J. Eur. Math. Soc. 24, 3031–3053 (2022) DOI 10.4171/JEMS/1137

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Uniform boundary controllability and homogenization of wave equations

Received April 28, 2019; revised June 29, 2020

Abstract. We obtain sharp convergence rates, using Dirichlet correctors, for solutions of wave equations in a bounded domain with rapidly oscillating periodic coefficients. The results are used to prove the exact boundary controllability that is uniform in ε (the scale of the microstructure) for the projection of solutions to the subspace generated by the eigenfunctions with eigenvalues less than $C\varepsilon^{-2/3}$.

Keywords. Boundary controllability, oscillating coefficient, wave equation, homogenization, convergence rate

1. Introduction

In this paper we study the exact boundary controllability, uniform in $\varepsilon > 0$, of the wave operator

$$\partial_t^2 + \mathcal{L}_\varepsilon \tag{1.1}$$

in a bounded domain, where the elliptic operator $\mathcal{L}_\varepsilon$ is given by

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla), \tag{1.2}$$

and $\varepsilon > 0$ is a small parameter. Throughout we will assume that the $d \times d$ coefficient matrix $A = A(y) = (a_{ij}(y))$ is real, bounded, measurable, satisfies the ellipticity condition

$$\mu|\xi|^2 \le \langle A\xi, \xi\rangle \le \frac{1}{\mu}|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^d, \tag{1.3}$$

where $\mu > 0$, the symmetry condition

$$a_{ij}(y) = a_{ji}(y) \quad \text{for } 1 \le i, j \le d, \tag{1.4}$$

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Mathematics Subject Classification (2020): 35Q93, 49J20, 35B27

and the periodicity condition

$$A(y+z) = A(y) \quad \text{for any } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1.5}$$

Let Ω be a bounded domain in $\mathbb{R}^d$. Given initial data $(\theta_{\varepsilon,0}, \theta_{\varepsilon,1}) \in L^2(\Omega) \times H^{-1}(\Omega)$, one is interested in finding $T > 0$ and a control $g_\varepsilon \in L^2(S_T)$ such that the weak solution of the evolution problem

$$\begin{cases} (\partial_t^2 + \mathcal{L}_\varepsilon)v_\varepsilon = 0 & \text{in } \Omega_T = \Omega \times (0,T], \\ v_\varepsilon = g_\varepsilon & \text{on } S_T = \partial\Omega \times [0,T], \\ v_\varepsilon(x,0) = \theta_{\varepsilon,0}(x), \quad \partial_t v_\varepsilon(x,0) = \theta_{\varepsilon,1}(x) & \text{for } x \in \Omega, \end{cases} \tag{1.6}$$

satisfies the conditions

$$v_\varepsilon(x,T) = \partial_t v_\varepsilon(x,T) = 0 \quad \text{for } x \in \Omega. \tag{1.7}$$

This classical control problem in highly heterogeneous media was proposed by J.-L. Lions [16]. Let u_ε be the solution of the initial-Dirichlet problem

$$\begin{cases} (\partial_t^2 + \mathcal{L}_\varepsilon)u_\varepsilon = 0 & \text{in } \Omega_T, \\ u_\varepsilon = 0 & \text{on } S_T, \\ u_\varepsilon(x,0) = \varphi_{\varepsilon,0}(x), \quad \partial_t u_\varepsilon(x,0) = \varphi_{\varepsilon,1}(x) & \text{for } x \in \Omega, \end{cases} \tag{1.8}$$

where $\varphi_{\varepsilon,0} \in H_0^1(\Omega)$ and $\varphi_{\varepsilon,1} \in L^2(\Omega)$. By the Hilbert Uniqueness Method (HUM), the existence of a control g_ε which is uniformly bounded in $L^2(S_T)$ for $\varepsilon > 0$ is equivalent to the following two estimates, usually called *observability inequalities*:

$$\frac{1}{T}\int_0^T \int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \, dt \le C\{\|\nabla \varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\}, \tag{1.9}$$

$$c\{\|\nabla \varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} \le \frac{1}{T}\int_0^T \int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \, dt, \tag{1.10}$$

with positive constants C and c independent of $\varepsilon > 0$ (see [16]). However, it has been known since the early 1990s that both (1.9) and (1.10) fail to hold uniformly in $\varepsilon > 0$, even in the case $d = 1$ [1]. We remark that for $\varepsilon = 1$ (without the periodicity condition), a fairly complete solution of the exact boundary controllability problem for second-order hyperbolic equations was found by C. Bardos, G. Lebeau, and J. Rauch [5], using microlocal analysis. See also related work in [4, 7] and references therein.

In this paper we shall show that estimates (1.9) and (1.10) hold uniformly if the initial data $(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1})$ in (1.8) are taken from a low-frequency subspace of $H_0^1(\Omega) \times L^2(\Omega)$. More precisely, let $\{\lambda_{\varepsilon,k} : k = 1, 2, \ldots\}$ denote the increasing sequence of Dirichlet eigenvalues for $\mathcal{L}_\varepsilon$ in Ω. Let $\{\psi_{\varepsilon,k} : k = 1, 2, \ldots\}$ be a set of Dirichlet eigenfunctions in $H_0^1(\Omega)$ for $\mathcal{L}_\varepsilon$ in Ω such that $\{\psi_{\varepsilon,k}\}$ forms an orthonormal basis for $L^2(\Omega)$ and $\mathcal{L}_\varepsilon(\psi_{\varepsilon,k}) = \lambda_{\varepsilon,k}\psi_{\varepsilon,k}$ in Ω. Define

$$\mathcal{A}_N = \Big\{h = \sum_{\lambda_{\varepsilon,k} \le N} a_k \psi_{\varepsilon,k} : a_k \in \mathbb{R}\Big\}. \tag{1.11}$$

Theorem 1.1. *Assume $A = A(y)$ satisfies conditions (1.3)–(1.5). Also assume that there exists $M > 0$ such that*

$$|A(x) - A(y)| \le M|x - y| \quad \text{for any } x, y \in \mathbb{R}^d. \tag{1.12}$$

Let Ω be a bounded C^3 domain in $\mathbb{R}^d$. Let u_ε be a solution of (1.8) with initial data $(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1}) \in \mathcal{A}_N \times \mathcal{A}_N$. If $N \le C_0 T^{-2/3}\varepsilon^{-2/3}$ for some $C_0 > 0$, then the inequality (1.9) holds with a constant C depending only on d, μ, C_0, M and Ω. Moreover, there exist $c_0 > 0$ and $T_0 > 0$, depending only on d, μ, M and Ω, such that if $N \le c_0 T^{-2/3}\varepsilon^{-2/3}$ and $T \ge T_0$, then (1.10) holds with a constant c depending only on d, μ, M and Ω.

Following [8], one may use Theorem 1.1 to prove the following result on uniform boundary controllability. Let $N \le \delta T^{-2/3}\varepsilon^{-2/3}$ and $T \ge T_0$, where $\delta = \delta(d, A, \Omega) > 0$ is sufficiently small. Given $(\theta_{\varepsilon,0}, \theta_{\varepsilon,1}) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists $g_\varepsilon \in L^2(S_T)$ such that the solution of (1.6) satisfies the conditions

$$P_N v_\varepsilon(x, T) = 0 \quad \text{and} \quad P_N \partial_t v_\varepsilon(x, T) = 0 \quad \text{for } x \in \Omega, \tag{1.13}$$

where P_N denotes the projection operator from $L^2(\Omega)$ or $H^{-1}(\Omega)$ to the space $\mathcal{A}_N$. Moreover, the control g_ε satisfies the uniform estimates

$$c\|g_\varepsilon\|_{L^2(S_T)} \le \|P_N\theta_{\varepsilon,0}\|_{L^2(\Omega)} + \|P_N\theta_{\varepsilon,1}\|_{H^{-1}(\Omega)} \le C\|g_\varepsilon\|_{L^2(S_T)}, \tag{1.14}$$

where $C > 0$ and $c > 0$ are independent of ε. See Section 4.

In the case $d = 1$, it was proved by C. Castro [10] that the estimates (1.9) and (1.10) hold uniformly if the initial data are taken from $\mathcal{A}_N \times \mathcal{A}_N$ and $N \le \delta\varepsilon^{-2}$, where $\delta > 0$ is sufficiently small. See also [9] for the case where the initial data are taken from a subspace generated by the eigenfunctions with eigenvalues greater than $C\varepsilon^{-2-\sigma}$ for some $\sigma > 0$. The approaches used in [9, 10] do not extend to the multi-dimensional case. To the best of the authors' knowledge, the only results in the case $d \ge 2$ are found in [3, 15]. M. Avellaneda and the first author [3] used the asymptotic expansion of the Poisson kernel for the elliptic operator $\mathcal{L}_\varepsilon$ in Ω to identify the weak limits of the controls. G. Lebeau [15] considered the wave operator with oscillating density, $\rho(x, x/\varepsilon)\partial_t^2 - \Delta_g$, where Δ_g is the Laplace operator for some fixed smooth metric, and the function $\rho(x, y)$ is periodic in y. Theorem 1.1 seems to be the first result on the observability inequalities (1.9) and (1.10) for wave operators with oscillating coefficients $A(x/\varepsilon)$ in higher dimensions.

Let

$$u_\varepsilon(x, t) = \cos(\sqrt{\lambda_{\varepsilon,k}}\, t)\psi_{\varepsilon,k}.$$

Then $(\partial_t^2 + \mathcal{L}_\varepsilon)u_\varepsilon = 0$ in Ω_T and $u_\varepsilon = 0$ on S_T. Also, $u_\varepsilon(x, 0) = \psi_{\varepsilon,k}(x)$ and $\partial_t u_\varepsilon(x, 0) = 0$ for $x \in \Omega$. Thus the inequalities (1.9) and (1.10) would imply that

$$c\lambda_{\varepsilon,k} \le \int_{\partial\Omega} |\nabla\psi_{\varepsilon,k}|^2\, d\sigma \le C\lambda_{\varepsilon,k}. \tag{1.15}$$

It was proved in [1, 10] that (1.15) cannot hold uniformly in $\varepsilon > 0$ and $k \ge 1$. Counter-examples were constructed using eigenfunctions with eigenvalues $\lambda_{\varepsilon,k} \sim \varepsilon^{-2}$—the wave length of the solutions is of the order of the size of the microstructure. See also related work by A. Hassell and T. Tao [12] for Dirichlet eigenfunctions on a compact Riemannian manifold with boundary. C. Kenig and the present authors [13] proved that for $d \geq 2$,

$$\int_{\partial\Omega} |\nabla \psi_{\varepsilon,k}|^2 \, d\sigma \leq C \lambda_{\varepsilon,k}(1 + \varepsilon \lambda_{\varepsilon,k}) \tag{1.16}$$

if $\varepsilon^2 \lambda_{\varepsilon,k} \leq 1$, where C is independent of ε and k. This in particular implies that the upper bound in (1.15) holds if $\varepsilon\lambda_{\varepsilon,k} \leq 1$. Furthermore, it is proved in [13] that if $\varepsilon\lambda_{\varepsilon,k} \leq \delta$, where $\delta > 0$ depends only on A and Ω, then the lower bound in (1.15) also holds uniformly in ε and k. These results suggest that one may be able to extend Theorem 1.1 to the case $N \leq C\varepsilon^{-1}$. But this remains unknown. In view of the one-dimensional results in [8,10], one may conjecture further that the main conclusion in Theorem 1.1 is valid when $N \leq \delta\varepsilon^{-2}$ and δ is sufficiently small.

We now describe our approach to Theorem 1.1, which is based on homogenization. Under the assumptions (1.3)–(1.5) as well as suitable conditions on F, $\varphi_{\varepsilon,0}$ and $\varphi_{\varepsilon,1}$, the solution u_ε of the initial-Dirichlet problem

$$\begin{cases} (\partial_t^2 + \mathcal{L}_\varepsilon) u_\varepsilon = F & \text{in } \Omega_T, \\ u_\varepsilon = 0 & \text{on } S_T, \\ u_\varepsilon(x,0) = \varphi_{\varepsilon,0}(x), \quad \partial_t u_\varepsilon(x,0) = \varphi_{\varepsilon,1}(x) & \text{for } x \in \Omega, \end{cases} \tag{1.17}$$

converges strongly in $L^2(\Omega_T)$ to the solution of the homogenized problem

$$\begin{cases} (\partial_t^2 + \mathcal{L}_0) u_0 = F & \text{in } \Omega_T, \\ u_0 = 0 & \text{on } S_T, \\ u_0(x,0) = \varphi_0(x), \quad \partial_t u_0(x,0) = \varphi_1(x) & \text{for } x \in \Omega, \end{cases} \tag{1.18}$$

where $\mathcal{L}_0$ is an elliptic operator with constant coefficients (see e.g. [6]). In the first part of this paper we shall investigate the problem of convergence rates.

Let

$$\Phi_\varepsilon = (\Phi_{\varepsilon,1}, \Phi_{\varepsilon,2}, \ldots, \Phi_{\varepsilon,d})$$

denote the Dirichlet corrector for the operator $\mathcal{L}_\varepsilon$ in Ω, where, for $1 \leq j \leq d$, the function $\Phi_{\varepsilon,j}$ is the solution in $H^1(\Omega)$ of the Dirichlet problem

$$\begin{cases} \mathcal{L}_\varepsilon(\Phi_{\varepsilon,j}) = 0 & \text{in } \Omega, \\ \Phi_{\varepsilon,j} = x_j & \text{on } \partial\Omega. \end{cases} \tag{1.19}$$

Theorem 1.2. *Assume $A = A(y)$ satisfies conditions (1.3)–(1.5). Let u_ε be a weak solution of (1.17), where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$. Let*

$$w_\varepsilon = u_\varepsilon - u_0 - (\Phi_{\varepsilon,k} - x_k)\frac{\partial u_0}{\partial x_k}, \tag{1.20}$$

where u_0 is the solution of (1.18). Then for any $t \in (0, T]$,

$$\begin{aligned}\Big(\int_\Omega (|\nabla w_\varepsilon(x,t)|^2 + |\partial_t w_\varepsilon(x,t)|^2)\,dx\Big)^{1/2}&\\ \le C\{\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0}) - \mathcal{L}_0(\varphi_0)\|_{H^{-1}(\Omega)} + \|\varphi_{\varepsilon,1}-\varphi_1\|_{L^2(\Omega)}\}&\\ + C\varepsilon\{\|\nabla^2\varphi_0\|_{L^2(\Omega)} + \|\nabla\varphi_1\|_{L^2(\Omega)}\}&\\ + C\varepsilon \sup_{t\in(0,T]} \|\nabla^2 u_0(\cdot,t)\|_{L^2(\Omega)}&\\ + C\varepsilon\sqrt{T} \sup_{t\in(0,T]} \big\||\partial_t\nabla^2 u_0(\cdot,t)| + |\partial_t^2\nabla u_0(\cdot,t)|\big\|^{1/2}_{L^2(\Omega)} \sup_{t\in(0,T]} \|\nabla^2 u_0(\cdot,t)\|^{1/2}_{L^2(\Omega)},&\end{aligned} \tag{1.21}$$

where C depends only on d and μ.

Theorem 1.2, together with Rellich identities, allows us to control the boundary integral

$$\int_0^T\int_{\partial\Omega} |\nabla u_\varepsilon - (\nabla\Phi_\varepsilon)(\nabla u_0)|^2\,d\sigma\,dt,$$

where the initial data (φ_0,φ_1) in (1.18) is chosen so that $\mathcal{L}_0(\varphi_0) = \mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})$ and $\varphi_1 = \varphi_{\varepsilon,1}$ in Ω (see [18] for the case $d = 1$). Since $|\nabla\Phi_\varepsilon| \le C$ (see [2]) and $|\det(\nabla\Phi_\varepsilon)| \ge c > 0$ on $\partial\Omega$ (see [13]), this reduces the problem to the estimates (1.9) and (1.10) for the homogenized operator $\partial_t^2 + \mathcal{L}_0$ with constant coefficients. We remark that the Rellich identities, which use the Lipschitz condition (1.12), are applied to the function w_ε in (1.20). We further point out that the power of ε in the condition $N \le C_0T^{-2/3}\varepsilon^{-2/3}$ is dictated by the highest-order term on the right-hand side of (1.21). Also, the C^3 condition on Ω is only used for estimates of the homogenized solutions.

The problem of convergence rates is of much interest in its own right in the theory of homogenization. Note that no smoothness condition on A is needed in Theorem 1.2. Let w_ε be given by (1.20). Since $\|\Phi_\varepsilon - x\|_{L^\infty(\Omega)} \le C\varepsilon$, it follows that

$$|\partial_t u_\varepsilon - \partial_t u_0| \le |\partial_t w_\varepsilon| + C\varepsilon|\partial_t\nabla u_0|, \tag{1.22}$$

$$|\nabla u_\varepsilon - (\nabla\Phi_\varepsilon)(\nabla u_0)| \le |\nabla w_\varepsilon| + C\varepsilon|\nabla^2 u_0|, \tag{1.23}$$

where C depends only on d and μ. As a result, Theorem 1.2 gives the $O(\varepsilon)$ convergence rates for both $\|\partial_t u_\varepsilon - \partial_t u_0\|_{L^2(\Omega)}$ and $\|\nabla u_\varepsilon - (\nabla\Phi_\varepsilon)(\nabla u_0)\|_{L^2(\Omega)}$. By Sobolev imbedding, we may also deduce an $O(\varepsilon)$ convergence rate for $\|u_\varepsilon(\cdot,t) - u_0(\cdot,t)\|_{L^2(\Omega)}$ directly from (1.21). However, a better estimate with lower-order derivatives required for u_0 is obtained at the end of Section 3 (see (3.15)). We mention that in the case $\Omega = \mathbb{R}^d$, the following estimate was proved by M. A. Dorodnyi and T. A. Suslina [11]:

$$\|u_\varepsilon(\cdot,t) - u_0(\cdot,t)\|_{L^2(\mathbb{R}^d)} \le C\varepsilon(t+1)\{\|\varphi_0\|_{H^{3/2}(\mathbb{R}^d)} + \|\varphi_1\|_{H^{1/2}(\mathbb{R}^d)}\} \tag{1.24}$$

for any $t \in \mathbb{R}$, where $(\partial_t^2 + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t^2 + \mathcal{L}_0)u_0 = 0$ in $\mathbb{R}^{d+1}$, and u_ε and u_0 have the same initial data (φ_0,φ_1). The results of [11] (see also [19]) are obtained by an operator-theoretic approach, using Floquet–Bloch theory. In the case of bounded domains, for a periodic hyperbolic system, Yu. M. Meshkova [17] obtained an $O(\varepsilon)$ estimate for $\|u_\varepsilon(\cdot,t)-u_0(\cdot,t)\|_{L^2(\Omega)}$, assuming the initial data (φ_0,φ_1) belong to some subspace of $H^4(\Omega)$. We note that the highest-order terms on the right-hand side of (3.15) involve $\|\varphi_0\|^{1/2}_{H^2(\Omega)}\|\nabla\varphi_0\|^{1/2}_{L^2(\Omega)}$ and $\|\nabla\varphi_1\|^{1/2}_{L^2(\Omega)}\|\varphi_1\|^{1/2}_{L^2(\Omega)}$, which are consistent with $\|\varphi_0\|_{H^{3/2}(\Omega)}$ and $\|\varphi_1\|_{H^{1/2}(\Omega)}$ respectively, in terms of scaling.

We point out that the symmetry condition (1.4) is essential in the proofs of Theorems 1.1 and 1.2, but the assumption that the equations are scalar is not. Theorem 1.1 continues to hold for elliptic systems $\partial_t-\text{div}(A(x/\varepsilon)\nabla)$ if $A(y)=(a_{ij}^{\alpha\beta}(y))$, with $1\le i,j\le d$ and $1\le\alpha,\beta\le m$, satisfies the ellipticity condition (1.3) for $\xi=(\xi_i^\alpha)\in\mathbb{R}^{m\times d}$, the periodicity condition (1.5), the Lipschitz condition (1.12), and the symmetry condition $a_{ij}^{\alpha\beta}=a_{ji}^{\beta\alpha}$. In the case of Theorem 1.2, the estimate (1.21) holds in a $C^{1,\eta}$ domain Ω if A satisfies (1.3), (1.5), the symmetry condition above, and is Hölder continuous. The additional smoothness conditions on A and Ω are used for the estimates of the correctors χ (see (2.1) below) and Φ_ε.

The summation convention that repeated indices (in a term) are summed is used throughout the paper.

2. Preliminaries

Throughout this section we will assume that $A=A(y)$ satisfies conditions (1.3)–(1.5). A function u in $\mathbb{R}^d$ is said to be 1-*periodic* if $u(y+z)=u(y)$ for a.e. $y\in\mathbb{R}^d$ and for any $z\in\mathbb{Z}^d$. Let $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d\cong[0,1)^d$. We use $H^1(\mathbb{T}^d)$ to denote the closure of the set of 1-periodic C^∞ functions in $\mathbb{R}^d$ in the space $H^1(Y)$, where $Y=(0,1)^d$.

Let $\chi(y)=(\chi_1(y),\dots,\chi_d(y))$ denote the first-order corrector for $\mathcal{L}_\varepsilon$, where, for $1\le j\le d$, the function $\chi_j=\chi_j(y)$ is the unique weak solution in $H^1(\mathbb{T}^d)$ of the cell problem

$$\begin{cases} -\text{div}(A(y)\nabla\chi_j)=\text{div}(A(y)\nabla y_j) & \text{in } \mathbb{T}^d,\\ \displaystyle\int_{\mathbb{T}^d}\chi_j\,dy=0. \end{cases} \tag{2.1}$$

Note that χ_j is 1-periodic and

$$\mathcal{L}_\varepsilon\{x_j+\varepsilon\chi_j(x/\varepsilon)\}=0 \quad \text{in } \mathbb{R}^d. \tag{2.2}$$

By the classical De Giorgi–Nash estimate, $\chi_j\in L^\infty(\mathbb{R}^d)$ and $\|\chi_j\|_\infty\le C$, where C depends only on d and μ. Let

$$\mathcal{L}_0=-\text{div}(\widehat{A}\nabla), \tag{2.3}$$

where $\widehat{A}=(\widehat{a}_{ij})_{d\times d}$ and

$$\widehat{a}_{ij}=\int_{\mathbb{T}^d}\left(a_{ij}+a_{ik}\frac{\partial\chi_j}{\partial y_k}\right)dy \tag{2.4}$$

(the summation convention is used). Under the conditions (1.3)–(1.5), one may show that the matrix $\widehat{A}$ is symmetric and satisfies the ellipticity condition

$$\mu|\xi|^2\le\langle\widehat{A}\xi,\xi\rangle\le\frac{1}{\mu}|\xi|^2 \quad \text{for any } \xi\in\mathbb{R}^d, \tag{2.5}$$

with the same constant μ as in (1.3). It is well known that the homogenized operator for $\partial_t^2 + \mathcal{L}_\varepsilon$ is given by $\partial_t^2 + \mathcal{L}_0$. In particular, if $\varphi_{\varepsilon,0} = \varphi_0$ and $\varphi_{\varepsilon,1} = \varphi_1$, the solution u_ε of the initial-Dirichlet problem (1.17) converges strongly in $L^2(\Omega_T)$ to the solution u_0 of the homogenized problem (1.18).

For $1 \le i, j \le d$, let

$$b_{ij} = a_{ij} + a_{ik}\frac{\partial \chi_j}{\partial y_k} - \widehat{a}_{ij}. \tag{2.6}$$

It follows from the definitions of χ_j and $\widehat{a}_{ij}$ that

$$\frac{\partial}{\partial y_i} b_{ij} = 0 \quad \text{and} \quad \int_{\mathbb{T}^d} b_{ij}\, dy = 0. \tag{2.7}$$

Lemma 2.1. *There exist 1-periodic functions ϕ_{kij} in $H^1(\mathbb{T}^d)$ for $1 \le i, j, k \le d$ such that $\int_{\mathbb{T}^d} \phi_{kij}\, dy = 0$,*

$$b_{ij} = \frac{\partial}{\partial y_k}\phi_{kij} \quad \textit{and} \quad \phi_{kij} = -\phi_{ikj}. \tag{2.8}$$

Moreover, $\phi_{kij} \in L^\infty(\mathbb{R}^d)$ and $\|\phi_{kij}\|_\infty \le C$, where C depends only on d and μ.

Proof. See [14, Remark 2.1]. ∎

Let $\Phi_\varepsilon(x)$ be the Dirichlet corrector for $\mathcal{L}_\varepsilon$ in Ω, defined by (1.19). Since

$$\mathcal{L}_\varepsilon\{\Phi_{\varepsilon,j} - x_j - \varepsilon\chi_j(x/\varepsilon)\} = 0 \quad \text{in } \Omega, \tag{2.9}$$

by the maximum principle we have

$$\|\Phi_{\varepsilon,j} - x_j - \varepsilon\chi_j(x/\varepsilon)\|_{L^\infty(\Omega)} = \|\varepsilon\chi_j(x/\varepsilon)\|_{L^\infty(\partial\Omega)}.$$

It follows that

$$\|\Phi_{\varepsilon,j} - x_j\|_{L^\infty(\Omega)} \le 2\varepsilon\|\chi_j\|_\infty \le C\varepsilon, \tag{2.10}$$

where C depends only on d and μ. If Ω is a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$ for some $\alpha > 0$ and A is Hölder continous, then by the boundary Lipschitz estimate for $\mathcal{L}_\varepsilon$ [2], we also have

$$\|\nabla\Phi_{\varepsilon,j}\|_{L^\infty(\Omega)} \le C, \tag{2.11}$$

where C depends only on d, A and Ω.

Lemma 2.2. *Suppose that*

$$(\partial_t^2 + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t^2 + \mathcal{L}_0)u_0 \quad \textit{in } \Omega \times (T_0, T_1). \tag{2.12}$$

Let

$$w_\varepsilon = u_\varepsilon - u_0 - (\Phi_{\varepsilon,k} - x_k)\frac{\partial u_0}{\partial x_k}. \tag{2.13}$$

Then

$$(\partial_t^2 + \mathcal{L}_\varepsilon) w_\varepsilon = -\varepsilon \frac{\partial}{\partial x_i} \left\{ \phi_{kij}(x/\varepsilon) \frac{\partial^2 u_0}{\partial x_k \partial x_j} \right\} + \frac{\partial}{\partial x_i} \left\{ a_{ij}(x/\varepsilon) [\Phi_{\varepsilon,k} - x_k] \frac{\partial^2 u_0}{\partial x_j \partial x_k} \right\} + a_{ij}(x/\varepsilon) \frac{\partial}{\partial x_j} [\Phi_{\varepsilon,k} - x_k - \varepsilon \chi_k(x/\varepsilon)] \frac{\partial^2 u_0}{\partial x_i \partial x_k} - (\Phi_{\varepsilon,k} - x_k) \partial_t^2 \frac{\partial u_0}{\partial x_k}. \tag{2.14}$$

Proof. Note that by (2.12),

$$\begin{aligned}(\partial_t^2 + \mathcal{L}_\varepsilon) w_\varepsilon &= (\mathcal{L}_0 - \mathcal{L}_\varepsilon) u_0 - \mathcal{L}_\varepsilon \left\{ (\Phi_{\varepsilon,k} - x_k) \frac{\partial u_0}{\partial x_k} \right\} - (\Phi_{\varepsilon,k} - x_k) \partial_t^2 \frac{\partial u_0}{\partial x_k} \\ &= \frac{\partial}{\partial x_i} \left\{ b_{ij}(x/\varepsilon) \frac{\partial u_0}{\partial x_j} \right\} + \frac{\partial}{\partial x_i} \left\{ a_{ij}(x/\varepsilon) \frac{\partial}{\partial x_j} [\Phi_{\varepsilon,k} - x_k - \varepsilon \chi_k(x/\varepsilon)] \frac{\partial u_0}{\partial x_k} \right\} \\ &\quad + \frac{\partial}{\partial x_i} \left\{ a_{ij}(x/\varepsilon) [\Phi_{\varepsilon,k} - x_k] \frac{\partial^2 u_0}{\partial x_j \partial x_k} \right\} - (\Phi_{\varepsilon,k} - x_k) \partial_t^2 \frac{\partial u_0}{\partial x_k},\end{aligned}$$

where $b_{ij}(y)$ is (2.6). Since $\frac{\partial}{\partial y_i} b_{ij} = 0$, we see that

$$\frac{\partial}{\partial x_i} \left\{ b_{ij}(x/\varepsilon) \frac{\partial u_0}{\partial x_j} \right\} = b_{ij}(x/\varepsilon) \frac{\partial^2 u_0}{\partial x_i \partial x_j} = -\varepsilon \frac{\partial}{\partial x_i} \left\{ \phi_{kij}(x/\varepsilon) \frac{\partial^2 u_0}{\partial x_k \partial x_j} \right\},$$

where we have used (2.8) for the last step. Finally, in view of (2.9), we have

$$\frac{\partial}{\partial x_i} \left\{ a_{ij}(x/\varepsilon) \frac{\partial}{\partial x_j} [\Phi_{\varepsilon,k} - x_k - \varepsilon \chi_k(x/\varepsilon)] \frac{\partial u_0}{\partial x_k} \right\} = a_{ij}(x/\varepsilon) \frac{\partial}{\partial x_j} [\Phi_{\varepsilon,k} - x_k - \varepsilon \chi_k(x/\varepsilon)] \frac{\partial^2 u_0}{\partial x_i \partial x_k}.$$

This completes the proof. ∎

We end this section with well known energy estimates for the initial-Dirichlet problem

$$\begin{cases} (\partial_t^2 + \mathcal{L}_0) u_0 = 0 & \text{in } \Omega_T, \\ u_0 = 0 & \text{on } S_T, \\ u_0(x, 0) = \varphi_0(x), \quad \partial_t u_0(x, 0) = \varphi_1(x) & \text{for } x \in \Omega. \end{cases} \tag{2.15}$$

Let Ω be a bounded domain in $\mathbb{R}^d$. Given $\varphi \in H_0^1(\Omega)$ and $\varphi_1 \in L^2(\Omega)$, the evolution problem (2.15) has a unique solution in $u_0 \in L^\infty(0, T; H_0^1(\Omega))$ with $\partial_t u_0 \in L^\infty(0, T; L^2(\Omega))$. Moreover, the solution satisfies

$$\|\nabla u_0(\cdot, t)\|_{L^2(\Omega)} + \|\partial_t u_0(\cdot, t)\|_{L^2(\Omega)} \le C \{ \|\nabla \varphi_0\|_{L^2(\Omega)} + \|\varphi_1\|_{L^2(\Omega)} \} \tag{2.16}$$

for any $t \in (0, T]$, where C depends only on d and μ. Let $\{\lambda_{0,k} : k = 1, 2, \dots\}$ denote the increasing sequence of eigenvalues for $\mathcal{L}_0$ in Ω. Let $\{\psi_{0,k}\}$ be a set of eigenfunctions in $H_0^1(\Omega)$ for $\mathcal{L}_0$ in Ω such that $\{\psi_{0,k}\}$ forms an orthonormal basis for $L^2(\Omega)$ and $\mathcal{L}_{\varepsilon,0}(\psi_{0,k}) = \lambda_{0,k}\psi_{0,k}$ in Ω. Suppose that

$$\varphi_0 = \sum_k a_k \psi_{0,k} \quad \text{and} \quad \varphi_1 = \sum_k b_k \psi_{0,k},$$

where $a_k, b_k \in \mathbb{R}$. Then the solution of (2.15) is given

$$u_0(x,t) = \sum_k \{a_k \cos(\sqrt{\lambda_{0,k}}\, t) + b_k \lambda_{0,k}^{-1/2} \sin(\sqrt{\lambda_{0,k}}\, t)\}\psi_{0,k}(x). \tag{2.17}$$

It follows that

$$\begin{aligned} &\|\mathcal{L}_0(u_0)(\cdot,t)\|_{L^2(\Omega)} + \|\partial_t \nabla u_0(\cdot,t)\|_{L^2(\Omega)} + \|\partial_t^2 u_0(\cdot,t)\|_{L^2(\Omega)} \\ &\qquad \le C\{\|\mathcal{L}_0(\varphi_0)\|_{L^2(\Omega)} + \|\nabla\varphi_1\|_{L^2(\Omega)}\} \end{aligned} \tag{2.18}$$

for any $t \in (0, T]$, where C depends only on d and μ.

If Ω is a bounded $C^{1,1}$ domain, $\varphi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\varphi_1 \in H_0^1(\Omega)$, one may use the H^2 estimate for the elliptic operator $\mathcal{L}_0$,

$$\|\nabla^2 u\|_{L^2(\Omega)} \le C\|\mathcal{L}_0(u)\|_{L^2(\Omega)} \quad \text{for } u \in H_0^1(\Omega) \cap H^2(\Omega),$$

and (2.18) to show that

$$\|\nabla^2 u_0(\cdot,t)\|_{L^2(\Omega)} \le C\{\|\mathcal{L}_0(\varphi_0)\|_{L^2(\Omega)} + \|\nabla\varphi_1\|_{L^2(\Omega)}\} \tag{2.19}$$

for any $t \in (0, T]$, where C depends only on d, μ, and Ω. Furthermore, if Ω is a bounded C^3 domain, $\varphi_0 \in H^3(\Omega) \cap H_0^1(\Omega)$ and $\varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\begin{aligned} &\|\nabla^3 u_0(\cdot,t)\|_{L^2(\Omega)} + \|\partial_t \nabla^2 u_0(\cdot,t)\|_{L^2(\Omega)} + \|\partial_t^2 \nabla u_0(\cdot,t)\|_{L^2(\Omega)} + \|\partial_t^3 u_0(\cdot,t)\|_{L^2(\Omega)} \\ &\qquad \le C\{\|\mathcal{L}_0(\varphi_0)\|_{H^1(\Omega)} + \|\mathcal{L}_0(\varphi_1)\|_{L^2(\Omega)}\} \end{aligned} \tag{2.20}$$

for any $t \in (0, T]$.

3. Convergence rates

Throughout this section we assume that $A = A(y)$ satisfies (1.3)–(1.5). No additional smoothness condition on A is needed.

For a function w in $\Omega \times [T_0, T_1]$, we introduce the energy functional

$$E_\varepsilon(t; w) = \frac{1}{2}\int_\Omega \{\langle A(x/\varepsilon)\nabla w(x,t), \nabla w(x,t)\rangle + (\partial_t w(x,t))^2\}\, dx \tag{3.1}$$

for $t \in [T_0, T_1]$.

Lemma 3.1. *Let u_ε, u_0, and w_ε be as in Lemma 2.2, with $u_\varepsilon = u_0$ on $\partial\Omega \times [T_0, T_1]$. Then*

$$
\begin{aligned}
&|E_\varepsilon(T_1; w_\varepsilon) - E_\varepsilon(T_0; w_\varepsilon)| \\
&\quad \le C\varepsilon \left(\int_{T_0}^{T_1} \int_\Omega (|\partial_t \nabla^2 u_0| + |\partial_t^2 \nabla u_0|)^2 \, dx \, dt \right)^{1/2} \left(\int_{T_0}^{T_1} E_\varepsilon(t; w_\varepsilon) \, dt \right)^{1/2} \\
&\qquad + C\varepsilon \|\nabla^2 u_0(\cdot, T_1)\|_{L^2(\Omega)} E_\varepsilon(T_1; w_\varepsilon)^{1/2} \\
&\qquad + C\varepsilon \|\nabla^2 u_0(\cdot, T_0)\|_{L^2(\Omega)} E_\varepsilon(T_0; w_\varepsilon)^{1/2},
\end{aligned}
\tag{3.2}
$$

where C depends only on d and μ.

Proof. Using the symmetry condition (1.4), we obtain

$$
E_\varepsilon(T_1; w_\varepsilon) - E_\varepsilon(T_0; w_\varepsilon) = \int_{T_0}^{T_1} \langle (\partial_t^2 + \mathcal{L}_\varepsilon) w_\varepsilon, \partial_t w_\varepsilon \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \, dt. \tag{3.3}
$$

We will use the formula (2.14) for $(\partial_t^2 + \mathcal{L}_\varepsilon) w_\varepsilon$ to bound the right-hand side of (3.3). The fact that $w_\varepsilon = 0$ on $\partial\Omega \times [T_0, T_1]$ is also used.

Let I_1 denote the first term on the right-hand side of (2.14). It follows by integration by parts (first in x and then in t) that

$$
\begin{aligned}
\left| \int_{T_0}^{T_1} \langle I_1, \partial_t w_\varepsilon \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \, dt \right| &= \varepsilon \left| \int_{T_0}^{T_1} \int_\Omega \phi_{kij}(x/\varepsilon) \frac{\partial^2 u_0}{\partial x_k \partial x_j} \cdot \partial_t \frac{\partial w_\varepsilon}{\partial x_i} \, dx \, dt \right| \\
&\le C\varepsilon \int_{T_0}^{T_1} \int_\Omega |\partial_t \nabla^2 u_0| \, |\nabla w_\varepsilon| \, dx \, dt \\
&\quad + C\varepsilon \int_\Omega |\nabla^2 u_0(x, T_1)| \, |\nabla w_\varepsilon(x, T_1)| \, dx \\
&\quad + C\varepsilon \int_\Omega |\nabla^2 u_0(x, T_0)| \, |\nabla w_\varepsilon(x, T_0)| \, dx.
\end{aligned}
$$

By the Cauchy inequality this leads to

$$
\begin{aligned}
\left| \int_{T_0}^{T_1} \langle I_1, \partial_t w_\varepsilon \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \, dt \right| &\le C\varepsilon \|\partial_t \nabla^2 u_0\|_{L^2(\Omega \times (T_0, T_1))} \left(\int_{T_0}^{T_1} E_\varepsilon(t; w_\varepsilon) \, dt \right)^{1/2} \\
&\quad + C\varepsilon \|\nabla^2 u_0(\cdot, T_1)\|_{L^2(\Omega)} E_\varepsilon(T_1; w_\varepsilon)^{1/2} \\
&\quad + C\varepsilon \|\nabla^2 u_0(\cdot, T_0)\|_{L^2(\Omega)} E_\varepsilon(T_0; w_\varepsilon)^{1/2},
\end{aligned}
\tag{3.4}
$$

where C depends only on d and μ. Let I_2 denote the second term on the right-hand side of (2.14). Since

$$
\|\Phi_{\varepsilon,k} - x_k\|_{L^\infty(\Omega)} \le C\varepsilon,
$$

it is easy to see that (3.4) also holds with I_2 in place of I_1.

Next, let I_3 denote the third term on the right-hand side of (2.14). Using integration by parts in the t variable, we see that

$$\begin{aligned}
&\left|\int_{T_0}^{T_1}\int_\Omega I_3 \cdot \partial_t w_\varepsilon \, dx\, dt\right| \\
&\qquad \le C\int_{T_0}^{T_1}\int_\Omega |\nabla[\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)]|\,|\partial_t \nabla^2 u_0|\,|w_\varepsilon|\, dx\, dt \\
&\qquad\quad + C\int_\Omega |\nabla[\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)]|\,|\nabla^2 u_0(x,T_1)|\,|w_\varepsilon(x,T_1)|\, dx \\
&\qquad\quad + C\int_\Omega |\nabla[\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)]|\,|\nabla^2 u_0(x,T_0)|\,|w_\varepsilon(x,T_0)|\, dx.
\end{aligned}$$

It follows from the Cauchy inequality that

$$\begin{aligned}
&\left|\int_{T_0}^{T_1}\int_\Omega I_3 \cdot \partial_t w_\varepsilon \, dx\, dt\right| \\
&\qquad \le C\|\nabla[\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)] w_\varepsilon\|_{L^2(\Omega\times(T_0,T_1))}\|\partial_t \nabla^2 u_0\|_{L^2(\Omega\times(T_0,T_1))} \\
&\qquad\quad + C\|\nabla[\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)] w_\varepsilon(\cdot,T_1)\|_{L^2(\Omega)}\|\nabla^2 u_0(\cdot,T_1)\|_{L^2(\Omega)} \\
&\qquad\quad + C\|\nabla[\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)] w_\varepsilon(\cdot,T_0)\|_{L^2(\Omega)}\|\nabla^2 u_0(\cdot,T_0)\|_{L^2(\Omega)}.
\end{aligned}$$

Since $\mathcal{L}_\varepsilon(\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)) = 0$ in Ω and $w_\varepsilon = 0$ on $\partial\Omega$, by Caccioppoli's inequality

$$\begin{aligned}
\|\nabla[\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)] w_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &\le C\|[\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)]\nabla w_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \\
&\le C\varepsilon\|\nabla w_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \qquad (3.5)
\end{aligned}$$

for $t \in [T_0, T_1]$. As a result, the estimate (3.4) continues to hold if we replace I_1 by I_3.

Finally, let I_4 denote the last term on the right-hand side of (2.14). By the Cauchy inequality, we obtain

$$\begin{aligned}
\left|\int_{T_0}^{T_1}\int_\Omega I_4 \cdot \partial_t w_\varepsilon \, dx\, dt\right| &\le C\varepsilon\|\partial_t^2 \nabla u_0\|_{L^2(\Omega\times(T_0,T_1))}\|\partial_t w_\varepsilon\|_{L^2(\Omega\times(T_0,T_1))} \\
&\le C\varepsilon\|\partial_t^2 \nabla u_0\|_{L^2(\Omega\times(T_0,T_1))}\left(\int_{T_0}^{T_1} E_\varepsilon(t; w_\varepsilon)\, dt\right)^{1/2}.
\end{aligned}$$

This completes the proof of (3.2). ∎

The next lemma gives an estimate of $E_\varepsilon(t; w_\varepsilon)$ for $t = 0$.

Lemma 3.2. *Let w_ε, $\varphi_{\varepsilon,0}$, φ_0, $\varphi_{\varepsilon,1}$ and φ_1 be as in Theorem 1.2. Then*

$$\begin{aligned}
E_\varepsilon(0; w_\varepsilon) &\le C\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0}) - \mathcal{L}_0(\varphi_0)\|^2_{H^{-1}(\Omega)} + C\|\varphi_{\varepsilon,1} - \varphi_1\|^2_{L^2(\Omega)} \\
&\quad + C\varepsilon^2\{\|\nabla^2\varphi_0\|^2_{L^2(\Omega)} + \|\nabla\varphi_1\|^2_{L^2(\Omega)}\}, \qquad (3.6)
\end{aligned}$$

where C depends only on d and μ.

Proof. Note that

$$\partial_t w_\varepsilon(x,0) = \partial_t u_\varepsilon(x,0) - \partial_t u_0(x,0) - (\Phi_{\varepsilon,k} - x_k)\partial_t \frac{\partial u_0}{\partial x_k}(x,0)$$
$$= \varphi_{\varepsilon,1} - \varphi_1 - (\Phi_{\varepsilon,k} - x_k)\frac{\partial \varphi_1}{\partial x_k}.$$

It follows that

$$\|\partial_t w_\varepsilon(\cdot,0)\|_{L^2(\Omega)} \le \|\varphi_{\varepsilon,1} - \varphi_1\|_{L^2(\Omega)} + C\varepsilon\|\nabla\varphi_1\|_{L^2(\Omega)}.$$

Next, to bound $\|\nabla w_\varepsilon(\cdot,0)\|_{L^2(\Omega)}$, we use

$$\int_\Omega \langle A(x/\varepsilon)\nabla w_\varepsilon, \nabla w_\varepsilon\rangle\, dx = \int_\Omega \langle \mathcal{L}_\varepsilon(w_\varepsilon), w_\varepsilon\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\, dx \tag{3.7}$$

and the formula

$$\begin{aligned}\mathcal{L}_\varepsilon(w_\varepsilon)(x,0) = \mathcal{L}_\varepsilon(\varphi_{\varepsilon,0}) - \mathcal{L}_0(\varphi_0) - \varepsilon\frac{\partial}{\partial x_i}\left\{\phi_{kij}(x/\varepsilon)\frac{\partial^2\varphi_0}{\partial x_k\partial x_j}\right\}\\ + \frac{\partial}{\partial x_i}\left\{a_{ij}(x/\varepsilon)[\Phi_{\varepsilon,k} - x_k]\frac{\partial^2\varphi_0}{\partial x_j\partial x_k}\right\}\\ + a_{ij}(x/\varepsilon)\frac{\partial}{\partial x_j}[\Phi_{\varepsilon,k} - x_k - \varepsilon\chi_k(x/\varepsilon)]\frac{\partial^2\varphi_0}{\partial x_i\partial x_k}.\end{aligned} \tag{3.8}$$

The proof of (3.8) is similar to that of (2.14). It follows from (3.7) and (3.8) that

$$\begin{aligned}\|\nabla w_\varepsilon(\cdot,0)\|^2_{L^2(\Omega)} &\le C\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0}) - \mathcal{L}_0(\varphi_0)\|_{H^{-1}(\Omega)}\|\nabla w_\varepsilon(\cdot,0)\|_{L^2(\Omega)}\\ &\quad + C\varepsilon\|\nabla^2\varphi_0\|_{L^2(\Omega)}\|\nabla w_\varepsilon(\cdot,0)\|_{L^2(\Omega)}\\ &\quad + C\|\nabla[\Phi_\varepsilon - x - \varepsilon\chi(x/\varepsilon)]w_\varepsilon(\cdot,0)\|_{L^2(\Omega)}\|\nabla^2\varphi_0\|_{L^2(\Omega)}\\ &\le C\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0}) - \mathcal{L}_0(\varphi_0)\|_{H^{-1}(\Omega)}\|\nabla w_\varepsilon(\cdot,0)\|_{L^2(\Omega)}\\ &\quad + C\varepsilon\|\nabla^2\varphi_0\|_{L^2(\Omega)}\|\nabla w_\varepsilon(\cdot,0)\|_{L^2(\Omega)},\end{aligned}$$

where we have used the Caccioppoli inequality (3.5) for the last step. This yields

$$\|\nabla w_\varepsilon(\cdot,0)\|_{L^2(\Omega)} \le C\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0}) - \mathcal{L}_0(\varphi_0)\|_{H^{-1}(\Omega)} + C\varepsilon\|\nabla^2\varphi_0\|_{L^2(\Omega)}$$

and completes the proof. ∎

We are now in a position to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let

$$\begin{aligned} M_0 &= \sup_{0\le t\le T} \left(\int_\Omega |\nabla^2 u_0(x,t)|^2\, dx \right)^{1/2}, \\ M_1 &= \sup_{0\le t\le T} \left(\int_\Omega \left(|\partial_t \nabla^2 u_0(x,t)| + |\partial_t^2 \nabla u_0(x,t)| \right)^2 dx \right)^{1/2}. \end{aligned} \tag{3.9}$$

Let w_ε be defined by (2.13). We will show that for any $t \in [0,T]$,

$$E_\varepsilon(t; w_\varepsilon) \le C\{E_\varepsilon(0; w_\varepsilon) + \varepsilon^2 M_0(M_0 + TM_1)\}, \tag{3.10}$$

where C depends only on d and μ. This, together with the estimate of $E_\varepsilon(0; w_\varepsilon)$ in Lemma 3.2, gives the inequality (1.21).

It follows from Lemma 3.1 that for $0 \le t_0 \le t \le t_0 + \delta \le T$,

$$E_\varepsilon(t; w_\varepsilon) \le E_\varepsilon(t_0; w_\varepsilon) + C\varepsilon(\delta M_1 + M_0) \sup_{t\in[t_0,t_0+\delta]} E_\varepsilon(t; w_\varepsilon)^{1/2},$$

where C depends only on d and μ. We now consider two cases. In the first case we assume $M_1 \le 2T^{-1} M_0$. By letting $t_0 = 0$ and $\delta = T$, we obtain

$$\begin{aligned} \sup_{t\in[0,T]} E_\varepsilon(t; w_\varepsilon) &\le E(0; w_\varepsilon) + C\varepsilon M_0 \sup_{t\in[0,T]} E_\varepsilon(t; w_\varepsilon)^{1/2} \\ &\le E_\varepsilon(0; w_\varepsilon) + C\varepsilon^2 M_0^2 + \tfrac{1}{2} \sup_{t\in[0,T]} E_\varepsilon(t; w_\varepsilon), \end{aligned}$$

from which the estimate (3.10) follows.

In the second case we assume $M_1 > 2T^{-1} M_0$. Using the Cauchy inequality, we obtain

$$E_\varepsilon(t; w_\varepsilon) \le E_\varepsilon(t_0; w_\varepsilon) + C\varepsilon^2 \gamma^{-1} (\delta M_1 + M_0)^2 + \gamma \sup_{t\in[t_0,t_0+\delta]} E_\varepsilon(t; w_\varepsilon)$$

for any $t \in [t_0, t_0 + \delta]$, where $\gamma \in (0,1)$. This gives

$$\sup_{t\in[t_0,t_0+\delta]} E_\varepsilon(t; w_\varepsilon) \le \frac{E_\varepsilon(t_0; w_\varepsilon)}{1-\gamma} + \frac{C\varepsilon^2(\delta M_1 + M_0)^2}{\gamma(1-\gamma)}. \tag{3.11}$$

Let $\delta = T/n \in (0, 1/2)$, where $n \ge 1$ is to be chosen later. Let $t_\ell = \ell\delta$, where $0 \le \ell \le n$. By using (3.11) repeatedly, we see that

$$\begin{aligned} \sup_{t\in[t_\ell,t_{\ell+1}]} E_\varepsilon(t; w_\varepsilon) + \frac{C\varepsilon^2(\delta M_1 + M_0)^2}{\gamma} &\le \frac{1}{1-\gamma} \left\{ E_\varepsilon(t_\ell; w_\varepsilon) + \frac{C\varepsilon^2(\delta M_1 + M_0)^2}{\gamma} \right\} \\ &\le \frac{1}{(1-\gamma)^\ell} \left\{ E_\varepsilon(0; w_\varepsilon) + \frac{C\varepsilon^2(\delta M_1 + M_0)^2}{\gamma} \right\}. \end{aligned}$$

This implies that

$$\sup_{t\in[0,T]} E_\varepsilon(t;w_\varepsilon) \le \frac{1}{(1-\gamma)^n}\left\{E_\varepsilon(0;w_\varepsilon) + \frac{C\varepsilon^2(\delta M_1 + M_0)^2}{\gamma}\right\}.$$

Choose $\gamma = \frac{\delta}{2T} = \frac{1}{2n}$. Note that

$$(1-\gamma)^n = \left(1 - \frac{1}{2n}\right)^n \sim e^{-1/2}.$$

It follows that

$$\sup_{t\in[0,T]} E_\varepsilon(t;w_\varepsilon) \le C\{E_\varepsilon(0;w_\varepsilon) + C\varepsilon^2 T\delta^{-1}(\delta M_1 + M_0)^2\}.$$

Finally, if $M_0 = 0$, we let $\delta \to 0$ to obtain the desired estimate. If $M_0 \ne 0$, we choose $\delta = cM_0M_1^{-1} < T$ and obtain

$$E(t;w_\varepsilon) \le CE(0;w_\varepsilon) + C\varepsilon^2 TM_0M_1$$

for any $t \in [0,T]$. This completes the proof. ∎

We end this section by establishing a convergence rate for $\|u_\varepsilon(\cdot,t) - u_0(\cdot,t)\|_{L^2(\Omega)}$ for $0 < t < T$. Consider the initial-Dirichlet problem

$$\begin{cases} (\partial_t^2 + \mathcal{L}_\varepsilon)u_\varepsilon = 0 & \text{in } \Omega_T = \Omega\times(0,T], \\ u_\varepsilon = 0 & \text{on } S_T = \partial\Omega\times[0,T], \\ u_\varepsilon(x,0) = \varphi_0(x), \quad \partial_t u_\varepsilon(x,0) = \varphi_1(x) & \text{for } x\in\Omega, \end{cases} \tag{3.12}$$

and its homogenized problem

$$\begin{cases} (\partial_t^2 + \mathcal{L}_0)u_0 = 0 & \text{in } \Omega_T, \\ u_0 = 0 & \text{on } S_T, \\ u_0(x,0) = \varphi_0(x), \quad \partial_t u_0(x,0) = \varphi_1(x) & \text{for } x\in\Omega, \end{cases} \tag{3.13}$$

where $\varphi_0 \in H_0^1(\Omega)\cap H^2(\Omega)$ and $\varphi_1 \in H_0^1(\Omega)$. Let

$$v_\varepsilon(x,t) = \int_0^t u_\varepsilon(x,s)\,ds \quad \text{and} \quad v_0(x,t) = \int_0^t u_0(x,s)\,ds.$$

Then

$$\begin{cases} (\partial_t^2 + \mathcal{L}_\varepsilon)v_\varepsilon = \varphi_1 & \text{in } \Omega_T, \\ v_\varepsilon = 0 & \text{on } S_T, \\ v_\varepsilon(x,0) = 0, \quad \partial_t v_\varepsilon(x,0) = \varphi_0(x) & \text{for } x\in\Omega, \end{cases}$$

and

$$\begin{cases} (\partial_t^2 + \mathcal{L}_0)v_0 = \varphi_1 & \text{in } \Omega_T, \\ v_0 = 0 & \text{on } S_T, \\ v_0(x,0) = 0, \quad \partial_t v_0(x,0) = \varphi_0(x) & \text{for } x\in\Omega. \end{cases}$$

By applying Theorem 1.2 to v_ε and v_0 and using (1.22), we see that for any $t \in (0, T]$,

$$\begin{aligned}
&\|u_\varepsilon(\cdot,t) - u_0(\cdot,t)\|_{L^2(\Omega)} \\
&\quad \le C\varepsilon\|\nabla\varphi_0\|_{L^2(\Omega)} + C\varepsilon \sup_{t\in[0,T]} \|\nabla^2 v_0(\cdot,t)\|_{L^2(\Omega)} + C\varepsilon \sup_{t\in[0,T]} \|\nabla u_0(\cdot,t)\|_{L^2(\Omega)} \\
&\qquad + C\varepsilon\sqrt{T} \sup_{t\in(0,T)} \big\| |\nabla^2 u_0(\cdot,t)| + |\partial_t \nabla u_0| \big\|^{1/2}_{L^2(\Omega)} \sup_{t\in(0,T)} \|\nabla^2 v_0(\cdot,t)\|^{1/2}_{L^2(\Omega)},
\end{aligned} \tag{3.14}$$

where we have used the fact $\partial_t v_0 = u_0$. Note that, if Ω is $C^{1,1}$,

$$\begin{aligned}
\|\nabla^2 v_0(\cdot,t)\|_{L^2(\Omega)} &\le C\|\mathcal{L}_0(v_0)(\cdot,t)\|_{L^2(\Omega)} \\
&\le C\|\partial_t u_0(\cdot,t)\|_{L^2(\Omega)} + C\|\varphi_1\|_{L^2(\Omega)} \\
&\le C\{\|\nabla\varphi_0\|_{L^2(\Omega)} + \|\varphi_1\|_{L^2(\Omega)}\},
\end{aligned}$$

where we have used (2.16) for the last inequality. This, together with (3.14), (2.18) and (2.19), yields

$$\begin{aligned}
&\|u_\varepsilon(\cdot,t) - u_0(\cdot,t)\|_{L^2(\Omega)} \\
&\quad \le C\varepsilon\{\|\nabla\varphi_0\|_{L^2(\Omega)} + \|\varphi_1\|_{L^2(\Omega)}\} \\
&\qquad + C\varepsilon\sqrt{T}(\|\varphi_0\|_{H^2(\Omega)} + \|\nabla\varphi_1\|_{L^2(\Omega)})^{1/2}(\|\nabla\varphi_0\|_{L^2(\Omega)} + \|\varphi_1\|_{L^2(\Omega)})^{1/2}
\end{aligned} \tag{3.15}$$

for any $t \in (0, T]$, where Ω is $C^{1,1}$ and the constant C depends only on d, μ and Ω.

4. Uniform boundary controllability

Throughout this section we will assume that $A = A(y)$ satisfies conditions (1.3)–(1.5) as well as the Lipschitz condition (1.12).

Let u_ε be the solution of the initial-Dirichlet problem

$$\begin{cases}
(\partial_t^2 + \mathcal{L}_\varepsilon)u_\varepsilon = 0 & \text{in } \Omega_T = \Omega \times (0,T], \\
u_\varepsilon = 0 & \text{on } S_T = \partial\Omega \times [0,T], \\
u_\varepsilon(x,0) = \varphi_{\varepsilon,0}(x), \quad \partial_t u_\varepsilon(x,0) = \varphi_{\varepsilon,1}(x) & \text{for } x \in \Omega.
\end{cases} \tag{4.1}$$

We are interested in the estimates (1.9) and (1.10) with positive constants C and c independent of $\varepsilon > 0$.

Let $h = (h_1, \dots, h_d)$ be a vector field in $C^1(\mathbb{R}^d; \mathbb{R}^d)$ and $n = (n_1, \dots, n_d)$ denote the outward unit normal to $\partial\Omega$. We start with the well known Rellich identity

$$\begin{aligned}
\int_0^T\!\!\int_{\partial\Omega} \langle h, n\rangle \cdot a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \cdot \frac{\partial u_\varepsilon}{\partial x_i}\, d\sigma\, dt &= 2\int_0^T\!\!\int_{\partial\Omega} h_k a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_i} \left\{ n_k \frac{\partial u_\varepsilon}{\partial x_j} - n_j \frac{\partial u_\varepsilon}{\partial x_k} \right\} d\sigma\, dt \\
&\quad - \int_0^T\!\!\int_\Omega \operatorname{div}(h) \cdot a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \cdot \frac{\partial u_\varepsilon}{\partial x_i}\, dx\, dt - \int_0^T\!\!\int_\Omega h_k \frac{\partial a_{ij}^\varepsilon}{\partial x_k} \cdot \frac{\partial u_\varepsilon}{\partial x_j} \cdot \frac{\partial u_\varepsilon}{\partial x_i}\, dx\, dt \\
&\quad + 2\int_0^T\!\!\int_\Omega \frac{\partial h_k}{\partial x_j} \cdot a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_k} \cdot \frac{\partial u_\varepsilon}{\partial x_i}\, dx\, dt - 2\int_0^T\!\!\int_\Omega h_k \frac{\partial u_\varepsilon}{\partial x_k} \cdot \mathcal{L}_\varepsilon(u_\varepsilon)\, dx\, dt,
\end{aligned} \tag{4.2}$$

where $a^\varepsilon_{ij} = a_{ij}(x/\varepsilon)$. The identity (4.2) follows from integration by parts (in the x variable). We remark that the symmetry condition (1.4), which is essential for (4.2) even in the case of constant coefficients, is used to obtain

$$a^\varepsilon_{ij}\frac{\partial}{\partial x_k}\left(\frac{\partial u_\varepsilon}{\partial x_i}\cdot\frac{\partial u_\varepsilon}{\partial x_j}\right) = 2\mathcal{L}_\varepsilon(u_\varepsilon)\cdot\frac{\partial u_\varepsilon}{\partial x_k} + 2\frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij}\frac{\partial u_\varepsilon}{\partial x_j}\cdot\frac{\partial u_\varepsilon}{\partial x_k}\right)$$

in the proof of (4.2). It also follows from integration by parts that

$$\begin{aligned}
&\int_0^T\int_\Omega h_k\frac{\partial u_\varepsilon}{\partial x_k}\cdot\partial_t^2 u_\varepsilon\,dx\,dt \\
&\quad= -\frac{1}{2}\int_0^T\int_{\partial\Omega}\langle h,n\rangle\cdot(\partial_t u_\varepsilon)^2\,d\sigma\,dt + \frac{1}{2}\int_0^T\int_\Omega \operatorname{div}(h)\cdot(\partial_t u_\varepsilon)^2\,dx\,dt \\
&\qquad+\int_\Omega h_k\frac{\partial u_\varepsilon}{\partial x_k}(x,T)\partial_t u_\varepsilon(x,T)\,dx - \int_\Omega h_k\frac{\partial u_\varepsilon}{\partial x_k}(x,0)\partial_t u_\varepsilon(x,0)\,dx.
\end{aligned}\tag{4.3}$$

Suppose $u_\varepsilon = 0$ on $\partial\Omega$. Since $n_k\frac{\partial u_\varepsilon}{\partial x_j} - n_j\frac{\partial u_\varepsilon}{\partial x_k} = 0$ and $\partial_t u_\varepsilon = 0$ on $\partial\Omega$, by combining (4.2) with (4.3), we obtain

$$\begin{aligned}
\int_0^T\int_{\partial\Omega}\langle h,n\rangle\cdot a^\varepsilon_{ij}\frac{\partial u_\varepsilon}{\partial x_j}\cdot\frac{\partial u_\varepsilon}{\partial x_i}\,d\sigma\,dt &= \int_0^T\int_\Omega \operatorname{div}(h)\cdot\left((\partial_t u_\varepsilon)^2 - a^\varepsilon_{ij}\frac{\partial u_\varepsilon}{\partial x_j}\cdot\frac{\partial u_\varepsilon}{\partial x_i}\right)dx\,dt \\
&\quad-\int_0^T\int_\Omega h_k\frac{\partial a^\varepsilon_{ij}}{\partial x_k}\cdot\frac{\partial u_\varepsilon}{\partial x_j}\cdot\frac{\partial u_\varepsilon}{\partial x_i}\,dx\,dt \\
&\quad+2\int_0^T\int_\Omega \frac{\partial h_k}{\partial x_j}\cdot a^\varepsilon_{ij}\frac{\partial u_\varepsilon}{\partial x_k}\cdot\frac{\partial u_\varepsilon}{\partial x_i}\,dx\,dt \\
&\quad-2\int_0^T\int_\Omega h_k\frac{\partial u_\varepsilon}{\partial x_k}\cdot(\partial_t^2+\mathcal{L}_\varepsilon)u_\varepsilon\,dx\,dt \\
&\quad+\int_\Omega h_k\frac{\partial u_\varepsilon}{\partial x_k}(x,T)\partial_t u_\varepsilon(x,T)\,dx \\
&\quad-\int_\Omega h_k\frac{\partial u_\varepsilon}{\partial x_k}(x,0)\partial_t u_\varepsilon(x,0)\,dx.
\end{aligned}\tag{4.4}$$

Lemma 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let u_0 be a weak solution of (3.13) for the homogenized operator $\partial_t^2 + \mathcal{L}_0$. Then*

$$\int_0^T\int_{\partial\Omega}|\nabla u_0|^2\,d\sigma\,dt \le C(Tr_0^{-1}+1)\{\|\nabla\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{L^2(\Omega)}\},\tag{4.5}$$

where r_0 denotes the diameter of Ω. Moreover, if $T \ge C_0 r_0$, then

$$Tr_0^{-1}\{\|\nabla\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{L^2(\Omega)}\} \le C\int_0^T\int_{\partial\Omega}|\nabla u_0|^2\,d\sigma\,dt.\tag{4.6}$$

The constants C and C_0 depend only on d, μ and the Lipschitz character of Ω.

Proof. This is well known and follows readily from (4.4) (with $\widehat{a}_{ij}$ in place of a_{ij}^{ε}) (see e.g. [16]). We include a proof here for the reader's convenience. To see (4.5), we choose a vector field $h \in C^1(\mathbb{R}^d;\mathbb{R}^d)$ such that $\langle h,n\rangle \geq c_0 > 0$ on $\partial\Omega$ and $|\nabla h| \leq C/r_0$. It follows from (4.4) with $\widehat{a}_{ij}$ in place of a_{ij}^{ε} that

$$
\begin{aligned}
c\int_0^T\int_{\partial\Omega} |\nabla u_0|^2\, d\sigma\, dt \leq \frac{C}{r_0}\int_0^T\int_\Omega (|\nabla u_0|^2 + |\partial_t u_0|^2)\, dx\, dt \\
+ C\int_\Omega |\nabla u_0(x,T)|\,|\partial_t u_0(x,T)|\, dx + C\int_\Omega |\nabla \varphi_0|\,|\varphi_1|\, dx \\
\leq C(Tr_0^{-1}+1)\{\|\nabla\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{L^2(\Omega)}\},
\end{aligned}
$$

where we have used the energy estimate (2.16) for the last step.

To prove (4.6), we choose $h(x) = x - x_0$, where $x_0 \in \Omega$. Note that $\operatorname{div}(h) = d$. It follows from (4.4) that

$$
\begin{aligned}
\left| -\int_0^T\int_{\partial\Omega} \langle h,n\rangle \cdot \widehat{a}_{ij}\frac{\partial u_0}{\partial x_j}\cdot\frac{\partial u_0}{\partial x_i}\, d\sigma\, dt + dX + (2-d)Y \right| \\
\leq Cr_0\{\|\nabla\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{L^2(\Omega)}\}, \qquad (4.7)
\end{aligned}
$$

where

$$
X = \int_0^T\int_\Omega (\partial_t u_0)^2\, dx\, dt, \qquad Y = \int_0^T\int_\Omega \widehat{a}_{ij}\frac{\partial u_0}{\partial x_j}\cdot\frac{\partial u_0}{\partial x_i}\, dx\, dt.
$$

Note that by conservation of energy,

$$
X + Y = T\int_\Omega \left(\varphi_1^2 + \widehat{a}_{ij}\frac{\partial\varphi_0}{\partial x_j}\cdot\frac{\partial\varphi_0}{\partial x_i}\right) dx,
$$

and

$$
\begin{aligned}
X - Y &= \int_0^T\int_\Omega \partial_t(u_0\partial_t u_0)\, dx\, dt \\
&= \int_\Omega u_0(x,T)\partial_t u_0(x,T)\, dx - \int_\Omega \varphi_0\varphi_1\, dx \\
&\leq Cr_0\{\|\nabla\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{L^2(\Omega)}\},
\end{aligned}
$$

where we have used Poincaré's inequality and the energy estimates for the last step. By writing $dX + (2-d)Y$ as $(X+Y) + (d-1)(X-Y)$, we deduce from (4.7) that

$$
\begin{aligned}
\left| \int_0^T\int_{\partial\Omega} \langle h,n\rangle \cdot \widehat{a}_{ij}\frac{\partial u_0}{\partial x_j}\cdot\frac{\partial u_0}{\partial x_i}\, d\sigma\, dt - T\int_\Omega \left(\varphi_1^2 + \widehat{a}_{ij}\frac{\partial\varphi_0}{\partial x_j}\cdot\frac{\partial\varphi_0}{\partial x_i}\right) dx \right| \\
\leq Cr_0\{\|\nabla\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{L^2(\Omega)}\},
\end{aligned}
$$

from which the inequality (4.6) follows if $T \geq C_0 r_0$. ∎

The argument used in the proof of Lemma 4.1 for $\partial_t^2 + \mathcal{L}_0$ does not work for the operator $\partial_t^2 + \mathcal{L}_\varepsilon$; the derivative of a_{ij}^{ε} is unbounded as $\varepsilon \to 0$. Our approach to Theorem 1.1 is to approximate the solution u_ε of (4.1) with initial data $(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1})$ by a solution of

(3.13) for the homogenized operator $\partial_t^2 + \mathcal{L}_0$ with initial data (φ_0, φ_1), where $\varphi_1 = \varphi_{\varepsilon,1}$ and φ_0 is the function in $H_0^1(\Omega)$ such that

$$\mathcal{L}_0(\varphi_0) = \mathcal{L}_\varepsilon(\varphi_{\varepsilon,0}) \quad \text{in } \Omega. \tag{4.8}$$

Lemma 4.2. *Let Ω be a bounded C^3 domain in $\mathbb{R}^d$. Let u_ε and u_0 be the solutions of (4.1) and (3.13) with initial data $(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1})$ and (φ_0, φ_1), respectively. Assume that $\varphi_1 = \varphi_{1,\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\varphi_0 \in H^3(\Omega) \cap H_0^1(\Omega)$ satisfies (4.8). Let w_ε be given by (2.13). Then for $0 < \varepsilon < \min(r_0, T)$,*

$$\begin{aligned}\int_0^T \int_{\partial\Omega} |\nabla w_\varepsilon|^2 \, d\sigma \, dt \le\; & C T\varepsilon \{\|\varphi_0\|^2_{H^2(\Omega)} + \|\varphi_1\|^2_{H^1(\Omega)}\} \\ & + C T^2 \varepsilon \{\|\varphi_0\|_{H^2(\Omega)} + \|\varphi_1\|_{H^1(\Omega)}\}\{\|\varphi_0\|_{H^3(\Omega)} + \|\varphi_1\|_{H^2(\Omega)}\} \\ & + C T \varepsilon^3 \{\|\varphi_0\|^2_{H^3(\Omega)} + \|\varphi_1\|^2_{H^2(\Omega)}\},\end{aligned} \tag{4.9}$$

where C depends only on d, μ, M, and Ω.

Proof. Let h be a vector field in $C^1(\mathbb{R}^d; \mathbb{R}^d)$ such that $\langle h, n\rangle \ge c_0 > 0$ on $\partial\Omega$ and $|\nabla h| \le C r_0^{-1}$. We apply the Rellich identity (4.4) with w_ε in place of u_ε. This gives

$$\begin{aligned}\int_0^T \int_{\partial\Omega} |\nabla w_\varepsilon|^2 \, d\sigma \, dt &\le \frac{C}{\varepsilon} \int_0^T \int_\Omega |\nabla w_\varepsilon|^2 \, dx \, dt + C \int_0^T \int_\Omega |\nabla w_\varepsilon||(\partial_t^2 + \mathcal{L}_\varepsilon) w_\varepsilon| \, dx \, dt \\ &\quad + C \sup_{t\in[0,T]} \|\nabla w_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \|\partial_t w_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \\ &\le C T \varepsilon^{-1} \sup_{t\in[0,T]} \{\|\nabla w_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \|\partial_t w_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}\} \\ &\quad + C\varepsilon \int_0^T \int_\Omega |(\partial_t^2 + \mathcal{L}_\varepsilon) w_\varepsilon|^2 \, dx \, dt,\end{aligned} \tag{4.10}$$

where we have used the Cauchy inequality for the last step. Since Ω is C^3 and A is Lipschitz, $\nabla \Phi_\varepsilon$ is bounded. Also, under the smoothness condition (1.12), the functions $\nabla \chi_j$ and $\nabla \phi_{kij}$ are bounded. Thus, in view of (2.14), we obtain

$$|(\partial_t^2 + \mathcal{L}_\varepsilon) w_\varepsilon| \le C\{|\nabla^2 u_0| + \varepsilon |\nabla^3 u_0| + \varepsilon |\partial_t^2 \nabla u_0|\}. \tag{4.11}$$

This, together with (4.10) and Theorem 1.2, gives

$$\begin{aligned}&\int_0^T \int_{\partial\Omega} |\nabla w_\varepsilon|^2 \, d\sigma \, dt \\ &\quad\le C T\varepsilon \{\|\varphi_0\|^2_{H^2(\Omega)} + \|\varphi_1\|^2_{H^1(\Omega)}\} + C T\varepsilon \sup_{t\in(0,T]} \|\nabla^2 u_0(\cdot,t)\|^2_{L^2(\Omega)} \\ &\qquad + C T^2 \varepsilon \sup_{t\in(0,T]} \|\nabla^2 u_0(\cdot,t)\|_{L^2(\Omega)} \sup_{t\in(0,T]} \| |\partial_t \nabla^2 u_0(\cdot,t)| + |\partial_t^2 \nabla u_0(\cdot,t)| \|_{L^2(\Omega)} \\ &\qquad + C T \varepsilon^3 \sup_{t\in(0,T]} \Big\| |\nabla^3 u_0(\cdot,t)| + |\partial_t^2 \nabla u_0(\cdot,t)| \Big\|^2_{L^2(\Omega)},\end{aligned}$$

from which the estimate (4.9) follows by using the energy estimates (2.19) and (2.20). ∎

The next theorem provides an upper bound for $\|\nabla u_\varepsilon\|_{L^2(S_T)}$.

Theorem 4.3. *Assume that A satisfies conditions (1.3)–(1.5), and (1.12). Let Ω be a bounded C^3 domain in $\mathbb{R}^d$. Let u_ε be a weak solution of (4.1) with initial data $\varphi_{\varepsilon,0} \in H^3(\Omega) \cap H_0^1(\Omega)$ and $\varphi_{\varepsilon,1} \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, for $0 < \varepsilon < \min(T, r_0)$,*

$$\begin{aligned}
&\int_0^T \int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \, dt \\
&\quad \le CT\{\|\nabla \varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} \\
&\qquad + CT\varepsilon\{\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|^2_{L^2(\Omega)} + \|\nabla\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} \\
&\qquad + CT^2\varepsilon\{\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|_{L^2(\Omega)} + \|\nabla\varphi_{\varepsilon,1}\|_{L^2(\Omega)}\}\{\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|_{H^1(\Omega)} + \|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,1})\|_{L^2(\Omega)}\} \\
&\qquad + CT\varepsilon^3\{\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|^2_{H^1(\Omega)} + \|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,1})\|^2_{L^2(\Omega)}\},
\end{aligned} \tag{4.12}$$

where C depends only on d, μ, M, and Ω.

Proof. Let u_0, w_ε be as in Lemma 4.2. Note that

$$\nabla w_\varepsilon = \nabla u_\varepsilon - (\nabla\Phi_\varepsilon)(\nabla u_0) - (\Phi_\varepsilon - x)\nabla^2 u_0. \tag{4.13}$$

It follows that

$$\begin{aligned}
\int_0^T \int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \, dt \le C \int_0^T \int_{\partial\Omega} |\nabla w_\varepsilon|^2 \, d\sigma \, dt + C \int_0^T \int_{\partial\Omega} |\nabla u_0|^2 \, d\sigma \, dt \\
+ C\varepsilon^2 \int_0^T \int_{\partial\Omega} |\nabla^2 u_0|^2 \, d\sigma \, dt.
\end{aligned} \tag{4.14}$$

To bound the first term on the right-hand side of (4.14), we use (4.10) as well as the fact that $\varphi_1 = \varphi_{\varepsilon,1}$ and $\mathcal{L}_0(\varphi_0) = \mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})$ in Ω. The second term on the right-hand side of (4.14) is handled by Lemma 4.1. Finally, to bound the third term, we use the inequality

$$\int_{\partial\Omega} |\nabla^2 u_0|^2 \, d\sigma \le C \int_\Omega |\nabla^2 u_0|^2 \, dx + C \int_\Omega |\nabla^2 u_0| \, |\nabla^3 u_0| \, dx. \tag{4.15}$$

To see (4.15), one chooses a vector field $h \in C_0^1(\mathbb{R}^d; \mathbb{R}^d)$ such that $\langle h, n\rangle \ge c_0 > 0$ on $\partial\Omega$, and applies the divergence theorem to the integral $\int_{\partial\Omega} |\nabla^2 u_0|^2 \langle h, n\rangle \, d\sigma$. ∎

We also obtain a lower bound for $\|\nabla u_\varepsilon\|_{L^2(S_T)}$.

Theorem 4.4. *Assume that A and Ω satisfy the same conditions as in Theorem 4.3. Let u_ε be a weak solution of (4.1) with initial data $\varphi_{\varepsilon,0} \in H^3(\Omega) \cap H_0^1(\Omega)$ and $\varphi_{\varepsilon,1} \in H^2(\Omega) \cap H_0^1(\Omega)$. If $T \ge C_0 r_0$ and $0 < \varepsilon < r_0$, then*

$$\begin{aligned}
&T r_0^{-1}\{\|\nabla \varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} \\
&\quad \le C \int_0^T \int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \, dt \\
&\qquad + CT\varepsilon\{\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|^2_{L^2(\Omega)} + \|\nabla\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} \\
&\qquad + CT^2\varepsilon\{\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|_{L^2(\Omega)} + \|\nabla\varphi_{\varepsilon,1}\|_{L^2(\Omega)}\}\{\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|_{H^1(\Omega)} + \|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,1})\|_{L^2(\Omega)}\} \\
&\qquad + CT\varepsilon^3\{\|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|^2_{H^1(\Omega)} + \|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,1})\|^2_{L^2(\Omega)}\},
\end{aligned} \tag{4.16}$$

where C depends only on d, μ, M, and Ω.

Proof. The proof uses (4.13) and the fact that

$$|\det(\nabla\Phi_\varepsilon)| \geq c_0 > 0 \quad \text{on } \partial\Omega, \tag{4.17}$$

which was proved in [13]. Let u_0, w_ε be as in Lemma 4.2. It follows from (4.6), (4.13) and (4.17) that

$$\begin{aligned} Tr_0^{-1}\{\|\nabla\varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} &\leq C\int_0^T\int_{\partial\Omega}|\nabla u_0|^2\,d\sigma\,dt \\ \leq C\int_0^T\int_{\partial\Omega}|\nabla u_\varepsilon|^2\,d\sigma\,dt + C\int_0^T\int_{\partial\Omega}|\nabla w_\varepsilon|^2\,d\sigma\,dt &+ C\varepsilon^2\int_0^T\int_{\partial\Omega}|\nabla^2 u_0|^2\,d\sigma\,dt. \end{aligned} \tag{4.18}$$

The last two terms on the right-hand side of (4.18) are treated exactly as in the proof of Theorem 4.3. ∎

Proof of Theorem 1.1. Let

$$\varphi_{\varepsilon,0} = \sum_{\lambda_{\varepsilon,k}\leq N} a_k\psi_{\varepsilon,k} \quad \text{and} \quad \varphi_{\varepsilon,1} = \sum_{\lambda_{\varepsilon,k}\leq N} b_k\psi_{\varepsilon,k},$$

where $\{\psi_{\varepsilon,k}\}$ forms an orthonormal basis for $L^2(\Omega)$, $\psi_{\varepsilon,k} \in H_0^1(\Omega)$ and $\mathcal{L}_\varepsilon(\psi_{\varepsilon,k}) = \lambda_{\varepsilon,k}\psi_{\varepsilon,k}$ in Ω. Then

$$\|\nabla\varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)} \sim \sum_{\lambda_{\varepsilon,k}\leq N}\{|a_k|^2\lambda_{\varepsilon,k} + |b_k|^2\}. \tag{4.19}$$

Also, note that

$$\begin{aligned} \|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|^2_{L^2(\Omega)} + \|\nabla\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)} &\leq C\sum_{\lambda_{\varepsilon,k}\leq N}\{|a_k|^2\lambda^2_{\varepsilon,k} + |b_k|^2\lambda_{\varepsilon,k}\} \\ &\leq CN\sum_{\lambda_{\varepsilon,k}\leq N}\{|a_k|^2\lambda_{\varepsilon,k} + |b_k|^2\}, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} \|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,0})\|^2_{H^1(\Omega)} + \|\mathcal{L}_\varepsilon(\varphi_{\varepsilon,1})\|^2_{L^2(\Omega)} &\leq C\sum_{\lambda_{\varepsilon,k}\leq N}\{|a_k|^2\lambda^3_{\varepsilon,k} + |b_k|^2\lambda^2_{\varepsilon,k}\} \\ &\leq CN^2\sum_{\lambda_{\varepsilon,k}\leq N}\{|a_k|^2\lambda_{\varepsilon,k} + |b_k|^2\}. \end{aligned} \tag{4.21}$$

In view of Theorem 4.3 we obtain

$$\begin{aligned} \int_0^T\int_{\partial\Omega}|\nabla u_\varepsilon|^2\,d\sigma\,dt &\leq CT\{1+\varepsilon N + T\varepsilon N^{3/2} + \varepsilon^3 N^2\}\{\|\nabla\varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} \\ &\leq CT\{\|\nabla\varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} \end{aligned}$$

if $N \le C_0 T^{-2/3}\varepsilon^{-2/3}$. This gives (1.9). The inequality (1.10) follows from Theorem 4.4 in a similar manner. Indeed, by Theorem 4.4 and (4.19)–(4.21), if $T \ge T_0$ then

$$T\{\|\nabla\varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\}$$
$$\le C\int_0^T\int_{\partial\Omega} |\nabla u_\varepsilon|^2\, d\sigma\, dt + CT\{\varepsilon N + T\varepsilon N^{3/2} + \varepsilon^3 N^2\}\{\|\nabla\varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\},$$

where T_0 and C depends only on d, μ, M, and Ω. As a result, we obtain (1.10) when $N \le c_0 T^{-2/3}\varepsilon^{-2/3}$, where $c_0 = c_0(d,\mu,M,\Omega) > 0$ is so small that

$$C\{\varepsilon N + T\varepsilon N^{3/2} + \varepsilon^3 N^2\} \le 1/2.$$

This completes the proof. ■

Remark 4.5. Let Γ be a subset of $\partial\Omega$. Suppose that there exist $T, c_0 > 0$ such that the inequality

$$c_0\{\|\nabla\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{L^2(\Omega)}\} \le \frac{1}{T}\int_0^T\int_\Gamma |\nabla u_0|^2\, d\sigma\, dt \tag{4.22}$$

holds for solutions u_0 of the homogenized problem (3.13). It follows from the proof of Theorem 1.1 that if $N \le \delta\varepsilon^{-2/3}$ and $\delta = \delta(c_0, T, \Omega, A) > 0$ is sufficiently small, the inequality

$$c\{\|\nabla\varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} \le \frac{1}{T}\int_0^T\int_\Gamma |\nabla u_\varepsilon|^2\, d\sigma\, dt \tag{4.23}$$

holds for solutions u_ε of (1.8) with initial data $(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1})$ in $\mathcal{A}_N \times \mathcal{A}_N$.

Given $(\theta_{\varepsilon,0}, \theta_{\varepsilon,1}) \in L^2(\Omega) \times H^{-1}(\Omega)$, to find a control $g_\varepsilon \in L^2(S_T)$ such that the solution of (1.6) satisfies the projection condition (1.13), one considers the functional

$$J_\varepsilon(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1})$$
$$= -\langle\theta_{\varepsilon,1}, u_\varepsilon(x,0)\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} + \int_\Omega \theta_{\varepsilon,0}\partial_t u_\varepsilon(x,0)\, dx + \frac{1}{2}\int_0^T\int_{\partial\Omega}\left(\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}\right)^2 d\sigma\, dt,$$

where $\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon} = n_i a_{ij}(x/\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}$ denotes the conormal derivative associated with $\mathcal{L}_\varepsilon$, and u_ε is the solution of

$$\begin{cases} (\partial_t^2 + \mathcal{L}_\varepsilon)u_\varepsilon = 0 & \text{in } \Omega_T, \\ u_\varepsilon = 0 & \text{on } S_T \\ u_\varepsilon(x,T) = \varphi_{\varepsilon,0}, \quad \partial_t u_\varepsilon(x,T) = \varphi_{\varepsilon,1} & \text{for } x \in \Omega. \end{cases} \tag{4.24}$$

Since time is reversible in the wave equation, it follows from Theorem 1.1 that if $(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1}) \in \mathcal{A}_N \times \mathcal{A}_N$ and $N \le \delta\varepsilon^{-2/3}$ then

$$\begin{aligned} c\{\|\nabla\varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\} &\le \int_0^T\int_{\partial\Omega}\left(\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}\right)^2 d\sigma\, dt \\ &\le C\{\|\nabla\varphi_{\varepsilon,0}\|^2_{L^2(\Omega)} + \|\varphi_{\varepsilon,1}\|^2_{L^2(\Omega)}\}, \end{aligned} \tag{4.25}$$

where the constants $C, c > 0$ are independent of $\varepsilon > 0$. As a functional on $\mathcal{A}_N \times \mathcal{A}_N \subset H_0^1(\Omega) \times L^2(\Omega)$, J_ε is continuous, strictly convex, and satisfies the coercivity estimate

$$J_\varepsilon(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1}) \geq c\{\|\nabla \varphi_{\varepsilon,0}\|_{L^2(\Omega)}^2 + \|\varphi_{\varepsilon,1}\|_{L^2(\Omega)}^2\} - C\{\|\theta_{\varepsilon,0}\|_{L^2(\Omega)}^2 + \|\theta_{\varepsilon,1}\|_{H^{-1}(\Omega)}^2\}.$$

This implies that J_ε has a unique minimum $J_\varepsilon(\phi_0, \phi_1)$ on $\mathcal{A}_N \times \mathcal{A}_N$. Let w_ε be the solution of (4.24) with data $(w_\varepsilon(x,T), \partial_t w_\varepsilon(x,T)) = (\phi_0, \phi_1)$. By the first variational principle,

$$-\langle \theta_{\varepsilon,1}, u_\varepsilon(x,0)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} + \int_\Omega \theta_{\varepsilon,0}\partial_t u_\varepsilon(x,0)\,dx + \int_0^T\int_{\partial\Omega} \frac{\partial w_\varepsilon}{\partial \nu_\varepsilon}\cdot\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\,d\sigma\,dt = 0 \tag{4.26}$$

for any solution u_ε of (4.24) with data $(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1}) \in \mathcal{A}_N \times \mathcal{A}_N$. As a result, the function $g_\varepsilon = \frac{\partial w_\varepsilon}{\partial \nu_\varepsilon}$ is a control that gives (1.13). Indeed, let v_ε be the solution of (1.6) with $g_\varepsilon = \frac{\partial w_\varepsilon}{\partial \nu_\varepsilon}$; then

$$\begin{aligned}
&\langle \partial_t v_\varepsilon(\cdot,T), \varphi_{\varepsilon,0}\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} - \int_\Omega v_\varepsilon(x,T)\varphi_{\varepsilon,1}(x)\,dx \\
&\quad = \langle \theta_{\varepsilon,1}, u_\varepsilon(\cdot,T)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} - \int_\Omega \theta_{\varepsilon,0}\partial_t u_\varepsilon(x,0)\,dx - \int_0^T\int_{\partial\Omega} g_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\,d\sigma\,dt = 0
\end{aligned}$$

for any $(\varphi_{\varepsilon,0}, \varphi_{\varepsilon,1}) \in \mathcal{A}_N \times \mathcal{A}_N$. This shows that $P_N v_\varepsilon(x,T) = 0$ and $P_N \partial_t v_\varepsilon(x,T) = 0$ for $x \in \Omega$. One may also use (4.26) to show that among all controls that give (1.13), $g_\varepsilon = \frac{\partial w_\varepsilon}{\partial \nu_\varepsilon}$ has minimal $L^2(S_T)$ norm.

Finally, using $J_\varepsilon(\phi_0, \phi_1) \leq J(0,0) = 0$ and (4.25), one may deduce that

$$\int_0^T\int_{\partial\Omega} |g_\varepsilon|^2\,d\sigma\,dt \leq C\{\|P_N\theta_{\varepsilon,0}\|_{L^2(\Omega)}^2 + \|P_N\theta_{\varepsilon,1}\|_{H^{-1}(\Omega)}^2\}.$$

By a duality argument [8] and (4.25), one may also show that

$$c\{\|P_N\theta_{\varepsilon,0}\|_{L^2(\Omega)}^2 + \|P_N\theta_{\varepsilon,1}\|_{H^{-1}(\Omega)}^2\} \leq \int_0^T\int_{\partial\Omega} |g_\varepsilon|^2\,d\sigma\,dt.$$

We omit the details and refer the reader to [8] for the one-dimensional case.

Acknowledgments. We thank the referees for their very helpful comments and suggestions.

Funding. Research supported in part by NSF grant DMS-1501000. Research of Zhongwei Shen supported in part by NSF grant DMS-1600520.

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