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(with an appendix by Dan Abramovich)

# On the hyperbolicity of base spaces for maximally variational families of smooth projective varieties

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**Abstract.** For maximal variational smooth families of projective manifolds whose general fibers have semi-ample canonical bundle, the Viehweg hyperbolicity conjecture states that the base spaces of such families are of log-general type. This deep conjecture was recently proved by Campana–Păun and was later generalized by Popa–Schnell. In this paper we prove that those base spaces are pseudo Kobayashi hyperbolic, as predicted by the Lang conjecture: any complex quasi-projective manifold is pseudo Kobayashi hyperbolic if it is of log-general type. As a consequence, we prove the Brody hyperbolicity of moduli spaces of polarized manifolds with semi-ample canonical bundle. This proves a 2003 conjecture by Viehweg–Zuo. We also prove the Kobayashi hyperbolicity of base spaces for effectively parametrized families of minimal projective manifolds of general type. This generalizes previous work by To–Yeung, who further assumed that these families are canonically polarized.

**Keywords.** Pseudo Kobayashi hyperbolicity, Brody hyperbolicity, moduli spaces, Viehweg–Zuo question, polarized variation of Hodge structures, Viehweg–Zuo Higgs bundles, Finsler metric, positivity of direct images, Griffiths curvature formula for Hodge bundles

## 0. Introduction

### *0.1. Main theorems*

A complex space $X$ is *Brody hyperbolic* if there is no non-constant holomorphic map $\gamma : \mathbb{C} \to X$. The first result in this paper is the affirmative answer to a conjecture by Viehweg–Zuo [63, Question 0.2] on the Brody hyperbolicity of moduli spaces for polarized manifolds with semi-ample canonical sheaf.

**Theorem A** (Brody hyperbolicity of moduli spaces). *Consider the moduli functor $\mathscr{P}_h$ of polarized manifolds with semi-ample canonical sheaf introduced by Viehweg* [61, §7.6],

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*where h is the Hilbert polynomial associated to the polarization $\mathscr{H}$. Assume that for some quasi-projective manifold V there exists a smooth family $(f_U : U \to V, \mathscr{H}) \in \mathscr{P}_h(V)$ for which the induced moduli map $\varphi_U : V \to P_h$ is quasi-finite over its image, where $P_h$ denotes the quasi-projective*[1] *coarse moduli scheme for $\mathscr{P}_h$. Then the base space V is Brody hyperbolic.*

A complex space $X$ is called *pseudo Kobayashi hyperbolic* if $X$ is hyperbolic modulo a proper Zariski closed subset $\Delta \subsetneq X$, that is, the Kobayashi pseudo-distance $d_X : X \times X \to [0, +\infty[$ of $X$ satisfies $d_X(p, q) > 0$ for any distinct $p, q \in X$ not both in $\Delta$. In particular, $X$ is *pseudo Brody hyperbolic*: any non-constant holomorphic map $\gamma : \mathbb{C} \to X$ has image $\gamma(\mathbb{C}) \subset \Delta$. When $\Delta$ is an empty set, this definition reduces to the usual definition of *Kobayashi hyperbolicity*, and the Kobayashi pseudo-distance $d_X$ is a distance.

In this paper we indeed prove a stronger result than Theorem A.

**Theorem B.** *Let $f_U : U \to V$ be a smooth projective morphism between complex quasi-projective manifolds with connected fibers. Assume that the general fiber of $f_U$ has semi-ample canonical bundle, and $f_U$ is of maximal variation, that is, the general fiber of $f_U$ can only be birational to at most countably many other fibers. Then the base space V is pseudo Kobayashi hyperbolic.*

As a byproduct, we reduce the pseudo Kobayashi hyperbolicity of varieties to the existence of certain negatively curved Higgs bundles (which we call *Viehweg–Zuo Higgs bundles* in Definition 1.1). This provides a main building block for our recent work [26] on the hyperbolicity of bases of log Calabi–Yau pairs.

Another aim of the paper is to confirm a folklore conjecture on the *Kobayashi hyperbolicity* for moduli spaces of minimal projective manifolds of general type, which can be thought of as an analytic refinement of Theorem A in the case when fibers have big and nef canonical bundle.

**Theorem C.** *Let $f_U : U \to V$ be a smooth projective family of minimal projective manifolds of general type over a quasi-projective manifold V. Assume that $f_U$ is* effectively parametrized, *that is, the Kodaira–Spencer map*

$$\rho_y : \mathscr{T}_{V,y} \to H^1(U_y, \mathscr{T}_{U_y}) \tag{0.1.1}$$

*is injective for each point $y \in V$, where $\mathscr{T}_{U_y}$ denotes the tangent bundle of the fiber $U_y := f_U^{-1}(y)$. Then the base space V is Kobayashi hyperbolic.*

### *0.2. Previous related results*

Theorem B is closely related to the *Viehweg hyperbolicity conjecture*: if $f_U : U \to V$ is a maximally variational smooth projective family of projective manifolds with semi-ample canonical bundle over a quasi-projective manifold $V$, then the base $V$ must be of log-general type. In the series of works [28, 62, 63], Viehweg–Zuo constructed in a first

[1] The quasi-projectivity of $P_h$ was proved by Viehweg [61].

step a big subsheaf of symmetric log differential forms of the base (so-called *Viehweg–Zuo sheaves*). Building on this result, the Viehweg hyperbolicity conjecture was shown by Kebekus–Kovács [35–37] when $V$ is a surface or threefold, by Patakfalvi [44] when $V$ is compact or admits a non-uniruled compactification, and it was completely solved by Campana–Păun [16], who proved a vast generalization of the famous generic semipositivity result of Miyaoka (see also [14,15,52] for other proofs). More recently, using a deep theory of Hodge modules, Popa–Schnell [47] constructed Viehweg–Zuo sheaves on the base space $V$ of the smooth family $f_U : U \to V$ of projective manifolds whose geometric generic fiber admits a good minimal model. Combining this with the aforementioned theorem of Campana–Păun, they proved that such base space $V$ is of log-general type. Therefore, Theorem B is predicted by a famous conjecture of Lang (cf. [42, Chapter VIII, Conjecture 1.4]), which stipulates that a complex quasi-projective manifold is pseudo Kobayashi hyperbolic if and only if it is of log-general type. To our knowledge, Lang's conjecture is by now known for the trivial case of curves, for general hypersurfaces $X$ in the complex projective space $\mathbb{C}P^n$ of high degrees [10, 22, 56] as well as their complements $\mathbb{C}P^n \setminus X$ [11], for projective manifolds whose universal cover carries a bounded strictly plurisubharmonic function [9], for quotients of bounded (symmetric) domains [12, 13, 49], and for subvarieties of abelian varieties [65]. Theorem B therefore provides some new evidence for Lang's conjecture.

Theorem A was first proved by Viehweg–Zuo [63, Theorem 0.1] for moduli spaces of *canonically polarized* manifolds. Combining the approaches by Viehweg–Zuo [63] with those by Popa–Schnell [47], very recently Popa–Taji–Wu [48, Theorem 1.1] proved Theorem A for moduli spaces of *polarized* manifolds with big and semi-ample canonical bundles. As we will see below, our work owes a lot to the general strategies and techniques in their work [48, 63].

The Kobayashi hyperbolicity of the moduli spaces $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geq 2$ has long been known owing to the work of Royden and Wolpert [50, 64]. The first important breakthrough on higher-dimensional generalizations was made by To–Yeung [58], who proved Kobayashi hyperbolicity of the base $V$ considered in Theorem C when the canonical bundle $K_{U_y}$ of each fiber $U_y := f_U^{-1}(y)$ of $f_U : U \to V$ is further assumed to be ample (see also [7, 54] for alternative proofs). Differently from the approaches in [48, 63], their strategy is to study the curvature of the generalized Weil–Petersson metric for families of canonically polarized manifolds, along the approaches initiated by Siu [55] and later developed by Schumacher [53]. For smooth families of Calabi–Yau manifolds (resp. orbifolds), Berndtsson–Păun–Wang [7] and Schumacher [54] (resp. To–Yeung [59]) proved the Kobayashi hyperbolicity of the base once the family is assumed to be effectively parametrized.

Recently, Lu, Sun, Zuo and the author [29] proved a big Picard-type theorem for moduli spaces of polarized manifolds with semi-ample canonical sheaf. A crucial step of the proof relies on the "generic local Torelli-type theorem" in Theorem D. Theorem D also inspired us a lot in our more recent work [27] on the big Picard theorem for varieties admitting variations of Hodge structure.

*0.3. Strategy of the proof*

For the smooth family $f_U : U \to V$ of canonically polarized manifolds with maximal variation, Viehweg–Zuo [63] constructed certain negatively twisted Higgs bundles (which we call *Viehweg–Zuo Higgs bundles* in Definition 1.1) $(\tilde{\mathscr{E}}, \tilde{\theta}) := (\bigoplus_{q=0}^{n} \mathscr{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^{n} \mathbb{1} \otimes \theta_{n-q,q})$, over some smooth projective compactification $Y$ of a certain birational model $\tilde{V}$ of $V$, where $\mathscr{L}$ is some big and nef line bundle on $Y$, and $(\bigoplus_{q=0}^{n} E^{n-q,q}, \bigoplus_{q=0}^{n} \theta_{n-q,q})$ is a Higgs bundle induced by a polarized variation of Hodge structure defined over a Zariski open set of $\tilde{V}$. In a recent paper [48], Popa–Taji–Wu introduced several new inputs to develop Viehweg–Zuo's strategy in [63], which enables them to construct those Higgs bundles on base spaces of smooth families whose geometric generic fiber admits a good minimal model (see also Theorem 1.2 for a weaker statement as well as a slightly different proof following the original construction by Viehweg–Zuo). As we will see in the main content, Viehweg–Zuo Higgs bundles (VZ Higgs bundles for short) are crucial tools in proving our main results.

When all fibers $U_y := f_U^{-1}(y)$ of the smooth family $f_U : U \to V$ considered in Theorem B have ample or big and nef canonical bundles, let us briefly recall the general strategy of proving the *pseudo Brody hyperbolicity* of $V$ in [48, 63]. A certain Higgs subbundle $(\mathscr{F}, \eta)$ of $(\tilde{\mathscr{E}}, \tilde{\theta})$ with log poles contained in the divisor $D := Y \setminus \tilde{V}$ gives rise to a morphism

$$\tau_{\gamma,k} : \mathscr{T}_{\mathbb{C}}^{\otimes k} \to \gamma^*(\mathscr{L}^{-1} \otimes E^{n-k,k}) \tag{0.3.1}$$

for any entire curve $\gamma : \mathbb{C} \to \tilde{V}$. If $\gamma : \mathbb{C} \to \tilde{V}$ is Zariski dense, then by the Kodaira–Nakano vanishing (when $K_{U_y}$ is ample) and the Bogomolov–Sommese vanishing theorems (when $K_{U_y}$ is big and nef), one can verify that $\tau_{\gamma,1}(\mathbb{C}) \not\equiv 0$. Hence there is some $m > 0$ (depending on $\gamma$) such that $\tau_{\gamma,m}$ factors through $\gamma^*(\mathscr{L}^{-1} \otimes N^{n-m,m})$, where $N^{n-m,m}$ is the kernel of the Higgs field $\theta_m : E^{n-m,m} \to E^{n-m-1,m+1} \otimes \Omega_Y(\log D)$. Applying Zuo's theorem [67] on the negativity of $N^{n-m,m}$, a certain positively curved metric for $\mathscr{L}$ can produce a singular hermitian metric on $\mathscr{T}_{\mathbb{C}}$ with *Gaussian curvature* bounded from above by a negative constant, which contradicts the (Demailly's) Ahlfors–Schwarz lemma [21, Lemma 3.2]. However, this approach did not provide enough information on the Kobayashi pseudo-distance of the base $V$. Moreover, the use of a vanishing theorem cannot show $\tau_{\gamma,1}(\mathbb{C}) \not\equiv 0$ when fibers of $f_U : U \to V$ are not minimal manifolds of general type.

One of the main results in the present paper is to apply the VZ Higgs bundle to construct a (possibly degenerate) Finsler metric $F$ on some birational model $\tilde{V}$ of the base $V$, whose holomorphic sectional curvature is bounded above by a negative constant (*negatively curved Finsler metric* in Definition 2.3(ii)). A bimeromorphic criterion for pseudo Kobayashi hyperbolicity in Lemma 2.4 states that the base is pseudo Kobayashi hyperbolic if $F$ is *positive definite* over a Zariski dense open set. Let us now briefly explain our idea of the constructions. By factorizing through some Higgs subsheaf $(\mathscr{F}, \eta) \subseteq (\tilde{\mathscr{E}}, \tilde{\theta})$ with logarithmic poles *only* along the boundary divisor $D := Y \setminus \tilde{V}$, one can define, for any $k = 1, \ldots, n$, a morphism

$$\tau_k : \mathrm{Sym}^k \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-k,k}, \tag{0.3.2}$$

where $\mathscr{L}$ is some big line bundle over $Y$ equipped with a *positively curved* singular hermitian metric $h_{\mathscr{L}}$. Then for each $k$, the hermitian metric $h_k$ on $\tilde{\mathscr{E}}_k := \mathscr{L}^{-1} \otimes E^{n-k,k}$ induced by the Hodge metric as well as $h_{\mathscr{L}}$ (see Proposition 1.3 for details) will give rise to a Finsler metric $F_k$ on $\mathscr{T}_Y(-\log D)$ by taking the $k$-th root of the pull-back $\tau_k^* h_k$. However, the holomorphic sectional curvature of $F_k$ might not be negatively curved. Inspired by the aforementioned work of Schumacher, To–Yeung and Berndtsson–Păun–Wang [7,53,54,58] on curvature computations for generalized Weil–Petersson metrics on families of canonically polarized manifolds, we define a convex sum of Finsler metrics

$$F := \Big(\sum_{k=1}^{n} \alpha_k F_k^2\Big)^{1/2} \quad \text{with } \alpha_1, \ldots, \alpha_n \in \mathbb{R}^+ \tag{0.3.3}$$

on $\mathscr{T}_Y(-\log D)$, to offset the unwanted positive terms in the curvature $\Theta_{\tilde{\mathscr{E}}_k}$ by negative contributions from the $\Theta_{\tilde{\mathscr{E}}_{k+1}}$ (the last order term $\Theta_{\tilde{\mathscr{E}}_n}$ is always semi-negative by the Griffiths curvature formula). We prove in Proposition 2.14 that for proper $\alpha_1, \ldots, \alpha_n > 0$, the holomorphic sectional curvature of $F$ is negative and bounded away from zero. To summarize, we establish an *algorithm* for the construction of Finsler metrics via VZ Higgs bundles.

To prove Theorem B, we first note that VZ Higgs bundles over some birational model $\tilde{V}$ of the base space $V$ were constructed by Popa–Taji–Wu [48]. Let $Y$ be some smooth projective compactification $\tilde{V}$ with simple normal crossing boundary $D := Y \setminus \tilde{V}$. By our construction of a negatively curved Finsler metric $F$ defined in (0.3.3) via VZ Higgs bundles, to show that $F$ is *positive definite* over some Zariski open set, it suffices to prove that $\tau_1 : \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-1,1}$ defined in (0.3.2) is *generically injective* (which we call *generic local Torelli* for VZ Higgs bundles in §1.1). This is proved in Theorem D, by using the degeneration of the Hodge metric and the curvature properties of Hodge bundles. In particular, we show that the generic injectivity of $\tau_1$ is indeed an intrinsic feature of all VZ Higgs bundles (unrelated to the Kodaira dimension of fibers of $f$!). By a standard inductive argument in [48, 63], one can easily show that Theorem B implies Theorem A.

Now we will explain the strategy to prove Theorem C. Note that VZ Higgs bundles are only constructed over some birational model $\tilde{V}$ of $V$, which is not Kobayashi hyperbolic in general. This motivates us first to establish a *bimeromorphic criterion for Kobayashi hyperbolicity* in Lemma 2.5. Based on this criterion, in order to apply VZ Higgs bundles to prove the Kobayashi hyperbolicity of the base $V$ in Theorem C, it suffices to show that

(♠) for any given point $y$ on the base $V$, there exists a VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ constructed over some birational model $\nu : \tilde{V} \to V$ such that $\nu^{-1} : V \dashrightarrow \tilde{V}$ is defined at $y$;

(♣) the negatively curved Finsler metric $F$ on $\tilde{V}$ defined in (0.3.3) induced by $(\tilde{\mathscr{E}}, \tilde{\theta})$ is positive definite at the point $\nu^{-1}(y)$.

Roughly speaking, the idea is to produce an abundant supply of *fine* VZ Higgs bundles to construct sufficiently many negatively curved Finsler metrics, which are obstructions to the degeneracy of Kobayashi pseudo-distance $d_V$ of $V$. This is much more demanding than the Brody hyperbolicity and Viehweg hyperbolicity of $V$, which can be shown by the existence of *only one* VZ Higgs bundle on an arbitrary birational model of $V$, as mentioned in [28, 47, 48, 63].

Let us briefly explain how we achieve both (♠) and (♣).

As far as we see in [48, 63], in their construction of VZ Higgs bundles, one has to blow up the base several times (indeed twice). Recall that the basic setup in [48, 63] is the following: after passing to some smooth birational model $f_{\tilde{U}} : \tilde{U} = U \times_V \tilde{V} \to \tilde{V}$ of $f_U : U \to V$, one can find a smooth projective compactification $f : X \to Y$ of $\tilde{U}^r \to \tilde{V}$,

$$\begin{tikzcd} U^r \arrow[d] & \tilde{U}^r \arrow[l, "\overset{\text{bir}}{\sim}"'] \arrow[r, "\subseteq"] \arrow[d] & X \arrow[d, "f"] \\ V & \tilde{V} \arrow[l, "\overset{\text{bir}}{\sim}"', "\nu"] \arrow[r, "\subseteq"] & Y \end{tikzcd} \tag{0.3.4}$$

so that there exists (at least) one hypersurface

$$H \in |\ell K_{X/Y} - \ell f^*\mathscr{L}| \quad \text{for some } \ell \gg 0 \tag{0.3.5}$$

which is *transverse* to the general fibers of $f$. Here $\mathscr{L}$ is some big and nef line bundle over $Y$, and $U^r := U \times_V \times \cdots \times_V U$ (resp. $\tilde{U}^r$) is the $r$-fold fiber product of $f_U : U \to V$ (resp. $f_{\tilde{U}} : \tilde{U} \to \tilde{V}$). The VZ Higgs bundle is indeed the logarithmic Higgs bundle associated to the Hodge filtration of an auxiliary variation of polarized Hodge structure constructed by taking the middle-dimensional relative de Rham cohomology on the cyclic cover of $X$ ramified along $H$.

In order to find an $H$ as in (0.3.5), a crucial step in [48, 63] is the use of *weakly semistable reduction* by Abramovich–Karu [2] so that, after changing the birational model $U \to V$ by performing a certain (uncontrollable) base change $\tilde{U} := U \times_V \tilde{V} \to \tilde{V}$, one can find a "good" compactification $X \to Y$ of $\tilde{U}^r \to \tilde{V}$ and a finite dominant morphism $W \to Y$ from a smooth projective manifold $W$ such that the base change $X \times_Y W \to W$ is birational to a *mild morphism* $Z \to W$, which is in particular flat with reduced fibers (even functorial under fiber products). For our goal (♠), we need a more refined control of the *alteration for the base* in the weakly semistable reduction [2, Theorem 0.3], which remains unknown at the moment. Fortunately, as was suggested to us and proved in Appendix A by Abramovich, using moduli of Alexeev stable maps one can establish a $\mathbb{Q}$-*mild reduction* for the family $U \to V$ in place of the *mild reduction* in [63], so that we can also find a "good" compactification $X \to Y$ of $U^r \to V$ without passing through the birational models $\tilde{V} \to V$ as in (0.3.4). This is the main theme of Appendix A.

Even if we can apply $\mathbb{Q}$-mild reduction to avoid the first blow-up of the base as in [48, 63], the second blow-up is in general inevitable. Indeed, the *discriminant* of the new family $Z_H \to Y \supset V$ obtained by taking the cyclic cover along $H$ in (0.3.5) is in general not normal crossing. One has thus to blow up this discriminant locus of $Z_H \to Y$ to make it normal crossing as in [48]. Therefore, to ensure (♠), it then suffices to show that there exists a compactification $f : X \to Y$ of the smooth family $U^r \to V$ such that for some sufficiently ample line bundle $\mathscr{A}$ over $Y$,

$$f_*(mK_{X/Y}) \otimes \mathscr{A}^{-m} \text{ is globally generated over } V \text{ for some } m \gg 0. \qquad (*)$$

Indeed, for any given point $y \in V$, by $(*)$ one can find $H$ transverse to the fiber $X_y := f^{-1}(y)$, and thus the new family $Z_H \to Y$ will be smooth over an open set containing $y$. To the best of our knowledge, $(*)$ was only known when the moduli is canonically polarized [28, Proposition 3.4]. §3.2 is devoted to the proof of $(*)$ for the family $U \to V$ in Theorem C (see Theorem 3.7(iii) below). This in turn achieves (♠).

To get (♣), our idea is to take *different cyclic coverings* by "moving" $H$ in (0.3.5), to produce different "fine" VZ Higgs bundles. For any given point $y \in V$, by (♠), one can take a birational model $\nu : \tilde{V} \to V$ such that $\nu$ is isomorphic at $y$, and there exists a VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ on the normal crossing compactification $Y \supset \tilde{V}$. To prove that the induced negatively curved Finsler metric $F$ is positive definite at $\tilde{y} := \nu^{-1}(y)$, by our definition of $F$ in (0.3.3), it suffices to show that $\tau_1$ defined in (0.3.2) is *injective* at $\tilde{y}$ in the sense of $\mathbb{C}$-linear map between complex vector spaces

$$\tau_{1,\tilde{y}} : \mathscr{T}_{\tilde{V},\tilde{y}} \xrightarrow{\simeq} \mathscr{T}_Y(-\log D)_{\tilde{y}} \xrightarrow{\rho_{\tilde{y}}} H^1(X_{\tilde{y}}, \mathscr{T}_{X_{\tilde{y}}}) \xrightarrow{\varphi_{\tilde{y}}} \tilde{\mathscr{E}}_{1,\tilde{y}}.$$

As we will see in §3.4, when $H$ in (0.3.5) is properly chosen (indeed, transverse to the fiber $X_y$) which is ensured by $(*)$, $\varphi_{\tilde{y}}$ is injective at $\tilde{y}$. Hence $\tau_{1,\tilde{y}}$ is injective by our assumption of *effective parametrization* (hence $\rho_{\tilde{y}}$ is injective) in Theorem C. This is our strategy to prove Theorem C.

*Notations and conventions*

Throughout this article we will work over the complex number field $\mathbb{C}$.

- An *algebraic fiber space*[2] (or *fibration* for short) $f : X \to Y$ is a surjective projective morphism between projective manifolds with connected geometric fibers. A $\mathbb{Q}$-divisor $E$ in $X$ is said to be $f$-*exceptional* if $f(E)$ is an algebraic variety of codimension at least 2 in $Y$.
- We say that a morphism $f_U : U \to V$ is a *smooth family* if $f_U$ is a surjective smooth projective morphism with connected fibers between quasi-projective varieties.
- For any surjective morphism $Y' \to Y$, and the algebraic fiber space $f : X \to Y$, we denote by $(X \times_Y Y')\tilde{}$ the (unique) irreducible component (say the *main component*) of $X \times_Y Y'$ which dominates $Y'$.
- Let $\mu : X' \to X$ be a birational morphism from a projective manifold $X'$ to a singular variety $X$. The morphism $\mu$ is called a *strong desingularization* if $\mu^{-1}(X^{\text{reg}}) \to X^{\text{reg}}$ is an isomorphism. Here $X^{\text{reg}}$ denotes the smooth locus of $X$.

[2]Here we follow the definition in [43].

- For any birational morphism $\mu : X' \to X$, the *exceptional locus* denoted by $\mathrm{Ex}(\mu)$ is the inverse image of the smallest closed set of $X$ outside of which $\mu$ is an isomorphism.
- Denote by $X^r := X \times_Y \cdots \times_Y X$ the $r$-fold fiber product of the fibration $f : X \to Y$, by $(X^r)\tilde{}$ the *main component* of $X^r$ dominating $Y$, and by $X^{(r)}$ a *strong desingularization* of $(X^r)\tilde{}$.
- For any quasi-projective manifold $Y$, a Zariski open subset $Y_0 \subset Y$ is called a *big open set* of $Y$ if $\mathrm{codim}_{Y \setminus Y_0}(Y) \geq 2$.
- A singular hermitian metric $h$ on the line bundle $L$ is said to be *positively curved* if the curvature current satisfies $\Theta_h(L) \geq 0$.

## 1. Brody hyperbolicity of the base

To begin, let us introduce the definition of *Viehweg–Zuo Higgs bundles* over quasi-projective manifolds in an abstract way following [48, 63]. Then we prove a generic local Torelli theorem for VZ Higgs bundles. We will show that based on the previous work by Viehweg–Zuo and Popa–Taji–Wu, this generic local Torelli theorem suffices to prove Theorem A.

### 1.1. Abstract Viehweg–Zuo Higgs bundles

The definition we present below follows from the formulation in [28, 63] and [48, Proposition 2.7].

**Definition 1.1** (Abstract Viehweg–Zuo Higgs bundles). Let $V$ be a quasi-projective manifold, and let $Y \supset V$ be a projective compactification of $V$ with the boundary $D := Y \setminus V$ simple normal crossing. A *Viehweg–Zuo Higgs bundle on* $V$ is a logarithmic Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ over $Y$ consisting of the following data:

(i) a divisor $S$ on $Y$ such that $D + S$ is simple normal crossing,
(ii) a big and nef line bundle $\mathscr{L}$ over $Y$ with $\mathbf{B}_+(\mathscr{L}) \subset D \cup S$,
(iii) a Higgs bundle $(\mathscr{E}, \theta) := (\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$ induced by the lower canonical extension of a polarized VHS defined over $Y \setminus (D \cup S)$,
(iv) a Higgs subsheaf $(\mathscr{F}, \eta) \subset (\tilde{\mathscr{E}}, \tilde{\theta})$,

which satisfy the following properties:

(1) $(\tilde{\mathscr{E}}, \tilde{\theta}) := (\mathscr{L}^{-1} \otimes \mathscr{E}, \mathbb{1} \otimes \theta)$. In particular, $\tilde{\theta} : \tilde{\mathscr{E}} \to \tilde{\mathscr{E}} \otimes \Omega_Y(\log(D + S))$, and $\tilde{\theta} \wedge \tilde{\theta} = 0$.
(2) The Higgs subsheaf $(\mathscr{F}, \eta)$ has log poles only on the boundary $D$, that is, $\eta : \mathscr{F} \to \mathscr{F} \otimes \Omega_Y(\log D)$.
(3) Write $\tilde{\mathscr{E}}_k := \mathscr{L}^{-1} \otimes E^{n-k,k}$, and denote $\mathscr{F}_k := \tilde{\mathscr{E}}_k \cap \mathscr{F}$. Then the first stage $\mathscr{F}_0$ of $\mathscr{F}$ is an *effective line bundle*. In other words, there exists a non-trivial morphism $\mathscr{O}_Y \to \mathscr{F}_0$.

As shown in [28], by iterating $\eta$ $k$ times, we obtain

$$\mathscr{F}_0 \xrightarrow{\overbrace{\eta\circ\cdots\circ\eta}^{k\text{ times}}} \mathscr{F}_k \otimes (\Omega_Y(\log D))^{\otimes k}.$$

Since $\eta \wedge \eta = 0$, the above morphism factors through $\mathscr{F}_k \otimes \mathrm{Sym}^k\,\Omega_Y(\log D)$, and by (3) one thus obtains

$$\mathscr{O}_Y \to \mathscr{F}_0 \to \mathscr{F}_k \otimes \mathrm{Sym}^k\,\Omega_Y(\log D) \to \mathscr{L}^{-1} \otimes E^{n-k,k} \otimes \mathrm{Sym}^k\,\Omega_Y(\log D).$$

Equivalently, we have a morphism

$$\tau_k : \mathrm{Sym}^k\,\mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-k,k}. \tag{1.1.1}$$

It was proven in [28, Corollary 4.5] that $\tau_1$ is always non-trivial. We say that a VZ Higgs bundle satisfies the *generic local Torelli* if $\tau_1 : \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-1,1}$ in (1.1.1) is generically injective. As we will see in §1.4 (in Theorem D) the generic local Torelli holds for any VZ Higgs bundle.

### *1.2. A quick tour of Viehweg–Zuo's construction*

For smooth families $U \to V$ of Theorems A and B, it was shown in [28] and [48, Proposition 2.7] that there is a VZ Higgs bundle over some birational model $\tilde{V}$ of $V$. Indeed, using the deep theory of mixed Hodge modules, Popa–Taji–Wu [48] can even construct VZ Higgs bundles over the bases of maximal variational smooth families whose geometric generic fiber admits a good minimal model. Since in the proof of Theorem C we need to study the precise loci where $\tau_1$ is injective, in this subsection we recollect Viehweg–Zuo's construction of VZ Higgs bundles over the base space $V$ (up to a birational model and a projective compactification) from the smooth projective family $f_U : U \to V$ of Theorem B. We refer the readers to [28] and [48] for more details. In §3.4, we show how to refine this construction to prove Theorem C.

**Theorem 1.2.** *Let $U \to V$ be the smooth family of Theorem B. Then after replacing $V$ by a birational model $\tilde{V}$, there is a smooth compactification $Y \supset \tilde{V}$ and a VZ Higgs bundle over $\tilde{V}$.*

*Proof.* By [48, 63], one can take a birational morphism $\nu : \tilde{V} \to V$ and a smooth compactification $f : X \to Y$ of $U^r \times_V \tilde{V} \to \tilde{V}$ such that there exists a hypersurface

$$H \in |\ell\Omega^n_{X/Y}(\log \Delta) - \ell f^*\mathscr{L} + \ell E|, \quad n := \dim X - \dim Y, \tag{1.2.1}$$

with $\mathscr{L}$ a big and nef line bundle over $Y$ such that

(1) the complement $D := Y \setminus \tilde{V}$ is simple normal crossing;

(2) the hypersurface $H$ is smooth over some Zariski open set $V_0 \subset \tilde{V}$ with $D + S := Y \setminus V_0$ simple normal crossing;

(3) the divisor $E$ is effective and $f$-exceptional with $f(E) \cap V_0 = \emptyset$;

(4) $\mathbf{B}_+(\mathscr{L}) \cap V_0$ is empty, where $\mathbf{B}_+(\mathscr{L})$ is the augmented base locus of $\mathscr{L}$.

Here we denote $\Delta := f^{-1}(D)$ so that $(X, \Delta) \to (Y, D)$ is a log morphism. Within this basic setup, let us now construct a VZ Higgs bundle over $\tilde{V}$ following [28]. Leaving out a codimension 2 subvariety of $Y$ supported on $D + S$, we assume that

- the morphism $f$ is flat, and $E$ in (1.2.1) disappears;
- the divisor $D + S$ is smooth, and both $\Delta$ and $\Sigma = f^{-1}S$ are relative normal crossing.

Set $\mathcal{L} := \Omega^n_{X/Y}(\log \Delta)$. Let $\delta : W \to X$ be a blow-up of $X$ with centers in $\Delta + \Sigma$ such that $\delta^*(H + \Delta + \Sigma)$ is a normal crossing divisor. One thus obtains a cyclic covering of $\delta^* H$ by taking the $\ell$-th root out of $\delta^* H$. Let $Z$ be a strong desingularization of this covering, which is smooth over $V_0$ by (2). We denote the compositions by $h : W \to Y$ and $g : Z \to Y$; their restrictions to $V_0$ are both smooth. Write $\Pi := g^{-1}(S \cup D)$, which can be assumed to be normal crossing. Leaving out a further codimension 2 subvariety supported on $D + S$, we assume that $h$ and $g$ are also flat, and both $\delta^*(H + \Delta + \Sigma)$ and $\Pi$ are relative normal crossing. Set

$$F^{n-q,q} := R^q h_* \left( \delta^*(\Omega^{n-q}_{X/Y}(\log \Delta)) \otimes \delta^* \mathcal{L}^{-1} \otimes \mathcal{O}_W \left( \left\lfloor \frac{\delta^* H}{\ell} \right\rfloor \right) \right) / \text{torsion}.$$

It was shown in [28, §4] that there exists a natural edge morphism

$$\tau_{n-q,q} : F^{n-q,q} \to F^{n-q-1,q+1} \otimes \Omega_Y(\log D), \tag{1.2.2}$$

which gives rise to the first Higgs bundle $(\bigoplus_{q=0}^n F^{n-q,q}, \bigoplus_{q=0}^n \tau_{n-q,q})$ defined over a big open subset of $Y$ containing $V_0$.

Write $Z_0 := Z \setminus \Pi$. Then the local system $R^n g_* \mathbb{C}_{\restriction Z_0}$ extends to a locally free sheaf $\mathcal{V}$ on $Y$ (here $Y$ is projective rather than a big open set!) equipped with a logarithmic connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega_Y(\log(D + S)),$$

whose eigenvalues of the residues lie in $[0, 1)$ (the so-called *lower canonical extension*). By Schmid's *nilpotent orbit theorem* [51], the Hodge filtration of $R^n g_* \mathbb{C}_{\restriction Z_0}$ extends to a filtration $\mathcal{V} := \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^n$ of *subbundles* so that their graded sheaves $E^{n-q,q} := \mathcal{F}^{n-q} / \mathcal{F}^{n-q+1}$ are also locally free, and there exists

$$\theta_{n-q,q} : E^{n-q,q} \to E^{n-q-1,q+1} \otimes \Omega_Y(\log(D + S)).$$

This defines the second Higgs bundle $(\bigoplus_{q=0}^n E^{n-q,q}, \theta_{n-q,q})$. As observed in [28, 63], $E^{n-q,q} = R^q g_* \Omega^{n-q}_{Z/Y}(\log \Pi)$ over a big open subset of $Y$ by the theorem of Steenbrink [57, 66]. By the construction of the cyclic cover $Z$, this in turn implies the following commutative diagram over a big open subset of $Y$:

$$\begin{CD}
\mathscr{L}^{-1} \otimes E^{n-q,q} @>{\mathbb{1} \otimes \theta_{n-q,q}}>> \mathscr{L}^{-1} \otimes E^{n-q-1,q+1} \otimes \Omega_Y(\log(D + S)) \\
@A{\rho_{n-q,q}}AA @AA{\rho_{n-q-1,q+1} \otimes \iota}A \\
F^{n-q,q} @>{\tau_{n-q,q}}>> F^{n-q-1,q+1} \otimes \Omega_Y(\log D)
\end{CD} \tag{1.2.3}$$

as shown in [63, Lemma 6.2] (cf. also [28, Lemma 4.4]). Note that all the objects are defined on a big open set of $Y$ except for $(\bigoplus_{q=0}^{n} E^{n-q,q}, \theta_{n-q,q})$, which is defined on the whole $Y$. Following [63, §6], for every $q = 0, \ldots, n$, we define $F^{n-q,q}$ to be the reflexive hull, and the morphisms $\tau_{n-q,q}$ and $\rho_{n-q,q}$ extend naturally.

To conclude that $(\bigoplus_{q=0}^{n} \mathscr{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^{n} \mathbb{1} \otimes \theta_{n-q,q})$ is a VZ Higgs bundle as in Definition 1.1, we have to introduce a Higgs subsheaf with log poles supported on $D$. Write $\tilde{\theta}_{n-q,q} := \mathbb{1} \otimes \theta_{n-q,q}$ for short. Following [28, Corollary 4.5] (cf. also [48]), for each $q = 0, \ldots, n$ we define a coherent torsion-free sheaf $\mathscr{F}_q := \rho_{n-q,q}(F^{n-q,q}) \subset E^{n-q,q}$. Since $F^{n,0} \supset \mathscr{O}_Y$, one has $\mathscr{F}_0 \supset \mathscr{O}_Y$. By (1.2.2) and (1.2.3),

$$\tilde{\theta}_{n-q,q}|_{\mathscr{F}_q} : \mathscr{F}_q \to \mathscr{F}_{q+1} \otimes \Omega_Y(\log D);$$

let $\eta_q$ be the restriction of $\tilde{\theta}_{n-q,q}$ to $\mathscr{F}_q$. Then $(\mathscr{F}, \eta) := (\bigoplus_{q=0}^{n} \mathscr{F}_q, \bigoplus_{q=0}^{n} \eta_q)$ is a Higgs subbundle of $(\tilde{\mathscr{E}}, \tilde{\theta}) := (\bigoplus_{q=0}^{n} \mathscr{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^{n} \tilde{\theta}_{n-q,q})$. ∎

### *1.3. Proper metrics for logarithmic Higgs bundles*

We adopt the same notations as Definition 1.1 in the rest of §1. As is well-known, $\mathscr{E}$ can be endowed with the Hodge metric $h$ induced by the polarization, which may blow up around the simple normal crossing boundary $D + S$. However, according to the work of Schmid, Cattani–Schmid–Kaplan and Kashiwara [20, 33, 51], $h$ has *mild singularities* (at most logarithmic), and as proved in [63, §7] (for unipotent monodromies) and [48, §3] (for quasi-unipotent monodromies), one can take a proper singular metric $g_\alpha$ on $\mathscr{L}$ such that the induced singular hermitian metric $g_\alpha^{-1} \otimes h$ on $\tilde{\mathscr{E}} := \mathscr{L}^{-1} \otimes \mathscr{E}$ is locally bounded from above. Before we summarize the above-mentioned results in [48, §3], we introduce some notations from *loc. cit.*

Write the simple normal crossing divisor $D = D_1 + \cdots + D_k$ and $S = S_1 + \cdots + S_\ell$. Let $f_{D_i} \in H^0(Y, \mathscr{O}_Y(D_i))$ and $f_{S_i} \in H^0(Y, \mathscr{O}_Y(S_i))$ be the canonical sections defining $D_i$ and $S_i$. We fix smooth hermitian metrics $g_{D_i}$ and $g_{S_i}$ on $\mathscr{O}_Y(D_i)$ and $\mathscr{O}_Y(S_i)$. Set

$$r_{D_i} := -\log \|f_{D_i}\|^2_{g_{D_i}}, \qquad r_{S_i} := -\log \|f_{S_i}\|^2_{g_{S_i}},$$

and define

$$r_D := \prod_{i=1}^{k} r_{D_i}, \qquad r_S := \prod_{i=1}^{\ell} r_{S_i}.$$

Let $g$ be a singular hermitian metric with analytic singularities on the big and nef line bundle $\mathscr{L}$ such that $g$ is smooth on $Y \setminus \mathbf{B}_+(\mathscr{L}) \supset Y \setminus (D \cup S)$, and the curvature current satisfies $\sqrt{-1}\,\Theta_g(\mathscr{L}) \geq \omega$ for some smooth Kähler form $\omega$ on $Y$. For $\alpha \in \mathbb{N}$, define

$$g_\alpha := g \cdot (r_D \cdot r_S)^\alpha.$$

The following proposition is a slight variant of [48, Lemma 3.1, Corollary 3.4].

**Proposition 1.3** ([48]). *When $\alpha \gg 0$, after rescaling $f_{D_i}$ and $f_{S_i}$, there exists a continuous, positive definite hermitian form $\omega_\alpha$ on $\mathscr{T}_Y(-\log D)$ such that*

(i) *over* $V_0 := Y \setminus D \cup S$, *the curvature form*

$$\sqrt{-1}\,\Theta_{g_\alpha}(\mathscr{L})_{\upharpoonright V_0} \geq r_D^{-2} \cdot \omega_{\alpha \upharpoonright V_0};$$

(ii) *the singular hermitian metric* $h_g^\alpha := g_\alpha^{-1} \otimes h$ *on* $\mathscr{L}^{-1} \otimes \mathscr{E}$ *is locally bounded on* $Y$, *and smooth outside* $D+S$; *moreover,* $h_g^\alpha$ *is degenerate on* $D+S$;

(iii) *the singular hermitian metric* $r_D^2 h_g^\alpha$ *on* $\mathscr{L}^{-1} \otimes \mathscr{E}$ *is also locally bounded on* $Y$. ■

**Remark 1.4.** It follows from Proposition 1.3 that both $h_g^\alpha$ and $r_D^2 h_g^\alpha$ can be seen as Finsler metrics on $\mathscr{L}^{-1} \otimes \mathscr{E}$ which are degenerate on $\mathrm{Supp}(D+S)$ and positive definite on $V_0$.

Although the last statement of Proposition 1.3(ii) is not explicitly stated in [48], it can be easily seen from the proof of [48, Corollary 3.4]. Proposition 1.3 mainly relies on the asymptotic behavior of the Hodge metric for *lower canonical extension* of a variation of Hodge structure (cf. Theorem 1.5 below) when the local monodromies around the boundaries are only quasi-unipotent.

**Theorem 1.5** ([48, Lemma 3.2]). *Let* $\mathscr{H} = \mathscr{F}^0 \supset \mathscr{F}^1 \supset \cdots \supset \mathscr{F}^N \supset 0$ *be a variation of Hodge structure defined over* $(\Delta^*)^p \times \Delta^q$, *where* $\Delta$ (*resp.* $\Delta^*$) *is the unit disk* (*resp. punctured unit disk*). *Consider the lower canonical extension* ${}^l\mathscr{F}^\bullet$ *over* $\Delta^{p+q} \supset (\Delta^*)^p \times \Delta^q$, *and denote by* $(\mathscr{E}, \theta)$ *the associated Higgs bundle. Then for any holomorphic section* $s \in \Gamma(U, \mathscr{E})$, *where* $U \subsetneq \Delta^{p+q}$ *is a relatively compact open set containing the origin, one has the norm estimate*

$$|s|_{\mathrm{hod}} \leq C\big((-\log|t_1|)\cdot \ldots \cdot (-\log|t_p|)\big)^\alpha, \tag{1.3.1}$$

*where* $\alpha$ *is some positive constant independent of* $s$, *and* $t = (t_1, \ldots, t_{p+q})$ *denotes the coordinates of* $\Delta^{p+q}$.

Let us mention that the estimates of the Hodge metric for *upper canonical extension* were obtained by Peters [46] in one variable, and by Catanese–Kawamata [19] in several variables, based on [20, 51]. We provide a slightly different proof of Theorem 1.5 for completeness sake, following closely the approaches in [19, 46].

*Proof of Theorem 1.5.* The fundamental group $\pi_1((\Delta^*)^p \times \Delta^q)$ is generated by elements $\gamma_1, \ldots, \gamma_p$, where $\gamma_j$ may be identified with the counterclockwise oriented generator of the fundamental group of the $j$-th copy of $\Delta^*$ in $(\Delta^*)^p$. Set $T_j$ to be the monodromy transformation with respect to $\gamma_j$; they pairwise commute and are known to be quasi-unipotent, that is, for any multivalued section $\underline{v}(t_1, \ldots, t_{p+q})$ of $\mathscr{H}$, one has

$$\underline{v}(t_1, \ldots, e^{2\pi i} t_j, \ldots, t_{p+q}) = T_j \cdot \underline{v}(t_1, \ldots, t_{p+q}),$$

and $[T_j, T_k] = 0$ for any $j, k = 1, \ldots, p$. Set $T_j = D_j \cdot U_j$ to be the (unique) Jordan–Chevalley decomposition, so that $D_j$ is diagonalizable and $U_j$ is unipotent with $[D_j, U_j] = 0$. Since $T_j$ is quasi-unipotent by the theorem of Borel, all the eigenvalues of $D_j$ are thus roots of unity. Set $N_j := \frac{1}{2\pi i}\sum_{k>0}(I-U_j)^k/k$. If $D_j = \mathrm{diag}(d_{j\ell})$ then we set

$S_j = \mathrm{diag}(\lambda_{j\ell})$ with $\lambda_{j\ell} \in (-2\pi i, 0]$ and $\exp(\lambda_{j\ell}) = d_{j\ell}$. Since $[T_j, T_k] = 0$, Jordan–Chevalley decomposition implies that

$$[S_j, S_k] = [S_j, N_k] = [N_j, N_k] = 0. \tag{1.3.2}$$

Fix a point $t_0 \in (\Delta^*)^p \times \Delta^q$, and take a basis $v_1, \ldots, v_r \in V_{t_0}$ such that $S_1, \ldots, S_p$ are simultaneously diagonal, that is,

$$S_j(v_\ell) = \lambda_{j\ell}. \tag{1.3.3}$$

Define $\underline{v}_1(t), \ldots, \underline{v}_r(t)$ to be the induced multivalued flat sections. Then

$$e_j(t) := \exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p}(S_i + N_i)\cdot \log t_i\Big)\underline{v}_j(t)$$

are single-valued and form a basis of holomorphic sections for the lower canonical extension ${}^{l}\mathcal{H}$.

Recall that the $d_{j\ell}$ are all roots of unity. One can thus take a positive integer $m$ such that $m_{j\ell} := -m\lambda_{j\ell}/(2\pi i)$ are all *non-negative integers*. Equivalently, each $T_j^m$ is unipotent. Define a ramified cover

$$\pi : \Delta^{p+q} \to \Delta^{p+q}, \qquad (w_1, \ldots, w_{p+q}) \mapsto (w_1^m, \ldots, w_p^m, w_{p+1}, \ldots, w_{p+q}),$$

and set $\pi'$ to be the restriction of $\pi$ to $(\Delta^*)^p \times \Delta^q$. Then $\pi'^*\mathscr{F}^\bullet$ is a variation of Hodge structure on $(\Delta^*)^p \times \Delta^q$ with unipotent monodromy, and we define ${}^{c}\pi'^*\mathcal{H}$ to be the canonical extension of $\pi'^*\mathcal{H}$. Set $\underline{u}_j(w) := \pi'^*\underline{v}_j$, which is a multivalued section for the local system $\pi'^*\mathcal{H}$. Then

$$\underline{u}_j(w_1, \ldots, e^{2\pi i}w_j, \ldots, w_{p+q}) = T_j^m \cdot \underline{u}_j(w_1, \ldots, w_{p+q}).$$

Define

$$\tilde{e}_j(w) := \exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mN_i \cdot \log w_i\Big)\underline{u}_j(w); \tag{1.3.4}$$

these elements form a basis of ${}^{c}\pi'^*\mathcal{H}$. Based on the work of [20, 51], it was shown in [63, Claim 7.8] that

$$|\tilde{e}_j(w)|_{\mathrm{hod}} \le C_0\big((-\log|w_1|)\cdot \ldots \cdot(-\log|w_p|)\big)^{\alpha} \tag{1.3.5}$$

for some positive constants $C_0$ and $\alpha$. On the other hand, we have

$$\begin{aligned}\pi'^*e_j(w) &= \exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p}(S_i + N_i)\cdot \log w_i^m\Big)\pi'^*\underline{v}_j(w)\\ &\overset{(1.3.2)}{=} \exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mN_i \cdot \log w_i\Big)\cdot \exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mS_i \log w_i\Big)\pi'^*\underline{v}_j(w)\end{aligned}$$

$$\overset{(1.3.3)}{=} \exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mN_i\cdot\log w_i\Big)\cdot\exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} m\lambda_{ij}\log w_i\Big)\pi'^{*}\underline{v}_j(w)$$
$$= \prod_{i=1}^{p} w_i^{m_{ij}}\cdot\exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mN_i\cdot\log w_i\Big)\cdot\underline{u}_j(w)$$
$$\overset{(1.3.4)}{=} \prod_{i=1}^{p} w_i^{m_{ij}}\cdot\tilde{e}_j(w).$$

By the definition of lower canonical extension, the $m_{ij}$ are all non-negative integers, and thus

$$\pi'^{*}|e_j|_{\mathrm{hod}}(w) = |\pi'^{*}e_j(w)|_{\mathrm{hod}} = \prod_{i=1}^{p}|w_i|^{m_{ij}}|\tilde{e}_j(w)|_{\mathrm{hod}}$$
$$\overset{(1.3.5)}{\le} C_0\big((-\log|w_1|)\cdot\ldots\cdot(-\log|w_p|)\big)^{\alpha}.$$

Hence

$$|e_j|_{\mathrm{hod}}(t) \le \frac{C_0}{m^p}\big((-\log|t_1|)\cdot\ldots\cdot(-\log|t_p|)\big)^{\alpha}.$$

Note that ${}^{l}\mathscr{H} \overset{\mathscr{C}^\infty}{\simeq} \mathscr{E}$. Therefore, for any holomorphic section $s\in\Gamma(U,\mathscr{E})$, there exist smooth functions $f_1,\ldots,f_r\in\mathscr{O}(U)$ such that $s=\sum_{j=1}^{r} f_j e_j$. This shows the estimate (1.3.1). ∎

**Remark 1.6.** For the Hodge metric of upper canonical extension, one makes the choice $\lambda_{j\ell}\in[0,2\pi i)$ instead of $\lambda_{j\ell}\in(-2\pi i,0]$ in the proof of Theorem 1.5. Then the same computation as above easily shows that

$$|e_j|_{\mathrm{hod}}(t) \le \prod_{i=1}^{p}|t_i|^{-\frac{\lambda_{ij}}{2\pi i}}\frac{C}{m^p}\big((-\log|t_1|)\cdot\ldots\cdot(-\log|t_p|)\big)^{\alpha},$$

which was obtained in [19].

### *1.4. Generic local Torelli for VZ Higgs bundles*

In this section we prove that the generic local Torelli holds for any VZ Higgs bundle, which is a crucial step in the proofs of Theorems A and B.

**Theorem D** (Generic local Torelli). *For all abstract Viehweg–Zuo Higgs bundles defined in Definition 1.1, the morphism $\tau_1:\mathscr{T}_Y(-\log D)\to\mathscr{L}^{-1}\otimes E^{n-1,1}$ defined in (1.1.1) is generically injective.*

*Proof.* By Definition 1.1, the non-zero morphism $\mathscr{O}_Y\to\mathscr{F}_0\to\mathscr{L}^{-1}\otimes E^{n,0}$ induces a global section $s\in H^0(Y,\mathscr{L}^{-1}\otimes E^{n,0})$, which is *generically* non-vanishing over $V_0:=Y\setminus(D\cup S)$. Set

$$V_1:=\{y\in V_0\mid s(y)\neq 0\}, \tag{1.4.1}$$

which is a non-empty Zariski open set of $V_0$. For the first stage of the VZ Higgs bundle $\mathscr{L}^{-1} \otimes E^{n,0}$, we equip it with a singular metric $h_g^\alpha := g_\alpha^{-1} \otimes h$ as in Proposition 1.3, so that items 1.3(i) and 1.3(ii) of that proposition are satisfied. Note that $h_g^\alpha$ is smooth over $V_0$. Let $D'$ be the $(1,0)$-part of its Chern connection over $V_0$, and $\Theta_0$ its curvature form. Then by the Griffiths curvature formula for Hodge bundles (see [31]), over $V_0$ we have

$$\begin{aligned}\Theta_0 &= -\Theta_{\mathscr{L},g_\alpha} \otimes \mathbb{1} + \mathbb{1} \otimes \Theta_h(E^{n,0}) = -\Theta_{\mathscr{L},g_\alpha} \otimes \mathbb{1} - \mathbb{1} \otimes (\theta_{n,0}^* \wedge \theta_{n,0}) \\ &= -\Theta_{\mathscr{L},g_\alpha} \otimes \mathbb{1} - \tilde{\theta}_{n,0}^* \wedge \tilde{\theta}_{n,0},\end{aligned} \tag{1.4.2}$$

where we set $\tilde{\theta}_{n-k,k} := \mathbb{1} \otimes \theta_{n-k,k} : \mathscr{L}^{-1} \otimes E^{n-k,k} \to \mathscr{L}^{-1} \otimes E^{n-k-1,k+1} \otimes \Omega_Y(\log(D+S))$, and define $\tilde{\theta}_{n,0}^*$ to be the adjoint of $\tilde{\theta}_{n,0}$ with respect to the metric $h_g^\alpha$. Hence over $V_1$ one has

$$\begin{aligned}&-\sqrt{-1}\,\partial\bar{\partial}\log|s|^2_{h_g^\alpha} \\ &\quad = \frac{\{\sqrt{-1}\,\Theta_0(s), s\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}} + \frac{\sqrt{-1}\,\{D's, s\}_{h_g^\alpha} \wedge \{s, D's\}_{h_g^\alpha}}{|s|^4_{h_g^\alpha}} - \frac{\sqrt{-1}\,\{D's, D's\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}} \\ &\quad \le \frac{\{\sqrt{-1}\,\Theta_0(s), s\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}}\end{aligned} \tag{1.4.3}$$

thanks to the Lagrange inequality

$$\sqrt{-1}\,|s|^2_{h_g^\alpha} \cdot \{D's, D's\}_{h_g^\alpha} \ge \sqrt{-1}\,\{D's, s\}_{h_g^\alpha} \wedge \{s, D's\}_{h_g^\alpha}.$$

Putting (1.4.2) into (1.4.3), over $V_1$ one has

$$\begin{aligned}\sqrt{-1}\,\Theta_{\mathscr{L},g_\alpha} - \sqrt{-1}\,\partial\bar{\partial}\log|s|^2_{h_g^\alpha} &\le -\frac{\{\sqrt{-1}\,\tilde{\theta}_{n,0}^* \wedge \tilde{\theta}_{n,0}(s), s\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}} \\ &= \frac{\sqrt{-1}\,\{\tilde{\theta}_{n,0}(s), \tilde{\theta}_{n,0}(s)\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}}\end{aligned} \tag{1.4.4}$$

where $\tilde{\theta}_{n,0}(s) \in H^0(Y, \mathscr{L}^{-1} \otimes E^{n-1,1} \otimes \Omega_Y(\log(D+S)))$. By Proposition 1.3(ii), for any $y \in D \cup S$, one has

$$\lim_{y' \in V_0, y' \to y} |s|^2_{h_g^\alpha}(y') = 0.$$

Therefore, it follows from the compactness of $Y$ that there exists $y_0 \in V_0$ such that $|s|^2_{h_g^\alpha}(y_0) \ge |s|^2_{h_g^\alpha}(y)$ for any $y \in V_0$. Hence $|s|^2_{h_g^\alpha}(y_0) > 0$, and by (1.4.1), $y_0 \in V_1$. Since $|s|^2_{h_g^\alpha}$ is smooth over $V_0$, $\sqrt{-1}\,\partial\bar{\partial}\log|s|^2_{h_g^\alpha}(y_0)$ is semi-negative. By Proposition 1.3(i), $\sqrt{-1}\,\Theta_{\mathscr{L},g_\alpha}$ is strictly positive at $y_0$. By (1.4.4) and $|s|^2_h(y_0) > 0$, we conclude that

$\sqrt{-1}\,\{\tilde{\theta}_{n,0}(s), \tilde{\theta}_{n,0}(s)\}_{h_g^\alpha}$ is strictly positive at $y_0$. In particular, $\tilde{\theta}_{n,0}(s)(\xi) \neq 0$ for any non-zero $\xi \in \mathscr{T}_{Y,y_0}$. Over $V_0$ the map

$$\tau_1 : \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-1,1}$$

in (1.1.1) is defined by $\tau_1(\xi) := \tilde{\theta}_{n,0}(s)(\xi)$, so it is *injective at* $y_0 \in V_1$. Hence $\tau_1$ is *generically injective*. The theorem is thus proved. ∎

**Remark 1.7.** Viehweg–Zuo [28] showed that $\tau_1 : \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-1,1}$ defined in (1.1.1) does not vanish on $V$ using a global argument relying on the Griffiths curvature computation for the Hodge metric and the bigness of direct image sheaves due to Kawamata and Viehweg. Moreover, by the work of Viehweg–Zuo [63] and Popa–Taji–Wu [48], it has already been known that when fibers in Theorem A have big and semi-ample canonical bundle, the VZ Higgs bundles constructed in Theorem 1.2 over the base always satisfy Theorem D.

Though Theorem A follows from our more general result in Theorem B, we are able to prove Theorem A by directly applying the results of Viehweg–Zuo [63] and Popa–Taji–Wu [48]. Since we need some efforts to prove Theorem B, let us quickly show how to combine their work with Theorem D to prove Theorem A.

*Proof of Theorem A.* By the stratified arguments of Viehweg–Zuo [63], it suffices to prove that there cannot exist a Zariski dense entire curve. Assume for contradiction that there exists such a $\gamma : \mathbb{C} \to V$. The existence of a VZ Higgs bundle on some birational model $\tilde{V}$ of $V$ is known by Theorem 1.2. Let $\tilde{\gamma} : \mathbb{C} \to \tilde{V}$ be the lift of $\gamma$, which is also Zariski dense. In [48, 63], the authors proved that the restriction of $\tau_1$ defined in (1.1.1) on $\mathbb{C}$, say $\tau_1|_{\mathbb{C}} : \mathscr{T}_{\mathbb{C}} \to \tilde{\gamma}^*(\mathscr{L}^{-1} \otimes E^{n-1,1})$, has to vanish identically, or else, they can construct a pseudo-hermitian metric on $\mathbb{C}$ with strictly negative Gaussian curvature which violates the Ahlfors–Schwarz lemma. By Theorem D, this cannot happen since $\tilde{\gamma} : \mathbb{C} \to \tilde{V}$ is Zariski dense. The theorem is proved. ∎

## 2. Pseudo Kobayashi hyperbolicity of the base

In this section we first establish an algorithm to construct Finsler metrics whose holomorphic sectional curvatures are bounded above by a negative constant via VZ Higgs bundles. By our construction and generic local Torelli Theorem D, those Finsler metrics are positive definite over a Zariski open set, and by the Ahlfors–Schwarz lemma, we prove that a quasi-projective manifold is pseudo Kobayashi hyperbolic once it is equipped with a VZ Higgs bundle, and thus prove Theorem B.

### 2.1. *Finsler metric and (pseudo) Kobayashi hyperbolicity*

Throughout this subsection, $X$ will denote a complex manifold of dimension $n$.

**Definition 2.1** (Finsler metric). Let $\mathscr{E}$ be a holomorphic vector bundle on $X$. A *Finsler metric*[3] on $\mathscr{E}$ is a real non-negative continuous function $F : \mathscr{E} \to [0, +\infty[$ such that

$$F(av) = |a|F(v)$$

for any $a \in \mathbb{C}$ and $v \in \mathscr{E}$. The Finsler metric $F$ is *positive definite* on some subset $S \subset X$ if for any $x \in S$ and any non-zero vector $v \in \mathscr{E}_x$, $F(v) > 0$.

When $F$ is a Finsler metric on $\mathscr{T}_X$, we also say that $F$ is a *Finsler metric on* $X$.

Let $\mathscr{E}$ and $\mathscr{G}$ be locally free sheaves on $X$, and suppose that there is a morphism

$$\varphi : \operatorname{Sym}^m \mathscr{E} \to \mathscr{G}.$$

Then for any Finsler metric $F$ on $\mathscr{G}$, $\varphi$ induces a pseudo-metric $(\varphi^* F)^{1/m}$ on $\mathscr{E}$ defined by

$$(\varphi^* F)^{1/m}(e) := F(\varphi(e^{\otimes m}))^{1/m} \tag{2.1.1}$$

for any $e \in \mathscr{E}$. It is easy to verify that $(\varphi^* F)^{1/m}$ is also a Finsler metric on $\mathscr{E}$. Moreover, if over some open set $U$, $\varphi$ is an injection as a morphism between vector bundles, and $F$ is positive definite over $U$, then $(\varphi^* F)^{1/m}$ is also positive definite over $U$.

**Definition 2.2.** (i) The *Kobayashi–Royden infinitesimal pseudo-metric* of $X$ is a length function $\kappa_X : \mathscr{T}_X \to [0, +\infty[$, defined by

$$\kappa_X(\xi) = \inf_{\gamma} \{\lambda > 0 \mid \exists \gamma : \mathbb{D} \to X,\ \gamma(0) = x,\ \lambda \cdot \gamma'(0) = \xi\} \tag{2.1.2}$$

for any $x \in X$ and $\xi \in \mathscr{T}_X$, where $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$.

(ii) The *Kobayashi pseudo-distance* of $X$, denoted by $d_X : X \times X \to [0, +\infty[$, is

$$d_X(p, q) = \inf_{\ell} \int_0^1 \kappa_X(\ell'(\tau))\, d\tau$$

for any $p, q \in X$, where the infimum is taken over all differentiable curves $\ell : [0, 1] \to X$ joining $p$ to $q$.

(iii) Let $\Delta \subsetneq X$ be a closed subset. A complex manifold $X$ is *Kobayashi hyperbolic modulo* $\Delta$ if $d_X(p, q) > 0$ for any distinct $p, q \in X$ not both in $\Delta$. When $\Delta$ is an empty set, the manifold $X$ is *Kobayashi hyperbolic*; when $\Delta$ is proper and Zariski closed, the manifold $X$ is *pseudo Kobayashi hyperbolic*.

By definition it is easy to show that if $X$ is Kobayashi hyperbolic (resp. pseudo Kobayashi hyperbolic), then $X$ is Brody hyperbolic (resp. algebraically degenerate). Brody's theorem says that when $X$ is compact, $X$ is Kobayashi hyperbolic if it is Brody hyperbolic. However unlike the case of Kobayashi hyperbolicity, no criterion is known for pseudo Kobayashi hyperbolicity of a compact complex space in terms of entire curves. Moreover, there are many examples of complex (quasi-projective) manifolds which are Brody hyperbolic but not Kobayashi hyperbolic.

[3] This definition is a bit different from the definition in [38], which requires *convexity* or *triangle inequality*, and the Finsler metric there can be upper semi-continuous.

For any holomorphic map $\gamma : \mathbb{D} \to X$, the Finsler metric $F$ induces a continuous hermitian pseudo-metric on $\mathbb{D}$,

$$\gamma^* F^2 = \sqrt{-1}\,\lambda(t) dt \wedge d\bar{t},$$

where $\lambda(t)$ is a non-negative continuous function on $\mathbb{D}$. The *Gaussian curvature* $K_{\gamma^* F^2}$ of the pseudo-metric $\gamma^* F^2$ is defined to be

$$K_{\gamma^* F^2} := -\frac{1}{\lambda}\frac{\partial^2 \log \lambda}{\partial t \partial \bar{t}}. \tag{2.1.3}$$

**Definition 2.3.** Let $X$ be a complex manifold endowed with a Finsler metric $F$.

(i) For any $x \in X$ and $v \in \mathscr{T}_{X,x}$, let $[v]$ denote the complex line spanned by $v$. We define the *holomorphic sectional curvature* $K_{F,[v]}$ *in the direction of* $[v]$ by

$$K_{F,[v]} := \sup K_{\gamma^* F^2}(0)$$

where the supremum is taken over all $\gamma : \mathbb{D} \to X$ such that $\gamma(0) = x$ and $[v]$ is tangent to $\gamma'(0)$.

(ii) We say that $F$ is *negatively curved* if there is a positive constant $c$ such that $K_{F,[v]} \leq -c$ for all $v \in \mathscr{T}_{X,x}$ for which $F(v) > 0$.

(iii) A point $x \in X$ is a *degeneracy point* of $F$ if $F(v) = 0$ for some non-zero $v \in \mathscr{T}_{X,x}$, and the set of such points is denoted by $\Delta_F$.

As mentioned in §0, our negatively curved Finsler metrics are only constructed on birational models of the base spaces in Theorems B and C, we thus have to establish bimeromorphic criteria for (pseudo) Kobayashi hyperbolicity to prove the main theorems.

**Lemma 2.4** (Bimeromorphic criteria for pseudo Kobayashi hyperbolicity). *Let* $\mu : X \to Y$ *be a bimeromorphic morphism between complex manifolds. If there exists a Finsler metric* $F$ *on* $X$ *which is negatively curved in the sense of Definition* 2.3(ii)*, then* $X$ *is Kobayashi hyperbolic modulo* $\Delta_F$*, and* $Y$ *is Kobayashi hyperbolic modulo* $\mu(\mathrm{Ex}(\mu) \cup \Delta_F)$*, where* $\mathrm{Ex}(\mu)$ *is the exceptional locus of* $\mu$*. In particular, when* $\Delta_F$ *is a proper analytic subvariety of* $X$*, both* $X$ *and* $Y$ *are pseudo Kobayashi hyperbolic.*

*Proof.* The first statement is a slight variant of [38, Theorem 3.7.4]. By normalizing $F$ we may assume that $K_F \leq -1$. By the Ahlfors–Schwarz lemma, one has $F \leq \kappa_X$. Let $\delta_F : X \times X \to [0, +\infty[$ be the distance function on $X$ defined by $F$ in a similar way to $d_X$:

$$\delta_F(p,q) := \inf_{\ell} \int_0^1 F(\ell'(\tau))\, d\tau$$

for any $p, q \in X$, where the infimum is over all differentiable curves $\ell : [0,1] \to X$ joining $p$ to $q$. Since $F$ is continuous and positive definite over $X \setminus \Delta_F$, for any $p \in X \setminus \Delta_F$, one has $d_X(p,q) \geq \delta_F(p,q) > 0$ for any $q \neq p$, which proves the first statement.

Let $\mathrm{Hol}(Y, y)$ be the set of holomorphic maps $\gamma : \mathbb{D} \to Y$ with $\gamma(0) = y$. Pick any point $y \in U := Y \setminus \mu(\mathrm{Ex}(\mu))$. Then there is a unique point $x \in X$ with $\mu(x) = y$. Hence

$\mu$ induces a bijection of sets

$$\mathrm{Hol}(X, x) \xrightarrow{\simeq} \mathrm{Hol}(Y, y)$$

defined by $\tilde{\gamma} \mapsto \mu \circ \tilde{\gamma}$. Indeed, observe that $\mu^{-1} : Y \dashrightarrow X$ is a meromorphic map, hence so is $\mu^{-1} \circ \gamma$ for any $\gamma \in \mathrm{Hol}(Y, y)$. Since $\dim \mathbb{D} = 1$, the map $\mu^{-1} \circ \gamma$ is moreover holomorphic. It follows from (2.1.2) that

$$\kappa_X(\xi) = \kappa_Y(\mu_*(\xi))$$

for any $\xi \in \mathscr{T}_{X,x}$. Hence

$$\mu^*\kappa_Y|_{\mu^{-1}(U)} = \kappa_X|_{\mu^{-1}(U)} \geq F|_{\mu^{-1}(U)}.$$

Let $G : \mathscr{T}_U \to [0, +\infty[$ be the Finsler metric on $U$ such that $\mu^*G = F|_{\mu^{-1}(U)}$. Then $G$ is continuous and positive definite over $U \setminus \mu(\Delta_F)$, and

$$\kappa_Y|_U \geq G.$$

Therefore, for any $y \in Y \setminus \mu(\Delta_F \cup \mathrm{Ex}(\mu))$, one has $d_Y(y, z) > 0$ for any $z \neq y$, which proves the second statement. ■

The above criterion can be refined further to show the Kobayashi hyperbolicity of the complex manifold.

**Lemma 2.5** (Bimeromorphic criterion for Kobayashi hyperbolicity). *Let $X$ be a complex manifold. Assume that for each point $p \in X$, there is a bimeromorphic morphism $\mu : \tilde{X} \to X$ with $\tilde{X}$ equipped with a negatively curved Finsler metric $F$ such that $p \notin \mu(\Delta_F \cup \mathrm{Ex}(\mu))$. Then $X$ is Kobayashi hyperbolic.*

*Proof.* It suffices to show that $d_X(p, q) > 0$ for any distinct $p, q \in X$. We take the bimeromorphic morphism $\mu : \tilde{X} \to X$ in the lemma with respect to $p$. By Lemma 2.4, $X$ is Kobayashi hyperbolic modulo $\mu(\Delta_F \cup \mathrm{Ex}(\mu))$, which shows that $d_X(p, q) > 0$ for any $q \neq p$. The lemma follows. ■

## 2.2. *Curvature formula*

Let $(\tilde{\mathscr{E}}, \tilde{\theta})$ be the VZ Higgs bundle on a quasi-projective manifold $V$ defined in §1.1. In the next two subsections, we will construct a negatively curved Finsler metric on $V$ via $(\tilde{\mathscr{E}}, \tilde{\theta})$. Our main result is the following.

**Theorem 2.6** (Existence of negatively curved Finsler metrics). *Same notations as Definition 1.1. Assume that $\tau_1$ is injective over a non-empty Zariski open set $V_1 \subseteq Y \setminus (D \cup S)$. Then there exists a Finsler metric $F$ (see (2.3.6) below) on $\mathscr{T}_Y(-\log D)$ such that*

(i) *$F$ is positive definite over $V_1$;*
(ii) *when $F$ is seen as a Finsler metric on $V = Y \setminus D$, it is negatively curved in the sense of Definition 2.3(ii).*

Let us first construct the desired Finsler metric $F$, and then prove the curvature property. By (1.1.1), for each $k = 1, \ldots, n$, there exists

$$\tau_k : \mathrm{Sym}^k \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-k,k}. \tag{2.2.1}$$

Then it follows from Proposition 1.3(ii) that the Finsler metric $h_g^\alpha$ on $\mathscr{L}^{-1} \otimes E^{n-k,k}$ induces a Finsler metric $F_k$ on $\mathscr{T}_Y(-\log D)$ defined as follows: for any $e \in \mathscr{T}_Y(-\log D)_y$,

$$F_k(e) := (\tau_k^* h_g^\alpha)^{1/k}(e) = h_g^\alpha(\tau_k(e^{\otimes k}))^{1/k}. \tag{2.2.2}$$

For any $\gamma : \mathbb{D} \to V$, one has

$$d\gamma : \mathscr{T}_{\mathbb{D}} \to \gamma^* \mathscr{T}_V \hookrightarrow \gamma^* \mathscr{T}_Y(-\log D)$$

and thus the Finsler metric $F_k$ induces a continuous hermitian pseudo-metric on $\mathbb{D}$, denoted by

$$\gamma^* F_k^2 := \sqrt{-1}\, G_k(t) dt \wedge d\bar{t}. \tag{2.2.3}$$

In general, $G_k(t)$ may be identically equal to zero for all $k$. However, if we further assume that $\gamma(\mathbb{D}) \cap V_1 \neq \emptyset$, by the assumption in Theorem 2.6 that the restriction of $\tau_1$ to $V_1$ is injective, one has $G_1(t) \not\equiv 0$. Denote by $\partial_t := \frac{\partial}{\partial t}$ the canonical vector field in $\mathbb{D}$, and $\bar{\partial}_t := \frac{\partial}{\partial \bar{t}}$ its conjugate. Set $C := \gamma^{-1}(V_1)$, and note that $\mathbb{D} \setminus C$ is a discrete set in $\mathbb{D}$.

**Lemma 2.7.** *Assume that $G_k(t) \not\equiv 0$ for some $k > 1$. Then the* Gaussian curvature *$K_k$ of the continuous pseudo-hermitian metric $\gamma^* F_k^2$ on $C$ satisfies*

$$K_k := -\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} / G_k \leq \frac{1}{k}\left(-\left(\frac{G_k}{G_{k-1}}\right)^{k-1} + \left(\frac{G_{k+1}}{G_k}\right)^{k+1}\right) \tag{2.2.4}$$

*over $C \subset \mathbb{D}$.*

*Proof.* For $i = 1, \ldots, n$, let us write $e_i := \tau_i(d\gamma(\partial_t)^{\otimes i})$, which can be seen as a section of $\gamma^*(\mathscr{L}^{-1} \otimes E^{n-i,i})$. Then by (2.2.2) one observes that

$$G_i(t) = \|e_i\|_{h_g^\alpha}^{2/i}. \tag{2.2.5}$$

Let $\mathscr{R}_k = \Theta_{h_g^\alpha}(\mathscr{L}^{-1} \otimes E^{n-k,k})$ be the curvature form of $\mathscr{L}^{-1} \otimes E^{n-k,k}$ on the set $V_0 := Y \setminus (D \cup S)$ induced by the metric $h_g^\alpha = g_\alpha^{-1} \cdot h$ defined in Proposition 1.3(ii), and let $D'$ be the $(1,0)$-part of the Chern connection $D$ of $(\mathscr{L}^{-1} \otimes E^{n-k,k}, h_g^\alpha)$. Then for $k = 1, \ldots, n$, one has

$$\begin{aligned}
-\sqrt{-1}\,\partial\bar{\partial} \log G_k &= \frac{1}{k}\left(\frac{\{\sqrt{-1}\,\mathscr{R}_k(e_k), e_k\}_{h_g^\alpha}}{\|e_k\|_{h_g^\alpha}^2} + \frac{\sqrt{-1}\,\{D'e_k, e_k\}_{h_g^\alpha} \wedge \{e_k, D'e_k\}_{h_g^\alpha}}{\|e_k\|_{h_g^\alpha}^4}\right. \\
&\qquad \left. - \frac{\sqrt{-1}\,\{D'e_k, D'e_k\}_{h_g^\alpha}}{\|e_k\|_{h_g^\alpha}^2}\right) \\
&\leq \frac{1}{k} \frac{\{\sqrt{-1}\,\mathscr{R}_k(e_k), e_k\}_{h_g^\alpha}}{\|e_k\|_{h_g^\alpha}^2}
\end{aligned}$$

thanks to the Lagrange inequality

$$\sqrt{-1}\,\|e_k\|^2_{h^\alpha_g}\cdot\{D'e_k, D'e_k\}_{h^\alpha_g}\geq\sqrt{-1}\,\{D'e_k, e_k\}_{h^\alpha_g}\wedge\{e_k, D'e_k\}_{h^\alpha_g}.$$

Hence

$$-\frac{\partial^2\log G_k}{\partial t\partial\bar{t}}\leq\frac{1}{k}\cdot\frac{\langle\mathscr{R}_k(e_k)(\partial_t,\bar{\partial}_t), e_k\rangle_{h^\alpha_g}}{\|e_k\|^2_{h^\alpha_g}}. \tag{2.2.6}$$

Recall that for the logarithmic Higgs bundle $(\bigoplus_{k=0}^n E^{n-k,k}, \bigoplus_{k=0}^n\theta_{n-k,k})$, the curvature $\Theta_k$ on $E^{n-k,k}_{\restriction V_0}$ induced by the Hodge metric $h$ is given by

$$\Theta_k=-\theta^*_{n-k,k}\wedge\theta_{n-k,k}-\theta_{n-k+1,k-1}\wedge\theta^*_{n-k+1,k-1},$$

where we recall that $\theta_{n-k,k}: E^{n-k,k}\to E^{n-k-1,k+1}\otimes\Omega_Y(\log(D+S))$. Set $\tilde{\theta}_{n-k,k}:=\mathbb{1}\otimes\theta_{n-k,k}:\mathscr{L}^{-1}\otimes E^{n-k,k}\to\mathscr{L}^{-1}\otimes E^{n-k-1,k+1}\otimes\Omega_Y(\log(D+S))$, and one has

$$\begin{tikzcd}[column sep=huge]
\mathscr{L}^{-1}\otimes E^{n-k+1,k-1} \arrow[r, bend left, "\tilde{\theta}_{n-k+1,k-1}(\partial_t)"] & \mathscr{L}^{-1}\otimes E^{n-k,k} \arrow[l, bend left, "\tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)"] \arrow[r, bend left, "\tilde{\theta}_{n-k,k}(\partial_t)"] & \mathscr{L}^{-1}\otimes E^{n-k-1,k+1} \arrow[l, bend left, "\tilde{\theta}^*_{n-k,k}(\bar{\partial}_t)"]
\end{tikzcd}$$

where $\tilde{\theta}^*_{n-k,k}$ is the adjoint of $\tilde{\theta}_{n-k,k}$ with respect to the metric $h^\alpha_g$ over $Y\setminus(D\cup S)$. Here we also write $\partial_t$ (resp. $\bar{\partial}_t$) for $d\gamma(\partial_t)$ (resp. $d\gamma(\bar{\partial}_t)$ ) abusively. Then over $V_0$, we have

$$\begin{aligned}\mathscr{R}_k&=-\Theta_{\mathscr{L},g_\alpha}\otimes\mathbb{1}+\mathbb{1}\otimes\Theta_k\\&=-\Theta_{\mathscr{L},g_\alpha}\otimes\mathbb{1}-\tilde{\theta}^*_{n-k,k}\wedge\tilde{\theta}_{n-k,k}-\tilde{\theta}_{n-k+1,k-1}\wedge\tilde{\theta}^*_{n-k+1,k-1}.\end{aligned} \tag{2.2.7}$$

By the definition of $\tau_k$ in (1.1.1), for any $k=2,\ldots,n$ one has

$$e_k=\tilde{\theta}_{n-k+1,k-1}(\partial_t)(e_{k-1}), \tag{2.2.8}$$

and we can derive the following curvature formula:

$$\begin{aligned}&\langle\mathscr{R}_k(e_k)(\partial_t,\bar{\partial}_t), e_k\rangle_{h^\alpha_g}\\&\quad=-\Theta_{\mathscr{L},g_\alpha}(\partial_t,\bar{\partial}_t)\|e_k\|^2_{h^\alpha_g}\\&\qquad+\langle\tilde{\theta}^*_{n-k,k}(\bar{\partial}_t)\circ\tilde{\theta}_{n-k,k}(\partial_t)(e_k)-\tilde{\theta}_{n-k+1,k-1}(\partial_t)\circ\tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)(e_k), e_k\rangle_{h^\alpha_g}\\&\quad\leq\langle\tilde{\theta}^*_{n-k,k}(\bar{\partial}_t)\circ\tilde{\theta}_{n-k,k}(\partial_t)(e_k), e_k\rangle_{h^\alpha_g}\\&\qquad-\langle\tilde{\theta}_{n-k+1,k-1}(\partial_t)\circ\tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)(e_k), e_k\rangle_{h^\alpha_g}\\&\quad\overset{(2.2.8)}{=}\|e_{k+1}\|^2_{h^\alpha_g}-\|\tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)(e_k)\|^2_{h^\alpha_g}\end{aligned}$$

$$\leq \|e_{k+1}\|^2_{h^\alpha_g} - \frac{|\langle \tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)(e_k), e_{k-1}\rangle_{h^\alpha_g}|^2}{\|e_{k-1}\|^2_{h^\alpha_g}} \qquad \text{(Cauchy–Schwarz inequality)}$$

$$= \|e_{k+1}\|^2_{h^\alpha_g} - \frac{|\langle e_k, \tilde{\theta}_{n-k+1,k-1}(\partial_t)(e_{k-1})\rangle_{h^\alpha_g}|^2}{\|e_{k-1}\|^2_{h^\alpha_g}}$$

$$\overset{(2.2.8)}{=} \|e_{k+1}\|^2_{h^\alpha_g} - \frac{\|e_k\|^4_{h^\alpha_g}}{\|e_{k-1}\|^2_{h^\alpha_g}} \overset{(2.2.5)}{=} G^{k+1}_{k+1} - \frac{G^{2k}_k}{G^{k-1}_{k-1}}$$

Putting this into (2.2.6), we obtain (2.2.4). ∎

**Remark 2.8.** For the final stage $E^{0,n}$ of the Higgs bundle $(\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$. We make the convention that $G_{n+1} \equiv 0$. Then the Gaussian curvature for $G_n$ in (2.2.6) is always semi-negative, which is similar to the Griffiths curvature formula for Hodge bundles in [31].

When $k = 1$, by (2.2.6) one has

$$-\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} / G_1 \leq \frac{\langle \mathscr{R}_1(e_1)(\partial_t, \bar{\partial}_t), e_1\rangle_{h^\alpha_g}}{\|e_1\|^4_{h^\alpha_g}}$$

$$\overset{(2.2.7)}{=} \frac{-\Theta_{\mathscr{L},g_\alpha}(\partial_t, \bar{\partial}_t)}{\|e_1\|^2_{h^\alpha_g}} + \frac{\langle \tilde{\theta}^*_{n-1,1}(\bar{\partial}_t) \circ \tilde{\theta}_{n-1,1}(\partial_t)(e_1) - \tilde{\theta}_{n,0}(\partial_t) \circ \tilde{\theta}^*_{n,0}(\bar{\partial}_t)(e_1), e_1\rangle_{h^\alpha_g}}{\|e_1\|^4_{h^\alpha_g}}$$

$$\overset{(2.2.8)}{\leq} \frac{-\Theta_{\mathscr{L},g_\alpha}(\partial_t, \bar{\partial}_t)\|e_1\|^2_{h^\alpha_g} + \|e_2\|^2_{h^\alpha_g}}{\|e_1\|^4_{h^\alpha_g}}$$

$$= \frac{-\Theta_{\mathscr{L},g_\alpha}(\partial_t, \bar{\partial}_t)}{\|e_1\|^2_{h^\alpha_g}} + \left(\frac{G_2}{G_1}\right)^2.$$

We need the following lemma to control the negative term in the above inequality.

**Lemma 2.9.** *When $\alpha \gg 0$, there exists a* universal *constant $c > 0$ such that for any $\gamma : \mathbb{D} \to V$ with $\gamma(\mathbb{D}) \cap V_0 \neq \emptyset$, one has*

$$\frac{\Theta_{\mathscr{L},g_\alpha}(\partial_t, \bar{\partial}_t)}{\|e_1\|^2_{h^\alpha_g}} \geq c.$$

*In particular,*

$$-\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} / G_1 \leq -c + \left(\frac{G_2}{G_1}\right)^2.$$

*Proof.* By Proposition 1.3(ii), it suffices to prove that

$$\frac{\gamma^*(r_D^{-2}\cdot\omega_\alpha)(\partial_t,\bar\partial_t)}{\|e_1\|^2_{h_g^\alpha}} \geq c. \tag{2.2.9}$$

Note that

$$\frac{\gamma^*(r_D^{-2}\cdot\omega_\alpha)(\partial_t,\bar\partial_t)}{\|e_1\|^2_{h_g^\alpha}} = \frac{\gamma^*(\omega_\alpha)(\partial_t,\bar\partial_t)}{\gamma^*(r_D^2)\cdot\|e_1\|^2_{h_g^\alpha}} = \frac{\gamma^*\omega_\alpha(\partial_t,\bar\partial_t)}{\gamma^*\tau_1^*(r_D^2\cdot h_g^\alpha)(\partial_t,\bar\partial_t)},$$

where $\tau_1^*(r_D^2\cdot h_g^\alpha)$ is the Finsler metric on $\mathscr{T}_Y(-\log D)$ defined by (2.1.1). By Proposition 1.3(iii), $\omega_\alpha$ is a positive definite hermitian metric on $\mathscr{T}_Y(-\log D)$. Since $Y$ is compact, there exists a *uniform constant* $c>0$ such that

$$\omega_\alpha \geq c\tau_1^*(r_D^2\cdot h_g^\alpha).$$

We have thus obtained the desired inequality (2.2.9). ∎

In summary, we have the following curvature estimate for the Finsler metrics $F_1,\dots,F_n$ defined in (2.2.2), which is similar to [54, Lemma 9] for the Weil–Petersson metric.

**Proposition 2.10.** *Let $\gamma:\mathbb{D}\to V$ be such that $\gamma(\mathbb{D})\cap V_1\neq\emptyset$. Assume that $G_k\not\equiv 0$ for $k=1,\dots,q$, and $G_{q+1}\equiv 0$ (thus $G_j\equiv 0$ for all $j>q+1$). Then $q\geq 1$, and over $C:=\gamma^{-1}(V_1)$, which is the complement of a discrete set in $\mathbb{D}$, one has*

$$-\frac{\partial^2\log G_1}{\partial t\partial\bar t}/G_1 \leq -c+\left(\frac{G_2}{G_1}\right)^2, \tag{2.2.10}$$

$$-\frac{\partial^2\log G_k}{\partial t\partial\bar t}/G_k \leq \frac{1}{k}\left(-\left(\frac{G_k}{G_{k-1}}\right)^{k-1}+\left(\frac{G_{k+1}}{G_k}\right)^{k+1}\right) \quad \forall 1<k\leq q. \tag{2.2.11}$$

*Here the constant $c>0$ does not depend on the choice of $\gamma$.*

### 2.3. *Construction of the Finsler metric*

By Proposition 2.10, we observe that none of the Finsler metrics $F_1,\dots,F_n$ defined in (2.2.2) is negatively curved. Following the similar strategies in [7, 54, 58], we construct a new Finsler metric $F$ (see (2.3.6) below) by defining a convex sum of all $F_1,\dots,F_n$, to cancel the positive terms in (2.2.10) and (2.2.11) by negative terms in the next stage. By Remark 2.8, we observe that the last highest order term is always semi-negative. We mainly follow the computations in [54], and try to make this subsection as self-contained as possible. Let us first recall the following basic inequalities by Schumacher.

**Lemma 2.11** ([53, Lemma 8]). *Let $V$ be a complex manifold, and let $G_1,\dots,G_n$ be non-negative $\mathscr{C}^2$ functions on $V$. Then*

$$\sqrt{-1}\,\partial\bar\partial\log\Big(\sum_{i=1}^n G_i\Big) \geq \frac{\sum_{j=1}^n G_j\sqrt{-1}\,\partial\bar\partial\log G_j}{\sum_{i=1}^n G_i} \tag{2.3.1}$$

**Lemma 2.12** ([54, Lemma 17]). *Let $\alpha_j > 0$ for $j = 1, \ldots, n$. Then for all $x_j \geq 0$,*

$$\sum_{j=2}^{n}(\alpha_j x_j^{j+1} - \alpha_{j-1}x_j^{j})x_{j-1}^2 \cdot \ldots \cdot x_1^2$$
$$\geq \frac{1}{2}\Big(-\frac{\alpha_1^3}{\alpha_2^2}x_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}x_n^2 \cdot \ldots \cdot x_1^2 + \sum_{j=2}^{n-1}\Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Big)x_j^2 \cdot \ldots \cdot x_1^2\Big) \tag{2.3.2}$$

Set $x_j := G_j/G_{j-1}$ for $j = 2, \ldots, n$ and $x_1 := G_1$ where $G_j \geq 0$ for $j = 1, \ldots, n$. Put these into (2.3.2) to obtain

$$\sum_{j=2}^{n}\Big(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1}\frac{G_j^{j}}{G_{j-1}^{j-2}}\Big)$$
$$\geq \frac{1}{2}\Big(-\frac{\alpha_1^3}{\alpha_2^2}G_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2 + \sum_{j=2}^{n-1}\Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Big)G_j^2\Big). \tag{2.3.3}$$

The following technical lemma is crucial in constructing our negatively curved Finsler metric $F$.

**Lemma 2.13** ([54, Lemma 10]). *Let $F_1, \ldots, F_n$ be Finsler metrics on a complex space $X$, with holomorphic sectional curvatures $K_1, \ldots, K_n$. Then for the holomorphic sectional curvature of the Finsler metric $F := (F_1^2 + \cdots + F_n^2)^{1/2}$, one has*

$$K_F \leq \frac{\sum_{j=1}^{n} K_j F_j^4}{F^4}. \tag{2.3.4}$$

*Proof.* For any holomorphic map $\gamma : \mathbb{D} \to X$, we denote by $G_1, \ldots, G_n$ the semi-positive functions on $\mathbb{D}$ such that

$$\gamma^* F_i^2 = \sqrt{-1}\, G_i dt \wedge d\bar{t}$$

for $i = 1, \ldots, n$. Then

$$\gamma^* F^2 = \sqrt{-1}\Big(\sum_{i=1}^{n} G_i\Big)dt \wedge d\bar{t},$$

and it follows from (2.1.3) that the Gaussian curvature of $\gamma^* F^2$ satisfies

$$K_{\gamma^* F^2} = -\frac{1}{\sum_{i=1}^{n} G_i}\frac{\partial^2 \log(\sum_{i=1}^{n} G_i)}{\partial t \partial \bar{t}}$$
$$\overset{(2.3.1)}{\leq} -\frac{1}{(\sum_{i=1}^{n} G_i)^2}\sum_{j=1}^{n} G_j \frac{\partial^2 \log G_j}{\partial t \partial \bar{t}} \leq \frac{\sum_{j=1}^{n} K_j G_j^2}{(\sum_{i=1}^{n} G_i)^2}.$$

The lemma follows from Definition 2.3(i). ∎

For any $\gamma : \mathbb{D} \to V$ with $C := \gamma^{-1}(V_1) \neq \emptyset$, we define a hermitian pseudo-metric $\sigma := \sqrt{-1}\, H(t) dt \wedge d\bar{t}$ on $\mathbb{D}$ by taking a convex sum in the following form:

$$H(t) := \sum_{k=1}^{n} k\alpha_k G_k(t),$$

where $G_k$ is defined in (2.2.3), and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ are some *universal constants* which will be fixed later. Following the similar estimate in [54, Proposition 11], one can choose those constants so that the Gaussian curvature $K_\sigma$ of $\sigma$ is uniformly bounded.

**Proposition 2.14.** *There exist* universal *constants* $0 < \alpha_1 \leq \cdots \leq \alpha_n$ *and* $K > 0$ (*independent of* $\gamma : \mathbb{D} \to V$) *such that*

$$K_\sigma \leq -K \quad \text{on } C.$$

*Proof.* It follows from (2.3.4) that

$$K_\sigma \leq \frac{1}{H^2} \sum_{j=1}^{n} j\alpha_j K_j G_j^2 \quad \text{with} \quad K_j := -\frac{\partial^2 \log G_j}{\partial t \partial \bar{t}} / G_j .$$

By Proposition 2.10, one has

$$\begin{aligned}
K_\sigma &\leq \frac{\alpha_1 G_1^2}{H^2}\left(-c + \left(\frac{G_2}{G_1}\right)^2\right) + \frac{1}{H^2}\sum_{j=2}^{n} \alpha_j G_j^2 \left(-\left(\frac{G_j}{G_{j-1}}\right)^{j-1} + \left(\frac{G_{j+1}}{G_j}\right)^{j+1}\right) \\
&\leq \frac{1}{H^2}\left(-c\alpha_1 G_1^2 - \sum_{j=2}^{n}\left(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1}\frac{G_j^j}{G_{j-1}^{j-2}}\right)\right) \\
&\overset{(2.3.3)}{\leq} \frac{1}{H^2}\left(\left(-c + \frac{1}{2}\frac{\alpha_1^2}{\alpha_2^2}\right)\alpha_1 G_1^2 + \frac{1}{2}\sum_{j=2}^{n-1}\left(\frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} - \frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}}\right) G_j^2 - \frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2\right) \\
&=: -\frac{1}{H^2}\sum_{j=1}^{n} \beta_j G_j^2.
\end{aligned}$$

One can take $\alpha_1 = 1$, and choose the further $\alpha_j > \alpha_{j-1}$ inductively such that $\min_j \beta_j > 0$. Set $\beta_0 := \min_j \frac{\beta_j}{(j\alpha_j)^2}$. Then

$$\begin{aligned}
K_\sigma &\leq -\frac{1}{H^2}\beta_0 \sum_{j=1}^{n} (j\alpha_j G_j)^2 \leq -\frac{\beta_0}{nH^2}\left(\sum_{j=1}^{n} j\alpha_j G_j\right)^2 \\
&= -\frac{\beta_0}{n} =: -K.
\end{aligned}$$

Note that $\alpha_1, \ldots, \alpha_n$ and $K$ are universal. The lemma is thus proved. ∎

It follows from Proposition 2.14 and (2.1.3) that

$$\frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} \geq K H(t) \geq 0 \tag{2.3.5}$$

over the Zariski dense open set $C \subseteq \mathbb{D}$, and in particular $\log H(t)$ is a subharmonic function over $C$. Since $H(t) \in [0, +\infty[$ is continuous (in particular locally bounded from above) over $\mathbb{D}$, $\log H(t)$ is a subharmonic function over $\mathbb{D}$, and the estimate (2.3.5) holds over the whole $\mathbb{D}$.

In summary, we construct a *negatively curved Finsler metric* $F$ on $Y \setminus D$, defined by

$$F := \Big(\sum_{k=1}^{n} k\alpha_k F_k^2\Big)^{1/2}, \tag{2.3.6}$$

where $F_k$ is defined in (2.2.2), such that $\gamma^* F^2 = \sqrt{-1}\, H(t) dt \wedge d\bar{t}$ for any $\gamma : \mathbb{D} \to V$. Since we assume that $\tau_1$ is injective over $V_0$, the Finsler metric $F_1$ is positive definite on $V_0$, and *a fortiori* so is $F$. Therefore, the proof of Theorem 2.6 is finished.

### 2.4. *Proof of Theorem B*

By Theorem 1.2, there is a VZ Higgs bundle over some birational model $\tilde{V}$ of $V$. By Theorem D and Theorem 2.6, we can associate to this VZ Higgs bundle a negatively curved Finsler metric which is positive definite over some Zariski dense open set of $\tilde{V}$. The theorem follows directly from the bimeromorphic criterion for pseudo Kobayashi hyperbolicity in Lemma 2.4.

**Remark 2.15.** Let me mention that Sun and Zuo also have a similar idea of constructing a Finsler metric over the base using Viehweg–Zuo Higgs bundles combined with To–Yeung's method [58].

## 3. Kobayashi hyperbolicity of the base

In this section we will prove Theorem C. We first refine Viehweg–Zuo's result on the positivity of direct images. We then apply this result to take a different branch covering in the construction of VZ Higgs bundles to prove the Kobayashi hyperbolicity of the base in Theorem C.

### 3.1. *Preliminaries on positivity of direct images*

We first recall a *pluricanonical extension theorem* due to Cao [17, Theorem 2.10]. Its proof is a combination of the Ohsawa–Takegoshi–Manivel $L^2$-extension theorem with the semi-positivity of the $m$-relative Bergman metric studied by Berndtsson–Păun [5, 6] and Păun–Takayama [45].

**Theorem 3.1** (Pluricanonical $L^2$-extension). *Let $f : X \to Y$ be an algebraic fiber space such that the Kodaira dimension of the general fiber is non-negative. Assume that $f$ is smooth over a dense Zariski open subset of $Y_0 \subset Y$ such that both $B := Y \setminus Y_0$ and $f^*B$ are normal crossing. Let $L$ be any pseudo-effective line bundle $L$ on $X$ equipped with a positively curved singular metric $h_L$ with algebraic singularities such that*

(i) *there exists some regular value $z \in Y$ of $f$ such that for some $m \in \mathbb{N}$, all the sections in $H^0(X_z, (mK_X + L)_{\restriction X_z})$ extend locally near $z$;*
(ii) $H^0(X_z, (mK_{X_z} + L_{\restriction X_z}) \otimes \mathscr{J}(h^{1/m}_{L\restriction X_z})) \neq \emptyset$.

*Then for any regular value $y$ of $f$ such that*

(i) *all sections in $H^0(X_y, mK_{X_y} + L_{\restriction X_y})$ extend locally near $y$;*
(ii) *the metric $h_{L\restriction X_y}$ is not identically equal to $+\infty$,*

*the restriction map*

$$H^0(X, mK_{X/Y} - m\Delta_f + L + f^*A_Y) \twoheadrightarrow H^0(X_y, (mK_{X_y} + L_{\restriction X_y}) \otimes \mathscr{J}(h^{1/m}_{L\restriction X_y}))$$

*is surjective. Here $A_Y$ is a* universal *ample line bundle on $Y$ which does not depend on $L$, $f$ or $m$, and*

$$\Delta_f := \sum_j (a_j - 1)V_j. \tag{3.1.1}$$

*where the sum is taken over all prime divisors $V_j$ of $f^*B$ with multiplicity $a_j$ and with image $f(V_j)$ being a divisor in $Y$.*

We will apply a technical result of [18, Claim 3.5] to prove Theorem 3.7(i). Let us first recall some definitions of singularities of divisors from [61, Chapter 5.3] in a slightly different language.

**Definition 3.2.** Let $X$ be a smooth projective variety, and let $\mathscr{L}$ be a line bundle such that $H^0(X, \mathscr{L}) \neq \emptyset$. One defines

$$e(\mathscr{L}) = \sup\{1/c(D) \mid D \in |\mathscr{L}| \text{ is an effective divisor}\} \tag{3.1.2}$$

where

$$c(D) := \sup\{c > 0 \mid (X, c \cdot D) \text{ is a klt divisor}\}$$

is the *log-canonical threshold* of $D$.

Viehweg showed that one can control the lower bound of $e(\mathscr{L})$.

**Lemma 3.3** ([61, Corollary 5.11]). *Let $X$ be a smooth projective variety equipped with a very ample line bundle $\mathscr{H}$, and let $\mathscr{L}$ be a line bundle such that $H^0(X, \mathscr{L}) \neq \emptyset$.*

(i) *There is a uniform estimate*

$$e(\mathscr{L}) \leq c_1(\mathscr{H})^{\dim X - 1} \cdot c_1(\mathscr{L}) + 1. \tag{3.1.3}$$

(ii) *Let $Z := X \times \cdots \times X$ ($r$ factors) and $\mathscr{M} := \bigotimes_{i=1}^r \mathrm{pr}_i^* \mathscr{L}$. Then $e(\mathscr{M}) = e(\mathscr{L})$.*

Let us recall the following result by Cao–Păun [18].

**Lemma 3.4** (Cao–Păun). *Let $f : X \to Y$ be an algebraic fiber space such that the Kodaira dimension of the general fiber is non-negative. Assume that $f$ is smooth over a dense Zariski open subset of $Y_0 \subset Y$ such that both $B := Y \setminus Y_0$ and $f^*B$ are normal crossing. Then there exists a positive integer $C \geq 2$ such that for any $m \geq m_0$ and $a \in \mathbb{N}$, any $y \in Y_0$ and any section*

$$\sigma \in H^0(X_y, amCK_{X_y}),$$

*there exists a section*

$$\Sigma \in H^0(X, f^*A_Y - af^* \det f_*(mK_{X/Y}) + amr_mCK_{X/Y} + a(P_m + F_m)) \tag{3.1.4}$$

*whose restriction to the fiber $X_y$ is equal to $\sigma^{\otimes r_m}$. Here $F_m$ and $P_m$ are effective divisors on $X$ (independent of $a$) such that $F_m$ is $f$-exceptional with $f(F_m) \subset \mathrm{Supp}(B)$, $\mathrm{Supp}(P_m) \subset \mathrm{Supp}(\Delta_f)$, $r_m := \mathrm{rank}\, f_*(mK_{X/Y})$, and $A_Y$ is the universal ample line bundle on $Y$ defined in Theorem 3.1.*

We recall the definition of a *Kollár family of varieties with semi-log-canonical singularities* (*slc family* for short).

**Definition 3.5** (slc family). An *slc family* is a flat proper morphism $f : X \to B$ such that

(i) each fiber $X_b := f^{-1}(b)$ is a projective variety with slc singularities;
(ii) $\omega_{X/B}^{[m]}$ is flat;
(iii) the family $f : X \to B$ satisfies the *Kollár condition*, which means that, for any $m \in \mathbb{N}$, the reflexive power $\omega_{X/B}^{[m]}$ commutes with arbitrary base change.

To make Definition 3.5(iii) precise, for every base change $\tau : B' \to B$, given the induced morphism $\rho : X' = X \times_B B' \to X$ we note that the natural homomorphism $\rho^*\omega_{X/B}^{[m]} \to \omega_{X'/B'}^{[m]}$ is an isomorphism.

Let us collect the basic properties of slc families, well-known to experts.

**Lemma 3.6.** *Let $g : Z \to W$ be a surjective morphism between quasi-projective manifolds with connected fibers, which is birational to an slc family $g' : Z' \to W$ whose generic fiber has at most Gorenstein canonical singularities.*

(i) *The total space $Z'$ is normal and has only canonical singularities at worst.*
(ii) *If $\nu : W' \to W$ is a dominant morphism with $W'$ smooth quasi-projective, then $Z' \times_W W' \to W'$ is still an slc family whose generic fiber has at most Gorenstein canonical singularities, and is birational to $(Z \times_W W')\tilde{} \to W'$.*
(iii) *Denote by $Z'^r$ the $r$-fold fiber product $Z' \times_W \cdots \times_W Z'$. Then $g'^r : Z'^r \to W$ is also an slc family whose generic fiber has at most Gorenstein canonical singularities. Moreover, $Z'^r$ is birational to the main component $(Z^r)\tilde{}$ of $Z^r$ dominating $W$.*
(iv) *Let $Z^{(r)}$ be a desingularization of $(Z^r)\tilde{}$. Then*

$$(g^{(r)})_*(\ell K_{Z^{(r)}/W}) \simeq (g'^r)_*(\ell K_{Z'^r/W})$$

*is reflexive for every sufficiently divisible $\ell > 0$.*

### *3.2. Positivity of direct images*

This section is devoted to proving Theorem 3.7 on positivity of direct images, which refines results by Viehweg–Zuo [28, Proposition 3.4] and [63, Proposition 4.3]. It will be crucially used to prove Theorem C.

**Theorem 3.7.** *Let $f_0 : X_0 \to Y_0$ be a smooth family of projective manifolds of general type. Assume that for any $y \in Y_0$, the set of $z \in Y_0$ with $X_z \overset{\text{bir}}{\sim} X_y$ is finite.*

(i) *For any smooth projective compactification $f : X \to Y$ of $f_0 : X_0 \to Y_0$ and any sufficiently ample line bundle $A_Y$ over $Y$, $f_*(\ell K_{X/Y})^{\star\star} \otimes A_Y^{-1}$ is globally generated over $Y_0$ for any $\ell \gg 0$. In particular, $f_*(\ell K_{X/Y})$ is ample with respect to $Y_0$.*

(ii) *In the setting of* (i), $\det f_*(\ell K_{X/Y}) \otimes A_Y^{-r_\ell}$ *is also globally generated over $Y_0$ for any $\ell \gg 0$, where $r_\ell = \operatorname{rank} f_*(\ell K_{X/Y})$. In particular, $\mathbf{B}_+(\det f_*(\ell K_{X/Y})) \subset Y \setminus Y_0$.*

(iii) *For some $r \gg 0$, there exists an algebraic fiber space $f : X \to Y$ compactifying $X_0^r \to Y_0$ such that $f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}$ is globally generated over $Y_0$ for $\ell$ large and divisible enough. Here $X_0^r$ denotes the $r$-fold fiber product of $X_0 \to Y_0$, and $A_Y$ is some sufficiently ample line bundle over $Y$.*

*Proof.* Let us first show that to prove (i) and (ii), one can assume that both $B := Y \setminus Y_0$ and $f^*B$ are normal crossing.

For an arbitrary smooth projective compactification $f' : X' \to Y'$ of $f_0 : X_0 \to Y_0$, we take a log resolution $\nu : Y \to Y'$ with centers supported on $Y' \setminus Y_0$ such that $B := \nu^{-1}(Y' \setminus Y_0)$ is a simple normal crossing divisor. Define $X$ to be a strong desingularization of the main component $(X' \times_{Y'} Y)\tilde{}$ dominant over $Y$

$$
\begin{tikzcd}
X \arrow[r] \arrow[dr, "f"'] & X' \times_{Y'} Y \arrow[r] \arrow[d] & X' \arrow[d, "f'"] \\
& Y \arrow[r, "\nu"] & Y'
\end{tikzcd}
\tag{3.2.1}
$$

so that $f^*B$ is normal crossing. By [60, Lemma 2.5.a], there is the inclusion

$$
\nu_* f_*(mK_{X/Y}) \hookrightarrow f'_*(mK_{X'/Y'}), \tag{3.2.2}
$$

which is an isomorphism over $Y_0$ for each $m \in \mathbb{N}$. Hence for any ample line bundle $A$ over $Y'$, once $f_*(mK_{X/Y})^{\star\star} \otimes (\nu^*A)^{-1}$ is globally generated over $\nu^{-1}(Y_0) \simeq Y_0$ for some $m \geq 0$, $f'_*(mK_{X'/Y'})^{\star\star} \otimes A^{-1}$ will also be globally generated over $Y_0$. As we will see, (ii) is a direct consequence of (i). This proves the above statement.

(i) Let us fix a sufficiently ample line bundle $A_Y$ on $Y$. Assume that both $B := Y \setminus Y_0$ and $f^*B$ are normal crossing. It follows from [60, Theorem 5.2] that one can take some $b \gg a \gg 0$, $\mu \gg m \gg 0$ and $s \gg 0$ such that $\mathscr{L} := \det f_*(\mu m K_{X/Y})^{\otimes a} \otimes \det f_*(mK_{X/Y})^{\otimes b}$ is ample over $Y_0$. In other words, $\mathbf{B}_+(\mathscr{L}) \subset \operatorname{Supp}(B)$. By the definition of augmented base locus, one can even find $a, b \gg 0$ such that there exists a singular hermitian metric $h_1$ on $\mathscr{L} - 4A_Y$ which is smooth over $Y_0$, and the curvature current satisfies $\sqrt{-1}\,\Theta_{h_{\mathscr{L}}}(\mathscr{L}) \geq \omega$ for some Kähler form $\omega$ in $Y$. Denote $r_1 := \operatorname{rank} f_*(\mu m K_{X/Y})$

and $r_2 := \operatorname{rank} f_*(mK_{X/Y})$. It follows from Lemma 3.4 that for any sections

$$\sigma_1 \in H^0(X_y, a\mu mCK_{X_y}), \quad \sigma_2 \in H^0(X_y, bmCK_{X_y}),$$

there exist effective divisors $\Sigma_1$ and $\Sigma_2$ such that

$$\Sigma_1 + af^* \det f_*(m\mu K_{X/Y}) - f^*A_Y \overset{\text{linear}}{\sim} am\mu r_1 CK_{X/Y} + P_1 + F_1,$$
$$\Sigma_2 + bf^* \det f_*(mK_{X/Y}) - f^*A_Y \overset{\text{linear}}{\sim} bmr_2 CK_{X/Y} + P_2 + F_2,$$

and

$$\Sigma_{1 \restriction X_y} = \sigma_1^{\otimes r_1}, \quad \Sigma_{2 \restriction X_y} = \sigma_2^{\otimes r_2}.$$

Here $F_i$ is $f$-exceptional with $f(F_i) \subset \operatorname{Supp}(B)$, $\operatorname{Supp}(P_i) \subset \operatorname{Supp}(\Delta_f)$ for $i = 1, 2$.

Write $N := am\mu r_1 C + bmr_2 C$, $P := P_1 + P_2$ and $F := F_1 + F_2$. Fix any $y \in Y_0$. Then the effective divisor $\Sigma_1 + \Sigma_2$ induces a singular hermitian metric $h_2$ on the line bundle $L_2 := NK_{X/Y} - f^*\mathscr{L} + 2f^*A_Y + P + F$ such that $h_{2\restriction X_y}$ is not identically equal to $+\infty$, and the singular hermitian metric $h := f^*h_1 \cdot h_2$ over $L_0 := L_2 + f^*\mathscr{L} - 4f^*A_Y = NK_{X/Y} - 2f^*A_Y + P + F$ is not identically $+\infty$ either. In particular, for $\ell$ sufficiently large, the multiplier ideal sheaf $\mathscr{J}(h^{1/\ell}_{\restriction X_y})$ equals $\mathscr{O}_{X_y}$. By Siu's invariance of plurigenera, all global sections in $H^0(X_y, (\ell K_X + L_0)_{\restriction X_y}) \simeq H^0(X_y, (\ell + N)K_{X_y})$ extend locally, and we can thus apply Theorem 3.1 to obtain the desired surjectivity

$$H^0(X, \ell K_{X/Y} + L_0 - \ell\Delta_f + f^*A_Y) \twoheadrightarrow H^0(X_y, (\ell + N)K_{X_y}). \tag{3.2.3}$$

Recall that $\operatorname{Supp}(P) \subset \operatorname{Supp}(\Delta_f)$. Then $\ell f^*B \geq P$ for $\ell \gg 0$, and one has the inclusion of sheaves

$$\ell K_{X/Y} + L_0 - \ell\Delta_f + f^*A_Y \hookrightarrow (N + \ell)K_{X/Y} - f^*A_Y + F,$$

which is an isomorphism over $X_0$ . By (3.2.3) this implies that the direct image sheaves $f_*(\ell K_{X/Y} - f^*A_Y + F)$ are globally generated over some Zariski open set $U_y \subset Y_0$ containing $y$ for $\ell \gg 0$. Since $y$ is an arbitrary point in $Y_0$, the direct image $f_*(\ell K_{X/Y} + F) \otimes A_Y^{-1}$ is globally generated over $Y_0$ for $\ell \gg 0$ by noetherianity. Recall that $F$ is $f$-exceptional with $f(F) \subset \operatorname{Supp}(B)$. Then there is an injection

$$f_*(\ell K_{X/Y} + F) \otimes A_Y^{-1} \hookrightarrow f_*(\ell K_{X/Y})^{\star\star} \otimes A_Y^{-1}$$

which is an isomorphism over $Y_0$. Hence $f_*(\ell K_{X/Y})^{\star\star} \otimes A_Y^{-1}$ is also globally generated over $Y_0$. Hence $f_*(\ell K_{X/Y})$ is ample with respect to $Y_0$ for $\ell \gg 0$. The first claim follows.

(ii) The trick proving the second claim has already appeared in [23] in proving a conjecture by Demailly–Peternell–Schneider. We first recall that $f_*(\ell K_{X/Y})$ is locally free outside a codimension 2 analytic subset of $Y$. By the proof of Theorem 3.7(i), for $\ell$ sufficiently large and divisible, $f_*(\ell K_{X/Y} + F) \otimes A_Y^{-1}$ is locally free and generated by global sections over $Y_0$, where $F$ is some $f$-exceptional effective divisor. Therefore, its determinant $\det f_*(\ell K_{X/Y} + F) \otimes A_Y^{-r_\ell}$ is also globally generated over $Y_0$, where $r_\ell := \operatorname{rank} f_*(\ell K_{X/Y})$. Since $F$ is $f$-exceptional and effective, one has

$$\det f_*(\ell K_{X/Y} + F) \otimes A_Y^{-r_\ell} = \det f_*(\ell K_{X/Y}) \otimes A_Y^{-r_\ell},$$

and therefore $\det f_*(\ell K_{X/Y}) \otimes A_Y^{-r\ell}$ is also globally generated over $Y_0$. By the very definition of the augmented base locus $\mathbf{B}_+(\bullet)$ we conclude that

$$\mathbf{B}_+(\det f_*(\ell K_{X/Y})) \subset \operatorname{Supp}(B).$$

The second claim is proved.

(iii) We combine the ideas in [63, Proposition 4.1] as well as the pluricanonical extension techniques of Theorem 3.1. By Corollary A.2, there exists a smooth projective compactification $Y$ of $Y_0$ with $B := Y \setminus Y_0$ simple normal crossing, a non-singular finite covering $\psi : W \to Y$, and an slc family $g' : Z' \to W$ which extends the family $X_0 \times_{Y_0} W$. By Lemma 3.6(iii), for any $r \in \mathbb{Z}_{>0}$, the $r$-fold fiber product $g'^r : Z'^r \to W$ is still an slc family, which compactifies the smooth family $X_0^r \times_{Y_0} W \to W_0$, where $W_0 := \psi^{-1}(Y_0)$. Note that $Z'^r$ has canonical singularities.

Take a smooth projective compactification $f : X \to Y$ of $X_0^r \to Y_0$ such that $f^*B$ is normal crossing. Let $Z \to Z'^r$ be a strong desingularization of $Z'^r$, which also resolves this birational map $Z'^r \dashrightarrow (X \times_Y W)\tilde{}$. Then $g : Z \to W$ is smooth over $W_0 := \psi^{-1}(Y_0)$.

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "g"] \arrow[rrr, bend left] & Z'^r \arrow[r, dashed] \arrow[d, "g'^r"] & X \arrow[d, "f"] & (X \times_Y W)\tilde{} \arrow[l] \arrow[d] \\
W \arrow[r, equal] & W \arrow[r, "\psi"] & Y & W \arrow[l, "\psi"']
\end{tikzcd}$$

Let $\tilde{Z}$ be a strong desingularization of $Z'$, which is thus smooth over $W_0 := \psi^{-1}(Y_0)$. For the new family $\tilde{g} : \tilde{Z} \to W$, we denote $\tilde{Z}_0 := \tilde{g}^{-1}(W_0)$. Then $\tilde{Z}_0 \to W_0$ is also a smooth family, and any fiber of $Z_w$ with $w \in W_0$ is a projective manifold of general type. By our assumption in the theorem, for any $w \in W_0$, the set of $w' \in W_0$ with $\tilde{Z}_{w'} \overset{\text{bir}}{\sim} \tilde{Z}_w$ is finite as $\psi : W \to Y$ is a finite morphism. We can thus apply Theorem 3.7(i, ii) to our new family $\tilde{g} : \tilde{Z} \to W$.

From now on, we will always assume that $\ell \gg 0$ is sufficiently divisible so that $\ell K_{Z'}$ is Cartier. Let $A_Y$ be a sufficiently ample line bundle over $Y$, so that $A_W := \psi^* A_Y$ is also *sufficiently ample*. Since $Z'$ has canonical singularity, $\tilde{g}_*(\ell K_{\tilde{Z}/W}) = g'_*(\ell K_{Z'/W})$. It follows from Theorem 3.7(ii) that, for any $\ell \gg 0$, the line bundle

$$\det \tilde{g}_*(\ell K_{\tilde{Z}/W}) \otimes A_W^{-r} = \det g'_*(\ell K_{Z'/W}) \otimes A_W^{-r} \tag{3.2.4}$$

is globally generated over $W_0$, where $r := \operatorname{rank} g'_*(\ell K_{Z'/W})$ depends on $\ell$. Then there exists a positively curved singular hermitian metric $h_{\det}$ on the line bundle $\det g'_*(\ell K_{Z'/W}) \otimes A_W^{-r}$ such that $h_{\det}$ is smooth over $W_0$.

By the base change properties of slc families (see [8, Proposition 2.12] and [40, Lemma 2.6]), one has

$$\omega_{Z'^r/W}^{[\ell]} \simeq \bigotimes_{i=1}^{r} \operatorname{pr}_i^* \omega_{Z'/W}^{[\ell]}, \qquad g_*'^r(\ell K_{Z'^r/W}) \simeq \bigotimes^r g'_*(\ell K_{Z'/W}),$$

where $\mathrm{pr}_i : Z'^r \to Z'$ is the $i$-th directional projection map. Hence $\ell K_{Z'^r}$ is Cartier as well, and we have

$$\bigotimes^r g'_*(\ell K_{Z'/W}) \simeq g'^r_*(\ell K_{Z'^r/W}) = g_*(\ell K_{Z/W}).$$

By Lemma 3.6(iv), $g_*(\ell K_{Z/W})$ is reflexive, and we thus have

$$\det g'_*(\ell K_{Z'/W}) \to \bigotimes^r g'_*(\ell K_{Z'/W}) \simeq g_*(\ell K_{Z/W}),$$

which induces a natural effective divisor

$$\Gamma \in |\ell K_{Z/W} - g^* \det g'_*(\ell K_{Z'/W})|$$

such that $\Gamma_{\restriction Z_w} \neq 0$ for any (smooth) fiber $Z_w$ with $w \in W_0$. By Lemma 3.3 for any $w \in W_0$ the log-canonical threshold

$$\begin{aligned} c(\Gamma_{\restriction Z_w}) &\geq \frac{1}{e(\ell K_{Z_w})} = \frac{1}{e(\bigotimes_{i=1}^r \mathrm{pr}_i^*(K_{Z'_w}^{\otimes \ell}))} = \frac{1}{e(\ell K_{Z'_w})} \\ &\geq \frac{1}{\ell \cdot c_1(\mathscr{A})^{d-1} \cdot c_1(K_{Z'_w}) + 1} \geq \frac{2}{(C-1)\ell} \end{aligned} \tag{3.2.5}$$

for some $C \in \mathbb{N}$ which does not depend on $\ell$ or $w \in W_0$. Denote by $h$ the singular hermitian metric on

$$\ell K_{Z/W} - g^* \det g'_*(\ell K_{Z'/W})$$

induced by $\Gamma$. By the definition of log-canonical threshold, the multiplier ideal sheaf $\mathscr{J}(h_{\restriction Z_w}^{\frac{1}{(C-1)\ell}})$ equals $\mathscr{O}_{Z_w}$ for any fiber $Z_w$ with $w \in W_0$. Let us define a positively curved singular metric $h_{\mathscr{F}}$ for the line bundle $\mathscr{F} := \ell K_{Z/W} - r g^* A_W$ by setting $h_{\mathscr{F}} := h \cdot g^* h_{\det}$. Then $\mathscr{J}(h_{\mathscr{F}\restriction Z_w}^{\frac{1}{(C-1)\ell}}) = \mathscr{O}_{Z_w}$ for any $w \in W_0$.

For any $n \in \mathbb{N}^*$, applying Theorem 3.1 to $n\mathscr{F}$ we obtain the surjectivity

$$H^0(Z, (C-1)n\ell K_{Z/W} + n\mathscr{F} + g^* A_W) \twoheadrightarrow H^0(Z_w, Cn\ell K_{Z_w}) \tag{3.2.6}$$

for all $w \in W_0$. In other words,

$$g_*(C\ell n K_{Z/W}) \otimes A_W^{-(nr-1)}$$

is globally generated over $W_0$ for any $\ell \gg 0$ and any $n \geq 1$.

Since $K_{X_y}$ is big, one thus has

$$r = r_\ell \sim \ell^d \quad \text{as } \ell \to +\infty$$

where $d := \dim Z_w \geq 2$ (if the fibers of $f$ are curves, one can take a fiber product to replace the original family). Recall that $C$ is a constant which does not depend on $\ell$. One can thus *a priori* take an $\ell \gg 0$ such that $r \gg C\ell$. In conclusion, for sufficiently large and divisible $m$,

$$g_*(mK_{Z/W}) \otimes A_W^{-2m} = g_*(mK_{Z/W}) \otimes \psi^* A_Y^{-2m}$$

is globally generated over $W_0$. Therefore, we have a morphism

$$\bigoplus_{i=1}^{N} \psi^* A_Y^m \to g_*(mK_{Z/W}) \otimes \psi^* A_Y^{-m}, \tag{3.2.7}$$

which is surjective over $W_0$. On the other hand, by [60, Lemma 2.5.b], one has the inclusion

$$g_*(mK_{Z/W}) \hookrightarrow \psi^* f_*(mK_{X/Y}),$$

which is an isomorphism over $W_0$. Now, (3.2.7) induces a morphism

$$\bigoplus_{i=1}^{N} \psi_*\mathscr{O}_W \otimes A_Y^m \to \psi_* g_*(mK_{Z/W}) \otimes A_Y^{-m} \to \psi_*\psi^*(f_*(mK_{X/Y})) \otimes A_Y^{-m}, \tag{3.2.8}$$

which is surjective over $Y_0$. Note that even if $f_*(mK_{X/Y})$ is merely a coherent sheaf, the projection formula $\psi_*\psi^*(f_*(mK_{X/Y})) = f_*(mK_{X/Y})) \otimes \psi_*\mathscr{O}_W$ still holds since $\psi$ is finite (see [4, Lemma 5.7]). The trace map

$$\psi_*\mathscr{O}_W \to \mathscr{O}_Y$$

splits the natural inclusion $\mathscr{O}_Y \to \psi_*\mathscr{O}_W$, and is thus surjective. Hence (3.2.8) gives rise to a morphism

$$\bigoplus_{i=1}^{N} \psi_*\mathscr{O}_W \otimes A_Y^m \to \psi_* g_*(mK_{Z/W}) \otimes A_Y^{-m} \xrightarrow{\Phi} f_*(mK_{X/Y}) \otimes A_Y^{-m}, \tag{3.2.9}$$

which is surjective over $Y_0$. By taking $m$ sufficiently large, we may assume that $\psi_*\mathscr{O}_W \otimes A_Y^m$ is generated by its global sections. Then $f_*(mK_{X/Y}) \otimes A_Y^{-m}$ is globally generated over $Y_0$, completing the proof. ∎

Let us prove the following Bertini-type result, which will be used in the proof of Theorem 3.11.

**Lemma 3.8** (A Bertini-type result). *Let $f : X \to Y$ be the projective family in Theorem 3.7*(iii). *Then for any given smooth fiber $X_y$ with $y \in Y_0$, there exists $H \in |\ell K_{X/Y} - \ell f^* A_Y|$ such that $H_{\upharpoonright X_y}$ is smooth. In particular, there is a Zariski open neighborhood $V_0 \supset y$ such that $H$ is smooth over $V_0$.*

*Proof.* By Siu's invariance of plurigenera and Grauert–Grothendieck's "cohomology and base change", we know that $f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}$ is locally free on $Y_0$, and the natural map

$$(f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell})_y \to H^0(X_y, \ell K_{X_y})$$

is an isomorphism for any $y \in Y_0$. Since $K_{X_y}$ is assumed to be semi-ample, one can take $\ell \gg 0$ such that $|\ell K_{X_y}|$ is base point free. By the Bertini theorem, one can take a section $s \in H^0(X_y, \ell K_{X_y})$ whose zero locus is a smooth hypersurface on $X_y$. By Theorem 3.7(iii), one has the surjection

$$H^0(Y, f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}) \twoheadrightarrow (f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell})_y \xrightarrow{\simeq} H^0(X_y, \ell K_{X_y}).$$

Hence there is

$$\sigma \in H^0(X, \ell K_{X/Y} - \ell f^* A_Y) = H^0(Y, f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell})$$

which extends the section $s$. In other words, for the zero divisor $H = (\sigma = 0)$, its restriction to $X_y$ is smooth. Hence there is a Zariski open neighborhood $V_0 \ni y$ such that $H$ is smooth over $V_0$. The lemma is proved. ■

**Remark 3.9.** We do not know how to find a hypersurface $H \in |\ell K_{X/Y} - \ell f^* A_Y|$ whose discriminant locus in $Y$ is normal crossing. We have to blow up the base, $\nu : Y' \to Y$, to achieve this. As $f : X \to Y$ is not flat in general, for the new family

$$\begin{array}{ccccccc} X' & \longrightarrow & (X \times_Y Y')\tilde{} & \longrightarrow & X \times_Y Y' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & = & Y' & = & Y' & \xrightarrow{\nu} & Y \end{array}$$

where $X'$ is a desingularization of $(X \times_Y Y')\tilde{}$, in general

$$\nu^* f_*(\ell K_{X/Y}) \not\subset f'_*(\ell K_{X'/Y'}).$$

In other words, although the discriminant locus of $\nu^* H \to Y'$ is a simple normal crossing divisor in $Y'$, $\nu^* H$ might not lie in $|\ell K_{X'/Y'} - \ell f'^* \nu^* A_Y|$. We will overcome this problem in Theorem 3.11 at the cost of appearance of some $f'$-exceptional divisors.

Since the $\mathbb{Q}$-*mild reduction* in Corollary A.2 holds for any smooth surjective projective morphism with connected fibers and smooth base, it follows from our proof of Theorem 3.7(iii) and Kawamata's theorem [34] that one still has the *generic* global generation as follows.

**Theorem 3.10.** *Let $f_U : U \to V$ be a smooth projective morphism between quasi-projective varieties with connected fibers. Assume that the general fiber $F$ of $f_U$ has semi-ample canonical bundle, and $f_U$ is of maximal variation. Then there is an integer $r \gg 0$ and a smooth projective compactification $f : X \to Y$ of $U^r \to V$ such that $f_*(mK_{X/Y}) \otimes \mathscr{A}^{-m}$ is globally generated over some Zariski open subset of $V$. Here $U^r \to V$ is the $r$-fold fiber product of $U \to V$, and $\mathscr{A}$ is some ample line bundle on $Y$.* ■

### *3.3. Sufficiently many "moving" hypersurfaces*

As we have seen in §1.2 on the construction of VZ Higgs bundles, one has to apply the branch cover trick to construct a negatively twisted Hodge bundle on the compactification of the base, which is well-defined outside a simple normal crossing divisor. This means that the hypersurface $H \in |\ell K_{X/Y} - \ell f^* A_Y|$ in constructing the cyclic cover is smooth over the complement of an SNC divisor of the base. As discussed in Lemma 3.8, in general we cannot perform a simple blow-up of the base to achieve this. In this subsection we will overcome this difficulty by applying the methods of [48, Proposition 4.4]. It will be our basic setup in constructing refined VZ Higgs bundles in §3.4.

**Theorem 3.11.** *Let $X_0 \to Y_0$ be a smooth family of minimal projective manifolds of general type over a quasi-projective manifold $Y_0$. Suppose that for any $y \in Y_0$, the set of $z \in Y_0$ with $X_z \overset{\text{bir}}{\sim} X_y$ is finite. Let $Y \supset Y_0$ be the smooth compactification in Corollary A.2. Fix any $y_0 \in Y_0$ and some sufficiently ample line bundle $A_Y$ on $Y$. Then there exist a birational morphism $\nu : Y' \to Y$ and a new algebraic fiber space $f' : X' \to Y'$ which is smooth over $\nu^{-1}(Y_0)$ and such that for any sufficiently large and divisible $\ell$, one can find a hypersurface*

$$H \in |\ell K_{X'/Y'} - \ell(\nu \circ f')^* A_Y + \ell E| \tag{3.3.1}$$

*with the following properties:*

- *the divisor $D := \nu^{-1}(Y \setminus Y_0)$ is simple normal crossing;*
- *there exists a reduced divisor $S$ in $Y'$ such that $D + S$ is simple normal crossing and $H \to Y'$ is smooth over $Y' \setminus (D \cup S)$;*
- *the exceptional locus* $\mathrm{Ex}(\nu)$ *is contained in* $\mathrm{Supp}(D + S)$*, and $y_0 \notin \nu(D \cup S)$;*
- *the divisor $E$ is effective and $f'$-exceptional with $f'(E) \subset$* $\mathrm{Supp}(D + S)$.

*Moreover, when $X_0 \to Y_0$ is* effectively parametrized *over some open set containing $y_0$, so is the new family $X' \to Y'$.*

*Proof.* The proof is a continuation of that of Theorem 3.7(iii), and we adopt the notations therein. By (3.2.9) and the isomorphism

$$H^0(Z, \ell K_{Z/W} - \ell g^* A_W) \simeq H^0(Z'^r, \ell K_{Z'^r/W} - \ell(g'^r)^* A_W),$$

the morphism $\Phi : \psi_* g_*(\ell K_{Z/W}) \otimes A_Y^{-\ell} \to f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}$ in (3.2.9) gives rise to a natural map

$$\Upsilon : H^0(Z'^r, \ell K_{Z'^r/W} - \ell(g'^r)^* A_W) \to H^0(Y, f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}) \tag{3.3.2}$$

whose image $I$ generates $f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}$ over $Y_0$. Note that $\Upsilon$ is functorial in the sense that it does not depend on the choice of the birational model $Z \to Z'^r$. By the base point free theorem, for any $y \in Y_0$, $K_{X_y}$ is semi-ample, and we can assume that $\ell \gg 0$ is sufficiently large and divisible so that $\ell K_{X/Y}$ is relatively semi-ample over $Y_0$. Hence we can take a section

$$\sigma \in H^0(Z'^r, \ell K_{Z'^r/W} - \ell(g'^r)^* A_W) \tag{3.3.3}$$

such that the zero divisor of

$$\Upsilon(\sigma) \in H^0(X, \ell K_{X/Y} - \ell f^* A_Y) = H^0(Y, f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}),$$

denoted by $H_1 \in |\ell K_{X^{(r)}/Y} - \ell(f^{(r)})^* A_Y|$, is *transverse* to the fiber $X_{y_0}$. Denote by $T$ the *discriminant locus* of $H_1 \to Y$, and set $B := Y \setminus Y_0$. Then $y_0 \notin T \cup B$. Take a log-resolution $\nu : Y' \to Y$ with centers in $T \cup B$ so that both $D := \nu^{-1}(B)$ and $D + S := \nu^{-1}(T \cup B)$ are simple normal crossing. Let $X'$ be a strong desingularization of $(X \times_Y Y')\tilde{}$, and write $f' : X' \to Y'$, which is smooth over $Y'_0 := \nu^{-1}(Y_0)$. Set $X'_0 := f'^{-1}(Y'_0)$. It suffices to show that there exists a hypersurface $H$ in (3.3.1) with $H_{\upharpoonright(\nu\circ f')^{-1}(V)} = H_{1\upharpoonright(f^{(r)})^{-1}(V)}$, where $V := Y \setminus (S' \cup B) \subset Y_0$. Since the birational morphism $\nu$ is isomorphic at $y_0$, we can write $y_0$ as $\nu^{-1}(y_0)$ abusively.

Now we use similar arguments to [48, Proposition 4.4] to prove the existence of $H$ (the authors of [48] apply their methods to *mild morphisms*). Let $W'$ be a strong desingularization of $W \times_Y Y'$ which is finite at $y_0 \in Y'$. Write $W'_0 := \nu'^{-1}(W_0)$. By Lemma 3.6(ii), the new family $Z'' := Z'^r \times_W W' \to W'$ is still an slc family, which compactifies the smooth family $X'_0 \times_{Y'_0} W' \to W'_0$. Let $M'$ be a desingularization of $Z''$ such that it resolves the rational maps to $X'$ as well as $Z$.

$$
\begin{tikzcd}
 & X \arrow[dd, "f"] & & Z \arrow[ll] \arrow[rr] \arrow[dd, "g"] & & Z'^r \arrow[dd, "g'^r"] \\
X' \arrow[ur, "\mu"] \arrow[dd, "f'"] & & M' \arrow[ll] \arrow[rr] \arrow[ur] \arrow[dd, "h'"] & & Z'' \arrow[ur, "\mu'"] \arrow[dd, "g''"] & \\
 & Y & & W \arrow[ll, "\psi"] \arrow[rr, equal] & & W \\
Y' \arrow[ur, "\nu"] & & W' \arrow[ll, "\psi'"] \arrow[rr, equal] \arrow[ur, "\nu'"] & & W' \arrow[ur, "\nu'"] &
\end{tikzcd}
$$

By the properties of slc families, $\mu'^* \omega^{[\ell]}_{Z'^r/W} = \omega^{[\ell]}_{Z''/W'}$, which induces a natural map

$$
\mu^* : H^0(Z'^r, \ell K_{Z'^r/W} - \ell(g'^r)^* A_W) \to H^0(Z'', \ell K_{Z''/W'} - \ell(\nu' \circ g'')^* A_W). \tag{3.3.4}
$$

Since both $Z'^r$ and $Z''$ have canonical singularities, one has the natural morphisms

$$
g_*(\ell K_{Z/W}) \simeq (g'^r)_*(\ell K_{Z'^r/W}), \qquad h'_*(\ell K_{M'/W'}) = g''_*(\ell K_{Z''/W'}).
$$

We can leave out a subvariety of codimension at least 2 in $Y'$ supported on $D + S$ (which thus avoids $y_0$ by our construction) so that $\psi' : W' \to Y'$ becomes a *flat finite* morphism. As discussed at the beginning of the proof, there is also a natural map

$$
\Upsilon' : H^0(Z'', \ell K_{Z''/W'} - \ell(\nu' \circ g'')^* A_W) \to H^0(X', \ell K_{X'/Y'} - \ell(\nu \circ f')^* A_Y) \tag{3.3.5}
$$

as in (3.3.2), obtained by factorizing through $M'$.

Note that for $V := Y \setminus (T \cup B)$, $\nu : \nu^{-1}(V) \xrightarrow{\simeq} V$ is also an isomorphism, and thus the restriction of $X \to Y$ to $V$ is isomorphic to the restriction of $X' \to Y'$ to $\nu^{-1}(V)$.

Hence by our construction, the restriction of $Z''' \to W$ to $\psi^{-1}(V)$ is isomorphic to the restriction of $Z'' \to W'$ to $(\nu \circ \psi')^{-1}(V) = (\nu' \circ \psi)^{-1}(V)$. In particular, under the above isomorphism, for the section $\sigma \in H^0(Z''', \ell K_{Z'''/W} - \ell(g''')^* A_W)$ in (3.3.3) with $\Upsilon(\sigma)$ defining $H_1$, one has

$$\Upsilon(\sigma)_{\restriction f^{-1}(V)} \simeq \Upsilon'(\mu^*\sigma)_{\restriction (\nu\circ f')^{-1}(V)}$$

where $\mu^*$ and $\Upsilon'$ are defined in (3.3.4) and (3.3.5). Denote by $\tilde{H}$ the zero divisor defined by

$$\Upsilon'(\mu^*\sigma) \in H^0(X', \ell K_{X'/Y'} - \ell(\nu \circ f')^* A_Y).$$

Recall that $H_1$ is smooth over $V$, then $\tilde{H}$ is also smooth over $\nu^{-1}(V)$.

Note that $\Upsilon'(\mu^*\sigma) \in H^0(Y', f'_*(\ell K_{X'/Y'}) \otimes \nu^* A_Y^{-\ell})$ is only defined over a big open subset of $Y'$ containing $\nu^{-1}(V)$. Hence it extends to a global section

$$s \in H^0(X', \ell K_{X'/Y'} - \ell(\nu \circ f')^* A_Y + \ell E),$$

where $E$ is an $f'$-exceptional effective divisor with $f'(E) \subset \mathrm{Supp}(D + S)$. Denote by $H$ the hypersurface in $X'$ defined by $s$. Hence $H_{\restriction (\nu\circ f')^{-1}(V)} = \tilde{H}_{\restriction (\nu\circ f')^{-1}(V)}$, which is smooth over $\nu^{-1}(V) = Y' \setminus (D \cup S) \simeq V \ni y_0$. Note that the property of effective parametrization is invariant under fiber product. The theorem follows. ■

### 3.4. *Kobayashi hyperbolicity of the moduli spaces*

In this subsection, for an effectively parametrized smooth family of minimal projective manifolds of general type, we refine the Viehweg–Zuo Higgs bundles of Theorem 1.2 so that we can apply Theorem 2.6 and the bimeromorphic criterion for Kobayashi hyperbolicity in Lemma 2.5 to prove Theorem C.

**Theorem 3.12.** *Let $U \to V$ be an effectively parametrized smooth family of minimal projective manifolds of general type over the quasi-projective manifold $V$. Then for any given point $y \in V$, there exists a smooth projective compactification $Y$ for a birational model $\nu : \tilde{V} \to V$, and a VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta}) \supset (\mathscr{F}, \eta)$ over $Y$ with the following properties:*

- (i) *there is a Zariski open set $V_0$ of $V$ containing $y$ such that $\nu : \nu^{-1}(V_0) \to V_0$ is an isomorphism;*
- (ii) *both $D := Y \setminus \tilde{V}$ and $D + S := Y \setminus \nu^{-1}(V_0)$ are simple normal crossing divisors in $Y$;*
- (iii) *the Higgs bundle $(\tilde{E}, \tilde{\theta})$ has log poles supported on $D \cup S$, that is, $\tilde{\theta} : \tilde{E} \to \tilde{E} \otimes (\log(D + S))$;*
- (iv) *the morphism*

$$\tau_1 : \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-1,1} \tag{3.4.1}$$

*induced by the Higgs subsheaf $(\mathscr{F}, \eta)$ is injective over $V_0$.*

*Proof.* The proof is a continuation of that of Theorem 1.2, and we will adopt the same notations.

We first prove that for any $y \in V$, the set of $z \in V$ with $X_z \overset{\text{bir}}{\sim} X_y$ is finite. Take a polarization $\mathscr{H}$ for $U \to V$ with Hilbert polynomial $h$. Denote by $\mathscr{P}_h(V)$ the set of such pairs $(U \to V, \mathscr{H})$, up to isomorphism and up to fiberwise numerical equivalence for $\mathscr{H}$. By [61, Section 7.6], there exists a coarse quasi-projective moduli scheme $P_h$ for $\mathscr{P}_h$, and thus the family induces a morphism $V \to P_h$. By the assumption that the family $U \to V$ is effectively parametrized, the induced morphism $V \to P_h$ is quasi-finite, which in turn shows that the set of $z \in V$ with $X_z$ isomorphic to $X_y$ is finite. Note that a projective manifold of general type has finitely many minimal models. Hence the set of $z \in V'$ with $X_z \overset{\text{bir}}{\sim} X_y$ is finite as well.

Now we will choose the hypersurface in (1.2.1) carefully so that the cyclic cover construction in Theorem 1.2 can provide the desired refined VZ Higgs bundle. Let $Y' \supset V$ be the smooth compactification in Corollary A.2. By Theorem 3.11, for any given point $y \in V$ and any sufficiently ample line bundle $\mathscr{A}$ on $Y'$, there exists a birational morphism $\nu : Y \to Y'$ and a new algebraic fiber space $f : X \to Y$ such that one can find a hypersurface

$$H \in |\ell\Omega^n_{X/Y}(\log \Delta) - \ell(\nu \circ f)^*\mathscr{A} + \ell E|, \quad n := \dim X - \dim Y, \tag{3.4.2}$$

such that

- the inverse image $D := \nu^{-1}(Y' \setminus V)$ is a simple normal crossing divisor;
- there exists a reduced divisor $S$ such that $D + S$ is simple normal crossing, and $H \to Y$ is smooth over $V_0 := Y \setminus (D \cup S)$;
- the restriction $\nu : \nu^{-1}(V_0) \to V_0$ is an isomorphism;
- the given point $y$ is contained in $V_0$;
- the divisor $E$ is effective and $f$-exceptional with $f(E) \subset \mathrm{Supp}(D + S)$;
- for any $z \in V := \nu^{-1}(V')$, the canonical bundle of the fiber $X_z := f^{-1}(z)$ is big and nef;
- the restricted family $f^{-1}(V_0) \to V_0$ is smooth and effectively parametrized.

Here we set $\Delta := f^*D$ and $\Sigma := f^*S$. Write $\mathscr{L} := \nu^*\mathscr{A}$. Now we take the cyclic cover with respect to $H$ in (3.4.2) instead of that in (1.2.1), and perform the same construction of VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta}) \supset (\mathscr{F}, \tau)$ as in Theorem 1.2. Theorem 3.12(i–iii) can be seen directly from the properties of $H$ and the cyclic construction.

Theorem 3.12(iv) has already appeared in [48, Proposition 2.11] implicitly, and we give a proof here for the sake of completeness. Recall that both $Z$ and $H$ are smooth over $V_0$. Denote

$$H_0 := H \cap f^{-1}(V_0),$$
$$f_0 : X_0 = f^{-1}(V_0) \to V_0, \quad g_0 : Z_0 = g^{-1}(V_0) \to V_0.$$

We have

$$
\begin{aligned}
F^{n,0}_{\restriction V_0} &= f_*(\Omega^n_{X/Y}(\log \Delta)\otimes \mathscr{L}^{-1})_{\restriction V_0} = \mathscr{O}_{V_0},\\
E^{n-1,1}_{\restriction V_0} &= R^1(g_0)_*(\Omega^{n-1}_{Z_0/V_0})\\
&= R^1(f_0)_*\Big(\Omega^{n-1}_{X_0/V_0}\oplus\bigoplus_{i=1}^{\ell-1}\Omega^{n-1}_{X_0/V_0}(\log H_0)\otimes(K_{X_0/V_0}\otimes f_0^*\mathscr{L}^{-1})^{-i}\Big) \qquad (3.4.3)\\
F^{n-1,1}_{\restriction V_0} &= R^1 f_*(\Omega^{n-1}_{X/Y}(\log \Delta)\otimes \mathscr{L}^{-1})_{\restriction V_0}\\
&= R^1(f_0)_*(\Omega^{n-1}_{X_0/V_0}\otimes K^{-1}_{X_0/V_0}) \simeq R^1(f_0)_*(\mathscr{T}_{X_0/V_0}).
\end{aligned}
$$

Hence $\tau_{1\restriction V_0}$ factors through

$$
\begin{aligned}
\tau_{1\restriction V_0} : \mathscr{T}_{V_0} &\xrightarrow{\rho} R^1(f_0)_*(\mathscr{T}_{X_0/V_0}) \xrightarrow{\simeq} R^1(f_0)_*(\Omega^{n-1}_{X_0/V_0}\otimes K^{-1}_{X_0/V_0})\\
&\to R^1(f_0)_*(\Omega^{n-1}_{X_0/V_0}(\log H_0)\otimes K^{-1}_{X_0/V_0}) \to R^1(g_0)_*(\Omega^{n-1}_{Z_0/V_0})\otimes \mathscr{L}^{-1},
\end{aligned}
$$

where $\rho$ is the Kodaira–Spencer map. Although the intermediate objects in the above factorization might not be locally free, the $\mathbb{C}$-linear map induced by the sheaf morphism $\tau_{1\restriction V_0}$ at $z\in V_0$,

$$\tau_{1,z} : \mathscr{T}_{Y,z} \to (\mathscr{L}^{-1}\otimes E^{n-1,1})_z,$$

coincides with the following composition of $\mathbb{C}$-linear maps between finite-dimensional complex vector spaces:

$$
\begin{aligned}
\tau_{1,z} : \mathscr{T}_{Y,z} &\xrightarrow{\rho_z} H^1(X_z,\mathscr{T}_{X_z}) \xrightarrow{\simeq} H^1(X_z,\Omega^{n-1}_{X_z}\otimes K^{-1}_{X_z})\\
&\xrightarrow{j_z} H^1(X_z,\Omega^{n-1}_{X_z}(\log H_z)\otimes K^{-1}_{X_z}) \to H^1(Z_z,\Omega^{n-1}_{Z_z}). \qquad (3.4.4)
\end{aligned}
$$

To prove Theorem 3.12(iv), it then suffices to prove that each linear map in (3.4.4) is injective for any $z\in V_0$.

By the effective parametrization assumption, $\rho_z$ is injective. The map $j_z$ in (3.4.4) is the same as the $H^1$-cohomology map of the short exact sequence

$$0\to K^{-1}_{X_z}\otimes\Omega^{n-1}_{X_z}\to K^{-1}_{X_z}\otimes\Omega^{n-1}_{X_z}(\log H_z)\to K^{-1}_{X_z\restriction H_z}\otimes\Omega^{n-2}_{H_z}\to 0.$$

Observe that $K_{X_z\restriction H_z}$ is big. Indeed, this follows from

$$\operatorname{vol}(K_{X_z\restriction H_z}) = c_1(K_{X_z\restriction H_z})^{n-1} = c_1(K_{X_z})^{n-1}\cdot H_z = \ell c_1(K_{X_z})^n = \ell\operatorname{vol}(K_{X_z}) > 0.$$

Hence $j_z$ is injective by the Bogomolov–Sommese vanishing theorem

$$H^0(H_z, K^{-1}_{X_z\restriction H_z}\otimes\Omega^{d-2}_{H_z}) = 0,$$

as observed in [48]. Since $\psi_z : Z_z\to X_z$ is the cyclic cover obtained by taking the $\ell$-th roots out of the smooth hypersurface $H_z\in|\ell K_{X_z}|$, the morphism $\psi$ is finite. It follows from the degeneration of the Leray spectral sequence that

$$
\begin{aligned}
H^1(Z_z,\Omega^{n-1}_{Z_z}) &\simeq H^1(X_z,(\psi_z)_*\Omega^{n-1}_{Z_z})\\
&= H^1(X_z,\Omega^{n-1}_{X_z})\oplus\bigoplus_{i=1}^{\ell-1}H^1(X_z,\Omega^{n-1}_{X_z}(\log H_z)\otimes K^{-i}_{X_z}). \qquad (3.4.5)
\end{aligned}
$$

The last map in (3.4.4) is therefore injective, because the cohomology group $H^1(X_z, \Omega^{n-1}_{X_z}(\log H_z) \otimes K^{-1}_{X_z})$ is a direct summand of $H^1(Z_z, \Omega^{n-1}_{Z_z})$ by (3.4.5). As a consequence, the composition $\tau_{1,z}$ in (3.4.4) is injective at each point $z \in V_0$. Theorem 3.12(iv) is thus proved. ∎

Let us explain how Lemma 2.5 and Theorems 2.6 and 3.12 imply Theorem C.

*Proof of Theorem C.* We first take a smooth compactification $Y \supset V$ as in Corollary A.2. By Theorem 3.12, for any given point $y \in V$, there exists a birational morphism $\nu : Y' \to Y$ which is isomorphic at $y$, so that $D := Y' \setminus \nu^{-1}(V)$ is a simple normal crossing divisor, and there exists a VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ whose log pole $D + S$ avoids $y' := \nu^{-1}(y)$. Moreover, by Theorem 3.12(iv), $\tau_1$ is injective at $y'$. Applying Theorem 2.6, we can associate to $(\tilde{\mathscr{E}}, \tilde{\theta})$ a Finsler metric $F$ on $\mathscr{T}_{Y'}(-D)$ which is positive definite at $y'$. Moreover, if we think of $F$ as a Finsler metric on $\nu^{-1}(V)$, it is negatively curved in the sense of Definition 2.3(ii). Hence the base $V$ satisfies the conditions in Lemma 2.5, and we conclude that $V$ is Kobayashi hyperbolic. ∎

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## Appendix A. $\mathbb{Q}$-mild reductions

Let us work over the complex number field $\mathbb{C}$.

The main result in this appendix is the following:

**Theorem A.1.** *Let $f_0 : S_0 \to T_0$ be a projective family of smooth varieties with $T_0$ quasi-projective.*

(i) *There are compactifications $S_0 \subset \mathcal{S}$ and $T_0 \subset \mathcal{T}$, with $\mathcal{S}$ and $\mathcal{T}$ Deligne–Mumford stacks with projective coarse moduli spaces, and a projective morphism $f : \mathcal{S} \to \mathcal{T}$ extending $f_0$ which is a Kollár family of slc varieties.*

(ii) *Given a finite subset $Z \subset T_0$, there is a projective variety $W$ and a finite surjective lci morphism $\rho : W \to \mathcal{T}$, unramified over $Z$, such that $\rho^{-1}\mathcal{T}^{sm} = W^{sm}$.*

Here the notion of Kollár family refers to the condition that the sheaf $\omega^{[m]}_{\mathcal{S}/\mathcal{T}}$ is flat and its formation commutes with arbitrary base change for each $m$. We refer the readers to [1, Definition 5.2.1] for further details.

Note that the pull-back family $\mathcal{S} \times_{\mathcal{T}} W \to W$ is a Kollár family of slc varieties compactifying the pull-back $S_0 \times_{T_0} W_0 \to W_0$ of the original family to $W_0 := W \times_{\mathcal{T}} T_0$.

This is applied in the present paper, where some mild regularity assumption on $T_0$ and $W$ is required:

**Corollary A.2** ($\mathbb{Q}$-mild reduction). *Assume further $T_0$ is smooth. For any given finite subset $Z \subset T_0$, there exist*

(i) *a compactification $T_0 \subset \underline{\mathcal{T}}$ with $\underline{\mathcal{T}}$ a regular projective scheme,*

(ii) *a simple normal crossings divisor $D \subset \underline{\mathcal{T}}$ containing $\underline{\mathcal{T}} \setminus T_0$ and disjoint from $Z$,*

(iii) *a finite morphism $W \to \underline{\mathcal{T}}$ unramified outside $D$, and*

(iv) *a Kollár family $S_W \to W$ of slc varieties extending the given family $S_0 \times_{\underline{\mathcal{T}}} W$.*

The significance of these extended families is due to their $\mathbb{Q}$-mildness property. Recall from [2] that a family $S \to T$ is *$\mathbb{Q}$-mild* if whenever $T_1 \to T$ is a dominant morphism with $T_1$ having at most Gorenstein canonical singularities, the total space $S_1 = T_1 \times_S T$ has canonical singularities. It was shown by Kollár–Shepherd-Barron [39, Theorem 5.1] and Karu [32, Theorem 2.5] that Kollár families of slc varieties whose generic fiber has at most Gorenstein canonical singularities are $\mathbb{Q}$-mild.

The main result is proved using moduli of Alexeev stable maps.

Let $V$ be a projective variety. A morphism $\phi : U \to V$ is a *stable map* if $U$ is slc and $K_U$ is $\phi$-ample. More generally, given $\pi : U \to T$, a morphism $\phi : U \to V$ is a *stable map over $T$* or a *family of stable maps parametrized by $T$* if $\pi$ is a Kollár family of slc varieties and $K_{U/T}$ is $\phi \times \pi$-ample. Note that this condition is very flexible and does not require the fibers to be of general type, although key applications in Theorems 3.7(iii) and 3.11 require some positivity of the fibers.

**Theorem A.3** ([30, Theorem 1.5]). *The stable maps form an algebraic stack $M(V)$ locally of finite type over $\mathbb{C}$, each of whose connected components is a proper global quotient stack with projective coarse moduli space.*

The existence of an algebraic stack satisfying the valuative criterion for properness was known to Alexeev, and can also be deduced directly from the results of [1], which presents it as a global quotient stack. The work [30] shows that the stack has bounded, hence proper components, admitting projective coarse moduli spaces. An algebraic approach to these statements is provided in [32, Corollary 1.2].

*Proof of Theorem* A.1. (i) Let $T_0 \subset T$ and $S_0 \subset S$ be projective compactifications with $\pi : S \to T$ extending $f_0$. The family $S_0 \to T_0$ with the injective morphism $\phi : S_0 \to S$ is a family of stable maps into $S$, providing a morphism $T_0 \to M(S)$ which is in fact injective. Let $\mathcal{T}$ be the closure of $T_0$. Since $M(S)$ is proper, $\mathcal{T}$ is proper. Let $\mathcal{S}$ be the pull-back of the universal family along $\mathcal{T} \to M(S/T)$. Then $\mathcal{S} \supset S_0$ is a compactification as needed.

(ii) The existence of $W$ follows from the main result of [41]. ■

*Proof of Corollary* A.2. Consider the coarse moduli space $\underline{\mathcal{T}}$ of the stack $\mathcal{T}$ provided by the first part of the main result. This might be singular, but by Hironaka's theorem we may replace it by a resolution of singularities such that $D_\infty := \underline{\mathcal{T}} \setminus T_0$ is a simple normal crossings divisor. Thus condition (i) is satisfied.

For each component $D_i \subset D_\infty$ denote by $m_i$ the ramification index of $\mathcal{T} \to \underline{\mathcal{T}}$. In particular any covering $W \to \underline{\mathcal{T}}$ whose ramification indices over $D_i$ are divisible by $m_i$ lifts along the generic point of $D_i$ to $\mathcal{T}$.

Choosing a Kawamata covering package [2] disjoint from $Z$ we obtain a simple normal crossings divisor $D$ as required by (ii), and a finite covering $W \to \underline{\mathcal{T}}$ as required by (iii), such that $W \to \underline{\mathcal{T}}$ factors through $\mathcal{T}$ at every generic point of $D_i$.

By the Purity Lemma [3, Lemma 2.4.1] the morphism $W \to \mathcal{T}$ extends over all of $W$, hence we obtain a family $S_W \to W$ as required by (iv). ■

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