J. Eur. Math. Soc. 24, 2443–2492 (2022) DOI 10.4171/JEMS/1147

© 2021 European Mathematical Society
Published by EMS Press. This work is licensed under a CC BY 4.0 license.

Jean-François Babadjian · Flaviana Iurlano · Antoine Lemenant

Partial regularity for the crack set minimizing the two-dimensional Griffith energy

Received May 25, 2019

Abstract. In this paper we prove a $\mathscr{C}^{1,\alpha}$ regularity result for minimizers of the planar Griffith functional arising from a variational model of brittle fracture. We prove that any isolated connected component of the crack, the singular set of a minimizer, is locally a $\mathscr{C}^{1,\alpha}$ curve outside a set of zero Hausdorff measure.

Keywords. Free discontinuity problems, regularity theory, crack propagation, Griffith

1. Introduction

Following the original Griffith theory of brittle fracture [27], the variational approach introduced in [24] rests on the competition between a bulk energy, the elastic energy stored in the material, and a dissipation energy which is proportional to the area (the length in 2D) of the crack. In a planar elasticity setting, the *Griffith energy* is defined by

$$\mathscr{G}(u,K) := \int_{\Omega\setminus K} \mathbf{A}e(u) : e(u)\,dx + \mathscr{H}^1(K),$$

where $\Omega \subset \mathbb{R}^2$, which is bounded and open, stands for the reference configuration of a linearized elastic body, and $\mathbf{A}$ is a suitable elasticity tensor. Here, $e(u) = (\nabla u + \nabla u^T)/2$ is the elastic strain, the symmetric gradient of the displacement $u : \Omega \setminus K \to \mathbb{R}^2$ which is defined outside the crack $K \subset \overline{\Omega}$. This energy functional falls within the framework of free discontinuity problems, and it is defined on pairs function/set

$$(u,K) \in \mathscr{A}(\Omega) := \{K \subset \overline{\Omega} \text{ is closed and } u \in LD(\Omega' \setminus K)\},$$

Jean-François Babadjian: Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, F-91405 Orsay, France; jean-francois.babadjian@universite-paris-saclay.fr

Flaviana Iurlano: Sorbonne Université, CNRS, Université de Paris, Laboratoire Jacques-Louis Lions, F-75005 Paris, France; iurlano@ljll.math.upmc.fr

Antoine Lemenant: Université Paris Diderot – Paris 7, CNRS, UMR 7598 Laboratoire Jacques-Louis Lions, F-75005 Paris, France; lemenant@ljll.univ-paris-diderot.fr

Mathematics Subject Classification (2020): 49Q20, 49Q05, 74R10

where $\Omega' \supset \overline{\Omega}$ is a bounded open set (see (2.1) for a precise definition of the space LD of functions of Lebesgue deformation).

Minimizers of the Griffith energy have attracted a lot of attention in the last years. Although very close to its scalar analogue, which is known as the Mumford–Shah functional, the existence of a global minimizer $(u, K) \in \mathcal{A}(\Omega)$ (with a prescribed Dirichlet boundary condition) was proved only very recently in [10–12, 15, 26] (see Section 2 for details). It was also established in the meantime that the crack set K is $\mathcal{H}^1$-rectifiable and Ahlfors regular.

The main result of this paper is the following partial regularity property for the crack K.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected open set with $\mathcal{C}^1$ boundary and let Ω' be a bounded open set such that $\overline{\Omega} \subset \Omega'$. Let $\psi \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ be a boundary data, and let $\mathbf{A}$ be a fourth order elasticity tensor of the form*

$$\mathbf{A}\xi = \lambda(\operatorname{tr}\xi)I + 2\mu\xi \quad \text{for all } \xi \in \mathbb{M}^{2\times 2}_{\rm sym},$$

where $\mu > 0$ and $\lambda + \mu > 0$. Let $(u, K) \in \mathcal{A}(\Omega)$ be a solution to the minimization problem

$$\inf\{\mathcal{G}(v, K') : (v, K') \in \mathcal{A}(\Omega),\ v = \psi \text{ a.e. in } \Omega' \setminus \overline{\Omega}\}.$$

Then there exists $\alpha \in (0, 1)$ (depending only on $\mathbf{A}$) satisfying the following property: for every isolated connected component Γ of $K \cap \Omega$ there exists an exceptional relatively closed set $Z \subset \Gamma$ such that $\mathcal{H}^1(Z) = 0$ and $\Gamma \setminus Z$ is locally a $\mathcal{C}^{1,\alpha}$ curve.

Comments about the main result

The strategy of our approach is inspired by the regularity theory for minimizers of the classical Mumford–Shah functional. However, the presence of the symmetric gradient in the bulk energy term prevents the standard theory from being applied directly. We will explain later the main differences with the classical theory, and how we overcome some of the difficulties in this paper. Before that, let us first list a few remarks about the main result.

Firstly, it would be desirable to obtain the analogue of Bonnet's result [7] for the Griffith energy, i.e. to prove that each isolated connected component of K is a finite union of curves and to classify the blow-up limits of minimizers. However, this seems difficult since the proof of [7] relies on the monotonicity formula for the Dirichlet energy, which is not known in the case of the elastic energy.

Secondly, we emphasize that our proof strongly uses the two-dimensional setting and cannot be easily generalized to higher dimensions. We will describe below the main ideas of the proof, highlighting where the 2D assumption is crucial.

Thirdly, the $\mathcal{C}^{1,\alpha}$ regularity can be used as a first step in order to get higher regularity of both u and the crack K. Indeed, once we know that K is locally the graph of a $\mathcal{C}^{1,\alpha}$ function, one can write the Euler equation (which is a priori not well justified without any regularity of K, even in a weak sense). As a consequence, one can obtain $\mathcal{C}^\infty$ regularity using standard techniques and then analyticity by applying the results of [29] (see for instance [22, Theorems 3.18 and 3.19] together with [29, comment after Remark 4.12]).

Fourthly, since any connected component of K is automatically uniformly rectifiable (because it is compact, connected, and Ahlfors regular [18, Theorem 31.5]), it is tempting to think that the exceptional negligible set Z of Theorem 1.1 could be taken such that $\dim_{\mathcal{H}}(Z) < 1$. For the classical Mumford–Shah problem this is true and it can be proved using the uniform rectifiability of K. Indeed, this property permits one to apply the so-called ε-regularity theorem in many balls, and not only almost everywhere, as is the case when using Carleson measure estimates (see for instance [36]). A similar strategy can also be applied to Griffith minimizers, and it is successfully performed in [30].

Yet, let us stress that it is not known in general how to control the connected components of the singular set of a minimizer. Even in the scalar case, this question is a big issue related to the Mumford–Shah conjecture. Of course the number of connected components with positive $\mathcal{H}^1$-measure has to be at most countable, but it seems difficult to exclude the possibility of uncountably many negligible connected components that accumulate to form a set with positive $\mathcal{H}^1$-measure. We could also imagine many small connected components of positive measure that accumulate near a given bigger component. The assumption that we consider an isolated connected component in our main theorem rules out these pathological situations. The precise role of this hypothesis will be explained later.

Finally, our main result is stated on an isolated connected component of a general minimizer. An alternative could be to minimize the Griffith energy under a connectedness constraint, or under a uniform bound on the number of connected components. Existence and Ahlfors regularity of a minimizer in the class

$$\mathcal{A}_N(\Omega) := \{(u, K) \in \mathcal{A}(\Omega) : K \text{ has at most } N \text{ connected components}\},$$

for some fixed $N \in \mathbb{N}$, are much easier to obtain, due to the Blaschke and Gołąb theorems (see e.g. [9, Lemma 4]), and our result in this case would imply that the singular set is $\mathcal{C}^{1,\alpha}$ regular $\mathcal{H}^1$-almost everywhere. Indeed, a careful inspection of our proof reveals that all the competitors that we use preserve the topology of K, thus they can still be used under connectedness constraints on the singular set, leading to the same estimates. We thus obtain the following corollary.

Corollary 1.1. *Under the same assumptions as in Theorem 1.1, if $(u, K) \in \mathcal{A}_N(\Omega)$ is a solution to the minimization problem*

$$\inf\{\mathcal{G}(v, K') : (v, K') \in \mathcal{A}_N(\Omega),\ v = \psi \text{ a.e. in } \Omega' \setminus \overline{\Omega}\},$$

then there exist $\alpha \in (0, 1)$ (depending only on $\mathbf{A}$) and an exceptional relatively closed set $Z \subset K$ such that $\mathcal{H}^1(Z) = 0$ and $K \setminus Z$ is locally a $\mathcal{C}^{1,\alpha}$ curve.

Finally, it is quite probable that most of the results contained in this paper could be applied to almost minimizers instead of minimizers (i.e. pairs that minimize the Griffith energy in all balls of radius r with its own boundary datum, up to an error excess controlled by some $Cr^{1+\alpha}$ term). For the sake of simplicity we decided to treat in this paper minimizers of the global functional only.

Comments about the proof

The Griffith energy is similar to the classical Mumford–Shah energy in some aspects, but it actually necessitates the introduction of new ideas and new techniques. For the classical Mumford–Shah problem, there are two main approaches. The first one, in dimension 2 (see [7], or [17], written a bit differently in the monograph [18]), is of purely variational nature. It was extended to higher dimensions in [32] with a more complicated geometrical stopping time argument. Alternatively, there is a PDE approach [3,5] (see also [4]), valid in any dimension, which consists in working with the Euler–Lagrange equation. However, none of the aforementioned approaches can be directly applied to the Griffith energy.

More precisely, when trying to develop the regularity theory for the Griffith energy, one faces the following main obstacles:

(i) No Korn inequality. The well-known Korn inequality in elasticity theory enables one to control the full gradient, ∇u, by the symmetric part of the gradient, $e(u)$. Unfortunately, it is not valid in the cracked domain $\Omega \setminus K$, due to the possible lack of regularity of K (see [13,25]). Therefore, one has to keep working with the symmetric gradient in all the estimates.

(ii) No Euler–Lagrange equation. A consequence of the failure of the Korn inequality is the lack of the Euler–Lagrange equation. Indeed, when computing the derivative of the Griffith energy with respect to inner variations, i.e. by a perturbation of u of the type $u \circ \Phi_t(x)$ where $\Phi_t = \mathrm{id} + t\Phi$, some mixtures of derivatives of u appear and these are not controlled by the symmetric gradient $e(u)$. Therefore, the so-called “tilt-estimate”, which is one of the key ingredients of the method in [3,5], cannot be used.

(iii) No coarea formula. A fundamental tool in calculus of variations and in geometric measure theory is the so-called coarea formula, which enables one to reconstruct the total variation of a scalar function by integrating the perimeter of its level sets. In our setting, on the one hand the displacement u is a vector field, and on the other hand even for each coordinate of u there would be no analogue of this formula with $e(u)$ replacing ∇u. In the approach of [18] or [32], the coarea formula is a crucial ingredient which ensures that, provided the energy of u is very small in some ball, one can use a suitable level set of u to “fill the holes" of K, with very small length. It permits one to reduce to the case where the crack K “separates” the ball into two connected components. This is essentially the reason why our regularity result only holds on (isolated) connected components of K.

(iv) No monotonicity formula for the elastic energy. One of the main ingredients to control the energy in [7] and [18] (in dimension 2) is the so-called monotonicity formula, which essentially says that a suitable renormalization of the bulk energy localized in a ball of radius r is a nondecreasing function of r. This is not known for the elastic energy, i.e. when ∇u is replaced by $e(u)$.

(v) No good extension techniques. To prove any kind of regularity result, one has to create convenient competitors, and the main competitor in dimension 2 is obtained by replacing K in some ball B where it is sufficiently flat, by a segment S which is nearly a diameter. While doing so, and in order to use the minimality of (u, K), one has to define a new function v which coincides with u outside B, which belongs to $LD(B \setminus S)$, and whose elastic energy is controlled by that of u. Denoting by $C^{\pm}$ both connected

components of $\partial B \setminus S$, the way this is achieved in the standard Mumford–Shah theory (see [18] or [33]) consists in introducing the harmonic extensions of $u|_{C^\pm}$ to B using the Poisson kernel. This provides two new functions $u^\pm \in H^1(B)$, whose Dirichlet energies in the ball B are controlled by that of u on the boundary $\partial B \setminus K$. For the Griffith energy, the same argument cannot be used since there is no natural "boundary" elastic energy on $\partial B \setminus K$.

Let us now explain the novelty of the paper and how we obtain a regularity result, in spite of the aforementioned problems. We do not have any hope to directly solve the general problem (i), which would probably be a way to solve all the other ones. We follow mainly the two-dimensional approach of [18], for which one has to face the main obstacles (iii)–(v) described above.

Due to the absence of the coarea formula, we cannot control the size of the holes in K at small scales, when K is very flat, as done in [18]. This is a first reason why our theorem restricts to a connected component of K only. There is a second reason related to the decay of the normalized energy by use of a compactness argument (in the spirit of [31] or [5]) in the absence of a monotonicity formula for the energy. In this argument, one of our main tools is the so-called Airy function w associated to a minimizer u, which can be constructed only in the two-dimensional case. This function has already been used in [6] to prove compactness and Γ-convergence results related to the elastic energy, and it is defined through the harmonic conjugate (see Proposition 5.2). The main property of w is that it is a scalar biharmonic function in $\Omega \setminus K$ which satisfies $|D^2 w| = |\mathbf{A}e(u)|$. What is important is the fact that $D^2 w$ is a full gradient, while $e(u)$ is only a symmetric gradient. The other interesting fact in terms of boundary conditions, at least under connectedness assumptions, is the transformation of a Neumann type problem for the displacement u into a Dirichlet problem for the Airy function w, which is usually easier to handle.

We then find that, provided K is sufficiently flat in some ball $B(x_0, r)$ and the normalized energy

$$\omega(x_0, r) := \frac{1}{r} \int_{B(x_0,r)} \mathbf{A}e(u) : e(u)\, dx$$

is sufficiently small, we can control the decay of the energy $r \mapsto \omega(x_0, r)$ as $r \to 0$ (see Proposition 3.3). This first decay estimate is proved by contradiction, applying a compactness and Γ-convergence argument to the elastic energy. In this argument, it is crucial that the starting point x_0 belongs to an isolated connected component of K.

The second part of the proof is a decay estimate on the flatness, namely the quantity

$$\beta(x_0, r) := \frac{1}{r} \inf_L \max\Big(\sup_{x \in K \cap \overline{B}(x_0,r)} \operatorname{dist}(x, L), \sup_{x \in L \cap \overline{B}(x_0,r)} \operatorname{dist}(x, K) \Big),$$

where the infimum is taken over all affine lines L passing through x_0, measuring how far K is from a reference line in $B(x_0, r)$. This quantity is particularly useful since a decay estimate of the type $\beta(x_0, r) \le C r^\alpha$ leads to a $\mathscr{C}^{1,\alpha}$ regularity result on K (see Lemma 6.4).

The excess of density, namely

$$\frac{\mathcal{H}^1(K \cap B(x_0, r)) - 1}{2r},$$

controls the quantity $\beta(x_0, r)^2$, as a consequence of the Pythagoras inequality (see Lemma 6.3). In order to estimate the excess of density, the standard technique consists in comparing K, already known to be very flat in a ball $B(x_0, r)$ (i.e. $\beta(x_0, r) \leq \varepsilon$), with the competitor given by a segment S in $B(x_0, r)$. When doing this, one has to define a suitable admissible function v in $B(x_0, r)$ associated to the competitor S, which coincides with u outside $B(x_0, r)$ and has elastic energy controlled by that of u. This is where we have to face problem (v) mentioned earlier. The way we overcome this difficulty is a technical extension result (see Lemma 4.5). Whenever $\beta(x_0, r) + \omega(x_0, r) \leq \varepsilon$ for ε sufficiently small (depending only on the Ahlfors regularity constant θ_0), one can find a rectangle U such that

$$\overline{B}(x_0, r/5) \subset U \subset B(x_0, r),$$

and a "wall set" $\Sigma \subset \partial U$ such that

$$K \cap \partial U \subset \Sigma \quad \text{and} \quad \mathcal{H}^1(\Sigma) \leq \eta r,$$

where η is small. Moreover, if K' is a competitor for K in U (which "separates"), then there exists a function $v \in LD(U \setminus K')$ such that

$$u = v \quad \text{on } \partial U \setminus \Sigma,$$

and

$$\int_{U \setminus K'} \mathbf{A}e(v) : e(v)\, dx \leq \frac{C}{\eta^6} \int_{B(x_0,r)\setminus K} \mathbf{A}e(u) : e(u)\, dx.$$

The main point is that the set $\Sigma \subset U$ where the values of u and v do not match has very small length, essentially of order $\eta > 0$, which can be taken arbitrarily small. The price to pay is a diverging factor as $\eta \to 0$ in the right-hand side of the previous inequality. A similar statement with $\mathcal{H}^1(\Sigma) \leq r\beta(x_0, r)$ is much easier to prove, and is actually used before as a preliminary construction (see Lemma 4.2). We believe Lemma 4.5 is one of the most original parts of the proof of Theorem 1.1.

With this extension result at hand, estimating the flatness through the excess of density as described before, and choosing η of order $\omega(x_0, r)^{1/7}$, enables one to obtain a decay estimate for the flatness of the type (see Proposition 3.2)

$$\beta(x_0, r/50) \leq C\omega(x_0, r)^{1/14}.$$

The previous decay estimate together with the decay of the renormalized energy constitute the main ingredients which lead to the $\mathscr{C}^{1,\alpha}$ regularity result.

Organization of the paper

The paper is organized as follows. In Section 2, we introduce the main notation used throughout, and we precisely define the variational problem of fracture mechanics we are interested in. In Section 3, we prove our main result, Theorem 1.1, concerning the partial

$\mathscr{C}^{1,\alpha}$-regularity of the isolated connected components of the crack. The proof relies on two fundamental results. The first one, Proposition 3.2, is a flatness estimate in terms of the renormalized bulk energy, which is established in Section 4. The second one, Proposition 3.3, is a bulk energy decay which is proved in Section 5. In the Appendix (Section 6) we gather several technical results.

2. Statement of the problem

2.1. Notation

The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathscr{L}^n$, and the k-dimensional Hausdorff measure by $\mathscr{H}^k$. If E is a measurable set, we will sometimes write $|E|$ instead of $\mathscr{L}^n(E)$. If a and $b \in \mathbb{R}^n$, we write $a \cdot b = \sum_{i=1}^n a_i b_i$ for the Euclidean scalar product, and we denote the norm by $|a| = \sqrt{a \cdot a}$. The open (resp. closed) ball of center x and radius r is denoted by $B(x,r)$ (resp. $\overline{B}(x,r)$).

We write $\mathbb{M}^{n\times n}$ for the set of real $n \times n$ matrices, and $\mathbb{M}^{n\times n}_{\rm sym}$ for the real symmetric $n \times n$ matrices. Given $A \in \mathbb{M}^{n\times n}$, we let $|A| := \sqrt{\operatorname{tr}(AA^T)}$ (A^T is the transpose of A, and tr A is its trace), which is the usual Frobenius norm over $\mathbb{M}^{n\times n}$.

Given an open subset U of $\mathbb{R}^n$, we denote by $\mathcal{M}(U)$ the space of all real valued Radon measures with finite total variation. We use standard notation for Lebesgue spaces $L^p(U)$ and Sobolev spaces $W^{k,p}(U)$ or $H^k(U) := W^{k,2}(U)$. If K is a closed subset of $\mathbb{R}^n$, we denote by $H^k_{0,K}(U)$ the closure of $\mathscr{C}^\infty_c(\overline{U} \setminus K)$ in $H^k(U)$. In particular, if $K = \partial U$, then $H^k_{0,\partial U}(U) = H^k_0(U)$.

Functions of Lebesgue deformation. Given a vector field (distribution) $u : U \to \mathbb{R}^n$, the *symmetrized gradient* of u is denoted by

$$e(u) := \frac{\nabla u + \nabla u^T}{2}.$$

In linearized elasticity, u stands for the displacement, while $e(u)$ is the elastic strain. The elastic energy of a body is given by a quadratic form of $e(u)$, so that it is natural to consider displacements such that $e(u) \in L^2(U; \mathbb{M}^{n\times n}_{\rm sym})$. If U has Lipschitz boundary, it is well known that u actually belongs to $H^1(U;\mathbb{R}^n)$ as a consequence of the Korn inequality. However, when U is not smooth, we can only assert that $u \in L^2_{\rm loc}(U;\mathbb{R}^n)$. This motivates the following definition of the *space of Lebesgue deformation*:

$$LD(U) := \{u \in L^2_{\rm loc}(U;\mathbb{R}^n) : e(u) \in L^2(U;\mathbb{M}^{n\times n}_{\rm sym})\}. \tag{2.1}$$

If U is connected and u is a distribution with $e(u) = 0$, then necessarily it is a rigid movement, i.e. $u(x) = Ax + b$ for all $x \in U$, for some skew-symmetric matrix $A \in \mathbb{M}^{n\times n}$ and some vector $b \in \mathbb{R}^n$. If, in addition, U has Lipschitz boundary, the following Poincaré–Korn inequality holds: there exists a constant $c_U > 0$ and a rigid movement r_U such that

$$\|u - r_U\|_{L^2(U)} \le c_U \|e(u)\|_{L^2(U)} \quad \text{for all } u \in LD(U). \tag{2.2}$$

According to [2, Theorem 5.2, Example 5.3], it is possible to make r_U more explicit in the following way: consider a measurable subset E of U with $|E| > 0$; then one can take

$$r_U(x) := \frac{1}{|E|}\int_E u(y)\,dy + \left(\frac{1}{|E|}\int_E \frac{\nabla u(y) - \nabla u(y)^T}{2}\,dy\right)\left(x - \frac{1}{|E|}\int_E y\,dy\right),$$

provided the constant c_U in (2.2) also depends on E.

Hausdorff convergence of compact sets. Let K_1 and K_2 be compact subsets of a common compact set $K \subset \mathbb{R}^n$. The *Hausdorff distance* between K_1 and K_2 is given by

$$d_{\mathcal{H}}(K_1, K_2) := \max\Big(\sup_{x\in K_1} \operatorname{dist}(x, K_2),\ \sup_{y\in K_2} \operatorname{dist}(y, K_1)\Big).$$

We say that a sequence (K_n) of compact subsets of K *converges in the Hausdorff distance* to the compact set K_∞ if $d_{\mathcal{H}}(K_n, K_\infty) \to 0$. Finally, let us recall *Blaschke's selection principle* which asserts that from any sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of K, one can extract a subsequence converging in the Hausdorff distance.

Capacities. We will use the notion of capacity for which we refer to [1,28]. We just recall the definition and several facts. The $(k,2)$-*capacity* of a compact set $K \subset \mathbb{R}^n$ is defined by

$$\operatorname{Cap}_{k,2}(K) := \inf\{\|\varphi\|_{H^k(\mathbb{R}^n)} : \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n),\ \varphi \ge 1 \text{ on } K\}.$$

This definition is then extended to open sets $A \subset \mathbb{R}^2$ by

$$\operatorname{Cap}_{k,2}(A) := \sup\{\operatorname{Cap}_{k,2}(K) : K \subset A,\ K \text{ compact}\},$$

and to arbitrary sets $E \subset \mathbb{R}^n$ by

$$\operatorname{Cap}_{k,2}(E) := \inf\{\operatorname{Cap}_{k,2}(A) : E \subset A,\ A \text{ open}\}.$$

One of the interests of capacity is that it enables one to give an accurate meaning to the pointwise value of Sobolev functions. More precisely, every $u \in H^k(\mathbb{R}^n)$ has a $(k,2)$-*quasicontinuous representative* $\tilde{u}$, which means that $\tilde{u} = u$ a.e. and, for each $\varepsilon > 0$, there exists a closed set $A_\varepsilon \subset \mathbb{R}^n$ such that $\operatorname{Cap}_{k,2}(\mathbb{R}^n \setminus A_\varepsilon) < \varepsilon$ and $\tilde{u}|_{A_\varepsilon}$ is continuous on A_ε (see [1, Section 6.1]). The $(k,2)$-quasicontinuous representative is unique, in the sense that two such representatives of the same function $u \in H^k(\mathbb{R}^n)$ coincide $\operatorname{Cap}_{k,2}$-quasieverywhere, i.e. outside a set of zero $\operatorname{Cap}_{k,2}$-capacity. In addition, if U is an open subset of $\mathbb{R}^n$, then $u \in H_0^k(U)$ if and only if for all $\alpha \in \mathbb{N}^n$ of length $|\alpha| \le k-1$, $\partial^\alpha u$ has a $(k-|\alpha|,2)$-quasicontinuous representative that vanishes $\operatorname{Cap}_{k-|\alpha|,2}$-quasieverywhere on ∂U (see [1, Theorem 9.1.3]). We will only be interested in the cases $k=1$ or $k=2$ in dimension $n=2$.

2.2. *Definition of the problem*

We now describe the underlying fracture mechanics model and the related variational problem.

Reference configuration. Let us consider a homogeneous isotropic linearly elastic body occupying $\Omega \subset \mathbb{R}^2$ in its reference configuration. The Hooke law associated to this material is given by

$$\mathbf{A}\xi = \lambda(\operatorname{tr}\xi)I + 2\mu\xi \quad \text{for all } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}},$$

where λ and μ are the Lamé coefficients satisfying $\mu > 0$ and $\lambda + \mu > 0$. Note that this expression can be inverted into

$$\mathbf{A}^{-1}\sigma = \frac{1}{2\mu}\sigma - \frac{\lambda}{4\mu(\lambda+\mu)}(\operatorname{tr}\sigma)I = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}(\operatorname{tr}\sigma)I \quad \text{for all } \sigma \in \mathbb{M}^{2\times 2}_{\mathrm{sym}},$$

where $E := 4\mu(\lambda+\mu)/(\lambda+2\mu)$ is the Young modulus and $\nu := \lambda/(\lambda+2\mu)$ is the Poisson ratio.

Admissible displacements/cracks pairs. Let $\Omega' \subset \mathbb{R}^2$ be a bounded open set such that $\operatorname{diam}(\Omega') \le 2\operatorname{diam}(\Omega)$ and $\overline{\Omega} \subset \Omega'$. We say that a pair set/function is *admissible*, and we write $(u, K) \in \mathcal{A}(\Omega)$, if $K \subset \overline{\Omega}$ is closed and $u \in LD(\Omega' \setminus K)$.

Griffith energy. For all $(u, K) \in \mathcal{A}(\Omega)$, we define the *Griffith energy* functional by

$$\mathcal{G}(u, K) := \int_{\Omega\setminus K} \mathbf{A}e(u) : e(u)\,dx + \mathcal{H}^1(K).$$

In this work, we are interested in (interior) regularity properties of the global minimizers of the Griffith energy under a Dirichlet boundary condition, i.e., solutions to the (strong) minimization problem

$$\inf\{\mathcal{G}(v, K') : (v, K') \in \mathcal{A}(\Omega),\ v = \psi \text{ a.e. in } \Omega' \setminus \overline{\Omega}\}, \tag{2.3}$$

where $\psi \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ is a prescribed boundary displacement. Note that this formulation of the Dirichlet boundary condition permits one to account for possible cracks on $\partial\Omega$, where the displacement does not match the prescribed displacement ψ.

The question of the existence of solutions to (2.3) has been addressed in [11] (see also [12, 26]), extending the regularity results up to the boundary [10, 15]. For this, by analogy with the classical Mumford–Shah problem, it is convenient to introduce a weak formulation of (2.3) as follows:

$$\inf\left\{\int_\Omega \mathbf{A}e(v) : e(v)\,dx + \mathcal{H}^1(J_v) : v \in GSBD^2(\Omega'),\ v = \psi \text{ a.e. in } \Omega' \setminus \overline{\Omega}\right\},$$

with $GSBD^2$ a suitable space of generalized special functions of bounded deformation (see [16]) where the previous energy functional is well defined. According to [12, Theorem 4.1], if Ω has Lipschitz boundary, the previous minimization problem has a solution, denoted by u. In addition, if Ω is of class $\mathcal{C}^1$, thanks to [11, Theorems 5.6 and 5.7], there exist $\theta_0, R_0 > 0$, only depending on $\mathbf{A}$, such that the following property holds: for all $x_0 \in \overline{J_u}$ and all $r \in (0, R_0)$ such that $B(x_0, r) \subset \Omega'$,

$$\mathcal{H}^1(J_u \cap B(x_0, r)) \ge \theta_0 r.$$

The previous property of J_u ensures that, setting $K := \overline{J_u}$, we have $\mathcal{H}^1(K \setminus (J_u \cap \overline{\Omega})) = 0$, so that the pair $(u, K) \in \mathcal{A}(\Omega)$ is a solution of the strong problem (2.3). In addition, the crack set K is $\mathcal{H}^1$-rectifiable and Ahlfors regular: for all $x_0 \in K$ and all $r \in (0, R_0)$ such that $B(x_0, r) \subset \Omega'$,

$$\theta_0 r \leq \mathcal{H}^1(K \cap B(x_0, r)) \leq Cr, \tag{2.4}$$

where C is a constant depending only on Ω. The second inequality is obtained by comparing (u, K) with the most standard competitor (v, K') where $v := u\mathbf{1}_{\Omega' \setminus (\Omega \cap B(x_0, r))}$ and $K' := [K \setminus (\Omega \cap B(x_0, r))] \cup \partial(\Omega \cap B(x_0, r))$.

Next, taking in particular $K' = K$ and any $v \in LD(\Omega \setminus K)$ as competitor implies that $u \in LD(\Omega \setminus K)$ is also a solution of the minimization problem

$$\min \left\{ \int_{\Omega \setminus K} \mathbf{A}e(v) : e(v)\, dx : v \in LD(\Omega \setminus K),\ v = \psi \text{ on } \partial\Omega \setminus K \right\}.$$

Note that u is unique up to an additive rigid movement in each connected component of $\Omega \setminus K$ disjoint from $\partial\Omega \setminus K$. It turns out that u satisfies the following variational formulation: for all test functions $\varphi \in H^1(\Omega \setminus K; \mathbb{R}^2)$ with $\varphi = 0$ on $\partial\Omega \setminus K$,

$$\int_{\Omega \setminus K} \mathbf{A}e(u) : e(\varphi)\, dx = 0. \tag{2.5}$$

In particular, u is a solution to the elliptic system

$$-\operatorname{div}(\mathbf{A}e(u)) = 0 \quad \text{in } \mathcal{D}'(\Omega \setminus K; \mathbb{R}^2),$$

and, as a consequence, elliptic regularity shows that $u \in \mathcal{C}^\infty(\Omega \setminus K; \mathbb{R}^2)$.

3. The main quantities and proof of $\mathcal{C}^{1,\alpha}$ regularity

We now introduce the main quantities that will be at the heart of our analysis.

3.1. The normalized energy

Let $(u, K) \in \mathcal{A}(\Omega)$. Then for any x_0 and $r > 0$ such that $\overline{B}(x_0, r) \subset \Omega$ we define the *normalized elastic energy* by

$$\omega(x_0, r) := \frac{1}{r} \int_{B(x_0,r) \setminus K} \mathbf{A}e(u) : e(u)\, dx.$$

Sometimes we will write $\omega_u(x_0, r)$ to emphasize the underlying displacement u.

Remark 3.1. By definition of the normalized energy, for all $0 < t < r$, we have

$$\omega(x_0, t) \leq \frac{r}{t} \omega(x_0, r). \tag{3.1}$$

If $K' = \frac{1}{r}(K - x_0)$ and $v = \frac{1}{\sqrt{r}} u(r(\cdot + x_0))$ then

$$\omega_u(x_0, r) = \omega_v(0, 1).$$

3.2. The flatness

Let K be a closed subset of $\mathbb{R}^2$. For any $x_0 \in \mathbb{R}^2$ and $r > 0$, we define the *flatness* of K by

$$\beta(x_0, r) := \frac{1}{r} \inf_L \max\Big(\sup_{y \in K \cap \overline{B}(x_0,r)} \operatorname{dist}(y, L), \sup_{y \in L \cap \overline{B}(x_0,r)} \operatorname{dist}(y, K) \Big),$$

where the infimum is taken over all affine lines L passing through x_0. Sometimes we will write $\beta_K(x_0, r)$ to emphasize the underlying crack K.

Remark 3.2. By definition of flatness, we always have, for all $0 < t < r$,

$$\beta_K(x_0, t) \leq \frac{r}{t} \beta_K(x_0, r), \tag{3.2}$$

and if $K' = \frac{1}{r}(K - x_0)$, then

$$\beta_K(x_0, r) = \beta_{K'}(0, 1).$$

We will consider the situation where

$$\beta_K(x_0, r) \leq \varepsilon \tag{3.3}$$

for $\varepsilon > 0$ small. This implies in particular that $K \cap \overline{B}(x_0, r)$ is contained in a narrow strip of thickness εr passing through the center of the ball.

Let $L(x_0, r)$ be a line containing x_0 and satisfying

$$\sup_{x \in K \cap \overline{B}(x_0,r)} \operatorname{dist}(x, L(x_0, r)) \leq r \beta_K(x_0, r). \tag{3.4}$$

We will often use a local basis (depending on x_0 and r) denoted by (e_1, e_2), where e_1 is a vector tangent to the line $L(x_0, r)$, while e_2 is orthogonal to $L(x_0, r)$. The coordinates of a point y in that basis will be denoted by (y_1, y_2).

Provided (3.3) is satisfied with $\varepsilon \in (0, 1/2)$, we can define two discs $D^+(x_0, r)$ and $D^-(x_0, r)$ of radius $r/4$ and such that $D^{\pm}(x_0, r) \subset B(x_0, r) \setminus K$. Indeed, using the notation introduced above, setting $x_0^{\pm} := x_0 \pm \frac{3}{4} r e_2$, we can check that

$$D^{\pm}(x_0, r) := B(x_0^{\pm}, r/4)$$

satisfy the above requirements.

A property that will be fundamental in our analysis is separation in a closed ball.

Definition 3.1. Let K be a closed subset of $\mathbb{R}^2$, and let $x_0 \in \mathbb{R}^2$ and $r > 0$ be such that $\beta_K(x_0, r) \leq 1/2$. We say that K *separates* $D^{\pm}(x_0, r)$ in $\overline{B}(x_0, r)$ if the balls $D^{\pm}(x_0, r)$ are contained in two different connected components of $\overline{B}(x_0, r) \setminus K$.

The following lemma guarantees that when passing from a ball $B(x_0, r)$ to a smaller one $B(x_0, t)$, and provided that $\beta_K(x, r)$ is relatively small, the property of separating is preserved for t varying in a range depending on $\beta_K(x, r)$.

Lemma 3.1. *Let $\tau \in (0, 1/16)$, let $K \subset \Omega$ be a relatively closed set, and let $x_0 \in K$, and $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$. Assume that $\beta_K(x_0, r) \leq \tau$ and K separates $D^{\pm}(x_0, r)$ in $\overline{B}(x_0, r)$. Then for all $t \in (16\tau r, r)$, we have $\beta_K(x_0, t) \leq 1/2$ and K still separates $D^{\pm}(x_0, t)$ in $\overline{B}(x_0, t)$.*

Proof. We will need the following elementary inequality resulting from the mean value theorem:

$$\arcsin(t) \leq \sup_{s \in [0,1/2]} \frac{1}{\sqrt{1-s^2}} t = \frac{2}{\sqrt{3}} t \leq 2t \quad \text{for all } t \in [0, 1/2]. \tag{3.5}$$

Using the notations introduced above, considering the local basis (e_1, e_2) such that e_1 is tangent to $L(x_0, r)$ and e_2 is normal to $L(x_0, r)$, we have

$$K \cap \overline{B}(x_0, r) \subset \{y \in \overline{B}(x_0, r) : |y_2| \leq \tau r\}. \tag{3.6}$$

For all $t \in (16\tau r, r)$, we have

$$\beta(x_0, t) \leq \frac{r}{t} \beta(x_0, r) \leq \frac{1}{16\tau} \beta(x_0, r) \leq \frac{1}{16} \leq \frac{1}{2}, \tag{3.7}$$

so that $D^{\pm}(x_0, t)$ are well defined. Denoting by $\nu(x_0, t)$ a normal vector to the line $L(x_0, t)$, we can assume that $\nu(x_0, t) \cdot e_2 > 0$.

We first note that, similarly to (3.7), we can estimate

$$\operatorname{dist}(x, L(x_0, t)) \leq t\beta(x_0, t) \leq r\beta(x_0, r) \leq \tau r \quad \text{for all } x \in K \cap \overline{B}(x_0, t). \tag{3.8}$$

From (3.8) and (3.6) we deduce

$$B(x_0, t) \cap L(x_0, t) \subset \{y \in \mathbb{R}^2 : |y_2| \leq 2\tau r\}.$$

Denoting by $\alpha = \arccos(\nu(x_0, t) \cdot e_2)$ the angle between $\nu(x_0, t)$ and e_2, the previous inclusion implies

$$\alpha \leq \arcsin(2\tau r/t) \leq 4\tau r/t \leq 1/4, \tag{3.9}$$

where we have used (3.5) and $t > 16\tau r$.

Let $y_0 := x_0 + \frac{3}{4}\nu(x_0, t)t$ be the center of the disc $D^+(x_0, t)$. We have $|(y_0 - x_0)_2| = \cos(\alpha) \cdot \frac{3}{4}t$. In particular, using the elementary inequality $|\cos(\alpha) - 1| \leq \alpha$ and (3.9) we get

$$\operatorname{dist}(y_0, L(x_0, r)) = |(y_0 - x_0)_2| = \cos(\alpha) \cdot \tfrac{3}{4}t \geq \tfrac{3}{4}(1-\alpha)t \geq t/2,$$

hence, since $t > 16\tau r$, we infer that for all $y \in B(y_0, t/4)$,

$$\operatorname{dist}(y, L(x_0, r)) \geq t/2 - t/4 = t/4 \geq 4\tau r.$$

All in all, we have proved that

$$D^+(x_0, t) = B(y_0, t/4) \subset \{y \in \mathbb{R}^2 : y_2 \geq 4\tau r\}.$$

Arguing similarly for $D^-(x_0,t)$, we get

$$D^-(x_0,t)\subset\{y\in\mathbb{R}^2 : y_2\le -4\tau r\}.$$

Since by (3.6) we have $K\cap\overline{B}(x_0,t)\subset\{y\in\overline{B}(x_0,t) : |y_2|\le\tau r\}$, we deduce that $D^\pm(x_0,t)$ must lie in two distinct connected components of $\overline{B}(x_0,t)\setminus K$, and thus K actually separates $D^\pm(x_0,t)$ in $\overline{B}(x_0,t)$. ∎

The following topological result is well known.

Lemma 3.2. *Let $K\subset\Omega$ be a relatively closed set, and let $x_0\in K$ and $r>0$ be such that $\overline{B}(x_0,r)\subset\Omega$. Assume that $\mathcal{H}^1(K\cap B(x_0,r))<\infty$, $\beta_K(x_0,r)\le 1/2$ and K separates $D^\pm(x_0,r)$ in $\overline{B}(x_0,r)$. Then there exists an injective Lipschitz curve $\Gamma\subset K$ that still separates $D^\pm(x_0,r)$ in $\overline{B}(x_0,r)$.*

Proof. Since K separates $D^\pm(x_0,r)$ in $\overline{B}(x_0,r)$, there exists a compact connected set $\tilde{K}\subset K\cap\overline{B}(x_0,r)$ which still separates (see [34, Theorem 14.3]). Since $\tilde{K}$ also has finite $\mathcal{H}^1$-measure, it follows from [18, Corollary 30.2] that $\tilde{K}$ is arcwise connected. We denote by (e_1,e_2) an orthonormal system such that $L(x_0,r)$ has direction e_1. Since $\tilde{K}$ separates $D^\pm(x_0,r)$ in $\overline{B}(x_0,r)$, $\tilde{K}$ must contain at least one point in each connected component of $\partial B(x,r)\cap\{y\in\mathbb{R}^2 : |y_2|\le r/2\}$. Denoting by z and z' those two points, there exists a Lipschitz injective curve Γ in $\tilde{K}\cap\overline{B}(x_0,r)$ joining z and z' which separates $D^\pm(x_0,r)$ in $\overline{B}(x_0,r)$ (see for example [18, Proposition 30.14]). ∎

3.3. Initialization of the main quantities

We prove that if (u,K) is a minimizer of the Griffith functional, one can find many balls $\overline{B}(x_0,r)\subset\Omega$ such that K separates $D^\pm(x_0,r)$ in $\overline{B}(x_0,r)$ and such that $\beta_K(x_0,r)$ and $\omega_u(x_0,r)$ are small for $r>0$ small enough and for $\mathcal{H}^1$-a.e. $x_0\in\Gamma$, where $\Gamma\subset K$ is any connected component of $K\cap\Omega$. The restriction to a connected component Γ is only to ensure the separation property on K. Notice that in the following proposition we do not need the connected component to be isolated.

Proposition 3.1. *Let $(u,K)\in\mathcal{A}(\Omega)$ be a minimizer of the Griffith functional and let $\Gamma\subset K$ be a connected component of $K\cap\Omega$ such that $\mathcal{H}^1(\Gamma)>0$. Then for every $\varepsilon\in(0,10^{-3})$ there exists an exceptional set $Z\subset\Gamma$ with $\mathcal{H}^1(Z)=0$ such that the following property holds. For every $x_0\in\Gamma\setminus Z$, there exists $r_0>0$ such that*

$$\beta_K(x_0,r_0)\le\varepsilon,\quad \omega_u(x_0,r_0)\le\varepsilon$$

and K separates $D^\pm(x_0,r_0)$ in $\overline{B}(x_0,r_0)$.

Proof. The initialization for the quantity β is standard (see for instance [18, Exercises 41.21.3 and 41.23.1]); we sketch the proof for the sake of completeness.

Since K is a rectifiable set, we know that there exists $Z_1\subset K$ with $\mathcal{H}^1(Z_1)=0$ such that, at every point $x_0\in K\setminus Z_1$, K admits an approximate tangent line T_{x_0}, that is,

$$\lim_{r\to 0}\frac{\mathcal{H}^1(K\cap B(x_0,r)\setminus T_{x_0,\varepsilon r})}{r}=0 \tag{3.10}$$

for all $\varepsilon \in (0,1)$, where $T_{x_0,\varepsilon r} := \{y \in \mathbb{R}^2 : \mathrm{dist}(y, T_{x_0}) \le \varepsilon r\}$. Since K is also Ahlfors-regular by assumption, it is easily seen that T_{x_0} is the usual tangent, in the sense that for all $\varepsilon \in (0,1)$ there exists $r_\varepsilon > 0$ such that

$$K \cap B(x_0, r) \subset T_{x_0,\varepsilon r}$$

for all $r \le r_\varepsilon$. Indeed, assume that there exist $\varepsilon_0 \in (0,1)$ and sequences $r_k \to 0$ and $y_k \in K \cap B(x_0, r_k)$ such that

$$\mathrm{dist}(y_k, T_{x_0}) > \varepsilon_0 r_k.$$

Then by Ahlfors regularity (2.4) we have

$$\mathcal{H}^1(K \cap B(x_0, 2r_k) \setminus T_{x_0,\varepsilon_0 r_k/2}) \ge \mathcal{H}^1(K \cap B(y_k, \varepsilon_0 r_k/2)) \ge \theta_0 \varepsilon_0 r_k/2.$$

We conclude that

$$\liminf_{k\to\infty} \frac{\mathcal{H}^1(K \cap B(x_0, 2r_k) \setminus T_{x_0,\varepsilon_0 r_k/2})}{2r_k} \ge \frac{\theta_0\varepsilon_0}{4} > 0,$$

which contradicts (3.10). Hence, by definition, for all $x_0 \in K \setminus Z_1$ and all $\varepsilon \in (0,1)$, there exists $r_1 > 0$ such that

$$\beta(x_0, r) \le \varepsilon \quad \text{for all } r \le r_1.$$

Now we consider $\omega(x_0, r)$, which again can be initialized by the same argument used for the standard Mumford–Shah functional (see for instance [4, Proposition 7.9]). Let us reproduce it here. We consider the measure $\mu := \mathbf{A}e(u) : e(u)\mathcal{L}^2$. For all $t > 0$, let

$$E_t := \left\{x \in K : \limsup_{r\to 0} \frac{\mu(B(x,r))}{r} > t\right\}.$$

By a standard covering argument (see [4, Theorem 2.56]) one has

$$t\mathcal{H}^1(E_t) \le \mu(E_t).$$

But $E_t \subset K$ and $\mu(K) = 0$, thus $\mathcal{H}^1(E_t) = 0$ for all $t > 0$. By taking a sequence $t_n \searrow 0^+$ and defining $Z_2 := \bigcup_n E_{t_n}$, we find that $\mathcal{H}^1(Z_2) = 0$ and, for all $x_0 \in K \setminus Z_2$,

$$\lim_{r\to 0} \omega(x_0, r) = 0.$$

In other words, for every $x_0 \in K \setminus (Z_1 \cup Z_2)$, there exists $r_2 < r_1$ such that

$$\beta(x_0, r) \le \varepsilon, \quad \omega(x_0, r) \le \varepsilon,$$

for all $r \le r_2$.

It remains to prove the separation property of K. To this end, let Γ be a connected component of $K \cap \Omega$ which is relatively closed in $K \cap \Omega$. Since $\overline{\Gamma}$ is a compact connected set in $\mathbb{R}^2$ with $\mathcal{H}^1(\overline{\Gamma}) < \infty$, according to [18, Proposition 30.1] it is the range of an injective Lipschitz mapping $\gamma : [0,1] \to \overline{\Gamma}$. This implies that $\overline{\Gamma}$ has an approximate tangent line L_{x_0} for $\mathcal{H}^1$-a.e. $x_0 \in \overline{\Gamma}$. In addition, according to [8, Proposition 2.2(iii)], there exists an exceptional set $Z_3 \subset \overline{\Gamma}$ with $\mathscr{H}^1(Z_3) = 0$ and with the property that, for all $x_0 \in \Gamma \setminus Z_3$, one can find $r_3 > 0$ such that

$$\pi(\overline{\Gamma} \cap B(x_0, r)) \supset L_{x_0} \cap B(x_0, (1 - 10^{-3})r) \quad \text{for all } r \leq r_3, \tag{3.11}$$

where $\pi : \mathbb{R}^2 \to L_{x_0}$ denotes the orthogonal projection onto the line L_{x_0}. In particular, if moreover $\beta(x_0, r) \leq \varepsilon \leq 10^{-3}$, then the balls $D^{\pm}(x_0, (1 - 10^{-3})r)$ are well defined and, thanks to (3.11), $\overline{\Gamma}$ must separate $D^{\pm}(x_0, (1 - 10^{-3})r)$ in $\overline{B}(x_0, (1 - 10^{-3})r)$, hence K must separate too. Set $Z := Z_1 \cup Z_2 \cup Z_3$. Then $\mathscr{H}^1(Z) = 0$, and we have proved that for all $x_0 \in \Gamma \setminus Z$ and all $r \leq r_0 := \min(r_1, r_2, (1 - 10^{-3})r_3)$,

$$\beta(x_0, r) \leq \varepsilon, \quad \omega(x_0, r) \leq \varepsilon,$$

and K separates $D^+(x_0, r)$ from $D^-(x_0, r)$ in $\overline{B}(x_0, r)$, as required. ■

3.4. *Proof of Theorem 1.1*

The proof of our main result, Theorem 1.1, rests on both the following results, whose proofs are postponed to the subsequent sections. The first one is a flatness estimate in terms of the renormalized energy, which will be established in Section 4.

Proposition 3.2. *There exist $\varepsilon_1 > 0$ and $C_1 > 0$ (only depending on θ_0, the Ahlfors regularity constant of K) such that the following property holds. Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional. For all $x_0 \in K$ and $r > 0$ such that $\overline{B}(x_0, r) \subset \Omega$ and*

$$\omega_u(x_0, r) + \beta_K(x_0, r) \leq \varepsilon_1,$$

and K separates $D^{\pm}(x_0, r)$ in $\overline{B}(x_0, r)$, we have

$$\beta_K(x_0, r/50) \leq C_1 \omega_u(x_0, r)^{1/14}.$$

The second result is the following normalized energy decay which will be proved in Section 5.

Proposition 3.3. *For all $\tau > 0$, there exist $a \in (0, 1)$ and $\varepsilon_2 > 0$ such that the following property holds. Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let Γ be an isolated connected component of $K \cap \Omega$ such that $\mathscr{H}^1(\Gamma) > 0$. Let $x_0 \in \Gamma$ and $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$ and*

$$K \cap \overline{B}(x_0, r) = \Gamma \cap \overline{B}(x_0, r), \quad \beta_K(x_0, r) \leq \varepsilon_2.$$

Then

$$\omega_u(x_0, ar) \leq \tau\, \omega_u(x_0, r).$$

With both previous results at hand, we are in a position to bootstrap the preceding decay estimates in order to get a $\mathscr{C}^{1,\alpha}$-regularity estimate. Indeed, the conclusion of Theorem 1.1 will follow from the following result.

Proposition 3.4. *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional and let Γ be an isolated connected component of $K \cap \Omega$. Then there exists a relatively closed set $Z \subset \Gamma$ with $\mathcal{H}^1(Z) = 0$ such that for every $x_0 \in \Gamma \setminus Z$, one can find $r_0 > 0$ such that $\Gamma \cap B(x_0, r_0)$ is a $\mathcal{C}^{1,\alpha}$ curve for some $\alpha \in (0, 1)$ depending only on* **A**.

Proof. Let $\varepsilon_1 > 0$ and $C_1 > 0$ be the constants given by Proposition 3.2, and let $a > 0$ and $\varepsilon_2 > 0$ be the constants given by Proposition 3.3 corresponding to $\tau = 10^{-2}$. We define

$$\delta_1 := \min(\varepsilon_1, \varepsilon_2, 10^{-3}a), \qquad \delta_2 := \min(\delta_1, (a\delta_1/C_1)^{14}).$$

We can assume that $\mathcal{H}^1(\Gamma) > 0$, otherwise the proposition is trivial. Using the assumptions that Γ is an isolated connected component of $K \cap \Omega$ and Proposition 3.1 (applied with $\varepsilon = \delta_2/2$), we can find an exceptional set $Z \subset \Gamma$ with $\mathcal{H}^1(Z) = 0$ such that the following property holds. For every $x_0 \in \Gamma \setminus Z$ there exists $r > 0$ such that

$$\begin{gathered} K \cap \overline{B}(x_0, r) = \Gamma \cap \overline{B}(x_0, r), \\ \omega(x_0, r) \le \delta_2/2, \quad \beta(x_0, r) \le \delta_1/2, \quad K \text{ separates } D^{\pm}(x_0, r) \text{ in } \overline{B}(x_0, r). \end{gathered} \tag{3.12}$$

We start by showing a first self-improving estimate which stipulates that the quantities $\omega(x_0, r)$ and $\beta(x_0, r)$ will remain small at all smaller scales.

Step 1. We define $b := a/50$, and we claim that if

$$\omega(x_0, r) \le \delta_2, \quad \beta(x_0, r) \le \delta_1, \quad K \text{ separates } D^{\pm}(x_0, r) \text{ in } \overline{B}(x_0, r),$$

then

$$\omega(x_0, br) \le \delta_2, \quad \beta(x_0, br) \le \delta_1, \quad K \text{ separates } D^{\pm}(x_0, br) \text{ in } \overline{B}(x_0, br), \tag{3.13}$$

$$\omega(x_0, br) \le \tfrac{1}{2}\omega(x_0, r), \tag{3.14}$$

$$\beta(x_0, br) \le \frac{C_1}{a}\omega(x_0, r)^{1/14}. \tag{3.15}$$

Let us start with the renormalized energy. By Proposition 3.3 we get

$$\omega(x_0, ar) \le 10^{-2}\omega(x_0, r),$$

which yields, using (3.1),

$$\omega(x_0, br) \le 50\, \omega(x_0, ar) \le \tfrac{1}{2}\omega(x_0, r) \le \delta_2.$$

For what concerns the flatness, we can apply Proposition 3.2, so that

$$\beta(x_0, r/50) \le C_1\omega(x_0, r)^{1/14}.$$

Thus by (3.2) we get

$$\beta(x_0, br) \le \frac{1}{a}\beta(x_0, r/50) \le \frac{C_1}{a}\omega(x_0, r)^{1/14} \le \delta_1,$$

because $\delta_2 \leq (a\delta_1/C_1)^{14}$. Finally, since $\delta_1 \leq 10^{-3}a$, we infer that $16\delta_1 r \leq br < r$, so that K still separates $D^{\pm}(x_0, br)$ in $\overline{B}(x_0, br)$ owing to Lemma 3.1 (applied with $\tau = \delta_1$), and thus the claim is proved.

Step 2. Iterating the decay estimate established in Step 1, we find that (3.13)–(3.15) hold true in each ball $B(x_0, b^k r)$, $k \in \mathbb{N}$. We thus obtain

$$\omega(x_0, b^k r) \leq 2^{-k}\omega(x_0, r),$$

and subsequently, using now (3.15),

$$\beta(x_0, b^k r) \leq \frac{C_1}{a} 2^{-(k-1)/14} \omega(x_0, r)^{1/14}.$$

If $t \in (0, 1)$ we let $k \geq 0$ be the integer such that

$$b^{k+1} \leq t < b^k.$$

Notice in particular that

$$k + 1 \geq \frac{\ln(1/t)}{\ln(1/b)} > k$$

and thus $2^{-(k+1)/14} \leq t^\alpha$ with $\alpha = \frac{\ln(2)}{14|\ln(b)|} \in (0, 1)$. We deduce that

$$\beta(x_0, tr) \leq \frac{b^k}{t}\beta(x_0, b^k r) \leq \frac{C_1}{ab} 2^{-(k-1)/14}\omega(x_0, r)^{1/14} \leq Ct^\alpha \omega(x_0, r)^{1/14}$$

for some constant $C > 0$ only depending on C_1 and a.

Step 3. We now conclude the proof of the proposition. Indeed, according to (3.12), for every $x \in K \cap B(x_0, r/2)$ we still have

$$\omega(x, r/2) \leq \delta_2, \qquad \beta(x, r/2) \leq \delta_1,$$

and K separates $D^{\pm}(x, r/2)$ in $\overline{B}(x, r/2)$. Thus, by Steps 1 and 2 applied in each ball $B(x, r/2)$ with $x \in K \cap B(x_0, r/2)$, we deduce that

$$\beta(x, tr) \leq C\delta_2^{1/14} t^\alpha \qquad \text{for all } t \in (0, 1/2),$$

and since this is true for all $x \in K \cap B(x_0, r/2)$, we deduce that $K \cap B(x_0, a_0 r)$ is a $\mathscr{C}^{1,\alpha}$ curve for some $a_0 \in (0, 1/2)$ thanks to Lemma 6.4 in the appendix. ∎

4. Proof of the flatness estimate

In order to prove Proposition 3.2, we need to construct a competitor in a ball $B(x_0, r)$ where the flatness $\beta(x_0, r)$ and the renormalized energy $\omega(x_0, r)$ are small enough. The main difficulty is to control how the crack behaves close to the boundary of the ball. A first rough competitor is constructed in Propositions 4.1 and 4.2 by introducing a wall set of length $r\beta(x_0, r)$ on the boundary. It leads to density estimates in balls (or alternatively in rectangles) which state that, provided the crack is flat enough, the energy density scales like the diameter of the ball (or the width of the rectangle), up to a small error depending on $\beta(x_0, r)$ and $\omega(x_0, r)$.

Unfortunately, this rough competitor is not sufficient to get a convenient flatness estimate leading to the desired regularity result. A better competitor is obtained by suitably localizing the crack in two almost opposite boxes of size $\eta > 0$, arbitrarily small (see Lemma 4.4). Then we can define a competitor inside a larger rectangle U whose vertical sides intersect both the small boxes. The crack competitor is then defined by taking an almost horizontal segment inside the rectangle, together with a new wall set $\Sigma \subset \partial U$ of arbitrarily small length, made up of the intersection of the rectangle with the boxes. It is then possible to introduce a displacement competitor (see Lemma 4.5) by extending the value of u on $\partial U \setminus \Sigma$ inside U. The price to pay is that the bound on the elastic energy associated to this competitor might diverge as the length of the wall set is small. It is however possible to optimize the competition between the flatness and the renormalized energy associated to this competitor by taking $\eta = \omega(x_0, r)^{1/7}$, leading to the conclusion of Proposition 3.2.

4.1. Density estimates

In this section we prove some density estimates for the set K. Such estimates will be useful to select good radii, in a way that the corresponding spheres intersect the set K in two almost opposite points. One of the main tools to construct competitors will be the following extension lemma.

Lemma 4.1 (Harmonic extension in a ball from an arc of circle). *Let $0 < \delta \leq 1/2$, $x_0 \in \mathbb{R}^2$, $r > 0$ and let $\mathscr{C}_\delta \subset \partial B(x_0, r)$ be the circle arc defined by*

$$\mathscr{C}_\delta := \{x \in \partial B(x_0, r) : (x - x_0)_2 > \delta r\}.$$

Then there exists a constant $C > 0$ (independent of δ, x_0, and r) such that every function $u \in H^1(\mathscr{C}_\delta; \mathbb{R}^2)$ extends to a function $g \in H^1(B(x_0, r); \mathbb{R}^2)$ with $g = u$ on $\mathscr{C}_\delta$ and

$$\int_{B(x_0,r)} |\nabla g|^2 \, dx \leq C r \int_{\mathscr{C}_\delta} |\partial_\tau u|^2 \, d\mathcal{H}^1.$$

Proof. Let $\Phi : \mathscr{C}_\delta \to \mathscr{C}_0$ be a bilipschitz mapping with Lipschitz constants independent of $\delta \in (0, 1/2]$, x_0, and $r > 0$. Since $u \circ \Phi^{-1} \in H^1(\mathscr{C}_0; \mathbb{R}^2)$, we can define the extension by reflection $\tilde{u} \in H^1(\partial B(x_0, r); \mathbb{R}^2)$ on the whole circle $\partial B(x_0, r)$, which satisfies

$$\int_{\partial B(x_0,r)} |\partial_\tau \tilde{u}|^2 \, d\mathcal{H}^1 \leq C \int_{\mathscr{C}_\delta} |\partial_\tau u|^2 \, d\mathcal{H}^1,$$

where $C > 0$ is a constant independent of δ.

We next define g as the harmonic extension of $\tilde{u}$ in $B(x_0, r)$. Using [18, Lemma 22.16], we obtain

$$\int_{B(x_0,r)} |\nabla g|^2 \, dx \leq C r \int_{\partial B(x_0,r)} |\partial_\tau \tilde{u}|^2 \, d\mathcal{H}^1 \leq C r \int_{\mathscr{C}_\delta} |\partial_\tau u|^2 \, d\mathcal{H}^1,$$

which completes the proof. ∎

Lemma 4.2 (Extension lemma, first version). *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$ and $\beta_K(x_0, r) \leq 1/10$. Let S be the strip defined by*

$$S := \{y \in \overline{B}(x_0, r) : \text{dist}(y, L(x_0, r)) \leq r\beta(x_0, r)\}.$$

Then there exist a universal constant $C > 0$, $\rho \in (r/2, r)$, and $v^\pm \in H^1(B(x_0, \rho); \mathbb{R}^2)$ such that $v^\pm = u$ on $\mathscr{C}^\pm$, $\mathscr{C}^\pm$ being the connected components of $\partial B(x_0, \rho) \setminus S$, and

$$\int_{B(x_0,\rho)} |e(v^\pm)|^2 \, dx \leq C \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx.$$

Proof. Let $A^\pm$ be the connected components of $B(x_0, r) \setminus S$. Since $K \cap A^\pm = \emptyset$, by the Korn inequality there exist two skew-symmetric matrices $R^\pm$ such that the functions $x \mapsto u(x) - R^\pm x$ belong to $H^1(A^\pm; \mathbb{R}^2)$ and

$$\int_{A^\pm} |\nabla u - R^\pm|^2 \, dx \leq C \int_{A^\pm} |e(u)|^2 \, dx,$$

where the constant $C > 0$ is universal since the domains $A^\pm$ are all uniformly Lipschitz for all possible values of $\beta(x_0, r) \leq 1/10$. Using the change of variables in polar coordinates, we infer that

$$\int_{A^\pm} |\nabla u - R^\pm|^2 \, dx = \int_0^r \left(\int_{\partial B(x_0,\rho)\cap A^\pm} |\nabla u - R^\pm|^2 \, d\mathcal{H}^1 \right) d\rho,$$

which allows us to choose a radius $\rho \in (r/2, r)$ satisfying

$$\begin{aligned}
&\int_{\partial B(x_0,\rho)\cap A^+} |\nabla u - R^+|^2 \, d\mathcal{H}^1 + \int_{\partial B(x_0,\rho)\cap A^-} |\nabla u - R^-|^2 \, d\mathcal{H}^1 \\
&\qquad \leq \frac{2}{r} \int_{A^+} |\nabla u - R^+|^2 \, dx + \frac{2}{r} \int_{A^-} |\nabla u - R^-|^2 \, dx \leq \frac{C}{r} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx.
\end{aligned}$$

Setting $\mathscr{C}^\pm := \partial A^\pm \cap \partial B(x_0, \rho)$, in view of Lemma 4.1 applied to the functions $u^\pm : x \mapsto u(x) - R^\pm x$, which belong to $H^1(\mathscr{C}^\pm; \mathbb{R}^2)$ since they are regular, for $\delta = \beta(x_0, r)$ we get two functions $g^\pm \in H^1(B(x_0, \rho); \mathbb{R}^2)$ satisfying $g^\pm(x) = u(x) - R^\pm x$ for $\mathcal{H}^1$-a.e. $x \in \mathscr{C}^\pm$ and

$$\int_{B(x_0,\rho)} |\nabla g^\pm|^2 \, dx \leq C\rho \int_{\mathscr{C}^\pm} |\partial_\tau u^\pm|^2 \, d\mathcal{H}^1 \leq C \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx.$$

Finally, the functions

$$x \mapsto v^\pm(x) := g^\pm(x) + R^\pm x$$

satisfy the required properties. ■

We now use the extension to prove two density estimates, first in smaller balls, then in smaller rectangles.

Proposition 4.1 (Density estimate in a ball). *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$ and $\beta_K(x_0, r) \leq 1/10$. Then there exist a universal constant $C > 0$ and a radius $\rho \in (r/2, r)$ such that*

$$\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u)\, dx + \mathcal{H}^1(K \cap B(x_0, \rho)) \leq 2\rho + C\rho\big(\omega_u(x_0, r) + \beta_K(x_0, r)\big).$$

Proof. We keep the same notation as in the proof of Lemma 4.2. Let $\rho \in (r/2, r)$ and $v^\pm \in H^1(B(x_0, \rho); \mathbb{R}^2)$ be given by the conclusion of Lemma 4.2. We now construct a competitor in $B(x_0, \rho)$ as follows. First, we consider a "wall set" $Z \subset \partial B(x_0, \rho)$ defined by

$$Z := \{y \in \partial B(x_0, \rho) : \mathrm{dist}(y, L(x_0, r)) \leq r\beta(x_0, r)\}.$$

Note that $K \cap \partial B(x_0, \rho) \subset Z$,

$$\partial B(x_0, \rho) = [\partial A^+ \cap \partial B(x_0, \rho)] \cup [\partial A^- \cap \partial B(x_0, \rho)] \cup Z = \mathscr{C}^+ \cup \mathscr{C}^- \cup Z,$$

and that

$$\mathcal{H}^1(Z) = 4\rho \arcsin\left(\frac{r\beta(x_0, r)}{\rho}\right) \leq 4r\beta(x_0, r).$$

We are now ready to define the competitor (v, K') by setting

$$K' := [K \setminus B(x_0, \rho)] \cup Z \cup [L(x_0, r) \cap B(x_0, \rho)],$$

and, denoting by $V^\pm$ the connected components of $B(x_0, \rho) \setminus L(x_0, r)$ which intersect $A^\pm$,

$$v := \begin{cases} v^\pm & \text{in } V^\pm, \\ u & \text{otherwise.} \end{cases}$$

Since $\mathcal{H}^1(K' \cap \overline{B}(x_0, \rho)) \leq 2\rho + 4r\beta(x_0, r)$, we deduce that

$$\begin{aligned} &\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u) : e(u)\, dx + \mathcal{H}^1(K \cap \overline{B}(x_0, \rho)) \\ &\qquad \leq \int_{B(x_0,\rho)\setminus K} \mathbf{A}e(v) : e(v)\, dx + \mathcal{H}^1(K' \cap \overline{B}(x_0, \rho)) \\ &\qquad \leq C \int_{B(x_0,r)\setminus K} |e(u)|^2\, dx + \rho(2 + C\beta(x_0, r)) \\ &\qquad \leq 2\rho + C\rho(\omega(x_0, r) + \beta(x_0, r)), \end{aligned}$$

and the proposition follows. ■

The following proposition is similar to Proposition 4.1, but here balls are replaced by rectangles. The assumption that K separates $D^\pm(x_0, r)$ in $\overline{B}(x_0, r)$ is not crucial here and could be removed. We will keep it to simplify the proof of the proposition.

Proposition 4.2 (Density estimates in a rectangle). *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$, $\beta_K(x_0, r) \leq 1/10$, and K separates $D^{\pm}(x_0, r)$ in $\overline{B}(x_0, r)$. Let (e_1, e_2) be an orthogonal system such that $L(x_0, r)$ has direction e_1. Then there exists a universal constant $C_* > 0$, such that*

$$\begin{aligned}\mathcal{H}^1(K \cap \{y \in B(x_0, r) : r/5 \leq (y - x_0)_1 \leq 2r/5\})\\ \leq r/5 + C_* r\big(\beta_K(x_0, r) + \omega_u(x_0, r)\big).\end{aligned}$$

Proof. We first apply Lemma 4.2 to get a radius $\rho \in (r/2, r)$ and functions $v^{\pm} \in H^1(B(x_0, \rho); \mathbb{R}^2)$ which satisfy the conclusion. To construct a competitor for K in $\overline{B}(x_0, \rho)$, we would like to replace the set K inside the rectangle

$$R := \{y \in \mathbb{R}^2 : (y - x_0)_1 \in [r/5, 2r/5] \text{ and } |(y - x_0)_2| \leq r\beta(x, r)\}$$

by the segment $L(x_0, r) \cap R$ which has length exactly $r/5$. Such a competitor may not separate the balls $D^{\pm}(x_0, \rho)$ in $\overline{B}(x_0, \rho)$. If $D^{\pm}(x_0, \rho)$ belonged to the same connected component, we could only take v^+ (or v^-) as a competitor of u, introducing a big jump on the boundary of $B(x_0, \rho)$ and removing completely the jump on K. To overcome this problem, we consider a "wall set" (inside the vertical boundaries of R)

$$Z' := \{y \in \mathbb{R}^2 : (y - x_0)_1 \in \{r/5, 2r/5\} \text{ and } |(y - x_0)_2| \leq r\beta(x, r)\},$$

as well as a second wall set on $\partial B(x_0, r)$ as before, defined by

$$Z := \{y \in \partial B(x_0, \rho) : \operatorname{dist}(y, L(x_0, r)) \leq r\beta(x_0, r)\}.$$

We define

$$K' := [K \cap \Omega \setminus R] \cup Z \cup Z' \cup [L(x_0, r) \cap R].$$

Note that K' is now separating the ball $\overline{B}(x_0, \rho)$ (thanks to the wall set Z') and

$$\mathcal{H}^1(K' \cap B(x_0, \rho)) \leq r/5 + 8r\beta(x_0, r) + \mathcal{H}^1(K \cap B(x_0, \rho) \setminus R).$$

Now we define the competitor for the function u in $\overline{B}(x_0, \rho)$. To this end, recalling that

$$\mathscr{C}^{\pm} = \{y \in \partial B(x_0, \rho) : \pm \operatorname{dist}(y, L(x_0, r)) > r\beta(x_0, r)\},$$

and using the fact that K' separates the ball $\overline{B}(x_0, \rho)$, we can find two connected components $V^{\pm}$ of $B(x_0, \rho) \setminus K'$ whose closures intersect $\mathscr{C}^{\pm}$ and define

$$v := \begin{cases} v^{\pm} & \text{in } V^{\pm}, \\ u & \text{otherwise.} \end{cases}$$

Recall that $u = v^{\pm}$ on $\partial B(x_0, \rho) \setminus Z$. Note that the presence of Z in the singular set K' is due to the fact that $v^{\pm}$ does not match u on Z. The pair (v, K') is then a competitor for

(u,K) in $\overline{B}(x_0,\rho)$, and thus

$$\begin{aligned}\int_{B(x_0,\rho)\setminus K} \mathbf{A}e(u):e(u)\,dx + \mathcal{H}^1(K\cap \overline{B}(x_0,\rho))\\ \le \int_{B(x_0,\rho)\setminus K'} \mathbf{A}e(v):e(v)\,dx + \mathcal{H}^1(K'\cap \overline{B}(x_0,\rho))\\ \le C\int_{B(x_0,r)\setminus K}|e(u)|^2\,dx + \frac{r}{5} + 8r\beta(x_0,r) + \mathcal{H}^1(K\cap B(x_0,\rho)\setminus R),\end{aligned}$$

from which we deduce that

$$\mathcal{H}^1(K\cap\{y\in B(x_0,r): r/5\le (y-x_0)_1\le 2r/5\})\le r/5 + Cr(\beta(x_0,r)+\omega(x_0,r)).$$

which completes the proof. ∎

An interesting consequence of the previous density estimates is a result on selection of good radii, where the corresponding spheres intersect the set K in only two almost opposite points.

Lemma 4.3 (Finding a good radius). *There exists a universal constant $\varepsilon_0>0$ such that the following property holds. Let $(u,K)\in\mathcal{A}(\Omega)$ be a minimizer of the Griffith functional and let $x_0\in K$ and $r>0$ be such that $\overline{B}(x_0,r)\subset\Omega$ and*

$$\omega_u(x_0,r)+\beta_K(x_0,r)\le\varepsilon_0.$$

If the set K separates $D^{\pm}(x_0,r)$ in $\overline{B}(x_0,r)$, then there exists $s\in(r/8,r)$ such that $\#(K\cap\partial B(x_0,s))=2$.

Proof. According to Proposition 4.1, there exist a universal constant $C>0$ and a radius $\rho\in(r/2,r)$ such that

$$\int_{B(x_0,\rho)\setminus K}\mathbf{A}e(u):e(u)\,dx + \mathcal{H}^1(K\cap B(x_0,\rho))\le 2\rho + C\rho\big(\omega(x_0,r)+\beta(x_0,r)\big).$$

We now fix

$$\varepsilon_0 := \min\left(\frac{1}{8C},\frac{1}{10}\right). \tag{4.1}$$

Using inequality (6.1) of Lemma 6.1 and (4.1), we get the estimate

$$\int_0^{\rho}\#(K\cap\partial B(x_0,s))\,ds\le\mathcal{H}^1(K\cap B(x_0,\rho))\le(2+1/8)\rho.$$

Moreover, thanks to the fact that $r\beta(x_0,r)\le\frac{1}{10}r\le\frac{1}{5}\rho<\frac{1}{4}\rho$, we infer that for all $s\in(\rho/4,\rho)$ the circle $\partial B(x_0,s)$ is not totally contained in the strip $\{x\in\overline{B}(x_0,r):\operatorname{dist}(x,L(x_0,r))\le\beta(x_0,r)r\}$. Therefore, since K is assumed to separate $D^{\pm}(x_0,r)$ in $\overline{B}(x_0,r)$, we deduce that for all $s\in(\rho/4,\rho)$,

$$\#(K\cap\partial B(x_0,s))\ge 2.$$

Setting

$$A := \{s \in (\rho/4, \rho) : \#(K \cap \partial B(x_0, s)) \geq 3\},$$

we obtain

$$3\mathcal{L}^1(A) + 2\mathcal{L}^1([\rho/4, \rho] \setminus A) \leq \int_{\rho/4}^{\rho} \#(K \cap \partial B(x_0, s))\, ds \leq (2 + 1/8)\rho,$$

and finally

$$\mathcal{L}^1(A) \leq \big(2 + 1/8 - 2\tfrac{3}{4}\big)\rho = 5\rho/8 < 3\rho/4 = \mathcal{L}^1((\rho/4, \rho)).$$

We then deduce the existence of some $s \in (\rho/4, \rho) \setminus A$, which thus satisfies $\#(K \cap \partial B(x_0, s)) = 2$. Since $\rho \in (r/2, r)$, the radius s then belongs to $(r/8, r)$. ∎

4.2. The main extension result

The first rough density estimate given by Proposition 4.1 is based on the property that the crack is always contained in a small strip of thickness $r\beta(x_0, r)$. This enables one to construct a competitor outside a wall set with height of order $r\beta(x_0, r)$. However, in order to bootstrap the estimates on our main quantities, β and ω, we need to slightly improve such a density estimate obtaining a remainder of order $r\eta$, with η well chosen (of order $\omega(x_0, r)^{1/7}$), instead of $r\beta(x_0, r)$. To this end, we need a refined version of the extension lemma 4.2, in which the boundary value of the competitor displacement is prescribed outside a wall set of height $r\eta$, instead of $r\beta(x_0, r)$. To construct such a suitable small wall set, we first find a nice region in the annulus $B(x_0, 2r/5) \setminus B(x_0, r/5)$ where to cut, i.e. we find some little boxes in which the set K is totally trapped. This is the purpose of the following lemma. Notice that in all this subsection and in the next one we never use any connectedness assumption on K; we use a separating assumption only.

Lemma 4.4 (Selection of cutting squares). *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$ and*

$$\omega_u(x_0, r) + \beta_K(x_0, r) \leq \frac{1}{5C_*} \min(1, 10^{-2}\theta_0),$$

where $\theta_0 > 0$ is the Ahlfors regularity constant of K, and $C_* > 0$ is the universal constant given in Proposition 4.2. Also assume that K separates $D^{\pm}(x_0, r)$ in $\overline{B}(x_0, r)$. Let (e_1, e_2) be an orthogonal system such that $L(x_0, r)$ has direction e_1. Then for every $\eta \in (0, 10^{-2})$ there exist $y_0, z_0 \in \mathbb{R}^2$ such that

$$(y_0 - x_0)_1 \in (r/5, 2r/5), \quad (z_0 - x_0)_1 \in (-2r/5, -r/5), \tag{4.2}$$

$$|(y_0 - x_0)_2| \leq \beta(x_0, r)r, \quad |(z_0 - x_0)_2| \leq \beta(x_0, r)r, \tag{4.3}$$

and

$$K \cap \{y \in \mathbb{R}^2 : |(y - y_0)_1| \leq \eta r\} \subset \{y \in \mathbb{R}^2 : |(y - y_0)_2| \leq 30\eta r\}, \tag{4.4}$$

$$K \cap \{y \in \mathbb{R}^2 : |(y - z_0)_1| \leq \eta r\} \subset \{y \in \mathbb{R}^2 : |(y - z_0)_2| \leq 30\eta r\}. \tag{4.5}$$

Proof. It is enough to prove the existence of y_0 since the argument for z_0 is similar. For simplicity, we write $\beta := \beta(x_0, r)$, $\omega := \omega(x_0, r)$.

We start by finding a good vertical strip in which K has small length. Let us define the vertical strip

$$S := \{y \in B(x_0, r) : r/5 \le (y - x_0)_1 \le 2r/5\}.$$

Let $\eta < 1/10$ and let $N \in \mathbb{N}$, $N \ge 2$, be such that $\frac{1}{5N} \le \eta < \frac{1}{5N-5}$. Then $(N-1)/N \ge 1/2$ and

$$\frac{\eta}{2} \le \frac{1}{5N} \le \eta. \tag{4.6}$$

We split S into the pairwise disjoint union of N smaller sets $S_1, \dots, S_N$ defined, for all $k \in \{1, \dots, N\}$, by

$$S_k := \left\{ y \in S : \frac{r}{5} + \frac{k-1}{5N} r \le (y - x_0)_1 < \frac{r}{5} + \frac{k}{5N} r \right\}.$$

Since $\beta \le 1/10$ we can apply Proposition 4.2, which implies

$$\sum_{k=1}^{N} \mathcal{H}^1(K \cap S_k) \le \mathcal{H}^1(K \cap S) \le \frac{(1+E)r}{5} \tag{4.7}$$

with $E := 5C_*(\beta + \omega)$, where we recall that θ_0 is the Ahlfors regularity constant of K, and $C_* > 0$ is the universal constant given in Proposition 4.2. As will be used later, we notice that under our assumptions we have in particular that

$$E \le \min(1, 10^{-2}\theta_0). \tag{4.8}$$

From (4.7) we deduce the existence of $k_0 \in \{1, \dots, N\}$ such that (see Figure 1)

$$\mathcal{H}^1(K \cap S_{k_0}) \le \frac{(1+E)r}{5N}. \tag{4.9}$$

By the separation property of K, one can find inside $K \cap S_{k_0}$ an injective Lipschitz curve Γ connecting both vertical sides of ∂S_{k_0} (see Lemma 3.2). In particular,

$$\frac{r}{5N} \le \mathcal{H}^1(\Gamma) \le \frac{(1+E)r}{5N}, \tag{4.10}$$

and thus (4.9) leads to

$$\mathcal{H}^1(K \cap S_{k_0} \setminus \Gamma) \le \frac{Er}{5N} \le \eta E r. \tag{4.11}$$

Thanks to the length estimate (4.10), if we denote by $z, z' \in \partial S_{k_0}$ the two points of Γ on the boundary of S_{k_0}, we have in particular, for every $y \in \Gamma$,

$$|y - z| \le \mathcal{H}^1(\Gamma) \le \frac{1+E}{5N} r \le \frac{2}{5N} r,$$

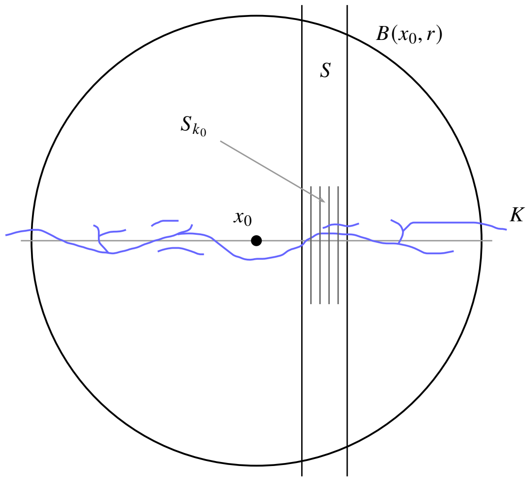

Fig. 1. The choice of S_{k_0}.

because $E \leq 1$. In other words,

$$\sup_{y \in \Gamma} |y - z| \leq \frac{2}{5N} r \leq 2\eta r. \tag{4.12}$$

We now finally give a bound on the distance from the points of K to the curve Γ in a strip slightly thinner than S_{k_0}, by use of the Ahlfors regularity of K. For that purpose we let

$$S' := \left\{ y \in S_{k_0} : \frac{r}{5} + \frac{k_0 - 1}{5N} r + \delta r \leq (y - x_0)_1 \leq \frac{r}{5} + \frac{k_0}{5N} r - \delta r \right\}$$

with $\delta := 2\eta E/\theta_0$. Since $E \leq 10^{-2}\theta_0$, we deduce that $\delta \leq \frac{10^{-1}}{5N}$, so that S' is not empty.

We claim that

$$\sup_{y \in K \cap S'} \operatorname{dist}(y, \Gamma) \leq \frac{2\eta E}{\theta_0} r. \tag{4.13}$$

Indeed, if $y \in K \cap S'$ is such that $d := \operatorname{dist}(y, \Gamma) > \delta r = \frac{2\eta E}{\theta_0} r$, then $B(y, \delta r) \subset S_{k_0} \setminus \Gamma$ and, by Ahlfors regularity,

$$\mathcal{H}^1(K \cap B(y, \delta r)) \geq \theta_0 \delta r = 2\eta E r,$$

which contradicts (4.11) and proves (4.13).

To conclude, gathering (4.13) and (4.12), we have obtained

$$\sup_{y \in K \cap S'} |y - z| \leq \frac{2\eta E}{\theta_0} r + 2\eta r \leq 3\eta r, \tag{4.14}$$

since $2E/\theta_0 \leq 1$. Therefore, if we define y_0 as being the middle point of the segment $\left[z, z + \frac{r}{5N} e_1\right]$ (in particular in the middle of S'), the conclusions (4.2) and (4.3) of the lemma are satisfied.

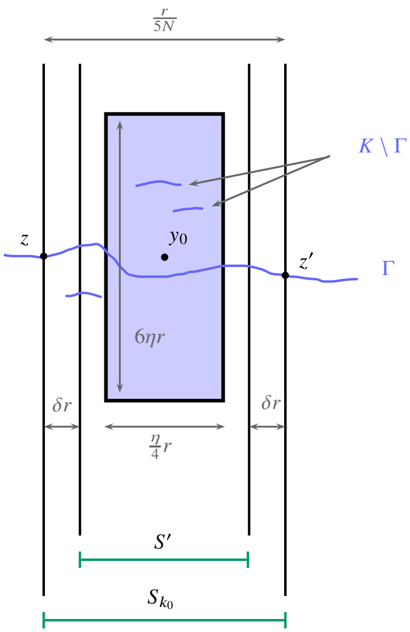

Fig. 2. The set K is trapped into a rectangle of size $\simeq \eta r$.

Next, we notice that by (4.6), the width of S' is exactly

$$\frac{1}{5N}r - 2\delta r = \frac{1}{5N}r - 4\frac{\eta E}{\theta_0}r \geq \left(\frac{\eta}{2} - 4\frac{\eta E}{\theta_0}\right)r \geq \frac{\eta r}{4}$$

provided that $E \leq \frac{\theta_0}{16}$, which is valid thanks to (4.8) (see Figure 2). Consequently, using (4.14) and $(y_0)_2 = z_2$, we deduce that with this choice of y_0,

$$K \cap \{y \in \mathbb{R}^2 : |(y - y_0)_1| \leq \eta/8r\} \subset K \cap S' \subset \{y \in \mathbb{R}^2 : |(y - y_0)_2| \leq 3\eta r\}.$$

The conclusion of the lemma follows by relabeling $\eta/8$ as η. ■

We are now in a position to establish an improved version of the extension lemma. Its proof is similar to that of Proposition 4.1, the difference being the definition of the wall set that has now size ηr instead of $r\beta(x_0, r)$.

Lemma 4.5 (Extension lemma). *Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$ and*

$$\omega_u(x_0, r) + \beta_K(x_0, r) \leq \frac{1}{5C_*}\min(1, 10^{-2}\theta_0),$$

where θ_0 is the Ahlfors regularity constant of K and $C_* > 0$ is the universal constant given in Proposition 4.2. Also assume that K separates $D^\pm(x_0, r)$ in $\overline{B}(x_0, r)$. Then for all $0 < \eta < 10^{-4}$ there exist:

- *an open rectangle U such that $\overline{B}(x_0, r/5) \subset U \subset B(x_0, r)$,*
- *a wall set (i.e. union of two vertical segments) $\Sigma \subset \partial U$ such that $K \cap \partial U \subset \Sigma$, $u \in H^1(\partial U \setminus \Sigma; \mathbb{R}^2)$ and $\mathcal{H}^1(\Sigma) \leq 120\eta r$.*

In addition, if $K' \subset \Omega$ is a closed set such that $K' \setminus U = K \setminus U$ and $D^\pm(x_0, r/5)$ are contained in two different connected components of $U \setminus K'$, then there exists a function $v \in H^1(\Omega \setminus K'; \mathbb{R}^2)$ such that

$$v = u \quad \text{on } (\Omega \setminus U) \setminus \Sigma$$

and

$$\int_{U \setminus K'} |e(v)|^2\, dx \leq \frac{C}{\eta^6} \int_{B(x_0,r) \setminus K} |e(u)|^2\, dx, \tag{4.15}$$

where $C > 0$ is universal.

Proof. We denote by (e_1, e_2) an orthogonal system such that $L(x_0, r)$ has direction e_1 and we apply Lemma 4.4, which gives the existence of $y_0, z_0 \in B(x_0, 2r/5) \setminus \overline{B}(x_0, r/5)$ satisfying (4.2)–(4.5). In order to construct the rectangle U and the wall set Σ, we need to introduce a domain A which is a "rectangular annulus" of thickness of order ηr.

Step 1: Construction of a rectangular annulus A. The vertical parts of A are defined to be the open rectangles

$$V_1 := \left\{x \in \mathbb{R}^2 : |(x - y_0)_1| < \eta r,\ |(x - x_0)_2| < \tfrac{1}{3} r\right\},$$
$$V_2 := \left\{x \in \mathbb{R}^2 : |(x - z_0)_1| < \eta r,\ |(x - x_0)_2| < \tfrac{1}{3} r\right\}.$$

Notice that $(y_0 - x_0)_1 \leq \frac{2}{5} r$ and $\eta r \leq 10^{-2} r$, so that

$$\sup_{y \in V_1} (y - x_0)_1 \leq \tfrac{2}{5} r + 10^{-2} r = \tfrac{41}{100} r,$$

which means that the right corners of V_1 have distance to x_0 bounded by $\sqrt{\frac{41^2}{100^2} + \frac{1}{9}}\, r < r$ and therefore $V_1 \subset B(x_0, r)$. By symmetry, $V_2 \subset B(x_0, r)$ as well.

Now the horizontal parts of A are the open rectangles

$$H_1 := \left\{x \in \mathbb{R}^2 : (z_0)_1 - \eta r < (x - x_0)_1 < (y_0)_1 + \eta r,\ \tfrac{1}{3} r - \eta r < (x - x_0)_2 < \tfrac{1}{3} r\right\},$$
$$H_2 := \left\{x \in \mathbb{R}^2 : (z_0)_1 - \eta r < (x - x_0)_1 < (y_0)_1 + \eta r,\ -\tfrac{1}{3} r < (x - x_0)_2 < -\tfrac{1}{3} r + \eta r\right\}.$$

Note that the four rectangles V_1, V_2, H_1, and H_2 are all contained in the ball $B(x_0, r)$. Finally, we define the "rectangular annulus" A by

$$A := V_1 \cup V_2 \cup H_1 \cup H_2,$$

which satisfies $\overline{B}(x_0, r/5) \subset A \subset B(x_0, r)$, because

$$\tfrac{1}{3} r - \eta r \geq \tfrac{1}{3} r - \frac{1}{100} r = \tfrac{97}{300} r > \tfrac{r}{5}$$

(see Figure 3).

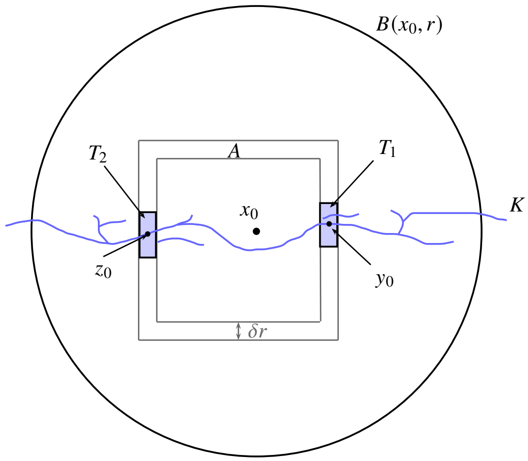

Fig. 3. The rectangular annulus A.

Next, we consider the two closed boxes

$$T_1 := \{x \in \mathbb{R}^2 : |(x - y_0)_1| \le \eta r \text{ and } |(x - y_0)_2| \le 30\eta r\} \subset V_1 \subset A,$$
$$T_2 := \{x \in \mathbb{R}^2 : |(x - z_0)_1| \le \eta r \text{ and } |(x - z_0)_2| \le 30\eta r\} \subset V_2 \subset A,$$

the main point being that $K \cap A \subset T_1 \cup T_2$.

Let us finally consider the subset of A outside the cutting boxes,

$$A' := A \setminus (T_1 \cup T_2),$$

and let $A^\pm$ be the two connected components of A'. The open sets $A^\pm$ are Lipschitz domains, and they are actually unions of vertical and horizontal rectangles of thickness of order η and lengths of order r (notice that $30\eta \le 10^{-2}$). In addition, since we have $K \cap A^\pm = \emptyset$ by construction, it follows that $u \in H^1(A^\pm; \mathbb{R}^2)$ and that the Korn inequality (see Lemma 6.2) applies in each rectangle composing $A^\pm$. Therefore, there exist skew-symmetric matrices $R^\pm$ such that

$$\int_{A^\pm} |\nabla u - R^\pm|^2 \, dx \le \frac{C}{\eta^5} \int_{A^\pm} |e(u)|^2 \, dx \le \frac{C}{\eta^5} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx \tag{4.16}$$

for some universal constant $C > 0$, where η^{-5} appears when estimating the distance between the skew-symmetric matrices in the intersection of two overlapping vertical and horizontal rectangles.

Step 2: Construction of the rectangle U. Let $R := R^+ \mathbf{1}_{A^+} + R^- \mathbf{1}_{A^-}$. Then

$$\int_{A'} |\nabla u - R|^2 \, dx \le \frac{C}{\eta^5} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx.$$

For any $t \in [-\eta r, \eta r]$ we denote the vertical line passing through $y(t) := y_0 + te_1$ by $L_t := y(t) + \mathbb{R}e_2$. According to Fubini's Theorem, we have

$$\int_{-\eta r}^{\eta r} \int_{L_t \cap A'} |\nabla u - R|^2 \, d\mathcal{H}^1 \, dt \leq \frac{C}{\eta^5} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx.$$

We can thus find $t_0 \in [-\eta r, \eta r]$ such that $u \in H^1(L_{t_0} \cap A'; \mathbb{R}^2)$ and

$$2\eta r \int_{L_{t_0} \cap A'} |\nabla u - R|^2 \, d\mathcal{H}^1 \leq \frac{C}{\eta^5} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx.$$

We perform the same argument at z_0, finding some $t_1 \in [-\eta r, \eta r]$ such that, denoting by L_{t_1} the line $z_0 + t_1 e_1 + \mathbb{R}e_2$, one has $u \in H^1(L_{t_1} \cap A'; \mathbb{R}^2)$ and

$$\int_{L_{t_1} \cap A'} |\nabla u - R|^2 \, d\mathcal{H}^1 \leq \frac{C}{r\eta^6} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx.$$

Arguing similarly for the top horizontal part of A^+, we get a horizontal line L_{H^+} such that $u \in H^1(L_{H^+} \cap A^+; \mathbb{R}^2)$ and

$$\int_{L_{H^+} \cap A^+} |\nabla u - R^+|^2 \, d\mathcal{H}^1 \leq \frac{C}{r\eta^6} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx.$$

The vertical line L_{t_0} intersects L_{H^+} in a single point a_0^+, and L_{t_1} intersects L_{H^+} in another single point a_1^+.

We perform a similar construction on the lower part A^- of A', which leads to another horizontal line L_{H^-} such that $u \in H^1(L_{H^-} \cap A^-; \mathbb{R}^2)$ and

$$\int_{L_{H^-} \cap A^-} |\nabla u - R^-|^2 \, d\mathcal{H}^1 \leq \frac{C}{r\eta^6} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx.$$

The vertical line L_{t_0} intersects L_{H^-} in a single point a_0^-, and L_{t_1} intersects L_{H^-} in another single point a_1^-.

Finally, we define U as the rectangle with vertices $(a_0^-, a_1^-, a_0^+, a_1^+)$ (see Figure 4) and we define

$$\Sigma := (T_1 \cup T_2) \cap \partial U,$$

so that $K \cap \partial U \subset \Sigma$, $\mathcal{H}^1(\Sigma) = 120\eta r$, and

$$\begin{aligned}\int_{\partial U \setminus \Sigma} |\nabla u - R|^2 \, d\mathcal{H}^1 \, dt &= \int_{\partial U \cap A'} |\nabla u - R|^2 \, d\mathcal{H}^1 \, dt \\ &\leq \frac{C}{r\eta^6} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx. \end{aligned} \tag{4.17}$$

Step 3: Construction of the competitor v. Since U is a rectangle with "uniform shape", there exists a bilipschitz mapping $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(U) = B := B(0,1)$, $\Phi(\partial U) = \partial B$ and $\Phi(\partial U^+) = \mathscr{C}_\delta$ for some $\delta < 1/2$, where $\partial U^+ := \partial U \cap A^+$ and $\mathscr{C}_\delta$ is as in the statement of Lemma 4.1. Note that the Lipschitz constants of Φ and Φ^{-1} are bounded by

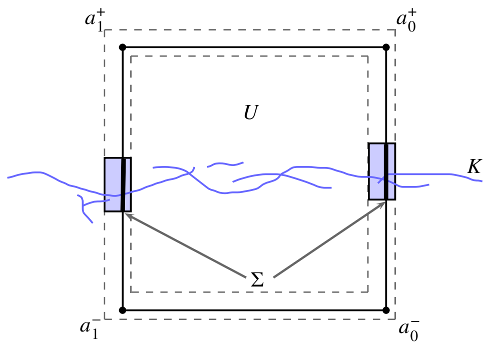

Fig. 4. The rectangular domain U and the wall set Σ.

Cr^{-1} and Cr where C is universal. Let R^+ be the skew-symmetric matrix appearing in (4.16). Since $u \in H^1(\partial U^+;\mathbb{R}^2)$, we infer that the function $x \mapsto u \circ \Phi^{-1}(x) - R^+\Phi^{-1}(x)$ belongs to $H^1(\mathscr{C}_\delta;\mathbb{R}^2)$. Applying Lemma 4.1, we obtain $h^+ \in H^1(B;\mathbb{R}^2)$ such that $h = u \circ \Phi^{-1} - R^+\Phi^{-1}$ on $\mathscr{C}_\delta$ and

$$\int_B |\nabla h^+|^2\,dx \le C \int_{\mathscr{C}_\delta} |\partial_\tau (u \circ \Phi^{-1} - R^+\Phi^{-1})|^2\,d\mathcal{H}^1,$$

where $C > 0$ is a universal constant. Then, defining $v^+ := h^+ \circ \Phi \in H^1(U;\mathbb{R}^2)$ and noticing that if τ is a tangent vector to ∂B, then $\nabla\Phi^{-1}\tau$ is a tangent vector to ∂U^+ $\mathcal{H}^1$-a.e. in ∂U^+, we infer that $v^+(x) = u(x) - R^+x$ for $\mathcal{H}^1$-a.e. $x \in \partial U^+$ and

$$\int_U |\nabla v^+|^2\,dx \le Cr \int_{\partial U \cap A^+} |\partial_\tau u - R^+\tau|^2\,d\mathcal{H}^1.$$

Arguing similarly for $\partial U^- := \partial U \cap A^-$ leads to a function $v^- \in H^1(U;\mathbb{R}^2)$ such that $v^-(x) = u(x) - R^-x$ for $\mathcal{H}^1$-a.e. $x \in \partial U^-$ and

$$\int_U |\nabla v^-|^2\,dx \le Cr \int_{\partial U \cap A^-} |\partial_\tau u - R^-\tau|^2\,d\mathcal{H}^1,$$

where R^- is the skew-symmetric matrix appearing in (4.16).

Let $K' \subset \Omega$ be as in the statement. We construct $v \in H^1(\Omega \setminus K';\mathbb{R}^2)$ by setting

$$v(x) := v^\pm(x) + R^\pm x$$

if x belongs to the connected component of $U \setminus K'$ containing $D^\pm(x_0, r/5)$, and $v := u$ otherwise.

Note that by construction $v = u$ on $\partial U \setminus \Sigma$, and

$$\int_{U\setminus K'} |e(v)|^2\,dx \le Cr \int_{\partial U\setminus\Sigma} |\nabla u - R^\pm|^2\,d\mathcal{H}^1.$$

Notice that $K \cap \partial U = K' \cap \partial U$ by assumption, because $K \setminus U = K' \setminus U$ and U is open. Thus, from (4.17) it follows that

$$\int_{U \setminus K'} |e(v)|^2 \, dx \leq \frac{C}{\eta^6} \int_{B(x_0,r)\setminus K} |e(u)|^2 \, dx,$$

as required. ∎

4.3. Proof of Proposition 3.2

In Lemma 4.5 we have constructed the key displacement competitor associated to a separating crack competitor, which will be employed to show the flatness estimate. The construction of the crack competitor will be similar to that of Proposition 4.1, i.e. it will be obtained by replacing K by a segment in some ball. The difference here will be in the error appearing in the density estimate, which will depend only on $\omega(x_0, r)$, and not on $\beta(x_0, r)$ anymore.

Proposition 4.3. *There exist $\varepsilon_0' > 0$ and $C' > 0$ such that the following holds. Let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional, and let $x_0 \in K$ and $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$,*

$$\omega_u(x_0, r) + \beta_K(x_0, r) \leq \varepsilon_0',$$

and K separates $D^\pm(x_0, r)$ in $\overline{B}(x_0, r)$. Then there exists $s \in (r/40, r/5)$ such that $K \cap \partial B(x_0, s) = \{z, z'\}$ for some $z \neq z'$, and

$$\mathcal{H}^1(K \cap B(x_0, s)) \leq |z - z'| + C' r \omega_u(x_0, r)^{1/7}. \tag{4.18}$$

Proof. We define

$$\varepsilon_0' := \min\left(10^{-28}, \frac{\varepsilon_0}{5}, \frac{10^{-2}\theta_0}{5C_*}, \frac{1}{5C_*}\right),$$

where $\varepsilon_0 > 0$ is the universal constant of Lemma 4.3, $\theta_0 > 0$ is the Ahlfors regularity constant, and $C_* > 0$ is the universal constant of Proposition 4.2. We notice that $\omega(x_0, r/5) + \beta(x_0, r/5) \leq \varepsilon_0$ and that K still separates $D^\pm(x_0, r/5)$ in $\overline{B}(x_0, r/5)$, since they are contained in two different connected components of $\overline{B}(x_0, r/5) \setminus \{y \in \mathbb{R}^2 : |(y - x_0)_2| > \beta(x_0, r)r\}$. Thus, according to Lemma 4.3 applied in $B(x_0, r/5)$, we can indeed find $s \in (r/40, r/5)$ such that $\#(K \cap \partial B(x_0, s)) = 2$, and we denote by z and z' the two points of $K \cap \partial B(x_0, s)$.

Fix now $\eta \in (0, 10^{-4})$. Let U be the rectangle satisfying $\overline{B}(x_0, r/5) \subset U \subset B(x_0, r)$, and Σ be the wall set satisfying $K \cap \partial U \subset \Sigma \subset \partial U$ and $\mathcal{H}^1(\Sigma) \leq 120\eta r$, given by Lemma 4.5 in $B(x_0, r)$ for $\eta \in (0, 10^{-4})$ fixed above.

Consider the set

$$K' := [z, z'] \cup (K \setminus B(x_0, s)).$$

By construction $K' \setminus U = K \setminus U$ and $D^\pm(x_0, r/5)$ are contained in different connected components of $U \setminus K'$. Then Lemma 4.5 provides $v \in H^1(\Omega \setminus K'; \mathbb{R}^2)$ which coincides with u on $(\Omega \setminus U) \setminus \Sigma$ and satisfies (4.15). The pair (v, K'') with

$$K'' := K' \cup \Sigma$$

is thus a competitor for (u, K), and by minimality of (u, K) we have

$$\int_{B(x_0,r)\setminus K} \mathbf{A}e(u) : e(u)\,dx + \mathcal{H}^1(K \cap \overline{B}(x_0, r)) \leq \int_{B(x_0,r)\setminus K''} \mathbf{A}e(v) : e(v)\,dx + \mathcal{H}^1(K'' \cap \overline{B}(x_0, r)).$$

Since $u = v$ outside of U and $K'' \cap \partial B(x_0, r) = K \cap \partial B(x_0, r)$, we deduce by (4.15) that

$$\begin{aligned}\mathcal{H}^1(K\cap B(x_0,r)) &\leq \mathcal{H}^1(K''\cap B(x_0,r)) + \frac{C}{\eta^6} r\omega(x_0,r)\\ &\leq \mathcal{H}^1(K\cap B(x_0,r)\setminus B(x_0,s)) + |z-z'| + \mathcal{H}^1(\Sigma) + \frac{C}{\eta^6} r\omega(x_0,r).\end{aligned}$$

Since $\mathcal{H}^1(\Sigma) \leq 120\eta r$ we get

$$\mathcal{H}^1(K \cap B(x_0, s)) \leq |z - z'| + 120\eta r + \frac{C}{\eta^6} r\omega(x_0, r).$$

Finally, since $\eta > 0$ was assumed to be arbitrary in $(0, 10^{-4})$, and $\omega(x_0, r) \leq 10^{-28}$ by assumption, we can choose $\eta = \omega(x_0, r)^{1/7} \leq 10^{-4}$ so that

$$\mathcal{H}^1(K \cap B(x_0, s)) \leq |z - z'| + Cr\omega(x_0, r)^{1/7},$$

as required. ∎

We are now ready to prove the main flatness estimate.

Proof of Proposition 3.2. Let us define

$$\varepsilon_1 = \min\left(\varepsilon_0', \left(\frac{\min(1, \theta_0)}{400C'}\right)^7\right), \tag{4.19}$$

where $\varepsilon_0' > 0$ is the threshold of Proposition 4.3 and $C' > 0$ is the universal constant in (4.18). By Proposition 4.3, there exists $s \in (r/40, r/5)$ such that $K \cap \partial B(x_0, s) = \{z, z'\}$ for some $z \neq z'$, and

$$\mathcal{H}^1(K \cap B(x_0, s)) \leq |z - z'| + C'r\omega(x_0, r)^{1/7}. \tag{4.20}$$

Notice that

$$\max(|(z - x_0)_2|, |(z' - x_0)_2|) \leq 40\varepsilon_0' r,$$

since $\beta(x_0, s) \leq 40\varepsilon_0'$, and

$$\mathcal{H}^1(K \cap B(x_0, s)) \leq 2r + C'r\omega(x_0, r)^{1/7} \leq 3r, \tag{4.21}$$

because $C'\omega(x_0, r)^{1/7} \leq 1$ by (4.19). Let L be the line passing through x_0 which is parallel to the segment $[z, z']$.

Step 1. We first prove that

$$\sup_{y\in K\cap\overline{B}(x_0,r/50)} \operatorname{dist}(y,L)\le C''r\omega(x_0,r)^{1/14}, \tag{4.22}$$

where $C''>0$ only depends on $\theta_0>0$.

Since $K\cap B(x_0,s)$ separates $D^{\pm}(x_0,s)$ in $\overline{B}(x_0,s)$, by Lemma 3.2 there exists an injective Lipschitz curve $\Gamma\subset K\cap\overline{B}(x_0,s)$ joining z and z'. As $\mathcal{H}^1(\Gamma)\ge|z-z'|$, according to estimate (4.20) we have

$$\mathcal{H}^1(K\cap B(x_0,s)\setminus\Gamma)\le\mathcal{H}^1(K\cap B(x_0,s))-\mathcal{H}^1(\Gamma)\le C'r\omega(x_0,r)^{1/7}. \tag{4.23}$$

We claim that for all $y\in K\cap\overline{B}(x_0,r/50)$,

$$\operatorname{dist}(y,\Gamma)\le\frac{2C'}{\theta_0}r\omega(x_0,r)^{1/7}. \tag{4.24}$$

Indeed, assume for contradiction that there exists $y_0\in K\cap\overline{B}(x_0,r/50)$ such that $\operatorname{dist}(y_0,\Gamma)>\delta r$ with $\delta=\frac{2C'}{\theta_0}r\omega(x_0,r)^{1/7}$. According to condition (4.19) we have $\delta<\frac{1}{200}$, so

$$B(y_0,\delta r)\subset B(x_0,r/40)\setminus\Gamma\subset B(x_0,s)\setminus\Gamma.$$

By Ahlfors regularity of K,

$$\mathcal{H}^1(K\cap B(x_0,s)\setminus\Gamma)\ge\mathcal{H}^1(K\cap B(y_0,\delta r))\ge\theta_0\delta r=2C'\omega(x_0,r)^{1/7}r,$$

which contradicts (4.23) and establishes the validity of (4.24).

Now, an application of Lemma 6.3 ensures that for all $w\in\Gamma$,

$$\begin{aligned}\operatorname{dist}(w,[z,z'])^2&\le\mathcal{H}^1(\Gamma)(\mathcal{H}^1(\Gamma)-|z'-z|)\\&\le\mathcal{H}^1(K\cap B(x_0,s))\big(\mathcal{H}^1(K\cap B(x_0,s))-|z'-z|\big)\\&\le 3C'r^2\omega(x_0,r)^{1/7},\end{aligned} \tag{4.25}$$

by (4.21) and (4.20).

According to (4.24), (4.25), and the triangle inequality, we infer that for all y in $K\cap\overline{B}(x_0,r/50)$,

$$\operatorname{dist}(y,[z,z'])\le\sqrt{3C'}\,r\omega(x_0,r)^{1/14}+\frac{2C'}{\theta_0}r\omega(x_0,r)^{1/7}\le\tilde{C}'r\omega(x_0,r)^{1/14} \tag{4.26}$$

with $\tilde{C}'>0$ depending on θ_0, where we have used $\omega(x_0,r)\le 1$ to estimate $\omega(x_0,r)^{1/7}\le\omega(x_0,r)^{1/14}$. Finally, if L denotes the line passing through x_0 which is parallel to the segment $[z,z']$, we deduce that (4.22) holds by the triangle inequality and (4.26) applied to $x_0\in L\cap K\cap\overline{B}(x_0,r/50)$.

Step 2. We now prove that

$$\sup_{x\in L\cap\overline{B}(x_0,r/50)}\operatorname{dist}(x,K)\le C'''r\omega(x_0,r)^{1/14}, \tag{4.27}$$

where $C''' > 0$ possibly depends on θ_0. For this purpose, we recall that K separates $D^\pm(x_0, r)$ in $\overline{B}(x_0, r)$, thus in particular, for every $x \in L \cap \overline{B}(x_0, r/50)$, the line orthogonal to L passing through x meets K at some point y. If $y \in \overline{B}(x_0, r/50)$, then by Step 1 we know that $|x - y| \leq C''\omega(x_0, r)^{1/14} r$, and then

$$\operatorname{dist}(x, K) \leq C'' r\omega(x_0, r)^{1/14}.$$

Now, if $y \notin \overline{B}(x_0, r/50)$, this is only possible for x very close to $\partial B(x_0, r/50)$, because $K \cap B(x_0, r)$ is contained in a strip around L of height $C'' r\omega(x_0, r)^{1/14}$, which is small. More precisely, one sees using Pythagoras' Theorem that the second case occurs only for points $x \in L$ satisfying

$$\operatorname{dist}(x, \partial B(x_0, r/50)) \leq \frac{r}{50} - \left(\left(\frac{r}{50}\right)^2 - \left(C'' r\omega(x_0, r)^{1/14}\right)^2\right)^{1/2} \leq rMC''\omega(x_0, r)^{1/14},$$

where $M > 0$ is a universal constant obtained from the elementary inequality

$$\frac{1}{50} - \left(\left(\frac{1}{50}\right)^2 - t^2\right)^{1/2} \leq Mt \quad \text{for all } 0 < t < 10^{-3},$$

which results from the mean value theorem. By the triangle inequality we then obtain

$$\operatorname{dist}(x, K) \leq (M + 1)C'' r\omega(x_0, r)^{1/14}.$$

Gathering (4.22) and (4.27), and using inequuality (3.4), we deduce that $\beta(x_0, r/50) \leq C_1 r\omega(x_0, r)^{1/14}$ for some constant $C_1 > 0$ depending on θ_0, which concludes the proof of the proposition. ■

5. Proof of the normalized energy decay

In this section we prove a decay estimate for the normalized energy of a Griffith minimizer. The strategy is based upon a compactness argument and a Γ-convergence type analysis where one shows the stability of the Neumann problem in planar elasticity along a sequence of sets K_n which converge in the Hausdorff sense to a diameter within a ball. It gives an alternative approach even for the scalar case (albeit only two-dimensional and under topological conditions) to the corresponding decay estimates of the normalized energy in the standard proofs of regularity for the Mumford–Shah minimizers ([4], [31, Theorem 1.10]).

We start by establishing some auxiliary results on the Airy function.

5.1. The Airy function

We state here a general result concerning the existence of the Airy function associated to a minimizer of the Griffith energy. The construction is similar to that in [6, Proposition 4.3], itself inspired by [9]. The Airy function will be used to get compactness results along a sequence of minimizers. The main difference with [6] is that now K is not assumed to be connected. The proofs are very similar to those of [6], and for that reason we do not write all the arguments but only point out the main changes with respect to the original proof.

First we recall the following result coming from De Rham's Theorem and proved in [6, Lemma 4.1] in the case where Ω is a ball. The extension to a general bounded open set with Lipschitz boundary is straightforward, since the only property used in that proof is the existence of traces of Sobolev functions on the boundary.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and let $L \subset \overline{\Omega}$ be a closed set. Consider the following subspaces of $L^2(\Omega;\mathbb{R}^2)$:*

$$X_L := \{\Psi \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^2) : \operatorname{supp}(\Psi) \cap L = \emptyset,\ \operatorname{div}\Psi = 0 \text{ in } \Omega\},$$
$$Y_L := \{\nabla v : v \in H^1(\Omega \setminus L),\ v = 0 \text{ on } \partial\Omega \setminus L\}.$$

Then $\overline{X_L} = Y_L^\perp$ in $L^2(\Omega;\mathbb{R}^2)$.

From the previous lemma, one can construct the "harmonic conjugate" v associated to a minimizer (u, K) of the Griffith functional. The proof follows the lines of that of [6, Proposition 4.2]. The main difference is that here the singular set K is not assumed to be connected. This implies that it is not in general possible to conclude that v vanishes on the full crack K. However, the proof makes it possible to ensure that, in a suitable weak sense, v is constant in each connected component of K, but the constants might depend on the component. That is why we renormalize the harmonic conjugate v to vanish only on an arbitrary connected component of the crack of positive length.

Proposition 5.1 (Harmonic conjugate). *Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected open set with Lipschitz boundary, and let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional. Then, for every connected component L of K with $\mathcal{H}^1(L) > 0$, there exists a function $v \in H^1_{0,L}(\Omega;\mathbb{R}^2) \cap \mathcal{C}^\infty(\Omega \setminus K;\mathbb{R}^2)$ such that*

$$\sigma := \mathbf{A}e(u) = \begin{pmatrix} -\partial_2 v_1 & \partial_1 v_1 \\ -\partial_2 v_2 & \partial_1 v_2 \end{pmatrix} \quad \text{a.e. in } \Omega. \tag{5.1}$$

Proof. Let L be a connected component of K with $\mathcal{H}^1(L) > 0$. According to the variational formulation (2.5) and the fact that $\sigma(x) \in \mathbb{M}^{2\times 2}_{\text{sym}}$ for a.e. $x \in \Omega \setminus K$, for any $v \in H^1(\Omega \setminus K;\mathbb{R}^2)$ with $v = 0$ on $\partial\Omega \setminus K$ we have

$$\int_\Omega \sigma : \nabla v \, dx = 0.$$

This is *a fortiori* true for any $v \in H^1(\Omega \setminus L;\mathbb{R}^2)$ with $v = 0$ on $\partial\Omega \setminus L$. Consequently, both lines of σ, denoted by

$$\sigma^{(1)} := \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \end{pmatrix}, \qquad \sigma^{(2)} := \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix},$$

lie in $Y_L^\perp$. Lemma 5.1 ensures the existence of a sequence $(\sigma_n^{(1)}) \subset X_L$ such that $\sigma_n^{(1)} \to \sigma^{(1)}$ in $L^2(\Omega;\mathbb{R}^2)$. Since $\operatorname{div}\sigma_n^{(1)} = 0$ in Ω, and Ω is simply connected and

$\mathrm{supp}(\sigma_n^{(1)}) \cap L = \emptyset$, it follows that

$$\sigma_n^{(1)} = \nabla^\perp p_n^{(2)} := \begin{pmatrix} -\partial_2 p_n^{(2)} \\ \partial_1 p_n^{(2)} \end{pmatrix}$$

for some $p_n^{(2)} \in \mathcal{C}^\infty(\overline{\Omega})$ with $\mathrm{supp}(\nabla p_n^{(2)}) \cap L = \emptyset$. Since L is connected, we can assume that, up to an additive constant, $p_n^{(2)} = 0$ on L. Consequently, since $\mathcal{H}^1(L) > 0$, Poincaré's inequality implies that $p_n^{(2)} \to p^{(2)}$ in $H^1(\Omega)$ for some $p^{(2)} \in H^1_{0,L}(\Omega)$ satisfying $\sigma^{(1)} = \nabla^\perp p^{(2)}$. We prove similarly the existence of $p^{(1)} \in H^1_{0,L}(\Omega)$ satisfying $\sigma^{(2)} = -\nabla^\perp p^{(1)}$. We then define

$$v := \begin{pmatrix} p^{(2)} \\ -p^{(1)} \end{pmatrix} \in H^1_{0,L}(\Omega; \mathbb{R}^2),$$

which satisfies (5.1). Finally, since $\sigma \in \mathcal{C}^\infty(\Omega \setminus K; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$, we have $v \in \mathcal{C}^\infty(\Omega \setminus K; \mathbb{R}^2)$. ■

We next construct the Airy function w associated to the displacement u in Ω following an approach similar to [6,9], but once more with the difference that K is no more assumed to be connected.

Proposition 5.2 (Airy function). *Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected open set with Lipschitz boundary, and let $(u, K) \in \mathcal{A}(\Omega)$ be a minimizer of the Griffith functional. If L is a connected component of K such that $\mathcal{H}^1(L) > 0$, then there exists a function $w \in H^2(\Omega) \cap H^1_{0,L}(\Omega)$ such that*

$$\Delta^2 w = 0 \quad \text{in } \mathcal{D}'(\Omega \setminus K) \tag{5.2}$$

and

$$\sigma = \begin{pmatrix} \partial_{22} w & -\partial_{12} w \\ -\partial_{12} w & \partial_{11} w \end{pmatrix} \quad \text{a.e. in } \Omega. \tag{5.3}$$

In addition, if $A \subset \mathbb{R}^2$ is an open set with $\overline{A} \subset \Omega$, then $w \in H^2_{0,L}(A)$.

Proof. Proposition 5.1 ensures the existence of $p^{(1)}, p^{(2)} \in H^1_{0,L}(\Omega)$ such that

$$\sigma^{(1)} = \nabla^\perp p^{(2)}, \quad \sigma^{(2)} = -\nabla^\perp p^{(1)} \quad \text{a.e. in } \Omega.$$

Since $p^{(1)} = p^{(2)} = 0$ on L, arguing as in [6, Proposition 4.3] it follows that

$$\begin{pmatrix} -p^{(2)} \\ p^{(1)} \end{pmatrix} \in Y_L^\perp = \overline{X}_L,$$

owing again to Lemma 5.1. Next, arguing as in the proof of Proposition 5.1, we deduce the existence of a function $w \in H^1_{0,L}(\Omega)$ such that

$$\nabla w = \begin{pmatrix} p^{(1)} \\ p^{(2)} \end{pmatrix} \in L^2(\Omega; \mathbb{R}^2).$$

By construction, the Airy function w satisfies (5.3), and arguing as in [6, Proposition 4.3], we see that it also satisfies (5.2).

It remains to show that if $A \subset \mathbb{R}^2$ is an open set with $\overline{A} \subset \Omega$, then $w \in H^2_{0,L}(A)$. We first note that $w \in H^1_{0,L}(\Omega) \cap H^2(\Omega)$ with $\nabla w \in H^1_{0,L}(\Omega; \mathbb{R}^2)$. In particular, since $w \in H^2(\Omega)$, it has a (Hölder) continuous representative, still denoted w, so that it makes sense to consider its pointwise values.

Since $A \setminus L$ is not smooth, in order to show that $w \in H^2_{0,L}(A)$, we will use a capacity argument similar to that used in [6, Proposition 4.3] and in [9, Theorem 1].

Let us consider a cut-off function $\eta \in \mathscr{C}^\infty_c(\Omega; [0,1])$ satisfying $\eta = 0$ on $\partial\Omega$ and $\eta = 1$ on A. Denote $z := \eta w$. Then $z \in H^2(\Omega) \cap H^1_0(\Omega \setminus L)$ and $\nabla z \in H^1_0(\Omega \setminus L; \mathbb{R}^2)$.

By [1, Theorem 9.1.3], $z \in H^2_0(\Omega \setminus L)$ if a $\mathrm{Cap}_{2,2}$-quasicontinuous representative of z vanishes on $\partial(\Omega \setminus L)$ $\mathrm{Cap}_{2,2}$-q.e., and a $\mathrm{Cap}_{1,2}$-quasicontinuous representative of ∇z vanishes on $\partial(\Omega \setminus L)$ $\mathrm{Cap}_{1,2}$-q.e.

Since $\nabla z \in H^1_0(\Omega \setminus L)$, the second property is a consequence of [28, Theorem 3.3.42]. As for the first, since z is continuous, it coincides with its $\mathrm{Cap}_{2,2}$-quasicontinuous representative. Moreover, since the empty set is the only set of zero $\mathrm{Cap}_{2,2}$-capacity, we are reduced to showing that $z = 0$ everywhere on $\partial(\Omega \setminus L)$. Let $F := \{x \in \partial(\Omega \setminus L) : z(x) = 0\}$. Then F is a compact set satisfying $\mathrm{Cap}_{1,2}(\partial(\Omega \setminus L) \setminus F) = 0$, because $z \in H^1_0(\Omega \setminus L)$. Let G be a connected component of $\partial(\Omega \setminus L) \setminus F$. Since a compact and connected set of positive diameter has a positive $\mathrm{Cap}_{1,2}$-capacity (see [28, Corollary 3.3.25]), we deduce that $\mathrm{diam}(G) = 0$, so that G is (at most) a singleton. Moreover, F being compact, its complement $\partial(\Omega \setminus L) \setminus F$ is open in the relative topology of $\partial(\Omega \setminus L)$, and thus G is (at most) an isolated point. Finally, since $\partial(\Omega \setminus L) = \partial\Omega \cup L$, because $L \subset \overline{\Omega}$ closed and $\mathscr{H}^1(L) < \infty$, and since neither $\partial\Omega$ or L has isolated points, we have $G = \emptyset$, and thus $z = 0$ on $\partial(\Omega \setminus L)$.

As a consequence of [1, Theorem 9.1.3], we conclude that $z \in H^2_0(\Omega \setminus L)$, or in other words, there exists a sequence $(z_n) \subset \mathscr{C}^\infty_c(\Omega \setminus L)$ such that $z_n \to z = \eta w$ in $H^2(\Omega \setminus L)$. Note in particular that $z_n \in \mathscr{C}^\infty(\overline{A})$ and z_n vanishes in a neighborhood of $L \cap \overline{A}$. Therefore, since $z = w$ and $\nabla z = \nabla w$ in A, we deduce that $w \in H^2_{0,L}(A)$. ∎

Remark 5.1. If $\Gamma \subset K \cap \Omega$ is a connected component of $K \cap \Omega$, then there exists a connected component L of K such that $\Gamma \subset L$. If we consider the Airy function given by Proposition 5.2 associated with this component L, then for all $A \subset \mathbb{R}^2$ open with $\overline{A} \subset \Omega$ we find that $w \in H^2_{0,\Gamma}(A)$ because $H^2_{0,L}(A) \subset H^2_{0,\Gamma}(A)$.

5.2. *Proof of Proposition* 3.3

Assume that the statement of the proposition is false. Then there exists $\tau_0 > 0$ such that for every $n \in \mathbb{N}$, one can find a minimizer $(\hat{u}_n, \hat{K}_n) \in \mathcal{A}(\Omega)$ of the Griffith functional (with the same Dirichlet boundary data ψ), an isolated connected component $\hat{\Gamma}_n$ of $\hat{K}_n \cap \Omega$, points $x_n \in \hat{\Gamma}_n$, and radii $r_n > 0$ with $\overline{B}(x_n, r_n) \subset \Omega$ such that

$$\hat{K}_n \cap \overline{B}(x_n, r_n) = \hat{\Gamma}_n \cap \overline{B}(x_n, r_n), \quad \beta_{\hat{K}_n}(x_n, r_n) \to 0,$$

and

$$\omega_{\hat{u}_n}(x_n, ar_n) > \tau_0\, \omega_{\hat{u}_n}(x_n, r_n)$$

for some $a \in (0, 1)$ (to be fixed later).

Rescaling and compactness. In order to prove compactness properties of the sequences of sets and displacements, we need to rescale them into a unit configuration. For simplicity, from now on, we write $B := B(0, 1)$. Let us first rescale the sets $\hat{K}_n$ and $\hat{\Gamma}_n$ by setting, for all $n \in \mathbb{N}$,

$$K_n := \frac{\hat{K}_n - x_n}{r_n}, \qquad \Gamma_n := \frac{\hat{\Gamma}_n - x_n}{r_n}.$$

Let $\hat{L}_n := L(x_n, r_n)$ be an affine line such that

$$d_{\mathcal{H}}(\hat{L}_n \cap \overline{B}(x_n, r_n), \hat{K}_n \cap \overline{B}(x_n, r_n)) \le 2r_n \beta_{\hat{K}_n}(x_n, r_n),$$

and $L_n := (\hat{L}_n - x_n)/r_n$ its rescaling. Up to a subsequence, and up to a change of orthonormal basis, we can assume that $L_n \cap \overline{B} \to T \cap \overline{B}$ in the Hausdorff distance, where $T := \mathbb{R}e_1$. Then, since

$$d_{\mathcal{H}}(L_n \cap \overline{B}, K_n \cap \overline{B}) = \frac{1}{r_n} d_{\mathcal{H}}(\hat{L}_n \cap \overline{B}(x_n, r_n), \hat{K}_n \cap \overline{B}(x_n, r_n)) \le 2\beta_{\hat{K}_n}(x_n, r_n) \to 0,$$

we deduce that $\Gamma_n \cap \overline{B} = K_n \cap \overline{B} \to T \cap \overline{B}$ in the Hausdorff distance.

We next rescale the displacements $\hat{u}_n$ by setting, for all $n \in \mathbb{N}$ and a.e. $y \in B$,

$$u_n(y) := \frac{\hat{u}_n(x_n + r_n y)}{\sqrt{\omega_{\hat{u}_n}(x_n, r_n) r_n}}.$$

Then

$$\int_{B \setminus K_n} \mathbf{A}e(u_n) : e(u_n)\, dx = 1, \tag{5.4}$$

$$\omega_{u_n}(0, a) > \tau_0. \tag{5.5}$$

Note that $u_n \in LD(B \setminus K_n)$ is a solution of

$$\inf \left\{ \int_{B \setminus K_n} \mathbf{A}e(z) : e(z)\, dx : z \in LD(B \setminus K_n),\ z = u_n \text{ on } \partial B \setminus K_n \right\},$$

and in particular

$$\int_{B \setminus K_n} \mathbf{A}e(u_n) : e(u_n)\, dx \le \int_{B \setminus K_n} \mathbf{A}e(u_n + \varphi) : e(u_n + \varphi)\, dx \tag{5.6}$$

for all $\varphi \in LD(B \setminus K_n)$ with $\varphi = 0$ on $\partial B \setminus K_n$.

According to the energy bound (5.4), up to a subsequence we have

$$e(u_n)\mathbf{1}_{B \setminus K_n} \rightharpoonup e \quad \text{weakly in } L^2(B; \mathbb{M}^{2\times 2}_{\text{sym}})$$

for some $e \in L^2(B; \mathbb{M}^{2\times 2}_{\rm sym})$. We next show that e is the symmetrized gradient of some displacement. To this end, we consider, for any $0 < \delta < 1/10$, the Lipschitz domain

$$A_\delta := \{x \in B : \mathrm{dist}(x, T) > \delta\} = A_\delta^+ \cup A_\delta^-,$$

where $A_\delta^\pm$ are the two connected components of A_δ. Note that for such δ, $D^\pm := B\big((0, \pm\frac{3}{4}), \frac{1}{4}\big) \subset A_\delta^\pm$ and $K_n \cap U_\delta = \emptyset$ for n large enough (depending on δ). Denoting by

$$r_n^\pm(x) := \frac{1}{|D^\pm|}\int_{D^\pm} u_n(y)\,dy + \left(\frac{1}{|D^\pm|}\int_{D^\pm} \frac{\nabla u_n(y) - \nabla u_n(y)^T}{2}\,dy\right)\left(x - \frac{1}{|D^\pm|}\int_{D^\pm} y\,dy\right)$$

the rigid body motion associated to u_n in $D^\pm$, by virtue of the Poincaré–Korn inequality [2, Theorem 5.2 and Example 5.3] we get

$$\|u_n - r_n^\pm\|_{H^1(A_\delta^\pm;\mathbb{R}^2)} \le c_\delta \|e(u_n)\|_{L^2(A_\delta^\pm;\mathbb{M}^{2\times 2}_{\rm sym})}$$

for some constant $c_\delta > 0$ depending on δ. Thanks to a diagonalization argument, for a further subsequence (not relabeled), we obtain a function $v \in LD(B \setminus T)$ such that $u_n - r_n^\pm \rightharpoonup v$ weakly in $H^1(A_\delta^\pm; \mathbb{R}^2)$ for any $0 < \delta < 1/10$. Necessarily we must have $e = e(v)$, and thus

$$e(u_n)\mathbf{1}_{B\setminus K_n} \rightharpoonup e(v) \quad \text{weakly in } L^2(B; \mathbb{M}^{2\times 2}_{\rm sym}).$$

Minimality. We next show that v satisfies the minimality property

$$\int_{B\setminus T} \mathbf{A}e(v) : e(v)\,dx \le \int_{B\setminus T} \mathbf{A}e(v+\varphi) : e(v+\varphi)\,dx$$

for all $\varphi \in LD(B \setminus T)$ such that $\varphi = 0$ on $\partial B \setminus T$. According to [9, Theorem 1], it is enough to consider competitors $\varphi \in H^1(B \setminus T; \mathbb{R}^2)$ such that $\varphi = 0$ on $\partial B \setminus T$.

For an arbitrary given competitor $\varphi \in H^1(B \setminus T; \mathbb{R}^2)$ such that $\varphi = 0$ on $\partial B \setminus T$, we construct a sequence of competitors for the minimization problems (5.6) using a jump transfer type argument (see [23] and [6]). To this end, we denote by $C_n^\pm$ the connected component of $B \setminus K_n$ which contains the point $(0, \pm 1/2)$, and we define

$$\varphi_n(x_1, x_2) = \begin{cases} \varphi(x_1, x_2) & \text{if } (x_1, x_2) \in [C_n^+ \cap \{x_2 \ge 0\}] \cup [C_n^- \cap \{x_2 \le 0\}], \\ \varphi(x_1, -x_2) & \text{if } (x_1, x_2) \in [C_n^+ \cap \{x_2 < 0\}] \cup [C_n^- \cap \{x_2 > 0\}], \\ 0 & \text{otherwise.} \end{cases}$$

Then one can check that $\varphi_n \in H^1(B \setminus K_n; \mathbb{R}^2)$ and $\varphi_n = 0$ on $\partial B \setminus K_n$. Moreover, $\varphi_n \to \varphi$ strongly in $L^2(B; \mathbb{R}^2)$ and $e(\varphi_n)\mathbf{1}_{B\setminus K_n} \to e(\varphi)$ strongly in $L^2(B; \mathbb{M}^{2\times 2}_{\rm sym})$. Therefore, thanks to the minimality property satisfied by u_n, we infer that

$$\int_{B\setminus K_n} \mathbf{A}e(u_n) : e(u_n)\,dx \le \int_{B\setminus K_n} \mathbf{A}e(u_n+\varphi_n) : e(u_n+\varphi_n)\,dx,$$

which implies, by expanding the squares, that

$$0 \le 2\int_{B\setminus K_n} \mathbf{A}e(u_n) : e(\varphi_n)\, dx + \int_{B\setminus K_n} \mathbf{A}e(\varphi_n) : e(\varphi_n)\, dx.$$

Since $e(\varphi_n)\mathbf{1}_{B\setminus K_n} \to e(\varphi)$ strongly in $L^2(B;\mathbb{M}^{2\times 2}_{\rm sym})$ and $e(u_n)\mathbf{1}_{B\setminus K_n} \rightharpoonup e(v)$ weakly in $L^2(B;\mathbb{M}^{2\times 2}_{\rm sym})$, we can pass to the limit as $n\to\infty$ to get

$$0 \le 2\int_{B\setminus T} \mathbf{A}e(v) : e(\varphi)\, dx + \int_{B\setminus T} \mathbf{A}e(\varphi) : e(\varphi)\, dx,$$

or still

$$\int_{B\setminus T} \mathbf{A}e(v) : e(v)\, dx \le \int_{B\setminus T} \mathbf{A}e(v+\varphi) : e(v+\varphi)\, dx.$$

As a consequence, v is a smooth function in $B\setminus T$. Moreover, due to the Korn inequality in both connected components of $B\setminus T$ (which are Lipschitz domains), we conclude that $v\in H^1(B\setminus T;\mathbb{R}^2)$.

Convergence of elastic energy. In order to pass to the limit in inequality (5.5), we need to show the convergence of the elastic energy, or in other words, the strong convergence of the sequence $(e(u_n))_{n\in\mathbb{N}}$ of elastic strains. This will be achieved by using Proposition 5.2 which provides an Airy function $\hat{w}_n$ associated to the displacement $\hat{u}_n$, satisfying $\hat{w}_n \in H^2(\Omega)\cap H^1_{0,\hat{\Gamma}_n}(\Omega)\cap H^2_{0,\hat{\Gamma}_n}(A)$ for all open sets $A\subset\mathbb{R}^2$ with $\overline{A}\subset\Omega$ (see also Remark 5.1) and such that

$$\Delta^2\hat{w}_n = 0 \quad \text{in } \Omega\setminus\hat{K}_n$$

and

$$\mathbf{A}e(\hat{u}_n) = \begin{pmatrix} \partial_{22}\hat{w}_n & -\partial_{12}\hat{w}_n \\ -\partial_{12}\hat{w}_n & \partial_{11}\hat{w}_n \end{pmatrix} \quad \text{in } \Omega.$$

Since $\hat{K}_n\cap\overline{B}(x_n,r_n) = \hat{\Gamma}_n\cap\overline{B}(x_n,r_n)$ and $\overline{B}(x_n,r_n)\subset\Omega$, we infer that $\hat{w}_n\in H^2_{0,\hat{K}_n}(B(x_n,r_n))$. We rescale the Airy function $\hat{w}_n$ by setting, for all $n\in\mathbb{N}$ and a.e. $y\in B$,

$$w_n(y) := \frac{\hat{w}_n(x_n+r_n y)}{\sqrt{\omega_{\hat{u}_n}(x_n,r_n)r_n}}$$

in such a way that $w_n\in H^2_{0,K_n}(B)$, and

$$\Delta^2 w_n = 0 \quad \text{in } B\setminus K_n$$

and

$$\mathbf{A}e(u_n) = \begin{pmatrix} \partial_{22}w_n & -\partial_{12}w_n \\ -\partial_{12}w_n & \partial_{11}w_n \end{pmatrix}.$$

In addition, since

$$\int_B |D^2 w_n|^2\, dx = \int_B |\mathbf{A}e(u_n)|^2\, dx \le C\int_B |e(u_n)|^2\, dx \le C,$$

Poincaré's inequality ensures that the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in $H^2(B)$, and thus up to a subsequence $w_n \rightharpoonup w$ weakly in $H^2(B)$, for some $w \in H^2(B)$. A similar capacity argument to that in [6, proof of Proposition 6.1] shows that $w \in H^2_{0,T}(B(0,r))$ for all $r < 1$, and

$$\Delta^2 w = 0 \quad \text{in } B \setminus T$$

and

$$\mathbf{A}e(u) = \begin{pmatrix} \partial_{22}w & -\partial_{12}w \\ -\partial_{12}w & \partial_{11}w \end{pmatrix}. \tag{5.7}$$

In addition, since the biharmonicity of w_n is equivalent to the minimality

$$\int_B |D^2 w_n|^2\,dx \le \int_B |D^2 z|^2\,dx$$

for all $z \in w_n + H^2_{0,K_n}(B)$, we can again reproduce the proof of [6, Proposition 6.1] to find that $w_n \to w$ strongly in $H^2(B(0,r))$ for all $r < 1$. In particular, it implies that $e(u_n)\mathbf{1}_{B\setminus K_n} \to e(v)$ strongly in $L^2(B(0,r);\mathbb{M}^{2\times 2}_{\mathrm{sym}})$, and thus passing to the limit in inequalities (5.4) and (5.5) yields

$$\omega_v(0,1) \le 1 \quad \text{and} \quad \omega_v(0,a) \ge \tau_0. \tag{5.8}$$

According to inequality (5.8), we infer that either

$$\frac{1}{a}\int_{B(0,a)\cap\{x_2>0\}} \mathbf{A}e(v) : e(v)\,dx \ge \frac{\tau_0}{2} \quad \text{or} \quad \frac{1}{a}\int_{B(0,a)\cap\{x_2<0\}} \mathbf{A}e(v) : e(v)\,dx \ge \frac{\tau_0}{2}.$$

Without loss of generality, we assume that

$$\frac{1}{a}\int_{B(0,a)\cap\{x_2>0\}} \mathbf{A}e(v) : e(v)\,dx \ge \frac{\tau_0}{2}. \tag{5.9}$$

Decay of elastic energy. We finally want to show a decay estimate for the elastic energy, which will contradict (5.9). To this end, denoting $B^\pm = B \cap \{\pm x_2 > 0\}$, we will work with the Airy function w to construct an extension of $v|_{B^+}$ onto B which still solves the elasticity system in B. According to [35, (3.28)] (see also [21]), since $w \in \mathcal{C}^\infty(B^+)$ is a solution of

$$\Delta^2 w = 0 \text{ in } B^+, \quad w = 0, \nabla w = 0 \text{ on } B \cap \{x_2 = 0\},$$

we can consider the biharmonic reflection $\tilde{w} \in \mathcal{C}^\infty(B)$ of $w|_{B^+}$ in B defined by

$$\tilde{w}(x) = \begin{cases} w(x) & \text{if } x \in B^+, \\ -w(x_1,-x_2) - 2x_2\partial_2 w(x_1,-x_2) - x_2^2 \Delta w(x_1,-x_2) & \text{if } x \in B^-, \end{cases}$$

which satisfies $\Delta^2 \tilde{w} = 0$ in B. Thanks to this biharmonic extension, we are going to extend the displacement $v|_{B^+}$ on the whole ball B to a function $\tilde{v}$ which minimizes the elastic energy. To this end, let us define the stress by

$$\tilde{\sigma} := \begin{pmatrix} \partial_{22}\tilde{w} & -\partial_{12}\tilde{w} \\ -\partial_{12}\tilde{w} & \partial_{11}\tilde{w} \end{pmatrix},$$

and the strain

$$\tilde{e} := \begin{pmatrix} \tilde{e}_{11} & \tilde{e}_{12} \\ \tilde{e}_{12} & \tilde{e}_{22} \end{pmatrix} := \mathbf{A}^{-1}\tilde{\sigma}$$

with

$$\tilde{e}_{11} = \frac{\tilde{\sigma}_{11}}{E} - \frac{\nu}{E}\tilde{\sigma}_{22}, \qquad \tilde{e}_{22} = \frac{\tilde{\sigma}_{22}}{E} - \frac{\nu}{E}\tilde{\sigma}_{11}, \qquad \tilde{e}_{12} = \frac{1+\nu}{E}\tilde{\sigma}_{12}.$$

Note that $\operatorname{div}\tilde{\sigma} = 0$ in B, and the compatibility condition

$$\partial_{22}\tilde{e}_{11} + \partial_{11}\tilde{e}_{22} - 2\partial_{12}\tilde{e}_{12} = 0 \quad \text{in } B$$

ensures the existence of some $\tilde{v} \in \mathcal{C}^\infty(B;\mathbb{R}^2)$ such that $\tilde{e} = e(\tilde{v})$ in B. In particular, according to (5.7), we have

$$\mathbf{A}e(v) = \begin{pmatrix} \partial_{22}w & -\partial_{12}w \\ -\partial_{12}w & \partial_{11}w \end{pmatrix} = \begin{pmatrix} \partial_{22}\tilde{w} & -\partial_{12}\tilde{w} \\ -\partial_{12}\tilde{w} & \partial_{11}\tilde{w} \end{pmatrix} = \mathbf{A}e(\tilde{v}) \quad \text{in } B^+,$$

which shows that $e(\tilde{v}) = e(v)$ in B^+, and thus v and $\tilde{v}$ only differ from a rigid body motion in B^+. We have thus constructed an extension $\tilde{v}$ of $v|_{B^+}$ which satisfies $-\operatorname{div}(\mathbf{A}e(\tilde{v})) = 0$ in B, or equivalently

$$\int_B \mathbf{A}e(\tilde{v}) : e(\tilde{v})\,dx \le \int_B \mathbf{A}e(\tilde{v}+\varphi) : e(\tilde{v}+\varphi)\,dx$$

for all $\varphi \in LD(B)$ such that $v = 0$ on ∂B.

According to (5.9), we have

$$\frac{1}{a}\int_{B(0,a)} \mathbf{A}e(\tilde{v}) : e(\tilde{v})\,dx \ge \frac{\tau_0}{2}. \tag{5.10}$$

Moreover, by standard decay energy estimates for elliptic systems (see e.g. [14, Proposition 3.4]), we infer that for all $\gamma \in (0,2)$ there exists a constant $c_\gamma = c(\gamma, \mathbf{A}) > 0$ such that for all $r \le 1$,

$$\int_{B(0,r)} \mathbf{A}e(\tilde{v}) : e(\tilde{v})\,dx \le c_\gamma r^{2-\gamma} \int_B \mathbf{A}e(\tilde{v}) : e(\tilde{v})\,dx \le c_\gamma r^{2-\gamma},$$

where the last inequality comes from (5.8), possibly with changed c_γ. Taking $\gamma = 1/2$ and $r = a$ yields

$$\frac{1}{a}\int_{B(0,a)} \mathbf{A}e(\tilde{v}) : e(\tilde{v})\,dx \le c_{1/2}a^{1/2},$$

which contradicts (5.10) provided we choose $a < \left(\frac{\tau_0}{2c_{1/2}}\right)^2$.

6. Appendix

The following lemma is an easy consequence of the coarea formula.

Lemma 6.1. *Let $K \subset \mathbb{R}^2$ be an $\mathcal{H}^1$-rectifiable set. Then for all $0 < s < r$ and $x_0 \in \mathbb{R}^2$ we have*

$$\int_s^r \#(K \cap \partial B(x_0, t))\, dt \le \mathcal{H}^1(K \cap B(x_0, r) \setminus B(x_0, s)). \tag{6.1}$$

Proof. Applying the coarea formula [4, Theorem 2.91] to the $\mathcal{H}^1$-rectifiable set $E := K \cap B(x_0, r) \setminus B(x, s)$ and the Lipschitz function $f : x \mapsto |x|$ we get

$$\int_s^r \#(K \cap \partial B(x_0, t))\, dt = \int_{\mathbb{R}} \mathcal{H}^0(E \cap f^{-1}(t))\, dt = \int_E \mathbf{J} d^E f \, d\mathcal{H}^1,$$

where $\mathcal{H}^1$-a.e. in E, $\mathbf{J}d^E f$ denotes the 1-dimensional coarea factor associated to the tangential differential df^E. Since E admits an approximate tangent line oriented by a unit vector τ at $\mathcal{H}^1$-a.e. points, we deduce that

$$\mathbf{J}d^E f_x = \left| \frac{x}{|x|} \cdot \tau \right| \le 1 \quad \mathcal{H}^1\text{-a.e. in } E,$$

which leads to (6.1). ■

We next recall a version of the Korn inequality in a rectangle.

Lemma 6.2 (Korn's constant in a rectangle). *For $h \in (0, 1)$, let $\Omega_h := (-1, 1) \times (-h, h)$ be a rectangle in $\mathbb{R}^2$ of height $2h$. There exists a constant $C > 0$ (independent of h) such that for all $u \in LD(\Omega_h)$ one can find a skew-symmetric matrix R_h for which the following Korn inequality holds:*

$$\int_{\Omega_h} |\nabla u - R_h|^2\, dx \le \frac{C}{h^4} \int_{\Omega_h} |e(u)|^2\, dx.$$

Proof. For $u \in LD(\Omega_h)$ we define the function $v \in LD(\Omega_1)$ by

$$\begin{cases} v_1(x_1, x_2) := u_1(x_1, hx_2) \\ v_2(x_1, x_2) := hu_2(x_1, hx_2) \end{cases} \quad \text{for a.e. } x = (x_1, x_2) \in \Omega_1.$$

We note that

$$\nabla v(x_1, x_2) = \begin{pmatrix} \partial_1 u_1 & h\partial_2 u_1 \\ h\partial_1 u_2 & h^2 \partial_2 u_2 \end{pmatrix} (x_1, hx_2),$$

so that

$$e(v)(x_1, x_2) = \begin{pmatrix} e_{11}(u) & he_{12}(u) \\ he_{21}(u) & h^2 e_{22}(u) \end{pmatrix} (x_1, hx_2).$$

Applying Korn's inequality to v in Ω_1 yields

$$\int_{\Omega_1} |\nabla v - R|^2\, dx \le C \int_{\Omega_1} |e(v)|^2\, dx$$

for some skew-symmetric matrix R, and where $C > 0$ is the Korn constant in the unit cube. In view of the above computations and using $h \in (0,1)$, we deduce that

$$|e(v)(x_1,x_2)|^2 \le |e(u)(x_1,hx_2)|^2,$$

while using $R_{11} = R_{22} = 0$ and $R_{12} = -R_{21}$ we get

$$\begin{aligned}
&|\nabla v(x_1,x_2) - R|^2 \\
&\quad = (|\partial_1 u_1|^2 + |h\partial_1 u_2 + R_{12}|^2 + |h\partial_2 u_1 - R_{12}|^2 + h^4|\partial_2 u_2|^2)(x_1,hx_2) \\
&\quad = (|\partial_1 u_1|^2 + h^2|\partial_1 u_2 + h^{-1}R_{12}|^2 + h^2|\partial_2 u_1 - h^{-1}R_{12}|^2 + h^4|\partial_2 u_2|^2)(x_1,hx_2) \\
&\quad \ge h^4|\nabla u(x_1,hx_2) - R_h|^2,
\end{aligned}$$

where

$$R_h := h^{-1}R = \begin{pmatrix} 0 & h^{-1}R_{12} \\ -h^{-1}R_{12} & 0 \end{pmatrix}$$

is still a skew-symmetric matrix. We then obtain

$$h^4 \int_{\Omega_1} |\nabla u(x_1,hx_2) - R_h|^2\,dx \le C \int_{\Omega_1} |e(u)(x_1,hx_2)|^2\,dx.$$

Finally, using the change of variables $(y_1,y_2) = (x_1,hx_2)$ we get

$$h^4 \int_{\Omega_h} |\nabla u - R_h|^2\,dy \le C \int_{\Omega_h} |e(u)|^2\,dy,$$

which completes the proof of the lemma. ∎

The next lemma is a standard flatness estimate for curves, coming from Pythagoras' Theorem.

Lemma 6.3. *Let* $\gamma : [0,1] \to \mathbb{R}^2$ *be a curve with endpoints* $z = \gamma(0)$ *and* $z' = \gamma(1)$*, with image* $\Gamma := \gamma([0,1])$. *Then*

$$\operatorname{dist}(y,[z,z'])^2 \le \frac{\mathcal{H}^1(\Gamma)(\mathcal{H}^1(\Gamma) - |z'-z|)}{2} \quad \textit{for all } y \in \Gamma. \tag{6.2}$$

Proof. Let $\bar{y}$ be a maximizer of the function $y \in \Gamma \mapsto \operatorname{dist}(y,[z,z'])$, i.e., $\bar{y}$ is the point in Γ furthest from the segment $[z,z']$, and define $d := \operatorname{dist}(\bar{y},[z,z'])$. Let $y' \in \mathbb{R}^2$ be the point making (z,z',y') an isosceles triangle with height d (see Figure 5). Denoting $a := |z-z'|/2$ and $L := |y'-z|$, according to Pythagoras' Theorem, we have

$$d^2 = L^2 - a^2 = (L-a)(L+a).$$

Thus $\mathcal{H}^1(\Gamma) \ge |z-\bar{y}| + |\bar{y}-z'| \ge 2L$ and $\mathcal{H}^1(\Gamma) \ge |z-z'|$ so that

$$d^2 \le \frac{1}{4}(\mathcal{H}^1(\Gamma) - |z-z'|)(\mathcal{H}^1(\Gamma) + |z-z'|) \le \frac{\mathcal{H}^1(\Gamma)(\mathcal{H}^1(\Gamma) - |z-z'|)}{2},$$

which proves (6.2). ∎

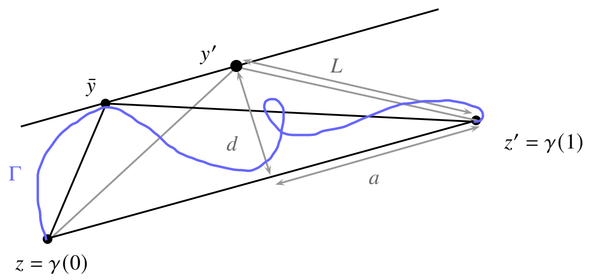

Fig. 5. The height estimate from Pythagoras' Theorem.

We conclude the appendix with the following standard lemma (see for instance [18, proof of Corollary 33.50], [20, proof of Theorem 5.5], [19, Section 10] for a nonexhaustive list of similar results). Unfortunately, we could not find a precise reference in the following elementary form, and therefore we provide an independent and complete proof for the reader's convenience.

Lemma 6.4. *Let $K \subset \mathbb{R}^2$ be a closed set containing the origin and satisfying the following property. There exist constants $C > 0$, $r_0 > 0$ and $\alpha > 0$ such that*

$$\beta_K(x,r) \le C r^\alpha \quad \text{for all } x \in K \cap B(0,1) \text{ and all } r \le r_0.$$

Then there exists $a \in (0,1)$ (only depending on C, r_0, and α) such that $K \cap B(0,a)$ is a 10^{-2}-Lipschitz graph as well as a $\mathscr{C}^{1,\alpha}$-regular curve.

Proof. For every $x \in K \cap B(0,1)$ and $0 < r \le r_0$, we denote as usual by $L(x,r)$ an affine line which approximates $K \cap B(x,r)$, i.e. such that

$$\max\Big(\sup_{z \in K \cap \overline{B}(x,r)} \operatorname{dist}(z, L(x,r)), \sup_{z \in L(x,r) \cap \overline{B}(x,r)} \operatorname{dist}(z,K)\Big) \le \beta_K(x,r) r \le C r^{1+\alpha}. \tag{6.3}$$

In addition, we denote by $\tau(x,r) \in \mathbb{S}^1/\{\pm 1\}$ an unoriented unit vector which is tangent to $L(x,r)$ and defined modulo ± 1. We use in $\mathbb{S}^1/\{\pm 1\}$ the complete distance defined, for all $\tau_1, \tau_2 \in \mathbb{S}^1/\{\pm 1\}$, by

$$d_S(\tau_1,\tau_2) := \min(|\tau_1 - \tau_2|, |\tau_1 + \tau_2|).$$

Step 1: Existence of tangents. For all $k \in \mathbb{N}$ we denote $r_k := 2^{-k} r_0$. We claim that $(\tau(x,r_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{S}^1/\{\pm 1\}, d_S)$. To see this, we show that for all $k \ge 0$ and all $x \in K \cap B(0,1)$ we have

$$d_S(\tau(x,r_{k+1}), \tau(x,r_k)) \le 9 C r_k^\alpha.$$

Indeed, let $z := x + \tau(x,r_{k+1}) r_{k+1} \in L(x,r_{k+1}) \cap \overline{B}(x,r_{k+1})$. By (6.3), there exists $y \in K \cap \overline{B}(x,r_{k+1})$ such that $|z-y| \le C r_{k+1}^{1+\alpha}$, and in particular

$$r_{k+1} - C r_{k+1}^{1+\alpha} \le |y - x| \le r_{k+1}. \tag{6.4}$$

Then, if we denote $v := \frac{y-x}{|y-x|}$, we have

$$\begin{aligned} d_S(v, \tau(x, r_{k+1})) &\le |v - \tau(x, r_{k+1})| = \left| \frac{y-x}{|y-x|} - \frac{z-x}{r_{k+1}} \right| \\ &\le \left| \frac{y-x}{|y-x|} - \frac{y-x}{r_{k+1}} \right| + \frac{1}{r_{k+1}} |z-y| \\ &\le \frac{\big| r_{k+1} - |y-x| \big|}{r_{k+1}} + C r_{k+1}^\alpha \le 2C r_{k+1}^\alpha, \end{aligned} \tag{6.5}$$

where we have used (6.4) to get the last inequality. Similarly, since $y \in \overline{B}(x, r_{k+1}) \cap K \subset \overline{B}(x, r_k) \cap K$, there exists $z' \in L(x, r_k) \cap \overline{B}(x, r_k)$ such that $|y - z'| \le C r_k^{1+\alpha}$. By (6.4) again we can estimate

$$|z' - x| \le |y - x| + |z' - y| \le r_{k+1} + C r_k^{1+\alpha}$$

and

$$|z' - x| \ge |y - x| - |z' - y| \ge r_{k+1} - C r_{k+1}^{1+\alpha} - C r_k^{1+\alpha} \ge r_{k+1} - 2C r_k^{1+\alpha},$$

thus a computation similar to the one in (6.5) leads to

$$\begin{aligned} d_S(v, \tau(x, r_k)) &\le |v - \frac{z'-x}{|z'-x|}| = \left| \frac{y-x}{|y-x|} - \frac{z'-x}{|z'-x|} \right| \\ &\le \left| \frac{y-x}{|y-x|} - \frac{y-x}{r_{k+1}} \right| + \left| \frac{y-x}{r_{k+1}} - \frac{z'-x}{r_{k+1}} \right| + \left| \frac{z'-x}{|z'-x|} - \frac{z'-x}{r_{k+1}} \right| \\ &\le C r_{k+1}^\alpha + C \frac{r_k^{1+\alpha}}{r_{k+1}} + 2C \frac{r_k^{1+\alpha}}{r_{k+1}} \le 7C r_k^\alpha. \end{aligned} \tag{6.6}$$

Gathering the above two inequalities, we obtain

$$d_S(\tau(x, r_k), \tau(x, r_{k+1})) \le d_S(\tau(x, r_k), v) + d_S(v, \tau(x, r_{k+1})) \le 9C r_k^\alpha = 9C r_0^\alpha 2^{-k\alpha},$$

as claimed. It follows that for all $k, l \ge k_0$,

$$d_S(\tau(x, r_k), \tau(x, r_l)) \le \sum_{i=k_0}^{\infty} 9C r_0^\alpha 2^{-i\alpha} = 2^{-k_0\alpha} \frac{9C r_0^\alpha}{1 - 2^{-\alpha}}.$$

Since the latter can be made arbitrarily small provided k_0 is large enough, we deduce that $\tau(x, r_k)$ is a Cauchy sequence in $\mathbb{S}^1/\{\pm 1\}$, and therefore it converges to some vector denoted by $\tau(x)$. In particular, letting $l \to \infty$, we get the following estimate for all $k \ge 0$:

$$d_S(\tau(x, r_k), \tau(x)) \le C' r_k^\alpha, \quad \text{where} \quad C' := \frac{9C}{1 - 2^{-\alpha}}.$$

Moreover, it can be easily seen through the distance estimate (6.3) that $T_x := x + \mathbb{R}\tau(x)$ is a tangent line to the set K at the point x.

Step 2: Hölder estimate for tangents. We now prove that the mapping $x \mapsto \tau(x)$ is Hölder continuous. Let x and y be two different points of $K \cap B(0,1)$ and let $\rho := |y-x|$. Assume first that $\rho \le r_0/4$ and let $k \in \mathbb{N}$ be such that

$$r_{k+2} \le \rho \le r_{k+1}.$$

We have

$$\begin{aligned} d_S(\tau(x),\tau(y)) &\le d_S(\tau(x),\tau(x,r_k)) + d_S(\tau(x,r_k),\tau(y,r_k)) + d_S(\tau(y,r_k),\tau(y)) \\ &\le 2C' r_k^\alpha + d_S(\tau(x,r_k),\tau(y,r_k)). \end{aligned} \tag{6.7}$$

In order to estimate $d_S(\tau(x,r_k),\tau(y,r_k))$, we notice that $y \in \overline{B}(x,r_k) \cap K$, thus there exists $z \in L(x,r_k) \cap \overline{B}(x,r_k)$ such that $|y-z| \le C r_k^{1+\alpha}$. Set $v := \frac{y-x}{|y-x|}$, so that a computation similar to that of (6.5) or (6.6) leads to

$$d_S(v,\tau(x,r_k)) \le 8C r_k^\alpha,$$

and inverting the roles of x and y,

$$d_S(v,\tau(y,r_k)) \le 8C r_k^\alpha.$$

Turning back to (6.7), we deduce that

$$\begin{aligned} d_S(\tau(x),\tau(y)) &\le 2C' r_k^\alpha + 16 C r_k^\alpha \le 16(C'+C)2^{2\alpha} r_{k+2}^\alpha \\ &\le 4^{\alpha+2}(C'+C)|x-y|^\alpha. \end{aligned} \tag{6.8}$$

When $\rho \ge r_0/4$, we can simply estimate

$$d_S(\tau(x),\tau(y)) \le 2 \le 2\frac{4^\alpha}{r_0^\alpha}|x-y|^\alpha,$$

which finally yields, for general $x, y \in K \cap B(0,1)$,

$$d_S(\tau(x),\tau(y)) \le C''|x-y|^\alpha, \tag{6.9}$$

with $C'' := \max(4^{\alpha+1} r_0^{-\alpha}, 4^{\alpha+2}(C'+C))$.

In other words, we have proved that K admits a tangent everywhere on $B(0,1)$ and that tangent lines behave nicely. We will prove now that $K \cap \overline{B}(0,a)$ is a curve for a small enough. Actually, a convenient way to prove this is to show the stronger property that $K \cap \overline{B}(0,a)$ is a Lipschitz graph for some $a \in (0,1)$ small enough.

Step 3: $K \cap \overline{B}(0,a)$ is a Lipschitz graph. We first show that for $a > 0$ small enough (to be fixed later), the set $K \cap \overline{B}(0,a)$ is a graph above the line $\mathbb{R}\tau(0)$, which we assume for simplicity to be oriented by $e_1 := \tau(0)$. Notice that for all $x \in K \cap \overline{B}(0,a)$,

$$d_S(\tau(x), e_1) \le C'' a^\alpha, \tag{6.10}$$

which means that for a small, all the tangents are oriented almost horizontally in $K \cap \overline{B}(0,a)$.

We assume for contradiction that there exist distinct points $x, y \in K \cap \overline{B}(0,a)$ such that $x_1 = y_1$. Let $a \le r_0/10$, $\rho := 10|x-y| = 10|x_2 - y_2| \le 20a$, and let $k \in \mathbb{N}$ be such that

$$r_{k+1} \le \rho \le r_k.$$

We denote by $\gamma_k \in [0, \pi/2]$ the angle between e_1 and $\tau(x, r_k)$. Since

$$d_S(e_1, \tau(x, r_k)) \le C'' a^\alpha + C' r_k^\alpha \le C'' a^\alpha + C'(40a)^\alpha,$$

for a small enough it is not restrictive to assume $\gamma_k \in [0, \pi/4]$. We deduce that

$$\operatorname{dist}(y, T_x) \le \frac{\operatorname{dist}(y, L(x, r_k))}{\cos \gamma_k} \le \sqrt{2}\operatorname{dist}(y, L(x, r_k)) \le \sqrt{2}\, C r_k^{1+\alpha} \le \sqrt{2}\, C(40a)^\alpha r_k.$$

Similarly, if $\gamma \in [0, \pi/2]$ stands for the angle between e_1 and $\tau(x)$, then, for a small enough and for a universal constant $C''' > 0$,

$$|x_2 - y_2| = |x - y| = \frac{\operatorname{dist}(y, T_x)}{\cos \gamma} \le 2C(40a)^\alpha r_k \le a^\alpha C''' |x_2 - y_2|,$$

which is a contradiction for a small enough (depending on C'''). Therefore, $K \cap \overline{B}(0,a)$ must be a graph above the segment $\overline{B}(0,a) \cap \tau(0)\mathbb{R}$ identified to $[-a, a]$. Now to prove that the graph is 10^{-3}-Lipschitz for a small enough, we can reproduce the same argument but for $x, y \in K \cap \overline{B}(0,a)$ satisfying now, for contradiction, $|x_2 - y_2| > 10^{-3}|x_1 - y_1|$.

Step 4: Conclusion. We have proved that $K \cap \overline{B}(0,a)$ is the 10^{-3}-Lipschitz graph of some function f on $[-a, a]$. Moreover, the tangent line to the graph of f at the point $(t, f(t))$, which exists for a.e. $t \in [-a, a]$, coincides with the tangent line $x + \mathbb{R}\tau(x)$ to K at the point $x = (t, f(t))$. Since the map $x \mapsto \tau(x)$ is α-Hölder continuous, it follows that the map $t \mapsto f'(t)$ coincides a.e. on $[-a, a]$ with an α-Hölder continuous function. A smoothing argument then implies that $f \in \mathcal{C}^{1,\alpha}([-a, a])$, and $K \cap \overline{B}(0,a)$ is a $\mathcal{C}^{1,\alpha}$ curve. ∎

Funding. F.I. has been a recipient of scholarships from the Fondation Sciences Mathématiques de Paris, Emergence Sorbonne Universités and from the Séphora-Berrebi Foundation. She gratefully acknowledges their support.

References

[1] Adams, D. R., Hedberg, L. I.: Function Spaces and Potential Theory. Grundlehren Math. Wiss. 314, Springer, Berlin (1996) Zbl 0834.46021 MR 1411441

[2] Alessandrini, G., Morassi, A., Rosset, E.: The linear constraints in Poincaré and Korn type inequalities. Forum Math. **20**, 557–569 (2008) Zbl 1151.26319 MR 2418206

[3] Ambrosio, L., Fusco, N., Pallara, D.: Partial regularity of free discontinuity sets. II. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24**, 39–62 (1997) Zbl 0896.49024 MR 1475772

[4] Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Math. Monographs, Clarendon Press, Oxford Univ. Press, New York (2000) Zbl 0957.49001 MR 1857292

[5] Ambrosio, L., Pallara, D.: Partial regularity of free discontinuity sets. I. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24**, 1–38 (1997) Zbl 0896.49023 MR 1475771

[6] Babadjian, J.-F., Chambolle, A., Lemenant, A.: Energy release rate for non-smooth cracks in planar elasticity. J. École Polytech. Math. **2**, 117–152 (2015) Zbl 1325.74126 MR 3366673

[7] Bonnet, A.: On the regularity of edges in image segmentation. Ann. Inst. H. Poincaré Anal. Non Linéaire **13**, 485–528 (1996) Zbl 0883.49004 MR 1404319

[8] Bonnivard, M., Lemenant, A., Santambrogio, F.: Approximation of length minimization problems among compact connected sets. SIAM J. Math. Anal. **47**, 1489–1529 (2015) Zbl 1319.49075 MR 3337998

[9] Chambolle, A.: A density result in two-dimensional linearized elasticity, and applications. Arch. Ration. Mech. Anal. **167**, 211–233 (2003) Zbl 1030.74007 MR 1978582

[10] Chambolle, A., Conti, S., Iurlano, F.: Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy. J. Math. Pures Appl. (9) **128**, 119–139 (2019) Zbl 1419.49054 MR 3980846

[11] Chambolle, A., Crismale, V.: Existence of strong solutions to the Dirichlet problem for the Griffith energy. Calc. Var. Partial Differential Equations **58**, art. 136, 27 pp. (2019) Zbl 1419.49055 MR 3984096

[12] Chambolle, A., Crismale, V.: Compactness and lower semicontinuity in *GSBD*. J. Eur. Math. Soc. **23**, 701–719 (2021) Zbl 07328122 MR 4210722

[13] Conti, S., Focardi, M., Iurlano, F.: Integral representation for functionals defined on SBD^p in dimension two. Arch. Ration. Mech. Anal. **223**, 1337–1374 (2017) Zbl 1398.49007 MR 3594357

[14] Conti, S., Focardi, M., Iurlano, F.: A note on the Hausdorff dimension of the singular set of solutions to elasticity type systems. Comm. Contemp. Math. **21**, art. 1950026, 58 pp. (2019) Zbl 1432.35079 MR 3996978

[15] Conti, S., Focardi, M., Iurlano, F.: Existence of strong minimizers for the Griffith static fracture model in dimension two. Ann. Inst. H. Poincaré Anal. Non Linéaire **36**, 455–474 (2019) Zbl 1458.74126 MR 3913194

[16] Dal Maso, G.: Generalised functions of bounded deformation. J. Eur. Math. Soc. **15**, 1943–1997 (2013) Zbl 1271.49029 MR 3082250

[17] David, G.: C^1-arcs for minimizers of the Mumford–Shah functional. SIAM J. Appl. Math. **56**, 783–888 (1996) Zbl 0870.49020 MR 1389754

[18] David, G.: Singular Sets of Minimizers for the Mumford–Shah Functional. Progr. Math. 233, Birkhäuser, Basel (2005) Zbl 1086.49030 MR 2129693

[19] David, G., De Pauw, T., Toro, T.: A generalization of Reifenberg's theorem in $\mathbb{R}^3$. Geom. Funct. Anal. **18**, 1168–1235 (2008) Zbl 1169.49040 MR 2465688

[20] De Pauw, T., Lemenant, A., Millot, V.: On sets minimizing their weighted length in uniformly convex separable Banach spaces. Adv. Math. **305**, 1268–1319 (2017) Zbl 1364.49058 MR 3570159

[21] Duffin, R. J.: Continuation of biharmonic functions by reflection. Duke Math. J. **22**, 313–324 (1955) Zbl 0064.35201 MR 79105

[22] Fonseca, I., Fusco, N., Leoni, G., Morini, M.: Equilibrium configurations of epitaxially strained crystalline films: existence and regularity results. Arch. Ration. Mech. Anal. **186**, 477–537 (2007) Zbl 1126.74029 MR 2350364

[23] Francfort, G. A., Larsen, C. J.: Existence and convergence for quasi-static evolution in brittle fracture. Comm. Pure Appl. Math. **56**, 1465–1500 (2003) Zbl 1068.74056 MR 1988896

[24] Francfort, G. A., Marigo, J.-J.: Revisiting brittle fracture as an energy minimization problem. J. Mech. Phys. Solids **46**, 1319–1342 (1998) Zbl 0966.74060 MR 1633984

[25] Friedrich, M.: A Korn-type inequality in SBD for functions with small jump sets. Math. Models Methods Appl. Sci. **27**, 2461–2484 (2017) Zbl 1386.74123 MR 3714634

[26] Friedrich, M., Solombrino, F.: Quasistatic crack growth in 2d-linearized elasticity. Ann. Inst. H. Poincaré Anal. Non Linéaire **35**, 27–64 (2018) Zbl 1386.74124 MR 3739927

[27] Griffith, A. A.: The phenomena of rupture and flow in solids. Philos. Trans. Roy Soc. London **221A**, 163–198 (1920)

[28] Henrot, A., Pierre, M.: Variation et optimisation de formes. Math. Appl. 48, Springer, Berlin (2005) Zbl 1098.49001 MR 2512810

[29] Koch, H., Leoni, G., Morini, M.: On optimal regularity of free boundary problems and a conjecture of De Giorgi. Comm. Pure Appl. Math. **58**, 1051–1076 (2005) Zbl 1082.35168 MR 2143526

[30] Labourie, C., Lemenant, A.: Regularity improvement for the minimizers of the two-dimensional Griffith energy. In preparation (2021)

[31] Lemenant, A.: Energy improvement for energy minimizing functions in the complement of generalized Reifenberg-flat sets. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **9**, 351–384 (2010) Zbl 1197.49050 MR 2731160

[32] Lemenant, A.: Regularity of the singular set for Mumford-Shah minimizers in $\mathbb{R}^3$ near a minimal cone. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **10**, 561–609 (2011) Zbl 1239.49062 MR 2905379

[33] Morel, J.-M., Solimini, S.: Variational Methods in Image Segmentation. Progr. Nonlinear Differential Equations Appl. 14, Birkhäuser Boston, Boston, MA (1995) Zbl 0827.68111 MR 1321598

[34] Newman, M. H. A.: Elements of the Topology of Plane Sets of Points. Cambridge Univ. Press, New York (1961) Zbl 0045.44003 MR 0132534

[35] Poritsky, H.: Application of analytic functions to two-dimensional biharmonic analysis. Trans. Amer. Math. Soc. **59**, 248–279 (1946) Zbl 0061.24009 MR 15630

[36] Rigot, S.: Big pieces of $C^{1,\alpha}$-graphs for minimizers of the Mumford–Shah functional. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29**, 329–349 (2000) Zbl 0960.49024 MR 1784178