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## Measurable Hall's theorem for actions of abelian groups

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**Abstract.** We prove a measurable version of the Hall marriage theorem for actions of finitely generated abelian groups. In particular, it implies that for free measure-preserving actions of such groups and measurable sets which are suitably equidistributed with respect to the action, if they are equidecomposable, then they are equidecomposable using measurable pieces. The latter generalizes a recent result of Grabowski, Máthé and Pikhurko on the measurable circle squaring and confirms a special case of a conjecture of Gardner.

**Keywords.** Circle squaring, Hall matching theorem, Mokobodzki medial means

### 1. Introduction

In 1925 Tarski famously asked if the unit square and the disk of the same area are equidecomposable by isometries of the plane, i.e. if one can partition one of them into finitely many pieces, rearrange them by isometries and obtain the second one. This problem became known as the Tarski circle squaring problem.

The question whether two sets of the same measure can be partitioned into congruent pieces has a long history. At the beginning of the 19th century Wallace, Bolyai and Gerwien showed that any two polygons in the plane of the same area are congruent by dissections (see [31, Theorem 3.2]) and Tarski [29] ([31, Theorem 3.9]) showed that such polygons are equidecomposable using pieces which are polygons themselves. Hilbert's 3rd problem asked if any two polyhedra of the same volume are equidecomposable using polyhedral pieces. The latter was solved by Dehn (see [1]). Banach and Tarski showed that in dimension at least 3, any two bounded sets in  $\mathbb{R}^n$  with nonempty interior, are equidecomposable, which leads to the famous Banach–Tarski paradox on doubling the ball. Back in dimension 2, the situation is somewhat different, as any two measurable subsets equidecomposable by isometries must have the same measure (see [31]) and this

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was one of the motivation for the Tarski circle squaring problem. Using isometries was also essential as von Neumann [32] showed that the answer is positive if one allows arbitrary area-preserving transformations. The crucial feature that makes the isometries of the plane special is the fact that the group of isometries of  $\mathbb{R}^2$  is amenable. Amenability was, in fact, introduced by von Neumann in the search of a combinatorial explanation of the Banach–Tarski paradox.

The first partial result on the Tarski circle squaring was a negative result of Dubins, Hirsch and Karush [5] who showed that pieces of such decompositions cannot have smooth boundary (which means that this cannot be performed using scissors). However, the full positive answer was given by Laczkovich in his deep paper [16]. In fact, in [17] Laczkovich proved a stronger result saying that whenever  $A$  and  $B$  are two bounded measurable subsets of  $\mathbb{R}^n$  of positive measure such that the upper box dimension of the boundaries of  $A$  and  $B$  is less than  $n$ , then  $A$  and  $B$  are equidecomposable. The assumption on the boundary is essential since Laczkovich [19] (see also [21]) found examples of two measurable sets of the same area which are not equidecomposable even though their boundaries have even the same Hausdorff dimension. The proof of Laczkovich, however, did not provide any regularity conditions on the pieces used in the decompositions. Given the assumption that  $A$  and  $B$  have the same measure, it was natural to ask if the pieces can be chosen to be measurable. Moreover, the proof of Laczkovich used the Axiom of Choice.

A major breakthrough was achieved recently by Grabowski, Máthé and Pikhurko [10] who showed that the pieces in Laczkovich’s theorem can be chosen to be measurable: whenever  $A$  and  $B$  are two bounded subsets of  $\mathbb{R}^n$  of positive measure such that the upper box dimension of the boundaries of  $A$  and  $B$  are less than  $n$ , then  $A$  and  $B$  are equidecomposable using measurable pieces. Another breakthrough came even more recently when Marks and Unger [25] showed that for Borel sets, the pieces in the decomposition can be even chosen to be Borel, and their proof did not use the Axiom of Choice.

The goal of the present paper is to give a combinatorial explanation of these phenomena. There are some limitations on how far this can go because already in Laczkovich’s theorem there is a restriction on the boundary of the sets  $A$  and  $B$ . Therefore, we are going to work in the measure-theoretic context and provide sufficient and necessary conditions for two sets to be equidecomposable almost everywhere. Recently, there has been a lot of effort to develop methods of the measurable and Borel combinatorics (see for instance the upcoming monograph by Marks and Kechris [23]) and we would like to work within this framework.

The classical Hall marriage theorem provides sufficient and necessary conditions for a bipartite graph to have a perfect matching. Matchings are closely connected with the existence of equidecompositions and both have been studied in this context. In 1996 Miller [27, Problem 15.10] asked whether there exists a Borel version of the Hall theorem. The question posed in such generality has a negative answer as there are examples of Borel graphs which admit perfect matchings but do not admit measurable perfect matchings. One example is provided already by the Banach–Tarski paradox (see [23]) and Laczkovich [15] constructed a closed graph which admits a perfect matching but does not have a measurable one. In the Baire category setting, Marks and Unger [24] proved that if

a bipartite Borel graph satisfies a stronger version of Hall’s condition with an additional  $\varepsilon > 0$ , i.e. if the set of neighbors of a finite set  $F$  is bounded from below by  $(1 + \varepsilon)|F|$ , then the graph admits a perfect matching with the Baire property (see also [22] and [4] for related results on matchings in this context). On the other hand, in all the results of Laczkovich [17], Grabowski, Máthé and Pikhurko [10] and Marks and Unger [25] on the circle squaring, a crucial role is played by the strong discrepancy estimates, with an  $\varepsilon > 0$  such that the discrepancies of both sets are bounded by  $C \frac{1}{n^{1+\varepsilon}}$  (for definitions see Section 2). Recall that given a finitely generated group  $\Gamma$  generated by a symmetric set  $S$  and acting freely on a space  $X$ , the *Schreier graph* of the action is the graph connecting two points  $x$  and  $y$  if  $\gamma \cdot x = y$  for one of the generators  $\gamma \in S$ .

**Definition 1.** Suppose  $\Gamma \curvearrowright (X, \mu)$  is a free pmp action of a finitely generated group on a space  $X$ . Write  $G$  for the Schreier graph of the action. A pair of sets  $A, B$  satisfies the *Hall condition* ( $\mu$ -a.e.) with respect to  $\Gamma$  (given a set of generators) if for every ( $\mu$ -a.e.)  $x \in X$  and for every finite subset  $F$  of  $\Gamma \cdot x$  we have

$$|F \cap A| \leq |N_G(F) \cap B|,$$

$$|F \cap B| \leq |N_G(F) \cap A|,$$

where  $N_G(F)$  means the neighborhood of  $F$  in the graph  $G$ .

This definition clearly depends on the choice of generators, and we say that  $A, B$  satisfy the *Hall condition* ( $\mu$ -a.e.) if the above holds for some choice of generators. For the case with a fixed set of generators (which will be more natural for us), we say that the action  $\Gamma \curvearrowright (X, \mu)$  satisfies *k-Hall condition* ( $\mu$ -a.e.) if for every ( $\mu$ -a.e., resp.)  $x \in X$  for every finite subset  $F$  of  $\Gamma \cdot x$  we have

$$|F \cap A| \leq |N_G^k(F) \cap B|,$$

$$|F \cap B| \leq |N_G^k(F) \cap A|,$$

where  $N_G^k(F)$  denotes the  $k$ -neighborhood of  $F$  in the graph  $G$ . Note that  $A, B$  satisfy the Hall condition if and only if  $A, B$  satisfy the  $k$ -Hall condition for some  $k > 0$ . We will work under the assumption that both sets  $A, B$  satisfy certain form of equidistribution on the orbits, namely that they are  $\Gamma$ -uniform (for definition see Section 2).

Our main result is the following.

**Theorem 2.** *Let  $\Gamma$  be a finitely generated abelian group and let  $\Gamma \curvearrowright (X, \mu)$  be a free pmp action. Suppose  $A, B \subseteq X$  are two measurable  $\Gamma$ -uniform sets. The following are equivalent:*

- (1) *The pair  $A, B$  satisfies the Hall condition with respect to  $\Gamma$   $\mu$ -a.e.*
- (2)  *$A$  and  $B$  are  $\Gamma$ -equidecomposable  $\mu$ -a.e. using  $\mu$ -measurable sets.*
- (3)  *$A$  and  $B$  are  $\Gamma$ -equidecomposable  $\mu$ -a.e.*

As a consequence, it gives the following.

**Corollary 3.** *Suppose  $\Gamma$  is a finitely generated abelian group and  $\Gamma \curvearrowright (X, \mu)$  is a free pmp Borel action on a standard Borel probability space. Let  $A, B \subseteq X$  be measurable*

*$\Gamma$ -uniform sets. If  $A$  and  $B$  are  $\Gamma$ -equidecomposable, then  $A$  and  $B$  are  $\Gamma$ -equidecomposable using measurable pieces.*

This generalizes the recent measurable circle squaring result [10] as Laczkovich [16] constructs an action of  $\mathbb{Z}^d$  satisfying the conditions above, for a suitably chosen  $d$  (big enough, depending on the box dimensions of the boundaries).

In fact, in 1991 Gardner [8, Conjecture 6] conjectured that whenever two bounded measurable sets in an Euclidean space are equidecomposable via an action of an amenable group of isometries, then they are equidecomposable using measurable sets. The above corollary confirms this conjecture in case of an abelian group  $\Gamma$  and  $\Gamma$ -uniform sets.

The main new idea in this paper is an application of measurable medial means. These have previously been used in descriptive set theory only in the work of Jackson, Kechris and Louveau [11] on amenable equivalence relations but that context was combinatorially different.<sup>1</sup> They are used together with a recent result of Conley, Jackson, Kerr, Marks, Seward and Tucker-Drob [3] on tilings of amenable group actions in averaging sequences of measurable matchings. This allows us to avoid using Laczkovich's discrepancy estimates that play a crucial role in both proofs of the measurable and Borel circle squaring. We also employ the idea of Marks and Unger in constructing bounded measurable flows. More precisely, following Marks and Unger we construct bounded integer-valued measurable flows from bounded real-valued measurable flows. However, instead of using Timár's result [30] for specific graphs induced by actions of  $\mathbb{Z}^d$ , we give a self-contained simple proof of the latter result, which works in the measurable setting for the natural Cayley graph of  $\mathbb{Z}^d$ . This is the only part of the paper which deals with abelian groups and we hope it could be generalized to a more general setting. On the other hand, the measurable averaging operators that we employ, cannot be made Borel and for this reason the results of this paper apply to the measurable setting and generalize only the results of [10].

While this paper deals with abelian groups (the crucial and only place which works under these assumptions is Section 6), a positive answer to the following question would confirm Gardner's conjecture [8, Conjecture 6].<sup>2</sup>

**Question 4.** Is the measurable version of Hall's theorem true for free pmp actions of finitely generated amenable groups?

## 2. Discrepancy estimates

Both proofs of Grabowski, Máthé and Pikhurko and of Marks and Unger use a technique that appears in Laczkovich's paper [16] and is based on discrepancy estimates. Laczkovich constructs an action of a group of the form  $\mathbb{Z}^d$  for  $d$  depending on the upper box dimension of the boundaries of the sets  $A$  and  $B$  such that both sets are very well equidistributed on orbits on this action. To be more precise, given an action  $\mathbb{Z}^d \curvearrowright (X, \mu)$

<sup>1</sup>As we have learnt recently, a similar idea can be also found in [20, 33].

<sup>2</sup>This has been recently answered in the negative by Kun [14], and the recent preprint [2] proves some optimal results the positive direction.

and a measurable set  $A \subseteq X$ , the *discrepancy* of  $A$  with respect to a finite subset  $F$  of an orbit of the action is defined as

$$D(F, A) = \left| \frac{|A \cap F|}{|F|} - \mu(A) \right|.$$

It is meaningful to compute the discrepancy with respect to finite cubes, i.e. subsets of orbits which are of the form  $[0, n]^d \cdot x$ , where  $x \in X$  and  $[0, n]^d \subseteq \mathbb{Z}^d$  is the  $d$ -dimensional cube with side  $\{0, \dots, n\}$ . The cube  $[0, n]^d$  has boundary, whose relative size with respect to the size of the cube is bounded by  $c \cdot \frac{1}{n}$  for a constant  $c$ .

A crucial estimation that appears in Laczkovich’s paper is that the action of  $\mathbb{Z}^d$  is such that for both sets  $A$  and  $B$  the discrepancy is estimated as

$$D([0, n]^d \cdot x, A), D([0, n]^d \cdot x, B) \leq c \frac{1}{n^{1+\varepsilon}} \tag{*}$$

for some  $\varepsilon > 0$  and some  $c > 0$ , which means that the discrepancies of both sets on cubes decay noticeably faster than the sizes of the boundaries of these cubes.

A slightly more natural condition on the equidistribution of a set  $A$  would be to remove the  $\varepsilon$  from (\*) and require that there exists a constant  $c$  such that for every  $n$  the discrepancy

$$D([0, n]^d \times \Delta \cdot x, A) \leq c \frac{1}{n} \tag{**}$$

for  $\mu$ -a.e.  $x$ . In fact, as shown in [18, Theorem 1.2], condition (\*) implies that (\*\*) is satisfied on every union of cubes in places of  $[0, n]^d$ . Sets satisfying the latter condition are called *uniformly spread* (see [18, Theorem 1.1]). However, in [18, Theorem 1.5], Laczkovich gives examples of sets which satisfy (\*\*) but are not uniformly spread.

In this paper, we work with even weaker assumption on equidistribution, given by the following definition.

**Definition 5.** Given a Borel free pmp action  $\Gamma \curvearrowright (X, \mu)$  of a finitely generated abelian group  $\Gamma = \mathbb{Z}^d \times \Delta$  with  $\Delta$  finite, and a measurable set  $A \subseteq X$ , we say that  $A$  is  $\Gamma$ -uniform if there exists a constant  $c > 0$  such that for  $\mu$ -a.e.  $X$  we have

$$|A \cap (F \cdot x)| \geq c|F|$$

whenever  $F$  is a set of the form  $F = [0, n]^d \times \Delta$ .

Note that this definition does not depend (up to changing the constant  $c$ ) on the way the group is written as  $\mathbb{Z}^d \times \Delta$  and a choice of generators for the group.

### 3. Measurable averaging operators

In this paper we use special kinds of measurable averaging operators. These can be constructed in different ways.

For the first construction, recall the definition of a medial mean.

**Definition 6.** A *medial mean* is a linear functional  $m : \ell_\infty \rightarrow \mathbb{R}$  which is positive, i.e.  $m(f) \geq 0$  if  $f \geq 0$ , normalized, i.e.  $m(1_{\mathbb{N}}) = 1$ , and shift invariant, i.e.  $m(Sf) = m(f)$ , where  $Sf(n) = f(n + 1)$ .

Medial means were studied already by Banach who showed their existence (the so-called *Banach limits*). In this paper we use a special kind of medial means  $m$  which is additionally measurable on  $[0, 1]^{\mathbb{N}}$  and we use it to take a measurable average of a sequence of functions  $f_n : (X, \mu) \rightarrow [0, 1]$  for a space  $(X, \mu)$  with a Borel probability measure  $\mu$ . It takes a bit more effort to construct medial means that are measurable but this can be done in a couple of ways.

Recall that Mokobodzki showed that under the assumption of the Continuum Hypothesis there exists a medial mean which is universally measurable as a function on  $[0, 1]^{\mathbb{N}}$ . For a proof the reader can consult the textbook of Fremlin [6, Theorem 538S] or the article [26]. However, a careful analysis of Mokobodzki's proof shows that for a single Borel probability measure  $\mu$ , the existence of a  $\mu$ -measurable medial mean does not require the Continuum Hypothesis. Nevertheless, some set-theoretical assumptions are still used even for a single  $\mu$ , such as the Hahn–Banach theorem.

Another construction of measurable averaging a sequence of measurable functions  $f_n : X \rightarrow [0, 1]$  (perhaps more familiar to the general mathematical audience than the Mokobodzki construction) can be done using the Banach–Saks theorem in  $L^2(X, \mu)$  by using weak\*-compactness of the unit ball (cf. [20]).

#### 4. Set-theoretical assumptions

In view of the foundational questions and the role of the Axiom of Choice in equidecompositions (e.g. recall that the Hahn–Banach theorem implies the Banach–Tarski paradox [28]), we argue below that for the measurable equidecompositions on co-null sets, we can remove any set theoretic assumptions beyond ZF and the Axiom of Dependent Choice (DC) that are needed to obtain a measurable medial mean.

Recall that Borel sets can be coded using a  $\Pi_1^1$  set (of Borel codes)  $BC \subseteq 2^{\mathbb{N}}$  in a  $\Delta_1^1$  way, i.e. there exists a subset  $C \subseteq BC \times X$  such that the family  $\{C_x : x \in BC\}$  consists of all Borel subsets of  $X$  and the set  $C$  can be defined using both  $\Sigma_1^1$  and  $\Pi_1^1$  definitions. For details the reader can consult the textbook of Jech [12, Chapter 25].

Given a Borel probability measure  $\mu$  on  $X$  and a subset  $P \subseteq X \times Y$ , we write  $\forall^\mu x P(x, y)$  to denote that  $\mu(\{x \in X : P(x, y)\}) = 1$ . It is well known [13, Chapter 29E] that if  $P$  is  $\Sigma_1^1$ , then  $\{y \in Y : \forall^\mu x P(x, y)\}$  is  $\Sigma_1^1$ .

The proposition below implies that if two Borel sets are coded using a real  $r$ , then we can argue about their measurable equidecomposition a.e. in  $L[r]$  (where AC and CH hold).

**Proposition 7.** *Let  $V \subseteq W$  be models of ZF+DC. Suppose in  $V$  we have a standard Borel space  $X$  with a Borel probability measure  $\mu$ , two Borel subsets  $A, B \subseteq X$  and  $\Gamma \curvearrowright (X, \mu)$  is a Borel pmp action of a countable group  $\Gamma$ . The statement that the sets  $A$  and  $B$  are  $\Gamma$ -equidecomposable  $\mu$ -a.e. using  $\mu$ -measurable pieces is absolute between  $V$  and  $W$ .*

*Proof.* Suppose that in  $W$  or  $V$  the sets  $A$  and  $B$  are  $\Gamma$ -equidecomposable  $\mu$ -a.e. Then there exist disjoint Borel subsets  $A_1, \dots, A_n$  of  $A$  and disjoint Borel subsets  $B_1, \dots, B_n$

of  $B$  such that  $\mu(A \setminus \bigcup_{i=1}^n A_i) = 0, \mu(B \setminus \bigcup_{i=1}^n B_i) = 0$  and  $\gamma_i A_i = B_i$  for some elements  $\gamma_1, \dots, \gamma_n \in \Gamma$ . This statement can be written as

$$\begin{aligned} &\exists x_1, \dots, x_n \bigwedge_{i \leq n} \text{BC}(x_i) \wedge \bigwedge_{i \neq j} C_{x_i} \cap C_{x_j} = \emptyset \\ &\wedge \forall^\mu x \left( x \in A \leftrightarrow \bigvee_{i=1}^n x \in C_{x_i} \right) \wedge \forall^\mu x \left( x \in B \leftrightarrow \bigvee_{i=1}^n x \in \gamma_i C_{x_i} \right) \end{aligned}$$

and thus is it  $\Sigma_2^1$ . By Shoenfield’s absoluteness theorem [12, Theorem 25.20], it is absolute between  $V$  and  $W$ . ■

**5. Measurable flows in actions of amenable groups**

Given a standard Borel space  $X$ , a Borel graph  $G$  on  $X$  and  $f : X \rightarrow \mathbb{R}$ , a function  $\varphi : G \rightarrow \mathbb{R}$  is an  $f$ -flow if

- $\varphi(x, y) = -\varphi(y, x)$  for every  $(x, y) \in G$  and
- $f(x) = \sum_{(x,y) \in G} \varphi(x, y)$  for every  $x \in X$ .

Let  $\Gamma$  be a finitely generated amenable group. Let  $\gamma_1, \dots, \gamma_d$  be a finite symmetric set of generators of  $\Gamma$ . Let  $X$  be a standard Borel space and let  $\mu$  be a Borel probability measure on  $X$ . Let  $\Gamma \curvearrowright (X, \mu)$  be a free pmp action. Recall that by the *Schreier graph* of the action we mean the graph  $\{(x, \gamma_i x) : x \in X, 1 \leq i \leq d\} \subseteq X \times X$ .

**Definition 8.** For finite sets  $F, K \subseteq \Gamma$  and  $\delta > 0$  we say that  $F$  is  $(K, \delta)$ -invariant if  $|KF \Delta F| < \delta|F|$ .

In the following lemma we assume that there exists a universally measurable medial mean  $m$ , which, by the remarks in the previous section, we can assume throughout this paper.

In order to make it a bit more general, let us define the Hall condition for functions: a function  $f : X \rightarrow \mathbb{Z}$  satisfies the  $k$ -Hall condition if for every finite set  $F$  contained in an orbit of  $\Gamma \curvearrowright X$  we have that

$$\sum_{\substack{x \in F \\ f(x) \geq 0}} f(x) \leq \sum_{\substack{x \in N_G^k(F) \\ f(x) \leq 0}} -f(x), \quad \sum_{\substack{x \in F \\ f(x) \leq 0}} -f(x) \leq \sum_{\substack{x \in N_G^k(F) \\ f(x) \geq 0}} f(x).$$

Note that a pair of sets  $A, B$  satisfies the  $k$ -Hall condition if and only if  $f = \chi_A - \chi_B$  satisfies the  $k$ -Hall condition.

**Proposition 9.** Let  $\Gamma$  be a finitely generated amenable group and  $\Gamma \curvearrowright (X, \mu)$  be a Borel free pmp action. Suppose  $f : X \rightarrow \mathbb{Z}$  is a measurable function such that

- $|f| \leq l$ ,
- $f$  satisfies the  $k$ -Hall condition,

for some  $k, l \in \mathbb{N}$ . Then there exists a  $\Gamma$ -invariant measurable subset  $X' \subseteq X$  of measure 1 and a measurable real-valued  $f$ -flow  $\phi$  on the Schreier graph of  $\Gamma \curvearrowright X'$  such

that

$$|\phi| \leq l \cdot d^k,$$

where  $d$  is the number of generators of  $\Gamma$ .

*Proof.* First, we are going to assume that  $|f| \leq 1$ , i.e. that  $f = \chi_A - \chi_B$  for two measurable subsets  $A, B \subseteq X$ . Indeed, replace  $X$  with  $X \times l$  and take the projection

$$\pi : X \times l \rightarrow X.$$

Then we can find two subsets  $A, B \subseteq X \times l$  such that

$$f(x) = |\pi^{-1}(\{x\}) \cap A| - |\pi^{-1}(\{x\}) \cap B|.$$

We can also induce the graph structure on  $X \times l$  by taking as edges the pairs  $((x, i), (y, j))$  such that  $(x, y)$  forms an edge in  $X$  as well as all pairs  $((x, i), (x, j))$  for  $i \neq j$ . Then  $A$  and  $B$  satisfy the  $k$ -Hall condition in  $X \times l$  for the above graph.

Let  $K = \{\gamma \in \Gamma : d(e, \gamma) \leq k\}$ . Fix  $\delta > 0$ . Use [3, Theorem 3.6] for  $K$  and  $\delta$  to get a  $\mu$ -conull  $\Gamma$ -invariant Borel set  $X' \subseteq X$ , a collection  $\{C_i : 1 \leq i \leq m\}$  of Borel subsets of  $X'$ , and a collection  $\{F_i : 1 \leq i \leq m\}$  of  $(K, \delta)$ -invariant subsets of  $\Gamma$  such that  $\mathcal{F} = \{F_i c : 1 \leq i \leq m, c \in C_i\}$  partitions  $X'$ .

For a finite set  $F \subseteq \Gamma$  define  $F(K) = \{f \in F : Kf \subseteq F\}$ . Note that if  $F'x = F''y$ , where  $F', F''$  are finite subsets of  $\Gamma$  and  $x, y \in X$ , then  $F'(K)x = F''(K)y$ . If  $F \subseteq X$  is a finite subset of a single orbit, then we let  $F(K) = F'(K)x$  where  $F' \subseteq \Gamma$  and  $x \in X$  satisfy  $F = F'x$ . This definition does not depend on the choice of representation  $F = F'x$  by the previous remark. Note that if  $F \subseteq X$  is  $(K, \delta)$ -invariant then

$$|F(K)| \geq |F| - |KF \Delta F| \cdot |K| > |F| \cdot (1 - \delta|K|).$$

Write

$$H = \{(x, \gamma x) \in A \times B : x \in F_i(K) \cdot c \text{ for some } 1 \leq i \leq m \text{ and } c \in C_i, \gamma \in K\}.$$

Then  $H$  is a locally finite Borel graph satisfying Hall's condition as  $A, B$  satisfy the  $k$ -Hall condition. By the Hall theorem, there exists a Borel injection

$$h : A \cap \bigcup_{F \in \mathcal{F}} F(K) \rightarrow B \cap \bigcup_{F \in \mathcal{F}} F.$$

Write  $G$  for the Schreier graph of  $\Gamma \curvearrowright X$ . For every  $x \in \text{dom } h$  let

$$p_x = \{(x_0, x_1), (x_1, x_2), \dots, (x_{j-1}, x_j)\}$$

be the shortest lexicographically smallest path in the graph  $G$  connecting  $x_0 = x$  with  $x_j = h(x)$ . Let  $\mathcal{P} = \{p_x : x \in \text{dom } h\}$ .

Define  $\phi : G \rightarrow \mathbb{R}$  by the formula

$$\phi(x, \gamma x) = |\{p \in \mathcal{P} : (x, \gamma x) \in p\}| - |\{p \in \mathcal{P} : (\gamma x, x) \in p\}|.$$

Note that  $\phi$  is Borel (by definition). Also,  $|\phi|$  is bounded by  $d^k$  (the number of paths of length not greater than  $k$  passing through a given edge in the graph  $G$ ). By definition,  $\phi$  is a  $(\chi_{\text{dom } h} - \chi_{\text{im } h})$ -flow.



Define

$$X'' = \bigcup_{F \in \mathcal{F}} F(K) \setminus (B \setminus h(A)).$$

Note that for every  $x \in X''$  we have  $\chi_A(x) - \chi_B(x) = \chi_{\text{dom } h}(x) - \chi_{\text{im } h}(x)$ .

For every  $1 \leq i \leq m$  let  $\{(A_{1,i}, B_{1,i}, h_{1,i}), (A_{2,i}, B_{2,i}, h_{2,i}), \dots, (A_{n_i,i}, B_{n_i,i}, h_{n_i,i})\}$  be the set of all triples  $(A', B', h')$  consisting of two subsets  $A', B' \subseteq F_i$  and a bijection  $h' : A' \rightarrow B'$ . For  $1 \leq j \leq n_i$  define

$$C_{j,i} = \{c \in C_i : (\text{dom } h_{j,i})c = A \cap (F_i c) \wedge \forall \gamma \in \text{dom } h_{j,i} \ h_{j,i}(\gamma)c = h(\gamma c)\}.$$

Then  $\{C_{1,i}, C_{2,i}, \dots, C_{n_i,i}\}$  is a partition of  $C_i$  into Borel sets.

Observe that for every  $F \in \mathcal{F}$  we have

$$|h(A) \cap F(K)| \geq |F(K) \cap A| - |F \setminus F(K)|$$

and

$$|B \cap F(K)| \leq |A \cap F| \leq |A \cap F(K)| + |F \setminus F(K)|.$$

Therefore

$$\begin{aligned} |F(K) \cap (B \setminus h(A))| &= |(F(K) \cap B) \setminus (F(K) \cap h(A))| \\ &= |F(K) \cap B| - |F(K) \cap h(A)| \\ &\leq |A \cap F(K)| + |F \setminus F(K)| - (|F(K) \cap A| - |F \setminus F(K)|) \\ &= 2|F \setminus F(K)|. \end{aligned}$$

It follows that

$$\begin{aligned} |F(K) \setminus (B \setminus h(A))| &= |F(K)| - |F(K) \cap (B \setminus h(A))| \\ &\geq |F(K)| - 2|F \setminus F(K)| \\ &= 3|F(K)| - 2|F| \\ &> |F|(1 - 3\delta|K|). \end{aligned}$$

Therefore

$$\begin{aligned} \mu(X'') &= \mu\left(\bigcup_{i=1}^m \bigcup_{j=1}^{n_i} (F_i(K)C_{j,i} \setminus (B \setminus h(A)))\right) \\ &= \sum_{i=1}^m \sum_{j=i}^{n_i} |F_i(K) \setminus (B_{j,i} \setminus A_{j,i})| \mu(C_{j,i}) \\ &> \sum_{i=1}^m \sum_{j=i}^{n_i} |F_i|(1 - 3\delta|K|) \mu(C_{j,i}) \\ &= \sum_{i=1}^m |F_i|(1 - 3\delta|K|) \mu(C_i) \\ &= 1 - 3\delta|K|. \end{aligned}$$

Now, for every  $n$  pick  $\delta_n > 0$  so that  $1 - 3\delta_n|K| > 1 - \frac{1}{2^n}$ . Denote  $h_n = h$ ,  $\phi_n = \phi$  and  $X_n = X''$  where  $h$ ,  $\phi$  and  $X''$  are constructed above for this particular  $\delta_n$ .

Let  $Y = \liminf X_n = \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty X_n$ . Then  $\mu(Y) = 1$ . We can assume that  $Y$  is  $\Gamma$ -invariant (by taking its subset if needed). Denote by  $G$  the Schreier graph of  $\Gamma \curvearrowright Y$ . Write  $\phi_\infty = (\phi_n)_{n \in \mathbb{N}} : G \rightarrow \ell^\infty$ . Define

$$\phi(x, y) = m(\phi_\infty(x, y)),$$

where  $m$  denotes the medial mean. Then for  $x \in Y$  we have

$$\begin{aligned} \sum_{\substack{y \\ (x,y) \in G}} \phi(x, y) &= \sum_{\substack{y \\ (x,y) \in G}} m((\phi_n(x, y))_{n \in \mathbb{N}}) \\ &= m\left(\left(\sum_{\substack{y \\ (x,y) \in G}} \phi_n(x, y)\right)_{n \in \mathbb{N}}\right) \\ &= m((\chi_{\text{dom } h_n}(x) - \chi_{\text{im } h_n}(x))_{n \in \mathbb{N}}) \\ &= \chi_A(x) - \chi_B(x) \end{aligned}$$

as the sequence  $\chi_{\text{dom } h_n}(x) - \chi_{\text{im } h_n}(x)$  is eventually constant and equal to  $\chi_A(x) - \chi_B(x)$ .

Therefore  $\phi$  is a  $(\chi_A - \chi_B)$ -flow in the Schreier graph  $G$  of  $\Gamma \curvearrowright Y$ . Moreover,  $|\phi|$  is bounded by  $d^k$ , which is a common bound for the flows  $\phi_n$ . For measurability of  $\phi$ , write  $\mu' = \phi_*(\mu \times \mu)$  for the pushforward to  $[-d^k, d^k]^{\mathbb{N}}$  of the measure  $\mu \times \mu$  on the graph  $G$  and note that since  $m$  is  $\mu'$ -measurable, it follows that  $\phi$  is  $\mu$ -measurable. ■

### 6. Flows in $\mathbb{Z}^d$

In this section we prove a couple of combinatorial lemmas which lead to a finitary procedure of changing a real-valued flow on a cube in  $\mathbb{Z}^d$  to an integer-valued flow on a cube in  $\mathbb{Z}^d$ . This gives an alternative proof of [25, Lemma 5.4] in the measurable setting. Also, this is the only part of the paper which deals with the groups  $\mathbb{Z}^d$  as opposed to arbitrary amenable groups.

Let

$$G = \{(x, x') \in \mathbb{Z}^d \times \mathbb{Z}^d : x' - x \in \{\pm e_1, \pm e_2, \dots, \pm e_d\}\}$$

be the Cayley graph of  $\mathbb{Z}^d$ . An edge  $(x, x')$  is called *positively oriented* if  $x' - x = e_j$  for some  $j$ .

**Definition 10.** For a set  $A \subseteq \mathbb{Z}^d$  we define

$$\begin{aligned} E(A) &= \{(x, x + e_j) : j \in \{1, 2, \dots, d\}, \{x, x + e_j\} \subseteq A\}, \\ E^+(A) &= \{(x, x + e_j) : j \in \{1, 2, \dots, d\}, \{x, x + e_j\} \cap A \neq \emptyset\}, \\ N(A) &= \{x + y : x \in A, y \in \{-1, 0, 1\}^d\}. \end{aligned}$$

So,  $E(A)$  is the set of positively oriented edges whose both endpoints are in  $A$ ,  $E^+(A)$  is the set of positively oriented edges whose at least one endpoint is in  $A$ , and  $N(A)$  is the neighborhood of  $A$  (in the sup-norm).

**Definition 11.** We say that a subset  $C$  of  $\mathbb{Z}^d$  is a *cube* if  $C$  is of the form

$$\{n_1, n_1 + 1, \dots, n_1 + k_1\} \times \dots \times \{n_d, n_d + 1, \dots, n_d + k_d\}$$

for some  $n_1, \dots, n_d, k_1, \dots, k_d \in \mathbb{Z}$  with  $k_1, \dots, k_d \geq 0$ . By the *upper face* of  $C$  we mean

$$\{n_1, n_1 + 1, \dots, n_1 + k_1\} \times \dots \times \{n_d + k_d\}.$$

**Definition 12.** For any  $x_1, x_2, x_3, x_4 \in \mathbb{Z}^d$  which are consecutive vertices of a unit square and a real number  $s$  we define a 0-flow  $\square_s^{x_1, x_2, x_3, x_4}$  by the following formula:

$$\square_s^{x_1, x_2, x_3, x_4}(y, z) = \begin{cases} s & \text{for } (y, z) \in \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1)\}, \\ -s & \text{for } (y, z) \in \{(x_2, x_1), (x_3, x_2), (x_4, x_3), (x_1, x_4)\}, \\ 0 & \text{otherwise.} \end{cases}$$

That is,  $\square_s^{x_1, x_2, x_3, x_4}$  is a flow sending  $s$  units through the path

$$x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_1.$$

Note that if  $\varphi : G \rightarrow \mathbb{R}$  is an  $f$ -flow and  $s = \varphi(x_1, x_4) - \lfloor \varphi(x_1, x_4) \rfloor$ , then

$$\psi = \varphi + \square_s^{x_1, x_2, x_3, x_4}$$

is an  $f$ -flow such that  $|\varphi - \psi| < 1$  and  $\psi(x_1, x_4)$  is an integer.

We will now prove a couple of lemmas stating that one can modify a flow so that it becomes integer-valued on certain sets of edges.

**Lemma 13.** Let  $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ . Let  $\varphi : G \rightarrow \mathbb{R}$  be a bounded  $f$ -flow. Let

$$C = \{n_1, n_1 + 1, \dots, n_1 + k_1\} \times \dots \times \{n_{d-1}, n_{d-1} + 1, \dots, n_{d-1} + k_{d-1}\} \times \{n_d, n_d + 1\}$$

for some  $n_1, \dots, n_d, k_1, \dots, k_{d-1} \in \mathbb{Z}$  with  $k_1, \dots, k_{d-1} \geq 0$ . Then for every  $1 \leq \ell < d$  there is an  $f$ -flow  $\psi$  such that:

- $\text{supp}(\varphi - \psi) \subseteq E(C)$ ,
- for every  $x = (x_1, \dots, x_{d-1}, n_d) \in C$  such that  $n_\ell \leq x_\ell < n_\ell + k_\ell$  we have

$$\psi(x, x + e_d) \in \mathbb{Z},$$

- $|\varphi - \psi| < 2$ .

*Proof.* Without loss of generality we may assume that  $n_1 = n_2 = \dots = n_d = 0$ .

For every  $j \leq k_\ell$  define

$$C_j = \{(x_1, \dots, x_{d-1}, 0) \in C : x_\ell = j\}.$$

We will define a sequence of  $f$ -flows  $\varphi_0, \varphi_1, \dots, \varphi_{k_\ell}$  such that

$$\varphi_j(x, x + e_d) \in \mathbb{Z} \quad \text{and} \quad \text{supp}(\varphi - \varphi_j) \subseteq E(C)$$

for all  $x \in \bigcup_{i < j} C_i$ .

So, let  $\varphi_0 = \varphi$ . Given  $\varphi_j$  we define  $\varphi_{j+1}$  in the following way. For every  $x \in C_j$  let  $\square_x = \square_s^{x,y,z,t}$ , where  $y = x + e_\ell, z = y + e_d, t = z - e_\ell = x + e_d$  and

$$s = \varphi_j(x, t) - \lfloor \varphi_j(x, t) \rfloor.$$

We define

$$\varphi_{j+1} = \varphi_j + \sum_{x \in C_j} \square_x.$$

Note that  $\text{supp}(\square_x)$  for  $x \in C_j$  are disjoint from  $\{(x, x + e_d) : x \in \bigcup_{i < j} C_i\}$ . Therefore,  $\varphi_{j+1}(x, x + e_d) = \varphi_j(x, x + e_d) \in \mathbb{Z}$  for  $x \in \bigcup_{i < j} C_i$ . Also, the sets  $\text{supp}(\square_x)$  are pairwise disjoint for  $x \in C_j$ , and therefore, by the definition of  $\varphi_{j+1}$  we have for  $x \in C_j$

$$\varphi_{j+1}(x, x + e_d) = \varphi_j(x, x + e_d) + \square_x(x, x + e_d) = \lfloor \varphi_j(x, x + e_d) \rfloor \in \mathbb{Z}.$$

It is also clear that  $\text{supp}(\square_x) \subseteq E(C)$ , so

$$\text{supp}(\varphi - \varphi_{j+1}) \subseteq \text{supp}(\varphi - \varphi_j) \cup \bigcup_{x \in C_j} \text{supp}(\square_x) \subseteq E(C).$$

Therefore  $\varphi_{j+1}$  satisfies all required properties.

We put  $\psi = \varphi_{k_\ell}$ . It remains to check that  $|\varphi - \psi| < 2$ . This is because

$$\psi = \varphi + \sum_{j=0}^{k_\ell-1} \sum_{x \in C_j} \square_x, \quad |\square_x| < 1,$$

and for every edge  $(y, z)$  there are at most two  $x \in \bigcup_{j < k_\ell} C_j$  for which  $\square_x(y, z) \neq 0$ . ■

**Lemma 14.** *Let  $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ . Let  $\varphi : G \rightarrow \mathbb{R}$  be a bounded  $f$ -flow. Let*

$C = \{n_1, n_1 + 1, \dots, n_1 + k_1\} \times \dots \times \{n_{d-1}, n_{d-1} + 1, \dots, n_{d-1} + k_{d-1}\} \times \{n_d, n_d + 1\}$  for some  $n_1, \dots, n_d, k_1, \dots, k_{d-1} \in \mathbb{Z}$  with  $k_1, \dots, k_{d-1} \geq 0$ . Then there is an  $f$ -flow  $\psi$  such that:

- $\text{supp}(\varphi - \psi) \subseteq E(C)$ ,
- if  $x = (x_1, \dots, x_{d-1}, n_d) \in C \setminus \{(n_1 + k_1, n_2 + k_2, \dots, n_{d-1} + k_{d-1}, n_d)\}$ , then  $\psi(x, x + e_d) \in \mathbb{Z}$ ,
- $|\varphi - \psi| < 2d$ .

*Proof.* Without loss of generality we may assume that  $n_1 = n_2 = \dots = n_d = 0$ .

Define

$$C_j = \{k_1\} \times \dots \times \{k_{\ell-1}\} \times \{0, 1, \dots, k_\ell\} \times \dots \times \{0, 1, \dots, k_{d-1}\} \times \{0, 1\}$$

and

$$D_j = \{(x_1, \dots, x_{d-1}, 0) : (x_1, \dots, x_j) \neq (k_1, \dots, k_j)\}.$$

By induction, construct  $f$ -flows  $\varphi_0, \varphi_1, \dots, \varphi_{d-1}$  such that

- (i)  $\text{supp}(\varphi - \varphi_j) \subseteq E(C)$ ,
- (ii)  $\varphi_j(x, x + e_d) \in \mathbb{Z}$  for every  $x \in D_j$ ,
- (iii)  $|\varphi_j - \varphi_{j-1}| < 2$ .

We define  $\varphi_0 = \varphi$ . Given  $\varphi_{j-1}$ , we obtain  $\varphi_j$  by applying Lemma 13 for  $\varphi_{j-1}$ ,  $f$ ,  $\ell = j$  and  $C_j$ . Then  $\varphi_j$  satisfies (i) as

$$\text{supp}(\varphi - \varphi_j) \subseteq \text{supp}(\varphi - \varphi_{j-1}) \cup \text{supp}(\varphi_{j-1} - \varphi_j) \subseteq E(C) \cup E(C_j) = E(C).$$

For (ii) observe that

$$D_j = D_{j-1} \cup \{(k_1, \dots, k_{j-1}, x_j, \dots, x_{d-1}, 0) \in C : x_j < k_j\}.$$

By Lemma 13,  $\varphi_j$  agrees with  $\varphi_{j-1}$  on  $\{(x, x + e_d) : x \in D_{j-1}\}$ , thus  $\varphi_j(x, x + e_d) \in \mathbb{Z}$  for  $x \in D_{j-1}$ . Moreover,  $\varphi_j(x, x + e_d) \in \mathbb{Z}$  for  $x \in D_j \setminus D_{j-1}$  again by Lemma 13. Also (iii) is immediate by Lemma 13. Therefore  $\varphi_j$  satisfies the required properties.

We define  $\psi = \varphi_{d-1}$ . By construction,  $\psi$  satisfies the first two conditions. For the third condition note that

$$|\varphi - \psi| \leq \sum_{j=1}^{d-1} |\varphi_j - \varphi_{j-1}| < 2d. \quad \blacksquare$$

**Lemma 15.** *Let  $C$  be a cube. Let  $\mathcal{C}$  be a collection of cubes such that:*

- $N(C') \subseteq C$  for every  $C' \in \mathcal{C}$ ,
- $N(C') \cap N(C'') = \emptyset$  for every distinct  $C', C'' \in \mathcal{C}$ .

Write

$$E = E^+(C) \setminus \bigcup \{E(N(C')) : C' \in \mathcal{C}\}.$$

Let  $f : \mathbb{Z}^d \rightarrow \mathbb{Z}$ . Let  $\varphi : G \rightarrow \mathbb{R}$  be a bounded  $f$ -flow. Then there exists an  $f$ -flow  $\psi : G \rightarrow \mathbb{R}$  such that:

- $\text{supp}(\varphi - \psi) \subseteq E(N(C))$ ,
- $\text{supp}(\varphi - \psi)$  is disjoint from  $E^+(C')$  for every  $C' \in \mathcal{C}$ ,
- $\psi(e)$  is integer for every edge  $e \in E$ ,
- $|\varphi - \psi| < 6d$ .

*Proof.* Without loss of generality we may assume that

$$C = \{1, 2, \dots, k_1\} \times \dots \times \{1, 2, \dots, k_d\}$$

for some positive integers  $k_1, \dots, k_d$  and

$$N(C) = \{0, 1, \dots, k_1 + 1\} \times \dots \times \{0, 1, \dots, k_d + 1\}.$$

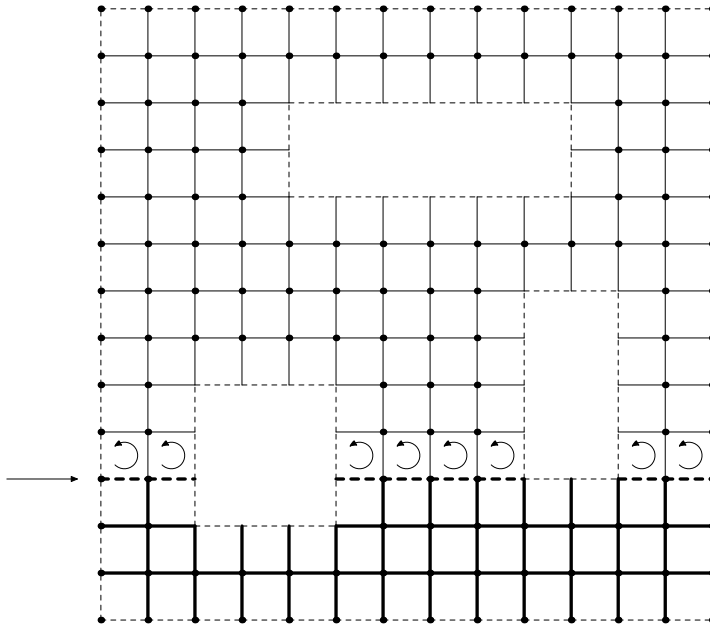
For any  $0 \leq k \leq k_d$  let  $H_k = \mathbb{Z}^{d-1} \times \{k\}$ . Let

$$E_{2k} = \{(x, x + e_d) \in E : x \in H_k\}$$

be the set of vertical edges from  $E$  having their starting point in  $H_k$  and let

$$E_{2k+1} = \{(x, x + e_j) \in E : x \in H_k, j < d\}$$

be the set of edges from  $E$  having both endpoints in  $H_k$ .



**Fig. 1.** Construction of  $\varphi_{2k+1}$ .

We construct a sequence  $\varphi_0, \varphi_1, \dots, \varphi_{2k_d}$  of  $f$ -flows so that

- $\text{supp}(\varphi - \varphi_k) \subseteq E(N(C))$  for every  $0 \leq k \leq 2k_d$ ,
- $\text{supp}(\varphi - \varphi_k)$  is disjoint from  $E^+(C')$  for every  $C' \in \mathcal{C}$  and  $0 \leq k \leq 2k_d$ ,
- $\varphi_k(y, z)$  is integer for every  $0 \leq k \leq 2k_d$  and  $(y, z) \in \bigcup_{i \leq k} E_i$ .

In the end we put  $\psi = \varphi_{2k_d}$ .

To define  $\varphi_0$  we use Lemma 13 for  $\varphi, f, \ell = 1$ , and the cube

$$\{1, 2, \dots, k_1 + 1\} \times \{1, 2, \dots, k_2\} \times \{1, 2, \dots, k_3\} \times \dots \times \{1, 2, \dots, k_{d-1}\} \times \{0, 1\}.$$

Suppose that the  $f$ -flow  $\varphi_{2k}$  is defined. Now we define  $\varphi_{2k+1}$  (cf. Figure 1). For every edge  $(x, y) \in E_{2k+1}$  let  $z = y + e_d, t = x + e_d, s = -\varphi_{2k}(x, y) + \lfloor \varphi_{2k}(x, y) \rfloor$  and  $\square_{(x,y)} = \square_s^{x,y,z,t}$ . Define

$$\varphi_{2k+1} = \varphi_{2k} + \sum_{(x,y) \in E_{2k+1}} \square_{(x,y)}.$$

Note that  $\varphi_{2k+1}$  assumes integer values on all  $(x, y) \in E_{2k+1}$ . Indeed, if  $(x', y') \in E_{2k+1}$  is distinct from  $(x, y)$ , then  $\square_{(x',y')}(x, y) = 0$  and so

$$\varphi_{2k+1}(x, y) = \varphi_{2k}(x, y) + \square_{(x,y)}(x, y) = \lfloor \varphi_{2k}(x, y) \rfloor \in \mathbb{Z}.$$

Moreover, by definition,  $\varphi_{2k+1}$  agrees with  $\varphi_{2k}$  on  $\bigcup_{i \leq 2k} E_i$ . It follows that  $\varphi_{2k+1}$  is integer-valued on  $\bigcup_{i \leq 2k+1} E_i$ .

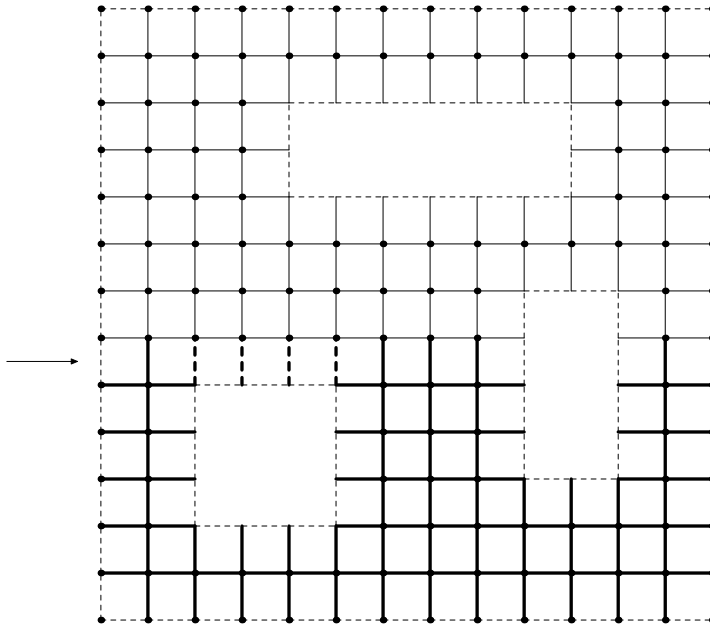


Fig. 2. Construction of  $\varphi_{2k}$ .

Since for every  $(x, y) \in E_{2k+1}$  we have

$$\text{supp}(\square_{(x,y)}) \subseteq E(N(C)) \quad \text{and} \quad \text{supp}(\square_{(x,y)}) \cap E^+(C') = \emptyset$$

for every  $C' \in \mathcal{C}$ , and  $\varphi_{2k}$  satisfies these as well by inductive hypothesis, we see that  $\varphi_{2k+1}$  also has these properties.

Thus  $\varphi_{2k+1}$  is as required.

Now suppose that  $\varphi_{2k+1}$  is defined. We construct  $\varphi_{2k+2}$  (cf. Figure 2). Let

$$D = \{x : (x, x + e_d) \in E_{2k+2}\}.$$

Note that every  $x \in D$  is either an element of  $C \setminus \bigcup\{N(C') : C' \in \mathcal{C}\}$  or lies on the upper face of some cube  $N(C')$  for  $C' \in \mathcal{C}$ . We also note that if  $C' \in \mathcal{C}$  then the upper face of  $N(C')$  is either contained in  $D$  or disjoint from  $D$ . So, let  $C_1, C_2, \dots, C_n$  be all elements of  $\mathcal{C}$  such that the upper faces  $D_1, D_2, \dots, D_n$  of  $N(C_1), N(C_2), \dots, N(C_n)$  are subsets of  $D$ .

Let  $(x, x + e_d) \in E_{2k+2}$ . Then either  $x \in D_j$  for some  $j \leq n$  or  $x \in D \setminus \bigcup_{j \leq n} D_j$ .

First we deal with the case  $x \in D \setminus \bigcup_{j \leq n} D_j$ . Then

$$(x - e_d, x) \in E_{2k} \quad \text{and} \quad (x, x + e_i), (x - e_i, x) \in E_{2k+1}$$

for every  $1 \leq i \leq d - 1$ . By inductive hypothesis,

$$\varphi_{2k+1}(x, x \pm e_1), \varphi_{2k+1}(x, x \pm e_2), \dots, \varphi_{2k+1}(x, x \pm e_{d-1}), \varphi_{2k+1}(x, x - e_d) \in \mathbb{Z}.$$

Since  $f(x) \in \mathbb{Z}$  and

$$f(x) = \sum_{i=1}^d \varphi_{2k+1}(x, x \pm e_i),$$

it follows that  $\varphi_{2k+1}(x, x + e_d) \in \mathbb{Z}$ .

Next we deal with the case  $x \in D_j$  for some  $j \leq n$ . Each  $D_j, j \leq n$  is dealt with separately. For every  $j \leq n$  we obtain an  $f$ -flow  $\varphi'_j$  by applying Lemma 14 for  $\varphi_{2k+1}, f$  and the cube

$$\begin{aligned} D'_j &= D_j \cup (D_j + e_d) \\ &= \{n'_1, \dots, n'_1 + k'_1\} \times \dots \times \{n'_{d-1}, \dots, n'_{d-1} + k'_{d-1}\} \times \{n'_d, n'_d + 1\}. \end{aligned}$$

Then  $\varphi'_j$  agrees with  $\varphi_{2k+1}$  outside of  $E(D'_j)$ , and  $\varphi'_j$  is also integer-valued on all edges of the form  $(x, x + e_d)$  with  $x \in D_j \setminus \{x'\}$ , where

$$x' = (n'_1 + k'_1, n'_2 + k'_2, \dots, n'_{d-1} + k'_{d-1}, n'_d).$$

The only problematic edge is the one  $(x', x' + e_d)$ . We claim that  $\varphi'_j(x', x' + e_d)$  is integer as well.

Indeed, observe that

$$\sum_{x \in N(C_j)} f(x) = \sum_{\substack{(x,y) \in E \\ x \in N(C_j), y \notin N(C_j)}} \varphi'_j(x, y)$$

Since  $f(x) \in \mathbb{Z}$  for every  $x$  and, by the properties of  $\varphi'_j$ , we have that  $\varphi'_j(x, y) \in \mathbb{Z}$  for all  $(x, y) \neq (x', x' + e_d)$  with  $x \in N(C_j)$  and  $y \notin N(C_j)$ , it follows that  $\varphi'_j(x', x' + e_d) \in \mathbb{Z}$  as well.

We define  $\varphi_{2k+2}$  by the formula

$$\varphi_{2k+2}(x, y) = \begin{cases} \varphi'_j(x, y) & \text{if } (x, y) \in E(D'_j) \text{ or } (y, x) \in E(D'_j) \text{ for some } j, \\ \varphi_{2k+1}(x, y) & \text{otherwise.} \end{cases}$$

Note that  $\varphi_{2k+2}$  is well-defined because  $E(D'_j)$  are pairwise disjoint. By definition, it is integer-valued on  $\bigcup_{i \leq 2k+2} E_i$ , and the conditions on  $\text{supp}(\varphi - \varphi_{2k+2})$  are clearly satisfied. Thus  $\varphi_{2k+2}$  is as required.

We put  $\psi = \varphi_{2k_d}$ . It remains to check that  $|\varphi - \psi| < 6d$ . This follows from the fact that the value on every edge was modified at most three times by at most  $2d$ . ■

### 7. Measurable bounded $\mathbb{Z}$ -flows a.e.

In this section we show how to turn a measurable bounded real-valued flow into a measurable bounded integer-valued flow on a set of measure 1. We only use Lemma 15 proved in the previous section and the Gao–Jackson tiling theorem for actions of  $\mathbb{Z}^d$ .

Suppose  $\mathbb{Z}^d \curvearrowright (X, \mu)$  is a free pmp action. We follow the notation from the previous section in the context of the action.



**Definition 16.** We say that a finite subset of  $X$  is a *cube* if it is of the form

$$\left(\prod_{i=1}^d k_i\right) \cdot x = (\{0, 1, \dots, k_1\} \times \dots \times \{0, 1, \dots, k_d\}) \cdot x$$

for some positive integers  $k_1, \dots, k_d$  and  $x \in X$ . We refer to the numbers  $k_1, \dots, k_d$  as to the lengths of the sides of the cube. A family of cubes  $\{(\prod_{i=1}^d k_i(x)) \cdot x : x \in C\}$  is *Borel* if the set  $C$  is Borel and the functions  $k_i$  are Borel. A family of cubes  $\{C_x : x \in C\}$  is a *tiling* of  $X$  if it forms a partition of  $X$ .

**Definition 17.** Let  $\mathcal{C} \subseteq [X]^{<\infty}$  be a collection of cubes. We say that it is *nested* if for every distinct  $C, C' \in \mathcal{C}$ :

- if  $C \cap C' = \emptyset$ , then  $N(C) \cap N(C') = \emptyset$ ,
- if  $C \cap C' \neq \emptyset$ , then either  $N(C) \subseteq C'$  or  $N(C') \subseteq C$ .

**Definition 18.** Given a cube of the form

$$C = \{(n_1, \dots, n_d) \cdot x : 0 \leq n_i \leq N_i\},$$

by its *interior* we mean the cube

$$\text{int } C = \{(n_1, \dots, n_d) \cdot x : 1 \leq n_i \leq N_i - 1\}$$

and its *boundary* is

$$\text{bd } C = C \setminus \text{int } C.$$

**Lemma 19.** Suppose  $\mathbb{Z}^d \curvearrowright (X, \mu)$  is a free pmp action. Then there is a sequence of families  $F_n$  of cubes such that each  $F_n$  consists of disjoint cubes,  $\bigcup F_n$  is nested and covers  $X$  up to a set of measure zero.

*Proof.* If  $S$  and  $T$  are families of sets, define

$$S \sqcap T = \{C \cap C' : C \in S, C' \in T, C \cap C' \neq \emptyset\}.$$

Note that  $\bigcup(S \sqcap T) = (\bigcup S) \cap (\bigcup T)$ . Also note that if  $S$  and  $T$  are families of cubes then  $S \sqcap T$  is a family of cubes as well. We also write  $\text{int } S = \{\text{int } C : C \in S\}$  and  $\text{int}^k$  for the  $k$ -th iterate of  $\text{int}$ .

Use the Gao–Jackson theorem [7] to obtain a sequence of partitions  $S_1, S_2, \dots$  of  $X$  so that  $S_n$  consists of cubes with sides  $n^3$  or  $n^3 + 1$ . Define  $S_n^1 = \text{int } S_n$  and

$$S_n^k = S_n^{k-1} \sqcap \text{int}^k S_{n+k} \quad \text{for } k > 1.$$

Note that each  $S_n^k$  consists of pairwise disjoint cubes.

Define

$$F_n = \liminf_m S_n^m = \{C : \exists m_0 \forall m \geq m_0 C \in S_n^m\}.$$

Note that if  $C \in F_n$ , then there exist unique cubes  $C_n \in S_n, C_{n+1} \in S_{n+1}, \dots$  such that  $C = \bigcap_{k \geq 0} \text{int}^{k+1} C_{n+k}$ . Also note that  $\bigcup F_n = \bigcap_{k=0}^\infty \bigcup \text{int}^{k+1} S_{n+k}$ .

We claim that  $F = \bigcup_n F_n$  is nested and covers a set of measure 1.

For nestedness, consider cubes  $C, C' \in F$ . Then  $C \in F_n, C' \in F_m$  for some  $n, m$ . We may assume that  $m \geq n$ . Write  $C = \bigcap_{k \geq 0} \text{int}^{k+1} C_{n+k}$  and  $C' = \bigcap_{k \geq 0} \text{int}^{k+1} C_{m+k}$  with  $C_k, C'_k \in S_k$ .

If  $m = n$  and  $C_k = C'_k$  for all  $k \geq m$ , then  $C = C'$ .

If  $m > n$  and  $C_k = C'_k$  for all  $k \geq m$ , then

$$C \subseteq \bigcap_{k \geq m} \text{int}^{k-n+1} C_k = \bigcap_{k \geq m} \text{int}^{k-n+1} C'_k \subseteq \bigcap_{k \geq m} \text{int}^{k-m+2} C_k,$$

so

$$N(C) \subseteq \bigcap_{k \geq m} \text{int}^{k-m+1} C_k = C'.$$

If  $C_k \neq C'_k$  for some  $k \geq m$ , then  $C_k \cap C'_k = \emptyset$ . Note that  $C \subseteq \text{int}^{k-n+1} C_k \subseteq \text{int} C_k$  so  $N(C) \subseteq C_k$ . Similarly,  $N(C') \subseteq C'_k$ . Since  $C_k, C'_k \in S_k$  are disjoint, it follows that  $C$  and  $C'$  are disjoint.

This shows that  $F$  is nested.

We will prove now that  $\mu(\bigcup F) = 1$ .

For a cube  $C$  let  $x_C$  to be the point  $x \in X$  such that  $C = (\prod_{i=1}^d [0, n_i]) \cdot x_C$ . For a positive integer  $n$  write  $X_n = \{x_C : C \in S_n\}$ . Note that for any  $0 \leq k < n$ ,

$$\begin{aligned} \mu\left(\bigcup \text{int}^k S_n\right) &\geq (n^3 - 2k)^d \mu(X_n) \\ &\geq \frac{(n^3 - 2k)^d}{(n^3 + 1)^d} = \left(1 - \frac{2k + 1}{n^3 + 1}\right)^d \\ &\geq 1 - d \cdot \frac{2k + 1}{n^3 + 1}. \end{aligned}$$

Since  $\bigcup F_n = \bigcap_{k=0}^\infty \bigcup \text{int}^{k+1} S_{n+k}$ , we have

$$\begin{aligned} \mu\left(X \setminus \bigcup F_n\right) &\leq \sum_{k=0}^\infty \mu\left(X \setminus \bigcup \text{int}^{k+1} S_{n+k}\right) \\ &\leq d \cdot \sum_{k=0}^\infty \frac{2k + 3}{(n + k)^3 + 1} \\ &\leq d \cdot \sum_{k=n}^\infty \frac{3}{k^2}. \end{aligned}$$

This implies that

$$\mu\left(X \setminus \bigcup F\right) = \lim_{n \rightarrow \infty} \mu\left(X \setminus \bigcup F_n\right) = 0.$$

Hence  $\mu(\bigcup F) = 1$ . ■

Marks and Unger [25, Lemma 5.4] showed that for every  $d \geq 2$ , any Borel, bounded real-valued flow on the Schreier graph of a free Borel action of  $\mathbb{Z}^d$  can be modified to

a bounded Borel integer-valued flow. Below we provide a short proof for the case  $d = 1$  and additionally an independent proof (based on Lemma 15) for  $d \geq 2$  in the case of a pmp action where we consider flows defined a.e.

**Proposition 20.** *Suppose  $\mathbb{Z}^d \curvearrowright (X, \mu)$  is a free pmp action and  $G$  is its Schreier graph. Let  $f : X \rightarrow \mathbb{Z}$  be a bounded measurable function. Then, for every measurable  $f$ -flow  $\varphi : G \rightarrow \mathbb{R}$ , there exists a measurable bounded  $\psi : G \rightarrow \mathbb{Z}$  such that:*

- $\psi$  is an  $f$ -flow  $\mu$ -a.e.,
- $|\psi| \leq |\varphi| + 12d$ .

*Proof.* First we deal with the case  $d = 1$ . In that case for every  $e \in G$  we simply put  $\psi(e) = \lfloor \varphi(e) \rfloor$ . Note that since  $G$  is a graph of degree 2, for every  $x \in X$ , the fractional parts of the two edges which contain  $x$  are equal because  $f$  is integer-valued. Thus,  $\psi$  is also an  $f$ -flow.

Now suppose  $d \geq 2$ . By Lemma 19, there exists an invariant subset  $X' \subseteq X$  of measure 1 and a sequence of families  $F_n$  of cubes such that  $\bigcup_{n \in \mathbb{N}} F_n$  is nested, each  $F_n$  consists of disjoint cubes,  $\bigcup_{n \in \mathbb{N}} F_n$  covers  $X'$ . By induction on  $n$  we construct measurable  $f$ -flows  $\varphi_n$  such that  $\varphi_0 = \varphi$  and

- $\text{supp}(\varphi_{n+1} - \varphi_n) \subseteq \bigcup \{E(N(C)) : C \in F_n\}$ ,
- $\varphi_m = \varphi_{n+1}$  for every  $m > n$  on every  $E^+(C)$  for  $C \in F_n$ ,
- $|\varphi_n| \leq |\varphi| + 12d$ .

Given the flow  $\varphi_n$  we apply Lemma 15 on each cube  $C \in F_n$  to obtain the flow  $\varphi_{n+1}$ . The bound on  $\varphi_n$  follows from the fact that the value of the flow on each edge is changed at most twice by at most  $6d$  along this construction.

The sequence  $\varphi_n$  converges pointwise on the edges of  $X'$  to a measurable  $f$ -flow  $\varphi_\infty$ , which is integer-valued on all edges in  $X'$  except possibly for the edges in  $\text{bd} C$  for cubes  $C \in \bigcup_n F_n$ . However, the family  $\{\text{bd} C : C \in \bigcup_n F_n\}$  consists of pairwise disjoint finite sets. By the integral flow theorem for finite graphs, we can further correct  $\varphi_\infty$  on each of these finite subgraphs without changing the bound  $|\varphi| + 12d$  to obtain a measurable integer-valued  $f$ -flow  $\psi$ , which is equal to  $\varphi_\infty$  on all edges from  $G \setminus \bigcup \{E(\text{bd} C) : C \in \bigcup_{n \in \mathbb{N}} F_n\}$ . ■

### 8. Hall’s theorem

In this section we prove Theorem 2. The proof of (1)  $\Rightarrow$  (2) is based on an idea of Marks and Unger [25].

*Proof of Theorem 2.* The implication (2)  $\Rightarrow$  (3) is obvious.

The implication (3)  $\Rightarrow$  (1) is true for every finitely generated group  $\Gamma$ . In general, if  $A$  and  $B$  are  $\Gamma$ -equidecomposable, and the group elements used in the decomposition are  $\gamma_1, \dots, \gamma_n$ , then  $A$  and  $B$  satisfy the  $k$ -Hall condition for  $k$  greater than the word lengths of the group elements  $\gamma_1, \dots, \gamma_n$ . If  $X' \subseteq X$  is a set of measure 1 such that  $A \cap X'$  and  $B \cap X'$  are  $\Gamma$ -equidecomposable, then  $A \cap X'$  and  $B \cap X'$  satisfy the  $k$ -Hall condition.

(1)  $\Rightarrow$  (2). Without loss of generality assume that the  $k$ -Hall condition and  $\Gamma$ -uniformity is satisfied everywhere. Let  $\Gamma = \mathbb{Z}^d \times \Delta$ , where  $\Delta$  is a finite group and  $d \geq 0$ .

If  $d = 0$ , then the group  $\Gamma$  is finite and the action has finite orbits (the discrepancy condition trivializes and we do not need to use it). On each orbit the Hall condition is satisfied, so on each orbit there exists a bijection between  $A$  and  $B$  on that orbit. Thus, the sets  $A$  and  $B$  are  $\Delta$ -equidecomposable using a Borel choice of bijections on each orbit separately.

Thus, we can assume for the rest of the proof that  $d \geq 1$ . Since  $\Delta$  is finite, we can quotient by its action and get a standard Borel space  $X' = X/\Delta$  with the probability measure induced by the quotient map  $\pi : X \rightarrow X'$ . We then have a free pmp action of  $\mathbb{Z}^d \curvearrowright X'$ . Consider the function  $f : X' \rightarrow \mathbb{Z}$  defined by

$$f(x') = |A \cap \pi^{-1}(\{x'\})| - |B \cap \pi^{-1}(\{x'\})|.$$

Note that  $f$  is bounded by  $|\Delta|$ . Using Proposition 9 and Proposition 20 we get an invariant subset  $Y' \subseteq X'$  of measure 1 and an integer-valued measurable  $f$ -flow  $\psi$  on the edges of the Schreier graph  $G$  of  $\mathbb{Z}^d \curvearrowright Y'$  on  $Y'$  such that  $|\psi| \leq |\Delta| d^k + 12d$ . Again, without loss of generality, we can assume  $Y' = X'$  by replacing  $X$  with  $Y = \pi^{-1}(Y')$ , if needed.

Note that there exists a constant  $r$ , depending only on  $d$  such that for every tiling of  $\mathbb{Z}^d$  with cubes with sides  $n$  or  $n + 1$ , every cube is adjacent to at most  $r$  many other cubes in the tiling.

Note that  $\Gamma$ -uniformity implies that for every set  $D$  such that  $D = D' \times \Delta$ , where  $D'$  is a cube with sides  $n$  or  $n + 1$  we have  $|A \cap D|, |B \cap D| \geq cn^d |\Delta|$ . Let  $n$  be such that  $cn^d |\Delta| \geq r(n + 1)^{d-1} (|\Delta| d^k + 12d)$ .

Using the Gao–Jackson theorem [7], find a Borel tiling  $T'$  of  $X'$  with cubes of sides  $n$  or  $n + 1$ . Pulling back the tiling to  $X$  via  $\pi$ , we get a Borel tiling  $T$  of  $X$  with cubes of the form  $D = (D' \times \Delta) \cdot x$  where  $D'$  has sides of length  $n$  or  $n + 1$ . Note that for every tile  $D$  in  $T$  we have

$$|A \cap D|, |B \cap D| \geq r(n + 1)^{d-1} (|\Delta| d^k + 12d). \tag{*}$$

Let  $H$  be the graph on  $T$  where two cubes are connected with an edge if they are adjacent and similarly let  $H'$  be the graph on  $T'$  with two cubes connected with an edge if they are adjacent. We have two functions  $F' : T' \rightarrow \mathbb{Z}$  defined as  $F'(C) = \sum_{x' \in C} f(x')$  and  $F : T \rightarrow \mathbb{Z}$  defined as

$$F(C) = |A \cap C| - |B \cap C|.$$

Define an  $F'$ -flow  $\Psi'$  on  $H'$  as

$$\Psi'(C, D) = \sum_{(x'_1, x'_2) \in G, x'_1 \in C, x'_2 \in D} \psi(x'_1, x'_2)$$

and let  $\Psi$  be an  $F$ -flow on  $H$  obtained by pulling back  $\Psi'$  via  $\pi$ . Note that any adjacent cubes in  $T'$  are connected by at most  $(n + 1)^{d-1}$  edges, so both  $\Psi$  and  $\Psi'$  are bounded by  $|\Psi|, |\Psi'| \leq (n + 1)^{d-1} (|\Delta| d^k + 12d)$ .

Note that each vertex in  $H'$  has degree at most  $r$  and the same is true in  $H$ .

Thus, by estimate (\*), for each  $C \in T$  and  $D \in T$  which are connected with an edge in  $H$ , we can find pairwise disjoint sets  $A(C, D), B(C, D) \subseteq C$  of size at least  $(n + 1)^{d-1}(|\Delta|d^k + 12d)$  such that  $A(C, D) \subseteq A \cap C, B(C, D) \subseteq B \cap C$ .

Now, the function which witnesses the equidecomposition is defined in two steps. First, we can find a map  $g : \text{dom}(g) \rightarrow B$  such that  $\text{dom}(g) \subset \bigcup_{(C,D) \in H} A(C, D)$ , for any two neighboring cubes  $C, D \in T$  satisfying  $\Psi(C, D) > 0$  we have

$$|\text{dom}(g) \cap A(C, D)| = \Psi(C, D)$$

and the points in  $\text{dom}(g) \cap A(C, D)$  are mapped injectively to  $B(C, D)$ . Note that for any cube  $C \in T$  the set  $C \cap (A \setminus \text{dom}(g))$  contains as many points as the set  $C \cap (B \setminus \text{im}(g))$ . Hence, one can extend  $g$  to a function  $g' : A \rightarrow B$  in such a way that its restriction to  $C \cap (A \setminus \text{dom}(g))$  is a bijection onto  $C \cap (B \setminus \text{im}(g))$  for any cube  $C \in T$ . Since  $\psi$  and hence  $\Psi'$  and  $\Psi$  are measurable, the function  $g'$  can be chosen to be measurable and it moves points by at most  $2(|\Delta| + (n + 1)^d)$  in the Schreier graph distance. Thus,  $g'$  witnesses that  $A$  and  $B$  are equidecomposable using measurable pieces. ■

### 9. Measurable circle squaring

In this section we comment on how Corollary 3 follows from Theorem 2. We use an argument which appears in a preprint of Grabowski, Máthé and Pikhurko [9] and provide a short proof for completeness.

**Lemma 21.** *Suppose that  $\Gamma \curvearrowright (X, \mu)$  is a free pmp action of a countable group  $\Gamma$ . If  $A, B \subseteq X$  are  $\Gamma$ -equidecomposable and  $X' \subseteq X$  is  $\Gamma$ -invariant, then  $A \cap X'$  and  $B \cap X'$  are also equidecomposable. If  $X'$  is additionally  $\mu$ -measurable and  $A$  and  $B$  are  $\Gamma$ -equidecomposable using  $\mu$ -measurable pieces, then  $A \cap X'$  and  $B \cap X'$  are  $\Gamma$ -equidecomposable using  $\mu$ -measurable pieces.*

*Proof.* The proof is the same in both cases. Let  $A_1, \dots, A_n$  and  $B_1, \dots, B_n$  be partitions of  $A$  and  $B$  such that  $\gamma_i A_i = B_i$  for some  $\gamma_i \in \Gamma$ . Put  $A'_i = A_i \cap X'$  and  $B'_i = B_i \cap X'$ . Then  $\gamma_i A'_i = B'_i$ , so  $A'_i$  and  $B'_i$  witness that  $A \cap X'$  and  $B \cap X'$  are equidecomposable. ■

**Lemma 22.** *Let  $\mu$  be a probability measure on  $X$  and  $\Gamma \curvearrowright X$  be a Borel pmp action of a countable group  $\Gamma$ . Suppose  $A, B \subseteq X$  are  $\Gamma$ -equidecomposable and there exists a measurable set  $Y \subseteq X$  of measure 1 such that  $A \cap Y, B \cap Y$  are equidecomposable using  $\mu$ -measurable pieces. Then  $A, B$  are equidecomposable using  $\mu$ -measurable pieces.*

*Proof.* Write  $X' = \bigcap_{\gamma \in \Gamma} \gamma X$ . Note that  $\mu(X') = 1$  and  $\gamma X' = X'$  for all  $\gamma \in \Gamma$ . By Lemma 21,  $A' = A \cap X'$  and  $B' = B \cap X'$  are  $\Gamma$ -equidecomposable using  $\mu$ -measurable pieces. Write  $X'' = X \setminus X'$  and note that  $\gamma X'' = X''$  for all  $\gamma \in \Gamma$ . By Lemma 21 again,  $A'' = A \cap X''$  and  $B'' = B \cap X''$  are  $\Gamma$ -equidecomposable. However, all pieces in the latter decomposition are  $\mu$ -null, hence  $\mu$ -measurable. This shows that  $A = A' \cup A''$  and  $B = B' \cup B''$  are  $\Gamma$ -equidecomposable using  $\mu$ -measurable pieces. ■

Finally, we give a proof of Corollary 3.

*Proof of Corollary 3.* Suppose  $\Gamma \curvearrowright (X, \mu)$  is a free pmp action of a finitely generated abelian group  $\Gamma$  and  $A$  and  $B$  are two measurable  $\Gamma$ -uniform sets which are  $\Gamma$ -equidecomposable. Note that since  $\Gamma$  is amenable,  $A$  and  $B$  must have the same measure (see [31, Corollary 10.9]). Let  $\gamma_1, \dots, \gamma_n$  be the elements of  $\Gamma$  used in the equidecomposition and let  $k$  be bigger than the lengths of  $\gamma_i$ . Then  $A$  and  $B$  satisfy the  $k$ -Hall condition. In particular,  $A$  and  $B$  satisfy the  $k$ -Hall condition  $\mu$ -a.e., so by Theorem 2 there is a  $\Gamma$ -invariant measurable set  $X' \subseteq X$  of measure 1 such that  $A \cap X'$  and  $B \cap X'$  are  $\Gamma$ -equidecomposable using  $\mu$ -measurable pieces. By Lemma 22,  $A$  and  $B$  are  $\Gamma$ -equidecomposable using  $\mu$ -measurable pieces as well. ■

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