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Corentin Boissy · Erwan Lanneau

Lengths spectrum of hyperelliptic components

To the memory of Jean-Christophe Yoccoz

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Abstract. We propose a general framework for studying pseudo-Anosov homeomorphisms on translation surfaces. This new approach, among other consequences, allows us to compute the systole of the Teichmüller geodesic flow restricted to the hyperelliptic connected components of the strata of Abelian differentials, settling a question of Farb [Problems on Mapping Class Groups and Related Topics, 11–55 (2006)]. These are the first results on the description of the systole of moduli spaces, outside some cases in low genera. We emphasize that all proofs and computations can be performed without the help of a computer. As a byproduct, our methods give a way to describe the bottom of the lengths spectrum of the hyperelliptic components, and we provide a picture of that for small genera.

Keywords. Pseudo-Anosov, translation surface

1. Introduction

Pseudo-Anosov maps of closed surfaces play an important role in Teichmüller dynamics. They were introduced by Thurston in 1988, and it is still a challenging problem to give explicit constructions of families of pseudo-Anosov mappings. Given a genus g closed topological surface S_g, a pseudo-Anosov map ϕ on S_g has an *expansion factor* $\lambda(\phi) \in \mathbb{R}_{>0}$ recording the exponential growth rate of the lengths of the curves under iteration of ϕ. The set of the logarithms of all expansion factors (when fixing the genus) is a discrete subset of $\mathbb{R}$: this is the lengths spectrum of the moduli space $\mathcal{M}_g$ for the Teichmüller metric. It has remained challenging to describe the set of expansion factors. In this paper, we address these two problems: a new construction of pseudo-Anosov mappings and a complete description of least expansion factors for an important class that we describe below.

Corentin Boissy: Institut de Mathématiques de Toulouse, Université Paul Sabatier, 31062 Toulouse, France; corentin.boissy@math.univ-toulouse.fr

Erwan Lanneau: Institut Fourier, Université Grenoble-Alpes, BP 74, 38402 Saint-Martin-d'Hères, France; erwan.lanneau@univ-grenoble-alpes.fr

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Half-translation surfaces are an important feature of pseudo-Anosov maps. Those surfaces are determined by a pair (X, q), where X is a Riemann surface and q a meromorphic quadratic differential (most of the time, we will write (X, ω) when $q = \omega^2$ is the global square of a holomorphic 1-form, and we will refer to (X, ω) as a translation surface). In this setting, there are other lengths spectra: $\mathrm{spec}(H)$ or $\mathrm{spec}(\mathcal{C}) \subset \mathrm{spec}(\mathrm{Mod}(g))$ for various subgroups $H < \mathrm{Mod}(g)$ of the mapping class group, or for various connected components $\mathcal{C}$ of the strata of the moduli spaces of quadratic differentials.

Describing $\mathrm{spec}(\mathcal{C})$ is a difficult problem, and determining a precise value of the *least element* (its systole) $L(\mathrm{spec}(\mathcal{C}))$ of this spectrum is a long-standing open question. Except few results in low genera, no systoles are known. Some bounds have been established [11, 17, 18],

$$\frac{\log(2)}{6} \leq |\chi(S_g)| \cdot L(\mathrm{spec}(\mathrm{Mod}(g))) \leq 2 \cdot \log\Big(\frac{3+\sqrt{5}}{2}\Big),$$

or [7]

$$L(\mathrm{Spec}(\mathcal{C}^{\mathrm{hyp}})) \in \Big]\sqrt{2}, \sqrt{2} + \frac{1}{2^{g-1}}\Big[$$

for every $g \geq 2$, but the question on the asymptotic behavior is widely open in general.

These objects have a long history and have been the subject of many recent investigations (see e.g. the work of McMullen [16, 17], Farb–Leininger–Margalit [10], Agol–Leininger–Margalit [2]). Different approaches have been used in order to tackle these questions, thanks to the work by Thurston, McMullen (via train-tracks and hyperbolic 3-manifolds) or by the work of Veech and Marmi–Moussa–Yoccoz (via the Rauzy–Veech induction). There is a recent approach by Agol [1] using veering triangulations (see also [4]).

The first result of this paper is a general framework for studying pseudo-Anosov homeomorphisms. This applies to many different connected components of the strata of Abelian and quadratic differentials. Among them, the *hyperelliptic components* are the simplest to understand (we refer to the work of Kontsevich and Zorich [13] for a description of these components): they consist entirely of hyperelliptic curves. These components are important since they are related to the braid group (or the mapping class group of the disc with n punctures $\mathrm{Mod}(0, n)$). Recently, a classification of affine invariant submanifolds in hyperelliptic connected components was obtained in [3]. Apisa proved that there is no primitive affine invariant submanifolds, except perhaps Teichmüller curves. In the present paper, our framework applies to this setting and completes the geometric description of these connected components. We will establish the following theorem.

Theorem A. *Let $g \geq 1$. The minimum value of the expansion factor $\lambda(\phi)$ over all affine pseudo-Anosov maps ϕ on a genus g translation surface $(X, \omega) \in \mathcal{C}^{\mathrm{hyp}}$ is given by the largest root of the polynomial*

$$\begin{aligned} &X^{2g+1} - 2X^{2g-1} - 2X^2 + 1 && \text{if } (X,\omega) \in \mathcal{H}^{\mathrm{hyp}}(2g-2), \\ &X^{2g+2} - 2X^{2g} - 2X^{g+1} - 2X^2 + 1 && \text{if } (X,\omega) \in \mathcal{H}^{\mathrm{hyp}}(g-1, g-1), \\ & && g \text{ is even}, \end{aligned}$$

$$X^{2g+2} - 2X^{2g} - 4X^{g+2} + 4X^{g} + 2X^2 - 1 \quad \textit{if } (X,\omega) \in \mathcal{H}^{\mathrm{hyp}}(g-1,g-1),\ g \textit{ is odd}.$$

Moreover, there are two conjugacy mapping classes realizing the minimum.

In particular, $L(\mathrm{spec}(\mathcal{C}^{\mathrm{hyp}}))$ is the logarithm of this largest root. This theorem settles a question of Farb [9, Problem 7.5] for hyperelliptic component and gives the very first instance of an explicit computation of the systole for an infinite family of strata of quadratic differentials.

In the case of $\mathcal{H}^{\mathrm{hyp}}(2)$, $\mathcal{H}^{\mathrm{hyp}}(1,1)$, this result was shown by Lanneau–Thiffeault [14] – this is the largest root of the polynomials

$$X^5 - 2X^3 - 2X^2 + 1 = (X+1)(X^4 - X^3 - X^2 - X + 1),$$
$$X^6 - 2X^4 - 2X^3 - 2X^2 + 1 = (X+1)^2(X^4 - 2X^3 + X^2 - 2X + 1),$$

respectively.

With more efforts, our method also gives a way to compute the other elements of the spectrum $\mathrm{Spec}(\mathcal{C}^{\mathrm{hyp}})$, where $\mathcal{C}^{\mathrm{hyp}}$ ranges over all hyperelliptic connected components, for any genus. As an instance, we will also prove (see Theorem 6.1) the following theorem.

Theorem B. *For any even g, $g \not\equiv 2 \bmod 3$, $g \geq 9$, the second least expansion factor of an affine pseudo-Anosov maps ϕ on a translation surface in $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ is given by the largest root of the polynomial*

$$X^{2g+1} - 2X^{2g-1} - 2X^{\lceil \frac{4g}{3} \rceil} - 2X^{\lfloor \frac{2g}{3} \rfloor + 1} - 2X^2 + 1.$$

Moreover, there are two conjugacy mapping classes realizing the minimum, up to replacing ϕ by ϕ^{-1}.

For small values of g, as an illustration of our construction, we are able to produce a complete description of the bottom of the spectrum of $\mathcal{C}^{\mathrm{hyp}}$. For $g \leq 10$, the lengths l of the closed Teichmüller geodesics on $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ satisfying, say, $l < 2$ are recorded in Table 1. For genus between 4 and 10, we only indicate the total number of different expansion factors.

Our techniques also provide a way to investigate the bottom of the spectrum for the stratum $\mathcal{H}^{\mathrm{hyp}}(g-1,g-1)$ for various g and various bounds (not necessarily 2). We also emphasize that this approach can be applied to a general stratum of quadratic differentials, not necessarily a hyperelliptic connected component. This will be done in a forthcoming work.

Organization of the paper

We conclude with a sketch of the proof of Theorem A and Theorem B. The strategy is to convert the computation of mapping classes and their expanding factors into a finite combinatorial problem. This is classical in pseudo-Anosov theory. In [1,4], the authors used the theory of veering triangulations to give a coding of the Teichmüller flow on connected components of strata of quadratic differentials. This may be used to study the topology

g	Perron root of polynomial	Numerical value
2	$X^5-2X^3-2X^2+1$	~ 1.72208380573904
3	$X^7-2X^5-2X^2+1$	~ 1.55603019132268
	$X^7-2X^5-X^4-X^3-2X^2+1$	~ 1.78164359860800
	$X^7-3X^5-3X^2+1$	~ 1.85118903363607
	$X^7-2X^5-2X^4-2X^3-2X^2+1$	~ 1.94685626827188
4	11 expansion factors	
5	22 expansion factors	
6	79 expansion factors	
7	142 expansion factors	
8	452 expansion factors	
9	1688 expansion factors	
10	4887 expansion factors	

Tab. 1. List of all expansion factors less than 2 for closed Teichmüller geodesics on $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ for $g \leq 10$.

of pseudo-Anosov mapping classes. Unfortunately, the underlying combinatorics is very complicated, and the computation of the precise systole with this approach seems to be out of reach.

The Rauzy–Veech induction in [20] involves much simpler combinatorial diagrams. However, the cost to pay is that this coding does not capture all pseudo-Anosov maps [7].

We will propose a new construction in order to solve these two difficulties at the same time.

(1) *Hyperelliptic connected components.* For any integer $n \geq 2$, let $\mathcal{D}_n$ be the hyperelliptic Rauzy diagram of size $2^{n-1}-1$ containing the permutation

$$\pi_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

Let $\mathcal{C}_n^{\mathrm{hyp}}$ be the associated connected component given by the Veech's zippered rectangle construction [20]. By the Kontsevich–Zorich classification [13],

$$\mathcal{C}_n^{\mathrm{hyp}} = \begin{cases} \mathcal{H}^{\mathrm{hyp}}(2g-2) & \text{if } n = 2g \text{ is even,} \\ \mathcal{H}^{\mathrm{hyp}}(g-1,g-1) & \text{if } n = 2g+1 \text{ is odd.} \end{cases}$$

(2) *Rauzy–Veech induction.* A pseudo-Anosov homeomorphism is *positive*, respectively *negative*, if it preserves an orientable measured foliation and if it fixes, respectively reverses, the orientation of the foliation. To any closed loop γ in the Rauzy diagram $\mathcal{D}_n$, under mild assumptions, one associates an affine pseudo-Anosov map $\phi(\gamma)$ on $(X,\omega) \in \mathcal{C}_n^{\mathrm{hyp}}$ (see [20] and § 2). Those maps are exactly positive pseudo-Anosov mappings fixing a separatrix.
In [7], it is shown that their expanding factors are bounded from below by 2. Thus this construction cannot achieve the bottom of the spectrum $\mathrm{Spec}(\mathcal{C}^{\mathrm{hyp}})$.

(3) *Symmetric Rauzy–Veech construction.* In § 3, we will associate to any path γ (not necessarily closed) in $\mathcal{D}_n$, from π to its symmetric $s(\pi)$ (see § 2.4) a *negative* pseudo-Anosov map $\phi(\gamma)$. We refer to this construction as the *symmetric Rauzy–Veech construction*. We will establish in Theorem 3.6 and Proposition 3.7 that any negative pseudo-Anosov map ϕ on $(X,\omega) \in \mathcal{C}_n^{\mathrm{hyp}}$ can be written as $\phi = \phi(\gamma)$ for some path γ. Observe that, on a hyperelliptic surface, up to composing with the hyperelliptic involution, one can assume that all pseudo-Anosov homeomorphisms are negative.
We also emphasize that Theorem 3.6 applies to a more general setting and not only to hyperelliptic surfaces.

(4) *Dynamics.* In § 4, we introduce a renormalization process, the ZRL renormalization (for Zorich Right Left induction). To any path γ in $\mathcal{D}_n$, under mild assumptions, we define a new path $\mathrm{ZRL}(\gamma)$ such that $\phi(\gamma)$ and $\phi(\mathrm{ZRL}(\gamma))$ define the same conjugacy class. We then establish that there exists $m \geq 0$ such that $\gamma' = \mathrm{ZRL}^{(m)}(\gamma)$ starts from one of the following $n-2$ permutations (see Theorem 4.1):

$$\begin{pmatrix} 1 & 2 & \cdots & \cdots & \cdots & \cdots & \cdots & n-1 & n \\ n & k & k-1 & \cdots & 1 & n-1 & n-2 & \cdots & k+1 \end{pmatrix}$$

for some $k = 1,\dots,n-2$ (i.e. obtained from π_n by applying k top Rauzy operations). This allows us to reduce considerably the range of paths to analyze in the Rauzy diagram $\mathcal{D}_n$. The particular shape of $\mathcal{D}_n$ will be strongly used in the proof.

(5) *Combinatorial part.* By using the geometry of the Rauzy class, in § 5, it is possible to further reduce the above set of paths to a *finite* set Γ_n. The technical computations are isolated to avoid overloading the main body of the paper. This is the content of the appendix.

(6) *Appendix* A. To any $\gamma \in \Gamma_n$, one associates a matrix $V(\gamma)$ corresponding to the action of $\phi(\gamma)$ on homology. Thus the expansion factor of $\phi(\gamma)$ is the Perron root of $V(\gamma)$. In this appendix, we compute the matrices $V(\gamma)$ and their characteristic polynomials by the help of the "rome" method (after [5]).

(7) *Appendix* B. We compare the top roots of each polynomial in order to establish the minimal expansion factor of $\mathrm{Spec}(\mathcal{C}^{\mathrm{hyp}})$.

As explained in the abstract, the computations are done without the help of the computer, except perhaps in the last step (Appendix B) for the concrete evaluation of roots for a few polynomials that appear in low genera.

For a first reading, the reader can skip Appendix A and Appendix B and go directly to Appendix D: it provides an elementary proof of Theorem A when n is even (systole of $\mathrm{Spec}(\mathcal{H}^{\mathrm{hyp}}(2g-2))$) without any technical computations. However, all the main steps of the strategy in this case are needed.

Appendices A and B are more technical and are mainly used for computing the systole of $\mathrm{Spec}(\mathcal{H}^{\mathrm{hyp}}(g-1,g-1))$ and the *second* least element of $\mathrm{Spec}(\mathcal{H}^{\mathrm{hyp}}(2g-2))$ (these two problems are of the same order of difficulty). In this situation, we have to deal with the existence of imprimitive matrices $V(\gamma)$.

2. Rauzy–Veech induction and pseudo-Anosov homeomorphism

In this section, we briefly recall the notions of interval exchange transformations, suspension data, Rauzy–Veech induction, and the associated construction of pseudo-Anosov homeomorphisms. We also provide a slight generalization of these standard notions. See for instance [7, 15, 21, 22] for references.

2.1. *Interval exchange transformation*

Let $I \subset \mathbb{R}$ be an open interval, and let us choose a finite subset Σ of I. Its complement is a union of $d \geq 2$ open subintervals I_j for $j = 1, \dots, d$. An interval exchange map is a one-to-one map T from $I \backslash \Sigma$ to a co-finite subset of I that is a translation on each subinterval of its definition domain. It is easy to see that T is precisely determined by the following data: a permutation that encodes how the intervals are exchanged by T and a vector with positive entries that encodes the lengths of the intervals.

We use the representation introduced first by Kerckhoff [12] and formalized later by Bufetov [8] and Marmi, Moussa and Yoccoz [15].

We will attribute a name to each interval I_j. In this case, we will speak of *labeled* interval exchange maps. One gets a pair of one-to-one maps (π_t, π_b) (t for "top" and b for "bottom") from a finite alphabet $\mathcal{A}$ to $\{1, \dots, d\}$ in the following way. In the partition of I into intervals, we denote the k-th interval, when counted from the left to the right, by $I_{\pi_t^{-1}(k)}$. Once the intervals are exchanged, the interval number k is $I_{\pi_b^{-1}(k)}$. Then, with this convention, the permutation encoding the map T is $\pi_b \circ \pi_t^{-1}$. We will denote the length of the intervals by a vector $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$.

We will call the pair (π_t, π_b) a *labeled* permutation, and $\pi_b \circ \pi_t^{-1}$ a permutation (or *reduced* permutation). One usually represents labeled permutations $\pi = (\pi_t, \pi_b)$ by a table:

$$\pi = \begin{pmatrix} \pi_t^{-1}(1) & \pi_t^{-1}(2) & \cdots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \pi_b^{-1}(2) & \cdots & \pi_b^{-1}(d) \end{pmatrix}.$$

2.2. *Flat surfaces*

Let S be a compact, connected genus g topological surface. A flat structure on S is a pair $(\mathcal{U}, \Sigma)$ such that Σ is a finite subset of S (whose elements are called *singularities*), and $\mathcal{U} = \{(U_i, z_i)\}$ is an atlas of charts of $S \backslash \Sigma$ such that the transition maps $z_j \circ z_i^{-1} : z_i(U_i \cap U_j) \to z_j(U_i \cap U_j)$ are translations or half-turns: $z_i = \pm z_j + \text{const}$, and for each $s \in \Sigma$, there is a neighborhood of s isometric to a Euclidean cone. Therefore, we get a *quadratic differential* defined locally in the coordinates z_i by the formula $q = dz_i^2$. This form extends to the points of Σ to zeroes, simple poles or marked points. A surface S with a flat structure is often called half-translation surface. We will sometimes use the notation (S, q) or simply S.

An important case is when there exists a sub-atlas such that all transition functions are translations or equivalently if the quadratic differential q is the global square of an Abelian differential. Choosing a square root ω of q (equivalently, a sub-atlas as above), we will then say that (X, ω) is a translation surface. Note that, on a translation surface, there is a well defined notion of left/right and up/down directions, while on a half-translation surface, we can only define horizontal and vertical directions.

A pseudo-Anosov homeomorphism ϕ on a surface S defines a pair of transverse measured foliations (one is contracted by the map, the other is expanded), and this data is equivalent to a flat structure on S, the two foliations become the horizontal and the vertical directions on the flat surface and ϕ is affine in the flat coordinates. By convention, we will *always* assume that the horizontal direction is the expanded one.

2.3. *Suspension data and weak suspension data*

The next construction provides a link between interval exchange transformations and translation surfaces, namely pairs (X, ω), where X is a compact Riemann surface and ω is a non-zero holomorphic 1-form on X. Following [15], a suspension datum for $T = (\pi, \lambda)$ is a vector $(\tau_\alpha)_{\alpha\in\mathcal{A}}$ such that

- $\sum_{\pi_t(\alpha)\le k} \tau_\alpha > 0$ for all $1 \le k \le d-1$,
- $\sum_{\pi_b(\alpha)\le k} \tau_\alpha < 0$ for all $1 \le k \le d-1$.

We will often use the notation $\zeta = \lambda + i\tau \in \mathbb{C}^{\mathcal{A}}$. To each suspension datum τ, we can associate a translation surface $(X, \omega) = X(\pi, \lambda, \tau)$ in the following way.

Consider the broken line L_t on $\mathbb{C} = \mathbb{R}^2$ defined by concatenation of the vectors $\zeta_{\pi_t^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin. Similarly, we consider the broken line L_b defined by concatenation of the vectors $\zeta_{\pi_b^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin. If the lines L_t and L_b have no intersections other than the endpoints, we can construct a Riemann surface X by identifying each side ζ_j on L_t with the side ζ_j on L_b by a translation. The form dz on $\mathbb{C}$ descends to a holomorphic 1-form ω on X. The resulting surface (X, ω) is a translation surface. Note that the lines L_t and L_b might have some other intersection points. But in this case, one can still define a translation surface by using the *zippered rectangle construction*, due to Veech [20] (see [6, Section 2.3] for more details).

In the sequel, we will need a slightly more general notion of a suspension datum.

Definition 2.1. Let $T = (\pi, \lambda)$ be an interval exchange map. A *weak suspension data* for T is an element $\tau \in \mathbb{R}^{\mathcal{A}}$ such that there exists $h \in \mathbb{R}$ satisfying

(i) $h + \sum_{\pi_t(\alpha)\le k} \tau_\alpha > 0$ for all $1 \le k \le d-1$,
(ii) $h + \sum_{\pi_b(\alpha)\le k} \tau_\alpha < 0$ for all $1 \le k \le d-1$,
(iii) if $\pi_t^{-1}(1) = \pi_b^{-1}(d)$, then $\sum_{\pi_t(\alpha)\neq 1} \tau_\alpha > 0$,
(iv) if $\pi_t^{-1}(d) = \pi_b^{-1}(1)$, then $\sum_{\pi_b(\alpha)\neq 1} \tau_\alpha < 0$.

The parameter h above will be called *height* of the weak suspension datum τ. Observe that a suspension datum is a weak suspension datum of height 0 (in this case, the first two conditions imply the two others).

In a very similar way to the case of suspension data, one can associate to a pair (π, λ, τ, h) a translation surface $X = X(\pi, \lambda, \tau)$ and a horizontal interval $I_h \subset X$ as follows. Let I_h be a horizontal interval of length $\sum_\alpha \lambda_\alpha$. For $\alpha \in \mathcal{A}$, we consider the

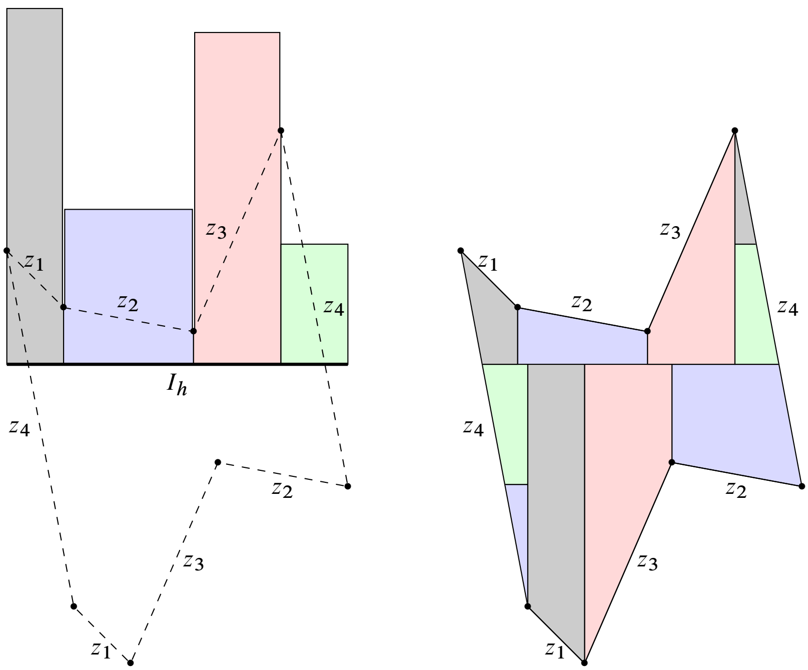

Fig. 1. Zippered rectangle construction for a weak suspension datum.

rectangle R_α of width λ_α and of height

$$h_\alpha = \sum_{\pi_t(\beta)\le\pi_t(\alpha)} \tau_\beta - \sum_{\pi_b(\beta)\le\pi_b(\alpha)} \tau_\beta .$$

By definition of weak suspension datum, $h_\alpha > 0$ for each α. Note that condition (iii) is needed to ensure that $h_a > 0$ for a permutation π of the form $\left(\begin{smallmatrix} a & ** & * \\ * & ** & a \end{smallmatrix}\right)$. Then we glue the rectangles $(R_\alpha)_\alpha$ in a similar way to the usual zippered rectangle construction (Figure 1). Two values h_1, h_2 give canonically isometric surfaces $X(\pi,\lambda,\tau)$, where the intervals I_{h_1} and I_{h_2} differ by a vertical translation (of length $h_2 - h_1$). In other words, there is an immersed Euclidean rectangle of height $|h_2 - h_1|$ whose horizontal sides are I_{h_1} and I_{h_2}. Also, vertical leaves starting from the endpoints of I_h satisfy the "classical conditions", i.e. each leaf will hit a singularity before intersecting I_h (in the positive or negative direction depending on h and the suspension data).

Conversely, let X be a translation surface and $I \subset X$ a horizontal interval with the same classical conditions as above. We assume that there are no vertical saddle connections. As in the classical case, there exists a (unique) weak suspension datum (π,λ,τ,h) such that $(X, I) = (X(\pi,\lambda,\tau), I_h)$. The datum (π,λ) is given by considering the interval exchange T defined by the first return map of the vertical flow on I, and h is the time (positive or negative) for which the vertical geodesic starting from the left end hits a singularity. The parameters τ_α are obtained by considering vertical geodesics starting from the discontinuities of T and the time where they hit singularities: the corresponding time t_k for the k-th discontinuity is $h + \sum_{\pi_t(\alpha)\le k} \tau_\alpha$. Also, if $\pi_t^{-1}(1) = \pi_b^{-1}(d)$, then condition (iii) corresponds to the fact that the vertical geodesic starting from the right end of I hits a singularity before intersecting I again (and similarly for condition (iv)).

2.4. *Rauzy–Veech induction and other Rauzy operations*

The Rauzy–Veech induction $\mathcal{R}(T)$ of T is defined as the first return map of T to a certain subinterval J of I (see [15, 19] for details).

We recall very briefly the construction. Following [15], we define the *type* of T by t if $\lambda_{\pi_t^{-1}(d)} > \lambda_{\pi_b^{-1}(d)}$ and b if $\lambda_{\pi_t^{-1}(d)} < \lambda_{\pi_b^{-1}(d)}$. When T is of type t (respectively b) we will say that the label $\pi_t^{-1}(d)$ (respectively $\pi_b^{-1}(d)$) is the winner and that $\pi_b^{-1}(d)$ (respectively $\pi_t^{-1}(d)$) is the loser. We define a subinterval J of I by

$$J = \begin{cases} I \backslash T(I_{\pi_b^{-1}(d)}) & \text{if } T \text{ is of type } t, \\ I \backslash I_{\pi_t^{-1}(d)} & \text{if } T \text{ is of type } b. \end{cases}$$

The image of T by the Rauzy–Veech induction $\mathcal{R}$ is defined as the first return map of T to the subinterval J. This is again an interval exchange transformation, defined on d letters (see e.g. [19]). The data of $\mathcal{R}(T)$ are very easy to express in term of those of T.

There are two cases to distinguish depending on whether T is of type t or b; the labeled permutations of $\mathcal{R}(T)$ only depends on π and on the type of T. If $\varepsilon \in \{t, b\}$ is the type of T, this defines two maps $\mathcal{R}_t$ and $\mathcal{R}_b$ by $\mathcal{R}(T) = (\mathcal{R}_\varepsilon(\pi), \lambda')$. We will often make use of the following notation: if $\varepsilon \in \{t, b\}$, we denote by $1 - \varepsilon$ the other element of $\{t, b\}$.

(1) T has type t. Let $k \in \{1, \dots, d-1\}$ such that $\pi_b^{-1}(k) = \pi_t^{-1}(d)$. Then

$$\mathcal{R}_t(\pi_t, \pi_b) = (\pi'_t, \pi'_b),$$

where $\pi_t = \pi'_t$ and

$$\pi_b'^{-1}(j) = \begin{cases} \pi_b^{-1}(j) & \text{if } j \le k, \\ \pi_b^{-1}(d) & \text{if } j = k+1, \\ \pi_b^{-1}(j-1) & \text{otherwise.} \end{cases}$$

(2) T has type b. Let $k \in \{1, \dots, d-1\}$ such that $\pi_t^{-1}(k) = \pi_b^{-1}(d)$. Then

$$\mathcal{R}_b(\pi_t, \pi_b) = (\pi'_t, \pi'_b),$$

where $\pi_b = \pi'_b$ and

$$\pi_t'^{-1}(j) = \begin{cases} \pi_t^{-1}(j) & \text{if } j \le k, \\ \pi_t^{-1}(d) & \text{if } j = k+1, \\ \pi_t^{-1}(j-1) & \text{otherwise.} \end{cases}$$

(3) Let us denote by $E_{\alpha\beta}$ the $d \times d$ matrix, where the α, β-th element is equal to 1, all others to 0. If T is of type t, then let $(\alpha, \beta) = (\pi_t^{-1}(d), \pi_b^{-1}(d))$; otherwise, let $(\alpha, \beta) = (\pi_b^{-1}(d), \pi_t^{-1}(d))$. Then $V_{\alpha\beta}\lambda' = \lambda$, where $V_{\alpha\beta}$ is the transvection matrix $\mathrm{Id} + E_{\alpha\beta}$.

If τ is a suspension data over (π, λ), then we define $\mathcal{R}(\pi, \lambda, \tau)$ by

$$\mathcal{R}(\pi, \lambda, \tau) = (\mathcal{R}_\varepsilon(\pi), V^{-1}\lambda, V^{-1}\tau),$$

where ε is the type of $T = (\pi, \lambda)$ and V is the corresponding transition matrix.

Remark 2.2. By construction, the two translation surfaces $X(\pi, \lambda, \tau)$ and $X(\pi', \lambda', \tau')$ are naturally isometric (as translation surfaces).

Remark 2.3. We can extend the Rauzy–Veech operation on the space of weak suspension data by using the same formulas. We easily see that $V^{-1}\zeta$ is a weak suspension data for $\mathcal{R}_\varepsilon(\pi)$. Moreover, $X(\mathcal{R}(\pi, \lambda, \tau))$ and $X(\pi, \lambda, \tau)$ are naturally isometric as translation surfaces.

If we iterate $n \geq 1$ times the Rauzy induction, we get a sequence $(\alpha_k, \beta_k)_{k=1,\dots,n}$ of winners/losers. By denoting $\mathcal{R}^{(n)}(\pi, \lambda) = (\pi^{(n)}, \lambda^{(n)})$, the transition matrix that relates $\lambda^{(n)}$ to λ is the product of the transition matrices. Thus we have

$$\left(\prod_{k=1}^{n} V_{\alpha_k\beta_k}\right)\lambda^{(n)} = \lambda. \tag{2.1}$$

Now we define other Rauzy moves that will be used later. Let π be a labeled permutation,

$$\pi = \begin{pmatrix} \pi_t^{-1}(1) & \pi_t^{-1}(2) & \cdots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \pi_b^{-1}(2) & \cdots & \pi_b^{-1}(d) \end{pmatrix}.$$

We define the *symmetric* of π, denoted by $s(\pi)$, the following labeled permutation:

$$s(\pi) = \begin{pmatrix} \pi_b^{-1}(d) & \pi_b^{-1}(d-1) & \cdots & \pi_b^{-1}(1) \\ \pi_t^{-1}(d) & \pi_t^{-1}(d-1) & \cdots & \pi_t^{-1}(1) \end{pmatrix}.$$

Observe that if τ is a weak suspension datum for (π, λ), then τ is also a weak suspension datum for $(s(\pi), \lambda)$ (it is not necessarily true for usual suspension data). For simplicity, we define $s(\pi, \lambda, \tau) = (s(\pi), \lambda, \tau)$. Recall that $\mathrm{SL}(2, \mathbb{R})$ acts on the set of translation surfaces (by post-composition of the chart maps with linear transformations from $\mathrm{SL}(2, \mathbb{R})$). We immediately check that

$$X(\pi, \lambda, \tau) = -\mathrm{Id} \cdot X(s(\pi, \lambda, \tau)).$$

Left Rauzy induction can be defined analogously to the Rauzy induction, by “cutting” the interval on the left. It can also be defined by $\mathcal{R}_L = s \circ \mathcal{R} \circ s$. From the above discussion, we see that $\mathcal{R}_L$ preserves weak suspension data, and that $X(\pi, \lambda, \tau)$ and $X(\mathcal{R}_L(\pi, \lambda, \tau))$ are naturally isometric as translation surfaces (see Remark 2.2).

2.5. *Labeled Rauzy diagrams*

Given a labeled permutation π, there are two possible Rauzy moves producing two (possibly the same) permutations. The moves depend on the type $T = (\pi, \lambda)$ can have.

We form a graph by taking all labeled permutations and by assigning an edge between π and π' if there exists a Rauzy move $\mathcal{R}_\varepsilon(\pi) = \pi'$ for some $\varepsilon \in \{t, b\}$. The *labeled Rauzy diagram* $\mathcal{D}(\pi)$ corresponds to the connected component of this graph containing π.

For an edge $\mathcal{R}_t(\pi) = \pi'$, we will use the notation $\pi \xrightarrow{\alpha,\beta} \pi'$, where

$$(\alpha, \beta) = (\pi_t^{-1}(d), \pi_b^{-1}(d))$$

are the winner-loser. Similarly, for an edge $\mathcal{R}_b(\pi) = \pi'$, we will use $\pi \xrightarrow{\alpha,\beta} \pi'$, where $(\alpha, \beta) = (\pi_b^{-1}(d), \pi_t^{-1}(d))$. Finally, we will denote by $\widetilde{V}(\gamma)$ the product of the transition matrices in equation (2.1).

Similarly, one can define the reduced Rauzy diagram $\mathcal{D}_{\mathrm{red}}(\pi)$ by considering (reduced) permutations as vertices. There is clearly a canonical map $\mathcal{D}(\pi) \to \mathcal{D}_{\mathrm{red}}(\pi)$.

2.6. *Hyperelliptic Rauzy diagrams and coordinates*

For any integer $n \geq 2$, we will consider the hyperelliptic Rauzy diagram $\mathcal{D}_n$ of size $2^{n-1} - 1$ containing the permutation

$$\pi_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$$

and denote by $\mathcal{C}_n^{\mathrm{hyp}}$ the associated connected component. We will refer to π_n as the *central permutation*. In this situation, labeled and reduced Rauzy diagrams coincide. When n is even, it corresponds to the hyperelliptic connected component $\mathcal{H}^{\mathrm{hyp}}(n-2)$, and when n is odd, it corresponds to the hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(\frac{n-1}{2}, \frac{n-1}{2})$. The precise description of these diagrams was given by Rauzy [19]; see also [7, Section 3.7.1]. An easy corollary is the following proposition.

Proposition 2.4. *Let $n \geq 2$. For any $\pi \in \mathcal{D}_n$, there exists a unique shortest path joining π_n to π.*

For a permutation π in $\mathcal{D}_n$ as above, we can write the path of the proposition as a unique sequence of $n_1 > 0$ Rauzy moves of type $\varepsilon \in \{t, b\}$, followed by $n_2 > 0$ Rauzy moves of type $1 - \varepsilon$, etc. (with the convention $1 - t = b$ and $1 - b = t$). The sequence of non-negative integers $n_1, \ldots, n_{k-1}$ uniquely defines the permutation π once ε is chosen. Moreover, one has $\sum_{i=1,\ldots,k-1} n_i < n - 1$.

Definition 2.5. Let π be a permutation in $\mathcal{D}_n$, and let $n_1, \ldots, n_{k-1}$ be as above. The *coordinates* of π are

$$\Big(n_1, \ldots, n_{k-1}, n - 1 - \sum_{i=1,\ldots,k-1} n_i\Big).$$

This definition is motivated by the following easy proposition.

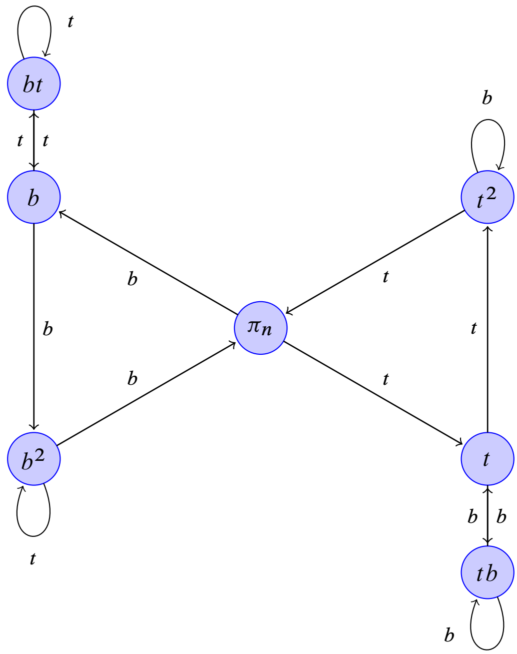

Fig. 2. Hyperelliptic Rauzy diagram for $n = 4$.

Proposition 2.6. *Let π be a permutation in $\mathfrak{D}_n$. The following holds.*

- *If π has coordinates $(n_1, \ldots, n_k)$, then $s(\pi)$ has coordinates $(n_k, \ldots, n_1)$.*
- *π and $s(\pi)$ belong to the same component of $\mathfrak{D}_n \backslash \{\pi_n\}$ if and only if k is even.*

Proof of Proposition 2.6. The proof is straightforward. To see that the last assertion holds, we can remark that π and $s(\pi)$ belong to the same connected component of $\mathfrak{D}_n \backslash \{\pi_n\}$ if and only if the minimal paths (from π_n) to π and $s(\pi)$ have the same starting Rauzy type. ∎

We will use the following terminology in Theorem 4.1.

Definition 2.7. The loop in $\mathfrak{D}_n$ formed by the vertices having coordinates $(k, n-1-k)$ for $k = 0, \ldots, n-2$ and $\varepsilon = t$ will be referred to as the *central loop*. The $n-2$ other loops, attached to the central loop, namely formed by the vertices having coordinates $(k, l, n-1-k-l)$ for $k = 1, \ldots, n-2$ and $l = 0, \ldots, n-k-1$ and $\varepsilon = t$, will be referred to as the *secondary loops*.

2.7. Pseudo-Anosov homeomorphism and Rauzy–Veech induction (after [20]*)*

Let γ be a path in a *labeled* Rauzy diagram. We assume that γ is closed in the *reduced* Rauzy diagram. One associates a matrix $V(\gamma)$ as follows. First let $\widetilde{V}(\gamma)$ be the matrix defined in Section 2.5.

Let (π_t, π_b), respectively (π'_t, π'_b), be the starting, respectively ending, labeled permutations of γ. By definition, they both define the same underlying permutation, namely $\pi_b \circ \pi_t^{-1} = \pi'_b \circ {\pi'}_t^{-1}$. Let P be the permutation matrix defined by permuting the columns of the $d \times d$ identity matrix according to the maps π_t, π'_t, i.e. $P = [p_{\alpha\beta}]_{\alpha,\beta\in\mathcal{A}^2}$, with $p_{\alpha\beta} = 1$ if $\beta = \pi_t^{-1}(\pi'_t(\alpha))$ and 0 otherwise. The transition matrix associated to the path γ is then

$$V(\gamma) = \widetilde{V}(\gamma) \cdot P.$$

A path γ in $\mathcal{D}$, that is closed in $\mathcal{D}_{\mathrm{red}}$ is *admissible* if the matrix $V(\gamma)$ is primitive (i.e. it as a power with only positive entries). To any such path, we associate the Perron–Frobenius eigenvalue $\theta > 1$ of $V(\gamma)$. We choose a positive eigenvector λ for θ. It can be shown that $V(\gamma)$ is symplectic [20]; thus let us choose an eigenvector τ for the eigenvalue θ^{-1} with $\tau_{\pi_t^{-1}(d)} > 0$. It turns out that τ defines a suspension data over $T = (\pi, \lambda)$. Indeed, the set of suspension data is an open cone, that is preserved by $V(\gamma)^{-1}$. Since the matrix $V(\gamma)^{-1}$ has a dominant eigenvalue θ (for the eigenvector τ), the vector τ must belong to this cone. If n is the length of γ, then

$$\mathcal{R}^{(n)}(\pi, \lambda, \tau) = (\pi, V(\gamma)^{-1}\lambda, V(\gamma)^{-1}\tau) = (\pi, \theta^{-1}\lambda, \theta\tau).$$

Hence the two surfaces $X(\pi, \lambda, \tau)$ and $g_t X(\pi, \lambda, \tau)$, where

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad t = \log(\theta) > 0,$$

differ by some element of the mapping class group (see Remark 2.2). In other words, there exists a pseudo-Anosov homeomorphism ϕ affine with respect to the translation surface $X(\pi, \lambda, \tau)$ and such that $D\phi = g_t$. The action of ϕ on the relative homology of (X, ω) is $V(\gamma)$; thus the expanding factor of ϕ is θ. This construction will be referred to as the *Rauzy–Veech construction*.

Definition 2.8. A pseudo-Anosov homeomorphism is positive (respectively negative) if it preserves an orientable measured foliation and if it fixes (respectively reverses) the orientation of the foliation.

By construction, ϕ fixes the zero on the left of the interval I, and thus it fixes a horizontal separatrix adjacent to this zero (namely, the oriented half line corresponding to the interval I) and therefore is positive. Conversely, one has the following theorem.

Theorem 2.9 (Veech). *Any positive pseudo-Anosov homeomorphism fixing a horizontal separatrix is obtained by the Veech construction.*

Recall that the main result of [7] was based on the following key proposition.

Proposition 2.10 ([7, Propositions 4.3 and 4.4]). *For any $n \geq 2$ and any positive pseudo-Anosov homeomorphism ϕ that is affine with respect to $(X, \omega) \in \mathcal{C}_n^{\mathrm{hyp}}$, if ϕ fixes a horizontal separatrix attached to a zero of ω, or a marked point, then $\lambda(\phi) > 2$.*

3. Obtaining all pseudo-Anosov homeomorphisms

The proof of Theorem A uses a generalization of the Rauzy–Veech construction of pseudo-Anosov homeomorphisms. We explain in Appendix C why a naive generalization of the Rauzy induction (obtained by combining left and right induction) does not work.

3.1. Construction of negative pseudo-Anosov maps: The symmetric Rauzy–Veech construction

The usual Rauzy–Veech construction naturally produces pseudo-Anosov maps that preserve the orientation of the stable and unstable foliation. The proposed generalization produces maps that reverse the orientation of the foliations.

We consider a path γ (not necessarily closed) in the labeled Rauzy diagram, from π to $s(\pi) = \pi'$ up to a relabeling. By a slight abuse of language, we say that γ is a path from π to $s(\pi)$. As in Section 2.7, one associates to γ a matrix $V(\gamma)$ by multiplying the corresponding product of the transition matrices with a suitable permutation matrix that corresponds to the relabeling between $s(\pi')$ and π. As in Section 2.7, $V(\gamma)$ is symplectic; thus one can choose an eigenvector λ for the eigenvalue θ and an eigenvector τ for the eigenvalue θ^{-1}. It turns out that τ is not necessarily a suspension datum, but it is a weak suspension datum, as is shown in the next proposition.

Proposition 3.1. *Let γ be a path in a Rauzy diagram from π to $s(\pi)$, and let $V := V(\gamma)$ be the corresponding matrix. We assume that V is primitive. If λ, τ are the eigenvectors as above, then*

- *up to multiplication by -1, τ is a weak suspension data for (π, λ);*
- *the surface $X(\pi, \lambda, \tau)$ admits an affine map ϕ whose derivative is*

$$\begin{pmatrix} -\theta & 0 \\ 0 & -\theta^{-1} \end{pmatrix},$$

  where θ is the maximal eigenvalue of V;
- *furthermore, ϕ admits a regular fixed point (with positive index).*

In the above situation, we will say that γ is admissible, and the associated negative pseudo-Anosov map will be referred to as $\phi(\gamma)$.

Proof of Proposition 3.1. Let $C(h)$ be the cone of weak suspension data of heights h over π. We have a map $V^{-1} : \mathbb{P}C(h) \to \mathbb{P}C(h')$. Observe that $C(h)$ is open and the union over all weak suspension data $C = \bigcup_h C(h)$ is an open convex cone. Hence there is an element $\tau \in C$ such that $[\tau] \in \mathbb{P}\overline{C}$ is fixed by V^{-1}. Hence there is $\theta \in \mathbb{R}$ such that $V^{-1}\tau = \theta\tau$.

We assume first that $\tau \in \mathring{C}$. We have that θ is the greatest eigenvalue of V^{-1}, hence of V. We construct the surface $(X,\omega) = X(\pi,\lambda,\tau)$. If $\zeta = \lambda + i\tau$, one has, for some integer k,

$$\mathcal{R}^k(\pi,\lambda,\tau) = (s(\pi), \theta^{-1}\lambda, \theta\tau)$$

Now there is a natural map f_1 from

$$(X_1,\omega_1) = X(\pi,\lambda,\tau) = X(s(\pi),\theta^{-1}\lambda,\theta\tau) \quad \text{to} \quad (X_2,\omega_2) = X(s(\pi),\lambda,\tau)$$

with $Df_1 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and a natural map f_2 from

$$(X_2,\omega_2) = X(s(\pi),\zeta) \quad \text{to} \quad (X_1,\omega_1) = X(s(\pi),\theta^{-1}\lambda + i\theta\tau)$$

with $Df_2 = \left(\begin{smallmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{smallmatrix}\right)$, and the composition $\psi = f_2 \circ f_1$ gives a pseudo-Anosov map affine on (X_1,ω_1) with θ as expansion factor and that reverse vertical and horizontal foliations. If we take $\phi = f_1 \circ f_2^{-1} = \psi^{-1}$, then $D\phi = \left(\begin{smallmatrix} -\theta & 0 \\ 0 & -\theta^{-1} \end{smallmatrix}\right)$ as required.

Now we prove that ϕ (or equivalently ψ) has a regular fixed point. Let h be a height of (π,λ,τ), and I_h the corresponding interval. Then the Rauzy–Veech induction gives a subinterval I'_h of I_h, and the image of I_h by ψ gives a interval I'_{h_2} defining the same weak suspension datum, with a different height (Figure 3). Hence there is a map from $\psi(I_h)$ to $I'_{h_1} \subset I_h$ with derivative $-\theta^{-1}$. It as a fixed point x; hence x and $\psi(x)$ are the endpoints of a vertical segment J_x that do not contain a singularity. There is a fixed point of ψ in J_x, which concludes the proof when $\tau \in \mathring{C}$.

Finally, we assume that $\tau \in \partial C$. By an argument similar to [6, Section 5.1], the data (π,λ,τ) still defines a translation surface, and in this case, there is a horizontal saddle connection. By the same argument as above, there is a pseudo-Anosov map ϕ preserving the horizontal and vertical foliation. This is a contradiction: the stable and unstable foliations of pseudo-Anosov maps are minimal. Proposition 3.1 is proved. ∎

3.2. An example

We use here the following convention (see Notation 1 and Figure 7): for any $k = 1,\dots,K_n$ and any $l = 1,\dots,n-2-k$, we define the path

$$\gamma_{n,k} : \pi := \pi_n.t^k \to s(\pi) : b^{n-1-k}t^{n-1-2k}$$

Lemma 3.2. *Let $n \geq 4$ and $1 \leq k \leq K_n$. Set $d = \gcd(n-1,k)$. We denote $n' = \frac{n-1}{d} + 1$ and $k' = \frac{k}{d}$. Then the matrix $V_{n',k'}$ is primitive and $\theta_{n,k} = \theta_{n',k'}$.*

Proof of Lemma 3.2. We first assume that k and $n-1$ are relatively prime. We compute the matrix $V_{n,k}$ associated to the path $\gamma_{n,k}$. Thereafter, in order to be able to compare the top eigenvalues of the matrices $V_{n,k}$, we will compute them with a labeling depending on k, with alphabet $\mathcal{A} = \{1,\dots,n\}$. To do this, we start from the central permutation, with the following labeling:

$$\pi_n = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ \alpha_n & \alpha_{n-1} & \cdots & \alpha_2 & \alpha_1 \end{pmatrix},$$

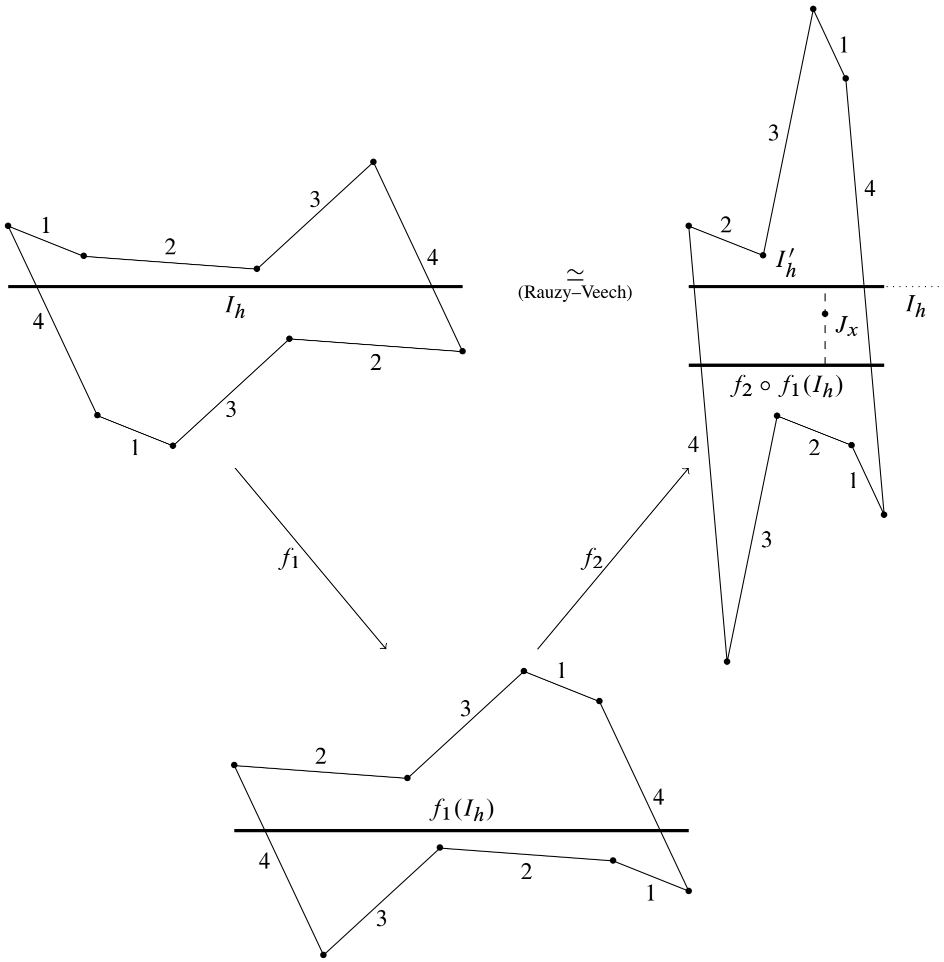

Fig. 3. The symmetric Rauzy–Veech construction.

where we have $\alpha_n = n$ and, for each $i \leq n-1$,

$$\alpha_i k = i - 1 \bmod (n-1) \quad \text{and} \quad \alpha_i \in \{1, \dots, n-1\}.$$

This is well defined since k and $n-1$ are relatively prime. In particular, $1 = \alpha_{k+1}$, $2 = \alpha_{2k+1}$, $\alpha_1 = n-1$, and more generally, $\alpha_{i+k} = \alpha_i + 1 \bmod n-1$. The starting point of $\gamma_{k,n}$ is

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_{n-1} & n \\ n & \cdots & \alpha_1 & \alpha_{n-1} & \cdots & \alpha_{k+1} \end{pmatrix}.$$

The path $\gamma_{n,k}$ consists of the Rauzy moves $b^{n-1-k}t^{n-1-2k}$; hence we have the following sequence of winners/losers.

- $1 = \alpha_{k+1}$ is successively winner against $\alpha_{k+2}, \dots, \alpha_{n-1}, n$.
- Then n is successively winner against $\alpha_{k+1}, \dots, \alpha_{n-1-k}$. Note that $k+1 < n-1-k$.

Also, the first line of the labeled permutation $s(\pi')$, where π' is the endpoint of $\gamma_{n,k}$, is

$$(\alpha'_1, \dots, \alpha'_{n-1}, n) = (\alpha_{n-k}, \alpha_{n-k+1}, \dots, \alpha_{n-1}, \alpha_0, \dots, \alpha_{n-1-k}, n),$$

i.e. $\alpha'_i = \alpha_{i-k}$ or $\alpha'_i = \alpha'_{i+n-1-k}$ depending on which of $i-k$ or $i+n-1-k$ is in $\{1, \dots, n-1\}$. In any case, we obtain $\alpha'_i = \alpha_i - 1$. Hence the matrix $V_{n,k}$ is obtained from the product of elementary Rauzy–Veech matrices by translating cyclically the first $n-1$ columns by 1 on the right. Finally, we have

$$V_{n,k} = \begin{pmatrix} a_{n-1} & 2 & a_2 & \cdots & \cdots & a_{n-2} & 1 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ 1 & 0 & & \cdots & & 0 & 0 \\ b_{n-1} & 1 & b_2 & \cdots & \cdots & b_{n-2} & 1 \end{pmatrix},$$

where, for $i \in \{2, \dots, n-2\}$, we have

$$a_i = \begin{cases} 2 & \text{if } i \in \{\alpha_{k+2}, \dots, \alpha_{n-1-k}\}, \\ 1 & \text{if } i \in \{\alpha_{n-k}, \dots, \alpha_{n-1}\}, \\ 0 & \text{if } i \in \{\alpha_1 \dots \alpha_k\}, \end{cases}$$

and $b_i = 1$ if and only if $a_i = 2$, and $b_i = 0$ otherwise. Note that $\alpha_1 = n-1$; hence $a_{n-1} = 0 = b_{n-1}$. Also, $\alpha_{n-k} = n-2$; hence $a_{n-2} = 1$ and $b_{n-2} = 0$. The matrix $V_{n,k}$ is clearly irreducible (this can be easily seen by inspecting its adjacency graph) and thus primitive since there is a non-zero diagonal element.

Now, if $d = \gcd(n-1, k) > 2$, we define α in the following way.

- $\alpha_{[1+ik]} = i$ for $i \le \frac{n-1}{d}$, where $[1+ik]$ is the representative modulo $n-1$ of $1+ik$, which is in $\{1, \dots, n-1\}$.
- $\alpha_n = \frac{n-1}{d} + 1$.
- The other α_i are chosen in any way.

The matrix $V_{n,k}$ with this labeling is

$$\left(\begin{array}{c|c} V_{\frac{n-1}{d}+1, \frac{k}{d}} & * \\ \hline 0 & * \end{array}\right),$$

where the bottom right block is a permutation matrix. This ends the proof of the lemma. ∎

Remark 3.3. When n is even and $k = \frac{n}{2} - 1$, we recover the expected systole, namely θ_{n,K_n} is the largest root of the polynomial $X^{n+1} - 2X^{n-1} - 2X^2 + 1$.

3.3. *Converse to the symmetric Rauzy–Veech construction*

Recall that, when considering pseudo-Anosov homeomorphism ϕ affine on a translation surface, we assume that ϕ expands the horizontal direction. As for the usual Rauzy–Veech construction, we have a converse.

Definition 3.4. Let ϕ be an affine negative pseudo-Anosov on a closed surface, having a regular fixed point p. A curve L is suitable for ϕ if the following conditions hold.

(1) It is made by a horizontal segment, starting from p, and then followed by a vertical segment, ending at a singular point. We do not allow L to have self-intersections.

(2) L and $\phi^{-1}(L)$ do not have intersections in their interior.

The horizontal part of $L \cup \phi^{-1}(L)$ will be called a base segment.

Proposition 3.5. *Given a base segment, the first return map of the vertical flow determines an interval exchange transformation and a weak suspension datum (π, λ, τ). Moreover, these data can be recovered by the symmetric Rauzy–Veech construction defined in Section 3.1.*

Proof. The fact that the base segment $I =]a, b[$ determines an interval exchange transformation $T = (\pi, \lambda)$ and a weak suspension datum τ, h for T is similar to the case for classical suspension data, except that the segment is not attached on the left to a singularity. This is left to the reader.

We only need to check that T defines a Rauzy path from π to $s(\pi)$ such that λ, τ are the corresponding eigenvectors.

The key remark is the following: the horizontal part of $\phi^{-1}(L) \cup \phi^{-2}(L)$ is a segment $I' \subset I$ that has the same left end as I. Observe that $\phi^{-1}(L)$ and $\phi^{-2}(L)$ do not have intersection on their interior; hence, by a classical argument, the interval exchange transformation T' associated to I' is obtained from T by applying a finite number of times the usual Rauzy induction (namely Rauzy induction on the right) to T. Similarly, for the weak suspension datum, we get $(\pi', \lambda', \tau') = (\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$.

Rotating the picture by 180°, we have, up to relabeling, $s(\pi^{(n)}) = \pi$, $\lambda^{(n)} = \frac{1}{\theta} \cdot \lambda$ and $\tau^{(n)} = \theta\tau$. This ends the proof. ∎

We have now all tools to prove the main theorem of this section.

Theorem 3.6. *Any affine negative pseudo-Anosov map having a regular fixed point P is obtained by the symmetric Rauzy–Veech construction.*

We emphasize that the underlying translation surface does not necessarily belong to a hyperelliptic locus.

Proof of Theorem 3.6. By Proposition 3.5, all we need to show is how to produce a suitable curve L for ϕ.

We start from any oriented curve L in X made by a finite horizontal segment, starting from P, and then followed by a finite vertical segment, ending at a singular point. We do not allow L to have self-intersections. Such a curve always exists (e.g. by using Veech's polygonal representation of translation surfaces). We denote the oriented horizontal and

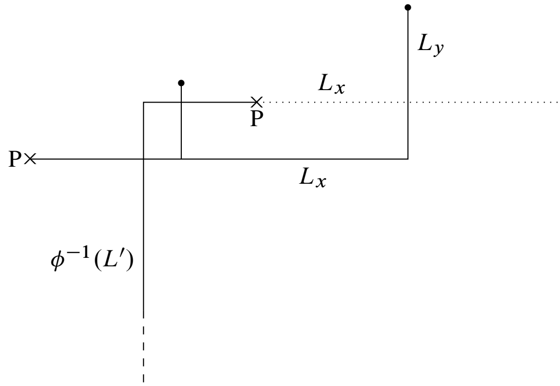

Fig. 4. Step 1: if $\phi^{-1}(L_x) \cap L_y = \emptyset$ and h is not minimal.

vertical components of L by L_x and L_y, respectively. They bound a rectangle R whose opposite corners are P and a zero σ of ω. We denote $\{c(L)\} = L_x \cap L_y$ and by h the length of L_y.

Now, if $\phi^{-1}(L) \cap \mathring{L} = \emptyset$, we are done. Otherwise, one of the following intersections is non-empty (possibly the two):

$$\phi^{-1}(L_x) \cap \mathring{L}_y \neq \emptyset \quad \text{or} \quad \phi^{-1}(L_y) \cap \mathring{L}_x \neq \emptyset.$$

We will perform several operations on L in order to obtain the required condition. The strategy is the following:

- 1st step: arranges that L bounds an immersed Euclidean rectangle $i(R)$,
- 2nd step: arranges that $\phi^{-1}(L_x) \cap \mathring{L}_y = \emptyset$,
- 3rd step: changes the fixed point in order to get a suitable curve.

1st step. We assume that h is *minimal* in the following sense: for each $q \in L_x$, the unit speed vertical geodesic starting from q (in the same direction as L_y) does not hit a singularity at a time less than h. Indeed, if h is not minimal, then there is a $q_0 \in L_x$ whose corresponding vertical geodesic hits a singularity for a time h' minimal. We then consider the new oriented curve L', starting from P such that $L'{}_x$ is the segment joining P to q_0 and $L'{}_y$ is the vertical segment of length h'. Note that L' still satisfies the no-self-intersection hypothesis (otherwise, we would find an element $q_1 \in L_x$ with $h_1 < h'$, contradicting the minimality assumption).

Now let $R \subset \mathbb{R}^2$ be the open rectangle of width $|L'_x|$ and of height h'. There is a natural translation map $i : R \to X$ that sends the bottom side of R to L'_x. By the assumption of minimality, $i(R)$ does not contain any singularity. Moreover, $P \notin i(R)$; otherwise, one easily see that the interior of L'_x intersects the interior of L'_y (see also Figure 4).

2nd step. Now we assume $\phi^{-1}(L_x) \cap \mathring{L}_y \neq \emptyset$. We first show that $\phi^{-1}(L_y) \cap \mathring{L}_x \neq \emptyset$. Let Q be the point in the intersection $\phi^{-1}(L_x) \cap \mathring{L}_y$ such that the vertical distance from

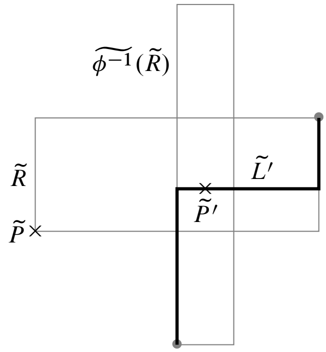

Fig. 5. Step 3: changing the fixed point.

$c(L)$ to Q is minimal. Since $|\phi^{-1}(L_x)| = \lambda^{-1}|L_x| < |L_x|$, one has $\phi^{-1}(c(L)) \in i(R)$. If $\phi^{-1}(L_y) \cap \mathring{L}_x = \emptyset$, then the vertical segment $\phi^{-1}(L_y)$ is contained in $i(R)$; in particular, $\phi^{-1}(\sigma) \in i(R)$: this contradicts the 1st step since there is no singularity inside $i(R)$.

Now we replace L by L' as follows: we choose Q' in $\phi^{-1}(L_y) \cap \mathring{L}_x$ such that the horizontal distance from P to Q' is minimal. Then we define L' by considering the horizontal segment, starting from P and ending at Q', and the vertical segment from Q' and ending at $\phi^{-1}(\sigma)$. Since

$$\phi^{-1}(L'_x) \cap L'_y \subset \phi^{-1}(L_x) \cap \phi^{-1}(L_y) = \{\phi^{-1}(c(L))\},$$

one has $\phi^{-1}(L'_x) \cap \mathring{L}'_y = \emptyset$ as required. Now, up to shortening L' as in the first step, we can assume that L'_x and L'_y bound an immersed rectangle R', and we still have

$$\phi^{-1}(L'_x) \cap \mathring{L}'_y = \emptyset.$$

3rd step. Let $\widetilde{X}$ be the universal covering of X. Choose $\widetilde{P}$ a preimage of P, $\widetilde{L}$ a preimage of L attached to $\widetilde{P}$. Now the rectangle R, as defined in the 1st step, embeds as a rectangle $\widetilde{R}$ in $\widetilde{X}$ with $\widetilde{L}$ as bottom and right sides. For any lift $\widetilde{\phi^{-1}}$ of ϕ^{-1} such that $\widetilde{\phi^{-1}}(\widetilde{L_y})$ intersects $\widetilde{R}$, $\widetilde{\phi^{-1}}(\widetilde{L_y})$ intersects the interior of $\widetilde{L_x}$ (in a unique point Q since we are working on the universal cover). Now we choose a lift $\widetilde{\phi^{-1}}$ of ϕ^{-1} that minimize the length d of the vertical segment joining Q to the singular point that is the end of $\widetilde{\phi^{-1}}(\widetilde{L_y})$. Now we easily see that $\widetilde{\phi^{-1}}(\widetilde{R})$ intersects $\widetilde{R}$ as in Figure 5. As in the proof of Proposition 3.1, we find $x \in L_x$ such that the corresponding vertical leaf is fixed by $\widetilde{\phi^{-1}}$. Then we find in $\widetilde{R}$ a fixed point $\widetilde{P}'$ for ϕ^{-1}.

Now we consider $\widetilde{L}'$ obtained as follows: take the horizontal segment with left end $\widetilde{P}'$ and whose right end is in $\widetilde{L_x}$, then consider the vertical segment that ends in the same singularity as $\widetilde{L}$ (Figure 5). We claim that $\widetilde{L}'$ projects in X into a suitable curve L'. Indeed, otherwise, $L'_x \cup \phi^{-1}(L'_x)$ either intersects the interior of L'_y or the interior of $\phi^{-1}(L'_y)$. In both cases, we find another intersection point between L_x and $\phi^{-1}(L'_y)$, which contradicts the minimality of d. Therefore, we have found a suitable curve; the theorem is proven. ■

For hyperelliptic surfaces, Theorem 3.6 reads as follows.

Proposition 3.7. *Let ϕ be an affine pseudo-Anosov map on a surface S in a hyperelliptic component, and let τ be the hyperelliptic involution. Then $\tau \circ \phi$ is also an affine pseudo-Anosov map on S. Denote $\{\phi, \tau \circ \phi\} = \{\phi^+, \phi^-\}$ such that ϕ^+ preserves the orientation of the vertical and horizontal foliations. We have the following.*

- *ϕ^- is obtained by the symmetric Rauzy–Veech construction.*
- *If ϕ^+ is not obtained by the usual Rauzy–Veech construction, then there are exactly two regular fixed points for ϕ^- that are interchanged by the hyperelliptic involution.*

Proof. We prove the first part. From the previous theorem, all we need to show is that ϕ^- has a regular fixed point. For a homeomorphism ϕ, we denote by ϕ_* the linear action of ϕ on the homology $H_1(S, \mathbb{R})$. We recall the Lefschetz formula

$$2 - \mathrm{Tr}(\phi_*) = \sum_{\phi(x)=x} \mathrm{Ind}(\phi, x).$$

When ϕ is of type pseudo-Anosov and x a fixed point, we can show that

- $\mathrm{Ind}(\phi, x) < 0$ if there is a fixed separatrix,
- $\mathrm{Ind}(\phi, x) = 1$ otherwise.

Assume that $\phi = \phi^+$ and the underlying translation surface is in $\mathcal{H}(2g-2)$. Then the unique singularity P is necessarily fixed and has index ≤ 1. All the other fixed points (possibly zero) are regular points, so have negative index. Hence $2 - \mathrm{Tr}(\phi_*) \leq 1$. Since $\mathrm{Tr}(\tau \circ \phi) = -\mathrm{Tr}(\phi)$, we conclude that $2 - \mathrm{Tr}(\phi^-_*) \geq 3$. Therefore, there must be at least two regular fixed points for ϕ^-.

Now assume that the underlying translation surface is in $\mathcal{H}(g-1, g-1)$. The two singularities P_1, P_2 are either fixed or interchanged by ϕ^+.

- If P_1, P_2 are fixed, then, as before, $2 - \mathrm{Tr}(\phi^+_*) \leq 2$; hence $2 - \mathrm{Tr}(\phi^-_*) \geq 2$. So there are a least two fixed points. But P_1, P_2 are interchanged by ϕ^-; hence the two fixed points are regular.
- If P_1, P_2 are interchanged by ϕ^+, then $2 - \mathrm{Tr}(\phi^+_*) \leq 0$; hence $2 - \mathrm{Tr}(\phi^-_*) \geq 4$. So there are at least four fixed points and hence at least two regular fixed points.

Now we see that, in the above proof, the case where ϕ^+ is not obtained by the usual Rauzy–Veech construction (i.e. when ϕ^+ does not have negative index fixed point) corresponds to the equality case. There is exactly one pair of regular fixed points $\{Q_1, Q_2\}$. Since ϕ^- and τ commute, we see that $\tau(Q_1)$ is a fixed point, hence Q_1 or Q_2. It cannot be Q_1; otherwise, $\tau \circ \phi^- = \phi^+$ has a regular fixed point, contradicting the hypothesis. Hence $\tau(Q_1) = Q_2$; this concludes the proof. ∎

We end this section with the following useful proposition.

Proposition 3.8. *Let γ be an admissible path from π to $s(\pi)$ passing through the central permutation, and let ϕ be the corresponding negative affine pseudo-Anosov map, obtained by the symmetric Rauzy–Veech construction. Then $\tau \circ \phi$ is obtained by the usual Rauzy–Veech construction, where τ is the hyperelliptic involution.*

Proof of Proposition 3.8. First the map $\psi = \tau \circ \phi$ is a positive pseudo-Anosov homeomorphism. Thus, according to Theorem 2.9, we only need to show that ψ fixes a separatrix. Let (π, λ, τ) be the weak suspension data associated to γ. Let h be a height, and $I = I_h \subset X(\pi, \lambda, \tau)$.

We claim that there is an immersed Euclidean rectangle whose horizontal sides are $\psi(I)$ and a subinterval of I. Assuming the claim, there is an isometry f from $\psi(I)$ to I obtained by following a vertical leaf (inside the Euclidean rectangle). The map $f \circ \psi$ is therefore a contracting map from I to itself (its derivative is θ^{-1}); hence f has a fixed point. It means that there is an element x in I whose image by ψ is in the vertical leaf l passing through x. Thus this vertical leaf l is preserved by ψ. Since ψ, restricted to l has derivative $\theta \neq 1$, there is a fixed point of ψ on l (perhaps this fixed point is a regular point). We conclude that ψ fixes a vertical separatrix (and thus also a horizontal separatrix) proving the proposition.

We now turn to the proof of the claim. By construction of ϕ, there is base segment I such that $I' = \phi(I) \subset I$. This subinterval I' is obtained from I after applying several steps of the Rauzy–Veech induction. By assumption, there is a step of the form $(\pi_n, \lambda'', \tau'')$, where π_n is the central permutation. This corresponds to an interval I'' satisfying $I' \subset I'' \subset I$. Now we easily see that $\tau(I'')$ and I'' are the horizontal sides of an immersed rectangle. ∎

4. Renormalization and changing the base permutation

A path γ is *pure* if $\tau \circ \phi(\gamma) = \phi(\gamma)^+$ is *not* obtained by the usual Rauzy–Veech construction. The aim of this section is, for pure admissible paths, to reduce our analysis to the set of paths of $\mathcal{D}_n$ starting from the central loop (see Definition 2.7).

Theorem 4.1. *For any pure admissible path γ in $\mathcal{D}_n$, there exists a path γ' starting from the central loop such that $\phi(\gamma)$ and $\phi(\gamma')$ are conjugated. In addition, the first step of γ' is on a secondary loop (i.e. of type b).*

The construction of γ' in terms of γ is not obvious. To prove the theorem, we appeal to the dynamics of the induction.

4.1. Renormalization: The ZRL acceleration

We fix a pure admissible path from π to $s(\pi)$. By Propositions 2.6 and 3.8, the coordinates $(n_1, \dots, n_k)$ of π satisfies k even (otherwise, γ passes through the central permutation, and thus γ is not a pure path).

Convention 1. We will always assume that the first Rauzy move is "t", i.e. π is obtained by applying the sequence $t^{n_1} b^{n_2} \dots t^{n_{k-1}}$ to π_n.

Observe that, up to conjugacy, we can always assume that Convention 1 holds.

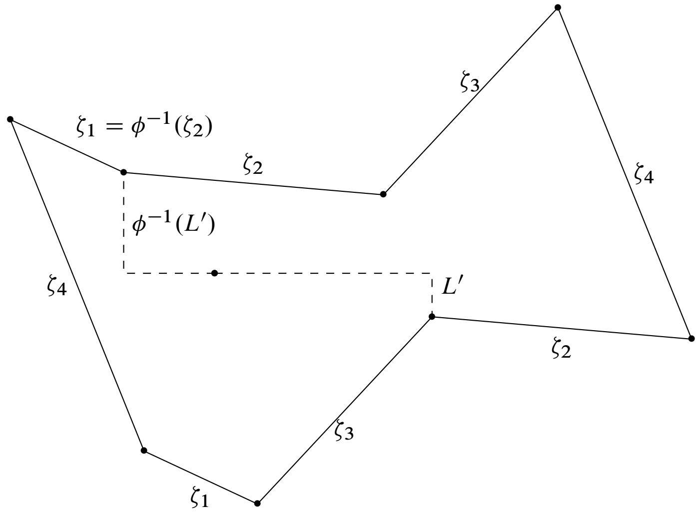

Fig. 6. Finding a new suitable curve: $\phi^{-1}(\zeta_2) = \zeta_1$.

Proposition 4.2. *Let $\alpha = \pi_b^{-1}(n)$ and $\beta = \pi_t^{-1}(1)$. Then $\phi^{-1}(\zeta_\alpha) = \zeta_\beta$. In particular, there exists a new uniquely defined base segment I' obtained as follows: we apply a right induction until α is loser (i.e. Rauzy path of type $b^l t$ for some $l \geq 0$) followed by a left induction until β is loser (i.e. left Rauzy path of type $\bar{t}^m \bar{b}$ for some $m \geq 0$). This segment I' defines a new Rauzy path γ' whose starting point π' satisfies Convention 1 (up to permuting top and bottom).*

Definition 4.3. The map ZRL from the space of pure admissible paths satisfying Convention 1 to itself is defined by $\mathrm{ZRL}(\gamma) = \gamma'$, where γ, γ' are as in Proposition 4.2.

Remark 4.4. Observe that, since the underlying translation surfaces obtained from γ and γ' are the same and since $\phi(\gamma)$ and $\phi(\gamma')$ are conjugated, the new path γ' is automatically pure so that ZRL is well defined.

Proof of Proposition 4.2. By assumption, the permutation π belongs to the connected component of $\mathcal{D}_n \backslash \{\pi_n\}$, where $\beta = \pi_t^{-1}(1)$ is never winner. As γ does not pass through the central permutation, α is never winner; hence the parameter ζ_β does not change during the steps of the Rauzy induction. In particular, by definition of ϕ (Figure 3), the segment corresponding to ζ_α is sent by ϕ^{-1} to the segment corresponding to ζ_β. Hence the curve L' as in Figure 6 is admissible and defines a new admissible path γ' from a permutation π' to $s(\pi')$. The horizontal part of $L' \cup \phi^{-1}(L')$ defines a new base segment I' and parameters (π', ζ') as described in Section 2.3. We obtain (π', ζ') by the sequence of right and left Rauzy moves described in the statement of the proposition. If π' does not satisfy Convention 1, we interchange the two lines of this permutation: this is equivalent to permuting "up" and "down" on the surface, and therefore to conjugate ϕ with an orientation reversing homeomorphism. ■

Remark 4.5. ZRL stands for Zorich acceleration of Right-Left induction. By construction, the two surfaces $X(\pi,\lambda,\tau)$ and $X(\pi',\lambda',\tau')$ belong to the same $\mathrm{GL}(2,\mathbb{R})$-orbit. It is difficult to simply express the path γ' in terms of the path γ. However, the coordinates for the permutations (see Definition 2.5) allow us to express the new starting point π' only in terms of π (see Proposition 4.7).

In view of considering iterates of the map ZRL, we will use the following lemma.

Lemma 4.6. *Assume that γ is a pure admissible path. Then the ZRL orbit of γ is infinite, and all the letters are winner and loser infinitely often.*

Proof. By Proposition 4.2, the new path γ' is a pure admissible path. Hence the ZRL orbit is infinite. In this proof, we do not interchange top and bottom line in order to follow Convention 1, but we consider a sequence of base segment $(I_n)_{n\in\mathbb{N}}$ on the same underlying surface X, with $I_{n+1}\subset I_n$. Note that the vertical flow on X is minimal (and uniquely ergodic) since X carries an affine pseudo-Anosov homeomorphism. We identify those segments as subsets of the real line, $I_n =]b_n,a_n[$, with (b_n) increasing and (a_n) decreasing. As for the usual Rauzy induction, all letters are winner infinitely often if and only if $|I_n| = a_n - b_n$ tends to zero. Let us assume that this is not the case. Then $a_n\to a$, $b_n\to b$ with $\phi^{-1}(a) = b$. We consider the sequence (λ_n,τ_n) of corresponding suspension data. Define t_n to be the time when the vertical unit speed geodesic starting from a_n reaches a singularity (t_n can be positive or negative). Observe that $|t_n|\to\infty$; otherwise, the set of singularities in X would not be discrete. Without loss of generality, we can assume that there is a subsequence (n_k) such that $t_{n_k} > 0$. There are two cases. We denote by g_a the geodesic in positive direction starting from a.

(1) If g_a is infinite, then it follows the (finite) one starts from a_{n_k} for a longer and longer time. By density, there is a time t such that $g_a(t)$ intersects $]b,a[$; hence, for k large enough, the geodesics starting from a_{n_k} intersect $]b,a[$, hence $I_{n,k}$ before reaching a singularity, a contradiction.

(2) If the geodesic g_a is finite in the positive direction, then it is necessarily infinite in the negative direction. If there is a subsequence $t_{m_k} < 0$, the same argument as above gives a contradiction. Hence, for n large enough, $t_n > 0$. The same argument also works for a non-accelerated right-left induction. Hence, for n large enough, the "right" part of the ZRL move is necessarily b. Similarly, the "left" part of the ZRL move is necessarily $\bar{t}$. But such move cannot hold for an infinite number of steps, a contradiction.

The lemma is proved. ■

4.2. Action of ZLR on the starting point

As announced, we now explain how ZRL acts on the starting point, in terms of the coding introduced in Definition 2.5.

Proposition 4.7. *Let γ be an admissible path, starting from a permutation π. For $k \geq 4$, let $(n_1, \dots, n_k)$ be the coding of π. The coding of the starting point of* ZRL(γ) *is obtained by the following rules.*

- *We replace (n_{k-1}, n_k) by*

$$\begin{cases} & (n_{k-1}+1, n_k-1), \\ or & (n_{k-1}, n_k-1, 1), \\ or & (n_{k-1}, \bar{l}, 1, n_k-1-\bar{l}). \end{cases}$$

- *We replace (n_1, n_2) by*

$$\begin{cases} & (n_1-1, n_2+1), \\ or & (1, n_1-1, n_2), \\ or & (n_1-1-\bar{m}, 1, \bar{m}, n_2), \end{cases}$$

where l, m are positive integers with $l < n_k - 1$, $m < n_1 - 1$, and $\bar{l}$, respectively $\bar{m}$, is the remainder of the Euclidean division of l, respectively m, by n_k, respectively n_1.

Proof of Proposition 4.7. By convention, the starting permutation π is of the form

$$t^{n_1} b^{n_2} \dots t^{n_{k-1}}.$$

The map ZRL acts on π by a sequence of Rauzy moves of the form $b^l t \bar{t}^m \bar{b}$ and then followed (perhaps) by a permutation of the lines (which does not change the coding).

The Rauzy moves $b^l t$ acts on the coding as

(1) $(n_1, \dots, n_k) \mapsto (n_1, \dots, n_{k-1}+1, n_k-1)$ if $l = 0 \bmod (n_k)$,

(2) $(n_1, \dots, n_k) \mapsto (n_1, \dots, n_{k-1}, n_k-1, 1)$ if $l = -1 \bmod (n_k)$,

(3) $(n_1, \dots, n_k) \mapsto (n_1, \dots, n_{k-1}, \bar{l}, 1, n_k-1-\bar{l})$ otherwise

(where $\bar{l}$ is the remainder of the Euclidean division of l by n_k). This gives the first part of the proposition.

The second part is obtained similarly: we must act on the left by the moves $\bar{t}^m \bar{b}$, which is equivalent to the moves $s.b^m.t.s$. This proves the proposition. ∎

We are now ready to prove the main theorem of this section.

Proof Theorem 4.1. Let γ be a pure admissible path. We need to show that there exists an iterate of ZRL which starts from a permutation π starting from the central loop, i.e. satisfying $\pi_t^{-1}(n) = \pi_b^{-1}(1)$.

We first observe the following: let $(l, \dots, l')$ be a coordinate of π; then $l \leq l'$. Indeed, if $l > l'$, then a path joining π to $s(\pi)$ must pass through the central permutation.

Now let $(n_1, , n_2, \dots, n_{k-1}, n_k) = (l, x, \dots, x', l')$ be the coding of a permutation, with $k \geq 4$ even. We prove by induction the following property $\mathcal{P}(l)$:

(1) $l = l'$.

(2) After applying a finite sequence of ZRL, we reach the permutation coded by

$$(l + x, \dots, x' + l'),$$

and during this sequence, the letters "between the blocks corresponding to x and x'", i.e. $\pi_\varepsilon^{-1}(j)$ for $j \in \{l+x+2, \dots, n-1-l'-x'\}$ are constant along the ZRL-orbit and non-winner. Note that, if $k \geq 4$, this set of letters is nonempty.

Initialization corresponds to the case $(1, x, \dots, x', l')$. Assume that $l' > 1$; then after one step of ZRL, the left part must be $(1+x, \dots)$, but in this case, in order to preserve the parity of the numbers of blocks, we must have

$$(x+1, \dots, x', l'-1, 1)$$

with $x+1 > 1$, which contradicts the initial observation. So we have $l' = 1$, and ZRL maps $(1, x, \dots, x', 1)$ to $(x+1, \dots, x'+1)$, and we see directly that the second condition is fulfilled.

Now let $1 < l \leq l'$, and let the initial permutation be coded by $(l, x, \dots, x', l')$. Assume $\mathcal{P}(l'')$ for any $l'' < l$. By Proposition 4.7 and observing again that the parity of the number of blocks is constant, after one step of ZRL, the coding is one of the following:

(1) $(1, l-1, x, \dots, x', l'-1, 1)$,
(2) $(l-1, x+1, \dots, x'+1, l'-1)$,
(3) $(l-1, x+1, \dots, x', l'_2, 1, l'_1)$ for some l'_1, l'_2 satisfying $l'_1 + l'_2 + 1 = l'$,
(4) $(l_1, 1, l_2, x, \dots, x'+1, l'-1)$ for some l_1, l_2 satisfying $l_1 + l_2 + 1 = l$,
(5) $(l_1, 1, l_2, x, \dots, x', l'_2, 1, l'_1)$ for l_1, l_2, l'_1, l'_2 as above.

Now we study these different cases.

(1) The following step is necessarily $(l, x, \dots, x', l')$.
(2) By the induction hypothesis, we have $l-1 = l'-1$, and after some steps, we obtain $(l+x, \dots, l'+x')$.
(3) By induction hypothesis, we have $l-1 = l'_1$, and after some steps of ZRL, we have $(l+x, \dots, x', l'_2, l)$, which again contradicts the first observation.
(4) By induction hypothesis, we have $l_1 = l'-1$. But $l_1 = l-1-l'_2 < l-1 \leq l'-1$, a contradiction.
(5) By induction hypothesis, we have $l'_1 = l_1$, and after some steps, we have

$$(l_1+1, l_2, x, \dots, x', l'_2, l_1+1),$$

and again after some steps, we have $(l, x, \dots, x', l')$. During these two sequences of ZRL, all the letters between the blocks corresponding to l_2 and l'_2 are unchanged and non-winner, and this set is nonempty.

Note that it is impossible to repeat infinitely steps (1) or (5) by Lemma 4.6; hence we will eventually get step (2), proving $\mathcal{P}(l)$.

Hence, after a finite number of ZRL steps, we obtain a permutation with a coding with two blocks ($k = 2$), which corresponds to a starting point in the central loop. This proves the first part of Theorem 4.1.

We now turn to the proof of the second part: the first step of the corresponding admissible Rauzy path leaves the central loop, i.e. is of type b. In this case, ZRL acts on the starting point π by $t\bar{b}$. We write π as

$$\begin{pmatrix} a & *** & b \\ b & *** & c \end{pmatrix}.$$

We easily see that $t\bar{b}$ preserves π, and λ_b becomes $\lambda_b - \lambda_a - \lambda_c$. Iterating ZRL, after a finite number of steps, b is not winner anymore. Theorem 4.1 is proved. ∎

5. Reducing to a finite number of paths

In view of Section 3, for a given n, one needs to control the spectral radii of matrices $V(\gamma)$ for all paths γ in $\mathcal{D}_n$. Section 4 shows that it is enough to consider paths γ starting from the central loop of $\mathcal{D}_n$. However, one still has to deal with infinitely many paths. In this section, we will show how one can restrict our problem to a *finite* number of paths.

Notation 1. For $n \geq 2$, we set $K_n = \lfloor \frac{n}{2} \rfloor - 1$ and $L_n = n - 2 - K_n$. Fix $k \in \{1, \ldots, K_n\}$. We define the path

$$\gamma_{n,k} : \pi \to s(\pi) : b^{n-1-k} t^{n-1-2k}.$$

If $l \in \{1, \ldots, 2n - 2 - 3k\}$, we also define the path

$$\gamma_{n,k,l} : \pi \to s(\pi) : \begin{cases} b^l t^{n-1-k-l} b^{n-1-k-l} t^{n-1-2k} \\ \qquad \text{if } 1 \leq l \leq n-2-k, \\ b^{n-1-k} t^{l-(n-1-k)} b^{2(n-1-k)-l} t^{2n-2-3k-l} \\ \qquad \text{if } n-2-k < l \leq 2n-2-3k. \end{cases}$$

Loosely speaking, $\gamma_{n,k}$ is the shortest path from π to $s(\pi)$ starting with a label "b", i.e. it is the concatenation of the small loop attached to π labeled "b" followed by the shortest path from π to $s(\pi)$. Similarly, $\gamma_{n,k,l}$ is a path joining π to $s(\pi)$ obtained from $\gamma_{n,k}$ by adding a loop: if $l \leq n-2-k$, then it is a "t" loop based at $\pi_n.t^k b^l$; if $l \geq n-1-k$, then it is a "b" loop based at $\pi_n.t^{l-(n-1-2k)}$ (Figure 7).

For any path γ in $\mathcal{D}_n$, we will denote by $\theta(\gamma)$ the maximal real eigenvalue of the matrix $V(\gamma)$. We will also use the notations $\theta_{n,k}$ (respectively $\theta_{n,k,l}$) for $\theta(\gamma_{n,k})$ (respectively $\theta(\gamma_{n,k,l})$). Observe that the matrix $V(\gamma)$ is not necessarily primitive, but the spectral radius is still an eigenvalue by the Perron–Frobenius theorem.

Theorem 5.1. *Let γ be an admissible path starting from a permutation $\pi = \pi_n.t^k$ in the central loop. We assume that the first step goes in the secondary loop.*

(1) *If $k > K_n$, then $\theta(\gamma) > 2$.*

(2) *If $k \leq K_n$, then $\theta(\gamma) \geq \theta_{n,k}$.*

(3) *If $V(\gamma_{n,k})$ is not primitive, either $\theta(\gamma) > 2$ or there exists $l \in \{1, \ldots, 2n-2-3k\}$, $l \neq n-1-k$, such that $\theta(\gamma) \geq \theta_{n,k,l}$.*

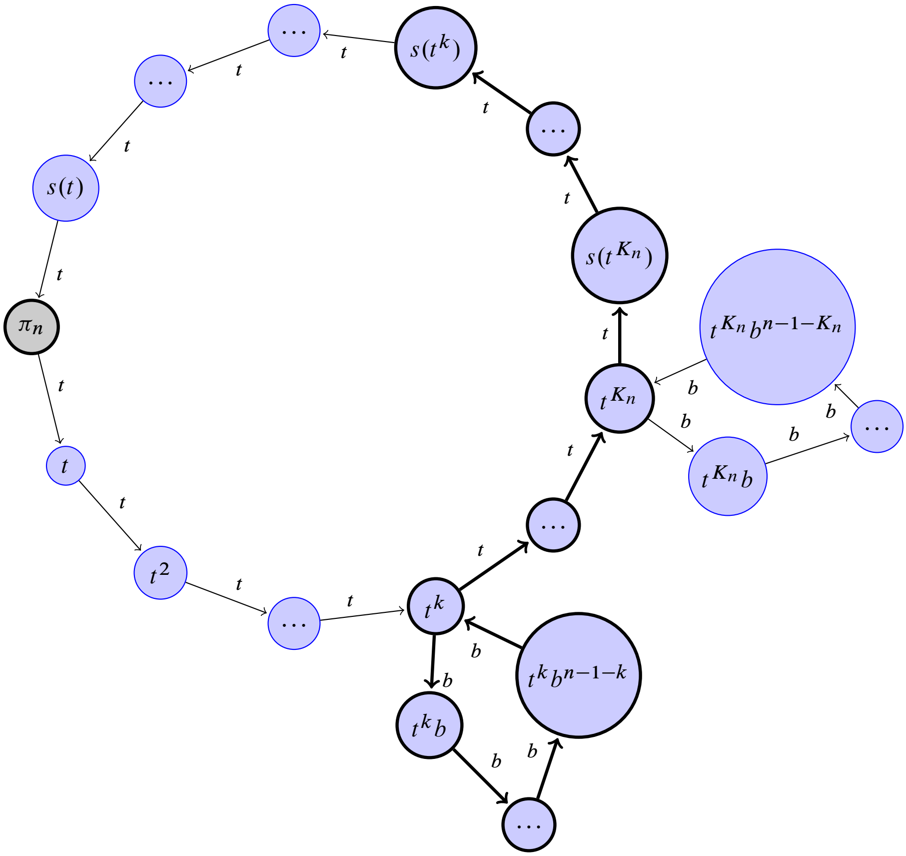

Fig. 7. Half of the Rauzy diagram $\mathcal{D}_n$. The path $\gamma_{n,k}$ is represented in bold style.

For two paths γ, γ', we will write $\gamma' \leq \gamma$ if the path γ' is a subset of the graph γ (viewed as an ordered collection of edges). For a real matrix $A \in M_n(\mathbb{R})$, we will write $A \geq 0$ (respectively $A \gg 0$) to mean that $A_{ij} \geq 0$ (respectively $A_{ij} > 0$) for all indices $1 \leq i, j \leq n$, and similarly for vectors $v \in \mathbb{R}^n$. The notation $A \geq B$ means $A - B \geq 0$. Before proving Theorem 5.1, we will use the following.

Proposition 5.2. *Let $\gamma' \leq \gamma$, i.e. the path γ is obtained from the path γ' by adding (possibly zero) closed loops. Then $V(\gamma') \leq V(\gamma)$ and $\theta(\gamma') \leq \theta(\gamma)$. Moreover, if $V(\gamma)$ is primitive and $\gamma \neq \gamma'$, then $\theta(\gamma') < \theta(\gamma)$.*

Proof of Proposition 5.2. If $V(\gamma') = V_1 \cdot V_2 \cdot \ldots \cdot V_l \cdot P'$ is the matrix associated to the path γ', where V_i are the elementary Rauzy–Veech matrices and P' is a permutation matrix, then the matrix associated to γ has the form

$$V(\gamma) = V_1 \cdot N_1 \cdot V_2 \cdot N_2 \cdot \ldots \cdot V_l \cdot N_l \cdot P,$$

where

- $N_1, \dots, N_l$ are products of (possibly empty) elementary Rauzy–Veech matrices, hence of the type $I + N_i'$, where I is the identity matrix and N_i' is a matrix with nonnegative coefficients,
- P is the permutation matrix corresponding to the end point of γ.

Since the labeled Rauzy diagram and the reduced Rauzy diagram coincide, the endpoints of γ and γ' also coincide in the labeled Rauzy diagram; hence $P = P'$. From these facts, we deduce that $V(\gamma') \leq V(\gamma)$. Let us show that $\theta(\gamma') \leq \theta(\gamma)$.

Recall that $V(\gamma')$ is not necessarily primitive. However, there is a permutation matrix P_σ such that

$$P_\sigma V(\gamma') P_{\sigma^{-1}} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ * & A_2 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ * & \cdots & * & A_s \end{pmatrix},$$

where the matrices A_i are primitive matrices. Up to a change of basis, one can assume that the spectral radius of A_s is achieved by $\theta(\gamma')$. Thus there is a non-negative vector w such that $A_s w = \theta(\gamma') w$. Now $v' := P_{\sigma^{-1}}(0 \ \dots \ 0 \ w)^T$ is a non-negative right eigenvector of $V(\gamma')$ for the eigenvalue $\theta(\gamma')$.

Let v be a positive left eigenvector of $V(\gamma)$ such that $vV(\gamma) = \theta(\gamma)v$ and $v > 0$. From $V(\gamma') \leq V(\gamma)$, one has

$$vV(\gamma)v' \geq vV(\gamma')v'.$$

Hence $\theta(\gamma)vv' \geq \theta(\gamma')vv'$, and since $vv' > 0$, we draw $\theta(\gamma) \geq \theta(\gamma')$ as desired.

We now prove that last claim: assume $V(\gamma') \leq V(\gamma)$ and $V(\gamma) \neq V(\gamma')$. Then there exists $k \in \mathbb{Z}_{>0}$ such that $V(\gamma)^k \gg V(\gamma')^k$. In particular, we can find $\alpha > 1$ such that $V(\gamma)^k \geq \alpha V(\gamma')^k$. Thus $\rho(\gamma)^k \geq \alpha \rho(\gamma')^k$ proving the proposition. ∎

Proof of Theorem 5.1. We prove the first assertion. Let k be any integer with $k > \frac{n-1}{2}$. Then $n - 1 - k < k$; hence $s(\pi) = \pi_n . t^{n-1-k}$ is "before" π in the central loop. Since γ is admissible, γ passes thought the central permutation. Now, if $k = \frac{n-1}{2}$, then up to relabeling, $\pi = s(\pi)$. By using the alphabet $\mathcal{A} = \{1, \dots, n\}$, we have

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & \frac{n-1}{2} & \cdots & \frac{n+1}{2} \end{pmatrix}.$$

Hence the relabeling does not changes the letter n. In particular, it must be winner at least once. This implies that γ contains the step $\pi \to t.\pi$, hence passes through the central permutation. In both situations, $\theta > 2$.

We now turn to the proof of the second part of the proposition. The Rauzy diagram that we consider has the particular property that removing any vertex disconnects it (except the particular ones that satisfy $t.\pi = \pi$ or $b.\pi = \pi$). This implies that the path γ is obtained from the path $\gamma_{n,k}$ by adding (possibly zero) closed loops, namely $\gamma_{n,k} \leq \gamma$. Hence we are in a position to apply Proposition 5.2; this finishes the proof of this part.

The proof of the last statement is similar once we remark that if $V_{n,k}$ is not primitive, then the path γ is obtained from $\gamma_{n,k}$ by adding at least one closed loop. Assuming that $\theta(\gamma) < 2$, this gives by definition a path of the form

$$\gamma_{n,k,l} \quad \text{for some } l \in \{1, \dots, 2n-2-3k\}.$$

Thus $\gamma_{n,k,l} \leq \gamma$, and Proposition 5.2 again applies. If $l = n-1-k$, then $\gamma_{n,k,l}$ is obtained from $\gamma_{n,k}$ by adding twice the same loop. Hence $V_{n,k,l}$ is *not* primitive and $\gamma \neq \gamma_{n,k,l}$. The same argument above applies, and γ must contain $\gamma_{n,k,l}$ for some $l \neq n-1-k$. We conclude with Proposition 5.2. ∎

Finally, to provide a clearer picture of the combinatorial part, we state the following theorem, corresponding to Proposition 5.1, Lemma 3.2 and Proposition A.4.

Theorem 5.3. *Let $n \geq 4$ be any integer. The following holds.*

(1) *Let γ be an admissible path starting from $\pi = \pi_n.t^k$ for some $k \leq n-1$ such that the first step is "b". We assume that $\theta(\gamma) < 2$. Then $k \leq K_n$ and $\theta(\gamma) \geq \theta_{n,k}$. Furthermore, if $\gamma \neq \gamma_{n,k}$, then there exists $l \in \{1, \dots, 2n-2-3k\}$ such that*

$$\theta(\gamma) \geq \theta_{n,k,l}.$$

If $V(\gamma_{n,k})$ is not primitive, we can also assume that $l \neq n-1-k$ in the previous statement.

(2) *For any $k \in \{1, \dots, K_n\}$, let $d = \gcd(n-1, k)$. If $n' = \frac{n-1}{d} + 1$ and $k' = \frac{k}{d}$, then the matrix $V_{n',k'}$ is primitive and $\theta_{n,k} = \theta_{n',k'}$.*

(3) *Let $n \equiv 3 \bmod 4$ and $l \in \{1, \dots, L_n + 3\}$. The matrix $V_{n,K_n,l}$ is primitive if and only if l is odd. Moreover, if l is even, then $\theta_{n,K_n,l} = \theta_{n',K_{n'},l'}$ with $n' = \frac{n+1}{2}$, $l' = \frac{l}{2}$ and $K_{n'} = \frac{K_n}{2}$.*

6. Proof of Theorem A and Theorem B

Theorem A and Theorem B will follow immediately from Theorems 6.1, 6.3 and 6.5. Recall that if $n = 2g$ is even, then $\mathcal{C}_n^{\mathrm{hyp}} = \mathcal{H}^{\mathrm{hyp}}(2g-2)$, and if $n = 2g+1$ is odd, then $\mathcal{C}_n^{\mathrm{hyp}} = \mathcal{H}^{\mathrm{hyp}}(g-1, g-1)$.

Theorem 6.1. *The following holds.*

(1) *If $n \geq 2$ is even, then $L(\mathrm{Spec}(\mathcal{C}_n^{\mathrm{hyp}})) = \log(\theta_{n,K_n})$.*

(2) *If $n \not\equiv 4 \bmod 6$ and $n \geq 18$ is even, then the second least element of $\mathrm{Spec}(\mathcal{C}_n^{\mathrm{hyp}})$ is $\log(\theta_{n,K_n-1})$.*

Moreover, there is a unique conjugacy negative pseudo-Anosov mapping class realizing the minimum.

Remark 6.2. Let $P_{n,k}$ be the characteristic polynomial of $V(\gamma_{n,k})$ *multiplied* by $X+1$. By definition, $\theta_{n,k}$ is the maximal real root of $P_{n,k}$. By Lemma A.2, we have, when $n=2g$,

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^2 + 1,$$

and when $n \not\equiv 4 \bmod 6$ and $\gcd(n-1, K_n-1) = 1$,

$$P_{n,K_n-1} = X^{n+1} - 2X^{n-1} - 2X^{\lceil \frac{2n}{3} \rceil} - 2X^{\lfloor \frac{n}{3} \rfloor + 1} - 2X^2 + 1,$$

which are the desired formulas for the case $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ of Theorem A and Theorem B.

Proof of Theorem 6.1. We will show that

$$L(\operatorname{Spec}(\mathcal{C}_n^{\mathrm{hyp}})) = \log(\theta_{n,K_n}).$$

Observe that $K_n = \frac{n}{2} - 1$; thus $\gcd(n-1, K_n) = 1$, and the matrix V_{n,K_n} is primitive (see Theorem 5.3). This shows $L(\operatorname{Spec}(\mathcal{C}_n^{\mathrm{hyp}})) \le \log(\theta_{n,K_n})$. A simple computation shows $\theta_{n,K_n} < 2$.

Now, by Proposition 3.7, $L(\operatorname{Spec}(\mathcal{C}_n^{\mathrm{hyp}})) = \log(\theta(\gamma))$ for some admissible path γ from a permutation π to $s(\pi)$. We must have $\theta(\gamma) < 2$; hence the path γ is pure. Now, from Theorem 4.1, we can assume that the path γ starts from $\pi = \pi_n.t^k$ with first step of type b. By Theorem 5.1, $\theta(\gamma) \ge \theta_{n,k}$ for some $k \in \{1, \dots, K_n\}$. Thus $\theta_{n,k} \le \theta_{n,K_n}$. We need to show that $k = K_n$. Let us assume that $k < K_n$.

(1) If $\gcd(k, n-1) = 1$, then Lemma B.3 implies $\theta_{n,k} > \theta_{n,K_n}$, which is a contradiction.

(2) If $\gcd(k, n-1) = d > 1$, then by the second point of Theorem 5.3, $\theta_{n,k} = \theta_{n',k'}$, where $n' = \frac{n-1}{d} + 1$ and $k' = \frac{k}{d}$. Since n' is even, $k' \neq K_{n'}$ and $\gcd(k', n'-1) = 1$, the previous step applies, and we have $\theta_{n',k'} > \theta_{n',K_{n'}}$. By Lemma B.2, the sequence $(\theta_{2n,K_{2n}})_n$ is a decreasing sequence, and hence $\theta_{n',K_{n'}} > \theta_{n,K_n}$. Again, we run into a contradiction.

In conclusion, $k = K_n$ and $L(\operatorname{Spec}(\mathcal{C}_n^{\mathrm{hyp}})) = \log(\theta_{n,K_n})$. By construction and since all inequalities above are strict, the conjugacy class of negative mapping classes realizing this minimum is unique.

We now finish the proof of the theorem with the second least expansion factor. The assumption $n \not\equiv 4 \bmod 6$ implies $\gcd(K_n - 1, n-1) = 1$. Hence V_{n,K_n-1} is primitive, and the second least expansion factor is less than or equal to θ_{n,K_n-1}. As before, a simple computation shows that $\theta_{n,K_n-1} < 2$ (see Lemma A.2).

Conversely, the second least expansion factor equals $\theta(\gamma)$ for some admissible path γ starting from $\pi = \pi_n.t^k$ for some $k \in \{1, \dots, K_n\}$, with "b" as the first step. If $k = K_n$, since $\gamma \neq \gamma_{n,K_n}$, Theorem 5.3 implies that $\theta(\gamma) \ge \theta_{n,K_n,l}$ for some

$$l \in \{1, \dots, 2n-2-3K_n\} = \{1, \dots, L_n + 2\}.$$

Again, Theorem 5.3 shows that if $k \le K_n - 1$, then $\theta(\gamma) \ge \theta_{n,k}$.

The theorem will follow from the following two assertions that are proven in Proposition B.8.

- For any $k = 1, \dots, K_n - 2$, one has $\theta_{n,k} > \theta_{n,K_n-1}$.
- For any $l = 1, \dots, L_n + 2$, one has $\theta_{n,K_n,l} > \theta_{n,K_n-1}$.

This ends the proof of Theorem 6.1. ∎

Theorem 6.3. *If $n \geq 5$ and $n \equiv 1 \bmod 4$, then $L(\mathcal{C}_n^{\mathrm{hyp}}) = \log(\theta_{n,K_n})$.*

Remark 6.4. By Lemma A.2, when $n = 1 \bmod 4$, we have that θ_{n,K_n} is the maximal real root of

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^{\frac{n+1}{2}} - 2X^2 + 1,$$

which is the desired formula for Theorem A, case $\mathcal{H}^{\mathrm{hyp}}(g-1, g-1)$ with g even (recall that $n = 2g+1$).

Proof of Theorem 6.3. We follow the same strategy as in the previous proof. Namely, $L(\mathcal{C}_n^{\mathrm{hyp}}) \leq \log(\theta_{n,K_n})$ since V_{n,K_n} is irreducible (see Theorem 5.3).

Now $L(\mathcal{C}_n^{\mathrm{hyp}}) = \log(\theta(\gamma))$ for some admissible path γ starting from $\pi_n.t^k$, where $k \in \{1, \dots, K_n\}$, and the path starts by a "b". Recall that we have shown $\theta(\gamma) \leq \theta_{n,K_n}$. We need to show that, for any $k \leq K_n - 1$, $\theta_{n,k} > \theta_{n,K_n}$. Again, by Theorem 5.3, we have $\theta_{n,k} = \theta_{n',k'}$, where $n' = \frac{n-1}{d} + 1$, $k' = \frac{k}{d}$ and $d = \gcd(n-1, k)$. By Proposition B.9, $\theta_{n',k'} > \theta_{n,K_n}$. This finishes the proof of Theorem 6.3. ∎

Theorem 6.5. *If $n \geq 7$ and $n \equiv 3 \bmod 4$, then $L(\mathcal{C}_n^{\mathrm{hyp}}) = \log(\theta_{n,K_n,L_n})$.*

Remark 6.6. Proposition A.4 gives that θ_{n,K_n,L_n} is the maximal real root of

$$P_{n,K_n,L_n} = X^{n+1} - 2X^{n-1} - 4X^{\frac{n-1}{2}+2} + 4X^{\frac{n-1}{2}} + 2X^2 - 1.$$

That gives the desired formula for Theorem A, case $\mathcal{H}^{\mathrm{hyp}}(g-1, g-1)$ with g odd (recall that $n = 2g+1$).

Proof of Theorem 6.5. As V_{n,K_n,L_n} is irreducible (see Theorem 5.3 where $n \equiv 3 \bmod 4$ and L_n is odd), we have $L(\mathcal{C}_n^{\mathrm{hyp}}) \leq \log(\theta_{n,K_n,L_n})$.

We also have $L(\mathcal{C}_n^{\mathrm{hyp}}) = \log(\theta(\gamma))$, where γ is an admissible path starting from $\pi_n.t^k$ for some $k \in \{1, \dots, K_n\}$, and the path starts by a "b" (recall that $\theta(\gamma) \leq \theta_{n,K_n,L_n}$).

Assume that $k \leq K_n - 1$ holds. Then Theorem 5.3 shows that $\theta(\gamma) \geq \theta_{n,k}$. Letting $n' = \frac{n-1}{d} + 1$ and $k' = \frac{k}{d}$, where $d = \gcd(n-1, k)$, we have $\theta_{n,k} = \theta_{n',k'}$. By Proposition B.6,

$$\theta_{n',k'} > \theta_{n,K_n,L_n}.$$

This is a contradiction with $\theta(\gamma) \leq \theta_{n,K_n,L_n}$. Hence we necessarily have $k = K_n$.

Now the matrix V_{n,K_n} is *not* primitive; hence $\gamma \neq \gamma_{n,K_n}$. Thus Theorem 5.3 implies that $\theta(\gamma) \geq \theta_{n,K_n,l}$ for some

$$l \in \{1, \dots, 2n - 2 - 3K_n\} = \{1, \dots, L_n + 3\}$$

with $l \neq n - K_n - 1 = L_n + 1$. We will discuss two different cases depending on the parity of l (recall that L_n is odd).

Case 1: l is even. By Theorem 5.3, one has $\theta_{n,K_n,l} = \theta_{n',K_{n'},l'}$ with $n' = \frac{n+1}{2}$, $l' = \frac{l}{2}$ and $K_{n'} = \frac{K_n}{2}$. If $l < L_n$, then $l' < L_{n'}$ and Lemma B.7 applies (since $n' \geq 4$ is even):

$$\theta_{n',K_{n'},l'} > \theta_{n',K_{n'},L_{n'}}.$$

If $l > L_n$, then $l = L_n + 3$. Again, Theorem 5.3 gives $\theta_{n,K_n,L_n+3} = \theta_{n',K_{n'},L_{n'}+2}$. In all cases, Proposition B.6 applies, and

$$\theta_{n',K_{n'},L_{n'}} > \theta_{n,K_n,L_n},$$

running into a contradiction.

Case 2: l is odd. If $l < L_n$, then by Proposition B.5, we have $\theta_{n,K_n,l} > \theta_{n,K_n,L_n}$. This is again a contradiction. The case $l > L_n$, namely $l = L_n + 2$, is ruled out by Lemma B.4: $\theta_{n,K_n,L_n+2} > 2 > \theta_{n,K_n,L_n}$.

In conclusion, $l = L_n$ and $\gamma = \gamma_{n,K_n,L_n}$. ∎

Appendix A. Matrix computations

The aim of this section is the computation of the different Rauzy–Veech matrices in $M_n(\mathbb{Z})$ and especially their characteristic polynomials. The rome technique (see below) reduces these computations to computations in $M_2(\mathbb{Z}[X])$ and $M_3(\mathbb{Z}[X])$ that are easier to handle.

In the sequel, we will denote by $P_{n,k}$ (respectively $P_{n,k,l}$) the characteristic polynomial of the matrices $V(\gamma_{n,k})$ (respectively $V(\gamma_{n,k,l})$) *multiplied* by $(X+1)$. Their maximal real root is $\theta_{n,k}$ and $\theta_{n,k,l}$, respectively.

A.1. The rome technique (after [5])

To compute the characteristic polynomial of matrices, we will use the rome method, developed in [5]. To this end, it is helpful to represent a matrix V into the form of a combinatorial graph which amounts to draw all paths of length 1 associated to V.

Given an $n \times n$ matrix $V = (v_{ij})$, a path $\eta = (\eta_i)_{i=0}^{l}$ of width $w(\eta)$ and length l is a sequence of elements of $\{1, 2, \ldots, n\}$ such that $w(\eta) = \prod_{j=1}^{l} v_{\eta_{j-1}\eta_j} \neq 0$. If $\eta_l = \eta_0$, we say that η is a loop.

A subset $R \subset \{1, 2, \ldots, n\}$ is called a "rome" if there are no loops outside R. Given $r_i, r_j \in R$, a path from r_i to r_j is a "first return path" if it does not intersect R, except at its starting and ending points. This allows us to define an $r \times r$ matrix $V_R(X)$ with coefficients in $\mathbb{Z}(X)$, where r is the size of R, by setting $V_R(X) = (a_{ij}(X))$, where $a_{ij}(X) = \sum_{\eta} w(\eta) \cdot X^{-l(\eta)}$, where the summation is over all first return paths beginning at r_i and ending at r_j.

Theorem ([5, Theorem 1.7]). *If R is a rome of cardinality r of an $n \times n$ matrix V, then the characteristic polynomial $\chi_V(X)$ of V is equal to*

$$(-1)^{n-r} X^n \det(V_R(X) - \mathrm{Id}_r).$$

Remark A.1. The matrices $V = V_{n,k}$ or $V = V_{n,k,l}$ can be seen as the action of homeomorphisms on absolute homology if n is even and relative homology otherwise. Thus their characteristic polynomials are reciprocal polynomials when n is even, and either reciprocal or anti-reciprocal polynomials when n is odd, depending on whether ϕ fixes or interchanges the two singularities), i.e. $\chi_V(X) = X^n\chi_V(X^{-1})$ or $\chi_V(X) = -X^n\chi_V(X^{-1})$. Thus $\chi_V = \pm\det(V_R(X^{-1}) - \mathrm{Id}_r)$.

A.2. The paths $\gamma_{n,k}$

We briefly recall Notation 1 (see also Figure 7): we define the path

$$\gamma_{n,k} : \pi := \pi_n.t^k \to s(\pi) : b^{n-1-k}t^{n-1-2k}.$$

for any $k = 1,\dots,K_n$ and any $l = 1,\dots,n-2-k$.

Lemma A.2. *Let $n \geq 4$ and $1 \leq k \leq K_n$. If $\gcd(n-1,k) = 1$, then*

$$P_{n,k} = X^{n+1} - 2X^{n-1} - 2\sum_{j\in J_{n,k}} X^j - 2X^2 + 1,$$

where $J_{n,k} = \{3,\dots,n-2\}\setminus\{\lceil\frac{i(n-1)}{k}\rceil + \varepsilon \mid i = 1,\dots,k-1$ and $\varepsilon = 0,1\}$. In particular, the following holds.

(1) *For even n,*

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^2 + 1,$$

and if $n \not\equiv 4 \bmod 6$, $\gcd(n-1, K_n - 1) = 1$ and

$$P_{n,K_n-1} = X^{n+1} - 2X^{n-1} - 2X^{\lceil\frac{2n}{3}\rceil} - 2X^{\lfloor\frac{n}{3}\rfloor+1} - 2X^2 + 1.$$

(2) *For $n = 1 \bmod 4$, we have $\gcd(n-1, K_n) = 1$ and*

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^{\frac{n+1}{2}} - 2X^2 + 1.$$

(3) *For $n = 3 \bmod 4$, we have $\gcd(n-1, K_n - 1) = 1$ and*

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^{n-2\lfloor\frac{n-5}{8}\rfloor-2} - 2X^{\frac{n+1}{2}} - 2X^{2\lfloor\frac{n-5}{8}\rfloor+3} - 2X^2 + 1.$$

(4) *More generally, for n even and $k \leq K_n - 1$, the highest nonzero monomial (except X^{n+1} and $-2X^{n-2}$) has degree at least $\lceil\frac{2n}{3}\rceil$.*

Proof of Lemma A.2. We will use the rome method as explained in Section A.1. We use the notation of the proof of Lemma 3.2. We will write $V_{n,k}$ as $V_{n,k} = A_n - B_{n,k}$, where

$$A_n = \begin{pmatrix} 0 & 2 & 2 & \cdots & 2 & 1 & 1 \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ 1 & 0 & & \cdots & & & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix},$$

and the only non-zero entries of $B_{n,k}$ are as follows: for any $i = 1, \dots, k-1$, set

$$l := \left\lfloor \frac{i(n-1)}{k} \right\rfloor + 1.$$

Then

$$\begin{cases} b_{1,l} = 1 \\ b_{1,l+1} = 2 \\ b_{n,l} = b_{n,l+1} = 1. \end{cases}$$

Observe that $1 \le k \le \frac{n}{2} - 1$; hence, for $i \in \{1, \dots, k-1\}$,

$$2 < 2i\frac{(n-1)}{n-2} \le \frac{i(n-1)}{k} \le \frac{(k-1)(n-1)}{k} = n - 1 - \frac{n-1}{k} < n - 3.$$

In particular, all integers of the form $\lceil \frac{i(n-1)}{k} \rceil + \varepsilon$ for $i \in \{1, \dots, k-1\}$ and $\varepsilon \in \{0, 1\}$ are mutually disjoint and in $\{3, \dots, n-2\}$.

Clearly, the set $R = \{1, n\}$ is a rome for A_n. Thus it is also a rome for $V_{n,k}$ (since we pass from A_n to $V_{n,k}$ by removing some paths). The 2×2 matrix $(A_n)_R$ is easily obtained as

$$(A_n)_R = \begin{pmatrix} X^2 + 2S_n & X \\ S_n & X \end{pmatrix}, \quad \text{where } S_n = \sum_{i=3}^{n-1} X^i.$$

To obtain the matrix $(V_{n,k})_R$, one has to subtract the polynomial corresponding to the paths passing through arrows with dashed lines (passing through vertices l and $l+1$, where $l = \lfloor \frac{i(n-1)}{k} \rfloor + 1$ for some $i = 1, \dots, k-1$). At this aim, we define

$$\begin{cases} T_{n,k} = \sum_{i=1}^{k-1} X^{n+1-(\lfloor \frac{i(n-1)}{k} \rfloor + 1)}, \\ Q_{n,k} = \sum_{i=1}^{k-1} X^{n-1-\lfloor \frac{i(n-1)}{k} \rfloor} = \sum_{i=1}^{k-1} X^{\lceil \frac{i(n-1)}{k} \rceil}. \end{cases}$$

The polynomial $T_{n,k}$ (respectively $2Q_{n,k}$) takes into account all simple paths from the vertex 1 to the vertex 1 passing through arrows with dashed lines connecting 1 to l (respectively to $l+1$). Hence

$$(V_{n,k})_R = \begin{pmatrix} X^2 + 2S_n - T_{n,k} - 2Q_{n,k} & X \\ S_n - T_{n,k} - Q_{n,k} & X \end{pmatrix}.$$

Note that $T_{n,k} = XQ_{n,k}$. By using [5], a straightforward computation gives

$$\chi_{V_{n,k}} = X^3 - X^2 - X + 1 + S_n \cdot (X-2) + 2Q_{n,k}.$$

Hence

$$P_{n,k} = X^{n+1} - 2\sum_{i=2}^{n-1} X^i + 1 + 2\sum_{i=1}^{k-1}(X^{\lceil \frac{i(n-1)}{k} \rceil} + X^{\lceil \frac{i(n-1)}{k} \rceil + 1}),$$

which implies the first statement of the lemma.

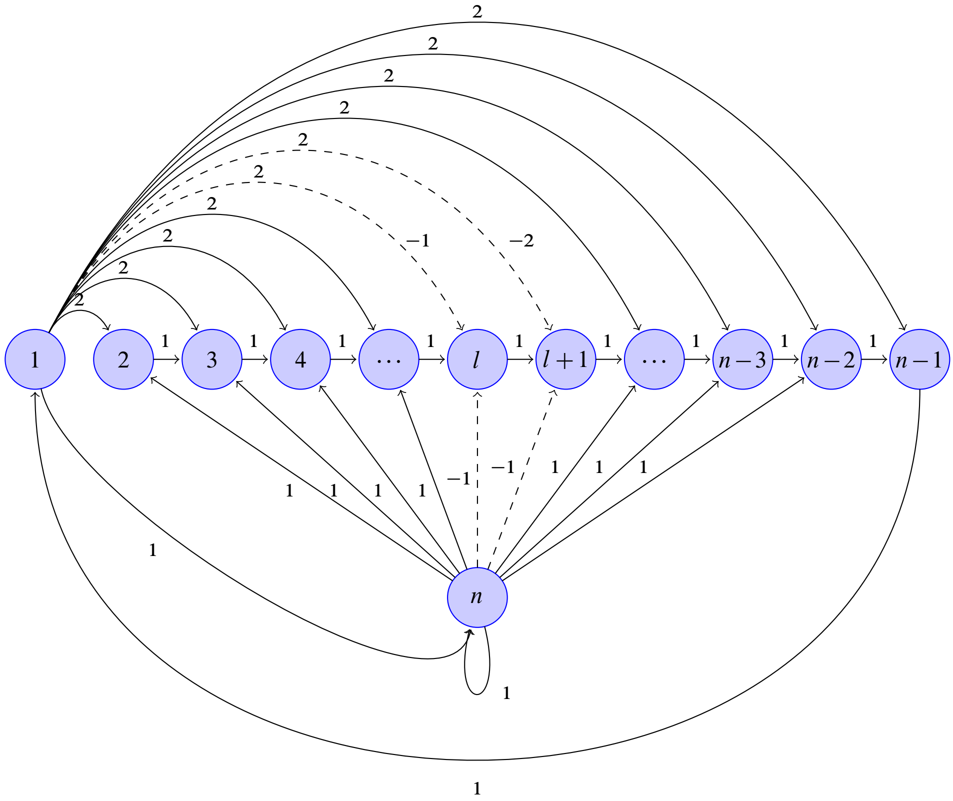

Fig. 8. The graph associated to $V_{n,k}$. By dashed lines, we have represented arrows that need to be removed from the graph in blue. Multiplicities are also indicated. To be more precise, there is one arrow from the vertex labeled 1 to the vertex labeled l and no arrow from the vertex labeled 1 to the vertex labeled $l+1$. In the graph, $l = \lfloor i(n-1)/k \rfloor + 1$ for any $i = 1, \dots, k-1$. Obviously, the graph associated to A_n is drawn in blue color.

We give a little more information on the set $J_{n,k}$. We write $J_{n,k} = \{j_1, \dots, j_r\}$ with $j_1 < \dots < j_r$ (possibly $r = 0$). We have

$$\left\lceil \frac{i(n-1)}{k} \right\rceil = 2i + \left\lceil \frac{i(n-1-2k)}{k} \right\rceil.$$

In particular, we see that $j_s = 2i_s + s$, where $i_s \geq 0$ is the smallest integer such that

$$\frac{i_s(n-1-2k)}{k} > s, \quad \text{i.e.} \quad i_s = \left\lfloor \frac{sk}{n-1-2k} \right\rfloor + 1.$$

Note also that $P_{n,k}$ must be reciprocal; hence $j \in J_{n,k}$ if and only if $n+1-j \in J_{n,k}$.

Now we compute particular cases.

(1) If n is even, $K_n = \frac{n}{2} - 1$. We have $n - 1 - 2K_n = 1$, $2i_1 + 1 = n - 1 > n - 2$, and hence $J_{n,K_n} = \emptyset$. For $k = K_n - 1 = \frac{n}{2} - 2$ and n even, $n \neq 4 \bmod 6$, we have

$$2i_1 + 1 = 2\left\lfloor \frac{n-4}{6} \right\rfloor + 3 = \left\lfloor \frac{n}{3} \right\rfloor + 1 = j_1,$$
$$2i_2 + 2 = 2\left\lfloor \frac{n-4}{3} \right\rfloor + 4 = \left\lceil \frac{2n}{3} \right\rceil = j_2 = n + 1 - j_1.$$

Hence $J_{n,K_n-1} = \{\lfloor \frac{n}{3} \rfloor + 1, \lceil \frac{2n}{3} \rceil\}$.

(2) If $n = 1 \bmod 4$, $K_n = \frac{n-1}{2} - 1 = \frac{n-3}{2}$, we have $n - 1 - 2K_n = 2$, and hence

$$2i_1 + 1 = 2\left\lfloor \frac{n-3}{4} \right\rfloor + 3 = \frac{n+1}{2} = j_1.$$

So $J_{n,K_n} = \{\frac{n+1}{2}\}$.

(3) If $n = 3 \bmod 4$, $k = K_n - 1 = \frac{n-5}{2}$, $n - 1 - 2k = 4$, and hence

$$2i_1 + 1 = 2\left\lfloor \frac{n-5}{8} \right\rfloor + 3 = j_1,$$
$$2i_2 + 2 = 2\left\lfloor \frac{n-5}{4} \right\rfloor + 4 = \frac{n+1}{2} = j_2,$$
$$2i_3 + 3 = 2\left\lfloor \frac{3(n-5)}{8} \right\rfloor + 5 = n + 1 - j_1.$$

So $J_{n,K_n} = \{2\lfloor \frac{n-5}{8} \rfloor + 3, \frac{n+1}{2}, n - 2\lfloor \frac{n-5}{8} \rfloor - 2\}$.

(4) More generally, for n even and $k \leq K_n - 1$, we evaluate j_1. If $n \not\equiv 4 \bmod 6$, we have

$$\begin{aligned} 2i_1 + 1 = 2\left\lfloor \frac{k}{n-1-2k} \right\rfloor + 3 &\leq 2\left\lfloor \frac{K_n - 1}{n - 1 - 2(K_n - 1)} \right\rfloor + 3 \\ &= 2\left\lfloor \frac{n-4}{6} \right\rfloor + 3 = \left\lfloor \frac{n}{3} \right\rfloor + 1 \end{aligned}$$

since $k \mapsto \frac{k}{n-1-2k}$ is increasing. If $n \equiv 4 \bmod 6$, the computation is a little different. In this case, we have $k < K_n - 1$, and hence

$$2i_1 + 1 < \left\lfloor \frac{K_n - 1}{n - 1 - 2(K_n - 1)} \right\rfloor + 3 = 2\left\lfloor \frac{n-4}{6} \right\rfloor + 3 = \left\lfloor \frac{n}{3} \right\rfloor + 2,$$

so $2i_1 + 1 \leq \lfloor \frac{n}{3} \rfloor + 1$. We conclude by using that the polynomial is reciprocal.

This finishes the proof. ∎

A.3. *The paths $\gamma_{n,K_n,l}$*

We recall Notation 1 (see also Figure 7) when $k = K_n$. Recall that $L_n = n - 2 - K_n$. For any $l \in \{1, \ldots, 2n - 2 - 3K_n\}$, we define the path

$$\gamma_{n,k,l} : \pi \to s(\pi) : \begin{cases} b^l t^{L_n+1-l} b^{L_n+1-l} t^{n-1-2K_n} \\ \qquad \text{if } 1 \leq l \leq L_n, \\ b^{L_n+1} t^{l-(L_n+1)} b^{2(L_n+1)-l} t^{2n-2-3K_n-l} \\ \qquad \text{if } L_n < l \leq 2n - 2 - 3K_n. \end{cases}$$

Note that, when n is even, $2n-2-3K_n = L_n+2$; otherwise, $2n-2-3K_n = L_n+3$.

Proposition A.3. *Let $n \geq 4$ be an even integer.*

(1) *If $l \in \{1,\dots,L_n\}$, then*

$$P_{n,K_n,l} = P_{n,K_n} - \frac{X^{n-2l+2}+X^{n-2l+4}+2X^{n-1}-X^{2l+1}-X^{2l-1}-2X^4}{(X+1)(X-1)}.$$

(2) *For $l = L_n$, one has*

$$P_{n,K_n,L_n} = X^{n+1}-2X^{n-1}-X^{n-3}-X^4-2X^2+1.$$

(3) *For $l = L_n+1$, $V_{n,K_n,L_n+1} = V_{n,K_n}+B_n$, where $B_n = (b_{i,j})$ and*

$$\begin{cases} b_{1,2j+1} = 1 & \text{for } 1 \leq j \leq K_n, \\ b_{1,2} = b_{1,n} = 1, & \\ b_{i,j} = 0 & \text{otherwise}. \end{cases}$$

(4) *For $l = L_n+2$, $V_{n,K_n,L_n+2} = V_{n,K_n}+C_n$, where $C_n = (c_{i,j})$ and*

$$\begin{cases} c_{1,2j+1} = c_{n-2,2j+1} = 1 & \text{for } 1 \leq j \leq K_n-1, \\ c_{1,n} = c_{n-2,n} = 1, & \\ c_{i,j} = 0 & \text{otherwise}. \end{cases}$$

Proof. Again, we will use the rome method in order to compute the characteristic polynomial. First we compute the matrix $V_{n,K,l}$. Note that n is even, so $n-1$ and $K_n = \frac{n}{2}-1$ are relatively prime. Following the proof of Lemma 3.2, we consider the central permutation π_n with the following labeling:

$$\pi_n = \begin{pmatrix} n-1 & n-3 & \cdots & 3 & 1 & n-2 & \cdots & 4 & 2 & n \\ n & 2 & 4 & \cdots & n-2 & 1 & 3 & \cdots & n-3 & n-1 \end{pmatrix}.$$

We have the following nice expression for the matrix V_{n,K_n} (see the proof of Lemma 3.2):

$$V_{n,K_n} = \begin{pmatrix} 0 & 2 & 1\,0 & \cdots & 1\,0 & 1 & 1 \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ 1 & 0 & & \cdots & & & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Let $l \in \{1,\dots,L_n\}$. By construction, $\gamma_{n,K_n,l}$ is a path obtained from γ_{n,K_n} by adding a closed loop (of type t and of length $n-1-K_n-l$) at $\pi_n.t^{K_n}b^l$. Hence the matrices V_{n,K_n} and $V_{n,K_n,l}$ differ by a non-negative matrix, i.e. $V_{n,K_n,l}-V_{n,K_n} = C_{n,l} = (c_{i,j})$, where

$$\begin{cases} c_{2l,2} = c_{2l,2l+1} = 1, \\ c_{2l,2j+3} = 2 \quad \text{for } l \leq j \leq K_n - 1, \\ c_{i,j} = 0 \quad \text{otherwise.} \end{cases}$$

On the way, the same argument applied to the matrix V_{n,K_n,L_n+2} applies, proving the last statement.

We are now in a position to compute $P_{n,K_n,l}$. The graph associated to the matrix V_{n,K_n} was already presented in the proof of Lemma A.2. We only need to add to this graph the edges corresponding to $C_{n,l}$.

Clearly, the set $R = \{1, 2l, n\}$ is a rome for $V_{n,K_n,l}$ and thus for V_{n,K_n} (since we pass from $V_{n,K_n,l}$ to V_{n,K_n} by removing some paths). The 3×3 matrix $(V_{n,K_n})_R$ is easily obtained as

$$(V_{n,K_n})_R = \begin{pmatrix} \sum_{i=2,\, i \text{ even}}^{n-2l} X^i & 2X^{2l-1} + \sum_{i=2,\, i \text{ even}}^{2l-2} X^i & X \\ X^{n-2l} & 0 & 0 \\ 0 & X^{2l-1} & X \end{pmatrix},$$

where $\sum_{i=2,\, i \text{ even}}^{2l-2} X^i = 0$ if $l = 1$.

Adding the matrix $C_{n,l}$ consists of adding two arrows from the vertex labeled $2l$ to the vertices 2, $2l + 1$ (with multiplicity 1) and $K_n - l$ arrows to the vertices $2l + 3, \dots, n - 3, n - 1$ (with multiplicity 2). In this situation, R is still a rome. To compute the matrix $(V_{n,K_n,l})_R$, we need to consider all paths passing through a dashed edge in the graph in Figure 9. Thus $(V_{n,K_n,l})_R(X) = (V_{n,K_n})_R(X) + C(X)$, where

$$C(X) = \begin{pmatrix} 0 & 0 & 0 \\ 2\sum_{i=2,\, i \text{ even}}^{n-2l-2} X^i + X^{n-2l} & X^{2l-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, by [5], we draw $\chi_{V_{n,K_n,l}} = -\det((V_{n,K_n})_R(X) + C(X) - \mathrm{Id}_3)$. By multi-linearity, one has

$$P_{n,K_n,l} = P_{n,K_n} - (1 + X) \cdot \det(W_{n,l}),$$

where

$$W_{n,l} = \begin{pmatrix} \sum_{i=2,\, i \text{ even}}^{n-2l} X^i - 1 & 2X^{2l-1} + \sum_{i=2,\, i \text{ even}}^{2l-2} X^i & X \\ 2\sum_{i=2,\, i \text{ even}}^{n-2l-2} X^i + X^{n-2l} & X^{2l-1} & 0 \\ 0 & X^{2l-1} & X - 1 \end{pmatrix}.$$

A direct computation gives

$$\det(W_{n,l}) = \frac{X^{n-2l+2} + X^{n-2l+4} + 2X^{n-1} - X^{2l+1} - X^{2l-1} - 2X^4}{(X + 1)^2 (X - 1)},$$

which is the desired result.

The second assertion comes for free since $2L_n = n - 2$, and by Lemma A.2, one has $P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^2 + 1$. Proposition A.3 is proved. ∎

Proposition A.4. *Let $n \geq 4$ be an integer with $n \equiv 3 \bmod 4$. Fix $l \in \{1, \dots, L_n + 3\}$.*

(1) *If l is even, then $V_{n,K_n,l}$ is reducible and $\theta_{n,K_n,l} = \theta_{n',K_{n'},l'}$ with $n' = \frac{n+1}{2}$, $l' = \frac{l}{2}$ and $K_{n'} = \frac{K_n}{2}$.*

(2) *If $l \leq L_n$ is odd, then $V_{n,K_n,l}$ is primitive and*

$$P_{n,K_n,l} = \frac{S_n(X) + 2(X^{\frac{n+7}{2}-l} - X^l + X^{l+\frac{n-1}{2}} - X^{n+3-l})}{(X-1)(X+1)},$$

where $S_n = 1 - 3X^2 - 2X^{\frac{n-1}{2}} + 8X^{\frac{n+3}{2}} - 2X^{\frac{n+7}{2}} - 3X^{n+1} + X^{n+3}$. In particular, for $l = L_n$, we have

$$P_{n,K_n,L_n} = X^{n+1} - 2X^{n-1} - 4X^{\frac{n+3}{2}} + 4X^{\frac{n-1}{2}} + 2X^2 - 1.$$

(3) *If $l = L_n + 2$, then*

$$V_{n,K_n,l} = \left(\begin{array}{c|c|c} 0_{K_n\times K_n} & \mathrm{Id}_{K_n\times K_n} & 0_{K_n\times 3} \\ \hline 2\cdots 2 & 0\cdots 0 & 2\;3\;2 \\ \hline 1\cdots 1 & 0\cdots 0 & 0\;2\;1 \\ \hline \mathrm{Id}_{K_n\times K_n} & 0_{K_n\times K_n} & 0_{K_n\times 3} \\ \hline 0\cdots 0 & 0\cdots 0 & 1\;1\;1 \end{array}\right),$$

and this matrix is primitive.

Proof. In the sequel, let $m = K_n + 1 = \frac{n-1}{2}$ (m is odd). We follow the strategy of the proof of the previous proposition. For the first point, we do the same as in the proof of Lemma 3.2 in the case when $d > 1$. The labeling of the central permutation is given by

$$\pi_n = \begin{pmatrix} m & * & m-2 & \cdots & 3 & * & 1 & * & m-1 & * & \cdots & 4 & * & 2 & m+1 \\ m+1 & 2 & * & 4 & \cdots & * & m-1 & * & 1 & * & 3 & \cdots & m-2 & * & m-1 \end{pmatrix}.$$

Then computation of $V_{n,K_n,l}$ gives an upper triangular 2×2 block matrix, with $V_{n',K_{n'},l'}$ for the top left block and a permutation matrix for the bottom right block.

In the next part, we compute $V_{n,K_n,l}$ by using the "standard" labeling of the central permutation, i.e.

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & & \cdots & 1 \end{pmatrix}.$$

We have $V_{n,K_n,l} = A_n + B_{n,l}$, where

$$A_n = \left(\begin{array}{c|c|c} 0_{K_n\times K_n} & \mathrm{Id}_{K_n\times K_n} & 0_{K_n\times 3} \\ \hline 1\cdots 1 & 0\cdots 0 & 2\;2\;1 \\ \hline 0\cdots 0 & 0\cdots 0 & 0\;1\;0 \\ \hline \mathrm{Id}_{K_n\times K_n} & 0_{K_n\times K_n} & 0_{K_n\times 3} \\ \hline 0\cdots 0 & 0\cdots 0 & 1\;1\;1 \end{array}\right),$$

and the only non-zero entries of $B_{n,l} = (b_{i,j})$ are

$$\begin{cases} b_{n-l,i} = 2 \quad \text{for } i = 1, \dots, m-l-1, \\ b_{n-l,m-l} = 1, \\ b_{n-l,n-1} = 2, \\ b_{n-l,n-2} = 1. \end{cases}$$

We will again use the rome method in order to compute the characteristic polynomial. It is helpful to represent the matrices in form of a combinatorial graph which amounts to draw all paths. From this, we see that $V_{n,K_n,l}$ is irreducible and therefore primitive since it has a nonzero diagonal element.

The graph associated to A_n is rather simple. Clearly, a rome is made of the subsets labeled $R = \{n, n-1, m\}$. The matrix $A_R(X)$ in this label is

$$A_R(X) = \begin{pmatrix} X & X & X^m \\ 0 & X^m & 0 \\ X & R & S \end{pmatrix},$$

with

$$R = 2X + \sum_{i=3,\, i \text{ odd}}^{m} X^i = 2X + X^3 \cdot \frac{1 - X^{m-1}}{1 - X^2},$$

$$S = 2X^m + \sum_{i=2,\, i \text{ even}}^{m-1} X^i = 2X^m + X^2 \cdot \frac{1 - X^{m-1}}{1 - X^2}.$$

Adding the matrix $B_{n,l}$ consists of adding arrows from the vertex labeled $n-l$ to the vertices $1, \dots, m-l, n-2$ with multiplicity 1 and to the vertex $n-1$ with multiplicity 2. In this situation, R is still a rome. To compute the matrix $(V_{n,K_n,l})_R$, we need to consider all paths passing through a dashed edge on the graph in Figure 9. Thus $(V_{n,K_n,l})_R(X) = A_R(X) + C(X)$, where

$$C(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P & Q \\ 0 & \sum_{i=0}^{(l-1)/2-1} \frac{P}{X^{2i}} & \sum_{i=0}^{(l-1)/2-1} \frac{Q}{X^{2i}} \end{pmatrix},$$

and

$$P = X^m + 2 \sum_{i=l,\, i \text{ odd}}^{m-2} X^i = X^m + 2X^l \cdot \frac{1 - X^{m-l}}{1 - X^2},$$

$$Q = X^{m+l-1} + 2 \sum_{i=l+1,\, i \text{ even}}^{m-1} X^i = X^{m+l-1} + 2X^{l+1} \cdot \frac{1 - X^{m-l}}{1 - X^2}.$$

Hence, by [5], we draw $\chi_{V_{n,K_n,l}} = -\det(A_R(X) + C(X) - \mathrm{Id}_3)$. By using the fact that

$$C(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P & Q \\ 0 & \frac{1-X^{1-l}}{1-X^{-2}} P & \frac{1-X^{1-l}}{1-X^{-2}} Q \end{pmatrix}$$

and $Q = X^m(X^{l-1} - X) + XP$, we easily obtain the desired equality.

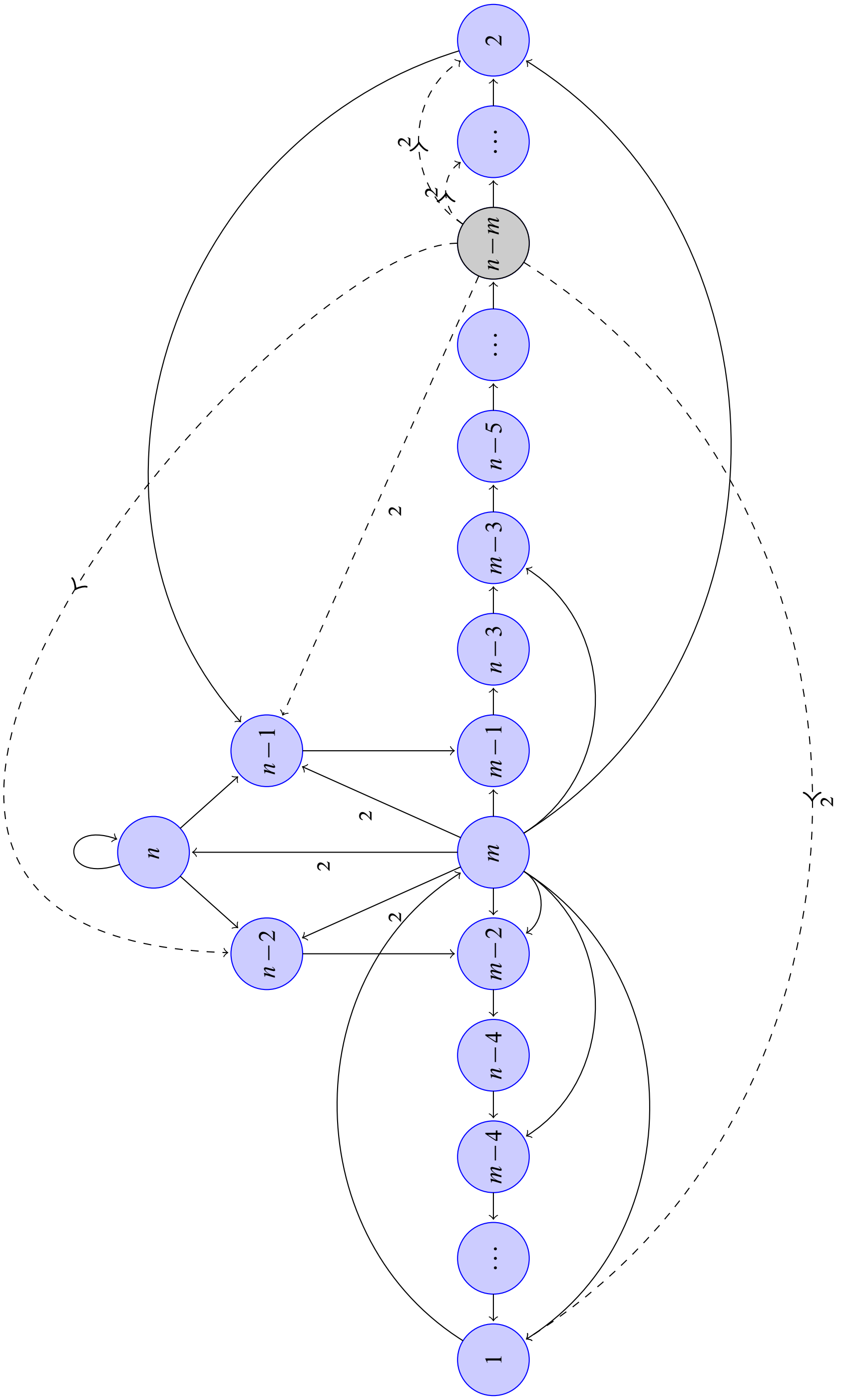

Fig. 9. The graph associated to $V_{n,K_{n,l}}$. By dashed lines, we have represented the arrows coming from the matrix $B_{n,l}$. The multiplicity is indicated only when it is 2; otherwise, it is 1.

The last assertion is also easy to derive from the fact that γ_{n,K_n,L_n+2} is obtained from γ_{n,K_n} by adding a "b" loop at $\pi_n.t^{K_n+1}$ and from the shape of A_n. This ends the proof of Proposition A.4. ∎

Appendix B. Comparing roots of polynomials

This section is devoted to comparing $\theta_{n,k}$ and $\theta_{n,k,l}$ (the maximal real roots of the polynomials $P_{n,k}$ and $P_{n,k,l}$ introduced in Appendix A) for various n, k, l. Observe that, by construction, $\theta_{n,k} > \sqrt{2}$ and $\theta_{n,k,l} > \sqrt{2}$.

One key ingredient for comparing maximal real roots of these polynomials is the following easy lemma.

Lemma B.1. *Let P_1, P_2 be two unitary polynomials of degree at least one such that, for $x > \sqrt{2}$, $P_1(x) - P_2(x) > 0$. We assume that P_1 has a root $\theta_1 > \sqrt{2}$. Then P_2 has a root $\theta_2 > \theta_1$.*

Proof. The assumption implies $P_2(\theta_1) < 0$. Since P_2 is unitary, the result follows by the mean value theorem. ∎

Lemma B.2. *The following holds.*

(1) *The sequence $(\theta_{2n,K_{2n}})_n$ is a decreasing sequence.*

(2) *The sequence $(\theta_{1+4n,K_{1+4n}})_n$ is a decreasing sequence.*

(3) *The sequence $(\theta_{3+4n,K_{3+4n},L_{3+4n}})_n$ is a decreasing sequence.*

Proof. We establish the lemma case by case.

Case 1. By Lemma A.2, $\theta_{2n,K_{2n}}$ is the largest root of

$$P_{2n} = X^{2n+1} - 2X^{2n-1} - 2X^2 + 1.$$

Observe that $P_{2n+2} - X^2 P_{2n} = 2X^4 - X^2 + 1$. Thus $P_{2n+2}(x) - x^2 P_{2n}(x) > 0$ for $x > \sqrt{2}$, and Lemma B.1 gives that $\theta_{2n,K_{2n}} < \theta_{2n+2,K_{2n+2}}$.

Case 2. By Lemma A.2, $\theta_{1+4n,K_{1+4n}}$ is the largest root of

$$P_{1+4n,K_{1+4n}} = X^{4n+2} - 2X^{4n} - 2X^{2n+1} - 2X^2 + 1.$$

Again, a simple computation establishes

$$P_{5+4n,K_{5+4n}} - X^4 P_{1+4n,K_{1+4n}} = 2X^{2n+5} - 2X^{2n+3} + 2X^6 - 2X^2 - X^4 + 1.$$

Hence $P_{5+4n,K_{5+4n}}(x) - x^4 P_{1+4n,K_{1+4n}}(x) > 0$ for $x > \sqrt{2}$, and by Lemma B.1,

$$\theta_{5+4n,K} < \theta_{1+4n,K}.$$

Case 3. By Proposition A.4,

$$P_{3+4n,K_{3+4n},L_{3+4n}} = X^{4n+4} - 2X^{4n+2} - 4X^{2n+3} + 4X^{2n+1} + 2X^2 - 1.$$

Hence, for $x > \sqrt{2}$,

$$P_{7+4n,K,L} - x^4 P_{3+4n,K,L} = (x^2-1)(4x^{2n+5} - 4x^{2n+3} - 2x^4 - x^2 + 1)$$
$$> 4x^{2n+3} - 2x^4 - x^2 + 1 > 0.$$

Hence Lemma B.1 applies, and $\theta_{7+4n,K,L} < \theta_{3+4n,K,L}$. The lemma is proved. ∎

Lemma B.3. *Let $n \geq 4$ and $1 \leq k < k' \leq K_n$. If $\gcd(n-1,k) = \gcd(n-1,k') = 1$, then $\theta_{n,k'} < \theta_{n,k}$.*

Proof. From the proof of Lemma A.2, we have

$$P_{n,k'} - P_{n,k} = 2(x+1)(Q_{n,k'} - Q_{n,k}), \quad \text{where } Q_{n,k} = \sum_{i=1}^{k-1} x^{\lceil \frac{i(n-1)}{k} \rceil}.$$

From Lemma B.1, we need to show that $P_{n,k'}(x) - P_{n,k}(x) > 0$ for $x > \sqrt{2}$. First we observe that

$$Q_{n,k'} - Q_{n,k} = \sum_{p=k}^{k'-1} x^{\lceil \frac{(k'-p)(n-1)}{k'} \rceil} + \sum_{p=1}^{k-1} (x^{\lceil \frac{(k'-p)(n-1)}{k'} \rceil} - x^{\lceil \frac{(k-p)(n-1)}{k} \rceil}).$$

Now, for any $p \in \{1, \ldots, k-1\}$,

$$\left\lceil \frac{(k-p)(n-1)}{k} \right\rceil \leq \left\lceil \frac{(k'-p)(n-1)}{k'} \right\rceil.$$

So, for any $x > 1$, $P_{n,k'}(x) - P_{n,k}(x) > x^{\lceil \frac{(k'-k)(n-1)}{k'} \rceil} > 0$, proving the lemma. ∎

Before comparing roots using polynomials, we end this subsection with a simple lemma.

Lemma B.4. *The following statements hold.*

- *Let $n \geq 7$ be an integer satisfying $n \equiv 3 \bmod 4$. Then $\theta_{n,K_n,L_n+2} > 2$.*
- *Let $n \geq 4$ be an even integer. Then $\theta_{n,K_n,L_n+1} > 3^{\frac{1}{2}}$.*
- *Let $n \geq 6$ be an even integer. Then $\theta_{n,K_n,L_n+2} > 6^{\frac{1}{4}}$.*

Proof. We will use the following classical inequality for the Perron root $\rho(A)$ of a non-negative primitive matrix $A = (a_{ij})_{i,j=1,\ldots,n}$:

$$\rho(A) > \delta(A), \quad \text{where } \delta(A) = \min_{j=1}^{n} \sum_{i=1}^{n} a_{ij}$$

(see e.g. [7, Proposition 4.2]).

We prove the first assertion. The matrix $V(\gamma_{n,K_n,L_n+2})$ is primitive (by Proposition A.4), and θ^2_{n,K_n,L_n+2} is the Perron root of V^2_{n,K_n,L_n+2}. It suffices to show that

$\delta(V^2_{n,K_n,L_n+2}) > 4$. By Proposition A.4, one has

$$V_{n,K_n,L_{n+2}} = \left(\begin{array}{c|c|c} 0_{K_n\times K_n} & \mathrm{Id}_{K_n\times K_n} & 0_{K_n\times 3} \\ \hline 2\cdots 2 & 0\cdots 0 & 2\;3\;2 \\ \hline 1\cdots 1 & 0\cdots 0 & 0\;2\;1 \\ \hline \mathrm{Id}_{K_n\times K_n} & 0_{K_n\times K_n} & 0_{K_n\times 3} \\ \hline 0\cdots 0 & 0\cdots 0 & 1\;1\;1 \end{array}\right).$$

The result then follows from an easy matrix computation.

For the second and third claim, the matrices $V(\gamma_{n,K_n,L_n+1})$ and $V(\gamma_{n,K_n,L_n+2})$ are primitive since $V\gamma_{n,K_n}$ is primitive (by Lemma 3.2). We remark that the sum of the elements of the column i of V^k is the number of paths of length k in the adjacency graph of V that ends in the vertex i. We conclude by a direct consideration of these paths for these two matrices (with $k = 2$ and $k = 4$ respectively) and by using Proposition A.3.

Lemma B.4 is proved. ∎

B.1. Comparing $\theta_{n,k,l}$ when $n \equiv 3 \bmod 4$

Lemma B.5. *Let $n \geq 7$ such that $n \equiv 3 \bmod 4$. If $l, l' \in \{1, \ldots, L_n\}$ are odd and $l < l'$, then $\theta_{n,K_n,l} > \theta_{n,K_n,l'}$.*

Proof. We follow the notation of the proof of Proposition A.4. A simple computation shows that

$$P_{n,K_n,l} - P_{n,K_n,l'} = \frac{2(X^l - X^{l'})(X^m - 1)(X^{l+l'} + X^{4+m})}{X^{l+l'}(X-1)(X+1)}.$$

In particular,

$$P_{n,K_n,l}(x) - P_{n,K_n,l'}(x) < 0 \quad \text{for } x > \sqrt{2}.$$

Lemma B.1 gives $\theta_{n,K_n,l} > \theta_{n,K_n,l'}$. ∎

Proposition B.6. *Let $n \geq 7$ be an integer satisfying $n \equiv 3 \bmod 4$.*

(1) *If $n' = \frac{n+1}{2}$, then $\theta_{n',K_{n'},L_{n'}} > \theta_{n,K_n,L_n}$.*

(2) *If $n' = \frac{n+1}{2}$, then $\theta_{n',K_{n'},L_{n'}+2} > \theta_{n,K_n,L_n}$.*

(3) *Suppose that $1 \leq k \leq K_n - 1$. If $d = \gcd(k, n-1)$, $n' = \frac{n-1}{d} + 1$ and $k' = \frac{k}{d}$, then $\theta_{n,K_n,L_n} < \theta_{n',k'}$.*

Proof. We establish the proposition case by case.

Case (1). We start with the first statement. Using Proposition A.3 and Proposition A.4, we have

$$\begin{aligned} P_{n',K_{n'},L_{n'}} &= x^{n'+1} - 2x^{n'-1} - x^{n'-3} - x^4 - 2x^2 + 1, \\ P_{n,K_n,L_n} &= x^{n+1} - 2x^{n-1} - 4x^{\frac{n+3}{2}} + 4x^{\frac{n-1}{2}} + 2x^2 - 1. \end{aligned}$$

Noticing that $n - n' = \frac{n-1}{2}$, hence $n - n' + 2 = \frac{n+3}{2}$, we have

$$P_{n,K_n,L_n} - x^{n-n'} P_{n',K_{n'},L_{n'}} = x^{n-3} + x^{4+n-n'} - 2x^{2+n-n'} + 3x^{n-n'} + 2x^2 - 1.$$

This polynomial clearly takes only positive values for $x > \sqrt{2}$, which proves the required inequality.

Case (2). Now we come to the second statement. Assume $n \geq 11$ (for $n = 7$, we directly prove the inequality). In this situation, $n' \geq 4$, and Lemma B.4 gives

$$\theta_{n',K_{n'},L_{n'}+2} > 6^{\frac{1}{4}}.$$

Let $\theta = \theta_{n,K_n,L_n}$ for simplicity. By Proposition A.4, we have

$$\theta^{n-1}(\theta^2 - 2) = 4\theta^{\frac{n+3}{2}} - 4\theta^{\frac{n-1}{2}} - 2\theta^2 + 1.$$

Hence

$$\theta - \sqrt{2} = \frac{1}{\theta + \sqrt{2}} \frac{4\theta^{\frac{n-1}{2}}(\theta^2 - 1) + 1 - 2\theta^2}{\theta^{n-1}} < \frac{4}{\theta^{\frac{n-1}{2}}} < \frac{1}{\sqrt{2}^{\frac{n-5}{2}}}.$$

Obviously,

$$\frac{1}{\sqrt{2}^{\frac{n-5}{2}}} < 6^{\frac{1}{4}} - \sqrt{2} \quad \text{for } n \geq 19.$$

Hence $\theta_{n,K_n,L_n} < 6^{\frac{1}{4}} < \theta_{n',K_{n'},L_{n'}+2}$. For $n < 19$, we check directly that this inequality holds.

Case (3). Finally, we prove the last statement. Assume first that $\gcd(n-1,k) = 1$; then $n' = n$ and $k' = k$. Note that $\gcd(n-1, K_{n-1} - 1) = 1$. From Lemma B.3, we have $\theta_{n,k} \geq \theta_{n,K_n-1}$ for any $k = 1, \dots, K_n - 1$. Since L is odd, we have, by Proposition A.4 and Lemma A.2,

$$\begin{aligned}
P_{n,K_n-1} &= x^{n+1} - 2x^{n-1} - 2x^{n-2\lfloor\frac{n-5}{8}\rfloor-2} - 2x^{\frac{n+1}{2}} - 2x^{2\lfloor\frac{n-5}{8}\rfloor+3} - 2x^2 + 1,\\
P_{n,K_n,L} &= x^{n+1} - 2x^{n-1} - 4x^{\frac{n+3}{2}} + 4x^{\frac{n-1}{2}} + 2x^2 - 1.
\end{aligned}$$

Hence, for $x > \sqrt{2}$,

$$\begin{aligned}
P_{n,K_n,L} - P_{n,K_n-1} &= 2x^{n-2\lfloor\frac{n-5}{8}\rfloor-2} - 4x^{\frac{n+3}{2}}\\
&\qquad + 2x^{\frac{n+1}{2}} + 4x^{\frac{n-1}{2}} + 2x^{2\lfloor\frac{n-5}{8}\rfloor+3} + 4x^2 - 2\\
&> 2x^{n-2\lfloor\frac{n-5}{8}\rfloor-2} - 4x^{\frac{n+3}{2}}.
\end{aligned}$$

For $n \geq 19$, we have $n - 2\lceil\frac{n-5}{8}\rceil - 2 \geq \frac{n+3}{2} + 2$, and hence $P_{n,K_n,L} - P_{n,K_n-1} > 0$. For $n = 7, 11, 15$, we compute directly the roots.

Now we assume that $\gcd(k, n-1) = d > 1$.

(1) If n' is odd (thus $n' \equiv 3 \bmod 4$), then by the above case, $\theta_{n',k'} > \theta_{n',K_{n'},L_{n'}}$. We conclude with Lemma B.2.

(2) If n' is even, then we need to show directly that $\theta_{n',k'} > \theta_{n,K_n,L_n}$. Note that n' even implies that d is an odd multiple of 2. There are two cases.

- $d \geq 6$. In this case, $\theta_{n',k'} \geq \theta_{n',K_{n'}}$ ($k' = K_{n'}$ is possible). We have

$$\begin{aligned} P_{n,K_n,L} - x^{n-n'} P_{n',K_{n'}} &= 2x^{2+n-n'} - x^{n-n'} - 4x^{\frac{n+3}{2}} + 4x^{\frac{n-1}{2}} + 2x^2 - 1 \\ &> 2x^{n-n'} - 4x^{\frac{n+3}{2}}. \end{aligned}$$

  We necessarily have $n > 15$, and hence $n - n' \geq \frac{n+3}{2} + 2$, so the above polynomial is positive for $x > \sqrt{2}$.

- $d = 2$. In this case, $\theta_{n',K_{n'}} = \theta_{n,K_n} < \theta_{n,K_n,L}$, and hence the previous strategy does not work. We remark that necessarily $k' < K_{n'}$. However, n' and $K_{n'} - 1$ are not necessarily relatively prime, so we need to compare directly $\theta_{n',k'}$ with $\theta_{n,K_n,L}$ by using statement (4) of Lemma A.2. We have

$$P_{n,K_n,L} - x^{n-n'} P_{n',k'} \geq 2x^{n-n'+\lceil\frac{2n'}{3}\rceil} - 4x^{\frac{n+3}{2}}.$$

  We necessarily have $n \geq 11$, and hence $n - n' + \lceil\frac{2n}{3}\rceil \geq \frac{n+3}{2} + 2$, implying the desired inequality.

This ends the proof of the proposition. ∎

B.2. *Comparing $\theta_{n,k,l}$ when $n \equiv 0 \bmod 2$*

Lemma B.7. *Let $n \geq 4$ be an even integer. If $l, l' \in \{1, \dots, L_n\}$ satisfy $l < l'$, then*

$$\theta_{n,K_n,l} > \theta_{n,K_n,l'}.$$

Proof. Proposition A.3 and a simple computation show

$$P_{n,K_n,l} - P_{n,K_n,l'} = \frac{(X^l - X^{l'})(X^2+1)(X^{2l+2l'} + X^{n+3})(X^l + X^{l'})}{X^{2l+2l'+1}(X+1)(X-1)}.$$

In particular,

$$P_{n,K_n,l}(x) - P_{n,K_n,l'}(x) < 0 \quad \text{for } x > \sqrt{2}.$$

Lemma B.1 gives $\theta_{n,K_n,l} > \theta_{n,K_n,l'}$. ∎

Proposition B.8. *Let $n \geq 18$ be an even integer satisfying $n \not\equiv 4 \bmod 6$. Then we have* $\gcd(n-1, K_n - 1) = 1$, *and the following holds.*

(1) *For any $k = 1, \dots, K_n - 2$, one has $\theta_{n,k} > \theta_{n,K_n-1}$.*

(2) *For any $l = 1, \dots, L_n$, one has $\theta_{n,K_n,l} > \theta_{n,K_n-1}$.*

(3) *For $l = L_n + 1$, one has $\theta_{n,K_n,L_n+1} > \theta_{n,K_n-1}$.*

(4) *For $l = L_n + 2$, one has $\theta_{n,K_n,L_n+2} > \theta_{n,K_n-1}$.*

Proof. Let us consider the first claim and set $d = \gcd(k, n-1)$. If $d = 1$, then we have $\theta_{n,k} > \theta_{n,K_n-1}$ by Lemma B.3. Otherwise, let $n' = \frac{n-1}{d} + 1 < n$ and $k' = \frac{k}{d}$. Note that $\gcd(k', n'-1) = 1$ and $\theta_{n,k} = \theta_{n',k'}$. By Lemma B.3, $\theta_{n',k'} > \theta_{n',K_{n'}}$. It suffices to show $\theta_{n',K_{n'}} > \theta_{n,K_n-1}$. We have, for $x > \sqrt{2}$,

$$\begin{aligned} P_{n,K_n-1} - x^{n-n'} P_{n',K_{n'}} &= 2x^{2+n-n'} - x^{n-n'} - 2x^{\lceil \frac{2n}{3} \rceil} - 2x^{\lfloor \frac{n}{3} \rfloor + 1} - 2x^2 + 1 \\ &> 2x^{n-n'} - 2x^{\lceil \frac{2n}{3} \rceil} + x^{n-n'} - 2x^{\lfloor \frac{n}{3} \rfloor + 1} - 2x^2 + 1. \end{aligned}$$

Note that $n-1$ is odd and not a multiple of 3, and hence $d \geq 5$. Since n is large enough, we have

$$n - n' - \frac{2n}{3} = (n-1)\left(1 - \frac{1}{d}\right) - \frac{2n}{3} \geq \frac{4}{5}(n-1) - \frac{2n}{3} \geq 0.$$

Hence $\lfloor n - n' - \frac{2n}{3} \rfloor = n - n' - \lceil \frac{2n}{3} \rceil \geq 0$. Similarly, $n - n' \geq 4 + \lfloor \frac{n}{3} \rfloor + 1$.

Hence

$$P_{n,K_n-1} - x^{n-n'} P_{n',K_{n'}} > 2x^{\lfloor \frac{n}{3} \rfloor + 1} - 2x^2 + 1 > 0.$$

The inequality $\theta_{n',K_{n'}} > \theta_{n,K_n-1}$ follows by Lemma B.1.

We now prove the second claim for $l = 1, \ldots, L_n$. Since we have $\theta_{n,K_n,l} > \theta_{n,K_n,L}$ by Lemma B.7, it suffices to show $\theta_{n,K_n,L} > \theta_{n,K_n-1}$. We have

$$P_{n,K_n-1} - P_{n,K_n,L} = x^{n-3} - 2x^{\lceil \frac{2n}{3} \rceil} - 2x^{\lfloor \frac{n}{3} \rfloor + 1} + x^4.$$

If $n \geq 21$, we get $n - 3 \geq 4 + \lceil \frac{2n}{3} \rceil$, and hence

$$P_{n,K_n-1} - P_{n,K_n,L} > 2x^{\lceil \frac{2n}{3} \rceil} - 2x^{\lfloor \frac{n}{3} \rfloor + 1} + x^4 > 0.$$

For $n \in \{18, 20\}$, we check directly that $P_{n,K_n-1} - P_{n,K_n,L} > 0$. Then $\theta_{n,K_n,L} > \theta_{n,K_n-1}$ follows by Lemma B.1.

Next we prove the third claim for $l = L_n + 1$ (in this case, the computation is different). By Lemma B.4, we have $\theta_{n,K_n,L_n+1} > 3^{\frac{1}{2}}$. For simplicity, let $\theta = \theta_{n,K_n-1}$. By Lemma A.2,

$$\theta^{n-1}(\theta^2 - 2) = 2\theta^{\lceil \frac{2n}{3} \rceil} + 2\theta^{\lfloor \frac{n}{3} \rfloor + 1} + 2\theta^2 - 1 < 2\theta^{\lceil \frac{2n}{3} \rceil} + 2\theta^{\lfloor \frac{n}{3} \rfloor + 1} + 2\theta^2.$$

Thus

$$\theta - \sqrt{2} < \frac{2}{\theta + \sqrt{2}} \left(\frac{1}{\theta^{n-1-\lceil \frac{2n}{3} \rceil}} + \frac{1}{\theta^{n-2-\lfloor \frac{n}{3} \rfloor}} + \frac{1}{\theta^{n-3}} \right) < \frac{3}{\sqrt{2}^{\frac{n}{3}-1}}$$

since $\theta > \sqrt{2}$ and n is large enough. Clearly,

$$\frac{3}{\sqrt{2}^{\frac{n}{3}-1}} < 3^{\frac{1}{2}} - \sqrt{2} \quad \text{for } n > 22,$$

and hence $\theta < 3^{\frac{1}{2}} < \theta_{n,K_n,L_n+1}$, which is the desired inequality. For $n \in \{18, 20\}$, we directly check the inequality.

Finally, we prove the last claim for $l = L_n + 2$. By Lemma B.4, $\theta_{n,K_n,L_n+2} > 6^{\frac{1}{4}}$. We can check that

$$\frac{3}{\sqrt{2}^{\frac{n}{3}-1}} < 6^{\frac{1}{4}} - \sqrt{2} \quad \text{for } n > 28,$$

and hence $\theta < 6^{\frac{1}{4}} < \theta_{n,K_n,L_n+2}$, which is the desired inequality (for $n < 30$, we directly check the inequality). The proposition is proved. ∎

B.3. Case $n \equiv 1 \bmod 4$

Proposition B.9. *Let $n \geq 5$ such that $n \equiv 1 \bmod 4$. For any $1 \leq k \leq K_n - 1$, we define $d = \gcd(k, n-1)$ and $n' = \frac{n-1}{d} + 1$, $k' = \frac{k}{d}$. Then $\theta_{n',k'} > \theta_{n,K_n}$.*

Proof. Let $k \in \{1, \ldots, K_n - 1\}$. If $\gcd(k, n-1) = 1$, Lemma B.3 implies $\theta_{n,k} > \theta_{n,K_n}$ as desired. If $\gcd(k, n-1) = d > 1$, then there are three cases depending on the value of $n' \bmod 4$.

(1) If $n' \equiv 1 \bmod 4$, the previous argument shows that $\theta_{n',k'} > \theta_{n',K_{n'}}$. By Lemma B.2, the sequence $(\theta_{n,K_n})_n$ is decreasing for $n \equiv 1 \bmod 4$, so we have $\theta_{n',K_{n'}} > \theta_{n,K_n}$ as desired.

(2) If n' is even, Lemma B.3 implies $\theta_{n',k'} > \theta_{n',K_{n'}}$. By Lemma A.2, for any $x > \sqrt{2}$,

$$\begin{aligned} P_{n,K_n} - x^{n-n'} P_{n',K_{n'}} &= 2x^{2+n-n'} - x^{n-n'} - 2x^{\frac{n+1}{2}} - 2x^2 + 1 \\ &> 3x^{n-n'} - 2x^{\frac{n+1}{2}} - 2x^2 > 0 \end{aligned}$$

(the last inequality comes from $d \geq 4$ and $n \geq 9$). By Lemma B.1, $\theta_{n',K_{n'}} > \theta_{n,K_n}$.

(3) If $n' \equiv 3 \bmod 4$, then $k' < K_{n'}$, and Proposition B.6 implies $\theta_{n',k'} > \theta_{n',K_{n'},L_{n'}}$. For $x > \sqrt{2}$,

$$\begin{aligned} P_{n,K_n} - x^{n-n'} P_{n',K_{n'},L_{n'}} &= 4x^{n-n'+\frac{n'+3}{2}} - 4x^{n-n'+\frac{n'-1}{2}} \\ &\quad - 2x^{2+n-n'} + x^{n-n'} - 2x^{\frac{n+1}{2}} - 2x^2 + 1 \\ &> 4x^{n-n'+\frac{n'-1}{2}} - 2x^{2+n-n'} - 2x^{\frac{n+1}{2}} + x^{n-n'} - 2x^2. \end{aligned}$$

The assumptions on n, n' implies that $n \neq 5, 9$, and hence $n \geq 13$ and $n' \geq 7$. This implies that $n - n' + \frac{n'-1}{2} = (n-1)(1 - \frac{1}{2d}) \geq \frac{n+1}{2}$, $n - n' + \frac{n'-1}{2} \geq 2 + n - n'$ and $n - n' \geq 4$. Thus $P_{n,K_n} - x^{n-n'} P_{n',K_{n'},L_{n'}} > 0$, and Lemma B.1 implies that $\theta_{n,K_n} < \theta_{n',K_{n'},L_{n'}}$.

The proof of Proposition B.9 is complete. ∎

Appendix C. A naive attempt to generalize the Rauzy–Veech construction

The classical construction of pseudo-Anosov homeomorphism by Rauzy induction necessarily produces maps that preserve a singularity, and a horizontal separatrix. This clearly comes from the fact that only right Rauzy induction is used. So it is natural to expect to produce pseudo-Anosov homeomorphisms that do not fix a separatrix by combining right and left induction. For instance, we consider a path γ in the labeled (extended) Rauzy diagram such that

- the image of γ in the reduced extended Rauzy diagram is closed,
- γ is the concatenation of a path γ_1 that consists only of right Rauzy moves, and a path γ_2 that consists only of left Rauzy moves.

As in Section 2.7, we associate to such path a matrix V by multiplying the corresponding product of the transition matrices by a suitable permutation matrix. Assume now that the matrix V is primitive. Let $\theta > 1$ be its Perron–Frobenius eigenvalue. We choose a positive eigenvector λ for θ. As before, V is symplectic; thus let us choose an eigenvector τ for the eigenvalue θ^{-1}. It turns out that τ is not necessarily a suspension datum, but it is a weak suspension datum (up to replacing it by its opposite). Indeed, the set of weak suspension data is an open cone W, which is by construction invariant by V^{-1}, and we conclude as previously (the proof is the same as in Proposition 3.1).

Proposition C.1. *A pseudo-Anosov homeomorphism affine on a translation surface and constructed as above fixes a vertical separatrix. In particular, it is obtained by the usual Rauzy–Veech construction.*

Proof. Let (π, λ, τ) be the weak suspension datum defined as above, and let h be a height. We will denote the associated surface by $X = X(\pi, \lambda, \tau)$ and by $I = I_h$ the corresponding horizontal interval. After the prescribed sequence of right and left Rauzy induction, we obtain the suspension datum $(\pi, \lambda', \tau') = (\pi, \frac{1}{\theta}\lambda, \theta\tau)$ defining the same surface X, with corresponding interval $I'_{h'} = I' \subset I$ (recall the Rauzy–Veech induction corresponds to cutting the interval on the right or on the left). Also, θh is an obvious height for $(\pi, \frac{1}{\theta}\lambda, \theta\tau)$, and the corresponding interval $I'_{\theta h}$ is the image by ϕ of the interval I_h.

Hence there is an isometry f from $I'_{\theta h}$ to $I'_{h'}$ obtained by following a vertical leaf (see Section 2.3). The map $f \circ \phi$ is therefore a contracting map from I_h to itself (its derivative is θ^{-1}), hence has a fixed point. This means that there is an element x in I_h whose image by ϕ is in the vertical leaf l passing through x. Thus this vertical leaf l is preserved by ϕ. Since ϕ, restricted to l, has derivative $\theta \neq 1$, there is a fixed point of ϕ on l. This fixed point is either a conical singularity or a regular point. In any case, ϕ fixes a vertical separatrix. Hence ϕ fixes also a horizontal separatrix. It is therefore obtained by the usual Rauzy–Veech construction. ∎

Appendix D. A direct proof of Theorem A when *n* is even

In this appendix, we will prove the following theorem.

Theorem D.1. *If $n \geq 4$ is even, then $L(\mathrm{Spec}(\mathcal{C}_n^{\mathrm{hyp}})) = \log(\theta_{n,K_n})$. Moreover, there are two conjugacy mapping classes realizing the minimum.*

Proof. By Remark 3.3, we have that $\log(\theta_{n,K_n})$ belongs to $\mathrm{Spec}(\mathcal{C}_n^{\mathrm{hyp}})$ when n is even. Theorem 4.1 implies $L(\mathrm{Spec}(\mathcal{C}_n^{\mathrm{hyp}})) = \log(\theta(\gamma))$ for some path γ starting from $\pi = \pi_n.t^k$

with first step of type b. By definition, $\theta(\gamma) \leq \theta_{n,K_n}$. Theorem 5.3 implies $\theta_{n,k} \leq \theta(\gamma)$ for some $k \in \{1,\dots,K_n\}$. Thus

$$\theta_{n,k} \leq \theta_{n,K_n}, \tag{D.1}$$

where $K_n := \frac{n}{2} - 1$. Thus we only have to show that $k = K_n$ (this will also prove that the conjugacy class of negative mapping class realizing this minimum is unique). Let us assume by contradiction that $k < K_n$. Let $d = \gcd(k, n-1)$. We will distinguish two cases: $d = 1$ and $d > 1$.

First case: k and $n-1$ are relatively prime. In the proof of Lemma 3.2, we computed the matrix $V_{n,k}$ associated to the path $\gamma_{n,k}$. Namely,

$$V_{n,k} = \begin{pmatrix} a_{n-1} & 2 & a_2 & \cdots & \cdots & a_{n-2} & 1 \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ 1 & 0 & & \cdots & & & 0 \\ b_{n-1} & 1 & b_2 & \cdots & \cdots & b_{n-2} & 1 \end{pmatrix},$$

where, for $i \in \{2,\dots,n-2\}$, we have

$$a_i = \begin{cases} 2 & \text{if } i \in \{\alpha_{k+2},\dots,\alpha_{n-1-k}\}, \\ 1 & \text{if } i \in \{\alpha_{n-k},\dots,\alpha_{n-1}\}, \\ 0 & \text{if } i \in \{\alpha_1 \dots \alpha_k\}, \end{cases}$$

and $b_i = 1$ if and only if $a_i = 2$, and $b_i = 0$ otherwise. Note that $a_{n-1} = 0$, and hence $b_{n-1} = 0$.

In the particular case of $k = \frac{n}{2} - 1$, we obtain $a_i = \frac{(-1)^i+1}{2}$, i.e.

$$(a_2,\dots,a_{n-2}) = (1,0,1,0,\dots,1).$$

For general k, the sequence $(a_i)_i$ has the following properties:

- if $a_i = 0$ or $a_i = 2$, then $a_{i+1} = 1$ or $a_{i+1} = 2$;
- if $a_i = 1$, then $a_{i+1} = 0$.

Using that $a_2 \geq 1$ and all other a_i, b_i are nonnegative, we easily see that $V_{n,k}$ is primitive. Let v be a Perron–Frobenius right eigenvector of V_{n,K_n}. By direct computation,

$$V_{n,k}v - V_{n,K_n}v = \begin{pmatrix} (a_2-1)v_3 + a_3v_4 + (a_4-1)v_5 + \cdots + (a_{n-2}-1)v_{n-1} \\ 0 \\ \vdots \\ 0 \\ b_2v_3 + \cdots + b_{n-2}v_{n-1} \end{pmatrix}.$$

We claim that $V_{n,k}v \geq V_{n,K_n}v$ (meaning that the inequality holds for each term and is strict for at least one term). Indeed, all rows, except the first one, are obviously non-negative. So we have to prove the claim for the first row only. Since $k \neq K_n$, we can define l to be the first index so that $a_l \neq \frac{(-1)^l+1}{2}$. From the above properties of the sequence (a_i), we see that l is necessarily even and that $a_l = 2$. Setting $\theta = \theta_{n,K_n}$ to improve readability, the first row is equal to

$$\begin{aligned} v_{l+1} &+ \sum_{i=l+1,\, i \text{ odd}}^{n-3} a_i v_{i+1} + \sum_{i=l+2,\, i \text{ even}}^{n-2} (a_i - 1) v_{i+1} \\ &= v_{l+1}\left(1 + \sum_{j=l+1,\, j \text{ odd}}^{n-3} a_j \theta^{j-l} + \sum_{i=l+2,\, i \text{ even}}^{n-2} (a_i - 1)\theta^{i-l}\right). \end{aligned}$$

Since $a_i \in \{0, 1, 2\}$, we only have to show that, when $a_i - 1 < 0$ (i is even), then

$$(a_i - 1)\theta^{i-l} + a_{i+1}\theta^{i+1-l} > 0.$$

This is clear from the definition of a_i: if $a_i = 0$, then $a_{i+1} = 1$ or $a_{i+2} = 2$. Thus

$$(a_i - 1)\theta^{i-l} + a_{i+1}\theta^{i+1-l} \geq -\theta^{i-l} + \theta^{i+1-l} = \theta^{i-l}(\theta - 1) > 0.$$

There is perhaps a problem with the last term, but we easily see that $a_{n-2} = 1$. The claim is proved.

Let w^T be a positive left eigenvector of $V_{n,k}$, $w^T V_{n,k} = \theta_{n,k} w^T$. From the inequality $V_{n,k} v \geq \theta_{n,K_n} v$, draw

$$w^T V_{n,k} v > \theta_{n,K_n} w^T v.$$

Hence $\theta_{n,k} w^T v > \theta_{n,K_n} w^T v$. In conclusion, $\theta_{n,k} > \theta_{n,K_n}$, which contradicts (D.1).

Second case: k and $n-1$ are not relatively prime. We have $d = \gcd(n-1, k) > 1$. Let $n' = \frac{n-1}{d} + 1$ and $k' = \frac{k}{d}$. Lemma 3.2 gives $\theta_{n,k} \geq \theta_{n',k'}$. By the previous discussion, the inequality $\theta_{n',k'} > \theta_{n',K_{n'}}$ holds (observe that k' and $n'-1$ are relatively prime and n' is an even integer).

By Remark 3.3, θ_{n,K_n} is the largest root of the polynomial $X^{n+1} - 2X^{n-1} - 2X^2 + 1$. In particular, the sequence $m \to \theta_{m,K_m}$, defined on even numbers, is decreasing. Since $n' < n$, we have $\theta_{n',K_{n'}} > \theta_{n,K_n}$. In conclusion, $\theta_{n,k} \geq \theta_{n,K_n}$; this again contradicts (D.1). The proof of Theorem A when n is even is now complete. ∎

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