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The core entropy for polynomials of higher degree

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Abstract. As defined by W. Thurston, the *core entropy* of a polynomial is the entropy of the restriction to its Hubbard tree. For each $d \geq 2$, we study the core entropy as a function on the parameter space of polynomials of degree d, and prove it varies continuously both as a function of the combinatorial data and of the coefficients of the polynomials. This confirms a conjecture of W. Thurston.

Keywords. Entropy, complex dynamics, complex polynomials, core entropy

1. Introduction

A classical way to measure the topological complexity of a dynamical system is its *entropy*. In particular, to each real polynomial map f one can associate the topological entropy of f as a dynamical system on the real line [22].

If $f : \mathbb{C} \to \mathbb{C}$ is a complex polynomial map, then the real line is no longer invariant, and it becomes less obvious how to define a notion of entropy for f. However, if f is *postcritically finite* (i.e., the forward orbits of the critical points are finite) then there is a canonical tree inside the complex plane, known as the *Hubbard tree* H_f, which is invariant under forward iteration [8].

In order to generalize the theory of entropy to complex polynomials, W. Thurston defined the *core entropy* of f as the topological entropy of the restriction of f to its Hubbard tree:

$$h(f) := h_{\text{top}}(f|_{H_f}).$$

Thurston conjectured that the core entropy is a continuous function of the polynomial. For quadratic polynomials, this was proven in [30] and [9].

In this paper, we generalize this result by developing the theory of core entropy for polynomials of any degree $d \geq 2$, and proving that it varies continuously over parameter space.

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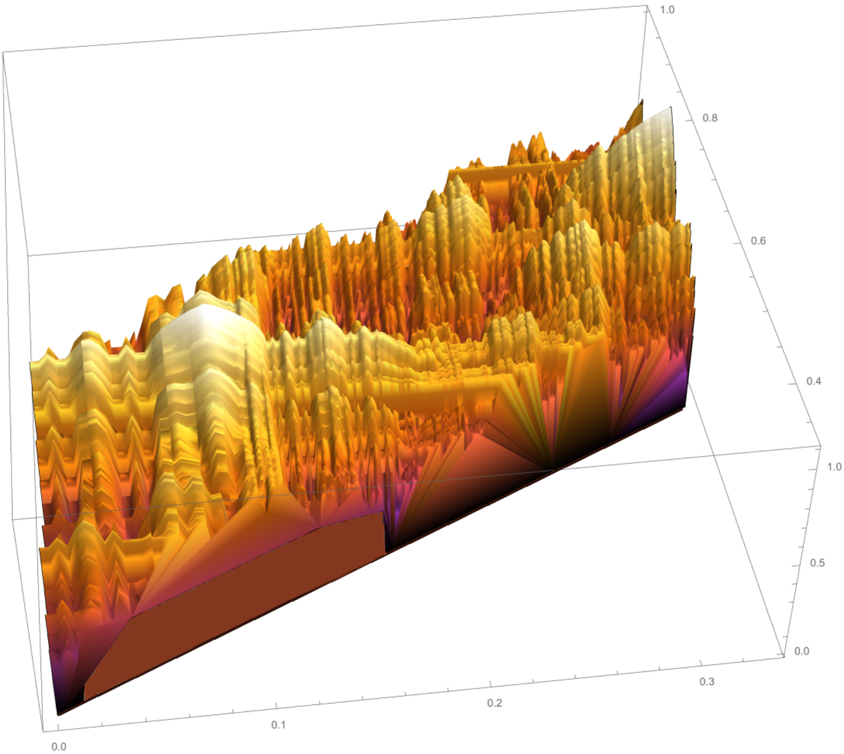


Fig. 1. The core entropy of cubic polynomials on the space of primitive majors. Because of the symmetries of parameter space, it is enough to restrict attention to the domain $0 \leq a \leq 1/3$, $1/3 \leq b \leq 1$.

In order to describe the global topology of the space of polynomials of a given degree, W. Thurston defined the space $\mathrm{PM}(d)$ of *primitive majors* of degree d [28]. The combinatorial parameter space $\mathrm{PM}(d)$ generalizes to higher degree the circle at infinity for the Mandelbrot set: it is always compact and has interesting topology (for instance, $\mathrm{PM}(d)$ is a $K(B_d, 1)$ where B_d is the braid group [28]; see also Section 1.3). Rational primitive majors are associated to postcritically finite polynomials: essentially, one records the major leaf for each critical point. Thus, we can assign to each primitive major m the *core entropy* $h(m)$ of the associated polynomial. We prove that the core entropy extends to a continuous function on the combinatorial parameter space:

Theorem 1.1. *The core entropy function $h(m)$ extends to a continuous function on the set* $\mathrm{PM}(d)$ *of primitive majors of degree d.*

In fact, we shall define a notion of core entropy for *any* primitive major, and show that it varies continuously over the parameter space $\mathrm{PM}(d)$. The previous result is purely combinatorial, as the core entropy can be computed from the combinatorial data of the primitive major m.

In the second part of the paper, we will address the dependence of the core entropy on the coefficients of the polynomial. Let us fix $d \geq 2$, and let us consider the space $\mathcal{P}_d$ of monic, centered, polynomials of degree d. A polynomial is *centered* if the barycenter of its roots is the origin. Inside $\mathcal{P}_d$ sits the *connectedness locus* $\mathcal{M}_d$, the set of polynomials with connected Julia set. While it is conjectured that the Mandelbrot set is locally connected [8], it is known that the connectedness locus for cubics is not locally connected ([16], [20], [10]).

We say that a sequence (f_n) of polynomials in $\mathcal{P}_d$ converges to a polynomial f if the coefficients of f_n converge to the coefficients of f. We will show that the core entropy is continuous not only as a function of the combinatorial parameters but also as a function of the coefficients:

Theorem 1.2. *Let $d \geq 2$. Then the core entropy $h(f)$ is a continuous function on the space of postcritically finite polynomials of degree d.*

To put our result in context, let us note that in going from the quadratic to the general degree case, the parameter space has complex dimension > 1, and there are several cases in dynamics where continuity fails in higher dimension. On the combinatorial side, recall that for cubic polynomials combinatorial rigidity fails, in the sense that, even among maps without indifferent cycles, the rational lamination is inadequate to determine the dynamics [13]. In a more analytic context, the action of the mapping class group on Teichmüller space $\mathcal{T}(S)$ extends continuously to the boundary if and only if $\dim_{\mathbb{C}} \mathcal{T}(S) = 1$ [15]. Similarly, Thurston's pullback map for polynomials fails to extend continuously to the Thurston boundary in higher dimensions ([25], [5]).

1.1. History

The study of topological entropy for real, quadratic polynomials goes back to the seminal work of Milnor–Thurston [22], who proved that it depends continuously and monotonically on the parameter. Alternative proofs are also given in [7], [33].

Two types of generalization of these results are possible: on the one hand, for complex quadratic polynomials generalizations of the real line are *veins* in the Mandelbrot set $\mathcal{M}$. In fact, it is known that the core entropy is monotone, increasing from the center of the Mandelbrot set to the tips ([17], [29], [36]). Thus, the core entropy function is intimately related to the topological structure of the Mandelbrot set: in fact, sublevels of the entropy function can be used to define wakes in $\mathcal{M}$.

On the other hand, a considerable amount of work has gone into understanding entropy for real polynomials of higher degree. In particular, it was conjectured by Milnor that the entropy for real polynomials of a given degree d is also monotone, in the sense that isentropic curves in parameter space are connected. This was proven by Milnor–Tresser [23] for cubics and by Bruin–van Strien [4] for general degrees.

The present paper is one of the first attempts to study the core entropy for higher-degree polynomials which are not real. Note that some of the techniques used in the quadratic case do not generalize, as we cannot use the vein structure of the Mandelbrot set. In fact, our proof of continuity does not rely at all on the understanding of the topology of the connectedness locus: on the other hand, we believe the core entropy may be a useful tool to define and investigate the hierarchical structure of the connectedness locus, which is much less understood than in the quadratic case.

Let us recall that the notion of core entropy can be extended to topologically finite polynomials, as in [29]. A polynomial f is *topologically finite* if it has connected, locally connected Julia set and its Hubbard tree H_f has finitely many ends (hence it is a finite topological tree). In that case, the Hubbard tree is a compact, forward invariant set, hence one can define in the usual way the topological entropy of the restriction of f to it. Note that topologically finite, not postcritically finite polynomials are abundant: for instance, every polynomial along a vein of the Mandelbrot set is topologically finite [29, Proposition 4.5].

In fact, for topologically finite polynomials, the core entropy is related by the simple formula (see [29], [3])

$$\text{H.dim } B(f) = \frac{h(f)}{\log d}$$

to the dimension of the set $B(f)$ of biaccessible angles, which has been studied extensively, e.g. in [34], [35], [26], [3], [19].

A more general way to define core entropy may be using Bowen's definition h_{Bow} of topological entropy for not necessarily compact sets [1]. Namely, if we denote by J_f^{bi} the set of points in the Julia set of f which have at least two rays landing on them, we can define $h(f) := h_{\text{Bow}}(f, J_f^{\text{bi}})$. For topologically finite polynomials, this coincides with the core entropy as defined above.

Finally, another definition of core entropy is given by [9], essentially as the growth rate of the number of precritical leaves in the lamination associated to f. It is fairly simple to show that all these definitions agree (in particular, they agree with the definition proposed and used in the current paper) for postcritically finite polynomials (see also [28]), and it would be very interesting to compare them for general polynomials.

1.2. The techniques

In our approach to core entropy, we avoid using the geometry or topology of parameter space, which can get extremely complicated, and rather we describe the combinatorics of polynomials through the use of laminations as in [27], [28].

In the degree 2 case, there is only one critical point, hence the combinatorics of a polynomial is captured by one *minor leaf*. In the higher degree case, however, there are more than one critical points, hence to each polynomial one associates a finite lamination, called a *critical portrait*, and to that a graph and a growth rate which we next describe.

The main difficulty to overcome when compared to the quadratic case is that the space of critical portraits is partitioned into *strata* given by the multiplicity and mutual position of the critical points, and each stratum is non-closed (see also Section 1.4). A recursive formula for the number of strata for each degree has been obtained by Tomasini [32].

For instance, a sequence of polynomials with distinct critical points can converge to a polynomial with critical points of higher multiplicity, and the number of critical leaves used to represent it can change. To overcome this, we develop the theory of *weak critical markings*, and we prove that *all* possible limit critical portraits have the same entropy, even though they belong to different strata.

In more detail, we use an algorithm devised by Thurston [28] in order to compute the core entropy without the need to understand the topology of the Hubbard tree. In fact, Gao Yan [11] proved that the algorithm yields the correct value of the core entropy for all postcritically finite polynomials of any degree (see also [14]).

(1) We define the *growth rate* for each critical portrait ξ as follows. Given a primitive major, we construct an infinite graph Γ_ξ (called a *wedge*) whose vertices are the pairs of postcritical angles, and whose edges are given by the action of the dynamics on the space of arcs in the Hubbard tree between postcritical points.

(2) We then associate to this infinite graph its *growth rate* $r(\xi)$ by considering the growth rate of the number of closed paths in the graph:

$$r(\xi) := \limsup_{n\to\infty} (\#\{\text{closed paths in } \Gamma_\xi \text{ of length } n\})^{1/n}$$

and prove that this number is a zero of a convergent power series $P(t)$, which we call the *spectral determinant*.

(3) We prove that the growth rate $r(\xi)$ is continuous on the space of critical portraits.

(4) For rational critical portraits ξ, we prove that the growth rate is related to the core entropy $h(\xi)$ given by Thurston's algorithm (Lemma 8.2), namely

$$h(\xi) = \log r(\xi).$$

This establishes Theorem 1.1 and concludes the combinatorial part of the paper.

To get the second main result (Theorem 1.2), let us consider a sequence $f_n \to f$ of postcritically finite polynomials of degree d. We study the variation in the landing points of the external rays corresponding to the portrait.

(1) Following Poirier [24], we construct for each postcritically finite polynomial a rational critical portrait, known as *critical marking*. Let us pick for each f_n a critical marking, which we denote by Θ_n.

(2) We then study the possible limits of the sequence (Θ_n): it turns out that limits of critical markings for f_n need not be critical markings for f, since more rays than expected can land on the same critical point (see Example 9.17). For this reason, we introduce the more general notion of *weak critical marking*, and prove (Proposition 9.16) that

$$\text{each limit } \Theta_\infty \text{ of the sequence } (\Theta_n) \text{ is a weak critical marking of } f.$$

This is the analytic part of the paper, as it requires controlling the convergence of landing rays as parameters change. In fact, the parts of the marking associated to Fatou critical points and to Julia critical points have to be dealt with separately.

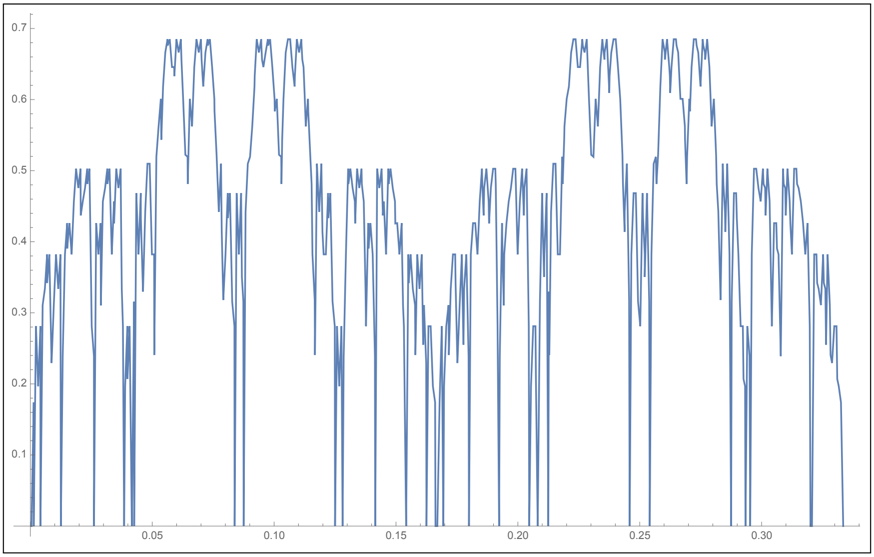


Fig. 2. The core entropy for *unicritical* cubic polynomials. The critical portraits are of the form $\{(a, a + 1/3, a + 2/3)\}$ with $0 \leq a \leq 1/3$. The maxima reach the value $\log 2$.

(3) We use [11] to conclude that Thurston's algorithm also gives the correct value of core entropy for weak critical markings, hence $h(\Theta_n) = h(f_n)$ and also $h(\Theta_\infty) = h(f)$.

(4) By the first combinatorial part, $h(\Theta_n) \to h(\Theta_\infty)$, hence

$$h(f_n) = h(\Theta_n) \to h(\Theta_\infty) = h(f),$$

which completes the proof of Theorem 1.2.

1.3. The space of primitive majors

Before delving into the proofs, we explore the structure and topology of the set $\mathrm{PM}(d)$ of primitive majors in degree 2 and 3. See also [2], and [28] for the general case.

Recall that a *leaf* in the Poincaré disk $\overline{\mathbb{D}}$ is a hyperbolic geodesic connecting two points of $\partial\mathbb{D}$. Moreover, an *ideal polygon* is a (filled-in) polygon with vertices on $\partial\mathbb{D}$ and with geodesic sides for the hyperbolic metric. (In fact, it will not be important to consider hyperbolic as opposed to euclidean geodesics, but we will draw pictures for the hyperbolic metric.)

Definition 1.3. A *critical portrait* of degree d is a collection

$$\xi = \{\ell_1, \ldots, \ell_s\}$$

of leaves and ideal polygons in $\overline{\mathbb{D}}$ fulfilling the following conditions:

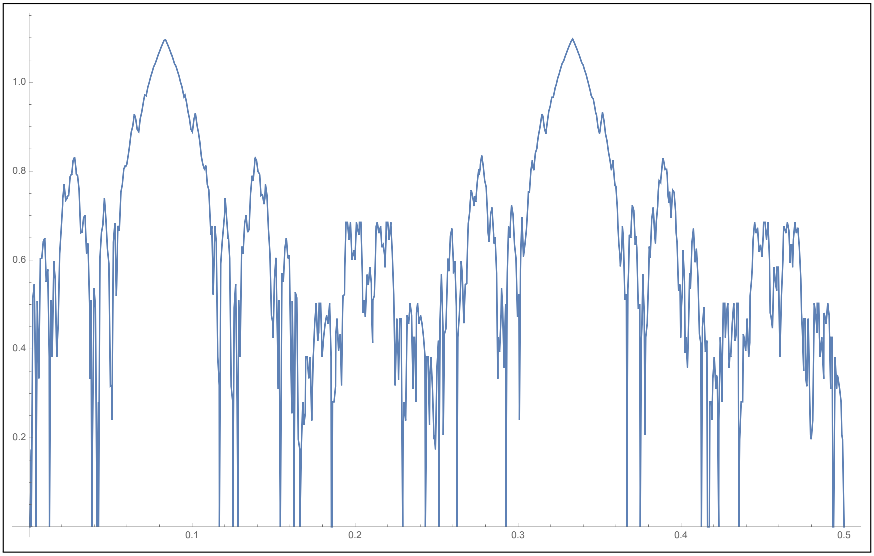


Fig. 3. The core entropy for *symmetric* cubic polynomials. The critical portraits are of the form $\{(a, a+1/3), (a+1/2, a+5/6)\}$ with $0 \le a \le 1/2$. There are two maxima for $a = 1/12$ and $a = 1/3$. These are also global maxima over the whole space PM(3), and the entropy equals log 3.

(1) any two distinct elements ℓ_k and ℓ_l of m either are disjoint or intersect in one point on $\partial\mathbb{D}$;
(2) the vertices of each ℓ_k are identified under $z \mapsto z^d$;
(3) $\sum_{k=1}^{s}(\#(\ell_k \cap \partial\mathbb{D}) - 1) = d - 1$.

Each ℓ_i will be called a *portrait element*, or just an *element* for brevity. The number s is called the *size* of the critical portrait.

The intuition is that ℓ_i contains (a subset of) the set of angles of external rays landing at the i^{th} critical point. Thus, (1) is a *non-crossing* condition, as rays landing at different critical points are disjoint, while (2) is a *critical marking* condition, since all such rays map under forward iteration to the same ray, which lands at some critical value. Finally, (3) is an analog of the *Riemann–Hurwitz* theorem.

A critical portrait m is said to be a *primitive major* if point (1) in the definition above is strengthened to the statement that the elements of m are pairwise disjoint. We remark that for a critical portrait m of degree d, the set $\mathbb{D} \setminus \bigcup_{k=1}^{s} \ell_k$ has d connected components, and each one takes a total arc length $1/d$ on the unit circle (see [11, Lemma 4.2]).

For $d = 2$, a primitive major is simply a diameter of the circle. Thus, each primitive major is parameterized by an angle $\theta \in \mathbb{R}/\mathbb{Z}$, and, since the angles θ and $\theta + 1/2$ determine the same diameter, the parameter space PM(2) is homeomorphic to $\mathbb{P}^1(\mathbb{R})$, hence to a circle.

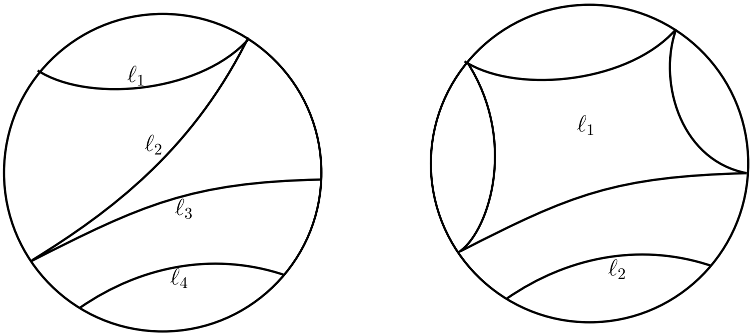


Fig. 4. Critical portraits of degree 5. The one on the right is a primitive major, while the one on the left is not a primitive major, but it *induces* the primitive major on the right (see Section 3.1).

For $d = 3$, a cubic polynomial has either one critical point of multiplicity 2, or two critical points of multiplicity 1. Hence, there are two types of primitive majors: either an ideal triangle, or two leaves. To each pair $(a, b) \in S^1 \times S^1$ one can associate the pair of leaves $\{(a, a + 1/3), (b, b + 1/3)\}$, and since the leaves cannot cross each other, one gets the strip

$$S := \{(a, b) \in S^1 \times S^1 : a + 1/3 \leq b \leq a + 2/3\},$$

which is the parameter space displayed in Figure 1. There are two particularly important slices:

(1) The *unicritical* slice S_1. It corresponds to the family $f(z) = z^3 + c$: each polynomial has only one critical point. Combinatorially, it is represented by the slice $b = a + 1/3$ and the primitive majors are of the form $\{(a, a + 1/3, a + 2/3)\}$. See Figure 2 for the graph of the restriction of the core entropy to this slice.

(2) The *symmetric* slice S_2. It corresponds to the family $f(z) = z^3 + cz$, and each polynomial is an odd function. Combinatorially, it is represented by $b = a + 1/2$, so the associated primitive majors are $\{(a, a + 1/3), (a + 1/2, a + 5/6)\}$. See Figure 3.

In order to account for the symmetries, note that a and b are interchangeable, so one can restrict to $a + 1/3 \leq b \leq a + 1/2$, getting the annulus

$$S' := \{(a, b) \in S^1 \times S^1 : a + 1/3 \leq b \leq a + 1/2\}.$$

The annulus S' has two boundary components, corresponding to the two slices S_1 and S_2. One sees by the above discussion that the slice S_1 is invariant under a rotation of period 3 on $S^1 \times S^1$, because the pairs $(a, a + 1/3)$, $(a + 1/3, a + 2/3)$ and $(a + 2/3, a)$ yield the same primitive major. Moreover, the pairs $(a, a + 1/2)$ and $(a + 1/2, a)$ also yield the same primitive major, hence the slice S_2 is invariant under a period 2 rotation.

Thus, the parameter space PM(3) is homeomorphic to the quotient of the annulus where one of the boundary components wraps around twice, and the other boundary circle wraps around three times. In formulas,

$$\mathrm{PM}(3) = \{(a, b) \in S^1 \times S^1 : a + 1/3 \leq b \leq a + 1/2\}/\sim$$

where $(a, a+1/3) \sim (a+1/3, a+2/3) \sim (a+2/3, a)$ and $(a, a+1/2) \sim (a+1/2, a)$. The resulting space is not quite a manifold, as a neighborhood of the unicritical locus S_1 contains three "sheets" which come together.

1.4. Stratification

It is clear from the above discussion that $\mathrm{PM}(d)$ has a natural stratification based on the size of the elements of the primitive major. Namely, let $s_1, \dots, s_r$ be integers with $\sum_{i=1}^{r}(s_i - 1) = d - 1$. Then one can define the stratum $\Pi(s_1, \dots, s_r)$ as the set of primitive majors in $\mathrm{PM}(d)$ which have elements of size $s_1, \dots, s_r$. In the above discussion of the cubic locus, the unicritical locus is the stratum $\Pi(3)$, while the generic stratum is $\Pi(2, 2)$. A natural question then becomes:

Question. What is the maximum of core entropy on each stratum? How many (and which) polynomials achieve the maximum?

In general, the global maximum on $\mathrm{PM}(d)$ equals $\log d$, while as an example in the unicritical locus $\Pi(3)$ the maximum is $\log 2$; this follows from the general upper bound of Proposition 8.3. In fact, a finer stratification of $\mathrm{PM}(d)$ can be given in terms of *major trees* (see Section 8.1).

As for the quadratic case, many other questions about core entropy are completely open in higher degree, and it would be of great interest to pursue them. For instance, the local maxima of the core entropy are expected to occur at dyadic angles, while a conjecture analogous to the one in the quadratic case (formulated in [29] and proven in [9]) on the maximum of entropy on the wakes has to be made precise. Moreover, we believe the local Hölder exponent of the entropy function is related to the value of the entropy (see [31] for real quadratic polynomials), and we expect self-similarity features in the graph of the entropy at preperiodic parameters (see [14] for discussion of the quadratic case).

1.5. Structure of the paper

We start in Section 2 by reviewing the techniques in graph theory needed to define the core entropy through the *spectral determinant*. In Section 3, we define the combinatorial parameter space using primitive majors, and recall Thurston's entropy algorithm to compute the entropy. Then (Section 4) we define the infinite graphs, called *wedges*, which we use to encode the combinatorial dynamics in the space of postcritical arcs. In order to study the limits of wedges as parameters vary, we define in Section 5 the concept of *weakly periodic* labeled wedge. In Section 7, we prove that any limit of a sequence of wedges which correspond to a sequence of convergent parameters actually yields the same entropy. Finally, in Section 8 we use this to establish the first main result, namely the continuity in the combinatorial parameter space (Theorem 1.1).

In the second part of the paper (Section 9) we transfer this combinatorial information to the analytic parameter space: there, we establish that if the coefficients of the polynomials converge, then the critical markings also converge in a suitable way. This is achieved by showing continuity properties of the landing points of certain rays, and introducing a generalization of the concept of critical marking (which we call *weak critical marking*), which captures the marking of a limit of postcritically finite polynomials. Using these tools we prove the second main result (Theorem 1.2).

2. Growth rates of graphs of bounded cycles

We start with some background material, following [30, Sections 2, 3].

2.1. Graphs of bounded cycles

In the following, by *graph* we mean a directed graph, i.e., a set $V(\Gamma)$ of *vertices* (which will be finite or countable) and a set $E(\Gamma)$ of *edges*, such that each edge e has a well-defined source $s(e) \in V$ and a target $t(e) \in V$ (thus, we allow edges between a vertex and itself, and multiple edges between two vertices). Given a vertex v, the set $\mathrm{Out}(v)$ of its outgoing edges is the set of edges with source v. The *outgoing degree* of v is the cardinality of $\mathrm{Out}(v)$; a graph has *bounded outgoing degree* if there is a uniform upper bound $d \geq 1$ on the outgoing degree of all its vertices.

A *path* in the graph *based at a vertex* v is a sequence $(e_1, \dots, e_n)$ of edges such that $s(e_1) = v$ and $t(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$. The *length* of the path is the number n of edges, and the set $\{s(e_1), \dots, s(e_n)\} \cup \{t(e_n)\}$ of vertices visited by the path is called its *support*. Similarly, a *closed path* based at v is a path $(e_1, \dots, e_n)$ such that $t(e_n) = s(e_1)$. Note that in this definition closed paths with different starting vertices will be considered to be different.

A *simple cycle* is a closed path which does not self-intersect, modulo cyclical equivalence; that is, a simple cycle is a closed path $(e_1, \dots, e_n)$ such that $s(e_i) \neq s(e_j)$ for $i \neq j$, and two such paths are considered the same simple cycle if the edges are cyclically permuted, i.e., $(e_1, \dots, e_n)$ and $(e_{k+1}, \dots, e_n, e_1, \dots, e_k)$ designate the same simple cycle. Finally, a *multi-cycle* is the union of finitely many simple cycles with pairwise disjoint (vertex-)supports. The length of a multi-cycle is the sum of the lengths of its components.

We say a graph has *bounded cycles* if it has bounded outgoing degree and for each integer $n \geq 1$ it has at most finitely many simple cycles of length n.

Note that if Γ has bounded cycles, then for each n it also has a finite number of closed paths of length n. We shall denote by $C(\Gamma, n)$ the number of closed paths of length n, and define the *growth rate* $r(\Gamma)$ as the exponential growth rate of the number of its closed paths, that is,

$$r(\Gamma) := \limsup_{n \to \infty} \sqrt[n]{C(\Gamma, n)}.$$

2.2. The spectral determinant

Let Γ be a graph with bounded cycles. Let $S(\Gamma, n)$ denote the number of simple multi-cycles of length n in Γ, and let us define

$$\sigma(\Gamma) := \limsup_{n\to\infty} \sqrt[n]{S(\Gamma, n)},$$

the growth rate of $S(\Gamma, n)$. Tiozzo [30, Section 2.1] defined a formal power series, called the *spectral determinant*, as

$$P(t) := \sum_{\gamma \text{ multi-cycle}} (-1)^{C(\gamma)} t^{\ell(\gamma)}, \tag{2.1}$$

where $\ell(\gamma)$ denotes the length of the multi-cycle γ, while $C(\gamma)$ is the number of connected components of γ, and proved that the inverse of the growth rate $r(\Gamma)$ is the minimal zero of $P(z)$:

Lemma 2.1 ([30, Theorem 2.3]). *Suppose $\sigma(\Gamma) \le 1$. Then (2.1) defines a holomorphic function $P(z)$ in the unit disk $|z| < 1$, and moreover the function $P(z)$ is non-zero in the disk $|z| < r(\Gamma)^{-1}$; if $r(\Gamma) > 1$, then also $P(r(\Gamma)^{-1}) = 0$.*

For a finite graph Γ with vertex set $V = \{v_1, \dots, v_q\}$, its *adjacency matrix* $A = (a_{ij})_{q\times q}$ is defined as

$$a_{ij} := \#(v_i \to v_j),$$

the number of edges from v_i to v_j. In this case, the spectral determinant $P(t)$ equals $\det(I - tA)$.

Lemma 2.2 ([30, Lemma 2.5]). *If Γ is a finite graph, then its growth rate $r(\Gamma)$ equals the largest real eigenvalue of its adjacency matrix.*

2.3. Weak cover of graphs

Let Γ_1, Γ_2 be two graphs with bounded cycles. A *graph map* from Γ_1 to Γ_2 is a map $\pi : V(\Gamma_1) \to V(\Gamma_2)$ on the vertex sets and a map on edges $\pi : E(\Gamma_1) \to E(\Gamma_2)$ which is compatible, in the sense that if the edge e connects v to w in Γ_1, then the edge $\pi(e)$ connects $\pi(v)$ to $\pi(w)$ in Γ_2. We shall usually denote such a map as $\pi : \Gamma_1 \to \Gamma_2$.

A *weak cover* of graphs is a graph map $\pi : \Gamma_1 \to \Gamma_2$ such that

- the map $\pi : V(\Gamma_1) \to V(\Gamma_2)$ is surjective;
- the induced map $\pi : \mathrm{Out}(v) \to \mathrm{Out}(\pi(v))$ between outgoing edges is a bijection for each $v \in V(\Gamma_1)$.

As a consequence of the definition of weak cover, one has the following facts:

Lemma 2.3 ([30, Lemmas 3.1 and 3.3]). *Let $\pi : \Gamma_1 \to \Gamma_2$ be a weak cover of graphs with bounded cycles. Then we have the following:*

(1) *The unique path lifting property: given $v \in \Gamma_1$ and $w = \pi(v) \in \Gamma_2$, for every path γ in Γ_2 based at w there is a unique path e in Γ_1 based at v such that $\pi(e) = \gamma$.*

(2) *Let S be a finite set of vertices of Γ_1, and suppose that every closed path in Γ_1 of length n passes through S. Then*

$$C(\Gamma_1, n) \leq n \cdot \#S \cdot C(\Gamma_2, n).$$

2.3.1. Quotient graphs. A general way to construct weak covers of graphs is the following. Suppose we have an equivalence relation $\sim$ on the vertex set V of a graph with bounded cycles, and denote by $V^{\mathcal{Q}}$ the set of equivalence classes of vertices. Such an equivalence relation is called *edge-compatible* if whenever $v_1 \sim v_2$, for any vertex w the total number of edges from v_1 to the members of the equivalence class of w equals that from v_2 to the members of the equivalence class of w. When we have such an equivalence relation, we can define a quotient graph $\Gamma^{\mathcal{Q}}$ with vertex set $V^{\mathcal{Q}}$. Namely, we denote for each $v, w \in V$ the respective equivalence classes as $[v]$ and $[w]$, and define the number of edges from $[v]$ to $[w]$ in the quotient graph to be

$$\#([v] \to [w]) = \sum_{u \in [w]} \#(v \to u).$$

By definition of edge-compatibility, the above sum does not depend on the representative v chosen inside the class $[v]$. Moreover, it is easy to see that the quotient map $\pi : \Gamma \to \Gamma^{\mathcal{Q}}$ is a weak cover of graphs.

3. Thurston's entropy algorithm for rational critical portraits

3.1. Primitive majors and critical portraits

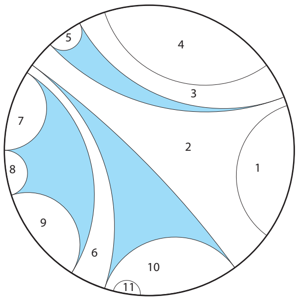
Recall the definition of *critical portrait* and *primitive major* from the introduction (Definition 1.3).

We now prove that each critical portrait induces a unique primitive major of the same degree. To see this, let $\xi = \{\ell_1, \dots, \ell_s\}$ be a critical portrait of degree d. We define an equivalence relation on ξ as the smallest equivalence relation such that if $\ell_i \cap \ell_j \neq \emptyset$, then ℓ_i and ℓ_j are equivalent. The portrait ξ is therefore divided into the equivalence classes $\mathcal{Q}_1, \dots, \mathcal{Q}_t$. For each $i \in \{1, \dots, t\}$, set

$$\Theta_i := \text{the convex hull in } \overline{\mathbb{D}} \text{ of } \bigcup_{\ell_j \in \mathcal{Q}_i} (\ell_j \cap \partial\mathbb{D}).$$

The collection $\{\Theta_1, \dots, \Theta_t\}$ of sets is easily checked to be a degree d primitive major, called the *primitive major induced by* ξ. For example, the primitive major in Figure 4 (right) is induced by the critical portrait on its left.

A critical portrait ξ induces an equivalence relation on $\partial\mathbb{D}$, namely the smallest equivalence relation $\sim$ such that $x \sim y$ whenever x and y belong to the same element of ξ. Two critical portraits are said to be *equivalent* if they induce the same equivalence relation on $\partial\mathbb{D}$. This is the same as saying that they induce the same primitive major.



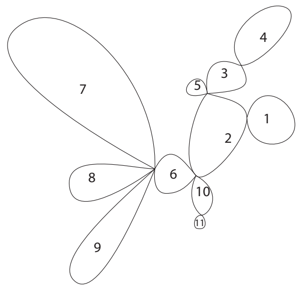


Fig. 5. The deformation of the unit circle by shrinking a degree 7 primitive major.

3.2. The topology on the space of primitive majors

For $d \geq 2$, we denote by $\mathrm{PM}(d)$ the space of all primitive majors of degree d. This space has a canonical metric md given by Thurston (see [28, Part I, Section 3]) as follows.

A primitive major m determines a quotient graph $\gamma(m)$ obtained from $\partial\mathbb{D}$ by identifying each element of m to a point (see Figure 5). The path metric on $\partial\mathbb{D}$ determines a path metric on $\gamma(m)$. Let $\mathrm{met}(m)$ be the pseudometric on $\partial\mathbb{D}$ obtained as pullback of the path metric on $\gamma(m)$ under the projection $\partial\mathbb{D} \to \gamma(m)$. Since pseudometrics on $\partial\mathbb{D}$ are elements of the space $C(\partial\mathbb{D} \times \partial\mathbb{D}, \mathbb{R})$, we may define the distance $d(\mathrm{met}(m), \mathrm{met}(m'))$ as just the usual distance on the space of continuous functions; the metric md on $\mathrm{PM}(d)$ is then defined as the distance between the corresponding pseudometrics: more precisely,

$$\mathrm{md}(m, m') := \sup_{x,y\in\partial\mathbb{D}} |\mathrm{met}(m)(x, y) - \mathrm{met}(m')(x, y)|.$$

We say that a sequence $(m_n)_{n\geq 0}$ of majors *converges* to m if the distance $\mathrm{md}(m_n, m)$ tends to zero.

Example 3.1. Let $d = 3$, and consider the following sequences of majors.

(1) Set $m_n = \left\{\ell_1^n := \left\{\frac{1}{n}, \frac{1}{3} + \frac{1}{n}\right\}, \ell_2^n := \left\{-\frac{1}{n}, \frac{2}{3} - \frac{1}{n}\right\}\right\}$. Then (m_n) converges to the primitive major $m = \{0, \frac{1}{3}, \frac{2}{3}\}$.

(2) Set $m_{2k} = \left\{\ell_1^{2k} := \left\{\frac{1}{2k}, \frac{1}{3} + \frac{1}{2k}\right\}, \ell_2^{2k} := \left\{-\frac{1}{2k}, \frac{2}{3} - \frac{1}{2k}\right\}\right\}$ and $m_{2k-1} = \left\{\ell_1^{2k-1} := \left\{\frac{1}{2k-1}, \frac{1}{3} + \frac{1}{2k-1}, \frac{2}{3} + \frac{1}{2k-1}\right\}\right\}$. Then (m_n) also converges to the major m.

From (2), we see that as (m_n) converges to m in $\mathrm{PM}(d)$, the size of m_n may vary. Hence the sizes of their induced labeled wedges (see Section 4) may also vary. This will cause difficulties in comparing the growth rates of the associated graphs. To solve this problem, we partition the sequence (m_n) into a finite number of convergent subsequences such that the majors in each subsequence have a common type.

For each critical portrait ξ, let $\xi^{\cup}$ denote the union of all elements of ξ, which is a compact subset of $\overline{\mathbb{D}}$. A sequence (ξ_n) of critical portraits is said to *Hausdorff converge* if the sequence $(\xi_n^{\cup})$ of compact sets converges in the Hausdorff distance.

Note that, if the sequence (ξ_n) of critical portraits Hausdorff converges to a compact set A, then for n sufficiently large we can label the elements of each ξ_n by $\ell_1^{(n)}, \dots, \ell_s^{(n)}$ so that for each $k \in \{1, \dots, s\}$ the sequence $(\ell_k^{(n)})_n$ of compact sets converges in the Hausdorff distance to some compact set ℓ_k, which is also the closure of either a leaf or a polygon. Moreover, one has $A = \bigcup_{k=1}^{s} \ell_k$. The set $\xi := \{\ell_1, \dots, \ell_s\}$ is called the *limit* of (ξ_n) in the sense of Hausdorff convergence.

The following proposition shows the relation between convergence and Hausdorff convergence for primitive majors.

Proposition 3.2. *Let (ξ_n) be a sequence of critical portraits which Hausdorff converge to ξ. Then*

(1) *the set ξ is a critical portrait of degree d; and*

(2) *if we let m_n, $n \geq 1$, be the primitive majors induced by ξ_n, and m the primitive major induced by ξ, then the majors (m_n) converge to m.*

Moreover,

(3) *if the majors (m_n) converge to m, the sequence (m_n) can be partitioned into a finite number of Hausdorff convergent subsequences.*

Proof. (1) Since each ℓ_k is the Hausdorff limit of $\ell_k^{(n)}$, we see that the vertices of $\ell_k^{(n)}$ converge to that of ℓ_k, which implies that the vertices of ℓ_k are identified under $z \mapsto z^d$ and $\sum_{k=1}^{s}(\#(\ell_k \cap \partial\mathbb{D}) - 1) = d - 1$, and that each pair ℓ_k, ℓ_l intersects at most on their boundary, which implies that the intersection is either one point on $\partial\mathbb{D}$ or a leaf. The latter case never happens because each component of $\mathbb{D} \setminus \bigcup_{k=1}^{s} \ell_k^{(n)}$ has total length $1/d$ on the unit circle. Then ξ is a critical portrait of degree d.

(2) Let ζ be any critical portrait of degree d, and let m_ζ denote its induced primitive major. Note that the portrait ζ also induces a quotient graph $\gamma(\zeta)$ obtained from $\partial\mathbb{D}$ by identifying each element of ζ to a point, and it coincides with $\gamma(m_\zeta)$. Let $\mathrm{met}(\zeta)$ be the pseudometric on the circle induced by the path metric on $\gamma(\zeta)$. We then get $\mathrm{met}(\xi) = \mathrm{met}(m_\xi)$. For any $\epsilon > 0$ and $x, y \in \partial\mathbb{D}$, as (ξ_n) Hausdorff converges to ξ, by the argument above we have, for n large,

$$|\mathrm{met}(m_n)(x, y) - \mathrm{met}(m)(x, y)| = |\mathrm{met}(\xi_n)(x, y) - \mathrm{met}(\xi)(x, y)| < \epsilon.$$

It follows that (m_n) converges to m.

(3) We consider the accumulation points of the sequence $(m_n^{\cup})$ in the Hausdorff topology. Let A be such an accumulation point and (m_n') a subsequence with $(m_n'^{\cup}) \to A$ in the Hausdorff distance. Then the majors m_n' Hausdorff converge to ξ with $\xi^{\cup} = A$. By assertions (1) and (2), ξ is a critical portrait which induces m. Note that the number of critical portraits that induce m is finite, so the accumulation set of $(m_n^{\cup})$ is finite. Since the space of all compact subsets of a compact set is compact (in the Hausdorff topology),

it follows that the sequence (m_n) can be subdivided into a finite number of Hausdorff convergent subsequences. ■

Note that the Hausdorff limit of primitive majors is not necessarily a primitive major: for example, the majors (m_n) in Example 3.1(1) Hausdorff converge to the critical portrait $\xi = \{\{0, 1/3\}, \{0, 2/3\}\}$. That is why we introduce the concept of critical portraits.

3.3. *Thurston's core entropy algorithm*

We will describe here how Thurston's entropy algorithm works on rational critical portraits (see also [11]).

By abuse of notation, we will identify a point of $\partial\mathbb{D}$ with its argument in $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Then all angles in the circle are considered to be mod 1, i.e. elements of $\mathbb{T}$. The map $\tau : \mathbb{T} \to \mathbb{T}$ is defined by $\tau(\theta) = d\theta \bmod \mathbb{Z}$.

Let $\xi = \{\ell_1, \dots, \ell_s\}$ be a critical portrait of degree d. Each ℓ_k is called an *element* of ξ. We set

$$x_k(i) := \tau^i(\ell_k \cap \mathbb{T}), \qquad i \geq 1,\ k \in \{1, \dots, s\}.$$

Note that the vertices of ℓ_k are identified by τ, so all $x_k(i)$ are points in $\mathbb{T}$. For each $k = 1, \dots, s$, the collection $\{x_k(i) : i \geq 0\}$ is an analog of the forward orbit of the k^{th} critical point.

To describe Thurston's entropy algorithm, we consider the positions of $x_k(i)$ and $x_l(j)$ with respect to ξ. We say that the element ℓ *separates* two points x_1 and x_2 of $\mathbb{T}$ if x_1 and x_2 lie in opposite connected components of $\mathbb{T} \setminus \ell$.

Given an ordered pair of points $(x_k(i), x_l(j))$ with $k, l \in \{1, \dots, s\}$ and $i, j \geq 1$, we say that their *separation vector* (with respect to ξ) is $(\alpha_1, \dots, \alpha_r)$, with $\alpha_i \in \{1, \dots, s\}$ for each i, if

(1) each ℓ_{α_i} belongs to ξ;

(2) the leaf joining $x_k(i)$ and $x_l(j)$ successively crosses the elements $\ell_{\alpha_1}, \dots, \ell_{\alpha_r}$ from $x_k(i)$ to $x_l(j)$;

(3) no other element of ξ separates $x_k(i)$ and $x_l(j)$.

We say that $x_k(i)$ and $x_l(j)$ are *not separated* if its separation vector is empty, and they are *separated* otherwise.

The following fact will be used in the proof of Proposition 5.4.

Lemma 3.3. *Let ξ be a critical portrait, and m the primitive major induced by ξ. Then two points of $\mathbb{T}$ are separated by an element Θ of m if and only if they are separated by an element ℓ of ξ which is contained in Θ.*

Proof. The sufficiency is obvious. For the necessity, let $x, y \in \mathbb{T}$ be separated by an element Θ of m. If $\#\Theta = 2$, then Θ is also an element of ξ, and the conclusion holds. Let $\#\Theta \geq 3$. Then the leaf $\overline{xy}$ intersects two boundary leaves of Θ, denoted by $\overline{\theta\eta}$ and $\overline{\theta'\eta'}$. Fixing θ, there is one angle in $\{\theta', \eta'\}$, say θ', such that any connected subset of $\overline{\mathbb{D}}$ joining θ and θ' intersects $\overline{xy}$. Let ℓ and ℓ' be two elements of ξ that contain θ and θ' respectively.

By the construction of m from ξ, there exist elements $\ell_{i_0} := \ell, \ell_{i_1}, \ldots, \ell_{i_t}, \ell_{i_{t+1}} := \ell'$ of ξ contained in Θ such that $\ell_{i_k} \cap \ell_{i_{k+1}} \neq \emptyset$ for $k \in \{0, \ldots, t\}$. It follows that the connected set $\bigcup_{k=0}^{t+1} \ell_{i_k}$ joins θ and θ', and hence intersects $\overline{xy}$. Consequently, an element of ξ among $\{\ell_{i_0}, \ldots, \ell_{i_{t+1}}\}$ intersects $\overline{xy}$. Then the conclusion holds. ∎

A critical portrait $\xi = \{\ell_1, \ldots, \ell_s\}$ is said to be *rational* if any $\ell \in \xi$ satisfies $\ell \cap \mathbb{T} \subseteq \mathbb{Q}/\mathbb{Z}$.

The algorithm. Let ξ be a rational critical portrait. Then the set

$$\mathcal{P}(\xi) := \{x_k(i) : 1 \leq k \leq s, i \geq 1\}$$

is finite. We define O_ξ as the set of all unordered pairs $\{x, y\}$ with $x \neq y \in \mathcal{P}(\xi)$ if $\#\mathcal{P}(\xi) \geq 2$, and consisting of only $\{x, x\}$ if $\mathcal{P}(\xi) = \{x\}$. Then O_ξ is finite but not empty.

The following is *Thurston's entropy algorithm* with input a rational critical portrait ξ.

(1) Let Σ_ξ be the abstract linear space over $\mathbb{R}$ generated by the elements of O_ξ.

(2) Define a linear map $\mathcal{A}_\xi : \Sigma_\xi \to \Sigma_\xi$ such that for any basis vector $\{x, y\} \in O_\xi$,

   (a) $\mathcal{A}_\xi(\{x, y\}) = 0$ if x, y belong to a common element ℓ_k of ξ;

   (b) $\mathcal{A}_\xi(\{x, y\}) = \{\tau(x), \tau(y)\}$ if x, y are not separated by ξ and do not belong to a common element of ξ; and

   (c) $\mathcal{A}_\xi(\{x, y\}) = \{\tau(x), \tau(\ell_{\alpha_1})\} + \{\tau(\ell_{\alpha_1}), \tau(\ell_{\alpha_2})\} + \cdots + \{\tau(\ell_{\alpha_{r-1}}), \tau(\ell_{\alpha_r})\} + \{\tau(\ell_{\alpha_r}), \tau(y)\}$ if x, y has separation vector $(\alpha_1, \ldots, \alpha_r) \neq \emptyset$.

(3) Denote by A_ξ the matrix of $\mathcal{A}_\xi$ in the basis O_ξ. It is a non-negative matrix. Compute its leading non-negative eigenvalue $\rho(\xi)$ (such an eigenvalue exists by the Perron–Frobenius theorem). It is easy to see that A_ξ is not nilpotent, therefore $\rho(\xi) \geq 1$.

The output of Thurston's entropy algorithm is then

$$h(\xi) := \log \rho(\xi),$$

which we define as the *core entropy* of the critical portrait ξ.

As proven by Gao [11], the algorithm gives the correct value of the core entropy for postcritically finite polynomials. For the definition of weak critical marking, see Section 9.2.

Theorem 3.4 ([11, Theorem 1.2]). *Let f be a postcritically finite polynomial with weak critical marking ξ. Then the core entropy $h(f)$ of f is given by Thurston's algorithm:*

$$h(f) = h(\xi).$$

Let us conclude the section with an example of the algorithm (see also Figure 6).

Example 3.5. Let $m = \{\{0, 1/3\}, \{7/15, 4/5\}\}$. Then the set $\mathcal{P}(m) = \{0, 1/5, 2/5, 3/5, 4/5\}$ gives rise to an abstract linear space Σ_m with basis

$$O_m = \left\{\left\{0, \tfrac{1}{5}\right\}, \left\{0, \tfrac{2}{5}\right\}, \left\{0, \tfrac{3}{5}\right\}, \left\{0, \tfrac{4}{5}\right\}, \left\{\tfrac{1}{5}, \tfrac{2}{5}\right\}, \left\{\tfrac{1}{5}, \tfrac{3}{5}\right\}, \left\{\tfrac{1}{5}, \tfrac{4}{5}\right\}, \left\{\tfrac{2}{5}, \tfrac{3}{5}\right\}, \left\{\tfrac{2}{5}, \tfrac{4}{5}\right\}, \left\{\tfrac{3}{5}, \tfrac{4}{5}\right\}\right\}$$

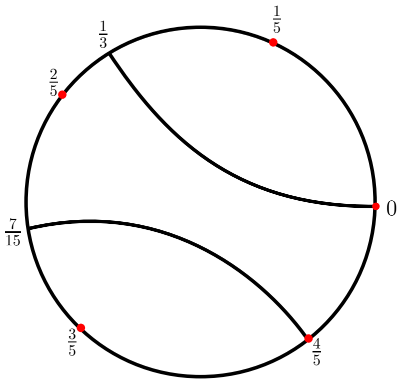


Fig. 6. An example of Thurston's entropy algorithm for a cubic polynomial.

The linear map $\mathcal{A}_m$ acts on the basis vectors as follows:

$$\{0,\tfrac{1}{5}\}\mapsto\{0,\tfrac{3}{5}\},\quad \{0,\tfrac{2}{5}\}\mapsto\{0,\tfrac{1}{5}\},\quad \{0,\tfrac{3}{5}\}\mapsto\{0,\tfrac{2}{5}\}+\{\tfrac{2}{5},\tfrac{4}{5}\},\quad \{0,\tfrac{4}{5}\}\mapsto\{0,\tfrac{2}{5}\},$$

$$\{\tfrac{1}{5},\tfrac{2}{5}\}\mapsto\{0,\tfrac{3}{5}\}+\{0,\tfrac{1}{5}\},\quad \{\tfrac{1}{5},\tfrac{3}{5}\}\mapsto\{0,\tfrac{3}{5}\}+\{0,\tfrac{2}{5}\}+\{\tfrac{2}{5},\tfrac{4}{5}\},\quad \{\tfrac{1}{5},\tfrac{4}{5}\}\mapsto\{0,\tfrac{3}{5}\}+\{0,\tfrac{2}{5}\},$$

$$\{\tfrac{2}{5},\tfrac{3}{5}\}\mapsto\{\tfrac{1}{5},\tfrac{2}{5}\}+\{\tfrac{2}{5},\tfrac{4}{5}\},\quad \{\tfrac{2}{5},\tfrac{4}{5}\}\mapsto\{\tfrac{1}{5},\tfrac{2}{5}\}\quad \{\tfrac{3}{5},\tfrac{4}{5}\}\mapsto\left\{\tfrac{4}{5},\tfrac{2}{5}\right\}.$$

We compute $h(m) := \log\rho(m) = 1.395$.

4. Labeled wedges and the associated graphs

We now turn to our general definition of core entropy for all critical portraits. In order to do so, we will generalize the transition matrix given by Thurston's algorithm to an infinite directed graph, which we call a *wedge*.

In this section, we shall first introduce a set of purely combinatorial objects, without reference to polynomial dynamics (or even to critical portraits), define abstractly their growth rate and prove some basic facts about it. Then in Section 4.4 we will see how to relate this abstract combinatorial objects to polynomial dynamics, by associating a labeled wedge to a critical portrait.

4.1. Wedges

We will now introduce a combinatorial object, called a *wedge*, to encode the dynamics on the set of "postcritical arcs" of a polynomial. In fact, we will construct a graph whose vertices are meant to represent all possible arcs in the Hubbard tree between the forward iterates of the critical points, and whose edges represent the transitions between arcs as given by Thurston's algorithm. We shall start with some abstract combinatorial notions.

First, the *basic wedge layer* is the subset of $\mathbb{Z}^2$ given by

$$\Sigma := \{(i, j) \in \mathbb{Z}^2 : 1 \le i \le j\}.$$

Fix $d \ge 2$ and $1 \le s \le d - 1$. Then the *wedge of size* s, denoted by Σ_s, is the union of s^2 copies of the basic wedge layer, which we imagine stacked on top of each other. Each copy is decorated with two symbols k, l with $1 \le k, l \le s$. As each element of Σ is denoted by two indices (i, j), we have

$$\Sigma_s = \big\{(y_k(i), y_l(j)) : k, l \in \{1, \dots, s\},\ 1 \le i \le j\big\}.$$

The pairs $(y_k(i), y_l(j))$ will be called *vertices*, as they will become the vertices of an infinite graph. Note that at this level the $y_k(i)$ are purely combinatorial labels. Given such d and s, and a critical portrait, in Section 4.4 we shall see how to decorate the wedge of size s to obtain a directed graph with labels on the vertices (see the example before Definition 4.5). These graphs will follow certain rules, which we introduce formally in Section 4.2. Assuming these rules, we prove in Section 4.3 facts about growth rates of these graphs.

In relation to polynomial dynamics, the element $y_k(i)$ is meant to represent the i^{th} iterate of the k^{th} critical point, while the vertex $(y_k(i), y_l(j))$ represents the arc in the Hubbard tree between the points of the filled Julia set determined by $y_k(i)$ and $y_l(j)$.

Moreover, given two elements $y_k(i)$ and $y_l(j)$ with $k, l \in \{1, \dots, s\}$ and $i, j \ge 1$, we denote by $\{y_k(i), y_l(j)\}$ the unique vertex in Σ_s which represents an ordering of the pair consisting of $y_k(i)$ and $y_l(j)$; that is, $\{y_k(i), y_l(j)\} = (y_k(i), y_l(j))$ if $i \le j$, and $\{y_k(i), y_l(j)\} = (y_l(j), y_k(i))$ otherwise.

				$\cdots$
			$(y_1(4), y_1(4))$	$\cdots$
		$(y_1(3), y_1(3))$	$(y_1(3), y_1(4))$	$\cdots$
	$(y_1(2), y_1(2))$	$(y_1(2), y_1(3))$	$(y_1(2), y_1(4))$	$\cdots$
$(y_1(1), y_1(1))$	$(y_1(1), y_1(2))$	$(y_1(1), y_1(3))$	$(y_1(1), y_1(4))$	$\cdots$

Fig. 7. The wedge Σ_1; each vertex represents an arc in the Hubbard tree between forward iterates of the critical point. For higher values of s, the wedge Σ_s has several "layers" given by the different values of k, l, which correspond to different critical points.

4.2. Labeled wedges and graphs

We then define a *labeling* of the wedge of size s as an assignment to each pair $(y_k(i), y_l(j))$ of a label, which can be either $\emptyset$ or

$$(\alpha_1, \dots, \alpha_r)$$

with $r \geq 1$ and $\alpha_1, \dots, \alpha_r \in \{1, \dots, s\}$ with the α_i's pairwise distinct. We call the wedge of size s with a labeling as above a *labeled wedge* of size s. A vertex of Σ_s is called *non-separated* if its label is $\emptyset$, and *separated* otherwise.

For each labeled wedge $\mathcal{W}$, we construct an *associated graph* Γ as follows. The vertices of Γ are the elements of $\mathcal{W}$, and for each vertex $v = (y_k(i), y_l(j))$ of Σ_s, we determine the set of edges with source v in the following way:

(1) if v is labeled $\emptyset$, then there is only one outgoing edge, namely

$$\{y_k(i), y_l(j)\} \to \{y_k(i+1), y_l(j+1)\};$$

such an edge will be called of *upward* type;

(2) if the ordered pair $(y_k(i), y_l(j))$ is labeled $(\alpha_1, \dots, \alpha_r) \neq \emptyset$, then there are exactly $r+1$ edges going out of v, and precisely the following:

$$\begin{array}{rcl} \{y_k(i), y_l(j)\} & \to & \{y_k(i+1), y_{\alpha_1}(1)\} \\ \{y_k(i), y_l(j)\} & \to & \{y_{\alpha_1}(1), y_{\alpha_2}(1)\} \\ \{y_k(i), y_l(j)\} & \to & \{y_{\alpha_2}(1), y_{\alpha_3}(1)\} \\ & \dots & \\ \{y_k(i), y_l(j)\} & \to & \{y_{\alpha_r}(1), y_l(j+1)\}. \end{array} \tag{4.1}$$

By the definition of labeled wedge, the set of edges going out of v is independent of the choice of the order of the elements of v.

For a vertex $v = \{y_k(i), y_l(j)\} \in \Sigma_s$, we call $\min\{i, j\}$ the *height* of v, and $\max\{i, j\}$ the *width* of v. Among the first and last edges in the list (4.1), the one whose target has smaller width will be called a *backward* edge, and the other one is a *forward* edge. All the other edges in (4.1) (the ones with targets of type $\{y_\alpha(1), y_\beta(1)\}$) will be called *central*. The names refer to the direction of the edge when the wedge is drawn as in Figure 8, with the width going from left to right and the height from bottom to top.

4.3. The growth of labeled wedges

Proposition 4.1. *Let Γ be the graph associated to a labeled wedge of size s. Then*

(1) *each vertex along any closed path of length n has height at most n;*

(2) *each vertex along any closed path of length n has width at most $2n$;*

(3) *the number of simple multi-cycles of length n is at most*

$$(2nk)^{k+\sqrt{2kn}},$$

where $k := s^2$. As a consequence, Γ has bounded cycles.

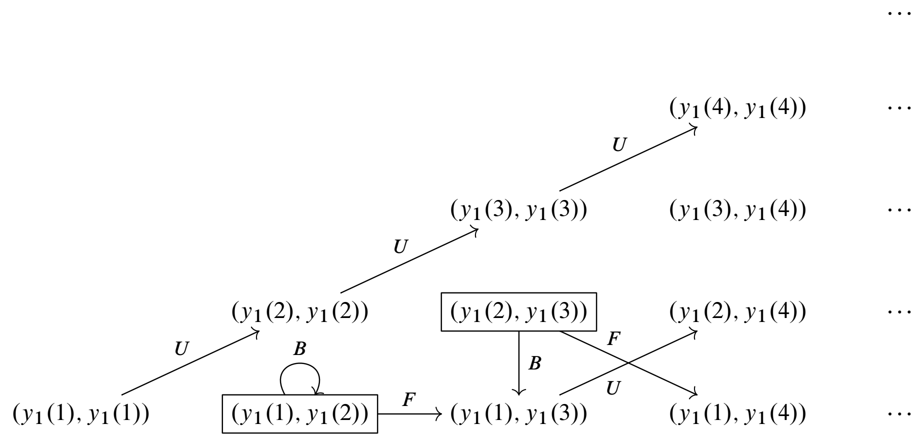


Fig. 8. A labeling of the wedge Σ_1, and its associated graph. The boxed vertices have separation vector α_1, representing the fact that the (first) critical point lies on the associated arc, while the other vertices have empty separation vector. The letters U, B, F represent the types (respectively, *upward*, *backward*, and *forward*) of the edges.

Proof. (1) Since upward edges always increase the height of a vertex, along each closed path there must be at least one backward, forward or central edge. Hence, since the target of a backward, forward or central edge has height 1, there must be at least one vertex of height 1 along the closed path. Since every edge increases the height at most by 1, the claim follows.

(2) Since forward and upward edges always increase the width of a vertex, along each closed path there must be at least one backward or central edge. By the previous point, the source of such an edge has height $\leq n$, hence its target has width $\leq n+1$. The claim follows since each edge increases the width by at most 1.

(3) Let γ a simple multi-cycle of length n. A vertex along the multi-cycle is called *central* of type (α, β) if the edge of the multi-cycle originating from it ends in the vertex $\{y_\alpha(1), y_\beta(1)\}$. Moreover, a vertex is called *backward relative to* γ if it is separated and the backward edge originating from it belongs to the multi-cycle γ.

We first note that γ is uniquely determined by the set of backward vertices, together with the set of central vertices and their type.

Since the multi-cycle is simple, first note that along the multi-cycle there are at most $s(s-1)/2 \leq k$ central vertices (at most one for each type).

We claim moreover that the number of backward vertices is at most $\sqrt{2kn}$. In fact, for each h the number of backward vertices along the multi-cycle of height h is at most $k := s^2$, since the target of a backward edge whose source has height h is of type $(y_\alpha(1), y_\beta(h+1))$, and there are at most s choices for α and s choices for β. Suppose now that the heights of the backward vertices along γ are $h_1, \dots, h_r$. Note that to each backward vertex of height h_i there corresponds a segment of γ of length h_i; indeed, each backward vertex v relative to γ is preceded, along γ, by a chain of $h_i - 1$ upward edges which connects a vertex of height 1 to v. As such segments are disjoint, the total length of their union is $h_1 + \cdots + h_r \le n$. Moreover, since for each value of h there are at most k values of i such that $h_i = h$, we get

$$\frac{r^2}{2k} \le \sum_{i=1}^{r} \frac{i}{k} \le \sum_{i=1}^{r} h_i \le n,$$

which proves the upper bound on the number of backward vertices.

Finally, since there are at most $2nk$ choices for each backward or central location (at most one for each diagonal), and γ is determined by its backward and central vertices, and their type, the total number of multi-cycles of length n is at most $(2nk)^{k+\sqrt{2kn}}$, as required. ∎

This proposition implies that the growth rate $\sigma(\Gamma)$ of the number $S(\Gamma, n)$ of simple multi-cycles is ≤ 1. Then the following result follows directly from Lemma 2.1.

Corollary 4.2. *Let $\mathcal{W}$ be a labeled wedge, and let Γ be its associated graph. Then the graph Γ has bounded cycles, and its spectral determinant $P(z)$ converges uniformly on compact subsets of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, defining a holomorphic function $P : \mathbb{D} \to \mathbb{C}$. Moreover, if the growth rate satisfies $r = r(\Gamma) > 1$, then the smallest real root of P is r^{-1}. If $r = 1$, then P does not have any zeros in the unit disk.*

We shall sometimes denote by $r(\mathcal{W})$ the growth rate of the graph associated to the labeled wedge $\mathcal{W}$. The following definition of (weak) convergence shall be crucial.

Definition 4.3. We say that a sequence $(\mathcal{W}_n)_{n\ge 1}$ of labeled wedges of size s *converges* if for each finite set of vertices $S \subseteq \Sigma_s$ there exists N such for each $n \ge N$ the labels of the elements of S for $\mathcal{W}_n$ are the same.

Let us recall a basic property of this notion of convergence.

Lemma 4.4 ([30, Lemma 4.4]). *If a sequence $(\mathcal{W}_n)_{n\ge 1}$ of labeled wedges of a fixed size $s \ge 1$ converges to $\mathcal{W}$, then the growth rate of $\mathcal{W}_n$ converges to that of $\mathcal{W}$.*

4.4. From critical portraits to labeled wedges

We now see how to associate to each critical portrait a labeled wedge. Then we will define the extension of the core entropy function as the growth of the associated infinite graph.

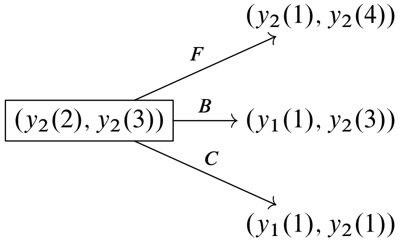
Let $d \ge 2$, and let $\xi = \{\ell_1, \dots, \ell_s\}$ be a critical portrait of degree d. Recall that $x_k(i) = \tau^i(\ell_k)$ for each $k = 1, \dots, s$ and each $i \ge 1$.

The portrait ξ induces a labeled wedge of size s as follows: for any vertex $\{y_k(i), y_l(j)\}$ of Σ_s, the ordered pair $(y_k(i), y_l(j))$ is labeled $(\alpha_1, \dots, \alpha_r)$ $(r \ge 0)$ if the ordered pair $(x_k(i), x_l(j))$ has separation vector $(\alpha_1, \dots, \alpha_r)$ with respect to ξ.

Example. As an example, consider the critical portrait displayed in Figure 6, with $d = 3$. We have

$$\xi = \{(0, \tfrac{1}{3}), (\tfrac{7}{15}, \tfrac{4}{5})\},$$

thus $\ell_1 = (0, 1/3)$ and $\ell_2 = (7/15, 4/5)$. As an example of labels, consider $x_2(2) = 3^2 \frac{4}{5} = \frac{1}{5} \mod 1$, and $x_2(3) = 3^3 \frac{4}{5} = \frac{3}{5} \mod 1$. As one can see from the picture, the pair $(1/5, 3/5)$ is separated by the leaf $\ell_1 = (0, 1/3)$ and $\ell_2 = (7/15, 4/5)$, hence the vertex $(y_2(2), y_2(3))$ has label (ℓ_1, ℓ_2). The edges going out of this vertex are



Again, the letters B, C, F identify, respectively, the backward, central, and forward edges. We denote by $\mathcal{W}_\xi$ the labeled wedge induced by ξ, and by Γ_ξ its associated graph. The growth rate of Γ_ξ is simply denoted by $r(\xi)$.

Definition 4.5. Let ξ be a critical portrait. Then the *growth rate* $r(\xi)$ of ξ is defined as the growth rate of the associated graph Γ_ξ.

4.5. Equivalence relation induced by a critical portrait

So far we have constructed an infinite graph whose vertices represent all possible arcs in the Hubbard tree joining forward iterates of the critical points. However, iterates of post-critical angles may coincide. Thus, any critical portrait induces an equivalence relation on the circle, where two pairs are defined to be equivalent if they represent the same pair of points on the circle. Let us see the details.

Let ξ be a critical portrait. We define an equivalence relation $\sim_\xi$ on the set

$$\left\{y_k(i) : k \in \{1, \dots, s\},\ i \geq 1\right\}$$

such that $y_k(i) \sim_\xi y_l(j)$ if $x_k(i) = x_l(j)$. This means that the two forward iterates of the critical angles coincide. This equivalence relation induces an equivalence relation, denoted by $\equiv_\xi$, on the vertices of the wedge Σ_s such that

$$\{y_{k_1}(i_1), y_{l_1}(j_1)\} \equiv_\xi \{y_{k_2}(i_2), y_{l_2}(j_2)\}$$

if they are equivalent as a pair: that is, either $y_{k_1}(i_1) \sim_\xi y_{k_2}(i_2)$ and $y_{l_1}(j_1) \sim_\xi y_{l_2}(j_2)$, or $y_{k_1}(i_1) \sim_\xi y_{l_2}(j_2)$ and $y_{l_1}(j_1) \sim_\xi y_{k_2}(i_2)$.

Finally, a vertex $v = \{y_k(i), y_l(j)\} \in \Sigma_s$ is called a *diagonal vertex* with respect to ξ if $y_k(i) \sim_\xi y_l(j)$: that is, the arc it represents is reduced to a single point.

5. Weakly periodic labeled wedges

We say that two vertices v_1, v_2 of a labeled wedge have *opposite labels* if the label of v_1 equals $(\alpha_1, \dots, \alpha_r)$ while the label of v_2 equals $(\alpha_r, \dots, \alpha_1)$.

Lemma 5.1. *Let ξ be a critical portrait of degree $d \geq 2$. Then the labeled wedge $\mathcal{W}_\xi$ satisfies*

(1) *its diagonal vertices are all labeled $\emptyset$;*

(2) *if $v_1 \equiv_\xi v_2 \in \mathcal{W}_\xi$, then v_1 and v_2 have either the same or opposite labels.*

Proof. It is easily checked by the definition of $\mathcal{W}_\xi$. ∎

It turns out that what we really deal with in the following context is a class of labeled wedges more general than $\mathcal{W}_\xi$, because the family of labeled wedges induced by critical portraits is not closed under convergence.

For example, let $(\xi_n = (\ell_1^n, \ell_2^n))_{n\geq 1}$ be a sequence of cubic critical portraits with

$$\ell_1^n = \{1/4, 7/12\} \quad \text{and} \quad \ell_2^n = \{1/8 - 1/n, 19/24 - 1/n\}.$$

Then ξ_n converges to the critical portrait $\xi = (\ell_1, \ell_2)$ in the Hausdorff topology with

$$\ell_1 = \{1/4, 7/12\}, \qquad \ell_2 = \{1/8, 19/24\},$$

and the corresponding labeled wedge $\mathcal{W}_{\xi_n}$ converges to a labeled wedge $\mathcal{W}$ as $n \to \infty$.

We consider the vertex $v = (y_2(1), y_2(2))$ in the wedge Σ_2 of size 2. It is easy to see that v has separation vector $(1, 2)$ as a vertex of $\mathcal{W}_{\xi_n}$ for all large n. Hence, in the limit wedge $\mathcal{W}$, v also has separation vector $(1, 2)$. On the other hand, since $x_2(1) = 3/8$ and $x_2(2) = 1/8$, the separation vector of v in $\mathcal{W}_\xi$ is $(1) \neq (1, 2)$. Note that the lack of 2 in the separation vector of $v \in \mathcal{W}_\xi$ is due to the fact that $x_2(2) \in \ell_2$.

Motivated by this example, we define *weakly periodic labeled wedges of type ξ* as a generalization of $\mathcal{W}_\xi$, which will be proved in Lemma 6.3 to include all limit labeled wedges of $(\mathcal{W}_{m_n})_{n\geq 1}$, where $(m_n)_{n\geq 1}$ is any sequence of primitive majors which converge to ξ in the Hausdorff topology.

Definition 5.2. We call a labeled wedge $\mathcal{W}$ *weakly periodic of type ξ* if the labels of its vertices satisfy the following conditions.

(1) Suppose that the separation vector of the ordered pair $(x_k(i), x_l(j))$ is $(\alpha_1, \dots, \alpha_r)$. Then the label in $\mathcal{W}$ of $(y_k(i), y_l(j))$ is of the form

$$(\beta_1, \dots, \beta_t, \alpha_1, \dots, \alpha_r, \gamma_1, \dots, \gamma_u) \tag{5.1}$$

where $x_k(i) \in \beta_\iota$ for $\iota = 1, \dots, t$ and $y_l(j) \in \gamma_\iota$ for $\iota = 1, \dots, u$.

As a consequence, in terms of the quotient equivalence relation, the set of endpoints of the β's all reduce to a single class; similarly for γ's. Note that t and u may be zero, which shows that the condition is satisfied by the standard labeled wedge associated to ξ.

(2) (a) Moreover, if the label of the ordered pair $(y_k(i), y_l(j))$ is

$$(\beta_1, \dots, \beta_t, \alpha_1, \dots, \alpha_r, \gamma_1, \dots, \gamma_u)$$

and $x_l(j) = x_{l'}(j')$, then the label of the ordered pair $(y_k(i), y_{l'}(j'))$ is

$$(\beta_1, \dots, \beta_t, \alpha_1, \dots, \alpha_r, \gamma_1', \dots, \gamma_{u'}') \tag{5.2}$$

(i.e., the β_ι are the same).

(b) Similarly, if the label of the ordered pair $(y_k(i), y_l(j))$ is

$$(\beta_1, \ldots, \beta_t, \alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_u)$$

and $x_k(i) = x_{k'}(i')$, then the label of the ordered pair $(y_{k'}(i'), y_l(j))$ is

$$(\beta'_1, \ldots, \beta'_{t'}, \alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_u). \tag{5.3}$$

(i.e., the γ_ι are the same).

In the label (5.1), we call the subvector $(\beta_1, \ldots, \beta_t)$ the *former-trivial labeled vector*, $(\gamma_1, \ldots, \gamma_u)$ the *latter-trivial labeled vector*, and $(\alpha_1, \ldots, \alpha_r)$ the *essential labeled vector* of the ordered pair $(y_k(i), y_l(j))$.

Let Γ be the graph associated to a weakly periodic labeled wedge of type ξ. We denote by Γ_ξ^{ND} the subgraph of Γ obtained by taking as vertices all pairs which are non-diagonal, and as edges all the edges of Γ which do not have either as a source or target a diagonal pair.

Lemma 5.3. *The equivalence relation $\equiv_\xi$ on Γ_ξ^{ND} is edge-compatible. Consequently, we get a quotient graph $\Gamma_\xi^Q := \Gamma^{ND}/_{\equiv_\xi}$, and the quotient map $\pi_\xi : \Gamma_\xi^{ND} \to \Gamma_\xi^Q$ is a weak cover of graphs.*

Proof. Let $v = \{y_k(i), y_l(j)\}$ and $v' = \{y_{k'}(i'), y_{l'}(j')\}$ be non-diagonal and $\equiv_\xi$-equivalent. We assume that $y_k(i) \sim_\xi y_{k'}(i')$ and $y_l(j) \sim_\xi y_{l'}(j')$, and that the ordered pair $(y_k(i), y_l(j))$ has label as in (5.1).

If the subvector $(\beta_1, \ldots, \beta_t)$ is not empty, we have

$$y_k(i+1) \sim_\xi y_{\beta_1}(1) \sim_\xi \cdots \sim_\xi y_{\beta_t}(1).$$

It follows that

$$\{y_k(i+1), y_{\beta_1}(1)\}, \{y_{\beta_1}(1), y_{\beta_2}(1)\}, \ldots, \{y_{\beta_{t-1}}(1), y_{\beta_t}(1)\}$$

are diagonal vertices, and $\{y_{\beta_t}(1), y_{\alpha_1}(1)\}$ is $\equiv_\xi$-equivalent to $\{y_k(i+1), y_{\alpha_1}(1)\}$. A similar argument holds for the subvector $(\gamma_1, \ldots, \gamma_u)$. Therefore, there are at most $r+1$ edges in Γ_ξ^{ND} going out of v, and precisely all those from the following list which do not end in a diagonal vertex:

$$\begin{aligned}
&\{y_k(i), y_l(j)\} \overset{e_1}{\to} \begin{cases} \{y_{\alpha_1}(1), y_k(i+1)\} & \text{if } (\beta_1, \ldots, \beta_t) = \emptyset, \\ \{y_{\alpha_1}(1), y_{\beta_t}(1)\} & \text{otherwise,} \end{cases} \\
&\{y_k(i), y_l(j)\} \overset{e_2}{\to} \{y_{\alpha_1}(1), y_{\alpha_2}(1)\}, \\
&\{y_k(i), y_l(j)\} \overset{e_3}{\to} \{y_{\alpha_2}(1), y_{\alpha_3}(1)\}, \\
&\qquad \ldots \\
&\{y_k(i), y_l(j)\} \overset{e_{r+1}}{\to} \begin{cases} \{y_{\alpha_r}(1), y_l(j+1)\} & \text{if } (\gamma_1, \ldots, \gamma_u) = \emptyset, \\ \{y_{\alpha_r}(1), y_{\gamma_1}(1)\} & \text{otherwise.} \end{cases}
\end{aligned} \tag{5.4}$$

With a similar argument, we find that the edges in Γ_ξ^{ND} going out of v' are precisely the non-diagonal ones among the following:

$$\begin{aligned}
\{y_{k'}(i'), y_{l'}(j')\} &\overset{e_1'}{\to} \begin{cases}\{y_{\alpha_1}(1), y_{k'}(i'+1)\} & \text{if } (\beta_1', \dots, \beta_{t'}') = \emptyset,\\ \{y_{\alpha_1}(1), y_{\beta_{t'}'}(1)\} & \text{otherwise,}\end{cases}\\
\{y_{k'}(i'), y_{l'}(j')\} &\overset{e_2'}{\to} \{y_{\alpha_1}(1), y_{\alpha_2}(1)\},\\
\{y_{k'}(i'), y_{l'}(j')\} &\overset{e_3'}{\to} \{y_{\alpha_2}(1), y_{\alpha_3}(1)\},\\
&\dots\\
\{y_{k'}(i'), y_{l'}(j')\} &\overset{e_{r+1}'}{\to} \begin{cases}\{y_{\alpha_r}(1), y_{l'}(j'+1)\} & \text{if } (\gamma_1', \dots, \gamma_{u'}') = \emptyset,\\ \{y_{\alpha_r}(1), y_{\gamma_1'}(1)\} & \text{otherwise.}\end{cases}
\end{aligned} \tag{5.5}$$

Note that, in any case, the target of each e_t $(1 \le t \le r+1)$ is $\equiv_\xi$-equivalent to the target of e_t'. This implies immediately that the relation $\equiv_\xi$ is edge compatible. ■

To simplify notation, from now on we shall make the dependence of both Γ^{ND} and Γ^Q on ξ implicit, by omitting the subscript.

Note that by construction the quotient graph Γ^Q can also be defined as follows. Consider the postcritical set $\mathcal{P}$ on the circle:

$$\mathcal{P} = \mathcal{P}(\xi) := \{\tau^i(\ell_k) : i \ge 1,\ 1 \le k \le s\}.$$

Then take the set

$$\mathcal{A} = \mathcal{A}(\xi) := \{\{x, y\} \in \mathcal{P} \times \mathcal{P} : x \ne y\}$$

of non-degenerate pairs of postcritical points (the label $\mathcal{A}$ is because one thinks of it as the set of *arcs* in the Hubbard tree between postcritical points in the filled Julia set, identifying an arc with its endpoints). The set of vertices of Γ^Q is precisely $\mathcal{A}$, while the set of edges is given by the dynamics.

Proposition 5.4. *Let ξ_1 and ξ_2 be two critical portraits, and Γ_1, Γ_2 be two weakly periodic labeled wedges of type, respectively, ξ_1 and ξ_2. If ξ_1 is equivalent to ξ_2 (see Section 3.1), then the quotient graphs Γ_1^Q and Γ_2^Q are isomorphic.*

Proof. Since any critical portrait is equivalent to exactly one primitive major, there exists a primitive major m which is equivalent to both ξ_1 and ξ_2. Thus, it is enough to prove the statement when one of the two critical portraits, say ξ_1, is a primitive major.

If two portrait elements intersect on the boundary, then they have the same image under τ. Hence the set of images $\mathcal{P} = \{\tau^i(\ell_k) : i \ge 1, 1 \le k \le s\}$ is the same for ξ_1 and ξ_2. Thus, the two graphs Γ_i^Q have the same vertex set.

In order to check the edges, consider now a vertex $v = (x_k(i), x_l(j))$ of Γ_1^Q (and also of Γ_2^Q, as seen above), and suppose that the separation vector of the two points $x = x_k(i)$ and $y = x_l(j)$ on the circle equals $(\alpha_1, \dots, \alpha_r)$. By definition of weakly periodic, the label of v in Γ_2^Q equals $(\beta_1, \dots, \beta_t, \alpha_1, \dots, \alpha_r, \gamma_1, \dots, \gamma_u)$ for some choice of β_i and γ_i. Then note that, since x belongs to all the β_i, the arcs $(\tau(\beta_i), \tau(\beta_{i+1}))$ for $i = 1, \dots, t-1$ are all degenerate. So are the arcs $(\tau(\gamma_i), \tau(\gamma_{i+1}))$ for $i = 1, \dots, u-1$

since y belongs to all the γ_i. Thus, the outgoing edges from v are the non-degenerate arcs among $(\tau(x), \tau(\alpha_1)), (\tau(\alpha_1), \tau(\alpha_2)), \dots, (\tau(\alpha_{r-1}), \tau(\alpha_r)), (\tau(\alpha_r), \tau(y))$. Now, by definition of the equivalence relation there exist equivalence classes $\Theta_1, \dots, \Theta_w$ such that $\alpha_1, \dots, \alpha_{i_1}$ belongs to Θ_1, $\alpha_{i_1+1}, \dots, \alpha_{i_2}$ belongs to Θ_2, etc. Then we note that the arcs $(\tau(\alpha_1), \tau(\alpha_2))$, up to $(\tau(\alpha_{i_1-1}), \tau(\alpha_{i_1}))$ are also degenerate, and so on, hence the outgoing edges from (x, y) are the non-degenerate arcs among

$$(\tau(x), \tau(\alpha_1)), (\tau(\alpha_1), \tau(\alpha_2)), \dots, (\tau(\alpha_{r-1}), \tau(\alpha_r)), (\tau(\alpha_r), \tau(y)).$$

This is by definition the list of outgoing edges from (x, y) in $\Gamma_1^{\mathcal{Q}}$, proving the claim. ■

6. The comparison of growth rates of weakly periodic labeled wedges

Throughout this section, we always assume that $\xi = \{\ell_1, \dots, \ell_s\}$ is a critical portrait. Let $\mathcal{W}$ be a weakly periodic labeled wedge of type ξ, and Γ its associated graph. The notations $\Gamma^{\mathcal{Q}}$ and Γ^{ND} follow Proposition 5.3.

Proposition 6.1. *Let $\mathcal{W}$ be a weakly periodic labeled wedge of type ξ, and Γ its associated graph. Then the growth rates of Γ^{ND} and $\Gamma^{\mathcal{Q}}$ are equal.*

Proof. Let S denote the set of vertices in Σ_s which have widths and heights at most $2n$. By Proposition 4.1, each closed path in Γ^{ND} of length n passes through S. Applying (2) of Lemma 2.3, we get the estimate

$$r(\Gamma^{ND}) \leq r(\Gamma^{\mathcal{Q}}).$$

We then need to show $r(\Gamma^{\mathcal{Q}}) \leq r(\Gamma^{ND})$.

Let $\gamma = (e_1, \dots, e_n)$ be a closed path in $\Gamma^{\mathcal{Q}}$ with $v_0 := s(e_1)$ and $v_t := t(e_t)$ for each $t \in \{1, \dots, n\}$. Each v_t represents an arc, hence it has two endpoints. By induction on t, we will declare certain endpoints of v_t as *marked*, according to the following rule:

(1) by definition, both endpoints of the arc v_0 are marked;

(2) recursively, an endpoint of v_{t+1} is marked if it is the image of a marked endpoint of v_t, in the following sense. Let $v_t = (x_k(i), x_l(j))$, and suppose the separation vector is $(\alpha_1, \dots, \alpha_r)$. Then, if the endpoint $x_k(i)$ is marked, we also mark the endpoint $x_k(i+1)$ of the arc $(x_k(i+1), x_{\alpha_1}(1))$. Similarly, if $x_l(j)$ is marked, then we mark the endpoint $x_l(j+1)$ of the arc $(x_{\alpha_r}(1), x_l(j+1))$. All other endpoints of v_{t+1} are no-marked.

We then say that a vertex v_t is marked if at least one of its endpoints is marked. A path γ with vertices $v_0, \dots, v_n$ is *peripheral* if all its vertices are marked, and *non-peripheral* otherwise.

Note that by construction, of all the edges going out of v_t, at most two are marked. As a consequence, for any v_0 and any n, there are at most two peripheral paths of length n which start at v_0.

Claim. *If γ is peripheral, then there is a vertex $\widetilde{v}_0 \in \pi^{-1}(v_0)$ which has width and height at most Cn, where C is a constant which depends only on ξ.*

Proof of the Claim. Pick a vertex $w_0 = \{y_k(i), y_l(j)\} \in \pi^{-1}(v_0)$, and consider the lift $\widetilde{\gamma} = (\widetilde{e}_1, \dots, \widetilde{e}_n)$ of γ based at w_0. Set $w_t := t(\widetilde{e}_t)$ for each $t \in \{1, \dots, n\}$. The claim will be checked by cases.

(1) Each vertex $w_0, \dots, w_{n-1}$ is labeled $\emptyset$. Then $w_n = \{y_k(i+n), y_l(j+n)\}$. Note that $w_0 \equiv_\xi w_n$, so we get either $x_k(i) = x_k(i+n)$ and $x_l(j) = x_l(j+n)$, or $x_k(i) = x_l(j+n)$ and $x_l(j) = x_k(i+n)$. In both cases ℓ_k and ℓ_l are eventually periodic. It follows that there are a constant $C_1 > 0$ and $i_0, j_0 < C_1$ such that $x_k(i_0) = x_k(i)$ and $x_l(j_0) = x_l(j)$. The point $\widetilde{v}_0 := \{y_k(i_0), y_l(j_0)\}$ thus satisfies the requirements.

(2) There is a separated vertex among $w_0, \dots, w_n$. To better show the argument, first assume that w_0 is separated. Then w_1 has height 1.

If there is a central or backward edge among $\widetilde{e}_2, \dots, \widetilde{e}_n$, then the width and height of w_n are both less than n, and $\widetilde{v}_0 := w_n$ satisfies the requirement.

Otherwise, w_1 equals $\{y_\alpha(1), y_k(i+1)\}$ or $\{y_\beta(1), y_l(j+1)\}$, and the edges $\widetilde{e}_2, \dots, \widetilde{e}_n$ are either forward or upward. By symmetry, we can assume $w_1 = \{y_\alpha(1), y_k(i+1)\}$. It follows that $w_n = \{y_{\alpha'}(p), y_k(i+n)\}$ with $p \le n$ and $w_n \equiv_\xi w_0$. If $y_k(i) \sim_\xi y_{\alpha'}(p)$ and $y_l(j) \sim_\xi y_k(i+n)$, we set

$$\widetilde{v}_0 := \{y_{\alpha'}(p), y_{\alpha'}(p+n)\},$$

which is $\equiv_\xi$-equivalent to w_0, and has width and height at most $2n$. If $y_k(i) \sim_\xi y_k(i+n)$ and $y_l(j) \sim_\xi y_{\alpha'}(p)$, then ℓ_k is eventually periodic. There is hence an integer i_0 less than a constant C_2 such that $y_k(i_0) \sim_\xi y_k(i)$. The vertex

$$\widetilde{v}_0 = \{y_{\alpha'}(p), y_k(i_0)\}$$

is what we want, with width and height at most $C_2 n$.

In the general case, let w_k be the first separated vertex among $w_0, \dots, w_n$. Then by the previous argument there exists a vertex $\widetilde{v}_k$ which projects to v_k and has height and width $\le \max\{C_1, C_2\} n$. By lifting the path $(e_{k+1}, \dots, e_n)$ starting from w_k one gets a vertex $\widetilde{v}_0$ which projects to v_0 and with height and width bounded above by $\max\{C_1, C_2\} n + n$, as required. ■

Now, note that the number of vertices of the wedge with both width and height bounded by Cn is at most $(sCn)^2$, hence the number of projections to Γ^Q of such vertices is also bounded above by $(sCn)^2$. Finally, as we previously observed the number of peripheral paths of length n starting at a given vertex v_0 is at most 2, we get

$$\#\{\text{peripheral closed paths of length } n\} \le 2s^2C^2n^2. \tag{6.1}$$

Claim. *If γ is a non-peripheral closed path in Γ^Q, then there exists a closed path $\widetilde{\gamma} \subseteq \Gamma^{ND}$ of length n which projects in Γ^Q to a cyclic permutation of γ.*

Proof of the Claim. By a cyclic permutation of $\gamma = (e_1, \ldots, e_n)$, we mean a path of the form $(e_k, \ldots, e_n, e_1, \ldots, e_{k-1})$ for some k. By definition of non-peripheral, there exists the least $n_1 \leq n$ for which at least one of the endpoints of v_{n_1} is not marked, and the least $n_2 \in [n_1, n]$ for which none of the endpoints of v_{n_2} is marked. Choose now a vertex $\widetilde{v}_0$ of Γ^{ND} which projects to v_0, and let us lift γ starting from there. Thus we get a sequence $\widetilde{v}_0, \widetilde{v}_1, \ldots, \widetilde{v}_n$ of vertices, and it is not necessarily true that $\widetilde{v}_0 = \widetilde{v}_n$. Now keep lifting γ starting from $\widetilde{v}_n$, obtaining a sequence $\widetilde{v}_{n+1}, \ldots, \widetilde{v}_{2n}$ which also projects to γ.

By definition of weakly periodic labeled wedge, the two vertices $\widetilde{v}_{n_1}$ and $\widetilde{v}_{n+n_1}$ have a common endpoint. Then, by applying successively conditions (2)(a, b) of the definition, one finds that the same is true for all the pairs $\widetilde{v}_t$, $\widetilde{v}_{t+n}$ with $n_1 \leq t < n_2$. Finally, this implies that $\widetilde{v}_{n_2} = \widetilde{v}_{n+n_2}$.

Thus, by lifting the path $v_{n_2}, \ldots, v_n, v_1, \ldots, v_{n_2-1}, v_{n_2}$ in Γ^Q one gets a closed path in Γ^{ND}, as required. ■

By (6.1) and the last claim, we have the estimates

$$\begin{aligned} C(\Gamma^Q, n) &= \#\left\{ \begin{matrix} \text{peripheral closed} \\ \text{paths of length } n \end{matrix} \right\} + \#\left\{ \begin{matrix} \text{non-peripheral closed} \\ \text{paths of length } n \end{matrix} \right\} \\ &\leq 2s^2C^2n^2 + nC(\Gamma^{ND}, n), \end{aligned}$$

from which it follows that $r(\Gamma^Q) \leq r(\Gamma^{ND})$ as required. ■

Lemma 6.2. *Let ξ be a critical portrait, and m its induced primitive major. Then*

$$r(\xi) = r(m).$$

Proof. Note that the labels of every diagonal of the labeled wedges $\mathcal{W}_\xi$, $\mathcal{W}_m$ are empty, so $r(\xi) := r(\Gamma_\xi) = r(\Gamma_\xi^{ND})$ and $r(m) := r(\Gamma_m) = r(\Gamma_m^{ND})$. It then follows from Propositions 6.1 and 5.4 that

$$r(\xi) = r(\Gamma_\xi^{ND}) \overset{6.1}{=} r(\Gamma_\xi^Q) \overset{5.4}{=} r(\Gamma_m^Q) \overset{6.1}{=} r(\Gamma_m^{ND}) = r(m). \qquad ■$$

Lemma 6.3. *Let (m_N) be a sequence of primitive majors which Hausdorff converge to a critical portrait ξ, and suppose that the associated sequence $(\mathcal{W}_N)$ of labeled wedges converges to some labeled wedge $\mathcal{W}$. Then $\mathcal{W}$ is weakly periodic of type ξ.*

Proof. In order to check (1) of the definition of weakly periodic labeled wedge, let the ordered pair $(x_k(i), x_l(j))$ have separation vector $(\alpha_1, \ldots, \alpha_r)$ with respect to ξ. Note that if $x_k(i) \notin \ell_\alpha$, then for N large the point $x_k^{(N)}(i)$ has the same position with respect to $\ell_\alpha^{(N)}$ as that of $x_k(i)$ with respect to ℓ_α. Thus, the portrait elements $\alpha_1, \ldots, \alpha_r$ must be part of the separation vector of $(x_k^{(N)}(i), x_l^{(N)}(j))$, and on the other hand the only other portrait elements which are part of this separation vector must contain either $x_k(i)$ or $x_l(j)$. Since $\mathcal{W}_N \to \mathcal{W}$, the label of $(y_k(i), y_l(j))$ in $\mathcal{W}$ equals the separation vector for $(x_k^{(N)}(i), x_l^{(N)}(j))$ for N large, so this argument proves property (1) in the definition of weakly periodic labeled wedge.

Let us now prove (2)(a). Let $v' = \{y_k(i), y_{l'}(j')\} \in \Sigma_s$ be a vertex $\equiv_\xi$-equivalent to v, i.e., $x_l(j) = x_{l'}(j')$, and let the separation vector of $(x_k(i), x_l(j))$ be $(\alpha_1, \dots, \alpha_r)$. Then the separation vector of $(x_k^{(N)}(i), x_l^{(N)}(j))$ is of type $(\beta_1, \dots, \beta_t, \alpha_1, \dots, \alpha_r, \gamma_1, \dots, \gamma_u)$, where the $\ell_{\beta_i}^{(N)}$ are precisely the portrait elements which separate $x_k^{(N)}(i)$ and $\ell_{\alpha_1}^{(N)}$. For the same reason, the separation vector of $(x_k^{(N)}(i), x_{l'}^{(N)}(j'))$ is of type $(\beta_1, \dots, \beta_t, \alpha_1, \dots, \alpha_r, \gamma'_1, \dots, \gamma'_{u'})$ (note the β_i are the same). Since $\mathcal{W}_N$ converges to $\mathcal{W}$, these are also the labels of, respectively, $(y_k(i), y_l(j))$ and $(y_k(i), y_{l'}(j))$ in $\mathcal{W}$, proving the claim. (2)(b) follows analogously. ■

7. The convergence of labeled wedges induced by primitive majors

To prove the continuity of the growth rate $r(m)$ (for primitive majors m in the metric md), we expect to apply Lemma 4.4. For this purpose, we need to know when the labeled wedges $\mathcal{W}_{m_N}$ converge as the majors m_N converge. Note that even if the m_N converge in the Hausdorff topology, the labeled wedges $\mathcal{W}_{m_N}$ may not converge. For example, in the quadratic case, if θ is periodic, then the labeled wedges $\mathcal{W}_{\theta'}$ do not converge as $\theta' \to \theta$. However, we will show that it is true for a subsequence.

Lemma 7.1. *Let $s \geq 1$. Then any sequence $(\mathcal{W}_N)$ of labeled wedges of size s has a convergent subsequence.*

Proof. This follows by our choice of (weak!) topology on the space of labeled wedges. Since any vertex of Σ_s has finitely many possible labels, for each finite set S of vertices of Σ_s there exists a subsequence $(\mathcal{W}_{N_k})$ of labeled wedges such that all vertices of S have the same label. The claim follows by picking an exhaustion (S_n) of Σ_s by finite sets and applying the usual diagonalization argument. ■

Lemma 7.2. *Let (m_N) be a sequence of primitive majors which converges to a critical portrait ξ in the Hausdorff topology, and so that the associated labeled wedges $\mathcal{W}_{m_N}$ converge to a wedge $\mathcal{W}_\infty$, with associated infinite graph Γ_∞. Then*

$$r(\Gamma_\infty) = r(\Gamma_\infty^{ND}).$$

Proof. Denote by $\ell_1, \dots, \ell_s$ the elements of ξ, and by $x_k(i)$ the point $\tau^i(\ell_k)$ on the circle. Recall that a vertex $v = \{y_k(i), y_l(j)\}$ of Γ_∞ is diagonal with respect to ξ if $x_k(i) = x_l(j)$.

For the graph Γ associated to a labeled wedge and any integer $n \geq 1$, we denote by Γ^n the finite subgraph of Γ such that $V(\Gamma^n)$ is the set of vertices of Γ with width and height at most n, and $E(\Gamma^n)$ is the set of all edges of Γ with both sources and targets in $V(\Gamma^n)$.

For a primitive major m, we will denote by $\mathcal{W}_m$ the associated labeled wedge, and by Γ_m its associated infinite graph. Now fix $n \geq 1$. Since $\mathcal{W}_{m_N} \to \mathcal{W}_\infty$, we can choose $m = m_N$ sufficiently close to ξ so that each vertex $v \in \Sigma_s$ with width and height at most $2n$ has a common label in $\mathcal{W}_m$ and $\mathcal{W}_\infty$. It follows that all graphs Γ_m^{2n} coincide with Γ_∞^{2n}.

Moreover, note that by Proposition 4.1 every closed path of length n in Γ_∞ actually lives in Γ_∞^{2n}, which is also equal to Γ_m^{2n}.

To prove $r(\Gamma_\infty) = r(\Gamma_\infty^{ND})$, we shall check the estimate

$$\#\begin{Bmatrix}\text{closed paths of length } n \text{ in } \Gamma_\infty^{2n},\\ \text{containing } \xi\text{-diagonal vertices}\end{Bmatrix} \leq 4s^2n^2,$$

which by the above discussion is equivalent to

$$\#\begin{Bmatrix}\text{closed paths of length } n \text{ in } \Gamma_m^{2n},\\ \text{containing } \xi\text{-diagonal vertices}\end{Bmatrix} \leq 4s^2n^2.$$

This result will follow from the following fact:

($\star$) For each n and each diagonal vertex v_0 of Γ_m, there exists at most one closed path of length n based at v_0 in Γ_m.

First note that the fact implies the claim. Indeed, recall that s is the size of the wedge, and the height and width of any diagonal vertex v_0 which lies on a closed path of length n are bounded above by $2n$, by Lemma 4.1. Hence, each such diagonal vertex is of the form $(y_k(i), y_l(j))$, with $k, l \leq s$ and $i, j \leq 2n$, thus the number of such vertices is at most $(2n \cdot s)^2 = 4s^2n^2$.

This yields the estimate

$$\#\begin{Bmatrix}\text{closed paths of length } n \text{ in } \Gamma_m^{2n},\\ \text{containing diagonal vertices}\end{Bmatrix} \leq 4s^2n^2$$

as required.

Let us now prove ($\star$). In order to do so, for each major m approximating ξ denote by $x_k^m(i) := \tau^i(\ell_k^m)$ the iterate of the approximating element ℓ_k^m.

Suppose that $v_0 = \{y_{k_0}(i_0), y_{l_0}(j_0)\}$ is a ξ-diagonal vertex, and let $\theta_0 = x_{k_0}(i_0) = x_{l_0}(j_0)$. Choose an interval I in the circle which contains θ_0 in its interior, and such that the map $\tau^n : I \to \tau^n(I)$ is a homeomorphism.

Now choose a primitive major $m = m_N$ close enough to the limit ξ so that $x_{k_0}^m(i_0)$ and $x_{l_0}^m(j_0)$ belong to I. Note that if $x_{k_0}^m(i_0)$ and $x_{l_0}^m(j_0)$ coincide, then the vertex v_0 is non-separated in Γ_m, and so are all its descendants in the graph Γ_m: thus, v_0 does not lie on any closed path. Thus, we can assume that the interval $[x_{k_0}^m(i_0), x_{l_0}^m(j_0)]$ is not a point.

For each vertex $v = \{y_k(i), y_l(j)\}$ and each approximating major m, let $J_v^m := [x_k^m(i), x_l^m(j)]$ denote the corresponding arc on the circle connecting the two iterates of the approximating major.

Suppose now that there is a path $v_0 \to v_1 \to \cdots \to v_{n-1} \to v_n$ in Γ_m. Note that by construction each interval $J_{v_{t+1}}^m$ is a subinterval of $\tau(J_{v_t}^m)$, and thus $J_{v_n}^m$ is a subinterval of $L = \tau^n(J_{v_0}^m)$. Moreover, distinct paths yield disjoint subintervals.

If the path is closed ($v_0 = v_n$), then the intervals $J_{v_0}^m$ and $J_{v_n}^m$ must coincide. However, as all intervals $J_{v_n}^m$ for different choices of paths are disjoint, there is at most one path for which $J_{v_n}^m$ coincides with $J_{v_0}^m$. Thus, there exists at most one closed path of length n based at v_0, proving ($\star$). ∎

8. The continuity of growth rate and core entropy

In this part, we will show the continuity of the growth rate function on the space of primitive majors, and then prove it coincides with the value given by Thurston's algorithm for rational majors. As a consequence, the core entropy extends to a continuous function on $\mathrm{PM}(d)$, which establishes Theorem 1.1.

Theorem 8.1. *The growth rate function $r : \mathrm{PM}(d) \to \mathbb{R}$ is continuous.*

Proof. On the contrary, assume that there exists $\epsilon_0 > 0$ and a sequence (m_N) of majors converging to m such that

$$|r(m_N) - r(m)| > \epsilon_0 \quad \text{for all } N.$$

According to Proposition 3.2, there exists a subsequence which Hausdorff converges to a critical portrait ξ, and ξ induces m. Moreover, by Lemma 7.1, by passing to a further subsequence (still denoted (m_N)) we can assume that the associated labeled wedges $\mathcal{W}_N$ converge to some labeled wedge $\mathcal{W}_\infty$. By Lemma 6.3, the limit wedge $\mathcal{W}_\infty$ is weakly periodic of type ξ. Let Γ_N be the infinite graph associated to $\mathcal{W}_N$, and Γ_∞ the graph associated to $\mathcal{W}_\infty$. As a consequence,

$$r(\Gamma_N) \to r(\Gamma_\infty) \quad \text{as } N \to \infty.$$

Now, combining Lemma 7.2 and Proposition 6.1 we get

$$r(\Gamma_\infty) = r(\Gamma_\infty^{ND}) = r(\Gamma_\infty^{Q}),$$

and similarly, if Γ_m denotes the infinite graph associated to the primitive major m (note that m, being a primitive major, is trivially the limit of a constant family of primitive majors)

$$r(\Gamma_m) = r(\Gamma_m^{ND}) = r(\Gamma_m^{Q}).$$

Now, since Γ_∞ is weakly periodic of type ξ and Γ_m is weakly periodic of type m, and since m and ξ are equivalent, by Proposition 5.4 we have

$$r(\Gamma_\infty^{Q}) = r(\Gamma_m^{Q}),$$

hence combining the previous equalities yields

$$r(\Gamma_\infty) = r(\Gamma_m),$$

which contradicts the assumption that $r(m_N) \nrightarrow r(m)$. ∎

To finish the proof of the main theorem, we need the following lemma.

Lemma 8.2. *Let ξ be a rational critical portrait of degree $d \geq 2$. Then the logarithm of the growth rate $r(\xi)$ of the infinite graph Γ_ξ coincides with the core entropy of ξ:*

$$h(\xi) = \log r(\xi).$$

Proof. Let ξ be a rational critical portrait, Γ_ξ the associated infinite graph, and $G_\xi := \Gamma_\xi^Q$ the quotient graph of Γ_ξ^{ND}. By unraveling the definition, the matrix A_ξ constructed in Section 3.3 is exactly the adjacency matrix of G_ξ. By Lemma 7.2 and Proposition 6.1, the growth rate of Γ_ξ coincides with that of G_ξ. Moreover, by Lemma 2.2, the growth rate of G_ξ coincides with the largest real eigenvalue of its adjacency matrix, that is, the largest real eigenvalue of A_ξ. Thus, its logarithm is the core entropy $h(\xi)$. ∎

Proof of Theorem 1.1. This follows directly from Lemma 8.2 and Theorem 8.1. ∎

8.1. *Major trees and an upper bound*

Let us conclude this section with an upper bound on the core entropy. Given a critical portrait ξ and $x, y \in \partial\mathbb{D}$, we denote by $\delta(\xi, x, y)$ the number of elements of ξ separating x and y, and let the *depth* of ξ be

$$\Delta(\xi) := \sup_{x,y\in\partial\mathbb{D}} \delta(\xi, x, y).$$

Note that by construction, $\Delta(\xi) \le s + 1$, where s is the size of ξ. Moreover, if ξ is the critical portrait of a real polynomial of degree d with $d - 1$ distinct real critical points, then $\Delta(\xi) = d$, while a unicritical polynomial will have $\Delta(\xi) = 2$.

Note that the depth of a primitive major m depends only on its *major tree*, as discussed in [32, Section 4.2.2]. The major tree $\mathcal{T}_m$ of m is defined as the bipartite graph which has

- a white vertex for every element of m;
- a black vertex for each connected component of $\overline{\mathbb{D}} \setminus \bigcup_i \overline{\ell_i}$;
- an edge between two vertices of different color if the corresponding portrait element and the complementary component intersect on the boundary of the circle.

By construction, if we equip the major tree with the simplicial distance, we have $\Delta(m) = \frac{1}{2} \operatorname{diam} \mathcal{T}_m$.

Proposition 8.3. *Suppose that a primitive major m has depth $\Delta(m)$. Then the extension h of the core entropy satisfies*

$$h(m) \le \log \Delta(m).$$

Proof. Let $\Delta = \Delta(m)$. By the construction of labeled wedges, each vertex in the infinite wedge Γ_m has separation vector of cardinality at most Δ. Hence, each vertex of the graph has at most Δ outgoing edges. Further, by Lemma 4.1, every vertex on a closed path of length n has at most height n and width $2n$, hence there are at most $n \cdot 2n \cdot s^2$ such vertices. Thus, the total number of closed paths of length n is at most $2n^2s^2\Delta^n$, hence the growth rate satisfies

$$r(m) = r(\Gamma_m) \le \limsup_{n\to\infty} \sqrt[n]{2n^2s^2\Delta^n} = \Delta,$$

which yields the proposition as the extended core entropy h satisfies $h(m) = \log r(m)$. ∎

9. Continuity of core entropy on the space of polynomials

Let $d \geq 2$ be an integer, and f a complex polynomial of degree d. The *filled-in Julia set* K_f is the set of points which do not escape to infinity under iteration, the *Julia set* J_f is the boundary of K_f, and the *Fatou set* is $F_f := \mathbb{C} \setminus J_f$. A point $c \in \mathbb{C}$ is called a *critical point* of f if $f'(c) = 0$. The *critical set* of f is defined to be

$$\mathrm{crit}(f) := \{c \in \mathbb{C} : f'(c) = 0\},$$

and the *postcritical set* is

$$\mathrm{post}(f) := \overline{\{f^n(c) : c \in \mathrm{crit}(f),\ n \geq 1\}}.$$

A polynomial is called *postcritically finite* if its postcritical set is finite. Any postcritically finite polynomial f has an f-invariant tree H_f containing the orbits of its critical points, called the *Hubbard tree*, which captures the dynamics of the polynomial. Following Thurston, the *core entropy* of f, denoted by $h(f)$, is defined to be the topological entropy of f on its Hubbard tree, i.e.,

$$h(f) := h_{\mathrm{top}}(f, H_f).$$

In the previous part, we showed the continuity of the core entropy of rational critical portraits. As an application, we will prove the continuity of the core entropy of postcritically finite polynomials of any given degree.

Let $\mathcal{P}_d$ denote the parameter space of monic centered polynomials of degree d. We say that a sequence $(f_n)_{n\geq 1} \subseteq \mathcal{P}_d$ *converges* to $f \in \mathcal{P}_d$ if the coefficients of f_n converge to the corresponding coefficients of f. The objective of this section is to prove the following result.

Theorem 9.1. *Let f_n, $n \geq 1$, and f be postcritically finite polynomials in $\mathcal{P}_d$. If $f_n \to f$ as $n \to \infty$, then $h(f_n) \to h(f)$.*

We outline the proof. Following Poirier [24], we associate to each polynomial f_n (resp. f) a rational formal critical portrait $\Theta_n = \{\mathcal{F}_n, \mathcal{J}_n\}$ (resp. $\Theta = \{\mathcal{F}, \mathcal{J}\}$), called a (weak) *critical marking* (see Section 9.2 below). By Theorem 3.4,

$$h(\Theta_n) = h(f_n) \text{ for } n \geq 1 \quad \text{and} \quad h(\Theta) = h(f). \tag{9.1}$$

Therefore, applying Theorem 1.1, one just needs to have a good choice of Θ_n and Θ such that Θ_n Hausdorff converges to Θ as $n \to \infty$ and Θ is a weak critical marking for f. This is accomplished in Proposition 9.16 by studying continuity properties of external rays.

9.1. *The dynamics of polynomials*

Let $f \in \mathcal{P}_d$. A point $z \in \mathbb{C}$ is called a *preperiodic point* of f if there exist integers $m \geq 0$ and $n \geq 1$ such that $f^m(z) = f^{m+n}(z)$. If $m = 0$, the point z is called *periodic*. The minimal m and n with this property are called the *preperiod* and *period* of z respectively.

Let f be a polynomial in $\mathcal{P}_d$ with connected filled-in Julia set. By Böttcher's theorem, there exists a unique conformal isomorphism $\phi_f : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus K_f$ with ϕ_f tangent to the identity at ∞ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C} \setminus K_f & \xrightarrow{f} & \mathbb{C} \setminus K_f \\ \phi_f \downarrow & & \downarrow \phi_f \\ \mathbb{C} \setminus \overline{\mathbb{D}} & \xrightarrow{z \mapsto z^d} & \mathbb{C} \setminus \overline{\mathbb{D}} \end{array} \tag{9.2}$$

The map ϕ_f is called the *Böttcher coordinate* of f. The *external ray of argument* θ, denoted by $R_f(\theta)$, is the image by ϕ_f^{-1} of the ray $\{z = re^{2\pi i\theta} : r > 1\}$. We say that it *lands* if the intersection

$$\bigcap_{r>1} \overline{\phi_f^{-1}((1,r]e^{2\pi i\theta})}$$

is a point, called the *landing point* of $R_f(\theta)$. Since a power map sends radial lines to radial lines, the polynomial f sends external rays to external rays. Set $U_f(\infty) := \mathbb{C} \setminus K_f$. The *Green function* G_f associated with f is the harmonic function equal to $\log |\phi_f(z)|$ on $U_f(\infty)$ and vanishing on K_f. The number $s = G_f(z) \geq 0$ is called the *potential* of $z \in \mathbb{C}$.

Now, assume that f is a postcritically finite polynomial. Then the Fatou set of f consists of attracting basins and all periodic points in J_f are repelling. The filled-in Julia set K_f is connected and locally connected, and each bounded Fatou component is a Jordan domain. By Böttcher's theorem, there is a system of Riemann mappings

$$\{\phi_U : \mathbb{D} \to U : U \text{ a bounded Fatou component}\},$$

each extending to a homeomorphism of $\overline{\mathbb{D}}$, such that the following diagram commutes for all U:

$$\begin{array}{ccc} \overline{\mathbb{D}} & \xrightarrow{\text{power map } z^{d_U}} & \overline{\mathbb{D}} \\ \phi_U \downarrow & & \downarrow \phi_{f(U)} \\ \overline{U} & \xrightarrow{f} & \overline{f(U)} \end{array}$$

The image $\phi_U(0)$ is called the *center* of the Fatou component U. It is easy to see that any center is mapped to a critical periodic point under finitely many iterations of f. The images in U under ϕ_U of closed radial lines in $\overline{\mathbb{D}}$ are, by definition, the *internal rays* of U. As with external rays, the polynomial f sends internal rays to internal rays.

Let f be a postcritically finite polynomial. Then any pair of points in the closure of a bounded Fatou component can be joined in a unique way by a Jordan arc consisting of (at most two) segments of internal rays. We call such arcs *regulated*. Since K_f is arc-connected, for any $z_1, z_2 \in K_f$ there is an arc $\gamma : [0, 1] \to K_f$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. In general, we will not distinguish between the map γ and its image. It is proved in [8] that such arcs can be chosen in a unique way so that the intersection with the closure of a Fatou component is regulated. We still call such arcs regulated and denote them by $[z_1, z_2]$. By [8, Proposition 2.7], the set

$$H_f := \bigcup_{p,q \in \mathrm{post}(f)} [p, q]$$

is a finite connected tree, called the *Hubbard tree* of f. A point $z \in J_f$ is called *biaccessible* if there are at least two rays landing at z. The following result is well-known.

Lemma 9.2. *Let f be a postcritically finite polynomial. Then every biaccessible point in J_f is mapped to the Hubbard tree of f under finitely many iterates of f.*

Definition 9.3 (Core entropy of polynomials). The *core entropy* of f, denoted by $h(f)$, is defined to be the topological entropy of the restriction of f to its Hubbard tree H_f, i.e.,

$$h(f) := h_{\mathrm{top}}(f, H_f).$$

9.2. Weak critical markings of postcritically finite polynomials

In order to classify all postcritically finite polynomials up to topological conjugacy, Poirier [24] defined for any postcritically finite polynomial a finite collection of combinatorial data, called a *critical marking*, considering the set of rays landing at the critical points of f.

In this section, we recall the definition of critical marking, and explain how it can be used to compute the core entropy. However, as we will see in Section 9.4, the set of critical markings of postcritically finite polynomials is not closed: indeed, if a sequence (f_n) of polynomials converges to a polynomial f and the corresponding critical markings Θ_n of f_n converge to Θ, then Θ is not necessarily a critical marking of f. To solve this problem, we also introduce the more general notion of *weak critical marking* (see also [11]).

This construction requires the definition of supporting rays/arguments.

Definition 9.4 (supporting rays/arguments). Let U be a bounded Fatou component of a postcritically finite polynomial f, and let $z \in \partial U$. The external rays landing at z divide the plane into finitely many regions. We label the arguments of these rays by $\theta_1, \dots, \theta_k$ in counterclockwise cyclic order, so that U belongs to the region delimited by $R(\theta_1)$ and $R(\theta_2)$ ($\theta_1 = \theta_2$ if there is a single ray landing at z). The ray $R(\theta_1)$ (resp. $R(\theta_2)$) is called the *left-supporting* (resp. *right-supporting*) *ray* of U at z, and the argument θ_1 (resp. θ_2) is called the *left-supporting* (resp. *right-supporting*) *argument* of U at z.

9.2.1. Critical Fatou markings. Let f be a postcritically finite polynomial of degree d, and let $U_1, \dots, U_n$ be its critical Fatou components (i.e., the Fatou components containing a critical point). Following Poirier [24], we now construct for each critical Fatou component U a finite set $\Theta(U)$, whose elements are angles of external rays which land on the boundary of U. Denote $\delta_U := \deg(f|_U)$.

Case 1: *U is a periodic, critical Fatou component.* Let

$$U \mapsto f(U) \mapsto \cdots \mapsto f^n(U) = U$$

be a critical Fatou cycle of period n. We will construct the associated set $\Theta(U')$ for all critical Fatou components U' in this cycle simultaneously. Let $z \in \partial U$ be a periodic point with period $\leq n$. Let θ denote the left-supporting argument of U at z. Clearly, θ is periodic with period n. We call θ a *preferred* angle for U. Note that this choice naturally determines a left-supporting argument of each Fatou component $f^k(U)$ for $k \in \{0, \ldots, n-1\}$, which is called a *preferred angle* of $f^k(U)$. Let U' be a critical Fatou component in the cycle and θ' its preferred angle. We now define $\Theta(U')$ as any set of $\delta_{U'}$ angles such that

(a) $\theta' \in \Theta'(U)$;

(b) the rays corresponding to the elements of $\Theta(U')$ land at $\delta_{U'}$ distinct points of $\partial U'$ and are inverse images of $f(R(\theta'))$.

Case 2: *U is a strictly preperiodic Fatou component.* Let k be minimal such that $U' = f^k(U)$ is a critical Fatou component. We may assume that $\Theta(U')$ is already chosen, according to the previous case. Choose an angle $\theta' \in \Theta(U')$. We define $\Theta(U)$ to be the set of arguments of the δ_U rays landing at δ_U distinct points of ∂U that are k^{th} inverse images of $R(\theta')$.

A *weak critical Fatou marking* is a collection

$$\mathcal{F} = \{\Theta(U_1), \ldots, \Theta(U_s)\}$$

given by the above construction such that the convex hulls in $\overline{\mathbb{D}}$ of $\Theta(U_1), \ldots, \Theta(U_n)$ have pairwise disjoint interiors, where $U_1, \ldots, U_n$ are the critical Fatou components of f. Weak critical Fatou markings are not uniquely determined by f, and there are finitely many choices. If all angles which appear in $\mathcal{F}$ are left-supporting ones, then we call $\mathcal{F}$ a *critical Fatou marking*, which is the original object considered by Poirier.

As an example, we consider the cubic polynomial $f(z) = z^3 + \frac{3}{2}z^2$. The critical point $z = 0$ is fixed, and the other critical point $z = -1$ is mapped to a repelling fixed point $z = 1/2$ (see Figure 9).

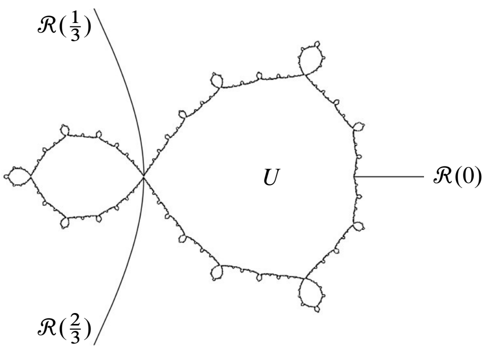


Fig. 9. The Julia set of $f(z) = z^3 + \frac{3}{2}z^2$.

Thus there is only one critical Fatou component U, which contains 0, and the point z of least period on ∂U is $z = 1/2$, hence the preferred angle is $\theta' = 0$. Then we have two choices for a weak critical Fatou marking of f: $\mathcal{F} = \{\Theta(U) = \{0, 1/3\}\}$ is a weak critical Fatou marking, but not a critical Fatou marking, since $R(1/3)$ is not left-supporting for U, while $\mathcal{F} = \{\Theta(U) = \{0, 2/3\}\}$ is a critical Fatou marking of f.

9.2.2. Critical Julia markings. Let c be a critical point which lies in the Julia set of f. Then a *critical Julia portrait element* landing at c is a finite subset Θ of the circle of cardinality ≥ 2 such that

(1) for each $\theta \in \Theta$, the external ray with angle θ lands at c;

(2) all rays $R(\theta)$ with $\theta \in \Theta$ are mapped by f to the same ray.

A *weak critical Julia marking* of f is a finite collection $\mathcal{J} = \{\Theta_1(c_1), \dots, \Theta_m(c_m)\}$ where

(1) each $\Theta_i(c_i)$ is a critical Julia portrait element landing at c_i;

(2) the set $\{c_1, \dots, c_m\}$ equals the set of all critical points of f which lie in the Julia set (however, the c_i need not be distinct!);

(3) any two of the convex hulls in the closed unit disk of $\Theta_1(c_1), \dots, \Theta_m(c_m)$ either are disjoint or intersect at one point on $\partial\mathbb{D}$;

(4) for each critical point $c \in J_f$, we have the formula

$$\deg(f|_c) - 1 = \sum_{c_j = c} (\#\Theta_j(c_j) - 1).$$

Once again, weak critical Julia markings are not uniquely determined by f, and there are finitely many choices. If all c_i are distinct, then we call $\mathcal{J}$ a *critical Julia marking*. Critical Julia markings are the original combinatorial objects defined by Poirier, while we relax the definition by allowing the same critical point in the Julia set to appear with multiplicity.

To show the non-uniqueness, let us consider the following example, which comes from [11]. We consider the postcritically finite polynomial $f_c(z) = z^3 + c$ with $c \approx 0.22036 + 1.18612i$. The critical value c receives two rays with arguments $11/72$ and $17/72$. Then

$$\Theta := \big\{\Theta_1(0) := \{11/216, 83/216\}, \Theta_2(0) := \{89/216, 161/216\}\big\}$$

is a weak critical marking, but not a critical marking, of f_c, and

$$\Theta := \big\{\Theta(0) := \{11/216, 83/216, 155/216\}\big\}$$

is a critical marking of f_c (see Figure 10).

Definition 9.5. A *weak critical marking* of f is a collection

$$\Theta = \{\mathcal{F}, \mathcal{J}\} \tag{9.3}$$

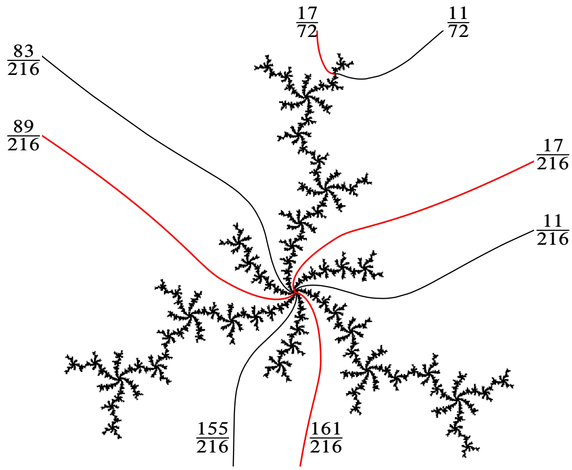


Fig. 10. The Julia set of $f_c(z) = z \mapsto z^3 + 0.22036 + 1.18612i$.

where $\mathcal{F}$ is a weak critical Fatou marking of f and $\mathcal{J}$ is a weak critical Julia marking of f, such that the convex hulls in $\overline{\mathbb{D}}$ of the elements of Θ have pairwise disjoint interiors.

Note that a weak critical marking Θ of a postcritically finite polynomial is a rational critical portrait (but not necessarily a primitive major! – see [24, Example 2.7]). Thus, we will denote by $h(\Theta)$ the core entropy associated to this critical portrait. Any such polynomial admits at least one, and in general finitely many weak critical markings. If $\mathcal{F}$ and $\mathcal{J}$ are actually a critical Fatou marking and a critical Julia marking (as opposed to weak ones), then we call Θ a *critical marking* of f.

9.3. Convergence of external rays

In the proof of Theorem 9.1, a result about the convergence of external rays (Lemma 9.15) will play an important role. The aim of this section is to prove this result via a sequence of lemmas (some of which are well-known).

Lemma 9.6. *Let $f \in \mathcal{P}_d$ and z be a repelling preperiodic point of f such that the forward orbit of z avoids the critical points of f. Then there exists a neighborhood Λ of f in $\mathcal{P}_d$ and a holomorphic map $\xi_z : \Lambda \to \mathbb{C}$ such that $\xi_z(f) = z$ and $\xi_z(f')$ is the unique repelling preperiodic point of f' near z with the same preperiod and period as z for all $f' \in \Lambda$. The point $\xi_z(f')$ is called the* continuation of z at f'.

This follows directly from the implicit function theorem. Let now $(S_n) \subseteq \mathbb{C}$ be a sequence of sets. We denote by $\limsup S_n$ the set of points $z \in \mathbb{C}$ such that every neighborhood of z intersects infinitely many S_n. It follows immediately from the definition that $\limsup S_n$ is closed.

Lemma 9.7 (Goldberg–Milnor [12]). *For $f \in \mathcal{P}_d$ consider an external ray $R_f(\theta)$ which lands at a repelling preperiodic point z whose orbit avoids the critical points of f. Then $R_{f'}(\theta)$ lands at the continuation of z at f', for all f' in a sufficiently small neighborhood of f. Moreover, if $f_n \to f$ as $n \to \infty$, then $\limsup_n \overline{R_{f_n}(\theta)} = \overline{R_f(\theta)}$.*

Assume that f_n, $n \geq 0$, are polynomials in $\mathcal{P}_d$ with connected Julia set. For each $n \geq 0$, we denote J_{f_n}, K_{f_n} simply by J_n, K_n respectively, the external ray $R_{f_n}(\theta)$ by $R_n(\theta)$ for all $\theta \in \mathbb{R}/\mathbb{Z}$, the infinite Fatou component $U_{f_n}(\infty)$ by $U_n(\infty)$, and the Böttcher coordinate ϕ_{f_n} given in (9.2) by ϕ_n. The following result is well-known.

Lemma 9.8. *Let f_n, $n \geq 0$, be polynomials in $\mathcal{P}_d$ with connected Julia set such that $f_n \to f_0$ as $n \to \infty$. Then the inverse ψ_n of the Böttcher coordinate ϕ_n converges to $\psi_0 := \phi_0^{-1}$ uniformly on any compact subset of $\mathbb{C} \setminus \overline{\mathbb{D}}$.*

Proof. Let $\psi_n : \mathbb{C} \setminus K(f_n) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ be the Riemann map of $K(f_n)$. Since the $K(f_n)$ are uniformly bounded, the image of ψ_n contains a ball around ∞ of uniform radius. Hence, the family (ψ_n) is precompact; let ψ_0 be any limit. Each ψ_n satisfies the Böttcher equation $\psi_n(z^d) = f_n(\psi_n)$, hence in the limit one gets $\psi_0(z^d) = f_0(\psi_0)$, so ψ_0 is the Böttcher map for f_0. ∎

Lemma 9.9. *Let f_n, $n \geq 0$, be polynomials in $\mathcal{P}_d$ with connected Julia set such that $f_n \to f_0$ as $n \to \infty$. For each argument t and any sequence $\sigma = (t_n)_{n\geq 1}$ of arguments converging to t, let $B_\sigma(t) := \limsup \overline{R_n(t_n)}$ and $K_\sigma(t) := B_\sigma(t) \cap K_0$, and also denote $\tau(\sigma) := (\tau(t_n))_{n\geq 1}$ (it converges to $\tau(t)$). Then*

(1) *$B_\sigma(t) \cap U_0(\infty) = R_0(t)$, so that $B_\sigma(t) = R_0(t) \cup K_\sigma(t)$;*

(2) *the sets $B_\sigma(t)$ and $K_\sigma(t)$ are connected, $f_0(B_\sigma(t)) \subseteq B_{\tau(\sigma)}(\tau(t))$ and $f_0(K_\sigma(t)) \subseteq K_{\tau(\sigma)}(\tau(t))$.*

Proof. (1) On the one hand, let $z_n \in \overline{R_n(t_n)}$, $n \geq 1$, converge to $z \in B_\sigma(t)$, and let the potential of z_n be s_n. By choosing a subsequence if necessary, we assume that $s_n \to s \geq 0$ as $n \to \infty$. It is known that the Green functions $G_n(z)$ uniformly converge to $G_0(z)$ on $\mathbb{C}$ [8, Proposition 8.1], so $z \in U_0(\infty)$ if and only if $s > 0$. In the case of $s > 0$, by Lemma 9.8, the points $z_n = \psi_n(e^{s_n+2\pi i t_n})$ converge to $z = \psi_0(e^{s+2\pi i t}) \in R_0(t)$. On the other hand, given any $s > 0$, by Lemma 9.8, we have

$$R_n(t_n) \ni \psi_n(e^{s+2\pi i t_n}) \to \psi_0(e^{s+2\pi i t}) \in R_0(t).$$

Since s is arbitrary, it follows that $\overline{R_0(t)}$ is contained in $B_\sigma(t)$.

(2) Let x be a point of $\overline{R_0(t)} \cap K_0$, which belongs to $K_\sigma(t) \subseteq B_\sigma(t)$ by (1). Let now y be another point of $B_\sigma(t) \cap K_0$: by definition, there exists a sequence y_{n_k} such that $y_{n_k} \in R_{n_k}(t_{n_k})$ and $y_{n_k} \to y$. By applying (1) to this subsequence, there exists a further subsequence (which we still denote by x_{n_k}) of points which converge to x and satisfy $x_{n_k} \in R_{n_k}(t_{n_k})$. Let c_{n_k} be the segment of the ray $R_{n_k}(t_{n_k})$ connecting x_{n_k} and y_{n_k}, and let c be a Hausdorff limit of the segments c_{n_k}. Then by construction the set c is a connected, compact set which contains x and y, and it is also a subset of $B_\sigma(t)$, proving that $B_\sigma(t)$ is connected. Note that x and y belong to K_0, so the potentials of x_{n_k}

and y_{n_k} with respect to f_{n_k} converge to 0. This implies that the limit c of c_n belongs to $B_\sigma(t) \cap K_0 = K_\sigma(t)$, proving that $K_\sigma(t)$ is connected.

Since $f_0(R_0(t)) = R_0(\tau(t))$, it remains to show that $f_0(K_\sigma(t)) \subseteq K_{\tau(\sigma)}(\tau(t))$. Let $z \in K_\sigma(t)$. Then there exist $z_n \in R_n(t_n)$ with potential s_n such that $z_n \to z$ and $s_n \to 0$ as $n \to \infty$. Since f_n uniformly converges to f_0, $w_n := f_n(z_n)$ converges to $w := f_0(z)$ as $n \to \infty$. On the one hand, note that $w_n \in R_n(\tau(t_n))$, so $w \in B_{\tau(\sigma)}(\tau(t))$. On the other hand, the potentials of w_n are ds_n, converging to 0, so $w \in K_0$. It follows that $w = f(z) \in K_{\tau(\sigma)}(\tau(t))$. ∎

The next lemma comes directly from [6, Lemma 6.3].

Lemma 9.10. *Let f_n, $n \geq 0$, be postcritically finite polynomials in $\mathcal{P}_d$ such that $f_n \to f_0$ as $n \to \infty$. Let U_0 be a Fatou component of f_0. Then the center of U_0 is contained in a Fatou component of f_n, denoted by U_n, for all sufficiently large n. Furthermore, any given compact subset of U_0 is contained in U_n for all sufficiently large n. The Fatou component U_n is called the* deformation of U_0 at f_n.

Lemma 9.11. *Let f_n, $n \geq 0$, be postcritically finite polynomials in $\mathcal{P}_d$ such that $f_n \to f_0$ as $n \to \infty$. Let U_0 be a Fatou component of f_0, and U_n the deformation of U_0 at f_n for each sufficiently large n. Then the centers of U_n converge to that of U_0 as $n \to \infty$, and* $\deg(f_n|_{U_n}) = \deg(f_0|_{U_0})$ *for all sufficiently large n. Furthermore, for any preperiodic point $z \in \partial U_0$, there is a unique point $z_n \in \partial U_n$, having the same preperiod and period as z, such that $z_n \to z$ as $n \to \infty$. The point z_n is called the* continuation of z at ∂U_n.

Proof. Let x_n be the center of U_n. If x_0 is periodic, the continuation y_n of x at f_n is an attracting periodic point contained in U_n (by the Implicit Function Theorem and Lemma 9.10). Hence $z_n = p_n$. Let us now deal with the preperiodic case by induction. Assume that $f_n(x_n) \to f_0(x_0)$ as $n \to \infty$: we need to show that $x_n \to x_0$ and $\deg(f_n|_{U_n}) = \deg(f_0|_{U_0})$ as $n \to \infty$. Set $\delta := \deg(f_0|_{U_0})$. By Rouché's theorem, any given small neighborhood of p_0 contains exactly δ preimages by f_n of $f_n(x_n)$ (counted with multiplicity) for every sufficiently large n. Note that all these preimages belong to U_n by Lemma 9.10, and are the centers of some Fatou components of f_n. So these preimages must coincide with x_n. It follows that $x_n \to x_0$ as $n \to \infty$ and $\deg(f_n|_{U_n}) = \delta$ for all sufficiently large n.

For the remaining result of this lemma, we first assume that z is periodic. Then z is repelling because f_0 is postcritically finite. In this case, the conclusion holds by Goldberg and Milnor's proof in [12, Appendix B]. Now, let $z \in \partial U_0$ be a preperiodic point. Set $v := f_0(z) \in \partial f(U_0)$. Inductively, we assume that v_n is the unique preperiodic point of f_n in $\partial f_n(U_n)$ such that v_n has the same preperiod and period as v, and $v_n \to v$ as $n \to \infty$. Since f_n uniformly converges to f_0, given any small disk neighborhood W_z of z, there is a disk neighborhood V_v of v such that the component of $f_n^{-1}(V_v)$ that contains z, denoted by $D_{n,z}$, belongs to W_z, for all sufficiently large n and $n = 0$. Given any sufficiently large n, choose a point $a_n \in D_{n,z} \cap U_n$ and set $b_n := f_n(a_n)$. Then $b_n \in V_v \cap f_n(U_n)$. By the inductive assumption, the point v_n belongs to $\partial f_n(U_n) \cap V_v$. One can then choose an arc $\gamma_n \subseteq f_n(U_n) \cap V_v$ joining b_n and v_n. Lifting γ_n by f_n with the starting point a_n, we get an arc $\widetilde{\gamma}_n \subseteq D_{n,z} \cap U_n$. Its endpoint, denoted by z_n, belongs to ∂U_n and satisfies $f_n(z_n) = v_n$. By the argument above, we in fact proved that for any point $z' \in \partial U_0$ with $f_0(z') = v$, and any small neighborhood $W_{z'}$ of z', there exists a point $z'_n \in \partial U_n$ such that $z'_n \in W_{z'}$ and $f_n(z'_n) = v_n$ for all sufficiently large n. Since $\deg(f_n|_{U_n}) = \deg(f_0|_{U_0})$, the points which have the same properties as z'_n are unique. This completes the proof of the lemma. ∎

Lemma 9.12. *Let $f_n, n \geq 0$, be postcritically finite polynomials in $\mathcal{P}_d$ such that $f_n \to f_0$ as $n \to \infty$. Let U_0 be a Fatou component of f_0, and U_n the deformation of U_0 at f_n for each large n. Suppose I_n is a preperiodic internal ray of f_n in U_n with fixed preperiod $k \geq 0$ and period $p \geq 1$. If the landing point z_n of I_n converges to z, then $\limsup_{n\to\infty} I_n = I$, where I is the internal ray of f in U_0 landing at z.*

Proof. If I is periodic, the conclusion holds by Goldberg and Milnor's proof in [12, Appendix B]. By induction on k, it then suffices to prove $\limsup I_n = I$ provided that $\limsup f_n(I_n) = f(I)$. Since $f_n \to f$, we can choose Böttcher coordinates φ_0 of U_0 and φ_n of U_n such that $\varphi_n^{-1} : D \to U_n$ converges uniformly on compact sets to $\varphi_0^{-1} : D \to U_0$. It follows that $I' := \limsup I_n \cap U_0$ is an internal ray of U. On the other hand, $\limsup I_n \cap \partial U_0$ is compact, connected and contains the point z. The map f sends this set into $\limsup f_n(I_n) \cap \partial f(U_0)$, which is by induction a singleton. Thus $\limsup I_n \cap \partial U_0 = \{z\}$, and hence $I' = I$. ∎

Lemma 9.13. *Let $f_n, n \geq 0$, be postcritically finite polynomials in $\mathcal{P}_d$ such that $f_n \to f_0$ as $n \to \infty$. Let U_0 be a Fatou component of f_0, and U_n the deformation of U_0 at f_n. If θ is the left-supporting* (*resp. right-supporting*) *angle of U_0 at a periodic point z, then θ is also the left-supporting* (*resp. right-supporting*) *angle of U_n at z_n for all large n, where z_n denotes the continuation of z at f_n.*

Proof. We just prove this lemma in the case where θ is a left-supporting angle for U_0. The proof of the right-supporting case is exactly the same. Let $\theta_1, \ldots, \theta_s$ be the external angles associated with z in the counterclockwise direction with $\theta_1 = \theta$. In this case, all $\theta_1, \ldots, \theta_s$ are periodic with a common period, and z is a repelling periodic point. By Lemma 9.11, the continuation z_n of z at f_n belongs to ∂U_n for all large n, and Lemma 9.7 implies that the external rays of f_n with arguments $\theta_1 \ldots, \theta_s$ land at z_n.

Pick a point $p \in U_0$. We denote by W the component of $\mathbb{C} \setminus (R_0(\theta_1) \cup R_0(\theta_2))$ that contains p. Since $\limsup R_n(\theta_i) = R_0(\theta_i)$ for all $i = 1, \ldots, s$, for each sufficiently large n there exists a unique component of $\mathbb{C} \setminus (R_n(\theta_1) \cup R_n(\theta_2))$ that contains p, which we denote by W_n. Note that $p \in U_n$ and U_n is contained in a component of $\mathbb{C} \setminus (R_n(\theta_1) \cup R_n(\theta_2))$, so $U_n \subseteq W_n$ for all sufficiently large n. We denote by (θ_1, θ_2) the set of arguments we meet when traveling on $\mathbb{R}/\mathbb{Z}$ from θ_1 to θ_2 in the counterclockwise direction.

For contradiction, and passing to a subsequence if necessary, one can assume that θ_1 is not the left-supporting angle of U_n at z_n for all sufficiently large n. For each n, we denote by η_n the left-supporting angle of U_n at z_n. By the argument in the last paragraph, each η_n belongs to (θ_1, θ_2). Note also that each η_n has the same period as θ_1, so by choosing a subsequence if necessary, one can assume $\eta_n = \eta \in (\theta_1, \theta_s)$ for all sufficiently large n. But then, by Lemma 9.7, the ray $R_0(\eta)$ also lands at z, contradicting the fact that θ_1 is the left-supporting angle for U_0 at z. ∎

Lemma 9.14. *Let f be a postcritically finite polynomial, and $S \subseteq J_f$ be a connected compact set with more than one point. Let $[z, w]$ denote the regulated arc in K_f joining $z \neq w \in S$.*

(1) *Every component of $S \cap [z, w]$ is an arc or a point in J_f; and every component of $[z, w] \setminus S$ is the union of two internal rays of a Fatou component U.*

(2) *If (a, b) is a component of $[z, w] \setminus S$ with $b \notin \{z, w\}$, then b is a preperiodic point in the boundary of the Fatou component containing (a, b).*

(3) *If $[z, w] \subseteq S$, then the open arc (z, w) contains a preperiodic point whose forward orbit avoids the critical points of f.*

Proof. (1) The first conclusion is obvious because $[z, w]$ is an arc. To prove the second one, let (a, b) be a component of $[z, w] \setminus S$. Then there exists a bounded component D of $\mathbb{C} \setminus ([z, w] \cup S)$ such that ∂D contains (a, b). Note that D belongs to the interior of K_f, so it belongs to a Fatou component U. It follows that $\partial D \setminus (a, b) \subseteq \overline{U} \cap S \subseteq \partial U$. Hence, $a, b \in \partial U$ and (a, b) is the union of the two internal rays in U landing at a and b.

(2) In this case, b is a biaccessible point, i.e., there are at least two external rays landing at b, and is in the boundary of a Fatou component according to (1). By Lemma 9.2, all its sufficiently high iterates by f are intersections of periodic Fatou components and the Hubbard tree. Since there are only finitely many such points, then b is preperiodic.

(3) As f is postcritically finite, then it is expanding in a neighborhood of J_f in the sense that, given a neighborhood W of J_f, there exist constants $\lambda > 1$ such that for any arc $\gamma \subseteq J_f$ with $f^n : \gamma \to \mathbb{C}$ injective,

$$\text{length}(f^n(\gamma)) \geq \lambda^n \,\text{length}(\gamma), \tag{9.4}$$

where length$(\cdot)$ denotes the length of arcs in the canonical orbifold metric of f (see [8, Section 4], [21, Section 19] and [18, Section A.3]).

We denote by $\mathcal{A}$ the set of open regulated arcs (c, ξ) satisfying the conditions

(a) c is a critical point of f and $[c, \xi] \subseteq H_f \cap J_f$;

(b) (c, ξ) avoids the postcritical points of f and the branching points of H_f;

(c) length$((c, \xi)) = \kappa$, where κ is a sufficiently small universal constant.

It is clear that $\mathcal{A}$ contains finitely many elements.

We claim:

($\star$) the preperiodic points whose forward orbit avoids the critical points of f are dense in each member of $\mathcal{A}$.

Given an arc γ, a point z on γ, and $\delta > 0$, we say that z is *δ-contained* in γ if γ contains an open arc of length 2δ with center z. To represent such an arc, we use the notation

$$D_\delta^\gamma(z) := \{w \in \gamma : \text{length}([z, w]) < \delta\}.$$

To prove ($\star$), let γ_1 be any element of $\mathcal{A}$, and pick a point $a \in \gamma_1$ and $\epsilon > 0$. Now choose a number $\delta_1 < \epsilon/2$ such that a is $2\delta_1$-contained in γ_1. Since f is expanding, the forward iterates of any open segment in γ_1 will eventually contain a critical point of f. It follows that there exists a sufficiently large integer n_1 with $\kappa/\lambda^{n_1} < \delta_1$ and a segment $[z_1, w_1] \subseteq D^{\gamma_1}_{\delta_1}(a)$ such that $[z_2, w_2] = f^{n_1}([z_1, w_1])$ belongs to an element of $\mathcal{A}$, denoted γ_2. Let $\delta_2 > 0$ be such that z is δ_2-contained in γ_2 for every $z \in [z_2, w_2]$. By shrinking $[z_1, w_1]$ if necessary, one can find an integer n_2 with $\kappa/\lambda^{n_2} < \delta_2$ such that $[z_3, w_3] := f^{n_2}([z_2, w_2])$ is contained in an element of $\mathcal{A}$, denoted γ_3. Repeating this process $N := \#\mathcal{A}$ times, we obtain the segments $[z_i, w_i]$ and the elements γ_i of $\mathcal{A}$ for $i = 1, \dots, N+1$, and the numbers n_i, δ_i for $i = 1, \dots, N$, such that

- $[z_i, w_i] \subseteq \gamma_i \in \mathcal{A}$;
- every $z \in [z_i, w_i]$ is δ_i-contained in γ_i;
- $\kappa/\lambda^{n_i} < \delta_i$;
- $f^{n_i}([z_i, w_i]) = [z_{i+1}, w_{i+1}]$.

For each $i \in \{1, \dots, N\}$, we denote by β_i the lift of γ_{i+1} by f^{n_i} that contains $[z_i, w_i]$.

We claim:

($\star\star$) $\beta_i \subseteq D^{\gamma_i}_{\delta_i}(z_i) \subseteq \gamma_i$.

Since f is uniformly expanding on J_f and by the choice of n_i and δ_i, the length of β_i satisfies

$$\text{length}(\beta_i) \le \text{length}(\gamma_{i+1})/\lambda^{n_i} = \kappa/\lambda^{n_i} < \delta_i.$$

So it is enough to prove that $\beta_i \subseteq \gamma_i$. Indeed, otherwise there must be a point $p \in \beta_i \cap \gamma_i$ which is a branch point of $\gamma_i \cup \beta_i$. By property (b) in the construction of $\mathcal{A}$, the first n_i terms in the orbit of p contain no critical points of f. Then $f^{n_i}(p)$ is a branch point of $f^{n_i}(\beta_i) \cup f^{n_i}(\gamma_i)$; now, $f^{n_i}(\beta_i) = \gamma_{i+1}$ is a subset of H_f, and moreover $\gamma_i \subseteq H_f$, so also $f^{n_i}(\gamma_i) \subseteq H_f$. Thus, $f^{n_i}(p)$ is a branch point of the Hubbard tree H_f, which contradicts property (b) and completes the proof of claim ($\star\star$).

Since $\#\mathcal{A} = N$, there exist $i < j \in \{1, \dots, N+1\}$ such that $\gamma_i = \gamma_j$. Denote by γ_i' the pullback of $\gamma_j = \gamma_i$ along the orbit from $[z_i, w_i]$ to $[z_j, w_j]$. It follows from ($\star\star$) that $\gamma_i' \subseteq D^{\gamma_i}_{\delta_i}(z_i) \subseteq \gamma_i$. Then the attracting map

$$(f^{n_i})^{-1} : \gamma_i \to \gamma_i' \subseteq \gamma_i$$

has a fixed point. Hence $D^{\gamma_i}_{\delta_i}(z_i)$ contains a periodic point, which is off the orbits of the critical points of f by (2). Consequently, $D^{\gamma_1}_{\delta_1}(z_1) \subseteq D^{\gamma_1}_{\epsilon}(a)$ contains a preperiodic point whose orbit avoids the critical points of f. Note that $\gamma_1, a \in \gamma_1$ and ϵ are all arbitrary, so claim ($\star$) is proven.

Since $[z, w] \subseteq J_f$, by shrinking $[z, w]$ if necessary, each of z, w receives at least two rays of f. By Lemma 9.2, z and w are eventually mapped into the Hubbard tree by iterations of f. By shrinking $[z, w]$ again if necessary, one can assume that $[z', w'] := f^n([z, w]) \subseteq H_f \cap J_f$. Since f is expanding, some iteration of $[z', w']$ must contain a critical point of f, and hence intersect some element of $\mathcal{A}$. It follows from ($\star$) that $[z, w]$ contains preperiodic points whose forward orbits avoid the critical points of f. ■

Using the previous lemmas, we can prove the following convergence result.

Lemma 9.15. *Let f_n, $n \geq 0$, be postcritically finite polynomials in $\mathcal{P}_d$ such that $f_n \to f_0$ as $n \to \infty$. If the angles θ_n converge to an angle θ, then $\limsup \overline{R_n(\theta_n)} = \overline{R_0(\theta)}$, and the landing points of $R_n(\theta_n)$ converge to that of $R_0(\theta)$.*

Proof. Note that if the first conclusion holds, then the second one follows directly. So we just need to prove $\limsup \overline{R_n(\theta_n)} = \overline{R_0(\theta)}$. We follow the notation of Lemma 9.9. Set $\sigma := (\theta_n)_{n\geq 1}$. It is enough to show that $K_\sigma(\theta)$ is a singleton.

If this is not the case, then by Lemma 9.9 the set $K_\sigma(\theta)$ is connected and contains a point w distinct from the landing point z of $R_0(\theta)$ which belongs to $K_\sigma(\theta)$. Moreover, $K_\sigma(\theta)$ is contained in J_0: indeed, if there exists $z \in K_\sigma(\theta)$ which belongs to a Fatou component U, then by Lemma 9.10 it also belongs to its deformation U_n for n large, hence it cannot be an accumulation point of the rays $R_n(\theta_n)$. Let $[z, w]$ denote the regulated arc in K_0.

In the case of $[z, w] \not\subseteq K_\sigma(\theta)$, the segment $[z, w]$ passes through a Fatou component U of f_0. We choose an arc Γ separating z, w as follows. Pick x, x' in different components of $\partial U \setminus [z, w]$ such that their orbits avoid the critical points of P. We denote by $R_0(t), R_0(t')$ the external rays landing at x, x' respectively, and by I, I' the internal rays in U landing at x, x' respectively. The arc Γ is defined as

$$\Gamma := R_0(t) \cup \overline{I} \cup \overline{I'} \cup R_0(t').$$

It clearly separates z and w. By Lemma 9.7, the rays $R_n(t), R_n(t')$ land at the continuations x_n, x_n' of x, x' respectively, and they belong to the boundary of U_n by Lemma 9.11, where U_n is the deformation of U at f_n. We thus obtain an arc $\Gamma_n := R_n(t) \cup \overline{I_n} \cup \overline{I_n'} \cup R_n(t')$ for each sufficiently large n, with I_n, I_n' the internal rays in U_n landing at x_n, x_n' respectively.

In the case of $[z, w] \subseteq K_\sigma(\theta)$ $(\subseteq J_f)$, by Proposition 9.14(3) there exists a preperiodic point $b \in (z, w)$ whose forward orbit avoids the critical points of f_0. We pick two rays $R_0(t), R_0(t')$ landing at b such that the simple curve $\Gamma := R_0(t) \cup \{b\} \cup R_0(t')$ separates z and w. By Lemma 9.7, the rays $R_n(\alpha)$ and $R_n(\beta)$ land at the continuation b_n of b at f_n for sufficiently large n. Then we get a sequence of simple curves $\Gamma_n := R_n(t) \cup \{b_n\} \cup R_n(t')$ for all large n.

In either case, according to Lemmas 9.7 and 9.12, we have $\limsup_{n\to\infty} \Gamma_n = \Gamma$. Note that Γ separates z, w. Then Γ_n separates z, w for all sufficiently large n. On the other hand, by taking a subsequence if necessary, we can assume that $R_n(\theta_n)$ is close to both z and w for large n. It follows that there exist infinitely many n for which $\Gamma_n \cap R_n(\theta_n) \neq \emptyset$, a contradiction. ■

9.4. The limit of critical markings

Proposition 9.16. *Let f_n, $n \geq 1$, and f be postcritically finite polynomials in $\mathcal{P}_d$ such that $f_n \to f$ as $n \to \infty$, and let Θ_n be a critical marking of f_n for each large n. If (Θ_n) Hausdorff converges to Θ as $n \to \infty$, then Θ is a weak critical marking of f.*

Before proving Proposition 9.16, we show by two examples that the limit Θ is not necessarily a critical marking of f. These examples are in fact the motivation for us to define weak critical markings.

Example 9.17. We first consider the cubic polynomial $f(z) = z^3 + 3z^2/2$ (see Figure 9). Let $(f_n)_{n\geq 1}$ be a sequence of postcritically-finite cubic polynomials in $\mathcal{P}_3$ converging to f such that the rays with argument $1/3 + \epsilon_n, 2/3 + \epsilon_n$ land at the unique Julia critical point of f_n, where $\epsilon_n \to 0^+$ as $n \to \infty$. We denote by Γ_n the union of the external rays of f_n with arguments $1/3 + \epsilon_n, 2/3 + \epsilon_n$, together with their common landing point. Thus, the deformation U_n of U is contained in the right-side component of $\mathbb{C} \setminus \Gamma_n$, and hence $R_n(1/3)$ lands at ∂U_n but $R_n(2/3)$ does not. As a consequence, we obtain a critical marking Θ_n of each f_n as

$$\Theta_n := \big\{\mathcal{F}_n = \{0, 1/3\}, \mathcal{J}_n = \{1/3 + \epsilon_n, 2/3 + \epsilon_n\}\big\}.$$

Clearly Θ_n Hausdorff converges to $\Theta = \{\{0, 1/3\}, \{1/3, 2/3\}\}$, and Θ is a weak critical marking of f but not a critical marking (since $R(1/3)$ is not left-supporting for U).

The second example is based on the cubic polynomial f_{c_0} given in Figure 10, which admits two critical Julia markings $\{11/216, 83/216, 155/216\}$ and $\{17/216, 89/216, 161/216\}$. Consider $\Theta = \{\Theta_1 := \{\frac{11}{216}, \frac{83}{216}\}, \Theta_2 := \{\frac{89}{216}, \frac{161}{216}\}\}$, a rational formal critical portrait of degree 3 which is a weak critical marking, but not a critical marking, of f_{c_0}. The forward orbits of arguments in Θ are

$$\Theta_1 \to \tfrac{11}{72} \to \tfrac{11}{24} \to \tfrac{3}{8} \rightleftarrows \tfrac{1}{8}, \qquad \Theta_2 \to \tfrac{17}{72} \to \tfrac{17}{24} \to \tfrac{1}{8} \rightleftarrows \tfrac{3}{8}.$$

By perturbing f_{c_0}, one can find a sequence (f_n) of postcritically finite polynomials (with two distinct critical points) with $f_n \to f_{c_0}$, and such that each f_n admits a critical marking of the form

$$\Theta_n := \{\Theta_{n,1} := \Theta_1 + \epsilon_{n,1}, \Theta_{n,2} := \Theta_2 + \epsilon_{n,2}\},$$

with $\epsilon_{n,1}, \epsilon_{n,2} \to 0$ as $n \to \infty$. Then each Θ_n is a critical marking of f_n and $\Theta_n \to \Theta$, but Θ is not a critical marking of f_{c_0}.

Proof of Proposition 9.16. Let U be a critical Fatou component of f and denote by U_n the deformation of U at f_n. Pick a critical marking for f_n, and let $\Theta(U_n)$ be the element associated to U_n in this marking.

In the periodic case, each $\Theta(U_n)$ contains a unique periodic angle θ_n with period equal to that of U_n and hence of U. By taking a subsequence if necessary, we can assume $\theta_n = \theta$ for large n. Note that any $\Theta(U_n)$ is a subset of $\tau^{-1}(\tau(\theta))$ and $\#\Theta(U_n) = \deg(f_n|_{U_n}) = \deg(f|_U)$ (by Lemma 9.11), so we can further assume by taking a subsequence that $\Theta(U_n)$ is constant for large n, so we can write $\Theta(U_n) = \Theta$. According to Lemmas 9.12 and 9.15, the rays with arguments in Θ land at the boundary of U. Furthermore, it follows from Lemma 9.13 that the periodic angle in Θ is left-supporting for U. In the strictly preperiodic case for U, by a similar argument and induction, we still find that $\Theta(U_n) = \Theta$ is constant for large n, so that $\#\Theta = \deg(f|_U)$ and the external rays of f with arguments in Θ land at the boundary of U.

Let $U_1, \dots, U_s$ be all the critical Fatou components of f. The discussion above shows that the collection $\{\Theta(U_1), \dots, \Theta(U_s)\}$ of sets is part of the critical marking Θ_n of f_n for all large n (by taking subsequences), and it is also a weak critical Fatou marking of f as defined in Subsection 9.2.1.

Now, we write each Θ_n as $\Theta_n := \{\mathcal{F}, \mathcal{L}_n\}$ with

$$\mathcal{F} := \{\Theta(U_1), \dots, \Theta(U_s)\} \quad \text{and} \quad \mathcal{L}_n := \{\Theta_{n,1}, \dots, \Theta_{n,m}\},$$

for all large n, such that $\Theta_{n,i} \to \Theta_i$ as $n \to \infty$ and $1 \le i \le m$. It follows immediately that $\#\Theta_{n,i} = \#\Theta_i$ for any $1 \le i \le m$. Note that each $\Theta_{n,i}$ corresponds to a critical point $c_{n,i}$ of f_n, and we can assume by taking subsequences that $c_{n,i}$ converges to a critical point c_i of f, which must belong to J_f.

We claim that if all $c_{n,i}$ are in the Fatou set of f_n, the sequence $(\overline{U(c_{n,i})})$ of closed disks Hausdorff converges to the common landing point of the rays of f_0 with arguments in Θ_i. It is enough to prove this result for any convergent subsequence of $(\overline{U(c_{n,i})})$, so we assume that $(\overline{U(c_{n,i})})$ converges in the Hausdorff metric to a connected compact set S. By Lemma 9.15, the rays of f_0 with arguments in Θ_i land at S. So, to prove the claim, we only need to check that S is a point.

For contradiction, we assume $\#S > 1$ and choose $x \neq y \in S$. For all $n \ge 1$, there exist $x_n \neq y_n \in \overline{U(c_{n,i})}$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. As in the proof of Lemma 9.15, we have $S \subseteq J_0$. Indeed, if there exists $z \in S$ which belongs to a Fatou component U, then by Lemma 9.10 it also belongs to its deformation U_n for n large, hence $U(c_{n,i})$ coincides with U_n, a contradiction.

Let $[x, y]$ be the regulated arc in K_f joining x, y. If $[x, y] \not\subseteq S$, then $[x, y]$ passes through a Fatou component U of f. One can find two preperiodic rays $R_0(\alpha), R_0(\beta)$ of f_0 landing at ∂U such that the orbits of their landing points avoid the critical points of f_0 and the set $R_0(\alpha) \cup U \cup R_0(\beta)$ separates $R_0(\theta)$ and $R_0(\eta)$. By Lemmas 9.7 and 9.11, the rays $R_n(\alpha)$ and $R_n(\beta)$ of f_n land at the boundary of U_n, the deformation of U at f_n, and converge to $R_0(\alpha)$ and $R_0(\beta)$ respectively. This implies that for all sufficiently large n, the simple curves Γ_n, consisting of the union of $R_n(\alpha), R_n(\beta)$ and the internal rays in U_n joining the landing points of $R_n(\alpha), R_n(\beta)$ separate x_n and y_n. Since x_n and y_n lie in the closure of the Fatou component $U(c_{n,i})$, this implies that $U_n = U(c_{n,i})$. This in turn implies that $c_i = \lim_{n\to\infty} c_{n,i}$ is a Fatou critical point, a contradiction to $c_i \in J_f$.

If $[x, y] \subseteq S$ ($\subseteq J_f$), then as shown in the proof of Lemma 9.15, for each sufficiently large n there exists a curve $\Gamma_n := R_n(\alpha) \cup \{z_n\} \cup R_n(\beta)$, consisting of two rays of f_n and their common landing point, separating x_n, y_n, and these Γ_n converge to the simple curve $\Gamma := R_0(\alpha) \cup \{z\} \cup R_0(\beta)$ which separates x and y. Therefore, for each sufficiently large n, we obtain a simple curve Γ_n which separates $x_n, y_n \in \overline{U(c_{n,i})}$ and is disjoint from $U(c_{n,i})$. This is impossible, so the claim is proven.

Now, to show $\Theta := \{\mathcal{F}, \mathcal{J}\}$ is a weak critical marking of f, we just need to check that $\mathcal{J}$ satisfies properties (1)–(4) in the definition of weak critical Julia marking (see Section 9.2.2). By the claim above and Lemma 9.15, for each $i \in \{1, \dots, m\}$, the rays of f with arguments in Θ_i land at the common point c_i of f. On the other hand, the set

$\{c_1, \ldots, c_m\}$ contains all the critical points of f in the Julia set. To see this, note that any critical point $c \in J_f$ is an accumulation point of the critical points of f_n according to Rouché's theorem. Furthermore, by Lemma 9.11, the point c cannot be an accumulation point of the critical points of f_n in the Fatou components corresponding to $\mathcal{F}$, hence it must be an accumulation point of $c_{n,1}, \ldots, c_{n,m}, n \geq 1$. It follows that $c \in \{c_1, \ldots, c_m\}$. The discussion above implies that properties (1)–(3) in the definition of weak critical Julia marking hold for $\mathcal{J}$, so we just need to check (4). Given a critical point $c \in J_f$, let

$$I_c := \big\{ i \in \{1, \ldots, m\} : c_i = c \big\}.$$

Then $c_{n,i} \to c$ as $n \to \infty$ if and only if $i \in I_c$. Note that c is a root of $f'(z)$ with multiplicity $\deg(f|_c) - 1$. Then, by Rouché's theorem, for each sufficiently large n the function f_n' has $\deg(f|_c) - 1$ roots near c, counting with multiplicity. On the other hand, for each sufficiently large n, the points $c_{n,i}$ with $i \in I_c$ are exactly the roots of f_n' near c, and each $c_{n,i}$ has multiplicity (as a root of f_n') equal to

$$\deg(f_n|_{c_{n,i}}) - 1 = \#\Theta(c_{n,i}) - 1 = \#\Theta(c_i) - 1.$$

It follows that the equation in property (4) holds. ∎

9.5. *The continuity of core entropy of polynomials*

Proof of Theorem 9.1. Let Θ_n be a critical marking of the polynomial f_n. Since f has only finitely many weak critical markings, by Proposition 9.16 the sequence (Θ_n) can be subdivided into finitely many Hausdorff convergent subsequences. So it is enough to prove the theorem in the case when (Θ_n) Hausdorff converges to Θ as $n \to \infty$. By Proposition 9.16, the critical portrait Θ is a weak critical marking of f. Note that Theorem 3.4 shows that $h(\Theta_n) = h(f_n)$ and $h(\Theta) = h(f)$. To complete the proof one needs to show $h(\Theta_n) \to h(\Theta)$ as $n \to \infty$. Let m_n denote the primitive major induced by Θ_n, and m the primitive major induced by Θ. By Proposition 3.2(2), the majors m_n converge to m. Moreover, by Lemmas 6.2 and 8.2 one gets the equalities $h(\Theta_n) = h(m_n)$ and $h(\Theta) = h(m)$. It follows from the above and Theorem 1.1 that

$$h(\Theta_n) = h(m_n) \xrightarrow{\text{Thm.1.1}} h(m) = h(\Theta) \quad \text{as } n \to \infty.$$

The theorem is proven. ∎

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