ-entropy estimate:

$$
\begin{aligned}
&E[\rho, u+2\kappa\nabla s(\rho), \nabla s(\rho)](t) + r_0\int_0^t\int_\Omega |u|^2 \\
&+\frac{r}{2}\int_\Omega\Big|\nabla\int_0^\rho \sqrt{K(s)}\,ds\Big|^2 dx + 2(1-\kappa)\int_0^t\int_\Omega \mu(\rho)|\mathbb{D}u|^2 + 20\kappa\delta\int_0^t\int_\Omega \mu'(\rho)\rho^8|\nabla\rho|^2 \\
&+2(1-\kappa)\int_0^t\int_\Omega (\mu'(\rho)\rho-\mu(\rho))(\operatorname{div} u)^2 + 2\kappa\int_0^t\int_\Omega \mu(\rho)|A(u+2\kappa\nabla s(\rho))|^2 \\
&+2\kappa\int_0^t\int_\Omega \frac{\mu'(\rho)p'(\rho)}{\rho}|\nabla\rho|^2 + r_1\int_0^t\int_\Omega \rho|u|^3 + \frac{r_2}{4}\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|u|^4 \\
&+\kappa r\int_0^t\int_\Omega \mu(\rho)|2\nabla^2 s(\rho)|^2 + \frac{1}{2}\kappa r\int_0^t\int_\Omega \lambda(\rho)|2\Delta s(\rho)|^2 \\
&\le \int_\Omega\Big(\rho_0\Big(\frac{|w_0|^2}{2}+(1-\kappa)\kappa\frac{|v_0|^2}{2}\Big)+\frac{\rho_0^\gamma}{\gamma-1}+\frac{\delta\rho_0^{10}}{9}+\frac{r}{2}\Big|\nabla\int_0^{\rho_0}\sqrt{K(s)}\,ds\Big|^2+G(\rho_0)\Big)dx \\
&+C\frac{r_1}{\delta}\int_\Omega E[\rho, u+2\kappa\nabla s(\rho), \nabla s(\rho)]. \qquad (2.22)
\end{aligned}
$$

It now suffices to remark that

$$
\begin{aligned}
&\int_0^t\int_\Omega \mu(\rho)|\mathbb{D}u|^2 + \int_0^t\int_\Omega (\mu'(\rho)\rho-\rho)|\operatorname{div} u|^2 \\
&\quad = \int_0^t\int_\Omega \mu(\rho)\big|\mathbb{D}u - \tfrac{1}{3}\operatorname{div} u\ \mathrm{Id}\big|^2 + \int_0^t\int_\Omega \big(\mu'(\rho)\rho-\mu(\rho)+\tfrac{1}{3}\mu(\rho)\big)|\operatorname{div} u|^2.
\end{aligned}
$$

Noting that $\alpha_1 > 2/3$, there exists $\varepsilon > 0$ such that

$$\mu'(\rho)\rho - \tfrac{2}{3}\mu(\rho) > \varepsilon\mu(\rho).$$

Such information and the control of $\sqrt{\mu(\rho)}\,|A(u)+2\kappa\nabla s(\rho)|$ in $L^2(0,T;L^2(\Omega))$ allow us, using the Grönwall Lemma and the constraints on the parameters, to get the uniform estimates (1.23)–(1.25).

Now we can show (1.26). First, we have

$$\nabla\mu(\rho) = \frac{\nabla\mu(\rho)}{\sqrt{\rho}}\sqrt{\rho} \in L^\infty(0,T;L^1(\Omega)),$$

due to mass conservation and uniform control on $\nabla\mu(\rho)/\sqrt{\rho}$ given in (1.23). Let us now write the equation satisfied by $\mu(\rho)$:

$$\partial_t\mu(\rho) + \operatorname{div}(\mu(\rho)u) + \frac{\lambda(\rho)}{2}\operatorname{div} u = 0.$$

Recalling that $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$ and the hypothesis on $\mu(\rho)$, we get

$$\frac{d}{dt}\int_\Omega \mu(\rho) \le C\Big(\int_\Omega |\lambda(\rho)|\,|\operatorname{div} u|^2 + \int_\Omega \mu(\rho)\Big),$$

and therefore

$$\mu(\rho) \in L^\infty(0,T;L^1(\Omega))$$

if $\mu(\rho_0) \in L^1(\Omega)$ due to the fact that $\sqrt{|\lambda(\rho)|}\,\operatorname{div} u \in L^2(0,T;L^2(\Omega))$.

Now, we observe that $\mu(\rho)/\sqrt{\rho}$ is smaller than 1 for $\rho \le 1$ because $\alpha_1 > 2/3$, and smaller than $\mu(\rho)$ for $\rho > 1$. Hence

$$\mu(\rho)/\sqrt{\rho} \in L^\infty(0,T;L^1(\Omega)).$$

Meanwhile, thanks to (1.9), we have

$$|\nabla(\mu(\rho)/\sqrt{\rho})| \le \left|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right| + \frac{\mu(\rho)}{2\rho\sqrt{\rho}}|\nabla\rho| \le \left(1+\frac{1}{\alpha_1}\right)\left|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right|.$$

By (1.23), $\nabla(\mu(\rho)/\sqrt{\rho})$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and finally $\mu(\rho)/\sqrt{\rho}$ is bounded in $L^\infty(0,T;L^6(\Omega))$. Thus,

$$\mu(\rho)u = \frac{\mu(\rho)}{\sqrt{\rho}}\sqrt{\rho}\,u$$

is uniformly bounded in $L^\infty(0,T;L^{3/2}(\Omega))$. Let us come back to the equation satisfied by $\mu(\rho)$ which reads

$$\partial_t\mu(\rho) + \operatorname{div}(\mu(\rho)u) + \frac{\lambda(\rho)}{2}\operatorname{div} u = 0.$$

Recalling that $\lambda(\rho)\operatorname{div} u \in L^\infty(0,T;L^1(\Omega))$, we get the conclusion on $\partial_t\mu(\rho)$.

Let us now prove that

$$Z(\rho) = \int_0^{\rho_n} \frac{\sqrt{\mu(s)}\,\mu'(s)}{s}\,ds \in L^{1+}((0,T)\times\Omega) \quad \text{uniformly.}$$

Note first that

$$0 \le \frac{\sqrt{\mu(s)}\,\mu'(s)}{s} \le \alpha_2\frac{\mu(s)^{3/2}}{s^2} \le c_2\alpha_2\left(s^{3\alpha_1/2-2}1_{s\le 1} + \frac{\mu(s)^{3/2-}}{s^{2-}}1_{s\ge 1}\right).$$

There exists $\varepsilon > 0$ such that $\alpha_1 > 2/3 + \varepsilon$, thus

$$0 \le \frac{\sqrt{\mu(s)}\,\mu'(s)}{s} \le c_2\alpha_2\left(s^{\varepsilon-1}1_{s\le 1} + \frac{\mu(s)^{3/2-}}{s^{2-}}1_{s\ge 1}\right).$$

Noting that $\mu'(s) > 0$ for $s > 0$ and using the definition of $Z(\rho)$, we get

$$0 \le Z(\rho) \le C(\rho^\varepsilon + \mu(\rho)^{3/2-})$$

with C independent of n. Thus $Z(\rho) \in L^\infty(0,T;L^{1+}(\Omega))$ uniformly with respect to n. Bounding $Z_1(\rho)$ follows similar lines.

2.3. Compactness lemmas

In this subsection, we provide general compactness lemmas which will be used several times in this paper.

Some uniform compactness

Lemma 2.2. *Assume we have a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ satisfying the estimates in Theorem 1.2, uniformly with respect to n. Then there exists a function $\rho \in L^\infty(0,T;L^\gamma(\Omega))$ such that, up to a subsequence,*

$$\mu(\rho_n) \to \mu(\rho) \quad \textit{in } C([0,T];L^{3/2}(\Omega) \textit{ weak}),$$

and

$$\rho_n \to \rho \quad \textit{a.e. in } (0,T)\times\Omega.$$

Moreover

$$\rho_n \to \rho \quad \textit{in } L^{(4\gamma/3)+}((0,T)\times\Omega),$$

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}}\,\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \rightharpoonup \sqrt{\frac{P'(\rho)\rho}{\mu'(\rho)}}\,\nabla\Big(\int_0^{\rho}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \quad \textit{in } L^1((0,T)\times\Omega)$$

and

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}}\,\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \in L^{1+}((0,T)\times\Omega).$$

If $\delta_n > 0$ is such that $\delta_n \to \delta \geq 0$, then

$$\delta_n\rho_n^{10} \to \delta\rho^{10} \quad \textit{in } L^{4/3}((0,T)\times\Omega).$$

Proof. From the estimate on $\mu(\rho_n)$ and the Aubin–Lions lemma, up to a subsequence, we have

$$\mu(\rho_n) \to \mu(\rho) \quad \text{in } C([0,T];L^{3/2}(\Omega) \text{ weak}),$$

and therefore using $\mu'(s) > 0$ on $(0,\infty)$ with $\mu(0) = 0$, we get the conclusion on ρ_n. Let us now recall that

$$\frac{\alpha_1}{\rho_n} \leq \frac{\mu'(\rho_n)}{\mu(\rho_n)} \leq \frac{\alpha_2}{\rho_n} \tag{2.23}$$

and therefore

$$c_1\rho_n^{\alpha_2} \leq \mu(\rho_n) \leq c_2\rho_n^{\alpha_1} \qquad \text{for } \rho_n \leq 1,$$

and

$$c_1\rho_n^{\alpha_1} \leq \mu(\rho_n) \leq c_2\rho_n^{\alpha_2} \qquad \text{for } \rho_n \geq 1,$$

with c_1 and c_2 independent of n. Note that

$$\sqrt{\frac{p'(\rho_n)\mu'(\rho_n)}{\rho_n}}\,\nabla\rho_n \in L^\infty(0,T;L^2(\Omega)) \quad \text{uniformly.} \tag{2.24}$$

Let us prove that there exists ε such that

$$I_0 = \int_0^T \int_\Omega \rho_n^{4\gamma/3+\varepsilon} < C$$

with C independent of n and the parameters. We first remark that it suffices to look at it when $\rho_n \geq 1$ and to observe that there exists ε such that $\varepsilon \leq (\gamma - 1)/3$. Taking such parameters we have

$$\int_0^T \int_\Omega \rho_n^{4\gamma/3+\varepsilon} 1_{\{\rho \geq 1\}} \leq \int_0^T \int_\Omega \rho_n^{2\gamma/3+\gamma-1/3} 1_{\{\rho\geq 1\}} \leq \int_0^T \int_\Omega \rho_n^{2\gamma/3+\gamma+\alpha_1-1} 1_{\{\rho \geq 1\}},$$

recalling that $\alpha_1 > 2/3$. Following [37], it remains to prove that

$$I_1 = \int_0^T \int_\Omega \rho_n^{[5\gamma+3(\alpha_1-1)]/3} 1_{\{\rho\geq 1\}} < \infty$$

uniformly. Denoting

$$I_2 = \int_0^T \int_\Omega \rho^{[5\gamma+3(\alpha_2-1)]/3} 1_{\{\rho \leq 1\}}$$

and using the bounds on $\mu(\rho_n)$ in terms of power functions in ρ, which are different for $\rho_n \geq 1$ and for $\rho_n \leq 1$, we can write

$$\begin{aligned} I_1 &\leq I_1 + I_2 \\ &\leq C_a \int_0^T \int_\Omega \rho_n^{2\gamma/3} P'(\rho_n)\mu(\rho_n) \leq C_a \int_0^T \|\rho_n^\gamma\|_{L^1(\Omega)}^{2/3} \|P'(\rho_n)\mu(\rho_n)\|_{L^3(\Omega)} \end{aligned}$$

where C does not depend on n. Using the Poincaré–Wirtinger inequality, one obtains

$$\begin{aligned} \|P'(\rho_n)\mu(\rho_n)\|_{L^3(\Omega)} &= \|\sqrt{P'(\rho_n)\mu(\rho_n)}\,\|_{L^6(\Omega)}^2 \\ &\leq \|\sqrt{P'(\rho_n)\mu(\rho_n)}\,\|_{L^1(\Omega)} + \|\nabla[\sqrt{P'(\rho_n)\mu(\rho_n)}\,]\|_{L^2(\Omega)}^2. \end{aligned}$$

Let us now check that the two terms are uniformly bounded in time. First we calculate

$$\nabla[\sqrt{P'(\rho_n)\mu(\rho_n)}\,] = \frac{P''(\rho_n)\mu(\rho_n) + P'(\rho_n)\mu'(\rho_n)}{\sqrt{P'(\rho_n)\mu(\rho_n)}} \nabla\rho_n.$$

Using (2.23), we can check that

$$\frac{P''(\rho_n)\mu(\rho_n) + P'(\rho_n)\mu'(\rho_n)}{\sqrt{P'(\rho_n)\mu(\rho_n)}} \leq \sqrt{\frac{P'(\rho_n)\mu'(\rho_n)}{\rho_n}}.$$

Therefore, using (2.24) we get, uniformly with respect to n,

$$\sup_{t\in[0,T]} \|\nabla[\sqrt{P'(\rho_n)\mu(\rho_n)}\,]\|_{L^2(\Omega)}^2 < \infty.$$

Let us now check that uniformly with respect to n,

$$\sup_{t\in[0,T]} \|\sqrt{P'(\rho_n)\mu(\rho_n)}\|_{L^1(\Omega)} < \infty. \tag{2.25}$$

Using the bounds on $\mu(\rho_n)$, we have

$$\int_\Omega \sqrt{P'(\rho_n)\mu(\rho_n)} \le C\int_\Omega [\rho_n^{(\gamma-1+\alpha_1)/2} 1_{\rho_n\le 1} + \rho_n^{(\gamma-1+\alpha_2)/2} 1_{\rho_n\ge 1}]$$

with C independent of n. Recalling that $\alpha_1 \ge 2/3$ and $\alpha_2 < 4$, we can check that

$$\int_\Omega \sqrt{P'(\rho_n)\mu(\rho_n)} \le C\int_\Omega [\rho_n^{\gamma/3} + \rho_n^{\gamma/2}\rho_n^{3/2}],$$

and therefore using $\rho_n^\gamma \in L^\infty(0,T;L^1(\Omega))$ and $\rho_n \in L^\infty(0,T;L^{10}(\Omega))$, we get (2.25). This ends the proof of the convergence of ρ_n to ρ in $L^{(4\gamma/3)+}((0,T)\times\Omega)$.

Let us now focus on the convergence of

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}}\,\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big). \tag{2.26}$$

First let us recall that

$$\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \in L^\infty(0,T;L^2(\Omega)) \quad \text{uniformly.}$$

Let us now prove that

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}} \in L^{2+}((0,T)\times\Omega). \tag{2.27}$$

Recall first that $\alpha_1 > 2/3$. We just have to consider $\rho_n \ge 1$. We write

$$\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)} 1_{\rho_n\ge 1} \le C\rho_n^{\gamma-\alpha_1+1} 1_{\rho_n\ge 1} \le C\rho_n^{\gamma+1/3} 1_{\rho_n\ge 1} \le C\rho_n^{4\gamma/3} 1_{\rho_n\ge 1}.$$

We can use the fact that $\rho_n^{(4\gamma/3)+} \in L^1((0,T)\times\Omega)$ uniformly to deduce (2.27). Thanks to

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}} \to \sqrt{\frac{P'(\rho)\rho}{\mu'(\rho)}} \quad \text{in } L^2((0,T)\times\Omega)$$

and

$$\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \to \nabla\Big(\int_0^{\rho}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \quad \text{weakly in } L^2((0,T)\times\Omega),$$

we have the weak convergence of (2.26) in $L^1((0,T)\times\Omega)$.

We now investigate limits of u independent of the parameters. We need to differentiate the case with hyperviscosity $\varepsilon_2 > 0$ from the case without it. In the case with hyperviscosity, the estimate depends on ε_1 because of the drag force r_1, while the estimate in the case $\varepsilon_2 = 0$ is independent of all the other parameters. That is why we will consider the limit as $\varepsilon_2 \to 0$ first.

Lemma 2.3. *Assume that $\varepsilon_1 > 0$ is fixed. Then there exists a constant $C > 0$ depending on ε_1 and C_{in}, but independent of all the other parameters (as long as they are bounded), such that for any initial values $(\rho_0, \sqrt{\rho_0}\, u_0)$ satisfying (1.29) for $C_{\text{in}} > 0$ we have*

$$\|\partial_t(\rho u)\|_{L^{1+}(0,T;W^{-s,2}(\Omega))} \le C, \qquad \|\nabla(\rho u)\|_{L^2(0,T;L^1(\Omega))} \le C.$$

Assume now that $\varepsilon_2 = 0$. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function, positive for $\rho > 0$, such that

$$\Phi(\rho) + |\Phi'(\rho)| \le \begin{cases} Ce^{-1/\rho} & \text{for } \rho \le 1, \\ Ce^{-\rho} & \text{for } \rho \ge 2. \end{cases}$$

Assume that the initial values $(\rho_0, \sqrt{\rho_0}\, u_0)$ satisfy (1.29) for a fixed $C_{\text{in}} > 0$. Then there exists a constant $C > 0$ independent of $\varepsilon_1, r_0, r_1, r_2, \delta$ (as long as they are bounded) such that

$$\|\partial_t[\Phi(\rho)u]\|_{L^{1+}(0,T;W^{-2,1}(\Omega))} \le C,$$
$$\|\nabla[\Phi(\rho)u]\|_{L^2(0,T;L^1(\Omega))} \le C.$$

Proof. We split the proof into two cases.

Case 1: $\varepsilon_1 > 0$. From the equation on ρu and the *a priori* estimates, we find directly that

$$\begin{aligned}\|\partial_t(\rho u)\|_{L^{1+}(0,T;W^{-s,2}(\Omega))} &\le C + \left(\frac{r_1}{\varepsilon_1}\right)^{1/4} \|\rho\|^{1/4}_{L^1(0,T)\times\Omega)} \left(r_1 \int_0^T \int_\Omega \frac{\rho}{\mu'(\rho)}|u|^4\right)^{3/4} \\ &\le C(1 + 1/\varepsilon_1).\end{aligned}$$

We have $\mu(\rho) \ge \varepsilon_1 \rho$, and from (1.23) we have the *a priori* estimate

$$\|\nabla\sqrt{\rho}\,\|^2_{L^\infty(0,T;L^2(\Omega))} \le C/\varepsilon_1.$$

Hence

$$\begin{aligned}\|\nabla(\rho u)\|_{L^2(0,T;L^1(\Omega))} &\le \left\|\frac{\rho}{\sqrt{\mu(\rho)}}\right\|_{L^\infty(0,T;L^2(\Omega))} \|\sqrt{\mu(\rho)}\,\nabla u\|_{L^2(0,T;L^2(\Omega)))} \\ &\quad + 2\|\nabla\sqrt{\rho}\,\|_{L^\infty(0,T;L^2(\Omega))}\|\sqrt{\rho}\,u\|_{L^\infty(0,T;L^2(\Omega))} \\ &\le C.\end{aligned}$$

Case 2: $\varepsilon_2 = 0$. Multiplying the equation for ρu by $\Phi(\rho)/\rho$ we get, just as for the renormalization,

$$\|\partial_t[\Phi(\rho)u]\|_{L^{1+}(0,T;W^{-2,1}(\Omega))} \le C.$$

Note that

$$\begin{aligned}\|\nabla[\Phi(\rho)u]\|_{L^2(0,T;L^1(\Omega))} &\le \left\|\frac{\Phi(\rho)}{\sqrt{\mu(\rho)}}\right\|_{L^\infty} \left\|\sqrt{\mu(\rho)}\,\nabla u\right\|_{L^2(L^2)} \\ + 2\left\|\frac{\Phi'(\rho)}{\mu'(\rho)}\right\|_{L^\infty((0,T)\times\Omega)} &\|\mu'(\rho)\nabla\sqrt{\rho}\,\|_{L^\infty(0,T;L^2(\Omega))}\|\sqrt{\rho}\,u\|_{L^\infty(0,T;L^2(\Omega))} \le C. \quad \blacksquare\end{aligned}$$

Lemma 2.4. *Assume that either $\varepsilon_{2,n} = 0$, or $\varepsilon_{1,n} = \varepsilon_1 > 0$. Let $(\rho_n, \sqrt{\rho_n}\, u_n)$ be a sequence of solutions for a family of bounded parameters with uniformly bounded initial values satisfying (1.29) with a fixed C_{in}. Assume that there exists $\alpha > 0$ and a smooth function $h : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ such that ρ_n^α is uniformly bounded in $L^p((0,T) \times \Omega)$ and $h(\rho_n, u_n)$ is uniformly bounded in $L^q((0,T) \times \Omega)$, with*

$$\frac{1}{p} + \frac{1}{q} < 1.$$

Then, up to a subsequence, ρ_n converges to a function ρ strongly in L^1 and $\sqrt{\rho_n}\, u_n$ converges weakly to a function q in L^2. Define $u = q/\sqrt{\rho}$ whenever $\rho \neq 0$, and $u = 0$ on the vacuum where $\rho = 0$. Then $\rho_n^\alpha h(\rho_n, u_n)$ converges strongly in L^1 to $\rho^\alpha h(\rho, u)$.

Proof. Thanks to the uniform bound on the kinetic energy $\int \rho_n |u_n|^2$, and to Lemma 2.2, up to a subsequence, ρ_n converges strongly in $L^1((0,T) \times \Omega)$ to a function ρ, and $\sqrt{\rho_n}\, u_n$ converges weakly in $L^2((0,T) \times \Omega)$ to a function q.

We want to show that, up to a subsequence, $u_n 1_{\{\rho>0\}}$ converges almost everywhere to $u 1_{\{\rho>0\}}$. We consider two cases. First, if $\varepsilon_{1,n} = \varepsilon_1 > 0$, then from Lemma 2.3 and the Aubin–Lions lemma, $\rho_n u_n$ converges strongly in $C^0(0,T; L^1(\Omega))$ to $\sqrt{\rho}\, q = \rho u$. Up to a subsequence, both ρ_n and $\rho_n u_n$ converge almost everywhere to, respectively, ρ and ρu. For almost every $(t,x) \in \{\rho > 0\}$, and n large enough, $\rho_n(t,x) > 0$, so $u_n = \rho_n u_n / \rho_n$ at this point converges to u. If $\varepsilon_{2,n} = 0$ we use the second part of Lemma 2.3 and thanks to the Aubin–Lions lemma, $\Phi(\rho_n) u_n$ converges strongly in $C^0(0,T; L^1(\Omega))$ to $\Phi(\rho) u$. We still have, up to a subsequence, both ρ_n and $\Phi(\rho_n) u_n$ converging almost everywhere to, respectively, ρ and $\Phi(\rho) u$ (we have used the fact that $\Phi(r)/\sqrt{r} = 0$ at $r = 0$). Since $\Phi(r) \neq 0$ for $r \neq 0$, for almost every $(t,x) \in \{\rho > 0\}$, and n large enough, $\Phi(\rho_n)(t,x) > 0$, so $u_n = \Phi(\rho_n) u_n / \Phi(\rho_n)$ at this point converges to u.

Note that

$$\rho_n^\alpha h(\rho_n, u_n) = \rho_n^\alpha h(\rho_n, u_n) 1_{\{\rho_n > 0\}} + \rho_n^\alpha h(\rho_n, u_n) 1_{\{\rho_n = 0\}}.$$

The first term converges almost everywhere to $\rho^\alpha h(\rho, u) 1_{\{\rho>0\}}$, and therefore to $\rho^\alpha h(\rho, u)$ in L^1 by the Lebesgue theorem. The second part can be estimated as follows:

$$\|\rho_n^\alpha h(\rho_n, u_n) 1_{\{\rho_n = 0\}}\|_{L^1} \leq \|h(\rho_n, u_n)\|_{L^q} \|\rho_n^\alpha 1_{\{\rho=0\}}\|_{L^{p-\varepsilon}}.$$

But $\rho_n^\alpha 1_{\{\rho=0\}}$ converges almost everywhere to 0, by the Lebesgue theorem, so the last term converges to 0. ■

Some compactness when the parameters are fixed. For any fixed positive δ, r_0, r_1, r_2 and r, to obtain a weak solution to (1.17), we only need to handle the compactness of the terms

$$r \rho_n \nabla\left(\sqrt{K(\rho_n)}\, \Delta\left(\int_0^{\rho_n} \sqrt{K(s)}\, ds\right)\right)$$

and

$$r_1 \frac{\rho_n}{\mu'(\rho_n)} |u_n|^2 u_n.$$

Indeed, due to the term $r_2\rho_n|u_n|u_n$ and the fact that $\inf_{s\in[0,\infty)}\mu'(s) > \varepsilon_1 > 0$, one obtains the compactness for all other terms in the same way as in [13, 40].

Capillarity term. To pass to the limits in

$$r\rho_n\nabla\Big(\sqrt{K(\rho_n)}\,\Delta\Big(\int_0^{\rho_n}\sqrt{K(s)}\,ds\Big)\Big),$$

we use the identity

$$\begin{aligned}
&\rho\nabla\Big(\sqrt{K(\rho_n)}\,\Delta\Big(\int_0^{\rho_n}\sqrt{K(s)}\;ds\Big)\Big)\\
&\quad= 4\big[2\,\mathrm{div}(\sqrt{\mu(\rho_n)}\,\nabla\nabla Z(\rho_n)) - \Delta(\sqrt{\mu(\rho_n)}\,\nabla Z(\rho_n))\big]\\
&\qquad+\Big[\nabla\Big[\Big(\frac{2\lambda(\rho_n)}{\sqrt{\mu(\rho_n)}}+k(\rho_n)\Big)\Delta Z(\rho_n)\Big]-\nabla\,\mathrm{div}[k(\rho_n)\nabla Z(\rho_n)]\Big]
\end{aligned}\tag{2.28}$$

where

$$Z(\rho_n)=\int_0^{\rho_n}[\mu(s)^{1/2}\mu'(s)]/s\,ds\quad\text{and}\quad k(\rho_n)=\int_0^{\rho_n}\frac{\lambda(s)\mu'(s)}{\mu(s)^{3/2}}\,ds.$$

It allows us to rewrite the weak form coming from the capillarity term as follows:

$$\begin{aligned}
&\int_0^t\int_\Omega\sqrt{K(\rho_n)}\,\Delta\Big(\int_0^{\rho_n}\sqrt{K(s)}\,ds\Big)\,\mathrm{div}(\rho_n\psi)\\
&\quad=4\int_0^t\int_\Omega\big(2\sqrt{\mu(\rho_n)}\,\nabla\nabla Z(\rho_n):\nabla\psi+\sqrt{\mu(\rho_n)}\,\nabla Z(\rho_n)\cdot\Delta\psi\big)\\
&\qquad+\int_0^t\int_\Omega\Big(\Big(\frac{2\lambda(\rho_n)}{\sqrt{\mu(\rho_n)}}+k(\rho_n)\Big)\Delta Z(\rho_n)\,\mathrm{div}\,\psi+k(\rho_n)\nabla Z(\rho_n)\cdot\nabla\,\mathrm{div}\,\psi\Big)\\
&\quad=A_1+A_2.
\end{aligned}$$

In fact, with Lemma 2.2 at hand, we are able to get compactness of A_1 and A_2 easily. Concerning A_1, we know that

$$\sqrt{\mu(\rho_n)}\to\sqrt{\mu(\rho)}\quad\text{ in } L^p((0,T);L^q(\Omega)) \text{ for all } p<\infty \text{ and } q<3.$$

Noting that $\nabla\nabla Z(\rho_n)$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$, we find that $\nabla Z(\rho_n)$ is uniformly bounded in $L^2(0,T;L^6(\Omega))$, because $\int_\Omega\nabla Z(\rho_n)=0$ due to periodicity. Thus we have the following weak convergence:

$$\int_\Omega\sqrt{\mu(\rho_n)}\,\nabla Z(\rho_n)\cdot\Delta\psi\to\int_\Omega\sqrt{\mu}\,\nabla Z\cdot\Delta\psi,$$

and

$$\int_\Omega\sqrt{\mu(\rho_n)}\,\nabla\nabla Z(\rho_n)\nabla\psi\to\int_\Omega\sqrt{\mu}\,\nabla\nabla Z:\nabla\psi,$$

thanks to Lemma 2.2. We conclude that $Z = Z(\rho)$, thanks to the bound on $Z(\rho_n)$ and the strong convergence of ρ_n. Thus using the compactness of ρ_n, the passage to the limit in A_1 is done. Concerning A_2, we just have to look at the coefficients

$$k(\rho_n) = \int_0^{\rho_n} \lambda(s)\mu'(s)/\mu(s)^{3/2}\, ds, \qquad j(\rho_n) = 2\lambda(\rho_n)/\sqrt{\mu(\rho_n)}\,.$$

Recalling the assumptions on $\mu(s)$ and the relation $\lambda(s) = 2(\mu'(s)s - \mu(s))$, we have

$$2(\alpha_1 - 1)\mu(s) \le \lambda(s) \le 2(\alpha_2 - 1)\mu(s),$$

and

$$\frac{\alpha_1}{\sqrt{\mu(s)}\, s} \le \frac{\mu'(s)}{\mu(s)^{3/2}} \le \frac{\alpha_2}{\sqrt{\mu(s)}\, s}.$$

This means that the coefficients $k(\rho_n)$ and $j(\rho_n)$ are comparable to $\sqrt{\mu(\rho_n)}$. Using the compactness of the density ρ_n and the information on $\mu(\rho_n)$ given in Corollary 2.2, we deduce the compactness of A_2 proceeding as for A_1.

Cubic non-linear drag term. We will use Lemma 2.4 to show the compactness of

$$\frac{\rho_n}{\mu'(\rho_n)}|u_n|^2 u_n.$$

More precisely, we write

$$\begin{aligned}\frac{\rho_n}{\mu'(\rho_n)}|u_n|^2 u_n &= \rho_n^{1/6}\sqrt{\frac{\rho_n}{\mu'(\rho_n)}}\,|u_n|^2 \rho_n^{1/3}|u_n|\frac{1}{\sqrt{\mu'(\rho_n)}}\\ &= \rho_n^{1/6} h(\rho_n, |u_n|).\end{aligned} \tag{2.29}$$

By Lemma 2.2, there exists $\varepsilon > 0$ such that $\rho_n^{1/6}$ is uniformly bounded in $L^\infty(0, T; L^{6\gamma+\varepsilon}(\Omega))$ and $\rho_n \to \rho$ a.e., so

$$\rho_n^{1/6} \to \rho^{1/6} \quad \text{in } L^{6\gamma+\varepsilon}((0,T)\times\Omega). \tag{2.30}$$

Noting that $\sqrt{\frac{\rho_n}{\mu'(\rho_n)}}\,|u_n|^2$ is uniformly bounded in $L^2(0, T; L^2(\Omega))$, and $\inf_{s\in[0,\infty)} \mu'(s) \ge \varepsilon_1 > 0$, we see that $\rho_n^{1/3}|u_n|\frac{1}{\sqrt{\mu'(\rho_n)}}$ is uniformly bounded in $L^3(0, T; L^3(\Omega))$, thus

$$\begin{aligned}h(\rho_n, |u_n|) &= \sqrt{\frac{\rho_n}{\mu'(\rho_n)}}\,|u_n|^2 \rho_n^{1/3}|u_n|\frac{1}{\sqrt{\mu'(\rho_n)}}\\ &\in L^{6/5}(0, T; L^{6/5}(\Omega)) \quad \text{uniformly.}\end{aligned} \tag{2.31}$$

By Lemma 2.4 and (2.29)–(2.31), we deduce that

$$\int_0^t \int_\Omega \frac{\rho_n}{\mu'(\rho_n)}|u_n|^2 u_n \to \int_0^t \int_\Omega \frac{\rho}{\mu'(\rho)}|u|^2 u. \quad ■$$

Relying on the compactness stated in this section and the compactness in [40], we can follow the argument in [13] to show Theorem 1.2. Thanks to the term $r_1\rho_n|u_n|u_n$, we have

$$\int_0^T\int_\Omega r_1\rho_n|u_n|^3 \le C.$$

This implies that

$$\sqrt{\rho_n}\,u_n \to \sqrt{\rho}\,u \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

With the above compactness results, we are able to pass to the limits to get a weak solution. In fact, to obtain a weak solution to (1.17), we have to pass to the limits as $\varepsilon_4 \to 0$, $n \to \infty$, $\varepsilon_3 \to 0$ and $\varepsilon \to 0$ respectively in the same spirit as in [13]. In particular, when passing to the limit as $\varepsilon_3 \to 0$, we also need to handle the identification of v with $2\nabla s(\rho)$. Following the argument in [13], one shows that v and $2\nabla s(\rho)$ satisfy the same moment equation. By the regularity and compactness of solutions, we can show the uniqueness of solutions. By the uniqueness, we have $v = 2\nabla s(\rho)$. This ends the proof of Theorem 1.2.

3. From weak solutions to renormalized solutions to the approximation

This section is dedicated to showing that a weak solution is a renormalized solution for our last level of approximation, in order to show Theorem 1.3. First, we introduce a new function

$$[f(t,x)]_\varepsilon = f * \eta_\varepsilon(t,x) \quad \text{for any } t > \varepsilon, \quad \text{and} \quad [f(t,x)]_\varepsilon^x = f * \eta_\varepsilon(x),$$

where

$$\eta_\varepsilon(t,x) = \frac{1}{\varepsilon^{N+1}}\eta\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right), \qquad \eta_\varepsilon(x) = \frac{1}{\varepsilon^N}\eta\left(\frac{x}{\varepsilon}\right),$$

with η a smooth non-negative even function compactly supported in the space-time ball of radius 1, and with integral 1. In this section, we will rely on the following two lemmas. Let ∂ be a partial derivative in one direction (space or time) in these two lemmas. The first one is the commutator lemma of DiPerna and Lions [38].

Lemma 3.1. *Let $f \in W^{1,p}(\mathbb{R}^N \times \mathbb{R}^+)$ and $g \in L^q(\mathbb{R}^N \times \mathbb{R}^+)$ with $1 \le p,q \le \infty$ and $1/p + 1/q \le 1$. Then*

$$\|[\partial(fg)]_\varepsilon - \partial(f([g]_\varepsilon))\|_{L^r(\mathbb{R}^N\times\mathbb{R}^+)} \le C\|f\|_{W^{1,p}(\mathbb{R}^N\times\mathbb{R}^+)}\|g\|_{L^q(\mathbb{R}^N\times\mathbb{R}^+)}$$

for some $C \ge 0$ independent of ε, f and g, and with r determined by $1/r = 1/p + 1/q$. In addition,

$$[\partial(fg)]_\varepsilon - \partial(f([g]_\varepsilon)) \to 0 \quad \text{in } L^r(\mathbb{R}^N\times\mathbb{R}^+)$$

as $\varepsilon \to 0$ if $r < \infty$. Moreover, in the same way, if $f \in W^{1,p}(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$ with $1 \le p,q \le \infty$ and $1/p + 1/q \le 1$, then

$$\|[\partial(fg)]_\varepsilon^x - \partial(f([g]_\varepsilon^x))\|_{L^r(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}$$

for some $C \geq 0$ *independent of* ε, f *and* g, *with* r *determined by* $1/r = 1/p + 1/q$. *In addition,*

$$[\partial(fg)]_\varepsilon^x - \partial(f([g]_\varepsilon^x)) \to 0 \quad \text{in } L^r(\mathbb{R}^N)$$

as $\varepsilon \to 0$ *if* $r < \infty$.

We also need another very standard lemma:

Lemma 3.2. *If* $f \in L^p(\Omega \times \mathbb{R}^+)$ *and* $g \in L^q(\Omega \times \mathbb{R}^+)$ *with* $1/p + 1/q = 1$ *and if* $H \in W^{1,\infty}(\mathbb{R})$, *then*

$$\begin{aligned}
&\int_0^T \int_\Omega [f]_\varepsilon g = \int_0^T \int_\Omega f[g]_\varepsilon,\\
&\lim_{\varepsilon \to 0} \int_0^T \int_\Omega [f]_\varepsilon g = \int_0^T \int_\Omega fg,\\
&\partial[f]_\varepsilon = [\partial f]_\varepsilon,\\
&\lim_{\varepsilon \to 0} \|H([f]_\varepsilon) - H(f)\|_{L^s_{\text{loc}}(\Omega \times \mathbb{R}^+)} = 0 \quad \text{for any } 1 \leq s < \infty.
\end{aligned}$$

We define a non-negative cut-off function ϕ_m for any fixed positive m as follows:

$$\phi_m(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1/(2m),\\ 2my - 1 & \text{if } 1/(2m) \leq y \leq 1/m,\\ 1 & \text{if } 1/m \leq y \leq m,\\ 2 - y/m & \text{if } m \leq y \leq 2m,\\ 0 & \text{if } y \geq 2m. \end{cases} \tag{3.1}$$

It enables us to define an approximate velocity for the density bounded away from zero and bounded away from infinity. This is crucial to our procedure, since the approximate velocity gradient is bounded in $L^2((0,T) \times \Omega)$. In particular, we introduce $u_m = u\phi_m(\rho)$ for any fixed $m > 0$. Thus, we can show ∇u_m is bounded in $L^2(0,T; L^2(\Omega))$ due to (3.1). In fact,

$$\begin{aligned}
\nabla u_m &= \phi_m'(\rho) u \otimes \nabla\rho + \phi_m(\rho) \frac{1}{\sqrt{\mu(\rho)}} \mathbb{T}_\mu\\
&= \left(\phi_m'(\rho) \frac{(\mu(\rho)\rho)^{1/4}}{(\mu'(\rho))^{3/4}}\right)\left(\left(\frac{\rho}{\mu'(\rho)}\right)^{1/4} u\right) \otimes \left(\frac{\mu'(\rho)}{\rho^{1/2}\mu(\rho)^{1/4}} \nabla\rho\right)\\
&\quad + \phi_m(\rho) \frac{1}{\sqrt{\mu(\rho)}} \mathbb{T}_\mu.
\end{aligned}$$

Similarly to [34], thanks to the cut-off function (3.1) and for m fixed, the functions $\phi_m'(\rho)(\mu(\rho)\rho)^{1/4}/(\mu'(\rho))^{3/4}$ and $\phi_m(\rho)/\sqrt{\mu(\rho)}$ are bounded. Then ∇u_m is bounded in $L^2((0,T) \times \Omega)$ using the estimates with $r > 0$ and $r_2 > 0$, and hence for $\varphi \in W^{2,\infty}(\mathbb{R})$, we find that $\nabla\varphi'((u_m)_j)$ is bounded in $L^2((0,T) \times \Omega)$ for $j = 1, 2, 3$.

The following estimates are necessary:

Lemma 3.3. *There exists a constant $C > 0$ depending only on the fixed solution $(\sqrt{\rho}, \sqrt{\rho}\, u)$, and C_m depending also on m, such that*

$$\begin{aligned}
&\|\rho\|_{L^\infty(0,T;L^{10}(\Omega))} + \|\rho u\|_{L^3(0,T;L^{5/2}(\Omega))} + \big\|\rho|u|^2\big\|_{L^2(0,T;L^{10/7}(\Omega))} \\
&\quad + \|\sqrt{\mu}\,(|\mathbb{S}_\mu| + r|\mathbb{S}_r|)\|_{L^2(0,T;L^{10/7}(\Omega))} + \left\|\frac{\lambda(\rho)}{\mu(\rho)}\right\|_{L^\infty((0,T)\times\Omega)} \\
&\quad + \left\|\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}}\,\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big)\right\|_{L^{1+}((0,T)\times\Omega)} \\
&\quad + \left\|\sqrt{\frac{P_\delta'(\rho_n)\rho_n}{\mu'(\rho_n)}}\,\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P_\delta'(s)\mu'(s)}{s}}\,ds\Big)\right\|_{L^{1+}((0,T)\times\Omega)} + \|r_0 u\|_{L^2((0,T)\times\Omega)} \le C,
\end{aligned}$$

and

$$\|\nabla\phi_m(\rho)\|_{L^4((0,T)\times\Omega)} + \|\partial_t\phi_m(\rho)\|_{L^2((0,T)\times\Omega)} \le C_m.$$

Proof. By (1.24), we have $\rho \in L^\infty(0,T;L^{10}(\Omega))$. Now $\nabla\sqrt{\rho} \in L^\infty(0,T;L^2(\Omega))$ because $\mu'(s) \ge \varepsilon_1$ and $\mu'(\rho)\nabla\rho/\sqrt{\rho} \in L^\infty(0,T;L^2(\Omega))$. Noting that

$$\rho u = \rho^{2/3}\rho^{1/3}u,$$

and $\rho^{2/3} \in L^\infty(0,T;L^{15}(\Omega))$ and $\rho^{1/3}u \in L^3(0,T;L^3(\Omega))$, we see that ρu is bounded in $L^3(0,T;L^{5/2}(\Omega))$.

By (1.24), we have $(\rho/\mu'(\rho))^{1/2}|u|^2 \in L^2((0,T)\times\Omega)$. Since

$$\rho|u|^2 = (\rho\mu'(\rho))^{1/2}\Big(\frac{\rho}{\mu'(\rho)}\Big)^{1/2}|u|^2,$$

it is bounded in $L^2(0,T;L^{10/7}(\Omega))$, where we have used the facts that $\mu(\rho) \in L^\infty(0,T;L^{5/2}(\Omega))$ (recalling that for $\rho \ge 1$ we have $\mu(\rho) \le c\rho^4$ and $\rho \in L^\infty(0,T;L^{10}(\Omega))$) and $\mu'(\rho)\rho \le \alpha_2\mu(\rho)$.

Similarly, we find that $\sqrt{\mu}\,(|\mathbb{S}_\mu| + r|\mathbb{S}_r|) \in L^2(0,T;L^{10/7}(\Omega))$ by (1.23). The $L^\infty((0,T)\times\Omega)$ bound for $\lambda(\rho)/\mu(\rho)$ may be obtained easily due to (1.3) and (1.9).

Concerning the estimates related to the pressures, we just have to look at the proof of Lemma 2.2. Noting that

$$\nabla\phi_m(\rho) = \phi_m'(\rho)\nabla\rho = \phi_m'(\rho)\frac{\rho^{1/2}\mu(\rho)^{1/4}}{\mu'(\rho)}\Big[\frac{\mu'(\rho)}{\rho^{1/2}\mu(\rho)^{1/4}}\nabla\rho\Big]$$

by (1.25), we conclude that $\nabla\phi_m(\rho)$ is bounded in $L^4((0,T)\times\Omega)$. It suffices to recall that thanks to the cut-off function ϕ_m, we have $\phi_m'(\rho)\rho^{1/2}\mu(\rho)^{1/4}/\mu'(\rho)$ bounded in $L^\infty((0,T)\times\Omega)$. Similarly, we write

$$\begin{aligned}
\partial_t\phi_m(\rho) &= \phi_m'(\rho)\partial_t\rho = -\phi_m'(\rho)\,\mathrm{div}(\rho u) \\
&= -\phi_m'(\rho)\frac{\rho}{\sqrt{\mu}}\,\mathrm{Tr}(\mathbb{T}_\mu) - \Big(\phi_m'(\rho)\frac{(\mu(\rho)\rho)^{1/4}}{(\mu'(\rho))^{3/4}}\Big)\Big(\frac{\rho^{1/4}}{(\mu'(\rho))^{1/4}}u\Big)\cdot\Big(\frac{\mu'(\rho)}{\rho^{1/2}\mu(\rho)^{1/4}}\nabla\rho\Big),
\end{aligned}$$

which shows that $\partial_t \phi_m(\rho)$ is bounded in $L^2(0,T;L^2(\Omega))$ thanks to (1.23), (1.24) and (1.25). and using the cut-off function property to bound the extra quantities in $L^\infty((0,T)\times\Omega)$ as previously. ■

Lemma 3.4. *The κ-entropy weak solution constructed in Theorem 1.2 is a renormalized solution; in particular, we have*

$$\begin{aligned}
&\int_0^T \int_\Omega \big(\rho\varphi(u)\psi_t + (\rho\varphi(u)\otimes u)\nabla\psi\big)\\
&-\int_0^T \int_\Omega \nabla\psi\varphi'(u)\Big[2\Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{S}_\mu + r\sqrt{\mu(\rho)}\,\mathbb{S}_r)\Big)\operatorname{Id}\Big]\\
&-\int_0^T \int_\Omega \psi\varphi''(u)\mathbb{T}_\mu\Big[2\Big((\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\mathbb{S}_\mu + r\mathbb{S}_r)\Big)\operatorname{Id}\Big]\\
&+\int_0^T \int_\Omega \psi\varphi'(u)F(\rho,u) = 0,
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
\sqrt{\mu(\rho)}\,\varphi_i'(u)[\mathbb{T}_\mu]_{jk} &= \partial_j(\mu\varphi_i'(u)u_k) - \sqrt{\rho}\,u_k\varphi_i'(u)\frac{\nabla\mu}{\sqrt{\rho}} + \bar{R}^1_\varphi,\\
\sqrt{\mu(\rho)}\,\varphi_i'(u)[\mathbb{S}_r]_{jk} &= 2\sqrt{\mu(\rho)}\,\varphi_i'(u)\partial_j\partial_k Z(\rho)\\
&\quad - 2\partial_j(\sqrt{\mu(\rho)}\,\partial_k Z(\rho)\varphi_i'(u)) + \bar{R}^2_\varphi,\\
\frac{\lambda(\rho)}{2\mu(\rho)}\varphi_i'(u)\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{T}_\mu) &= \operatorname{div}\Big(\frac{\lambda(\rho)}{\mu(\rho)}\sqrt{\rho}\,u\frac{\mu(\rho)}{\sqrt{\rho}}\varphi'(u)\Big)\\
&\quad - \sqrt{\rho}\,u\cdot\sqrt{\rho}\,\nabla s(\rho)\frac{\rho\mu''(\rho)}{\mu(\rho)}\varphi'(u) + \bar{R}^3_\varphi,\\
\frac{\lambda(\rho)}{\mu(\rho)}\varphi'(u)\operatorname{Tr}(\sqrt{\mu(\rho)}\,\mathbb{S}_r) &= \varphi_i'(u)\Big(\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} + \frac{1}{2}k(\rho)\Big)\Delta Z(\rho)\\
&\quad - \frac{1}{2}\operatorname{div}(k(\rho)\varphi_i'(u)\nabla Z(\rho)) + \bar{R}^4_\varphi,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\bar{R}^1_\varphi &= \varphi_i''(u)\mathbb{T}_\mu\sqrt{\mu(\rho)}\,u,\\
\bar{R}^2_\varphi &= 2\varphi_i''(u)\mathbb{T}_\mu\nabla Z(\rho),\\
\bar{R}^3_\varphi &= -\varphi_i''(u)\mathbb{T}_\mu\cdot\sqrt{\mu(\rho)}\,u\frac{\lambda(\rho)}{2\mu(\rho)},\\
\bar{R}^4_\varphi &= \frac{k(\rho)}{2\sqrt{\mu(\rho)}}\varphi_i''(u)\mathbb{T}_\mu\cdot\nabla Z(\rho).
\end{aligned} \tag{3.4}$$

Proof. We choose $[\phi_m'([\rho]_\varepsilon)\psi]_\varepsilon$ as a test function for the continuity equation with $\psi \in C_c^\infty((0,T)\times\Omega)$. Using Lemma 3.2, we have

$$\begin{aligned}
0 &= \int_0^T\int_\Omega \big(\partial_t\big[\phi_m'([\rho]_\varepsilon)\psi\big]_\varepsilon\rho + \rho u\cdot\nabla\big[\phi_m'([\rho]_\varepsilon)\psi\big]_\varepsilon\big)\\
&= -\int_0^T\int_\Omega \big(\phi_m'([\rho]_\varepsilon)\psi\,\partial_t[\rho]_\varepsilon + \operatorname{div}([\rho u]_\varepsilon)\phi_m'([\rho]_\varepsilon)\psi\big)\\
&= \int_0^T\int_\Omega \left(\psi_t\phi_m([\rho]_\varepsilon) - \psi\phi_m'([\rho]_\varepsilon)\left[\frac{\rho}{\sqrt{\mu(\rho)}}\operatorname{Tr}(\mathbb{T}_\mu) + 2\sqrt{\rho}\,u\cdot\nabla\sqrt{\rho}\right]_\varepsilon\right).
\end{aligned} \tag{3.5}$$

Using Lemmas 3.3 and 3.2, and passing to the limit as $\varepsilon\to 0$, from (3.5) we get

$$\begin{aligned}
0 &= \int_0^T\int_\Omega \left(\psi_t\phi_m(\rho) - \psi\phi_m'(\rho)\left[\frac{\rho}{\sqrt{\mu}}\operatorname{Tr}(\mathbb{T}_\mu) + 2\sqrt{\rho}\,u\cdot\nabla\sqrt{\rho}\right]\right)\\
&= \int_0^T\int_\Omega \left(\psi_t\phi_m(\rho) - \psi\left[\phi_m'(\rho)\frac{\rho}{\sqrt{\mu}}\operatorname{Tr}(\mathbb{T}_\mu) + u\cdot\nabla\phi_m(\rho)\right]\right),
\end{aligned} \tag{3.6}$$

since $\psi\nabla\phi_m(\rho)\in L^4((0,T)\times\Omega)$, $u\in L^2((0,T)\times\Omega)$, and ψ is compactly supported.

Similarly, we can choose $[\psi\phi_m(\rho)]_\varepsilon$ as a test function for the momentum equation. In particular, we have the following lemma.

Lemma 3.5.

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega [\psi\phi_m(\rho)]_\varepsilon\big(\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u)\big)\\
&\quad = -\int_0^T\int_\Omega \Big(\psi_t\rho u_m + \nabla\psi\cdot\big(\rho u\otimes u_m + \psi(\partial_t\phi_m(\rho) + u\cdot\nabla\phi_m(\rho))\big)\rho u\Big).
\end{aligned}$$

Proof. By Lemma 3.1, we can show that

$$\int_0^T\int_\Omega [\psi\phi_m(\rho)]_\varepsilon\partial_t(\rho u) \to -\int_0^T\int_\Omega \big(\partial_t\psi\rho u_m + \psi\partial_t\phi_m(\rho)\rho u\big).$$

For the second term, we have

$$\begin{aligned}
\int_0^T\int_\Omega [\psi\phi_m(\rho)]_\varepsilon \operatorname{div}(\rho u\otimes u) &= \int_0^T\int_\Omega \psi\phi_m(\rho)[\operatorname{div}(\rho u\otimes u)]_\varepsilon\\
&= \left(\int_0^T\int_\Omega \psi\phi_m(\rho)[\operatorname{div}(\rho u\otimes u)]_\varepsilon - \int_0^T\int_\Omega \psi\phi_m(\rho)[\operatorname{div}(\rho u\otimes u)]_\varepsilon^x\right)\\
&\quad + \int_0^T\int_\Omega \psi\phi_m(\rho)[\operatorname{div}(\rho u\otimes u)]_\varepsilon^x\\
&= R_1 + R_2,
\end{aligned}$$

where $[f(t,x)]_\varepsilon = f(t,x) * \eta_\varepsilon(t,x)$ and $[f(t,x)]_\varepsilon^x = f * \eta_\varepsilon(x)$ with $\varepsilon > 0$ small enough. We write

$$\begin{aligned} R_1 &= \int_0^T \int_\Omega \psi\phi_m(\rho)[\operatorname{div}(\rho u \otimes u)]_\varepsilon - \int_0^T \int_\Omega \psi\phi_m(\rho)[\operatorname{div}(\rho u \otimes u)]_\varepsilon^x \\ &= \int_0^T \int_\Omega \psi\nabla\phi_m(\rho) : [\rho u \otimes u]_\varepsilon - \int_0^T \int_\Omega \psi\nabla\phi_m(\rho) : [\rho u \otimes u]_\varepsilon^x. \end{aligned}$$

Thanks to Lemma 3.3, $\rho|u|^2 \in L^2(0,T;L^{10/7}(\Omega))$ and $\psi\nabla\phi_m(\rho) \in L^4((0,T)\times\Omega)$, and we conclude that $R_1 \to 0$ as $\varepsilon \to 0$. Meanwhile, we can apply Lemma 3.1 to R_2 directly, thus

$$\begin{aligned} &\int_0^T \int_\Omega \psi\phi_m(\rho)[\operatorname{div}(\rho u \otimes u)]_\varepsilon^x \\ &\quad = \left(\int_0^T \int_\Omega \psi\phi_m(\rho)[\operatorname{div}(\rho u \otimes u)]_\varepsilon^x - \int_0^T \int_\Omega \psi\phi_m(\rho)\operatorname{div}(\rho u \otimes [u]_\varepsilon^x)\right) \\ &\qquad + \int_0^T \int_\Omega \psi\phi_m(\rho)\operatorname{div}(\rho u \otimes [u]_\varepsilon^x) \\ &\quad = R_{21} + R_{22}. \end{aligned}$$

By Lemma 3.1, we have $R_{21} \to 0$ as $\varepsilon \to 0$. The term R_{22} will be calculated in the following way:

$$\begin{aligned} &\int_0^T \int_\Omega \psi\phi_m(\rho)\operatorname{div}(\rho u \otimes [u]_\varepsilon^x) \\ &\qquad = \int_0^T \int_\Omega \psi\phi_m(\rho)\operatorname{div}(\rho u)[u]_\varepsilon^x + \int_0^T \int_\Omega \psi\phi_m(\rho)\rho u \cdot \nabla[u]_\varepsilon^x \\ &\qquad = \int_0^T \int_\Omega \psi \operatorname{div}(\rho u)[u_m]_\varepsilon^x + \int_0^T \int_\Omega \psi\rho u \nabla(\phi_m(\rho)[u]_\varepsilon^x) \\ &\qquad\quad - \int_0^T \int_\Omega \psi[u]_\varepsilon^x \cdot \nabla\phi_m(\rho)\rho u \\ &\qquad = -\int_0^T \int_\Omega \nabla\psi\rho u \otimes [u_m]_\varepsilon^x - \int_0^T \int_\Omega \psi \cdot [u]_\varepsilon^x \nabla\phi_m(\rho)\rho u, \end{aligned}$$

which tends to

$$-\int_0^T \int_\Omega \nabla\psi\rho u \otimes u_m - \int_0^T \int_\Omega \psi \cdot u\nabla\phi_m(\rho)\rho u$$

as $\varepsilon \to 0$. ■

For the other terms in the momentum equation, we can use the same method as for (3.6) to get

$$\int_0^T \int_\Omega \Big(\psi_t \rho u_m + \nabla\psi \cdot \Big(\rho u \otimes u_m - 2\phi_m(\rho) \Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)} \operatorname{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\,\mathrm{Id}\Big)\Big)\Big) + \int_0^T \int_\Omega \psi\big(\partial_t\phi_m(\rho) + u\cdot\nabla\phi_m(\rho)\big)\rho u - \int_0^T \int_\Omega \Big(2\psi \Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)} \operatorname{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\,\mathrm{Id}\Big)\nabla\phi_m(\rho) + \psi\phi_m(\rho)F(\rho,u)\Big) = 0.$$

Thanks to (3.6), we have

$$\int_0^T \int_\Omega \Big(\psi_t \rho u_m + \nabla\psi \cdot \Big(\rho u \otimes u_m - 2\phi_m(\rho) \Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)} \operatorname{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\,\mathrm{Id}\Big)\Big)\Big) - \int_0^T \int_\Omega \Big(\psi\phi_m'(\rho)\frac{\rho}{\sqrt{\mu(\rho)}} \operatorname{Tr}(\mathbb{T}_\mu)\rho u - \psi\phi_m(\rho)F(\rho,u)\Big) - \int_0^T \int_\Omega \Big(2\psi \Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)} \operatorname{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\,\mathrm{Id}\Big)\nabla\phi_m(\rho)\Big) = 0. \tag{3.7}$$

The goal of this subsection is to derive a formulation of a renormalized solution following the idea in [34]. We choose $[\psi\varphi'([u_m]_\varepsilon)]_\varepsilon$ as a test function in (3.7). Using the same argument as for Lemma 3.5, we can show that

$$\int_0^T \int_\Omega \big(\partial_t\big[\psi\varphi'([u_m]_\varepsilon)\big]_\varepsilon \rho u_m + \nabla\big[\psi\varphi'([u_m]_\varepsilon)\big]_\varepsilon : (\rho u \otimes u_m)\big) \to \int_0^T \int_\Omega \big(\rho\varphi(u_m)\psi_t + \rho u \otimes \varphi(u_m)\nabla\psi\big)$$

and

$$
\begin{aligned}
&\int_0^T \int_\Omega \nabla\big[\psi\varphi'([u_m]_\varepsilon)\big]_\varepsilon\Big(-2\phi_m(\rho)\Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) \\
&\qquad\qquad + \frac{\lambda(\rho)}{2\mu(\rho)}\,\mathrm{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\Big)\,\mathrm{Id}\Big) \\
&+ \big[\psi\varphi'([u_m]_\varepsilon)\big]_\varepsilon\Big(-\phi_m'(\rho)\frac{\rho}{\sqrt{\mu(\rho)}}\,\mathrm{Tr}(\mathbb{T}_\mu)\rho u \\
&-2\Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\,\mathrm{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r)\,\mathrm{Id})\Big)\nabla\phi_m(\rho) + \phi_m(\rho)F(\rho,u)\Big) \\
&\to \int_0^T \int_\Omega \nabla(\psi\varphi'(u_m))\,\Big(-2\phi_m(\rho)\Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) \\
&\qquad\qquad + \frac{\lambda(\rho)}{2\mu(\rho)}\,\mathrm{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\,\mathrm{Id}\Big)\Big) \\
&+ \psi\varphi'(u_m)\Big(-\phi_m'(\rho)\frac{\rho}{\sqrt{\mu(\rho)}}\,\mathrm{Tr}(\mathbb{T}_\mu)\rho u \\
&- 2\Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\,\mathrm{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\Big)\nabla\phi_m(\rho) + \phi_m(\rho)F(\rho,u)\Big)
\end{aligned}
$$

as $\varepsilon \to 0$. Putting these two limits together, we have

$$
\begin{aligned}
&\int_0^T \int_\Omega \Big(\big(\rho\varphi(u_m)\psi_t + \rho u \otimes \varphi(u_m)\nabla\psi\big) \\
&+ \nabla\psi\varphi'(u_m)\Big(-2\phi_m(\rho)\Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\,\mathrm{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\Big)\Big) \\
&+ \psi\varphi''(u_m)\nabla u_m\Big(-\phi_m(\rho)2\Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\,\mathrm{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\Big)\Big) \\
&+ \psi\varphi'(u_m)\Big(-\phi_m'(\rho)\frac{\rho}{\sqrt{\mu(\rho)}}\,\mathrm{Tr}(\mathbb{T}_\mu)\rho u - 2\Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) \\
&\qquad + \frac{\lambda(\rho)}{2\mu(\rho)}\,\mathrm{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\Big)\nabla\phi_m(\rho) + \phi_m(\rho)F(\rho,u)\Big)\Big) = 0.
\end{aligned}
\tag{3.8}
$$

Now we should pass to the limit in (3.8) as $m \to \infty$. To this end, we should keep the following in mind:

$$
\begin{aligned}
&\phi_m(\rho) \to 1 \quad \text{for almost every } (t,x) \in \mathbb{R}^+ \times \Omega, \\
&u_m \to u \quad \text{for almost every } (t,x) \in \mathbb{R}^+ \times \Omega, \\
&|\rho\phi_m'(\rho)| \le 2, \quad |\rho\phi_m'(\rho)| \to 0 \quad \text{for almost every } (t,x) \in \mathbb{R}^+ \times \Omega.
\end{aligned}
\tag{3.9}
$$

We can find that

$$
\begin{aligned}
\sqrt{\mu(\rho)}\,\nabla u_m &= \sqrt{\mu(\rho)}\,\nabla(\phi_m(\rho)u) = \phi_m(\rho)\sqrt{\mu(\rho)}\,\nabla u + \phi_m'(\rho)\sqrt{\mu(\rho)}\,u\nabla\rho \\
&= \frac{\phi_m(\rho)}{\sqrt{\mu(\rho)}}\left(\nabla(\mu(\rho)u) - \sqrt{\rho}\,u\cdot\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right) \\
&\quad + \frac{\sqrt{\rho}}{\mu(\rho)^{3/4}}\left(\frac{\sqrt{\mu(\rho)}}{\rho}\mu'(\rho)\nabla\rho\right)\left(\frac{\rho^{1/4}}{(\mu'(\rho))^{1/4}}u\right)\left(\phi_m'(\rho)\frac{\mu(\rho)^{3/4}\rho^{1/4}}{(\mu'(\rho))^{3/4}}\right) \\
&= \phi_m(\rho)\mathbb{T}_\mu + \frac{\sqrt{\rho}}{\mu(\rho)^{3/4}}\left(\frac{\sqrt{\mu(\rho)}}{\rho}\mu'(\rho)\nabla\rho\right)\left(\frac{\rho^{1/4}}{(\mu'(\rho))^{1/4}}u\right)\left(\phi_m'(\rho)\frac{\mu(\rho)^{3/4}\rho^{1/4}}{(\mu'(\rho))^{3/4}}\right) \\
&= A_{1m} + A_{2m}.
\end{aligned}
$$

Note that

$$
\left|\phi_m'(\rho)\frac{\mu(\rho)^{3/4}\rho^{1/4}}{(\mu'(\rho))^{3/4}}\right| \le C|\phi_m'(\rho)\rho|,
$$

so $\phi_m'(\rho)\mu(\rho)^{3/4}\rho^{1/4}/(\mu(\rho)')^{3/4} \to 0$ for almost every (t,x). Thus, the dominated convergence theorem shows that $A_{2m} \to 0$ as $m \to \infty$. Meanwhile, the dominated convergence theorem also implies $A_{1m} \to \mathbb{T}_\mu$ in $L^2_{t,x}$. Hence, with (3.9) at hand, letting $m \to \infty$ in (3.8), one obtains

$$
\begin{aligned}
\int_0^T\int_\Omega\Big(&\big(\rho\varphi(u)\psi_t + \rho u\otimes\varphi(u)\nabla\psi\big) - 2\nabla\psi\varphi'(u) : \Big(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r) \\
&+ \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\,(\mathbb{S}_\mu + r\mathbb{S}_r))\,\mathrm{Id}\Big) - 2\psi\varphi''(u)\mathbb{T}_\mu : \Big((\mathbb{S}_\mu + r\mathbb{S}_r) \\
&\qquad + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\mathbb{S}_\mu + r\mathbb{S}_r)\,\mathrm{Id}\Big) + \psi\varphi'(u)F(\rho,u)\Big) = 0.
\end{aligned}
$$

From now on, we denote $R_\varphi = 2\psi\varphi''(u)\mathbb{T}_\mu((\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\mathbb{S}_\mu + r\mathbb{S}_r)\,\mathrm{Id})$. This ends the proof of Theorem 1.3. ∎

4. Renormalized solutions and weak solutions

The main goal of this section is the proof of Theorem 1.1 that states the existence of renormalized solutions of the Navier–Stokes equations without the additional terms, thus the existence of weak solutions of the Navier–Stokes equations.

4.1. Renormalized solutions

In this subsection, we will show the existence of renormalized solutions. To this end, we need the following stability lemma.

Lemma 4.1. *Take any fixed $\alpha_1 < \alpha_2$ as in (1.9) and consider sequences δ_n, r_{0n}, r_{1n} and r_{2n} such that $r_{i,n} \to r_i \geq 0$ with $i = 0, 1, 2$ and $\delta_n \to \delta \geq 0$. Consider a sequence $\mu_n : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (1.9) and (1.10) for fixed α_1 and α_2 such that*

$$\mu_n \to \mu \quad \text{in } C^0(\mathbb{R}^+).$$

If (ρ_n, u_n) satisfies (1.23)–(1.26), then up to a subsequence, still denoted n, the following convergences hold.

(1) $\rho_n \to \rho$ *strongly in $C^0(0, T; L^p(\Omega))$ for any $1 \leq p < \gamma$.*

(2) $\mu_n(\rho_n)u_n \to \mu(\rho)u$ *in $L^\infty(0, T; L^p(\Omega)$ for $p \in [1, 3/2)$.*

(3) $(\mathbb{T}_\mu)_n \to \mathbb{T}_\mu$ *weakly in $L^2(0, T; L^2(\Omega))$.*

(4) *For every $H \in W^{2,\infty}(\overline{\mathbb{R}^d})$ and $0 < \alpha < 2\gamma/\gamma + 1$, $\rho_n^\alpha H(u_n) \to \rho^\alpha H(u)$ strongly in $L^p(0, T; \Omega)$ for $1 \leq p < \frac{2\gamma}{(\gamma+1)\alpha}$. In particular, $\sqrt{\mu(\rho_n)}\, H(u_n) \to \sqrt{\mu(\rho)}\, H(u)$ strongly in $L^\infty(0, T; L^2(\Omega))$.*

Proof. Using (1.26), the Aubin–Lions lemma gives, up to a subsequence,

$$\mu_n(\rho_n) \to \tilde{\mu} \quad \text{in } C^0(0, T; L^q(\Omega))$$

for any $q < 3/2$. But $\sup |\mu_n - \mu| \to 0$ as $n \to \infty$. Thus,

$$\mu_n(\rho_n) \to \tilde{\mu}(t, x) \quad \text{in } C^0([0, T]; L^q(\Omega)), \tag{4.1}$$

so up to a subsequence,

$$\mu(\rho_n) \to \tilde{\mu}(t, x) \quad \text{a.e.}$$

Note that μ is an increasing function, so it is invertible, and μ^{-1} is continuous. This implies that $\rho_n \to \rho$ a.e. with $\mu(\rho) = \tilde{\mu}(t, x)$. Together with (4.1) and ρ_n being uniformly bounded in $L^\infty(0, T; L^\gamma(\Omega))$, we get part (1).

Note that

$$\nabla \frac{\mu(\rho_n)}{\sqrt{\rho_n}} = \frac{\sqrt{\rho_n}\, \nabla\mu(\rho_n)}{\rho_n} - \frac{\mu(\rho_n)\nabla\rho_n}{2\rho_n\sqrt{\rho_n}},$$

thus

$$\left|\nabla \frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right| \leq C|\sqrt{\rho_n}|\left|\frac{\nabla\mu(\rho_n)}{\sqrt{\rho_n}}\right|,$$

so $\nabla \frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0, T; L^2(\Omega))$, thanks to (1.23). Using (1.26), we find that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0, T; W^{1,2}(\Omega))$, thus uniformly bounded in $L^\infty(0, T; L^6(\Omega))$.

On the other hand, $\sqrt{\rho_n}\, u_n$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$. From Lemma 2.4, we have

$$\mu(\rho_n)u_n = \frac{\mu(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}\, u_n \to \mu(\rho)u \quad \text{in } L^\infty(0, T; L^q(\Omega))$$

for any $1 \leq q < 3/2$. Since $(\mathbb{T}_\mu)_n$ is bounded in $L^2(0, T; L^2(\Omega))$, up to a subsequence it converges weakly in $L^2(0, T; L^2(\Omega))$ to a function $\mathbb{T}_\mu$. In view of Lemma 2.4, this gives part (4). ∎

With Lemma 4.1, we are able to obtain the renormalized solutions of Navier–Stokes equations without any additional term by letting $n \to \infty$ in (3.2). We state this result in the following lemma, where we fix μ such that $\varepsilon_1 > 0$.

Lemma 4.2. *For any fixed $\varepsilon_1 > 0$, there exists a renormalized solution $(\sqrt{\rho}, \sqrt{\rho}\, u)$ to the initial value problem* (1.1)–(1.2).

Proof. We can use Lemma 4.1 to pass to the limits in the extra terms. We will have to follow this order: $r_2 \to 0$, then $r_1 \to 0$, and finally $r_0, \delta, r \to 0$ together.

• If $r_2 = r_2(n) \to 0$, we just write

$$r_2 \frac{\rho_n}{\mu'(\rho_n)} |u_n|^2 u_n = r_2^{1/4} \left(\frac{\rho_n}{\mu'(\rho_n)} \right)^{1/4} \left(\frac{\rho_n}{\mu'(\rho_n)} \right)^{3/4} |u_n|^2 u_n,$$

and $\mu'(\rho_n) \geq \varepsilon_1 > 0$, so $\left(\frac{\rho_n}{\mu'(\rho_n)}\right)^{1/4} \leq C|\rho_n|^{1/4}$, and thus

$$r_2 \frac{\rho_n}{\mu'(\rho_n)} |u_n|^2 u_n \to 0 \quad \text{in } L^{4/3}(0,T; L^{6/5}(\Omega)).$$

• For $r_1 = r(n) \to 0$,

$$|r_1 \rho_n |u_n| u_n| \leq r^{1/3} \rho_n^{1/3} r^{2/3} \rho_n^{2/3} |u_n|^2,$$

which converges to zero in $L^{3/2}(0,T; L^{9/7}(\Omega))$ using the drag term control in the energy and the information on the pressure law $P(\rho) = a\rho^{\gamma}$.

• For $r_0 = r_0(n) \to 0$, it is easy to conclude that

$$r_0 u_n \to 0 \quad \text{in } L^2((0,T) \times \Omega).$$

• We now consider the limit as $r \to 0$ of the term

$$r\rho_n \nabla \left(\sqrt{K(\rho_n)}\, \Delta \left(\int_0^{\rho_n} \sqrt{K(s)}\, ds \right) \right).$$

Noting the identity

$$\rho_n \nabla \left(\sqrt{K(\rho_n)}\, \Delta \left(\int_0^{\rho_n} \sqrt{K(s)}\, ds \right) \right) = 2\, \mathrm{div}\big(\mu(\rho_n) \nabla^2 (2s(\rho_n))\big) + \nabla\big(\lambda(\rho_n) \Delta(2s(\rho_n))\big),$$

we only need to focus on $\mathrm{div}(\mu(\rho_n)\nabla^2(2s(\rho_n)))$ since the same argument holds for the other term. Since

$$\begin{aligned} r \int_\Omega & \mathrm{div}\big(\mu(\rho_n) \nabla^2 (2s(\rho_n))\big) \psi \\ &= r \int_\Omega \frac{\rho_n}{\mu_n} \nabla Z(\rho_n) \otimes \nabla Z(\rho_n) \nabla \psi + r \int_\Omega \mu_n \nabla s(\rho_n) \Delta \psi \\ &= r \int_\Omega \frac{\rho_n}{\mu_n} \nabla Z(\rho_n) \otimes \nabla Z(\rho_n) \nabla \psi \, dx + r \int_\Omega \sqrt{\mu_n}\, \nabla Z(\rho_n) \Delta \psi, \end{aligned}$$

the first term can be controlled as

$$\left| r \int_\Omega \sqrt{\mu_n}\, \nabla Z(\rho_n) \Delta\psi \right| \\ \leq C r^{1/2} \| \sqrt{\mu(\rho_n)}\, \|_{L^2(0,T;L^2(\Omega))} \| \sqrt{r}\, \nabla Z(\rho_n) \|_{L^2(0,T;L^2(\Omega))} \to 0,$$

thanks to (1.25) and (1.26); and the second term as

$$r\left| \int_\Omega \frac{\rho_n}{\mu_n} \nabla Z(\rho_n) \otimes \nabla Z(\rho_n) \nabla\psi \right| \leq \sqrt{r}\,\sqrt{r} \int_\Omega \sqrt{\mu(\rho_n)}\, \frac{\rho_n}{\mu(\rho_n)^{3/2}} |\nabla Z(\rho_n)|^2 |\nabla\psi| \\ \leq C \left\| \sqrt{r}\, \frac{\rho_n}{\mu(\rho_n)^{3/2}} |\nabla Z(\rho_n)|^2 \right\|_{L^2(0,T;L^2(\Omega))} \| \sqrt{\mu(\rho_n)} \|_{L^2(0,T;L^2(\Omega))} r^{1/2} \to 0.$$

• Concerning the quantity $\delta\rho^{10}$, thanks to $\mu'_{\varepsilon_1}(\rho) \geq \varepsilon_1 > 0$, $\sqrt{\delta}\, |\nabla\rho^5|$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$. This shows that $\delta^{1/30}\rho$ is uniformly bounded in $L^{10}(0,T;L^{30}(\Omega))$. Thus, we have

$$\left| \int_0^T \int_\Omega \delta\rho^{10} \nabla\psi \right| \leq C(\psi)\delta^{2/3} \| \delta^{1/3}\rho^{10} \|_{L^1(0,T;L^3(\Omega))} \to 0$$

as $\delta \to 0$.

With Lemma 4.1 at hand, we are ready to obtain the renormalized solutions to (1.1)–(1.2). By parts (1) and (2) of Lemma 4.1, we can pass to the limits in the continuity equation. Thanks to part (4) of Lemma 4.1,

$$\sqrt{\mu(\rho_n)}\, \varphi'(u_n) \to \sqrt{\mu(\rho)}\, \varphi'(u) \quad \text{in } L^\infty(0,T;L^2(\Omega)).$$

With the help of Lemma 2.2, we can pass to the limit in the pressure, and thus we get the renormalized solutions. ∎

4.2. *Recovering weak solutions from renormalized solutions*

In this part, we get weak solutions from renormalized solutions constructed in Lemma 4.2. Now we show that Lemma 4.2 is valid without the condition $\varepsilon_1 > 0$. For such a μ, we construct a sequence μ_n converging to μ in $C^0(\mathbb{R}^+)$ and such that $\varepsilon_{1n} = \inf \mu'_n > 0$. Lemma 4.1 shows that, up to a subsequence,

$$\rho_n \to \rho \qquad \text{in } C^0(0,T;L^p(\Omega)), \tag{4.2}$$

$$\rho_n u_n \to \rho u \qquad \text{in } L^\infty(0,T;L^{\frac{p+1}{2p}}(\Omega)), \tag{4.3}$$

for any $1 \leq p < \gamma$, where $(\rho, \sqrt{\rho}\, u)$ is a renormalized solution to (1.1).

Now, we want to show that this renormalized solution is also a weak solution in the sense of Definition 1.2. To this end, we introduce a non-negative smooth function

$\Phi : \mathbb{R} \to \mathbb{R}$ with compact support and $\Phi(s) = 1$ for any $-1 \le s \le 1$. Let $\tilde{\Phi}(s) = \int_0^s \Phi(r)\,dr$ and define

$$\varphi_n(y) = n\tilde{\Phi}(y_1/n)\Phi(y_2/n)\Phi(y_3/n)$$

for any $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. Note that φ_n is bounded in $W^{2,\infty}(\mathbb{R}^3)$ for any fixed $n > 0$, $\varphi_n(y)$ converges everywhere to y_1 as $n \to \infty$, $\nabla\varphi_n$ is uniformly bounded in n and converges everywhere to the unit vector $(1, 0, 0)$, and

$$\|\nabla\nabla\varphi_n\|_{L^\infty(\mathbb{R}^3)} \le C/n \to 0$$

as $n \to \infty$. This allows us to control the measures in Definition 1.1 as follows:

$$\|R_{\varphi_n}\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|\overline{R}^1_{\varphi_n}\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|\overline{R}^2_{\varphi_n}\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} \le C\|\nabla\nabla\varphi_n\|_{L^\infty(\mathbb{R}^3)} \to 0$$

as $n \to \infty$. Using this function φ_n in the equation of Definition 1.1, the Lebesgue Theorem gives us the equation on ρu_1 in Definition 1.2 by letting $n \to \infty$. In this way, we are able to get the full vector equation on ρu by permuting the directions. Applying the Lebesgue dominated convergence theorem, one obtains (1.4) by passing to the limit in (1.12) with $i = 1$ and the function φ_n. Thus, we have shown that the renormalized solution is also a weak solution.

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