J. Eur. Math. Soc. 24, 3215–3332 (2022) DOI 10.4171/JEMS/1159

Cheuk Yu Mak · Ivan Smith

Fukaya–Seidel categories of Hilbert schemes and parabolic category $\mathcal{O}$

Received July 18, 2019

Abstract. We realise Stroppel's extended arc algebra [13, 51] in the Fukaya–Seidel category of a natural Lefschetz fibration on the generic fibre of the adjoint quotient map on a type A nilpotent slice with two Jordan blocks, and hence obtain a symplectic interpretation of certain parabolic two-block versions of Bernstein–Gel'fand–Gel'fand category $\mathcal{O}$. As an application, we give a new geometric construction of the spectral sequence from annular to ordinary Khovanov homology. The heart of the paper is the development of a cylindrical model to compute Fukaya categories of (affine open subsets of) Hilbert schemes of quasi-projective surfaces, which may be of independent interest.

Keywords. Symplectic Khovanov homology, arc algebra, nilpotent slice, BGG category $\mathcal{O}$, Fukaya category, Lefschetz fibration, Hilbert scheme

Contents

Cheuk Yu Mak: Centre for Mathematical Sciences, University of Cambridge, Cambridge, CB3 0WB, UK; mathmakmak@gmail.com

Ivan Smith: Centre for Mathematical Sciences, University of Cambridge, Cambridge, CB3 0WB, UK; is200@dpmms.cam.ac.uk

Mathematics Subject Classification (2020): 53D40, 57M25

1. Introduction

1.1. Summary

This paper has three parts:

- We give a 'cylindrical' formulation of the Fukaya category of (an affine open subset of) the Hilbert scheme of an affine algebraic surface (Sections 2–4).
- We compute the Fukaya–Seidel category of a natural Lefschetz fibration on the generic fibre of the adjoint quotient map on the nilpotent / Slodowy slice of Jordan type $(n, m-n)$ arising from its inclusion in the n-th Hilbert scheme of the Milnor fibre of the A_{m-1}-surface singularity, establishing a Morita equivalence to the principal block $\mathcal{O}^{n,m}$ of BGG parabolic category $\mathcal{O}$ associated to the partition $m = n + (m-n)$ (Sections 5–9).
- We exploit this equivalence to describe a new semi-orthogonal decomposition of the dg-category of perfect modules perf-$\mathcal{O}^{n,2n}$, from which we derive a geometric construction of the spectral sequence from annular to ordinary Khovanov homology (Sections 10–12).

1.2. Context

Fix a semisimple complex Lie algebra $\mathfrak{g}$. The Bernstein–Gel'fand–Gel'fand category $\mathcal{O}$ is an abelian (Noetherian and Artinian) category of finitely generated $\mathfrak{g}$-modules, which plays a central role in many parts of representation theory. It contains all the highest-weight modules, is closed under the operations of taking submodules, quotient modules and under tensoring with finite-dimensional modules, and is "minimal" with respect to those properties. Concretely for $\mathfrak{sl}_m$, fixing the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_m$ of diagonal matrices with respect to a choice of basis, it comprises the finitely generated $U(\mathfrak{sl}_m)$-modules which are $\mathfrak{h}$-semisimple and which are locally finite for the nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{sl}_m$ given by the upper triangular matrices. Given a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ containing the Borel $\mathfrak{n} \oplus \mathfrak{h}$, there is a parabolic subcategory $\mathcal{O}^{\mathfrak{p}}$ of $\mathcal{O}$ consisting of those modules on which $\mathfrak{p}$ acts locally finitely. Again for $\mathfrak{sl}_m$, a partition $m = m_1 + \cdots + m_k$ determines a parabolic $\mathfrak{p}$; the extreme cases $m = 1 + \cdots + 1$ and $m = m$ respectively define $\mathcal{O}^{\mathfrak{p}} = \mathcal{O}$ and $\mathcal{O}^{\mathfrak{p}} = \mathcal{O}^{\mathrm{fd}}$, the semisimple category of finite-dimensional $\mathfrak{sl}_m$-modules, so the parabolic subcategories can be viewed as intepolating between these. The central characters decompose $\mathcal{O}^{\mathfrak{p}}$ into blocks and the block containing the trivial modules is called the *principal block* $\mathcal{O}_0^{\mathfrak{p}}$.

It is known that for any $\mathfrak{p} \subset \mathfrak{g}$, the category $\mathcal{O}_0^{\mathfrak{p}}$ is equivalent to the category of modules over a finite-dimensional associative algebra; there are algorithmic descriptions of these algebras via quivers with relations [53], but nonetheless working with them concretely is often rather non-trivial. For parabolics $\mathfrak{p} \subset \mathfrak{sl}_m$ associated to two-block partitions $m = n + (m-n)$, the corresponding category $\mathcal{O}_0^{\mathfrak{p}} = \mathcal{O}^{n,m}$ has several other descriptions: it is Morita equivalent to the category of perverse sheaves on the Grassmannian $\mathrm{Gr}(n, m)$ constructible with respect to the Schubert stratification [10], and the underlying associative algebra[1] $K^{\text{alg}}_{n,m}$ has a diagrammatic description due to Stroppel [51] and Brundan–Stroppel [12, 13], in which context it is known as the 'extended arc algebra'. This paper gives a new interpretation of these algebras in terms of Fukaya–Seidel categories associated to natural Lefschetz fibrations on the generic fibre of the adjoint quotient map of the nilpotent (Jacobson–Morozov type) slices associated to nilpotent matrices with two Jordan blocks, and hence a symplectic-geometric construction of these particular principal blocks of parabolic category $\mathcal{O}$. This fits into the general dictionary between symplectic geometry and aspects of representation theory related to categorification [11, 26], and simultaneously extends the symplectic viewpoint on Khovanov homology [1, 2, 47] to a corresponding viewpoint on annular Khovanov homology, complementing recent work of [8, 21].

1.3. Main result

Let $\pi : A_{m-1} \to \mathbb{C}$ be the standard Lefschetz fibration on the Milnor fibre of the A_{m-1} surface singularity, so π is an affine conic fibration with m Lefschetz critical points; the symplectic topology of this fibration has been extensively studied; see for instance [26, 33]. The Hilbert scheme $\text{Hilb}^n(A_{m-1})$ of zero-dimensional length n subschemes has a distinguished divisor D_r of subschemes whose projection under π has length $< n$, the complement of which is an affine variety $\mathcal{Y}_{n,m}$. (When $2n \leq m$, this space is isomorphic to the generic fibre of the restriction of the adjoint quotient map $\mathfrak{sl}_m \to \mathfrak{h}/W$ to a transverse slice meeting the nilpotent cone at a matrix with Jordan blocks of size $\{n, m-n\}$, as studied in classical Springer theory [50].) The map π induces a map $\pi_{n,m} : \mathcal{Y}_{n,m} \to \mathbb{C}$ which is a Lefschetz fibration in the weak sense that its only critical points are of Lefschetz type (it may, however, have critical points at infinity when $n > 1$). Nonetheless, there is a well-defined Fukaya–Seidel A_∞ category $\mathcal{FS}(\pi_{n,m})$, governing the Floer theory of the Lefschetz thimbles associated to any collection of vanishing paths. We write $D^\pi(\mathcal{C})$ for the derived category of $\mathcal{C}$ (split-closure of twisted complexes), working over a field $\mathbb{K}$.

Theorem 1.1. *If $n \leq m$ and $\mathbb{K}$ has characteristic zero, $D^\pi\mathcal{FS}(\pi_{n,m})$ is quasi-equivalent to the dg-category of perfect modules* perf-$K^{\text{alg}}_{n,m}$ *over $K^{\text{alg}}_{n,m}$, hence Morita equivalent to parabolic category $\mathcal{O}^{n,m}$.*

Remark 1.2. For the extreme case $n = m$, we have $D^\pi\mathcal{FS}(\pi_{n,m}) = D^b(\mathbb{K})$ (see Remark 5.18) and $K^{\text{alg}}_{n,m} := \mathbb{K}$. When $n > m$, the map $\pi_{n,m}$ has no critical point and $\mathcal{FS}(\pi_{n,m})$ is an empty category; while $K^{\text{alg}}_{n,m}$ is not well-defined. When $n = 0$, our convention is $\mathcal{FS}(\pi_{n,m}) := \mathbb{K}$ and $K^{\text{alg}}_{n,m} := \mathbb{K}$. Under this convention, Theorem 1.1 is true for all non-negative integers n, m.

[1] Our $K^{\text{alg}}_{n,m}$ is K^{m-n}_n in [13]. The superscript 'alg' (for algebraic) distinguishes it from a symplectic sibling which is introduced later in the paper.

Remark 1.3. The hypothesis on the ground field arises in two places. The first one is a formality theorem for the A_∞ structure on the symplectic side, which we prove, following [1], via methods of non-commutative algebra which rely on inverting all primes. The second one comes from the construction of an auxiliary A_∞ category $\mathcal{F}\mathcal{S}^{\mathrm{cyl}}$, which is a cylindrical model of the Fukaya–Seidel category, for which the definition of the A_∞ operations involves dividing out by symmetry groups arising from re-ordering marked points on domains. *We work over a characteristic zero field unless stated otherwise.* All Lagrangians in the paper are assumed to be orientable and to admit (and be equipped with) spin structures.

Remark 1.4. It is reasonable to expect that symplectic models for the categories $\mathcal{O}^{\mathfrak{p}}$, for parabolics $\mathfrak{p}$ corresponding to partitions with more parts (or indeed in other simple $\mathfrak{g}$), can also be obtained from the symplectic geometry of nilpotent slices. In general, however, such slices are not known to be re-interpretable in terms of Hilbert schemes of surfaces, so different techniques would be needed to analyse them. Note that the map $\pi_{n,m} : \mathcal{Y}_{n,m} \to \mathbb{C}$ is itself obtained from the embedding of $\mathcal{Y}_{n,m}$ into a Hilbert scheme; it would be interesting to characterise $\pi_{n,m}$ intrinsically Lie-theoretically.

There is a well-known 'tautological correspondence' between a holomorphic disc in the symmetric product of a complex manifold X and a map of a branched cover of the disc to X itself [5, 16, 17, 29]. Following [29] such correspondences are referred to as 'cylindrical models' for computing Floer cohomology. Theorem 1.1 uses the embedding $\mathcal{Y}_{n,m} \hookrightarrow \mathrm{Hilb}^n(A_{m-1})$ [31], and the development of a cylindrical model for computing Fukaya–Seidel categories of Hilbert schemes. Whilst the tautological correspondence has been widely exploited before, the proof nonetheless requires numerous technical innovations: we require a model for the whole Fukaya category, our Lagrangian submanifolds are not compact and not cylindrical at infinity, and the Hilbert scheme is related to the symmetric product by a non-trivial crepant resolution. A brief summary of the relevant issues is given in Section 2.1 (our treatment requires only classical transversality theory, but an appeal to virtual perturbation theory would not bypass many of the difficulties).

1.4. Consequences

As commented previously, $K^{\mathrm{alg}}_{n,m}$ is Morita equivalent to a category of perverse sheaves on the Grassmannian $\mathrm{Gr}(m,n)$. From the (Schubert-compatible) isomorphism $\mathrm{Gr}(n,m) = \mathrm{Gr}(m-n,m)$, one obtains a geometrically non-trivial:

Corollary 1.5. *$D^\pi\mathcal{F}\mathcal{S}(\pi_{n,m})$ is quasi-equivalent to $D^\pi\mathcal{F}\mathcal{S}(\pi_{m-n,m})$.*

Note that the categories appearing in Corollary 1.5 are associated to Hilbert schemes of n respectively $m-n$ points, so take place in symplectic manifolds of different dimension. For another non-trivial symmetry, the identity $K^{\mathrm{alg}}_{n,m} \cong (K^{\mathrm{alg}}_{n,m})^{\mathrm{op}}$ of the arc algebra with its opposite – which has an easy diagrammatic proof – is non-trivial on the geometric side, because of the non-trivial boundary conditions and 'wrapping' involved in setting up a Fukaya–Seidel category (cf. Section 9.2).

The Fukaya–Seidel category of any Lefschetz fibration has a full exceptional collection. Additivity of Hochschild homology under semi-orthogonal decomposition [28] immediately yields

Corollary 1.6. *The Hochschild homology of the extended arc algebra, $HH_*(K^{\mathrm{alg}}_{n,m})$, vanishes in degree $* \neq 0$, and in degree zero is free of rank $\binom{m}{n}$.*

The corollary was previously established by Beliakova et al. [8] by an ingenious and involved algebraic argument involving a 'quantum deformation' of the Hochschild complex.

Underlying and extending Theorem 1.1, and the previous corollaries, there is a rich dictionary between objects in representation theory and in symplectic geometry, which is useful in both directions. The blocks of parabolic category $\mathcal{O}$ are renowned instances of highest-weight categories, i.e. ones with a full exceptional collection where the exceptional objects generate rather than just split-generate. A key point in the proof of Theorem 1.1 is that the endomorphism algebra of the collection of Lagrangian submanifolds associated to projective modules is formal. Crucially, the 'projective' Lagrangians are predicted by the diagrammatic combinatorics on the algebra side; away from Milnor fibres of surface singularities, they have no obvious counterpart for Hilbert schemes of general Lefschetz fibrations on quasi-projective surfaces.

Remark 1.7. In contrast to a number of recent formality results which hold (for instance) for degree reasons, the formality of the symplectic extended arc algebra holds for non-trivial geometric reasons, and is established, following [1], by building a non-commutative vector field $b \in HH^1(\mathcal{FS}(\pi_{n,m}))$ counting certain holomorphic curves with prescribed behaviour at infinity. The endomorphism algebra of the Lefschetz thimbles is in general *not* formal (cf. Appendix A), and it seems hard to prove Theorem 1.1 by directly comparing the A_∞-algebras associated to distinguished bases of exceptional objects.

In the other direction, the geometry gives rise rather directly to the following:

Theorem 1.8 (see Theorem 11.9). *There is a semi-orthogonal decomposition* perf-$K^{\mathrm{alg}}_{n,m} = \langle A_n, \ldots, A_0\rangle$ *where A_j is quasi-equivalent to* perf-$(K^{\mathrm{alg}}_{j,n} \otimes K^{\mathrm{alg}}_{n-j,m-n})$ *for all $j = 0, \ldots, n$.*

The equivalence of Theorem 1.8 does not send tensor products of projectives to projectives, which may make it less transparent from the viewpoint of the extended arc algebras themselves.

1.5. Invariants of braids

The braid group Br_m acts by symplectomorphisms on $\mathcal{Y}_{n,m}$; a braid β defines a symplectomorphism $\phi^{(n)}_\beta$ and a corresponding bimodule $P^{(n)}_\beta$ over the Fukaya–Seidel category $D^\pi\mathcal{FS}(\pi_{n,m})$. The Hochschild homology $HH_*(P^{(n)}_\beta)$ is an algebraic analogue of the natural symplectic invariant $HF^*(\phi_\beta^{(n)})$ given by taking fixed-point Floer cohomology.[2] We define the *symplectic annular Khovanov homology* of $\beta \in \mathrm{Br}_m$ by

$$\mathrm{AKh}^{\mathrm{symp}}(\beta) := \bigoplus_{j=0}^{m} HH_*(P_\beta^{(j)}).$$

This is by definition an invariant of the braid. Recall from [47] that there is a distinguished closed exact Lagrangian submanifold $L_\wp \subset \mathcal{Y}_{n,2n}$ with the property that

$$\mathrm{Kh}^{\mathrm{symp}}(\kappa(\beta)) := HF^*(L_\wp, \phi_\beta^{(n)}(L_\wp))$$

is an invariant of the link closure $\kappa(\beta)$ of $\beta \in \mathrm{Br}_n$, known as 'symplectic Khovanov cohomology'. This was introduced in [47] as a singly graded sibling to combinatorial Khovanov homology; working over a characteristic zero field $\mathbb{K}$, the isomorphism $\mathrm{Kh}(\kappa(\beta)) \cong \mathrm{Kh}^{\mathrm{symp}}(\kappa(\beta))$ of $\mathbb{Z}$-graded[3] vector spaces was established in [1,2].

Theorem 1.9. *There is a spectral sequence* $\mathrm{AKh}^{\mathrm{symp}}(\beta) \Rightarrow \mathrm{Kh}^{\mathrm{symp}}(\kappa(\beta))$ *from the symplectic annular Khovanov homology of* β *to the symplectic Khovanov cohomology of the link closure* $\kappa(\beta)$ *of* β*.*

Annular Khovanov homology was introduced by Asaeda, Przytycki and Sikora in [4], via a diagrammatic calculus for links in a solid torus. Roberts [39] showed there is a spectral sequence $A\mathrm{Kh}(\beta) \Rightarrow \mathrm{Kh}(\kappa(\beta))$, but the fact that $A\mathrm{Kh}(\beta)$ splits into summands which can be identified with Hochschild homology groups is non-trivial; this was conjectured by Auroux, Grigsby and Wehrli [6] and proven very recently by Beliakova, Putyra and Wehrli [8]. By contrast, if one had *defined* the annular Khovanov invariant as such a direct sum of Hochschild homologies, the existence of the spectral sequence would seem rather mysterious: the bimodules $P_\beta^{(j)}$ over the categories $\mathcal{FS}(\pi_{j,m})$ do not in themselves have enough information to determine the differentials in the spectral sequence (which do not preserve the decomposition by j). Theorem 1.9 gives a geometric explanation for the existence of a spectral sequence from a direct sum of Hochschild homologies to (symplectic) Khovanov cohomology; the crucial input is the semi-orthogonal decomposition from Theorem 1.8. We remark that there is an analogous spectral sequence from knot Floer homology, viewed as Hochschild homology of a suitable bimodule via [30], to Heegaard Floer homology; it would be interesting to see if that can be derived following the methods of this paper.

[2] There are numerous results relating fixed-point Floer cohomology and Hochschild homology (see e.g. [43]), which for instance show that the two invariants coincide when $n = 1$. When $n > 1$, since $\pi_{n,m}$ has critical points at infinity, the established results do not apply in our case; for simplicity we take our basic invariant of a braid to be the bimodule.

[3] In this paper we will not discuss the grading; the methods of [1] give rise to a second grading by elements of $\overline{\mathbb{K}}$, which is conjecturally integral, Markov-invariant and lifts the $\mathbb{Z}$-graded equivalence to a bigraded equivalence.

1.6. *Outline of the paper*

Let $\pi_E : E \to \mathbb{C}$ be a Lefschetz fibration on a quasi-projective surface. The map π_E defines a map $\pi_E^{[n]} : \mathrm{Hilb}^n(E) \to \mathbb{C}$, by taking the map induced from the sum of copies of π_E on the product E^n. The map $\pi_E^{[n]}$ is a Lefschetz fibration when restricted to the affine open subvariety of the Hilbert scheme $\mathrm{Hilb}^n(E)$ comprising subschemes whose projection to $\mathbb{C}$ has length n. Section 2 develops a cylindrical model $\mathcal{FS}^{\mathrm{cyl}}$ for the Fukaya(–Seidel) category of this associated Lefschetz fibration; Section 3 establishes some basic properties of $\mathcal{FS}^{\mathrm{cyl}}$ analogous to the usual Fukaya(–Seidel) category; Section 4 relates the cylindrical model to the usual model $\mathcal{FS}$, yielding in general an embedding $\mathcal{FS} \hookrightarrow \mathcal{FS}^{\mathrm{cyl}}$; and Section 5 applies this framework to type A Milnor fibres. Sections 6–7 construct a symplectic version of the extended arc algebra, which is a priori an A_∞ algebra, and prove that its cohomology agrees with its combinatorial sibling; Sections 8–9 then prove the A_∞ structure is actually formal in characteristic zero. Section 10 illustrates the dictionary between natural objects in the representation theory of the extended arc algebras and their geometric counterparts. Sections 11 and 12 introduce symplectic annular Khovanov homology, and relate this to ordinary Khovanov homology via a particular semi-orthogonal decomposition of the Fukaya–Seidel category associated to the $(n, 2n)$-nilpotent slice (associated to a nilpotent with Jordan blocks of size (n, n)). The Appendix shows that the A_∞ endomorphism algebra of the thimbles in the Fukaya–Seidel category is in general not formal; this underscores the important contribution of the relationship to representation theory, which picks out a different set of generators for the category (as studied in Sections 5–9) which do have formal endomorphism algebra.

2. The cylindrical Fukaya–Seidel category

2.1. *Overview*

The aim of this section is to define, given a Lefschetz fibration π_E on a complex surface E equipped with an exact symplectic structure (that satisfies some mild additional hypothesis, see Section 2.2.4) and a positive integer n, an A_∞ category $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$, which we call the *n-fold cylindrical Fukaya–Seidel category*. Objects in this category are, roughly speaking, unordered n-tuples $\underline{L} = \{L_1, \dots, L_n\}$ of pairwise disjoint exact Lagrangians with $L_i \subset E$. The product of the L_i defines a Lagrangian (which descends to one) in $\mathrm{Sym}^n(E)$, which can be lifted to a Lagrangian $\mathrm{Sym}(\underline{L})$ in $\mathrm{Hilb}^n(E)$.

Let $\pi_E^{[n]} : \mathrm{Hilb}^n(E) \to \mathbb{C}$ be the map induced from the sum of n copies of π_E. There is a divisor D_r of $\mathrm{Hilb}^n(E)$ such that $\mathcal{Y}_E := \mathrm{Hilb}^n(E) \setminus D_r$ admits an exact symplectic structure, and such that

$$\pi_{\mathcal{Y}_E} := \pi_E^{[n]}|_{\mathcal{Y}_E} : \mathcal{Y}_E \to \mathbb{C} \tag{2.1}$$

is a Lefschetz fibration in the weak sense that all the critical points of $\pi_{\mathcal{Y}_E}$ are of Lefschetz type.[4] Every Lefschetz thimble of $\pi_{\mathcal{Y}_E}$ is given by $\mathrm{Sym}(\underline{L})$ for some $\underline{L}$. If E is the A_{m-1} Milnor fibre and π_E is the conic fibration with m critical points, then $\mathcal{Y}_E$ and $\pi_{\mathcal{Y}_E}$ coincide with $\mathcal{Y}_{n,m}$ and $\pi_{n,m}$, respectively.

The A_∞ structure of $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ is carefully defined so that it has the property that the category of Lefschetz thimbles of $\pi_{\mathcal{Y}_E}$ embeds into $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ as a full subcategory, where on the object level, it is given by $\mathrm{Sym}(\underline{L}) \mapsto \underline{L}$. This reduces calculations in the Fukaya–Seidel category of $\pi_{\mathcal{Y}_E}$ to more accessible calculations in $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$, which involve holomorphic curve counts in E itself.

Ideally, we would like to have a choice of perturbation scheme such that there is a bijective correspondence between (perturbed-)holomorphic polygons $u : S \to \mathcal{Y}_E$ contributing to the A_∞ structure of $\mathcal{FS}(\pi_{\mathcal{Y}_E})$ and pairs (π_Σ, v) such that $\pi_\Sigma : \Sigma \to S$ is an n-fold branched covering and $v : \Sigma \to E$ is a solution to a (perturbed-)holomorphic equation. In this case, if we define the A_∞ structure of $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ as a signed rigid count of the pairs (π_Σ, v), then it would be tautological that the A_∞ categories $\mathcal{FS}(\pi_{\mathcal{Y}_E})$ and $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ would be equivalent. Problems arise when one implements this idea in practice:

- The domain-dependent almost complex structure on E has to be complex (and hence induce an almost complex structure on $\mathrm{Hilb}^n(E)$) to have any hope of a bijective correspondence between u and (π_Σ, v), which puts some restrictions on the perturbation scheme;
- Hamiltonian perturbations in $\mathrm{Hilb}^n(E)$ that are simultaneously induced from Hamiltonian perturbations in E and which preserve D_r are not general enough to achieve transversality; strictly speaking, there is no perturbation scheme that can allow us to define the A_∞ structure of $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ by merely counting (π_Σ, v);
- We need to include some $\underline{L}$ for which $\mathrm{Sym}(\underline{L})$ is not cylindrical in $\mathcal{Y}_E$ (e.g. to obtain formal A_∞ Floer cochain algebras in the setting of Theorem 1.1), so compactness of the moduli involved has to be carefully addressed.

Our main contribution to the construction of $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ is to overcome the above difficulties, roughly as follows:

- Instead of using moduli of polygons $\mathcal{R}^{d+1}$, we use moduli of polygons with ordered interior marked points $\mathcal{R}^{d+1,h}$ to define the A_∞ structure. The interior marked points keep track of the branch points of $\pi_\Sigma : \Sigma \to S$, and the domain-dependent almost complex structure is only required to be integrable near the interior marked points. This gives us more flexibility for the perturbation scheme and at the same time partially recovers the bijective correspondence between u and (π_Σ, v).

[4] There may be critical points at infinity, so this is not a symplectic Lefschetz fibration in the usual sense.

- Whilst the A_∞ structure of $\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ is defined by counting certain solutions u mapping to $\mathrm{Hilb}^n(E) \setminus D_r$, rather than pairs (π_Σ, v), our set-up is sufficiently flexible that the Floer differential and product can be computed by counting solutions (π_Σ, v).
- To achieve compactness, we need to avoid that solutions in $\mathrm{Hilb}^n(E) \setminus D_r$ escape to the vertical boundary, horizontal boundary or D_r. No escape along the vertical boundary is achieved by modifying Seidel's ingenious set-up in [46]; one takes the base of the Lefschetz fibration to be the upper half-plane, and uses hyperbolic isometries to gauge back perturbed holomorphic curves to actual holomorphic curves. No escape along the horizontal boundary and into D_r are each achieved by positivity of intersection, which relies on a delicate choice of Hamiltonian perturbation scheme and some particular geometric features of D_r. Familiarity with [46] may be helpful.

2.2. *Definitions and the set-up*

2.2.1. Domain moduli. Let $\mathcal{R}^{d+1,h}$ be the moduli space of discs with $d+1$ punctures on the boundary and h ordered and pairwise distinct interior marked points. Let $\mathcal{S}^{d+1,h}$ be the universal family of $\mathcal{R}^{d+1,h}$. We fix a distinguished puncture ξ^0 for the elements in $\mathcal{R}^{d+1,h}$ consistently, and order the remaining punctures $\xi^1, \dots, \xi^d$ counterclockwise along the boundary. For $S \in \mathcal{R}^{d+1,h}$, we denote the ordered interior marked points by $\xi_+^1, \dots, \xi_+^h$ and we use $\partial_j S$ to denote the boundary component of S between ξ^j and ξ^{j+1} for $j = 0, \dots, d$ (ξ^{d+1} is understood as ξ^0). We use $\mathrm{mk}(S)$ to denote the set of interior marked points of S.

The moduli space $\mathcal{R}^{d+1,h}$ can be compactified to the moduli space of stable discs, $\overline{\mathcal{R}}^{d+1,h}$. The latter moduli space is used to define bulk deformation in [20], [49], to which readers are referred for the details of its construction. We denote the universal family over $\overline{\mathcal{R}}^{d+1,h}$ by $\overline{\mathcal{S}}^{d+1,h}$.

2.2.2. Strip-like ends and marked-points neighbourhoods. For each $\mathcal{R}^{d+1,h}$, we make a choice of strip-like ends $\epsilon = \{\epsilon^0, \dots, \epsilon^d\}$ for elements in $\mathcal{R}^{d+1,h}$ such that ξ^0 is an output and ξ^j is an input for $j = 1, \dots, d$. Thus, for each $S \in \mathcal{R}^{d+1,h}$, we have holomorphic embeddings varying smoothly with respect to S,

$$\begin{cases} \epsilon^0 : \mathbb{R}^{\le 0} \times [0,1] \to S, \\ \epsilon^1, \dots, \epsilon^d : \mathbb{R}^{\ge 0} \times [0,1] \to S, \\ \lim_{s \to \pm\infty} \epsilon^j(s, \cdot) = \xi^j, \\ (\epsilon^j)^{-1}(\partial S) = \{(s,t) : t = 0, 1\}. \end{cases}$$

We denote $\bigcup_{S \in \mathcal{R}^{d+1,h}} \mathrm{Im}(\epsilon_j)$ in $\mathcal{S}^{d+1,h}$ by $N_{\epsilon_j}^{d+1,h}$.

Furthermore, we choose a 'marked-points neighbourhood' $\nu(\mathrm{mk}(S))$ for each $S \in \mathcal{R}^{d+1,h}$, which is a (possibly disconnected) open subset of S containing $\mathrm{mk}(S)$, such that

$$N_{\mathrm{mk}}^{d+1,h} := \bigcup_{S \in \mathcal{R}^{d+1,h}} \overline{\nu(\mathrm{mk}(S))} \tag{2.2}$$

is a smooth submanifold (with boundary) in $\mathcal{S}^{d+1,h}$. We require that $N_{\mathrm{mk}}^{d+1,h} \cap N_{\epsilon_j}^{d+1,h} = \emptyset$ for all j.

We fix a choice of cylindrical ends for the interior marked points of $S \in \mathcal{R}^{d+1,h}$, and for the interior marked points for the elements in the moduli of spheres with ordered marked points. Given that choice, one obtains a smooth structure on $\overline{\mathcal{R}}^{d+1,h}$ (see [49, Section 4], [40, Section (9g)]).

A choice of strip-like ends for all elements of $\mathcal{R}^{d+1,h}$, for all d and h, is called *consistent* if it extends to a choice of strip-like ends (smooth up to the boundary) for all elements in $\overline{\mathcal{R}}^{d+1,h}$. A choice of marked-points neighbourhoods for all elements of $\mathcal{R}^{d+1,h}$, for all d and h, is called *consistent* if the closure of their union, denoted by $\overline{N}_{\mathrm{mk}}^{d+1,h}$, is a smooth submanifold with boundary and corners in $\overline{\mathcal{S}}^{d+1,h}$. We fix such a consistent choice of strip-like ends and marked-points neighbourhoods.

Remark 2.1. Note that $\mathrm{mk}(S) \subset \nu(\mathrm{mk}(S))$ for all $S \in \mathcal{R}^{d+1,h}$ implies that if S in $\overline{\mathcal{R}}^{d+1,h}$ has a stable sphere component Q, then $Q \subset \nu(\mathrm{mk}(S))$.

2.2.3. An isometry group. Let G_{aff} be the group of orientation preserving affine transformations of the real line and $\mathfrak{g}_{\mathrm{aff}}$ be its Lie algebra. Let $\mathbb{H}$ be the closed upper half-plane, whose interior $\mathbb{H}^\circ$ is equipped with the hyperbolic area form $\omega_{\mathbb{H}^\circ} = \frac{d\,\mathrm{re}(w) \wedge d\,\mathrm{im}(w)}{\mathrm{im}(w)^2}$ and primitive $\theta_{\mathbb{H}^\circ} = -d^c(\log(\mathrm{im}(w)))$. We write $\partial_\infty \mathbb{H} := \mathbb{H} \setminus \mathbb{H}^\circ$.

The G_{aff}-action on the real line extends to $\mathbb{H}$ and we have a Lie algebra homomorphism

$$\mathfrak{g}_{\mathrm{aff}} \to C^\infty(\mathbb{H}^\circ, T\mathbb{H}^\circ) \tag{2.3}$$

which sends $\gamma \in \mathfrak{g}_{\mathrm{aff}}$ to a Hamiltonian vector field X_γ. We define

$$H_\gamma := \theta_{\mathbb{H}^\circ}(X_\gamma), \tag{2.4}$$

which is a Hamiltonian on $\mathbb{H}^\circ$ generating X_γ.

2.2.4. Target space. Our set-up is modified from [46]. Let $(E^\urcorner, J_{E^\urcorner})$ be a complex surface with boundary and let

$$\pi_{E^\urcorner} : E^\urcorner \to \mathbb{H} \tag{2.5}$$

be a proper Lefschetz fibration such that $\partial E^\urcorner = \pi_{E^\urcorner}^{-1}(\partial \mathbb{H})$. Let $\omega_{\overline{E}}$ be a symplectic form on $\overline{E} := E^\urcorner \setminus \partial E^\urcorner$ which tames $J_{\overline{E}} := J_{E^\urcorner}|_{\overline{E}}$ and makes

$$\pi_{\overline{E}} := \pi_{E^\urcorner}|_{\overline{E}} \to \mathbb{H}^\circ \tag{2.6}$$

into a symplectic Lefschetz fibration.

Let D_E be a smooth and reduced (but possibly disconnected) divisor of $E^\urcorner$. Let $E := \overline{E} \setminus D_E$, $J_E := J_{E^\urcorner}|_E$, $\omega_E := \omega_{\overline{E}}|_E$ and $\pi_E := \pi_{\overline{E}}|_E$. We assume that there is a primitive θ_E for ω_E on E (so, in particular, it implies that $\pi_{E^\urcorner}|_{D_E}$ is surjective).

We assume that there is a contractible compact subset $C_{\mathbb{H}} \subset \mathbb{H}^\circ$ such that $\pi_{\overline{E}}|_{\pi_{\overline{E}}^{-1}(\mathbb{H}^\circ \setminus C_{\mathbb{H}})}$ is symplectically locally trivial. It means that for all $x \in \pi_{\overline{E}}^{-1}(\mathbb{H}^\circ \setminus C_{\mathbb{H}})$,

(1) x is a regular point of $\pi_{\overline{E}}$,

(2) the horizontal distribution $T^h_x\overline{E} := (T^v_x\overline{E})^{\perp\omega_{\overline{E}}}$ is integrable in a neighbourhood of x, and

(3) $\omega_{\overline{E}}|_{T^h_x\overline{E}} = (\pi_{\overline{E}})^*\omega_{\mathbb{H}^\circ}|_{T^h_x\overline{E}}$.

We also require that for $x \in D_E \cap \pi_{\overline{E}}^{-1}(\mathbb{H}^\circ \setminus C_{\mathbb{H}})$, we have $T^h_x\overline{E} = T_x D_E$.

If we pick a point $* \in \mathbb{H}^\circ \setminus C_{\mathbb{H}}$ and define

$$(F, \omega_F, \theta_F) := (\pi_E^{-1}(*), \omega_E|_{\pi_E^{-1}(*)}, \theta_E|_{\pi_E^{-1}(*)}),$$

then the conditions above imply that for every $z \in \mathbb{H}^\circ \setminus C_{\mathbb{H}}$, there is an open neighbourhood U of z, and a symplectomorphism $(\pi_E^{-1}(U), \omega_E) \to (U \times F, \omega_{\mathbb{H}^\circ}|_U + \omega_F)$ compatible with the projection to U. Therefore, there is a natural way to extend the symplectic form $\omega_{\overline{E}}$ to $\omega_{E^\urcorner}$ on $E^\urcorner$.

Let $\mathcal{J}(E)$ be the space of ω_E-tamed almost complex structures J on E such that

$$J = J_E \text{ outside a compact subset in } E, \tag{2.7}$$

$$\pi_E \text{ is } (J, j_{\mathbb{H}^\circ})\text{-holomorphic}. \tag{2.8}$$

Condition (2.7) implies that every $J \in \mathcal{J}(E)$ can be smoothly extended to an $\omega_{E^\urcorner}$-tamed almost complex structure $J^\urcorner$ on $E^\urcorner$, and we assume that

$$\text{every } J^\urcorner\text{-holomorphic map } \mathbb{CP}^1 \to E^\urcorner \text{ has positive algebraic intersection number with } D_E. \tag{2.9}$$

We also assume that $c_1(E) = 0$ and a trivialization of the canonical bundle is chosen. For $\gamma \in \mathfrak{g}_{\mathrm{aff}}$, let

$$\mathcal{H}_\gamma(\mathbb{H}^\circ) := \{H \in C^\infty(\mathbb{H}^\circ, \mathbb{R}) \mid H \text{ is a constant in a neighbourhood of } C_{\mathbb{H}}, \text{ and } H = H_\gamma \text{ outside a compact subset of } \mathbb{H}^\circ\}, \tag{2.10}$$

$$\mathcal{H}_\gamma(E) := \{H \in C^\infty(E, \mathbb{R}) \mid H = \pi_E^* H' \text{ for some } H' \in \mathcal{H}_\gamma(\mathbb{H}^\circ)\}. \tag{2.11}$$

Then for $H \in \mathcal{H}_\gamma(\mathbb{H}^\circ)$, we have

$$X_H|_{C_{\mathbb{H}}} = 0 \quad \text{and} \quad X_H = X_{H_\gamma} \text{ outside a compact set.} \tag{2.12}$$

Since π_E is symplectically locally trivial outside $\pi_E^{-1}(C_{\mathbb{H}})$, for $H \in \mathcal{H}_\gamma(E)$, X_H is uniquely determined by the property that

$$(\pi_E)_*(X_H|_x) = X_{H'}|_{\pi_E(x)} \tag{2.13}$$

for all $x \in E$. If we extend X_H to a smooth vector field on $\overline{E}$, we also have

$$X_H|_x \in T_x D_E \quad \text{for all } x \in D_E. \tag{2.14}$$

Next, we consider the class $\mathcal{L}$ of properly embedded, oriented and spin Lagrangian submanifolds $L \subset E$ such that

$$\pi_E|_L \text{ is proper and } \partial L = \emptyset, \tag{2.15}$$

$$\begin{aligned}&\text{there is a compact subset } C_L \subset \mathbb{H}^\circ \text{ such that } \pi_E(L) \setminus C_L \text{ is either empty}\\&\text{or is a properly embedded arc } \gamma_L \text{ in } \mathbb{H}^\circ\text{, and if } \pi_E(L) \setminus C_L \neq \emptyset \text{ then we}\\&\text{denote the point } \overline{\gamma_L} \setminus \gamma_L \text{ by } \lambda_L \in \mathbb{R}\text{, where } \overline{\gamma_L} \text{ is the closure of } \gamma_L \text{ in } \mathbb{H},\end{aligned} \tag{2.16}$$

$$L \text{ is exact with respect to } \theta_E. \tag{2.17}$$

Note that if

$$\begin{cases} A = a_t dt \in \Omega^1([0,1], \mathfrak{g}_{\text{aff}}), \\ H = (H_t)_{t \in [0,1]} \text{ with } H_t \in \mathcal{H}_{a_t}(E), \end{cases} \tag{2.18}$$

and ϕ_H is the associated Hamiltonian diffeomorphism (which is well-defined everywhere because $\phi_H|_{\pi_E^{-1}(C_{\mathbb{H}})}$ is the identity), then $\phi_H^{-1}(L) \in \mathcal{L}$ and $\lambda_{\phi_H^{-1}(L)} = g_A^{-1}\lambda_L$, where $g_A \in G_{\text{aff}}$ is the parallel transport from 0 to 1 with respect to A (see Figure 2.1).

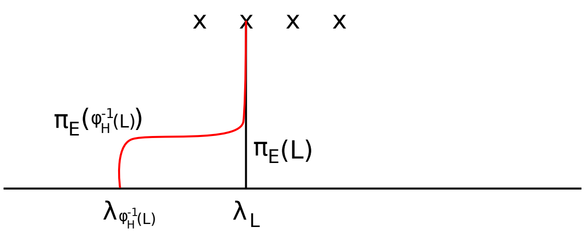


Fig. 2.1. An example of $\pi_E(L)$ (black) and $\pi_E(\phi_H^{-1}(L))$ (red); crosses are critical values.

Finally, we consider the set $\mathcal{L}^{\text{cyl},n}$ of (unordered) n-tuples of Lagrangians $\underline{L} = \{L_1, \dots, L_n\}$ such that

$$L_k \in \mathcal{L} \text{ for } k = 1, \dots, n, \tag{2.19}$$

$$\begin{aligned}&\text{there are pairwise disjoint contractible open sets } U_{L_k} \text{ in } \mathbb{H} \text{ such that}\\&\pi_E(L_k) \subset U_{L_k} \text{ and } U_{L_k} \cap \partial\mathbb{H} \text{ is either empty or contractible.}\end{aligned} \tag{2.20}$$

In particular, (2.20) implies that the L_k are pairwise disjoint. We define

$$\text{Sym}(\underline{L}) := q_{S_n}(L_1 \times \cdots \times L_n) \subset \text{Conf}^n(E) \subset \text{Sym}^n(E), \tag{2.21}$$

$$\text{Sym}(U_{\underline{L}}) := q_{S_n,\mathbb{H}^\circ}(U_{L_1} \times \cdots \times U_{L_n}) \subset \text{Sym}^n(\mathbb{H}^\circ), \tag{2.22}$$

$$\lambda_{\underline{L}} := \left[\min_k \lambda_{L_k}, \max_k \lambda_{L_k}\right] \subset \mathbb{R}, \tag{2.23}$$

where $q_{S_n} : E^n \to \text{Sym}^n(E)$ and $q_{S_n,\mathbb{H}^\circ} : (\mathbb{H}^\circ)^n \to \text{Sym}^n(\mathbb{H}^\circ)$ are the quotient maps by the symmetric group, and Conf^n is the configuration space of n points. When $\lambda_{\underline{L}} = \emptyset$ (i.e. when $\pi_E(L_k)$ is compact for all $k = 1, \dots, n$), we define $\lambda_{\underline{L}} := \{0\}$ so $\lambda_{\underline{L}}$ is a non-empty closed interval for all $\underline{L} \in \mathcal{L}^{\text{cyl},n}$.

2.2.5. Hilbert scheme of points. Let $\mathrm{Hilb}^n(E)$ be the Hilbert scheme of zero-dimensional length n subschemes on E, defined with respect to the complex structure J_E on E. We denote the Hilbert–Chow divisor by D_{HC} and the relative Hilbert scheme with respect to π_E by D_r, i.e. this is the divisor of subschemes whose projection under π_E has length $< n$. (We will sometimes write $\mathcal{Y}_E$ for the complement $\mathrm{Hilb}^n(E) \backslash D_r$ when the particular value of n is implicit or plays no role.) Let $\pi_{HC} : \mathrm{Hilb}^n(E) \to \mathrm{Sym}^n(E)$ be the contraction of D_{HC} and $\mathrm{Sym}^n(\pi_E) : \mathrm{Sym}^n(E) \to \mathrm{Sym}^n(\mathbb{H}^\circ)$ be the natural map induced by π_E. Let $\Delta_{\mathbb{H}^\circ} \subset \mathrm{Sym}^n(\mathbb{H}^\circ)$ be the big diagonal (i.e. all unordered tuples $\{q_1, \dots, q_n\}$ of points in $\mathbb{H}^\circ$ such that $q_i = q_j$ for some $i \neq j$).

Lemma 2.2. $(\mathrm{Sym}^n(\pi_E) \circ \pi_{HC})^{-1}(\Delta_{\mathbb{H}^\circ}) = D_{HC} \cup D_r$.

Proof. It is clear that $\mathrm{Sym}^n(\pi_E) \circ \pi_{HC}(D_{HC} \cup D_r) \subset \Delta_{\mathbb{H}^\circ}$. For the converse, if the support of $z \in \mathrm{Hilb}^n(E) \cap (\mathrm{Sym}^n(\pi_E) \circ \pi_{HC})^{-1}(\Delta_{\mathbb{H}^\circ})$ is a union of n distinct points, then $z \in D_r$. If the support consists of $k < n$ points instead, then $z \in D_{HC}$. ∎

Note that $\mathrm{Hilb}^n(E) \setminus D_{HC} = \mathrm{Conf}^n(E)$, and we have a trivialization of the canonical bundle of $\mathrm{Conf}^n(E)$ induced by that of E.

Lemma 2.3. *The trivialization of the canonical bundle of* $\mathrm{Conf}^n(E)$ *extends smoothly to a trivialization of the canonical bundle of* $\mathrm{Hilb}^n(E)$.

Proof. It follows from the fact that π_{HC} is a crepant resolution of $\mathrm{Sym}^n(E)$. ∎

We equip $\mathrm{Conf}^n(E)$ with the product symplectic form $\omega_{\mathrm{Conf}^n(E)}$ from (E, ω_E). This is smooth, but cannot be smoothly extended to a 2-form on $\mathrm{Hilb}^n(E)$.

Lemma 2.4. *For every open neighbourhood $U \subset \mathrm{Hilb}^n(E)$ of D_{HC}, there is a symplectic form on $\mathrm{Hilb}^n(E)$ which tames the complex structure, and coincides with $\omega_{\mathrm{Conf}^n(E)}$ outside U.*

Proof. This follows essentially from [52]; see also [36, 37] in the Kähler case. Let $D_{HC}^\rceil$ be the Hilbert–Chow divisor of $\mathrm{Hilb}^n(E^\rceil)$. Let $U, V \subset \mathrm{Hilb}^n(E^\rceil)$ be open neighbourhoods of $D_{HC}^\rceil$ such that U contains the closure of V. In [52], Voisin constructed two smooth closed 2-forms χ and Ψ on $\mathrm{Hilb}^n(E^\rceil)$ such that for some $\lambda_0 > 0$ and for all $\lambda \in (0, \lambda_0)$, $\chi + \lambda\Psi$ is a symplectic form on $\mathrm{Hilb}^n(E^\rceil)$ that tames the complex structure. Moreover, $\chi|_{\mathrm{Hilb}^n(E^\rceil)\backslash V} = \omega_{\mathrm{Conf}^n(E^\rceil)}|_{\mathrm{Conf}^n(E^\rceil)\backslash V}$ and $\Psi|_{\mathrm{Hilb}^n(E^\rceil)\backslash D_{HC}^\rceil} = d\Theta$ for some $\Theta \in \Omega^1(\mathrm{Hilb}^n(E^\rceil) \setminus D_{HC}^\rceil)$.

Let $\rho : \mathrm{Hilb}^n(E^\rceil) \to [0, 1]$ be a cut-off function such that $\rho|_V = 1$ and $\rho = 0$ outside U. Then $(\chi + \lambda d(\rho\Theta))|_{\mathrm{Hilb}^n(E^\rceil)\backslash U} = \omega_{\mathrm{Conf}^n(E^\rceil)}|_{\mathrm{Conf}^n(E^\rceil)\backslash U}$ and $(\chi + \lambda d(\rho\Theta))|_V = (\chi + \lambda\Psi)|_V$.

Since χ is non-degenerate and tames the complex structure outside V, and being non-degenerate and taming are both open conditions, for sufficiently small $\lambda > 0$, we know that $\chi + \lambda d(\rho\Theta)$ is a symplectic form which tames the complex structure outside V. Moreover, this is also true inside V because $\chi + \lambda d(\rho\Theta) = \chi + \lambda\Psi$ inside V. Therefore, we can restrict this symplectic form to $\mathrm{Hilb}^n(E)$ to get the result. ∎

Lemma 2.5. *If C is the image of a non-constant rational curve in* $\mathrm{Hilb}^n(E)$*, then* $[C] \cdot [D_r] > 0$.

Proof. From Lemma 2.4, one sees that the symplectic form on $\mathrm{Hilb}^n(E)$ is Poincaré dual to the relative cycle $-\epsilon[D_{HC}]$ for some $\epsilon > 0$ (recall that ω_E is exact). It is proved in [1, Lemma 5.4] that $[D_r]$ is a positive multiple of $-[D_{HC}]$ so $[C] \cdot [D_r] > 0$ follows from positivity of the symplectic area of C. ∎

Let $D_r^\circ = D_r \cap \mathrm{Conf}^n(E)$. For $J \in \mathcal{J}(E)$, we define $J^{[n]}$ to be the almost complex structure on $\mathrm{Conf}^n(E)$ descended from the product almost complex structure J^n on E^n. Note that $J^{[n]}$ is $\omega_{\mathrm{Conf}^n(E)}$-tamed and when $J = J_E$, $J^{[n]}$ extends smoothly to $\mathrm{Hilb}^n(E)$. We define the following space of almost complex structures on $\mathrm{Conf}^n(E)$:

$$\mathcal{J}^n(E) := \{J \mid J = (J')^{[n]} \text{ for some } J' \in \mathcal{J}(E)\} \tag{2.24}$$

For $H \in C^\infty(E, \mathbb{R})$, we define $H^{[n]} \in C^\infty(\mathrm{Conf}^n(E), \mathbb{R})$ to be

$$H^{[n]}(\underline{z}) := \sum_{i=1}^{n} H(z_i) \tag{2.25}$$

for $\underline{z} = \{z_1, \ldots, z_n\} \in \mathrm{Conf}^n(E)$. For $\gamma \in \mathfrak{g}_{\mathrm{aff}}$, let

$$\mathcal{H}_\gamma^{n,\mathrm{pre}}(E) := \{H \in C^\infty(\mathrm{Conf}^n(E), \mathbb{R}) \mid H = (H')^{[n]} \text{ for some } H' \in \mathcal{H}_\gamma(E)\}, \tag{2.26}$$

$$\mathcal{H}_\gamma^{n}(E) := \{H \in C^\infty(\mathrm{Conf}^n(E), \mathbb{R}) \mid H = H' \text{ outside a compact subset of } \mathrm{Conf}^n(E) \setminus D_r^\circ \text{ for some } H' \in \mathcal{H}_\gamma^{n,\mathrm{pre}}(E)\}. \tag{2.27}$$

Note that if

$$\begin{cases} A = a_t dt \in \Omega^1([0,1], \mathfrak{g}_{\mathrm{aff}}), \\ H = (H_t)_{t \in [0,1]} \text{ with } H_t = (H_t')^{[n]} \in \mathcal{H}_{a_t}^{n,\mathrm{pre}}(E) \text{ for } H_t' \in \mathcal{H}_{a_t}(E), \end{cases} \tag{2.28}$$

then ϕ_H is a well-defined Hamiltonian diffeomorphism of $\mathrm{Conf}^n(E)$ and the Hamiltonian vector field X_{H_t} satisfies

$$X_{H_t}|_p \in T_p D_r^\circ \tag{2.29}$$

for all $p \in D_r^\circ$ and all $t \in [0,1]$. It implies that $\phi_H(D_r^\circ) = D_r^\circ$. Moreover, we have $\phi_H^{-1}(\mathrm{Sym}(\underline{L})) = \mathrm{Sym}(\phi_{H'}^{-1}(\underline{L})) \in \mathrm{Sym}^n(\mathcal{L})$ and

$$\lambda_{\phi_{H'}^{-1}(\underline{L})} = g_A^{-1} \lambda_{\underline{L}}. \tag{2.30}$$

As a result, if $H = (H_t)_{t \in [0,1]}$ and $H_t \in \mathcal{H}_{a_t}^n(E)$ for all t, the associated Hamiltonian vector field satisfies (2.29) and ϕ_H is also a well-defined Hamiltonian diffeomorphism of $\mathrm{Conf}^n(E)$. Moreover, $\phi_H^{-1}(\mathrm{Sym}(\underline{L})) = \mathrm{Sym}(\underline{L'})$ outside a compact set of $\mathrm{Conf}^n(E) \setminus D_r^\circ$ for some $\underline{L'} \in \mathcal{L}^{\mathrm{cyl},n}$. Therefore, we can define $\lambda_{\phi_H^{-1}(\mathrm{Sym}(\underline{L}))} = \lambda_{\underline{L'}}$.

Remark 2.6. We introduce both $\mathcal{H}^{n,\mathrm{pre}}_{\gamma}(E)$ and $\mathcal{H}^{n}_{\gamma}(E)$ because, on the one hand, $\mathcal{H}^{n}_{\gamma}(E)$ gives us more freedom to achieve transversality, but on the other, working with $\mathcal{H}^{n,\mathrm{pre}}_{\gamma}(E)$ simplifies explicit computations for cases that transversality can be achieved within that more restricted class.

2.2.6. Floer cochains. Let $\mathcal{G}_{\mathrm{aff}}([0,1]) := C^{\infty}([0,1], G_{\mathrm{aff}})$, which is weakly contractible. Let $\mathcal{I}_{\mathrm{aff}}$ be the set of non-empty closed intervals in $\mathbb{R}$. Let

$$\mathcal{C}_{\mathrm{aff}}([0,1]) := \{(\lambda_0,\lambda_1) \in \mathcal{I}^2_{\mathrm{aff}} \mid \lambda_0 > \lambda_1\} \tag{2.31}$$

where the strict inequality means $\min \lambda_0 > \max \lambda_1$. For each $\Phi \in \mathcal{G}_{\mathrm{aff}}([0,1])$ and $(\lambda_0,\lambda_1) \in \mathcal{I}^2_{\mathrm{aff}}$, we define

$$\Phi_*(\lambda_0,\lambda_1) := (\Phi_*(0_{\mathrm{aff}}), \Phi_0(\lambda_0), \Phi_1(\lambda_1)) \in \Omega^1([0,1], \mathfrak{g}_{\mathrm{aff}}) \times \mathcal{I}^2_{\mathrm{aff}} \tag{2.32}$$

where 0_{aff} is the trivial connection, $\Phi_*(0_{\mathrm{aff}})$ is the gauge transformation by Φ and $\Phi_i := \Phi(i)$ for $i = 0, 1$. We define

$$\mathcal{P}_{\mathrm{aff}}([0,1]) := \{\Phi_*(\lambda_0,\lambda_1) \mid (\lambda_0,\lambda_1) \in \mathcal{C}_{\mathrm{aff}}([0,1])\}. \tag{2.33}$$

Note that, if $(A,\lambda_0,\lambda_1) \in \mathcal{P}_{\mathrm{aff}}([0,1])$, then by (2.31) and (2.32), we have

$$g_A^{-1}\lambda_1 < \lambda_0. \tag{2.34}$$

Lemma 2.7 (cf. [46, Section (2a)]). *$\mathcal{P}_{\mathrm{aff}}([0,1])$ is non-empty and weakly contractible. The projection*

$$\mathcal{P}_{\mathrm{aff}}([0,1]) \to \mathcal{I}^2_{\mathrm{aff}} \tag{2.35}$$

is a surjective weak fibration. Therefore, the fibres of (2.35) *are also weakly contractible.*

Proof. Since $\Phi_*(\lambda_0,\lambda_1) = \Phi'_*(\lambda'_0,\lambda'_1)$ if and only if $\Phi \circ (\Phi')^{-1}$ is a constant and $\lambda_j = \Phi \circ (\Phi')^{-1}(\lambda_j)$ for $j = 0, 1$, $\mathcal{P}_{\mathrm{aff}}([0,1])$ can be identified with $\mathcal{G}_{\mathrm{aff}}([0,1]) \times_{G_{\mathrm{aff}}} \mathcal{C}_{\mathrm{aff}}([0,1])$. That implies that $\mathcal{P}_{\mathrm{aff}}([0,1])$ is non-empty and weakly contractible.

On the other hand, given $\lambda_0, \lambda_1 \in \mathcal{I}_{\mathrm{aff}}$, there exists $g \in G_{\mathrm{aff}}$ such that $g^{-1}\lambda_1 < \lambda_0$. Moreover, there exists $\Phi \in \mathcal{G}_{\mathrm{aff}}([0,1])$ such that for $A := \Phi_*(0_{\mathrm{aff}})$, we have $g_A = g$. It follows that (2.35) is surjective. We leave it to the readers to check that (2.35) is a weak fibration (cf. [46, Section (2a)]). ∎

We denote the fibre of (2.35) at (λ_0,λ_1) by $\mathcal{A}_{\mathrm{aff}}([0,1],\lambda_0,\lambda_1)$.

Definition 2.8. For $\underline{L}_0 = \{L_{0,k}\}_{k=1}^n$ and $\underline{L}_1 = \{L_{1,k}\}_{k=1}^n$ in $\mathcal{L}^{\mathrm{cyl},n}$, a *perturbation pair* is a pair $(A_{\underline{L}_0,\underline{L}_1}, H_{\underline{L}_0,\underline{L}_1})$ such that

$$\begin{cases} A_{\underline{L}_0,\underline{L}_1} = a_{\underline{L}_0,\underline{L}_1,t}\,dt \in \mathcal{A}_{\mathrm{aff}}([0,1],\lambda_{\underline{L}_0},\lambda_{\underline{L}_1}), \\ H_{\underline{L}_0,\underline{L}_1} = (H_{\underline{L}_0,\underline{L}_1,t})_{t\in[0,1]}, \quad H_{\underline{L}_0,\underline{L}_1,t} \in \mathcal{H}^n_{a_{\underline{L}_0,\underline{L}_1,t}}(E) \end{cases} \tag{2.36}$$

and

$$\phi^{-1}_{H_{\underline{L}_0,\underline{L}_1}}(\mathrm{Sym}(\underline{L}_1)) \pitchfork \mathrm{Sym}(\underline{L}_0). \tag{2.37}$$

Lemma 2.9. *For any $\underline{L}_0, \underline{L}_1 \in \mathcal{L}^{\mathrm{cyl},n}$, the set of perturbation pairs is non-empty.*

Proof. For any choice of $(A_{\underline{L}_0,\underline{L}_1}, H_{\underline{L}_0,\underline{L}_1})$ satisfying (2.36), by (2.30) and (2.34) we have

$$\lambda_{\phi^{-1}_{H_{\underline{L}_0,\underline{L}_1}}(\mathrm{Sym}(\underline{L}_1))} < \lambda_{\underline{L}_0}, \tag{2.38}$$

so the transversality condition (2.37) is satisfied outside a compact subset. By definition (2.27), we are free to perturb $H_{\underline{L}_0,\underline{L}_1}$ inside a compact subset, so the result follows (cf. Remark 2.6). ∎

Let $\mathcal{X}(H_{\underline{L}_0,\underline{L}_1}, \underline{L}_0, \underline{L}_1) = \phi^{-1}_{H_{\underline{L}_0,\underline{L}_1}}(\mathrm{Sym}(\underline{L}_1)) \pitchfork \mathrm{Sym}(\underline{L}_0)$, which is identified with the set of $X_{H_{\underline{L}_0,\underline{L}_1}}$-chords from $\mathrm{Sym}(\underline{L}_0)$ to $\mathrm{Sym}(\underline{L}_1)$. By (2.29) and the fact that $\mathrm{Sym}(\underline{L}) \subset \mathrm{Conf}^n(E) \setminus D_r^\circ$ for every $\underline{L} \in \mathcal{L}^{\mathrm{cyl},n}$, we know that

$$\text{the } X_{H_{\underline{L}_0,\underline{L}_1}}\text{-chords from } \mathrm{Sym}(\underline{L}_0) \text{ to } \mathrm{Sym}(\underline{L}_1) \text{ are disjoint from } D_r^\circ. \tag{2.39}$$

Example 2.10. If (2.37) can be achieved by $H_{\underline{L}_0,\underline{L}_1,t} = (H'_{\underline{L}_0,\underline{L}_1,t})^{[n]} \in \mathcal{H}^{n,\mathrm{pre}}_{a_{\underline{L}_0,\underline{L}_1,t}}(E)$, then the set $\mathcal{X}(H_{\underline{L}_0,\underline{L}_1}, \underline{L}_0, \underline{L}_1)$ can be identified with the set of *unordered* n-tuples $\underline{x} = (x_1, \dots, x_n)$ such that $x_k \in \phi^{-1}_{H'_{\underline{L}_0,\underline{L}_1}}(L_{1,b_k}) \pitchfork L_{0,a_k}$, where $\{a_k \mid k = 1, \dots, n\} = \{b_k \mid k = 1, \dots, n\} = \{1, \dots, n\}$ (see Figure 2.2).

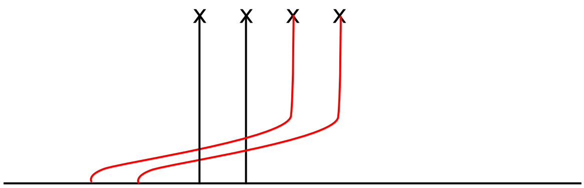


Fig. 2.2. Lagrangian tuples $\pi_E(\underline{L}_0)$ (black) and $\pi_E(\phi^{-1}_{H'_{\underline{L}_0,\underline{L}_1}}(\underline{L}_1))$ (red) for $H'_{\underline{L}_0,\underline{L}_1} \in H_\gamma(E)$.

Given $\underline{L}_0, \underline{L}_1 \in \mathcal{L}^{\mathrm{cyl},n}$, a perturbation pair $(A_{\underline{L}_0,\underline{L}_1}, H_{\underline{L}_0,\underline{L}_1})$ and a smooth family $J = (J_t)_{t\in[0,1]}$, $J_t \in \mathcal{J}^n(E)$, we define the Floer cochains (as a vector space over a chosen coefficient field of characteristic zero[5]) by

$$CF(\underline{L}_0, \underline{L}_1) := \bigoplus_{\underline{x} \in \mathcal{X}(H_{\underline{L}_0,\underline{L}_1}, \underline{L}_0, \underline{L}_1)} o_{\underline{x}} \tag{2.40}$$

where $o_{\underline{x}}$ is the orientation line at $\underline{x}$. In this case, we call $(A_{\underline{L}_0,\underline{L}_1}, H_{\underline{L}_0,\underline{L}_1}, J)$ a Floer cochain datum for $\underline{L}_0, \underline{L}_1 \in \mathcal{L}^{\mathrm{cyl},n}$.

[5] Recall that to work in characteristic zero requires that we fix spin structures on the Lagrangians; in the main examples studied later in this paper, the Lagrangians are products $(S^2)^j \times (\mathbb{R}^2)^k$ for $j, k \geq 0$ and admit unique spin structures.

For a fixed choice of grading functions $\eta_{0,k}$ on $L_{0,k}$ and $\eta_{1,k}$ on $L_{1,k}$, for $k = 1, \dots, n$, Lemma 2.3 implies that the product $\prod_k \eta_{i,k}$ descends to a grading function on $\mathrm{Sym}(\underline{L}_i)$, and we use that to grade $\mathcal{X}(H_{\underline{L}_0,\underline{L}_1}, \underline{L}_0, \underline{L}_1)$.

Example 2.11. In the situation of Example 2.10, the grading of $\underline{x} = (x_1, \dots, x_n)$ is

$$|\underline{x}| = \sum_{j=1}^{n} |x_j| \tag{2.41}$$

where $|x_j|$ is the Floer grading of x_j in E.

2.2.7. Floer data. We define the following group of gauge transformations:

$$\begin{aligned}\mathcal{G}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}) := \{\Phi \in C^\infty(\overline{\mathcal{S}}^{d+1,h}, G_{\mathrm{aff}}) \mid \Phi|_{\overline{N}_{\mathrm{mk}}^{d+1,h}} = \mathrm{id}_{G_{\mathrm{aff}}} \text{ and}\\ \Phi(\epsilon_j(s,t)) \text{ is independent of both } s \text{ and } S \in \overline{\mathcal{R}}^{d+1,h}\},\end{aligned} \tag{2.42}$$

where $\overline{N}_{\mathrm{mk}}^{d+1,h}$ is defined in Section 2.2.2.

Lemma 2.12. *$\mathcal{G}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h})$ is weakly contractible.*

Proof. Recalling that G_{aff} is contractible and $\overline{N}_{\mathrm{mk}}^{d+1,h}$ is a codimension 0 smooth submanifold of $\overline{\mathcal{S}}^{d+1,h}$ with boundary and corner, the result follows from the weak fibration

$$\mathcal{G}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}) \to C^\infty(\overline{\mathcal{S}}^{d+1,h}, G_{\mathrm{aff}}) \to C^\infty(\overline{N}_{\mathrm{mk}}^{d+1,h}, G_{\mathrm{aff}}) \times \prod_{j=0}^{d} C^\infty(\epsilon_j(0,[0,1]), G_{\mathrm{aff}}).$$

∎

Similarly to (2.31), we define

$$\mathcal{C}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}) := \{(\lambda_0, \dots, \lambda_d) \in \mathcal{I}_{\mathrm{aff}}^{d+1} \mid \lambda_0 > \lambda_1 > \dots > \lambda_d\}. \tag{2.43}$$

Consider a trivial G_{aff}-bundle over $\overline{\mathcal{S}}^{d+1,h}$ with fibre $\mathbb{R}$ and equip it with the trivial connection 0_{aff}. For $\Phi \in \mathcal{G}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h})$ and $\lambda \in C^\infty(\partial\overline{\mathcal{S}}^{d+1,h}, \mathcal{I}_{\mathrm{aff}})$, we define

$$\begin{aligned}\Phi_*\lambda &:= (\Phi_* 0_{\mathrm{aff}}, \Phi \circ \lambda) \in \Omega^1(\overline{\mathcal{S}}^{d+1,h}, \mathfrak{g}_{\mathrm{aff}}) \times C^\infty(\partial\overline{\mathcal{S}}^{d+1,h}, \mathcal{I}_{\mathrm{aff}}),\\ \mathcal{P}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}) &:= \{\Phi_*\lambda \mid \lambda \text{ is locally constant and}\\ &\qquad (\lambda|_{\partial_0\overline{\mathcal{S}}^{d+1,h}}, \dots, \lambda|_{\partial_d\overline{\mathcal{S}}^{d+1,h}}) \in \mathcal{C}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h})\},\end{aligned}$$

where $\partial_j\overline{\mathcal{S}}^{d+1,h} := \bigcup_{S \in \overline{\mathcal{R}}^{d+1,h}} \partial_j S$ for all j. We have the identification

$$\mathcal{P}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}) \simeq \mathcal{G}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}) \times_{G_{\mathrm{aff}}} \mathcal{C}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}) \tag{2.44}$$

so $\mathcal{P}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h})$ is weakly contractible. By (2.42), if $(A,\lambda) \in \mathcal{P}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h})$, then $A|_{\epsilon_j(s,t)}$ and $\lambda|_{\epsilon_j(s,k)}$ (for $k=0,1$) are independent of s and $S \in \overline{\mathcal{R}}^{d+1,h}$. Therefore, over strip-like ends, we have a well-defined projection

$$\mathcal{P}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}) \to \prod_{j=0}^{d} \mathcal{P}_{\mathrm{aff}}([0,1]), \tag{2.45}$$

which is surjective and is a weak fibration (because G_{aff} is contractible so one can extend $\Phi_j \in \mathcal{G}_{\mathrm{aff}}([0,1])$ over strip-like ends smoothly to a $\Phi \in \mathcal{G}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h})$ consistently). For a choice of $(A_j, \lambda_{j,0}, \lambda_{j,1}) \in \mathcal{P}_{\mathrm{aff}}([0,1])$ for each $j = 0,\dots,d$, we denote the fibre by

$$\mathcal{P}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}, \{(A_j,\lambda_{j,0},\lambda_{j,1})\}_j), \tag{2.46}$$

which is also weakly contractible. Note also that if $(A,\lambda) \in \mathcal{P}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h})$, then by (2.42),

$$A \text{ is flat everywhere and vanishes in } \overline{N}^{d+1,h}_{\mathrm{mk}}. \tag{2.47}$$

Now, for $S \in \mathcal{R}^{d+1,h}$, we define $\mathcal{P}_{\mathrm{aff}}(S, \{(A_j,\lambda_{j,0},\lambda_{j,1})\}_j)$ to be

$$\{(A_S,\lambda) \mid (A_S,\lambda) = (A,\lambda')|_S \text{ for some } (A,\lambda') \in \mathcal{P}_{\mathrm{aff}}(\overline{\mathcal{S}}^{d+1,h}, \{(A_j,\lambda_{j,0},\lambda_{j,1})\}_j)\}.$$

A *cylindrical Lagrangian label* is a choice of an element $\underline{L}_j \in \mathcal{L}^{\mathrm{cyl},n}$ associated to $\partial_j S$ for all j. We choose a cylindrical Lagrangian label and Floer cochain data (A_0, H_0, J_0) and (A_j, H_j, J_j) for $(\underline{L}_0, \underline{L}_d)$ and $(\underline{L}_{j-1}, \underline{L}_j)$ for $j = 1,\dots,d$, respectively.

Fix $A_S \in \mathcal{P}_{\mathrm{aff}}(S, \{(A_j, \lambda_{\underline{L}_{j-1}}, \lambda_{\underline{L}_j})\}_{j=0}^d)$, where for $j=0$, it should be understood as $(A_j, \lambda_{\underline{L}_{j-1}}, \lambda_{\underline{L}_j}) := (A_0, \lambda_{\underline{L}_0}, \lambda_{\underline{L}_d})$. Recall that we have chosen $J_j = (J_{j,t})_{t\in[0,1]}$, $J_{j,t} \in \mathcal{J}(E)$ for $j=0,\dots,d$. We have also chosen strip-like ends $\epsilon_0(s,t) : (-\infty,0]\times[0,1] \to S$ and $\epsilon_j(s,t) : [0,\infty)\times[0,1] \to S$ for $j=1,\dots,d$. We equip S with the following additional data:

$$\text{a smooth family } J = (J_z)_{z\in S},\ J_z \in \mathcal{J}^n(E), \text{ such that } J_z = J_E^{[n]} \text{ in } \nu(\mathrm{mk}(S)) \text{ and } J_{\epsilon_j(s,t)} = J_{j,t} \text{ for all } j, \tag{2.48}$$

$$K \in \Omega^1(S, C^\infty(\mathrm{Conf}^n(E),\mathbb{R})) \text{ such that for each } w \in TS,\ K(w) \in \mathcal{H}^n_{A_S(w)}(E), \text{ and for each } j \text{ and } r > 0, \text{ there is } c_j > 0 \text{ for which } \|(\epsilon_j^* K - H_j\,dt)e^{c_j|s|}\|_{C^r} \text{ converges to 0 as } s \text{ goes to } \pm\infty; \text{ moreover, } K \text{ vanishes in } \nu(\mathrm{mk}(S)). \tag{2.49}$$

Let $X_K \in \Omega^1(S, C^\infty(\mathrm{Conf}^n(E), TE))$ be the associated 1-form with values in Hamiltonian vector fields.

Remark 2.13. We require $\epsilon_j^* K$ to converge to $H_j\,dt$ exponentially fast, instead of coinciding with it, because it makes it easier to achieve regularity of the moduli whilst maintaining compatibility with gluing (see [46, Remark 4.7]).

Remark 2.14. Note that J can be extended smoothly to a family of tamed almost complex structures in $\mathrm{Conf}^n(E^{\rceil})$ and K can be extended smoothly to an element in $\Omega^1(S, C^\infty(\mathrm{Conf}^n(E^{\rceil}), \mathbb{R}))$.

We choose a smooth family of (A_S, J, K) for S varying in $\mathcal{R}^{d+1,h}$. Given $\underline{x}_0 \in \mathcal{X}(H_0, \underline{L}_0, \underline{L}_d)$ and $\underline{x}_j \in \mathcal{X}(H_j, \underline{L}_{j-1}, \underline{L}_j)$ for $j = 1, \dots, d$, we define $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)_{\mathrm{pre}}$ to be the moduli space of all maps $u : S \to \mathrm{Hilb}^n(E)$ such that

$$\begin{cases} u^{-1}(D_{HC}) \subset \nu(\mathrm{mk}(S)), \\ (Du|_z - X_K|_{u(z)})^{0,1} = 0 \text{ with respect to } (J_z)_{u(z)} \text{ for } z \in S, \\ u(z) \in \mathrm{Sym}(\underline{L}_j) \text{ for } z \in \partial_j S, \\ \lim_{s \to \pm\infty} u(\epsilon^j(s, \cdot)) = \underline{x}_j(\cdot) \text{ uniformly.} \end{cases} \tag{2.50}$$

Note that the conditions that X_K vanishes in $\nu(\mathrm{mk}(S))$, $J = J_E^{[n]}$ in $\nu(\mathrm{mk}(S))$, and $u^{-1}(D_{HC}) \subset \nu(\mathrm{mk}(S))$ guarantee that $(Du|_z - X_K|_{u(z)})^{0,1} = 0$ is a well-defined equation for all $z \in S$, by identifying $\mathrm{Conf}^n(E)$ as a subset of $\mathrm{Hilb}^n(E)$.

Next, we define $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ to be the subset of $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)_{\mathrm{pre}}$ consisting of all u such that

$$u(\xi_+^i) \in D_{HC} \quad \text{for all } i = 1, \dots, h. \tag{2.51}$$

Lemma 2.15. *For generic* (J, K) *such that* (2.48) *and* (2.49) *are satisfied, every solution* $u \in \mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)_{\mathrm{pre}}$ *is regular.*

Proof. For the ease of exposition, we discuss the case when $S \in \mathcal{R}^{d+1,h}$ is fixed. We refer readers to [40, Section (9k)] for the discussion when S is allowed to vary in $\mathcal{R}^{d+1,h}$, and to [45, Section 9] for the role of Remark 2.13 in achieving regularity compatibly with gluing.

Let $\mathcal{B}$ be the space of smooth maps $u : S \to \mathrm{Hilb}^n(E)$ such that $u^{-1}(D_{HC}) \subset \nu(\mathrm{mk}(S))$. Note that $\mathcal{B}$ is an open subset of $C^\infty(S, \mathrm{Hilb}^n(E))$. Let $V^1 \supset V^2 \supset \cdots$ be a sequence of neighbourhoods of D_{HC} such that $\bigcap_k V^k = D_{HC}$. Let $\mathcal{B}^k = \{u \in \mathcal{B} \mid \mathrm{Im}(u|_{S \setminus \nu(\mathrm{mk}(S))}) \cap V^k = \emptyset\}$. Note that $\mathcal{B} = \bigcup_k \mathcal{B}^k$. We want to run the Fredholm theory for appropriate Sobolev completions of $\mathcal{B}^k$ for each k.

For each k, using Lemma 2.4, we pick a symplectic form ω_k on $\mathrm{Hilb}^n(E)$ which agrees with $\omega_{\mathrm{Conf}^n(E)}$ outside V^k and tames the complex structure on $\mathrm{Hilb}^n(E)$. This induces a family of Riemannian metrics $g_k = (g_{k,z})_{z \in S}$ on $\mathrm{Hilb}^n(E)$ which agree with the metric induced from $\omega_{\mathrm{Conf}^n(E)}$ and J_z outside V^k. We use g_k to form an appropriate Sobolev completion of $\mathcal{B}^k$ (or the corresponding function space of the graphs of the maps).

The boundary conditions ensure that every solution u of (2.50) has non-empty intersection with a compact subset of $\mathrm{Conf}^n(E) \setminus D_r^\circ$ outside $\mathrm{mk}(S)$. For $u \in \mathcal{B}^k$, Gromov's graph trick applies, because on the one hand, $X_K|_{S \setminus \nu(\mathrm{mk}(S))}$ is the Hamiltonian field with respect to ω_k, and on the other, $X_K|_{\nu(\mathrm{mk}(S))} = 0$, so it is tautologically the Hamiltonian field with respect to ω_k. Hence regularity of u can be achieved by choosing K generically amongst functions satisfying (2.49), i.e. although we require that $K|_{\nu(\mathrm{mk}(S))} = 0$, the freedom of K outside $\nu(\mathrm{mk}(S))$ is sufficient to achieve regularity.

The outcome is a sequence of residual sets $\mathcal{S}^k$ in the space of all K satisfying (2.49) such that for every $K \in \mathcal{S}^k$, every $u \in \mathcal{B}^k$ satisfying (2.50) is regular. Therefore, we can take $\mathcal{S} := \bigcap_k \mathcal{S}^k$, which is still dense and every $u \in \mathcal{B}$ satisfying (2.50) is regular for every $K \in \mathcal{S}$.

More precisely, it means that for every $K \in \mathcal{S}$, and for every $u \in \mathcal{B}$ satisfying (2.50), there exists $N > 0$ such that for all $k > N$, the Fredholm operator D_u at u, with domain a Sobolev completion of $\mathcal{B}^k$ and codomain a Sobolev completion of $\Omega^{0,1}(S, u^*T\,\mathrm{Hilb}^n(E))$ with respect to the metric g_k, is surjective. This gives a manifold structure on $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)_{\mathrm{pre}}$. ∎

Lemma 2.16. *For generic (J, K) such that (2.48) and (2.49) are satisfied, every solution $u \in \mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ is regular.*

Proof. It suffices to show that for generic (J, K), the evaluation map

$$\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)_{\mathrm{pre}} \ni u \mapsto (u(\xi_+^1), \dots, u(\xi_+^h)) \in (\mathrm{Hilb}^n(E))^h \tag{2.52}$$

is transverse to $(D_{HC})^h$ (i.e. the algebraic intersection, which is well-defined as u is holomorphic near $\mathrm{mk}(S)$, is 1 at all the intersection points; note this implies that each such intersection point belongs to the smooth locus of D_{HC}). This can be achieved by combining the argument in Lemma 2.15 with standard transversality results for evaluation maps (see [34, Sections 6 & 7] and note that D_{HC} is the image of a smooth pseudocycle). ∎

There is a correspondence of maps as follows (see [35, Lemma 3.6] or [29, Section 13]). An n-fold branched covering $\pi_\Sigma : \Sigma \to S$ and a continuous map $v : \Sigma \to E$ together uniquely determine a continuous map $u : S \to \mathrm{Sym}^n(E)$, given by $u(z) = v(\pi_\Sigma^{-1}(z))$, counted with multiplicity. Conversely, if $u : S \to \mathrm{Sym}^n(E)$ is complex analytic near the big diagonal Δ_E and $\mathrm{Im}(u)$ is not contained in Δ_E, then we can form the fibre product

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\ \tilde{v}\ } & E^n \\ {\scriptstyle \pi_{\tilde{\Sigma}}}\big\downarrow & & \big\downarrow {\scriptstyle q_{S_n}} \\ S & \xrightarrow{\ u\ } & \mathrm{Sym}^n(E) \end{array}$$

and the map $\tilde{v}$ is S_n-equivariant. Let $\pi_1 : E^n \to E$ be the projection to the first factor. Consider the subgroup S_{n-1} of S_n which fixes the first element. The map $\pi_1 \circ \tilde{v} : \tilde{\Sigma} \to E$ factors through $\Sigma := \tilde{\Sigma}/S_{n-1} \to E$ and we denote the latter map by v. The map $\pi_{\tilde{\Sigma}}$ also induces a map $\pi_\Sigma : \Sigma \to S$. One can check that π_Σ is an n-fold branched covering such that $u(z) = v(\pi_\Sigma^{-1}(z))$. Moreover,

$$z \in S \text{ is a critical value of } \pi_\Sigma \text{ only if } u(z) \in \Delta_E. \tag{2.53}$$

We call this the ‘tautological correspondence’.

Remark 2.17. If the algebraic intersection number between u and Δ_E at z is 1, then $u(z)$ lies in the top stratum of Δ_E and there is exactly one critical point p of π_E such that z is its critical value. Moreover, p is Morse.

Lemma 2.18 (Tautological correspondence). *Every solution u of (2.50) determines uniquely an n-fold branched covering (Σ, π_Σ) of S and a map $v : \Sigma \to E$ such that $\pi_{HC} \circ u(z) = v(\pi_\Sigma^{-1}(z)) \in \mathrm{Sym}^n(E)$ for all $z \in S$.*

Proof. Every solution u of (2.50) is complex analytic near D_{HC} and $\mathrm{Im}(u)$ is not contained in D_{HC}. Therefore, by the tautological correspondence, we get an n-fold branched covering $\pi_\Sigma : \Sigma \to S$ and a continuous map $v : \Sigma \to E$ with $\pi_{HC} \circ u(z) = v(\pi_\Sigma^{-1}(z))$. ■

Remark 2.19. Suppose that we are in the situation of Example 2.10 for all the pairs $(\underline{L}_0, \underline{L}_d)$ and $(\underline{L}_{j-1}, \underline{L}_j)$. Suppose also that $J_z = (J_z')^{[n]} \in \mathcal{J}^n(E)$ and $K(w) = (K'(w))^{[n]} \in \mathcal{H}^{n,\mathrm{pre}}_{A_S(v)}(E)$ in (2.48) and (2.49), respectively. Then (2.50) and (2.51) imply that the maps $v : \Sigma \to E$ and $\pi_\Sigma : \Sigma \to S$ satisfy

$$(Dv|_z - X_{\pi_\Sigma^* K'}|_{v(z)})^{0,1} = 0 \quad \text{with respect to } (J'_{\pi_\Sigma(z)})_{v(z)} \text{ for all } z \in \Sigma \tag{2.54}$$

and the critical values of π_Σ are contained in $\mathrm{mk}(S)$.

2.2.8. Homotopy classes of maps. Let $B(\overline{S})$ be the real blow-up of $\overline{S}$ at the boundary punctures. In other words, we replace the punctures of S by closed intervals, which can be identified with $\{\epsilon^j(\pm\infty, t) \mid t \in [0,1]\}$. For $u \in \mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$, we define $G(u) := (\mathrm{Sym}^n(\pi_E) \circ \pi_{HC} \circ u, \mathrm{id}_S) : S \to \mathrm{Sym}^n(\mathbb{H}^\circ) \times S$, which can be continuously extended to

$$\overline{G}(u) : B(\overline{S}) \to \mathrm{Sym}^n(\mathbb{H}^\circ) \times B(\overline{S}) \tag{2.55}$$

by sending $\epsilon^j(\pm\infty, t)$ to $(\mathrm{Sym}^n(\pi_E) \circ \pi_{HC} \circ \underline{x}_j(t), \epsilon^j(\pm\infty, t))$ for all j. Note that, $\overline{G}(u)(\partial B(\overline{S}))$ lies inside

$$\begin{aligned} \partial_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1} := &\Big(\bigcup_j \mathrm{Sym}(U_{\underline{L}_j}) \times \partial_j S\Big) \\ &\cup \Big(\bigcup_j \{(\mathrm{Sym}^n(\pi_E) \circ \pi_{HC} \circ \underline{x}_j(t), \epsilon^j(\pm\infty, t)) \mid t \in [0,1]\}\Big) \\ &\subset \mathrm{Sym}^n(\mathbb{H}^\circ) \times B(\overline{S}). \end{aligned}$$

In particular, $\overline{G}(u)$ descends to a class in the space $\mathrm{Map}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ of homotopy class of continuous maps from $(B(\overline{S}), \partial B(\overline{S}))$ to $(\mathrm{Sym}^n(\mathbb{H}^\circ) \times B(\overline{S}), \partial_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1})$. In other words,

$$\begin{aligned} \overline{G}(u) \in \mathrm{Map}&(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1) \\ &:= [(B(\overline{S}), \partial B(\overline{S})), (\mathrm{Sym}^n(\mathbb{H}^\circ) \times B(\overline{S}), \partial_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1})] \end{aligned} \tag{2.56}$$

By (2.20), (2.39) and Lemma 2.2, $(\Delta_{\mathbb{H}^\circ} \times B(\overline{S})) \cap \partial_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1} = \emptyset$ so we have an intersection pairing (with respect to the obvious orientations)

$$\cdot[\Delta_{\mathbb{H}^\circ} \times B(\overline{S})] : \mathrm{Map}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1) \to \mathbb{Z}. \tag{2.57}$$

Lemma 2.20. *Given $\underline{x}_0, \dots, \underline{x}_d$, there is $I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1} \in \mathbb{Z}$ such that for every $u \in \mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$, we have*

$$[\overline{G}(u)] \cdot [\Delta_{\mathbb{H}^\circ} \times B(\overline{S})] = I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1}. \tag{2.58}$$

Moreover, $I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1}$ is independent of h.

Proof. Since $\mathrm{Sym}(U_{\underline{L}_j})$ is a contractible open set, we have $\mathrm{Map}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1) = \pi_2(\mathbb{C}^{n+1}, S^1) = \pi_1(S^1) = \mathbb{Z}$. In this case, the intersection pairing is a multiple of the winding number along the boundary and the winding number of $[\overline{G}(u)]$ is 1, by definition. ■

Lemma 2.21 (Positivity of intersection). *If $u \in \mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$, then*

$$[\overline{G}(u)] \cdot [\Delta_{\mathbb{H}^\circ} \times B(\overline{S})] \geq h, \tag{2.59}$$

and equality holds if and only if the image of u is disjoint from D_r, $u^{-1}(D_{HC}) = \mathrm{mk}(S)$ and the multiplicity of intersection between u and D_{HC} is 1 for all $z \in \mathrm{mk}(S)$.

Proof. Let $\overline{u} := \mathrm{Sym}^n(\pi_E) \circ \pi_{HC} \circ u : S \to \mathrm{Sym}^n(\mathbb{H}^\circ)$. There are two kinds of intersections between $\overline{u}$ and $\Delta_{\mathbb{H}^\circ}$, namely, at or away from $\mathrm{mk}(S)$. If $z \in \mathrm{mk}(S)$, then (2.48)–(2.51) imply that $\overline{u}$ is $(j_{\mathrm{Sym}^n(\mathbb{H}^\circ)}, j_S)$-holomorphic near z. It implies that the contribution of the algebraic intersection at z is at least 1 and z is the only intersection with $\Delta_{\mathbb{H}^\circ}$ in a small neighbourhood of z. Summing over all $z \in \mathrm{mk}(S)$, we find that the contribution to the algebraic intersection is at least h.

We need to show that the contribution from other intersections with $\Delta_{\mathbb{H}^\circ}$ is positive. Let $z_0 \in S \setminus \mathrm{mk}(S)$ be such that $\overline{u}(z_0) \in \Delta_{\mathbb{H}^\circ}$. Let $B(z_0) \subset S$ be a small disc centred at z_0. By Lemma 2.2, we must have $u(z_0) \in D_{HC} \cup D_r$. To show that the contribution at $\overline{u}(z_0)$ is positive, it suffices to show that the algebraic intersection number between $u(B(z_0))$ and $D_{HC} \cup D_r$ at $u(z_0)$ is positive. If $u(z_0) \in D_{HC}$, this follows from $u^{-1}(D_{HC}) \subset \nu(\mathrm{mk}(S))$ and $u|_{\nu(\mathrm{mk}(S))}$ being complex analytic. If $u(z_0) \in D_r$, then it can be achieved by Gromov's graph trick, and the fact that our choice of X_K is tangential to D_r at $u(z_0)$.

We give a detailed explanation of the last sentence. By the definitions (2.24) and (2.27), near D_r, we have $J_{z_0} = (J')^{[n]}$ and $K(w) = (\pi_E^* H^w)^{[n]}$ for some $H^w \in \mathcal{H}_{A_S(w)}(\mathbb{H}^\circ)$ for $w \in TB(z_0)$. It means that, by (2.8) and by shrinking $B(z_0)$ if necessary, we have

$$(X_K(\eta))_\tau \in T_\tau D_r \tag{2.60}$$

for all $\eta \in TB(z_0)$ and $\tau \in D_r$, where $T_\tau D_r$ is understood to be the tangent space of the smallest stratum of D_r that contains τ.

We consider the graph trick. Let $F = (u, \mathrm{id}) : B(z_0) \to \mathrm{Conf}^n(E) \times B(z_0)$. Let J_F be the almost complex structure of $\mathrm{Conf}^n(E) \times B(z_0)$ characterized by

$$\begin{cases} J_F|_{\mathrm{Conf}^n(E)\times\{z\}} = J_z, \\ d\pi_{B(z_0)} \circ J_F = j_S \circ d\pi_{B(z_0)}, \\ J_F(\partial_s + X_K(\partial_s)) = \partial_t + X_K(\partial_t), \end{cases} \tag{2.61}$$

where $\pi_{B(z_0)} : \mathrm{Conf}^n(E) \times B(z_0) \to B(z_0)$ is the projection. In holomorphic coordinates (s, t), we have

$$\begin{aligned}(DF + J_F \circ DF \circ j_S)(\partial_s) &= Du(\partial_s) + \partial_s + J_F(Du(\partial_t) - X_K(\partial_t)) + J_F(\partial_t + X_K(\partial_t)) \\ &= Du(\partial_s) + \partial_s - (Du(\partial_s) - X_K(\partial_s)) - (\partial_s + X_K(\partial_s)) = 0\end{aligned}$$

where the last line uses $(Du - X_K)^{0,1} = 0$. Similarly, $(DF + J_F \circ DF \circ j_S)(\partial_t) = 0$ and hence $(DF)^{0,1} = 0$ and F is (J_F, j_S)-holomorphic.

Notice that all strata of $D_r^\circ \times B(z_0)$ are J_F-holomorphic due to (2.60), (2.61) and the fact that $\Delta_{\mathbb{H}^\circ}$ is a J_z-holomorphic subvariety. As a result, the intersection $F(z_0)$ between $F(B(z_0))$ and $D_r^\circ \times B(z_0)$ is positive. This completes the proof. ∎

Corollary 2.22. *If $u \in \mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ and $I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1} = h$, then $\mathrm{Im}(u) \cap D_r = \emptyset$, $u^{-1}(D_{HC}) = \mathrm{mk}(S)$ and the multiplicity of intersection between u and D_{HC} is 1 for all $z \in \mathrm{mk}(S)$.*

Corollary 2.23. *If $I_{\underline{x}} = 0$, then $\mathcal{R}^{0+1,h}(\underline{x}) = \emptyset$ for all $h \geq 1$.*

Lemma 2.24. *For every $\underline{L} \in \mathcal{L}^{\mathrm{cyl},n}$, there is a perturbation pair $(A_{\underline{L}}, H_{\underline{L}})$ for $(\underline{L}, \underline{L})$ such that $I_{\underline{x}} = 0$ for all $\underline{x} \in \mathcal{X}(H_{\underline{L}}, \underline{L}, \underline{L})$.*

Proof. Let $A = a_t dt \in \mathcal{A}_{\mathrm{aff}}([0,1], \lambda_{\underline{L}}, \lambda_{\underline{L}})$. Since $g_A^{-1}\lambda_{\underline{L}} < \lambda_{\underline{L}}$, we can find $H' = (H'_t)_{t\in[0,1]}$, $H'_t \in \mathcal{H}_{a_t}(E)$, and an ordering of the Lagrangians in $\underline{L}$ such that $L_i \cap \phi_{H'}(L_j) \neq \emptyset$ only if $i \geq j$. In this case, the (possibly non-transverse) $X_{(H')^{[n]}}$-chords $\underline{x}(t)$ from $\mathrm{Sym}(\underline{L})$ to $\mathrm{Sym}(\underline{L})$ are given by unordered tuples of $X_{H'}$-chords from L_i to L_i for $i = 1, \dots, n$ (see Example 2.10). Moreover, by choosing H' appropriately, we can assume that $\mathrm{Sym}^n(\pi_E)(\pi_{HC}(\underline{x}(t))) \in \mathrm{Sym}(U_{\underline{L}})$ for all $X_{(H')^{[n]}}$-chords $\underline{x}(t)$.

We can pick H to be a compactly supported perturbation of $(H')^{[n]}$ such that every X_H-chord $\underline{x}(t)$ from $\mathrm{Sym}(\underline{L})$ to $\mathrm{Sym}(\underline{L})$ is transverse and $\mathrm{Sym}^n(\pi_E)(\pi_{HC}(\underline{x}(t))) \in \mathrm{Sym}(U_{\underline{L}})$ for all t. By taking $(A_{\underline{L}}, H_{\underline{L}})$ to be (A, H), the result follows, because the boundary conditions also project to $\mathrm{Sym}(U_{\underline{L}})$, which is contractible. ∎

2.2.9. Energy. The product symplectic form on $\mathrm{Conf}^n(E)$ cannot be smoothly extended to $\mathrm{Hilb}^n(E)$, and neither can the induced metric. Therefore, we will define energy and discuss compactness with the help of the corresponding maps $v : \Sigma \to E$ obtained from Lemma 2.18.

By (2.48) and (2.49), there exists a compact subset C of $\mathrm{Conf}^n(E) \setminus D_r^\circ$ and

$$\text{a smooth family } J' = (J'_z)_{z \in S},\ J'_z \in \mathcal{J}(E), \quad \text{and}$$
$$K' \in \Omega^1(S, C^\infty(E)) \text{ and } K'(w) \in \mathcal{H}_{A_S(w)}(E) \text{ for all } w \in TS$$

such that outside C, we have (cf. Remark 2.19)

$$(J'_z)^{[n]} = J_z \text{ for all } z \in S, \quad \text{and} \tag{2.62}$$

$$K'(w)^{[n]} = K(w) \text{ for all } w \in TS. \tag{2.63}$$

Let u be a solution of (2.50) and (2.51) and v be the map obtained by Lemma 2.18. Let $U \subset \mathrm{Conf}^n(E)$ be a relatively compact open neighbourhood of C, and define

$$S^{\mathrm{in}} := u^{-1}(\overline{U}), \qquad S^{\mathrm{out}} := u^{-1}(\mathrm{Hilb}^n(E) \setminus U), \tag{2.64}$$

where $\overline{U}$ is the closure of U. By adjusting U, we assume that $u(S)$ is transverse to $\partial\overline{U}$ so that $u^{-1}(\partial\overline{U}) \subset S$ is a smooth manifold with boundary. Note that, by definition, $\mathrm{mk}(S) \subset S^{\mathrm{out}}$ and S^{in} contains small neighbourhoods of the punctures. We also define $\Sigma^{\mathrm{in}} = \pi_\Sigma^{-1}(S^{\mathrm{in}})$ and $\Sigma^{\mathrm{out}} = \pi_\Sigma^{-1}(S^{\mathrm{out}})$.

Definition 2.25 (Energy). The *energy* of u is defined to be

$$E(u) := \frac{1}{2}\int_{S^{\mathrm{in}}} \|Du - X_K\|_g^2 \, d\mathrm{vol} + \frac{1}{2}\int_{\Sigma^{\mathrm{out}}} \|Dv - X_{\pi_\Sigma^* K'}\|_{g'}^2 \, d\mathrm{vol} \tag{2.65}$$

where g is the metric induced by J and $\omega_{\mathrm{Conf}^n(E)}$ and g' is the metric induced by J' and ω_E.

The energy is defined this way, and not as $\frac{1}{2}\int_S \|Du - X_K\|_g^2 \, d\mathrm{vol}$, because $u^* X_K$ is not defined at $u^{-1}(D_{HC}) \subset \mathrm{mk}(S)$, so it is not *a priori* clear that the latter expression is related to the action of the asymptotes of u. However, we show the following (see also (2.68) below).

Lemma 2.26. *The energy $E(u)$ is independent of the choice of U.*

Proof. It suffices to show that for every open subset $G \subset S \setminus \mathrm{mk}(S)$ such that $u(G) \cap C = \emptyset$, we have

$$\int_G \|Du - X_K\|_g^2 \, d\mathrm{vol} = \int_{\pi_\Sigma^{-1}(G)} \|Dv - X_{\pi_\Sigma^* K'}\|_{g'}^2 \, d\mathrm{vol}. \tag{2.66}$$

This is in turn clear because by (2.62) and (2.63), both J and X_K split as products. Since $\omega_{\mathrm{Conf}^n(E)}$ is also a product, the metric g is the product metric.

More precisely, let $G \subset S \setminus \mathrm{mk}(S)$ be a small open set such that $u(G) \cap C = \emptyset$ and $\pi_\Sigma^{-1}(G)$ is a disjoint union of open sets $G_1, \dots, G_n \subset \Sigma$. For $z \in G$ and $z_j := G_j \cap \pi_\Sigma^{-1}(z)$ for $j = 1, \dots, n$, we have canonical identifications

$$T_z G \simeq T_{z_j} G_j, \qquad T_{u(z)} \mathrm{Conf}^n(E) \simeq \bigoplus_{j=1}^n T_{v(z_j)} E. \tag{2.67}$$

By (2.62) and (2.63), both J_z and X_K (and $\omega_{\mathrm{Conf}^n(E)}$) respect this decomposition and every summand is given by J'_z and $X_{\pi_\Sigma{}^*K'}$, respectively. Therefore, (2.66) is true for G. Now, the result follows by summing over these small open subsets $G \subset S \setminus \mathrm{mk}(S)$. ■

By taking a sequence of larger and larger U and applying Lemma 2.26, we have

$$E(u) = \frac{1}{2}\int_{S\setminus \mathrm{mk}(S)} \|Du - X_K\|_g^2 \, d\mathrm{vol}. \tag{2.68}$$

Our next task is to derive a uniform upper bound for $E(u)$ that depends only on (A_S, J, K) and the Lagrangian boundary condition.

Consider again the graph construction. Let $\hat{v} := (v, \mathrm{id}) : \Sigma \to E \times \Sigma$ and define on $E \times \Sigma$ the following 2-forms:

$$\begin{aligned}\omega^{\mathrm{geom}}_{\pi_\Sigma^* K'} &:= \omega_E + \omega_E(X_{\pi_\Sigma^* K'}(\partial_s), \cdot) \wedge ds + \omega_E(X_{\pi_\Sigma^* K'}(\partial_t), \cdot) \wedge dt \\ &\qquad - \omega_E(X_{\pi_\Sigma^* K'}(\partial_s), X_{\pi_\Sigma^* K'}(\partial_t)) ds \wedge dt, \end{aligned}\tag{2.69}$$

$$\omega^{\mathrm{top}}_{\pi_\Sigma^* K'} := \omega_E - d(\pi_\Sigma^* K'(\partial_s) ds) - d(\pi_\Sigma^* K'(\partial_t) dt) = \omega^{\mathrm{geom}}_{\pi_\Sigma^* K'} + R_{\pi_\Sigma^* K'}, \tag{2.70}$$

where $R_{\pi_\Sigma^* K'}$ is the curvature defined by

$$\begin{aligned}R_{\pi_\Sigma^* K'} &:= (\partial_t \pi_\Sigma^* K'(\partial_s) - \partial_s \pi_\Sigma^* K'(\partial_t) + \{\pi_\Sigma^* K'(\partial_s), \pi_\Sigma^* K'(\partial_t)\}) ds \wedge dt \\ &\qquad \in \Omega^2(\Sigma, C^\infty(E)).\end{aligned}\tag{2.71}$$

It is clear that (cf. Remark 2.19)

$$(Dv|_{\Sigma^{\mathrm{out}}} - X_{\pi_\Sigma^* K'}|_{\Sigma^{\mathrm{out}}})^{0,1} = 0 \tag{2.72}$$

so, by tameness of J', we have

$$\frac{1}{2}\int_{\Sigma^{\mathrm{out}}} \|Dv - X_{\pi_\Sigma^* K'}\|^2_{g'} \, d\mathrm{vol} = \int_{\Sigma^{\mathrm{out}}} \hat{v}^* \omega^{\mathrm{geom}}_{\pi_\Sigma^* K'}. \tag{2.73}$$

Lemma 2.27. *There is a constant $T > 0$ such that for any choice of solution u of (2.50) and (2.51) (and hence the corresponding v) and U, we have $|\int_{\Sigma^{\mathrm{out}}} \hat{v}^* R_{\pi_\Sigma^* K'}| < T$.*

Proof. By (2.11), there exists $K'' \in \Omega^1(S, C^\infty(\mathbb{H}^\circ))$ such that $K' = \pi_E^* K''$ and $K''(w) \in \mathcal{H}_{A_S(w)}(\mathbb{H}^\circ)$ for all $w \in TS$. Recall also that $K'' = 0$ near $\mathrm{mk}(S)$. It implies that

$$R_{\pi_\Sigma^* K''} = \pi_\Sigma^* R_{K''} \quad \text{and} \quad R_{\pi_\Sigma^* K'} = \pi_E^* R_{\pi_\Sigma^* K''} \tag{2.74}$$

where the two terms $R_\bullet$ on the LHS of these equalities are defined using (2.71) and its K''-analogue.

Note that by (2.36) and (2.49), K'' converges exponentially fast in any C^r topology with respect to s over strip-like ends of S, so there is a constant $T' > 0$ such that for every j and any section $f : S \to \mathbb{H}^\circ \times S$, we have $|\int_{\mathrm{Im}(\epsilon_j)} f^* R_{K''}| < T'$. Moreover, (2.10), (2.36) and the flatness of A_S imply that $R_{K''}$ takes values in functions on $\mathbb{H}^\circ$ that are supported in a compact subset of $\mathbb{H}^\circ$. Therefore, by (2.74), $R_{\pi_\Sigma^* K'}$ takes values in functions on E that are uniformly bounded (the bound only depends on K'' but not on u), and there is a constant $T'' > 0$ such that for every j and any section $f : \Sigma \to E \times \Sigma$, we have $|\int_{\pi_{\Sigma-1}(\mathrm{Im}(\epsilon_j))} f^* R_{\pi_\Sigma^* K'}| < T''$.

Let $\Sigma^e \subset \Sigma$ be the closure of the complement of the strip-like ends. The discussion in the previous paragraph implies that $|\int_{\Sigma^{\mathrm{out}}} \hat{v}^* R_{\pi_\Sigma^* K'}|$ is bounded above by $(d+1)T''$ plus the integral of a bounded function over Σ^e, where the bound on that function depends only on K''. The result follows. ∎

Similarly, let $\hat{u} := (u, \mathrm{id}) : S \to \mathrm{Conf}^n(E) \times S$ and define on $\mathrm{Conf}^n(E) \times S$ the following 2-forms:

$$\omega_K^{\mathrm{geom}} := \omega_{\mathrm{Conf}^n(E)} + \omega_{\mathrm{Conf}^n(E)}(X_K(\partial_s), \cdot) \wedge ds + \omega_{\mathrm{Conf}^n(E)}(X_K(\partial_t), \cdot) \wedge dt - \omega_{\mathrm{Conf}^n(E)}(X_K(\partial_s), X_K(\partial_t)) ds \wedge dt^{\mathrm{in}}, \tag{2.75}$$

$$\omega_K^{\mathrm{top}} := \omega_{\mathrm{Conf}^n(E)} - d(K(\partial_s)ds) - d(K(\partial_t)dt) = \omega_K^{\mathrm{geom}} + R_K \tag{2.76}$$

where $R_K \in \Omega^2(S, C^\infty(\mathrm{Conf}^n(E)))$ is the curvature of K. We have

$$\frac{1}{2} \int_{S^{\mathrm{in}}} \|Du - X_K\|_g^2 \, d\mathrm{vol} = \int_{S^{\mathrm{in}}} \hat{u}^* \omega_K^{\mathrm{geom}}. \tag{2.77}$$

We have the parallel lemma.

Lemma 2.28. *There is a constant $T > 0$ such that for any choice of solution u of* (2.50) *and* (2.51)*, and any choice of U, we have $|\int_{S^{\mathrm{in}}} \hat{u}^* R_K| < T$.*

Proof. As before, there exists $T' > 0$ such that for every section $S \to \mathrm{Conf}^n(E) \times S$ and every j, we have $|\int_{\mathrm{Im}(\epsilon_j)} f^* R_K| < T'$. Ouside U, R_K takes values in bounded functions on $\mathrm{Hilb}^n(E) \setminus U$ with bound determined by K'. Since U is relatively compact, there is also a bound for the function-values of R_K inside U that is independent of u. Overall, if S^e is the closure of the complement of strip-like ends in S, then $|\int_{S^{\mathrm{in}}} \hat{u}^* R_K|$ is bounded above by $(d+1)T'$ plus an integration of a bounded function over S^e, so the result follows. ∎

The primitive 1-form θ_E for ω_E induces a primitive 1-form $\theta_{\mathrm{Conf}^n(E)}$ for $\omega_{\mathrm{Conf}^n(E)}$. It is clear that

$$\theta_{\pi_\Sigma^* K'}^{\mathrm{top}} := \theta_E - \pi_\Sigma^* K' \quad \text{and} \quad \theta_K^{\mathrm{top}} := \theta_{\mathrm{Conf}^n(E)} - K \tag{2.78}$$

are primitives of $\omega_{\pi_\Sigma^* K'}^{\mathrm{top}}$ and ω_K^{top}, respectively.

Lemma 2.29. *We have*

$$\int_{\Sigma^{\mathrm{out}}} \hat{v}^* \omega_{\pi_\Sigma^* K'}^{\mathrm{top}} + \int_{S^{\mathrm{in}}} \hat{u}^* \omega_K^{\mathrm{top}} = \int_{\partial S} \hat{u}^* \theta_K^{\mathrm{top}} \tag{2.79}$$

so there is a constant $T > 0$ such that for all u satisfying (2.50) *and* (2.51)*, we have $E(u) < T$.*

Proof. Apply the Stokes theorem to the terms on the left hand side of (2.79). Note that

$$\int_{\pi_\Sigma^{-1}(u^{-1}(\partial\overline{U}))} \hat{v}^*\theta^{\mathrm{top}}_{\pi_\Sigma^* K'} + \int_{u^{-1}(\partial\overline{U})} \hat{u}^*\theta^{\mathrm{top}}_K = 0 \tag{2.80}$$

because the Floer data splits into a product outside U and the orientation of the curves in the two summands are opposite to one another. Similarly,

$$\int_{\pi_\Sigma^{-1}(\partial S^{\mathrm{out}}\setminus u^{-1}(\partial\overline{U}))} \hat{v}^*\theta^{\mathrm{top}}_{\pi_\Sigma^* K'} = \int_{\partial S^{\mathrm{out}}\setminus u^{-1}(\partial\overline{U})} \hat{u}^*\theta^{\mathrm{top}}_K. \tag{2.81}$$

Therefore, we get (2.79).

Moreover, there is a constant $T' > 0$ (independent of u) such that $\int_{\partial S} \hat{u}^*\theta^{\mathrm{top}}_K < T'$ (see [46, Lemma 4.8]) so the result follows from Lemmas 2.27 and 2.28, and the equalities (2.70), (2.73), (2.76), (2.77) and (2.79). ■

Finally, we address the L^2-norm of v when the domain is not restricted to Σ^{out}.

Lemma 2.30. *For every relatively compact open subset $C_\Sigma \subset \Sigma$, there is a constant $T_{C_\Sigma} > 0$ such that for all v arising from applying Lemma 2.18 to u satisfying (2.50) and (2.51), we have*

$$\frac{1}{2}\int_{C_\Sigma} \|Dv - X_{\pi_\Sigma^* K'}\|^2_{g'}\, d\mathrm{vol} < T_{C_\Sigma}. \tag{2.82}$$

Proof. Let $G \subset S$ be a relatively compact open subset such that $C_\Sigma \subset \pi_\Sigma^{-1}(G)$. Since $J_z = (J'_z)^{[n]}$ and $X_K = X_{(K')^{[n]}}$ outside a compact subset of $\mathrm{Conf}^n(E) \setminus D_r^\circ$, there is a constant $T_G > 0$ (independent of u) such that

$$\int_{\pi_\Sigma^{-1}(G)} \|Dv - X_{\pi_\Sigma^* K'}\|^2_{g'}\, d\mathrm{vol} < T_G \int_G \|Du - X_K\|_g\, d\mathrm{vol}. \tag{2.83}$$

The right hand side is in turn bounded above by a constant independent of u, by Lemma 2.29. ■

Remark 2.31. The term $\frac{1}{2}\int_\Sigma \|Dv - X_{\pi_\Sigma^* K'}\|^2_{g'}\, d\mathrm{vol}$ may be infinite, because $X_K \neq X_{(K')^{[n]}}$ everywhere, so the symmetric product of the asymptotes of v corresponding to a fixed puncture of S is not necessarily an X_K-Hamiltonian chord. This implies that the integral over the strip-like ends might diverge.

2.2.10. Compactness. Let $\underline{x}_0, \dots, \underline{x}_d$ be as before. We next discuss compactness of the solution spaces $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$.

Lemma 2.32. *Fix $S \in \mathcal{R}^{d+1,h}$. Let $u_k : S_k \to \mathrm{Hilb}^n(E)$ be a sequence in $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ such that S_k converges to $S \in \mathcal{R}^{d+1,h}$. If there exists $z_k \in S_k \setminus \nu(\mathrm{mk}(S_k))$ such that $\|Du_k(z_k) - X_K(z_k)\|_g$ diverges to infinity, then for any compact subset C_S in the universal family over $\mathcal{R}^{d+1,h}$ and $N > 0$, there exists $k > N$ such that $z_k \notin C_S$.*

Proof. For simplicity, we assume $S_k = S$ for all k; the reasoning below adapts directly to the general case. Suppose the lemma were false; then there is a subsequence of z_k which converges to $z_\infty \in S$.

First assume that z_∞ is an interior point of S. Let $B \subset S$ be a small ball centred at z_∞ and conformally identify B with a 3ϵ-ball in $\mathbb{C}$ centred at the origin. Under this identification, we can assume that for all k, z_k lies in the ϵ-ball centred at z_∞. By assumption and the uniform bound of X_K near z_∞, $\|Du_k(z_k)\|_g$ diverges to infinity.

We apply a rescaling trick to the ϵ-ball B_k centred at z_k in the following sense (see [34, Section 4.2]). Let $r_k = \sup_{z \in B_k} \|Du_k(z)\|_g$ and $r_k B_k$ be the ϵr_k-ball centred at the origin. Let $\phi_k : r_k B_k \to B_k$ be $\phi_k(z) = z_k + z/r_k$. The sequence of maps $u_k \circ \phi_k : r_k B_k \to \mathrm{Conf}^n(E)$ satisfies $\sup_{r_k B_k} \|D(u_k \circ \phi_k)\| < 2$ for all k. Since $\pi_\Sigma^{-1}(B)$ is a disjoint union of n discs in Σ, the $u_k \circ \phi_k$ induce corresponding maps $V_{k,j} : r_k B_k \to E$ for $j = 1, \ldots, n$ as in Lemma 2.18. By the same reasoning as in Lemma 2.30, we know that there exists $T > 0$ such that $\sup_{r_k B_k} \|DV_{k,j}\|_{g'} < T$ for all k.

Since the metric on E extends to a metric on $E^\rceil$ and $V_{k,j} \in W^{1,p}$ (for $p > 2$) with uniformly bounded $W^{1,p}$-norm, by applying compactness of $W^{1,p}$ in C^0 to $V_{k,j}$ for each j in turn, we get a subsequence which converges uniformly on compact subsets to continuous functions $V_{\infty,j} : \mathbb{C} \to E^\rceil$ for all $j = 1, \ldots, n$. Moreover, $V_{\infty,j}$ satisfies the $(j_\mathbb{C}, J'_{z_\infty})$-holomorphic equation for all j, because of the stipulation that $J_z = (J'_z)^{[n]}$ for all z. Elliptic bootstrapping shows that $V_{\infty,j}$ is smooth. Moreover, since $\|Du_k(z_k) - X_K(z_k)\|_g$ diverges to infinity, at least one of $V_{\infty,j}$ is not a constant map. By Lemma 2.30, $DV_{\infty,j}$ has finite L^2-norm so we can apply removal of singularities to conclude that every $V_{\infty,j}$ extends to a J'_{z_∞}-holomorphic map $\mathbb{CP}^1 \to E^\rceil$ (and at least one of them is not a constant).

By assumption (2.9), every non-constant J'_{z_∞}-holomorphic map $\mathbb{CP}^1 \to E^\rceil$ has strictly positive algebraic intersection with D_E, which in turn implies that this is true for v_k for large k (because all other sphere bubbles, if any, also contribute positively to the algebraic intersection). However, $\mathrm{Im}(v_k)$ is contained in E, giving a contradiction.

Now, if instead $z_\infty \in \partial S$, then we can apply the same rescaling trick and the outcome is a J'_{z_∞}-holomorphic disc with appropriate Lagrangian boundary for each j, and at least one of them is non-constant. By exactness of the Lagrangian boundary, the J'_{z_∞}-holomorphic disc has strictly positive algebraic intersection with D_E, which in turn implies that that is true for v_k for large k, yielding the same contradiction. ■

Next, we assume $\sup_{z_k \in S_k \setminus \nu(\mathrm{mk}(S_k))} \|Du_k(z_k) - X_K(z_k)\|_g$ is uniformly bounded.

Proposition 2.33. *Fix $S \in \mathcal{R}^{d+1,h}$. Let $u_k : S_k \to \mathrm{Hilb}^n(E)$ be a sequence in $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \ldots, \underline{x}_1)$ such that S_k converges to $S \in \mathcal{R}^{d+1,h}$. Assume that $h = I_{\underline{x}_0;\underline{x}_d,\ldots,\underline{x}_1}$. If there exists $T > 0$ such that*

$$\sup_{z_k \in S_k \setminus \nu(\mathrm{mk}(S_k))} \|Du_k(z_k) - X_K(z_k)\|_g < T \tag{2.84}$$

for all k, then there exists $u_\infty \in \mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \ldots, \underline{x}_1)$ such that a subsequence of u_k converges (uniformly on compact subsets) to u_∞.

Proof. Without loss of generality, assume $S_k = S$ for all k. Let $v_k : \Sigma_k \to E$ be the corresponding maps. By Corollary 2.22, (2.53) and Remark 2.17, $\pi_{\Sigma_k} : \Sigma_k \to S$ is an n-fold branched covering such that its critical values are precisely $\mathrm{mk}(S)$ and all of its critical points are Morse. Since there are only finitely many such n-fold branched coverings of S, by passing to a subsequence we can assume $\Sigma_k = \Sigma$ for all k.

First, Lemma 2.30 and (2.84) imply that there is no energy concentration outside $\pi_\Sigma^{-1}(\nu(\mathrm{mk}(S)))$. Since $v_k|_{\pi_\Sigma^{-1}(\nu(\mathrm{mk}(S)))}$ is $(J'_{\pi_\Sigma(z)})_{z\in\pi_\Sigma^{-1}(\nu(\mathrm{mk}(S)))}$-holomorphic, energy concentration inside $\pi_\Sigma^{-1}(\nu(\mathrm{mk}(S)))$ would result in a non-constant map $\mathbb{CP}^1 \to E^\rceil$, which is impossible. Therefore, there exists $T > 0$ such that

$$\sup_{z_k\in\Sigma} \|Dv_k(z_k) - X_{\pi_\Sigma^* K}(z_k)\|_{g'} < T \tag{2.85}$$

for all k.

In particular, for every relatively compact open subset G of S, we have $v_k|_{\pi_\Sigma^{-1}(G)} \in W^{1,p}$ for $p > 2$. Therefore, $v_k|_{\pi_\Sigma^{-1}(G)}$ converges uniformly to a continuous function $v_\infty : \pi_\Sigma^{-1}(G) \to E^\rceil$.

There exists a compact set C_E of E such that outside C_E, $v_k|_{\pi_\Sigma^{-1}(G)}$ satisfies (2.72) (cf. Remark 2.19). Therefore, so does v_∞, and elliptic bootstrapping implies that v_∞ is smooth outside C_E and satisfies (2.72). Let $u'_\infty : G \to \mathrm{Sym}^n(E^\rceil)$ be the corresponding continuous map induced from v_∞. We can apply elliptic bootstrapping away from $\Delta_{E^\rceil}$ so u'_∞ is actually smooth away from $\Delta_{E^\rceil}$, so v_∞ is smooth everywhere.

We will prove that

$$\mathrm{Im}(v_\infty) \cap D_E = \emptyset, \tag{2.86}$$

$$\mathrm{Im}(v_\infty) \cap \pi_E^{-1}(\partial\mathbb{H}) = \emptyset, \tag{2.87}$$

$$(u'_\infty)^{-1}(\Delta_{E^\rceil}) = G \cap \nu(\mathrm{mk}(S)). \tag{2.88}$$

Given these, u'_∞ uniquely lifts to a map $u_\infty : G \to \mathrm{Hilb}^n(E)$ such that $u_\infty^{-1}(D_{HC}) \subset G \cap \nu(\mathrm{mk}(S))$. Note that this will be true for every relatively compact open subset G of S.

On the other hand, we will also prove that, by possibly passing to a subsequence,

$$\text{there is a compact subset } C \text{ of } \mathrm{Conf}^n(E) \text{ such that } u_k(\mathrm{Im}(\epsilon_j)) \subset C \text{ for all } k \text{ and } j, \tag{2.89}$$

so we can apply compactness of u_k over strip-like ends inside C. Combining all this information, and by a diagonal subsequence argument, we obtain u_∞ in $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ such that u_k has a subsequence that is uniformly converging to u_∞ over $S \setminus \nu(\mathrm{mk}(S))$. Note that we cannot guarantee the convergence is also uniform over $\nu(\mathrm{mk}(S))$, because u_∞ is lifted from u'_∞.

However, we can argue as follows. If there is a subsequence of u_k which converges uniformly over $\nu(\mathrm{mk}(S))$, then it converges pointwise to u_∞ over $\nu(\mathrm{mk}(S)) \setminus \mathrm{mk}(S)$. By continuity, the limit must be u_∞ and we are done. If there is no such subsequence, then for all $T > 0$ and $N > 0$, there exists $k > N$ such that $\sup_{\nu(\mathrm{mk}(S))} \|Du_k\|_{g''} > T$, where g'' is the induced metric from a choice of a symplectic form on $\mathrm{Hilb}^n(E)$ that tames $J_E^{[n]}$. In this case, bubbling occurs and results in a $J_E^{[n]}$-holomorphic sphere mapping to $\mathrm{Hilb}^n(E)$. However, every $J_E^{[n]}$-holomorphic sphere intersects D_r strictly positively (see Lemma 2.5), so $\mathrm{Im}(u_k) \cap D_r \neq \emptyset$ for large k. This contradicts our assumption that $h = I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1}$, by Corollary 2.22. ∎

Now, we need to justify the claims in the previous proof.

Lemma 2.34. (2.86) *is true.*

Proof. Suppose not; then there exists $z_0 \in \pi_\Sigma^{-1}(G)$ such that $v_\infty(z_0) \in D_E$.

First, if $\pi_{E^\rceil} \circ v_\infty(z_0) \in C_{\mathbb{H}}$, then v_∞ is J'-holomorphic near z_0 so it has strictly positive algebraic intersection with D_E, a contradiction.

If $\pi_{E^\rceil} \circ v_\infty(z_0) \notin C_{\mathbb{H}}$, then (2.72) implies that v_∞ satisfies a perturbed pseudo-holomorphic equation near z_0 with perturbation term having values in Hamiltonian vector fields tangent to D_E (see (2.14)). It means that the intersection also contributes strictly positively to the algebraic intersection by the graph trick (cf. the proof of Lemma 2.21), a contradiction. ∎

The following is a modification of the corresponding result of Seidel [46, Lemma 4.9].

Proposition 2.35. (2.87) *is true.*

Proof. We can make (2.72) more akin to the situation in [46], namely, if we define $A_\Sigma = \pi_\Sigma^* A_S$, $K_\Sigma = \pi_\Sigma^* K'$ and $(J_\Sigma)_z = J'_{\pi_\Sigma(z)}$ for $z \in \Sigma$, then outside a compact set of E, (2.72) becomes

$$(Dv_k - X_{K_\Sigma})^{0,1} = 0 \quad \text{with respect to } ((J_\Sigma)_z)_{u(z)}. \tag{2.90}$$

Note also that our choice of Floer data (A_S, K, J) (see (2.23) and (2.34)) and the induced Floer data $(A_\Sigma, K_\Sigma, J_\Sigma)$ guarantee that over each strip-like end, we have

$$g_{A_\Sigma}^{-1}(\lambda_{L_{j,b_i}}) = g_{A_j}^{-1}(\lambda_{L_{j,b_i}}) < \lambda_{L_{j-1,a_i}} \tag{2.91}$$

for all a_i and b_i and all $j = 1, \dots, d$. For $j = 0$, we have

$$g_{A_\Sigma}^{-1}(\lambda_{L_{d,b_i}}) = g_{A_0}^{-1}(\lambda_{L_{d,b_i}}) < \lambda_{L_{0,a_i}} \tag{2.92}$$

for all a_i and b_i,

In [46, Lemma 4.9], Seidel shows that if $\mathrm{Im}(v_\infty) \cap \pi_E^{-1}(\partial\mathbb{H}) \neq \emptyset$, then v_∞ lies entirely in $\pi_E^{-1}(\partial\mathbb{H})$. In that case it will satisfy

$$\begin{cases} \partial_s(\pi_E \circ v_\infty \circ \epsilon_0) = 0, \\ \partial_t(\pi_E \circ v_\infty \circ \epsilon_0) = X_{a_{\underline{L}_0,\underline{L}_d},t}, \\ \pi_E \circ v_\infty \circ \epsilon_0(s,0) \in \lambda_{\underline{L}_0}, \\ \pi_E \circ v_\infty \circ \epsilon_0(s,1) \in \lambda_{\underline{L}_d}. \end{cases} \tag{2.93}$$

These conditions imply $g_{A_0}^{-1}(\lambda_{\underline{L}_d}) \cap \lambda_{\underline{L}_0} \neq \emptyset$, violating (2.92). Therefore $\mathrm{Im}(v_\infty) \cap \pi_E^{-1}(\partial\mathbb{H}) = \emptyset$. ∎

Lemma 2.36. (2.88) *is true.*

Proof. Suppose not; then there is a point $z_0 \in G \setminus \mathrm{mk}(S)$ such that $u'_\infty(z_0) \in \Delta_E$ (here, we have applied Lemma 2.34 and Proposition 2.35 to replace $\Delta_{E^\rceil}$ by Δ_E).

Then, by the same reasoning as in Lemma 2.21, $\mathrm{Sym}^n(\pi_E) \circ u'_\infty(G)$ and $\Delta_{\mathbb{H}}$ intersect positively at z_0. That would violate the assumption that $h = I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1}$. ∎

Lemma 2.37. (2.89) *is true.*

Proof. Suppose not; then there exists $j \in \{0,\dots,d\}$ and a sequence $z_k = \epsilon_j(s_k,t_k)$ such that $u_k(z_k)$ goes to the infinite end of $\mathrm{Conf}^n(E)$. Moreover, we know that t_k has a subsequence converging to some $t_\infty \in [0,1]$ and s_k goes to infinity. Pick a small neighbourhood $D \subset \mathbb{C}$ of it_∞ and consider the maps $\hat{u}_k : D \to \mathrm{Conf}^n(E)$ given by $\hat{u}_k(s,t) = u_k(\epsilon_j(s+s_k,t))$. Let $\hat{v}_{k,a} : D \to E$ be the corresponding maps for $a = 1,\dots,n$.

Since there is no energy concentration, there is a subsequence of k such that for each $a = 1,\dots,n$, $\hat{v}_{k,a}$ converges uniformly to a continuous map $\hat{v}_{\infty,a} : D \to E^\rceil$; these in turn induce a continuous map $\hat{u}'_\infty : D \to \mathrm{Sym}^n(E^\rceil)$. By applying Lemma 2.34, 2.36 and Proposition 2.35, we get the corresponding results (2.86), (2.87) and (2.88) for $\hat{v}_{\infty,a}$ and $\hat{u}'_\infty$. This contradicts the assumption that $u_k(z_k)$ goes to infinity in $\mathrm{Conf}^n(E)$. ∎

We now assume that (A_S, J, K) is chosen to vary smoothly in $\overline{\mathcal{R}}^{d+1,h}$. That implies that $J = J_E^{[n]}$ and $K = 0$ on the sphere components of $S \in \overline{\mathcal{R}}^{d+1,h}$ (see Remark 2.1). We call the (interior/boundary) special points of S which connect different irreducible components of S (interior/boundary) *nodes*. For an interior node z on a disc component, we define the *multiplicity* $\mathrm{mult}(z)$ of z to be the total number of marked points (not including nodes) on the tree of sphere components to which z is connected.

Lemma 2.32 can be directly generalized by replacing $\mathcal{R}^{d+1,h}$ with $\overline{\mathcal{R}}^{d+1,h}$. The generalization of Proposition 2.33 is as follows.

Proposition 2.38. *Fix $S \in \overline{\mathcal{R}}^{d+1,h}$. Let $u_k : S_k \to \mathrm{Hilb}^n(E)$ be a sequence in $\mathcal{R}^{d+1,h}(\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1)$ such that S_k converges to $S \in \overline{\mathcal{R}}^{d+1,h}$. Assume that $h = I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1}$. If there exists $T > 0$ such that*

$$\sup_{z_k \in S_k \setminus \nu(\mathrm{mk}(S_k))} \|Du_k(z_k) - X_K(z_k)\|_g < T \tag{2.94}$$

for all k, then there exists a subsequence of u_k which converges (uniformly on compact subsets) to a stable map $u_\infty : S \to \mathrm{Hilb}^n(E)$ such that $u|_Q$ is a constant map for every sphere component Q of S. Moreover, interior nodes on disc components of S are mapped to D_{HC} under u_∞ and the algebraic intersection number between u_∞ and D_{HC} at an interior node z is given by $\mathrm{mult}(z)$.

Proof. For ease of exposition, we only consider the case $S = S_\alpha \cup S_\beta$, where S_α is a disc component and S_β is a sphere component. Let $z_{\alpha\beta} \in S_\alpha$ and $z_{\beta\alpha} \in S_\beta$ be the nodes. There exists an open subset $S_{\alpha,k}$ of S_k such that $S_{\alpha,k}$ converges to $S_\alpha \setminus z_{\alpha\beta}$. The proof of Proposition 2.33 can be applied to $u_k|_{S_{\alpha,k}}$ to conclude that there is a subsequence which converges (uniformly on compact subsets) to a map $u^\alpha_\infty : S_\alpha \setminus z_{\alpha\beta} \to \mathrm{Hilb}^n(E)$ which satisfies the conditions (2.50). By removal of singularities, we can extend this to a map $u^\alpha_\infty : S_\alpha \to \mathrm{Hilb}^n(E)$.

On the other hand, there exists an open subset $S_{\beta,k}$ of S_k such that $S_{\beta,k}$ converges to $S_\beta \setminus z_{\beta\alpha}$. Gromov compactness can be applied to $u_k|_{S_{\beta,k}}$, and after removal of singularities we obtain a stable $J^{[n]}_E$-holomorphic map u^β_∞ from a tree of spheres to $\mathrm{Hilb}^n(E)$.

We know that every non-constant $J^{[n]}_E$-holomorphic sphere intersects D_r strictly positively, by Lemma 2.5, so if u^β_∞ is not a constant map, then $\mathrm{Im}(u_k) \cap D_r \neq \emptyset$ for large k (cf. the last paragraph in the proof of Proposition 2.33). This is a contradiction, so u^β_∞ is a constant map to a point in D_{HC}. That implies $u^\alpha_\infty(z_{\alpha\beta}) \in D_{HC}$, and the algebraic intersection number with D_{HC} at this point is precisely $\mathrm{mult}(z_{\alpha\beta})$. ■

Corollary 2.39. *Suppose $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ has virtual dimension 1 and $h = I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1}$. For generic (J, K) satisfying (2.48) and (2.49), the moduli $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ can be compactified by adding stable maps $u : S \to \mathrm{Hilb}^n(E)$ for $S \in \overline{\mathcal{R}}^{d+1,h}$ for which S has no sphere components.*

Proof. By Lemma 2.32 and Proposition 2.38, $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ can be compactified by stable maps $u : S \to \mathrm{Hilb}^n(E)$ such that every sphere component of u is mapped to a constant. If S has a sphere component, then some disc component would have algebraic intersection number > 1 with D_{HC} at an interior node. However, this is a phenomenon of codimension at least 2, so it has virtual dimension at most -1 and hence can be avoided for generic (J, K).

More precisely, just as $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ is the fibre product between (a smooth pseudocycle replacement for) the inclusion $D^h_{HC} \to (\mathrm{Hilb}^n(E))^h$ and the evaluation map $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)_{\mathrm{pre}} \to (\mathrm{Hilb}^n(E))^h$, we can define a subset of $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)_{\mathrm{pre}}$ consisting of those u for which the interior marked points have prescribed algebraic intersection numbers with D_{HC}. In this case, some factors of the target of the evaluation map should be taken to be the appropriate jet bundles of $\mathrm{Hilb}^n(E)$ (cf. [15, Section 6]). The regularity argument in Lemmas 2.15 and 2.16 applies to this case to conclude that whenever there is an interior node on a disc component that is required to be mapped to D_{HC} with algebraic intersection number > 1, the restriction of u to the disc component does not exist for generic (J, K). Therefore, $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ can be compactified by stable maps without sphere components. ■

2.2.11. The definition. An object of $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ is an element $\underline{L} \in \mathcal{L}^{\mathrm{cyl},n}$. Given two objects $\underline{L}_0 = \{L_{0,k}\}^n_{k=1}$ and $\underline{L}_1 = \{L_{1,k}\}^n_{k=1}$, we choose a Floer chain datum $(A_{\underline{L}_0,\underline{L}_1}, H_{\underline{L}_0,\underline{L}_1}, J_{\underline{L}_0,\underline{L}_1})$ under the additional assumption that if $\underline{L}_0 = \underline{L}_1$, the perturbation pair chosen is given by Lemma 2.24. The corresponding morphism space is

$$\hom(\underline{L}_0, \underline{L}_1) := CF(\underline{L}_0, \underline{L}_1) = \bigoplus_{\underline{x} \in \mathcal{X}(H_{\underline{L}_0,\underline{L}_1}, \underline{L}_0, \underline{L}_1)} o_{\underline{x}}. \tag{2.95}$$

Recall that we have chosen strip-like ends, cylindrical ends and marked-points neighbourhoods in $\overline{\mathcal{S}}^{d+1,h}$ that are smooth up to the boundary and corners. Following [40, Section (9i)] (see also [45, Section 9]), we call a choice of Floer data *smooth and consistent* if for any Lagrangian labels $\underline{L}_0, \dots, \underline{L}_d$, the Floer data depends smoothly on $\overline{\mathcal{R}}^{d+1,h}$ up to the boundary and corner. Note that smoothness near the boundary and corner strata is defined with respect to a collar neighbourhood obtained from gluing the lower-dimensional strata. Therefore, a smooth and consistent choice of Floer data is obtained by an inductive procedure from the lower-dimensional strata. In our case, we proceed as follows:

(1) We equip the unique element $S \in \mathcal{R}^{1+1,0}$ with the (s-invariant) data

$$(A_{\underline{L}_0,\underline{L}_1}, J_{\underline{L}_0,\underline{L}_1}, H_{\underline{L}_0,\underline{L}_1}).$$

(2) For every element of $\mathcal{R}^{d+1,0}$ with Lagrangian labels being $\underline{L}_0, \dots, \underline{L}_d$, we equip it with (A_S, J, K) satisfying (2.47)–(2.49) such that (A_S, J, K) varies smoothly and consistently with respect to gluing (this is done inductively in d).

(3) Equip any sphere with any number of interior marked points with $(A_S, J, K) = (0, J_E^{[n]}, 0)$.

(4) Equip the element of $\mathcal{R}^{0+1,1}$ with Lagrangian label being $\underline{L}_0$ or $\underline{L}_1$ with (A_S, J, K) satisfying (2.47)–(2.49).

(5) For every element of $\mathcal{R}^{d+1,1}$ with Lagrangian labels being $\underline{L}_0, \dots, \underline{L}_d$, we equip it with (A_S, J, K) satisfying (2.47)–(2.49) such that (A_S, J, K) varies smoothly and consistently with respect to gluing.

(6) Repeat the procedure with increasing h.

For a generic consistent choice of Floer data, all elements in $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ are regular (here, the genericity is with respect to Floer's C_ϵ^∞-topology [40, Remark 9.9]).

The A_∞ structure is now defined by

$$\begin{aligned}\mu^d(\underline{x}_d, \dots, \underline{x}_1) \\ = \sum_{\underline{x}_0 \in \mathcal{X}(H_{\underline{L}_0,\underline{L}_d}, \underline{L}_0, \underline{L}_d)} \frac{1}{I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1}!} (\#\mathcal{R}^{d+1, I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1}}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1))\underline{x}_0\end{aligned}$$

where, when $h = 0$ and $d = 1$, $\#\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ is understood as the signed count after dividing out by $\mathbb{R}$-symmetry (and where we suppress the discussion of signs).

Lemma 2.40. *The collection of maps* $\{\mu^d\}$ *satisfies the* A_∞ *relation.*

Proof. By Corollary 2.39, the relevant 1-dimensional moduli spaces can be compactified by stable broken maps without sphere components. By Lemma 2.24 and exactness of the individual Lagrangians, every component of such a stable broken map has at least two boundary punctures. The rest follows from a well-established argument (see e.g. [49, Section 4.1.8]) together with the additivity property of $I_{\underline{x}_0;\underline{x}_d,\dots,\underline{x}_1}$ under gluing of maps. ∎

3. Properties of the cylindrical Fukaya–Seidel category

At this point, given a 4-dimensional exact Lefschetz fibration $\pi_E : E \to \mathbb{H}$, we have constructed for each $n \geq 1$ an A_∞ category $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ which captures certain Floer-theoretic computations in the Hilbert scheme of E, or rather its affine subset $\mathcal{Y}_E$. Unsurprisingly, the resulting A_∞ category satisfies typical properties of Fukaya-type categories, with proofs which are minor modifications of those which pertain to more familiar settings. This short section briefly elaborates upon some of these; to keep the exposition of manageable length, the proofs are only sketched.

3.1. Unitality and Hamiltonian invariance

Lemma 3.1 (Cohomological unit). *$\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ is cohomologically unital.*

Sketch of proof. To give a cochain representative of the cohomological unit of $HF(\underline{L}, \underline{L})$, we need to consider $S \in \mathcal{R}^{0+1,0}$. We put the Floer cochain datum $(A_{\underline{L},\underline{L}}, H_{\underline{L},\underline{L}}, J_{\underline{L},\underline{L}})$ on the outgoing strip-like end and stabilize the disc S by picking (A_S, K, J) satisfying (2.47)–(2.49). The Lagrangian boundary condition is a moving one from $\phi_{H_{\underline{L},\underline{L}}}(\mathrm{Sym}(\underline{L}))$ to $\mathrm{Sym}(\underline{L})$.

We then form the moduli space of maps $u : S \to \mathrm{Hilb}^n(E) \setminus (D_r \cup D_{HC})$ such that u is asymptotic to $\underline{x}_0 \in \mathcal{X}(H_{\underline{L},\underline{L}}, \underline{L}, \underline{L})$ near the strip-like end. The signed rigid count of this moduli gives a cochain level representative of the cohomological unit.

Note that we only need to consider $\mathcal{R}^{0+1,h}$ for $h = 0$ because, by Lemma 2.24, the corresponding moduli space for $h > 0$ is necessarily empty. ■

Lemma 3.2. *For any $\underline{L} \in \mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$, there is a choice of Floer cochain datum such that $CF^*(\underline{L}, \underline{L})$ is non-negatively graded and $CF^0(\underline{L}, \underline{L})$ is rank 1. As a result, the Hamiltonian chord generating $CF^0(\underline{L}, \underline{L})$ gives a chain level representative of the cohomological unit.*

Proof. As in the proof of Lemma 2.24, we can choose A and H' such that the (possibly non-transverse) $X_{(H')^{[n]}}$-chord from $\mathrm{Sym}(\underline{L})$ to itself is given by the unordered tuple of $X_{H'}$-chords $x(t)$ from L_i to itself for $i = 1, \dots, n$. Recall that H' is chosen such that $\pi_E(x(t)) \in U_{L_i}$ for all $x(t)$. For each i, we pick H'_i to be a perturbation of H' that is supported in a compact subset of $\pi_E^{-1}(U_{L_i})$ and such that the $X_{H'_i}$-chords from L_i to itself satisfy: (i) they are transverse, (ii) they are non-negatively graded, and (iii) there exists exactly one such chord of grading 0.

Recall the definition of q_{S_n} in (2.21). Note that there is a compact subset K of $q_{S_n}(\pi_E^{-1}(U_{L_1}) \times \cdots \times \pi_E^{-1}(U_{L_n}))$ which contains all the $X_{(H')^{[n]}}$-chords from $\mathrm{Sym}(\underline{L})$ to itself. Since $q_{S_n}(\pi_E^{-1}(U_{L_1}) \times \cdots \times \pi_E^{-1}(U_{L_n}))$ is symplectomorphic to the product symplectic manifold $\pi_E^{-1}(U_{L_1}) \times \cdots \times \pi_E^{-1}(U_{L_n})$, we can find a perturbation $H = (H_t)_{t \in [0,1]}$ of $(H')^{[n]}$ such that $H_t \in \mathcal{H}^n_{a_t}(E)$ for all t and H is the product type Hamiltonian $H(\underline{z}) = H'_1(z_1) + \cdots + H'_n(z_n)$ locally near K, where $\underline{z} \in q_{S_n}(\pi_E^{-1}(U_{L_1}) \times \cdots \times \pi_E^{-1}(U_{L_n}))$ is identified with $(z_1, \dots, z_n) \in \pi_E^{-1}(U_{L_1}) \times \cdots \times \pi_E^{-1}(U_{L_n})$.

In particular, it means that the X_H-Hamiltonian chords from $\mathrm{Sym}(\underline{L})$ to itself are precisely given by unordered tuples of $X_{H_i'}$-chords from L_i to itself for $i = 1, \dots, n$. Therefore, the X_H-chords are non-negatively graded and precisely one of them is in grading 0. ∎

Lemma 3.3 (Hamiltonian invariance). *If $\underline{L}_0, \underline{L}_1 \in \mathcal{L}^{\mathrm{cyl},n}$ can be connected by a smooth family $(\underline{L}_t)_{t\in[0,1]}$ of elements in $\mathcal{L}^{\mathrm{cyl},n}$, then $\underline{L}_0$ is quasi-isomorphic to $\underline{L}_1$ in $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$.*

Sketch of proof. Without loss of generality, we can assume that $\underline{L}_1$ is C^1-close to $\underline{L}_0$. In particular, we can assume that $\pi_E(L_{0,i}) \cap \pi_E(L_{1,j}) = \emptyset$ if $j \neq i$.

Pick a Floer cochain datum $(A_{\underline{L}_0,\underline{L}_1}, H_{\underline{L}_0,\underline{L}_1}, J_{\underline{L}_0,\underline{L}_1})$ for $(\underline{L}_0, \underline{L}_1)$. For each integer $h \geq 0$ and $S \in \mathcal{R}^{0+1,h}$, we equip S with the moving Lagrangian boundary label $\underline{L}_t$ on ∂S. Then, we pick (A_S, K, J) such that (2.47), (2.48) and (2.49) are satisfied and compatible with gluing with elements in $\mathcal{R}^{1+1,k}$ for all k. The rigid count of the corresponding solutions of (2.50) and (2.51) give us an element α in $CF^0(\underline{L}_0, \underline{L}_1)$.

We claim that α is a cocycle. Indeed, by the same argument as in Lemma 2.24, when the Floer cochain datum is carefully chosen, the rigid count of solutions for $h > 0$ vanishes. In this case, α is a sum of continuation elements $\alpha_i \in CF^0(L_{0,i}, L_{1,i})$.

Similarly, we can construct a cocycle β in $CF^0(\underline{L}_1, \underline{L}_0)$. By gluing, one checks that $\mu^2(\alpha, \beta)$ and $\mu^2(\beta, \alpha)$ are cohomological units (for suitable choices of Floer cochain data as above, these cohomological units agree with the classical units). ∎

3.2. *Canonical embeddings*

Let W be a contractible (hence connected) open subset of $\mathbb{H}$ such that

$$W \cap \partial\mathbb{H} \text{ equals } (R, \infty) \text{ or } (-\infty, R) \text{ for some } R \in \mathbb{R}. \tag{3.1}$$

Let $W^\circ := W \cap \mathbb{H}^\circ$. We can define an A_∞ full subcategory $\mathcal{FS}_W^{\mathrm{cyl},n}(\pi_E)$ of $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ with objects given by $\underline{L} = \{L_1, \dots, L_n\}$ such that $\pi_E(L_j) \subset W^\circ$ for all j.

The conditions that W is contractible and satisfies (3.1) imply that $W^c := \mathrm{Int}(\mathbb{H} \setminus W)$ is also contractible and satisfies (3.1), where $\mathrm{Int}(-)$ stands for interior.

Let $\underline{K}$ be an object of $\mathcal{FS}_{W^c}^{\mathrm{cyl},k}(\pi_E)$. Each object $\underline{L}$ of $\mathcal{FS}_W^{\mathrm{cyl},n}(\pi_E)$ determines an object $\underline{L} \sqcup \underline{K}$ in $\mathcal{FS}^{\mathrm{cyl},n+k}(\pi_E)$ given by adding to $\underline{L}$ the Lagrangians in $\underline{K}$.

Lemma 3.4. *There is a cohomologically faithful A_∞ functor $\sqcup \underline{K} : \mathcal{FS}_W^{\mathrm{cyl},n}(\pi_E) \to \mathcal{FS}^{\mathrm{cyl},n+k}(\pi_E)$, which on objects is given by sending $\underline{L}$ to $\underline{L} \sqcup \underline{K}$.*

Proof. By (3.1), W contains a neighbourhood of $[R, \infty)$ or $(-\infty, R]$. Therefore, we can choose the Hamiltonian term in the Floer data such that each element in $\mathcal{X}(\underline{L}_0 \sqcup \underline{K}, \underline{L}_1 \sqcup \underline{K})$ is a tuple given by adjoining an element in $\mathcal{X}(\underline{L}_0, \underline{L}_1)$ with another element in $\mathcal{X}(\underline{K}, \underline{K})$. It establishes a bijective correspondence between the generators in $CF(\underline{L}_0 \sqcup \underline{K}, \underline{L}_1 \sqcup \underline{K})$ and $CF(\underline{L}_0, \underline{L}_1) \otimes CF(\underline{K}, \underline{K})$. By Lemma 3.2, we can arrange the Floer data so that $CF^0(\underline{K}, \underline{K})$ is rank 1 and generated by a chain level representative e of the cohomological unit.

The A_∞ functor $\sqcup \underline{K}$ has first order term sending $x \in CF(\underline{L_0}, \underline{L_1})$ to the generator in $CF(\underline{L_0} \sqcup \underline{K}, \underline{L_1} \sqcup \underline{K})$ corresponding to $x \otimes e$, and has no higher order terms as an A_∞ functor. To check that $\sqcup \underline{K}$ is indeed an A_∞ functor, it suffices to show that there is a bijective correspondence between the moduli governing the A_∞ structures in $\mathcal{FS}_W^{\text{cyl},n}(\pi_E)$ and the moduli governing the A_∞ structures in $\mathcal{FS}^{\text{cyl},n+k}(\pi_E)$ with all inputs and outputs being of the form $x \otimes e$.

Let $u : S \to \text{Hilb}^n(E)$ be a solution contributing to the A_∞ structure of $\mathcal{FS}_W^{\text{cyl},n}(\pi_E)$. Let $v : \Sigma \to E$ be the map which tautologically corresponds to u. By projecting to $\mathbb{H}^\circ$ via π_E and appealing to the open mapping theorem, we know that the image of v lies inside W.

The proof of Lemma 3.2 shows that e is given by the unordered tuple of grading zero $X_{H_i'}$-Hamiltonian chords e_i from K_i to itself, for $i = 1, \ldots, k$. Moreover, we can assume that e_i is a constant chord (i.e. $X_{H_i'}(e_i(t)) = 0$ for all t).

We now separate the discussion into two cases, namely, the stable case $(d, h) \neq (1, 0)$ and the semi-stable case $(d, h) = (1, 0)$. We start with $(d, h) \neq (1, 0)$.

In this case, for each $S \in \mathcal{R}^{d+1,h}$, the constant map v_i from S to e_i satisfies

$$\begin{cases} (Dv_i - X_{H_j'})^{0,1} = 0, \\ v_i(\partial S) \subset L_i', \\ \lim_{s \to \pm\infty} v_i(\epsilon_j(s, \cdot)) = e_i \text{ uniformly for all } j = 0, \ldots, d. \end{cases} \tag{3.2}$$

The moduli space of solutions to (3.2) has virtual dimension 0 (because the conformal structure of S is fixed). Moreover, v_i is a regular solution to (3.2). Note also that $\pi_E \circ v_i \notin W$ for all i.

Now we define $\widetilde{\Sigma} = \Sigma \sqcup \bigsqcup_{i=1}^k S$, and we define $\pi_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to S$ to be $\pi_{\widetilde{\Sigma}} = \pi_\Sigma \sqcup \bigsqcup_{i=1}^k \text{Id}_S$, where $\text{Id}_S : S \to S$ is the identity map. Let $\widetilde{v} = v \sqcup \bigsqcup_{i=1}^k v_i$ and $\widetilde{u} : S \to \text{Hilb}^{n+k}(E)$ be the map tautologically corresponding to $\widetilde{v}$. Notice that $\text{Im}(v)$ and $\text{Im}(v_i)$ are pairwise disjoint. This means that the Fredholm operator associated to $\widetilde{u}$ splits into the direct sum of those associated to u and v_i. Therefore, $\widetilde{u}$ is a regular solution contributing to the A_∞ structure of $\mathcal{FS}^{\text{cyl},n+k}(\pi_E)$ with all inputs and output being of the form $x \otimes e$. Conversely, by the Lagrangian boundary conditions, every such solution is of the form $\widetilde{u}$ for some u contributing to the A_∞ structure of $\mathcal{FS}_W^{\text{cyl},n}(\pi_E)$.

Next, we consider the case $(d, h) = (1, 0)$. In this case, $\Sigma = \bigsqcup_{i=1}^n S$ and $\pi_\Sigma = \bigsqcup_{i=1}^n \text{Id}_S$. Since u is rigid, it means that the moduli space containing u is 1-dimensional before dividing out by $\mathbb{R}$-translation-symmetry. We define $\widetilde{\Sigma}$, $\pi_{\widetilde{\Sigma}}$, $\widetilde{v}$ and $\widetilde{u}$ as above. The moduli space containing $\widetilde{u}$ still has virtual dimension 1 before dividing by $\mathbb{R}$-symmetry, because the constant maps have virtual dimension 0. Moreover, $\widetilde{u}$ is regular because one can split the Fredholm operator of $\widetilde{u}$ to those of u and the constant maps, which are all regular. Conversely, rigid solutions contributing to the A_∞ structure of $\mathcal{FS}^{\text{cyl},n+k}(\pi_E)$ with $(d, h) = (1, 0)$ and both input and output being of the form $x \otimes e$ are necessarily of the form $\widetilde{u}$ for some u.

This finishes the proof. ∎

Corollary 3.5. *If* $\underline{L}_0, \underline{L}_1, \underline{L}_2$ *form an exact triangle in* $\mathcal{FS}_W^{\mathrm{cyl},n}(\pi_E)$*, then* $\underline{L}_0 \sqcup \underline{K}$, $\underline{L}_1 \sqcup \underline{K}, \underline{L}_2 \sqcup \underline{K}$ *form one in* $\mathcal{FS}^{\mathrm{cyl},n+k}(\pi_E)$.

Using Corollary 3.5, we can inductively construct a plethora of exact triangles. For example, we will prove in Section 5.1 that our set-up applies to the standard Lefschetz fibration π_E on the A_{m-1} Milnor fibre E. In that case, there are well-known exact triangles [40, Lemma 18.20] relating matching spheres and thimbles in $\mathcal{FS}^{\mathrm{cyl},1}(\pi_E) = \mathcal{FS}(\pi_E)$. We can obtain exact triangles in $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ by adjoining matching spheres/thimbles to the exact triangles in $\mathcal{FS}^{\mathrm{cyl},1}(\pi_E)$ (see Figure 3.1).

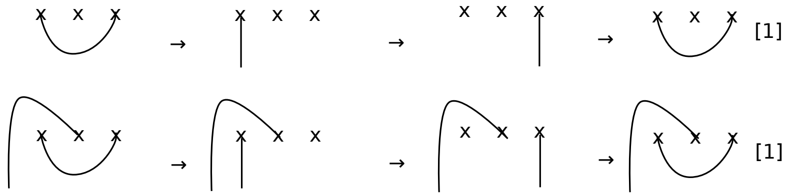


Fig. 3.1. The top row represents an exact triangle in $\mathcal{FS}(\pi_E)$ when E is the A_2 Milnor fibre. The bottom row is exact in $\mathcal{FS}^{\mathrm{cyl},2}(\pi_E)$ by Corollary 3.5.

3.3. Serre functor

An important observation due to Kontsevich and Seidel is that the global monodromy τ should induce the Serre functor on the Fukaya–Seidel category up to degree shift [22, 41, 42, 44]. In our context, the global monodromy of π_E induces an auto-equivalence of $\mathcal{FS}^{\mathrm{cyl},n}(E)$ for each n. By formally the same argument, we get

Claim 3.6. *The global monodromy of* π_E *induces the Serre functor on* $\mathcal{FS}^{\mathrm{cyl},n}(E)$ *up to a degree shift by* $-2n$.

Even for Lefschetz fibrations, a complete proof that the auto-equivalence induced by global monodromy agrees with the Serre functor (up to shift) does not seem to appear in the literature.[6] Any argument is likely to apply to our case. For the reader's convenience, we outline the essential geometric input underlying the claim.

Sketch of proof of Claim 3.6. As a graded symplectomorphism, we require that τ acts as the identity on the trivialization of the bicanonical bundle in a compact region containing the critical points. This means that for each compact exact graded Lagrangian L in E, $\tau(L) = L$ as graded objects. On the chain level, there is a canonical isomorphism (see the left of Figure 3.2)

$$CF^i(L_0, L_1) \simeq (CF^{-i}(L_1, \tau(L_0)[-2]))^\vee = (CF^{-i}(\tau^{-1}(L_1)[2], L_0))^\vee \tag{3.3}$$

given by sending an intersection point of $L_0^+ \cap L_1$ to the corresponding intersection point

[6]Forthcoming work of Abouzaid and Ganatra lays the general foundations for the treatment of Serre functors in the context of Fukaya categories for Landau–Ginzburg models, which generalize Fukaya–Seidel categories.

Fig. 3.2

of $\tau(L_0) \cap L_1^+$ but viewing the latter as a generator in the dual group $CF(L_1, \tau(L_0))$. Here, for two Lagrangians L, K, we use $L^+ \cap K$ (or $K \cap L^+$) to mean that we pick a Hamiltonian diffeomorphism ϕ such that $\lambda_K < \lambda_{\phi(L)}$, $\phi(L) \pitchfork K$ and we use $L^+ \cap K$ to denote $\phi(L) \pitchfork K$.

In particular, when L_0 is compact, a generator x of $CF^i(L_0, L_1)$ is mapped to the linear dual of the corresponding generator $x^\vee \in CF^{2-i}(L_1, L_0) = CF^{-i}(L_1, L_0[-2]) = CF^{-i}(L_1, \tau(L_0)[-2])$. The canonical isomorphism (3.3) lifts to the canonical isomorphism for Lagrangian tuples

$$CF^i(\underline{L}_0, \underline{L}_1) \simeq (CF^{-i}(\underline{L}_1, \tau(\underline{L}_0)[-2n]))^\vee = (CF^{-i}(\tau^{-1}(\underline{L}_1)[2n], \underline{L}_0))^\vee. \quad (3.4)$$

A key claim is that, under the canonical isomorphism (3.4), one can arrange to have a bijective correspondence between the moduli computing higher A_∞ operations:

$$CF(\underline{L}_{d-1}, \underline{L}_d) \times CF(\underline{L}_{d-2}, \underline{L}_{d-1}) \times \cdots \times CF(\underline{L}_0, \underline{L}_1) \to CF(\underline{L}_0, \underline{L}_d), \quad (3.5)$$

$$CF(\tau^{-1}(\underline{L}_d)[2n], \underline{L}_{d-1})^\vee \times CF(\underline{L}_{d-2}, \underline{L}_{d-1}) \times \cdots \times CF(\underline{L}_0, \underline{L}_1) \to CF(\tau^{-1}(\underline{L}_d)[2n], \underline{L}_0)^\vee, \quad (3.6)$$

where (3.6) is obtained by dualizing $CF(\tau^{-1}(\underline{L}_d)[2n], \underline{L}_{d-1})$ and $CF(\tau^{-1}(\underline{L}_d)[2n], \underline{L}_0)$ in the structural map

$$CF(\underline{L}_{d-2}, \underline{L}_{d-1}) \times \cdots \times CF(\underline{L}_0, \underline{L}_1) \times CF(\tau^{-1}(\underline{L}_d)[2n], \underline{L}_0) \to CF(\tau^{-1}(\underline{L}_d)[2n], \underline{L}_{d-1}).$$

The right side of Figure 3.2 gives a schematic indication of why such a bijection exists, in a simple case in which the Lagrangians are pairwise distinct (and the A_∞-products are governed by the same set of holomorphic curves projecting to the unique triangle in the base). Together, these claims imply that $CF(\tau^{-1}(\underline{L})[2n], -)^\vee$ is isomorphic as a right A_∞-module to $CF(-, \underline{L})$. In other words, on the object level, the global monodromy $\tau[-2n]$ sends $\underline{L}$ to $\tau(\underline{L})[-2n]$, whose Yoneda image is in turn isomorphic to $CF(\underline{L}, -)^\vee$. This is the first piece of geometric information that enters into Seidel's argument. ∎

Remark 3.7. There is an embedding from an appropriate Fukaya–Seidel category to the cylindrical version $\mathcal{FS}^{\mathrm{cyl},n}$ (see Proposition 4.12). Claim 3.6 is only used to prove that this embedding is essentially surjective for type A Milnor fibre (see Proposition 5.16), and to compare certain Lagrangian tuples with certain modules over the extended arc algebra in Section 10. It is not needed to derive any of the results mentioned in Section 1.

4. Comparing Fukaya–Seidel categories

In this section, we discuss the relation between $\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ and the Fukaya–Seidel category of the Lefschetz fibration on $\mathrm{Hilb}^n(E) \setminus D_r$ induced by π_E (we recall that this is a 'weak' Lefschetz fibration, in the sense that it may have critical points at infinity, but it still has a Fukaya–Seidel category of Lagrangians proper over the base, as constructed in [40]). This allows us to translate our subsequent study of $\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ back to the usual Fukaya–Seidel category.

4.1. *A directed subcategory*

Let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_m$ be the set of critical values of π_E. By applying a diffeomorphism of E covering a compactly supported diffeomorphism of $\mathbb{H}^\circ$ and using push-forward $J_{E^\urcorner}, \omega_{E^\urcorner}$, etc., we assume that $\boldsymbol{c}_k := k + \sqrt{-1} \in \mathbb{H}^\circ$ for $k = 1, \dots, m$. For simplicity, we assume that there is exactly one critical point lying above a critical value. We also assume that the symplectic parallel transport is well-defined everywhere. This holds for type A Milnor fibre (see [26] or the subsequent discussion in Section 5). In fact we only apply parallel transport to Lagrangians that are proper over $\mathbb{H}^\circ$ and one can avoid this hypothesis at the cost of having more notations.

A *matching path* $\gamma : [0,1] \to \mathbb{H}^\circ$ of $\pi_E : E \to \mathbb{H}^\circ$ is a smooth path from $\boldsymbol{c}_a$ to $\boldsymbol{c}_b$, for some $a, b \in \{1, \dots, m\}$ with $a \neq b$, such that $\gamma(t)$ is not a critical value of π_E for all $t \neq 0, 1$, and such that the vanishing cycles from the critical points lying above $\boldsymbol{c}_a$ to $\boldsymbol{c}_b$ match up under symplectic parallel transport along γ to give a Lagrangian matching sphere L_γ in E.

A *thimble path* $\gamma : [0,1] \to \mathbb{H}$ of $\pi_E : E \to \mathbb{H}^\circ$ is a smooth path from $\boldsymbol{c}_a$, for some $a \in \{1, \dots, m\}$, to a point on the real line such that $\gamma(t) \in \mathbb{H}^\circ \setminus \{\boldsymbol{c}_b \mid b = 1, \dots, m\}$ for $t \neq 0, 1$, and the symplectic parallel transport of the vanishing cycle from $\boldsymbol{c}_a$ gives a Lagrangian disc (thimble) L_γ in E.

Definition 4.1. For an n-tuple $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of pairwise disjoint embedded curves in $\mathbb{H}^\circ$ such that each curve is either a matching path or a thimble path of $\pi_E : E \to \mathbb{H}^\circ$, we can define the corresponding n-tuple of Lagrangians

$$\underline{L}_\Gamma := \{L_{\gamma_1}, \dots, L_{\gamma_n}\} \in \mathcal{L}^{\mathrm{cyl},n} \tag{4.1}$$

In this case, we call Γ an *admissible tuple*.

For $r \in \mathbb{R}$ and $k \in \{1, \dots, m\}$, let $l_{r,k}$ be the straight line joining r and $\boldsymbol{c}_k$. Every $l_{r,k}$ is a thimble path.

Let $\mathcal{I}$ be the set of cardinality n subsets of $\{1, \dots, m\}$. We define a partial ordering on $\mathcal{I}$, called the *Bruhat order*, as follows: For $I_0, I_1 \in \mathcal{I}$, $I_0 \leq I_1$ if and only if there is a bijection $f : I_0 \to I_1$ such that $x \leq f(x)$ for all $x \in I_0$. Strict inequality $I_0 < I_1$ is defined by $I_0 \leq I_1$ and $I_0 \neq I_1$.

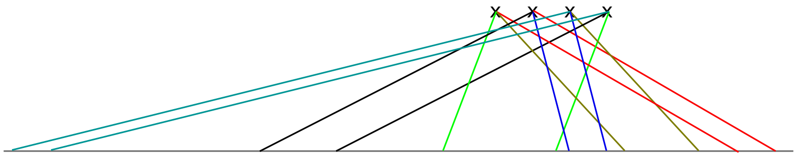


Fig. 4.1. The case when $s = 4$ and $n = 2$.

Let $f : \mathcal{I} \to \mathcal{I}_{\mathrm{aff}}$ (see Section 2.2.6) be a function such that

$$f(I_0) \cap f(I_1) = \emptyset \quad \text{for all } I_0 \neq I_1 \in \mathcal{I}, \tag{4.2}$$

$$f(I_0) > f(I_1) \quad \text{if } I_0 < I_1, \tag{4.3}$$

$$\mathrm{length}(f(I)) = m, \tag{4.4}$$

where $\mathrm{length}([a,b]) := b - a$.

Given f and $I = \{i_1 < \cdots < i_n\} \in \mathcal{I}$, we define

$$\gamma^{I,k} := l_{\min(f(I))+i_k, i_k} \tag{4.5}$$

so $\Gamma^I := \{\gamma^{I,k} \mid k = 1, \ldots, n\}$ is a collection of parallel lines in $\mathbb{H}^\circ$. For generic f, no three pairwise distinct lines in $\bigcup_{I \in \mathcal{I}} \Gamma^I$ intersect at the same point in $\{z \in \mathbb{H}^\circ \mid \mathrm{im}(z) < 1\}$ (see Figure 4.1).

Let $\underline{T}^I := \underline{L}_{\Gamma^I} \in \mathcal{L}^{\mathrm{cyl},n}$. For a generic perturbation of ω_E inside a compact subset, which changes the symplectic connection but keeps the symplectic Lefschetz fibration structure, we have

$$L_{\gamma^{I,k}} \pitchfork L_{\gamma^{I',k'}} \quad \text{for all } (I,k) \neq (I',k'). \tag{4.6}$$

Moreover, by applying a diffeomorphism of E covering a compactly supported diffeomorphism of $\mathbb{H}^\circ$ again (and using the push-forward $J_{E^\urcorner}$, $\omega_{E^\urcorner}$, etc.), we can assume that there exists $0 < \eta < 1$ such that π_E is symplectically locally trivial in $\pi_E^{-1}(\{z \in \mathbb{H}^\circ \mid \mathrm{im}(z) < 1 - \eta\})$ and for any $\gamma^{I,k} \neq \gamma^{I',k'} \in \Gamma$, we have

$$\gamma^{I,k} \cap \gamma^{I',k'} \subset \{z \in \mathbb{H}^\circ \mid \mathrm{im}(z) < 1 - \eta \text{ or } \mathrm{im}(z) = 1\}.$$

We are interested in the subcategory of $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ split-)generated by $\{\underline{T}^I \mid I \in \mathcal{I}\}$.

Lemma 4.2. *Let $I_0, I_1 \in \mathcal{I}$. Then $HF(\underline{T}^{I_0}, \underline{T}^{I_1}) \neq 0$ only if $I_0 < I_1$. Moreover, if $I_0 = I_1$, then $HF(\underline{T}^{I_0}, \underline{T}^{I_1})$ is generated by the identity element.*

Proof. For the first statement, it suffices to note that if $I_0 \not< I_1$ then we can find an isotopy of thimbles $\underline{T}_t^{I_0} \in \mathcal{L}^{\mathrm{cyl},n}$ such that $\underline{T}_0^{I_0} = \underline{T}^{I_0}$, $\lambda_{\underline{T}_1^{I_0}} > \lambda_{\underline{T}^{I_1}}$ and $\mathrm{Sym}(\underline{T}_1^{I_0}) \cap \mathrm{Sym}(\underline{T}^{I_1}) = \emptyset$ (see Figure 4.2). It implies that $CF(\underline{T}_1^{I_0}, \underline{T}^{I_1}) = HF(\underline{T}_1^{I_0}, \underline{T}^{I_1}) = 0$, but by Lemma 3.3, $\underline{T}_1^{I_0}$ is quasi-isomorphic to $\underline{T}^{I_0}$ as objects in $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$, so the first statement follows.

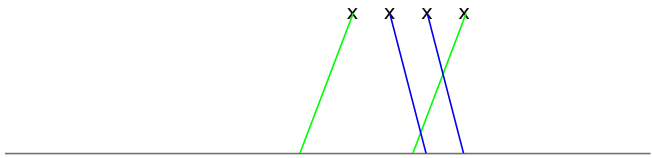


Fig. 4.2. $\mathrm{Sym}(\underline{T}_1^{I_0}) \cap \mathrm{Sym}(\underline{T}^{I_1}) = \emptyset$.

For the second one, when $I_0 = I_1 = I$, we can find an isotopy of thimbles $\underline{T}_t^I \in \mathcal{L}^{\mathrm{cyl},n}$ such that $\underline{T}_0^I = \underline{T}^I$, $\lambda_{\underline{T}_1^I} > \lambda_{\underline{T}^I}$ and $\mathrm{Sym}(\underline{T}_1^I) \cap \mathrm{Sym}(\underline{T}^I)$ is a singleton. Lemma 3.3 now implies that $HF(\underline{T}^I, \underline{T}^I)$ has rank 1, so is generated by the cohomological unit. ∎

Lemma 4.3. *Let $I_0 < I_1 < \cdots < I_d$. For the pairs $(\underline{T}^{I_0}, \underline{T}^{I_d})$, and $(\underline{T}^{I_{j-1}}, \underline{T}^{I_j})$ for $j = 1, \ldots, d$, we can choose Floer cochain data (A, H, J) such that $A = 0$ and $H \equiv 0$. Moreover, for $\underline{x}_0 \in \mathcal{X}(\underline{T}^{I_0}, \underline{T}^{I_d})$ and $\underline{x}_j \in \mathcal{X}(\underline{T}^{I_{j-1}}, \underline{T}^{I_j})$ for $j = 1, \ldots, d$, we can also choose the Floer data (A_S, K, J) such that $A_S = 0$ and $K \equiv 0$. In consequence, when $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \ldots, \underline{x}_1)$ has virtual dimension zero, its regularity can be achieved by generic J.*

Proof. First note that if $i < j < k$ and $\underline{x} = \{x_1, \ldots, x_n\} \in \mathrm{Sym}(\underline{T}^{I_i}) \cap \mathrm{Sym}(\underline{T}^{I_j}) \cap \mathrm{Sym}(\underline{T}^{I_k})$, then since there is no triple intersection in $\{z \in \mathbb{H}^\circ \mid \mathrm{im}(z) < 1\}$, it is necessary that for all $t = 1, \ldots, n$, $\pi_E(x_t) = \boldsymbol{c}_{l_t}$ for some $l_t \in \{1, \ldots, m\}$. This in turn implies that $i = j = k$, a contradiction. Therefore, if $i < j < k$ then $\mathrm{Sym}(\underline{T}^{I_i}) \cap \mathrm{Sym}(\underline{T}^{I_j}) \cap \mathrm{Sym}(\underline{T}^{I_k}) = \emptyset$.

We first discuss how to achieve regularity of elements in $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \ldots, \underline{x}_1)_{\mathrm{pre}}$. Let $u : S \to \mathrm{Hilb}^n(E)$ be an element in this space. Let $v : \Sigma \to E$ be the map associated to u by the tautological correspondence. Let $\Sigma_1, \ldots, \Sigma_m$ be the connected components of Σ and $v_j = v|_{\Sigma_j}$. By reordering $\Sigma_1, \ldots, \Sigma_m$ if necessary, we can assume that there is $0 \le a \le m$ such that $\pi_E \circ v_i$ is a constant for $i \le a$, and is a non-constant map otherwise.

By the boundary conditions, $\pi_\Sigma|_{\Sigma_i}$ must have degree 1 for $i \le a$, so $\Sigma_i = S$ is a disc with $d+1$ boundary punctures. We want to discuss the Fredholm operator associated to $v_i : S \to E$ for $i \le a$. Let us assume $a \neq 0$ and consider v_1. We denote the Lagrangian boundary label on $\partial\Sigma_1$ by $L_0, \ldots, L_d$ so that $L_j \in \underline{T}^{I_j}$ for $j = 0, \ldots, d$. The map $\pi_E \circ v_1$ is either a constant map to a point in $\{z \in \mathbb{H}^\circ \mid \mathrm{im}(z) < 1 - \eta\}$ or to a point $\boldsymbol{c}_k$ for some k.

We first suppose that $\pi_E \circ v_1$ is a constant map to a point in $\{z \in \mathbb{H}^\circ \mid \mathrm{im}(z) < 1 - \eta\}$. Then we must have $d = 1$ because no three pairwise distinct lines in Γ intersect at the same point in $\{z \in \mathbb{H}^\circ \mid \mathrm{im}(z) < 1\}$. If we view $\pi_E \circ v_1$ as a holomorphic map with boundary on $\pi_E(L_j)$ and both asymptotic conditions are given by uniform convergence to the intersection point, then $\pi_E \circ v_1$ is a regular rigid solution because the input and output are the same (so have the same gradings). The cokernel of the Fredholm operator D_{v_1}

of v_1 sits inside a short exact sequence

$$\operatorname{coker}(D_{\text{fibre}}) \to \operatorname{coker}(D_{v_1}) \to \operatorname{coker}(D_{\pi_E \circ v_1}) \to 0 \tag{4.7}$$

where $D_{\pi_E \circ v_1}$ is the Fredholm operator for $\pi_E \circ v_1$ and D_{fibre} is the Fredholm operator of v_1 viewed as a map to the fibre. By regularity of $\pi_E \circ v_1$, we have $\operatorname{coker}(D_{\pi_E \circ v_1}) = 0$. On the other hand, depending on the virtual dimension of v_1, either $\operatorname{coker}(D_{\text{fibre}})$ can also be made 0 by a generic choice of J, or v_1 does not exist for generic J. In the former case, D_{v_1} is surjective.

Next, we consider the case that $\pi_E \circ v_1$ is a constant map to a point $\boldsymbol{c}_k$ for some k. In this case, d is not necessarily 1 but, by (4.3), we have

$$\lambda_{L_0} > \cdots > \lambda_{L_d}. \tag{4.8}$$

We can assume that $L_i \cap L_j = p$ for all i, j, where p is the critical point lying above $\boldsymbol{c}_k$. Let B_p be a Darboux ball centred at p. We can assume that J is integrable near p so that B_p can be identified with a ball in $\mathbb{C}^2$. Moreover, by (4.8), we can assume that $T_pL_i \cap T_pB_p$ are pairwise transversally intersecting Lagrangian planes in T_pB_p with strictly decreasing Kähler angles. More explicitly, a local model is given by

$$\pi_E|_{B_p}(z_1, z_2) = z_1 z_2, \quad z_1, z_2 \in \mathbb{C}, \tag{4.9}$$

$$L_i \cap B_p = \{(z_1, z_2) = (re^{i\theta_i + t}, re^{i\theta_i - t}) \in B_p \mid r \geq 0,\ t \in [0, 2\pi]\} \text{ for some } \theta_i \in [0, 2\pi), \tag{4.10}$$

and (4.8) translates to $\theta_0 > \cdots > \theta_d$. As a result, there are choices of grading functions on $\{L_i\}_{i=0}^d$ such that for all i, the point p as a generator of $CF(L_{i-1}, L_i)$ has grading 0 (when $i = 0$, $CF(L_{i-1}, L_i)$ should be understood as $CF(L_0, L_d)$). Therefore, for a fixed $S \in \mathcal{R}^{d+1}$, the moduli space of solutions to the equation

$$\begin{cases} w : S \to E, \\ (Dw)^{0,1} = 0, \\ w(\partial_j S) \subset L_j \text{ for all } j, \\ \lim_{s \to \pm\infty} w(\epsilon_j(s, \cdot)) = p \text{ uniformly for all } j, \end{cases} \tag{4.11}$$

has virtual dimension 0. Moreover, the constant map from S to p is regular and rigid. On the other hand, v_1 must be the constant map from S to p, so v_1 is regular.

We can now address the regularity of u. Recall that the key point is to show that if η is an element in an appropriate Sobolev completion of $\Omega^{0,1}(S, u^*T\operatorname{Hilb}^n(E))$ which annihilates the image of the Fredholm operator associated to u, then η vanishes identically. Since η lies inside the kernel of the adjoint operator, η has the unique continuation property, so it suffices to show that η vanishes on an open subset G of S.

Let $z \in S \setminus \nu(\operatorname{mk}(S))$ be such that $u(z) \in \operatorname{Conf}^n(E)$. Let G be an open neighbourhood of z such that $\pi_\Sigma^{-1}(G)$ consists of n disjoint open sets $G_1, \ldots, G_n$. There is a neighbourhood U of $u(z)$ that is symplectomorphic to a product $U_1 \times \cdots \times U_n$ for open sets $U_i \subset E$ satisfying $U_i \cap U_j = \emptyset$ if $i \neq j$. Moreover, we can assume the image of $v|_{G_i}$ lies inside U_i, so we have $u|_G(z) = (v|_{G_1}(z), \dots, v|_{G_n}(z))$ under the identifications between U and $U_1 \times \cdots \times U_n$, and between G and G_i. Having this product type local model for $u|_G$, we can write $\eta|_G$ as $(\eta_1, \dots, \eta_n)$, where $\eta_i \in \Omega^{0,1}(G_i, v|_{G_i}^* TU_i)$. If $\eta \neq 0$, then at least one η_i is not 0; relabel it as η_1.

If $\pi_E \circ v|_{G_1}$ is not a constant map, then there is an infinitesimal deformation Y of J in the space of almost complex structures satisfying (2.48) (which only deforms in the first factor of the product $U_1 \times \cdots \times U_n$) such that

$$\int_G \langle \eta, Y \circ (du - X_K) \circ j_S \rangle = \int_{G_1} \langle \eta_1, Y \circ (dv_1 - X_{\pi_\Sigma^* K}) \circ j_S \rangle \neq 0 \tag{4.12}$$

where j_S is the complex structure on S. That contradicts the assumption that η annihilates the image of the Fredholm operator associated to u.

If $\pi_E \circ v|_{G_1}$ is a constant, then we must have $G_1 \subset \Sigma_i$ for some $i \leq a$. Without loss of generality, we assume $G_1 \subset \Sigma_1$ and $v|_{G_1} = v_1|_{G_1}$. From the discussion of the surjectivity of the Fredholm operator associated to v_1 above, we know that there exists an infinitesimal deformation Y of J supported in G_1, and an element ξ in an appropriate Sobolev completion of $C^\infty(S, v_1^* TE)$ supported in G_1 such that

$$\int_{G_1} \big\langle \eta_1, D_{v_1}\xi + \tfrac{1}{2} Y \circ (dv_1 - X_{\pi_\Sigma^* K}) \circ j_S \big\rangle \neq 0. \tag{4.13}$$

Therefore, if we identify G_1 with G and think of $v_1|_{G_1}^* TE = v_1|_{G_1}^* TU_1$ as a component of $u|_G^* TU$, then we have

$$\int_G \big\langle \eta, D_u\xi + \tfrac{1}{2} Y \circ (du - X_K) \circ j_S \big\rangle \neq 0. \tag{4.14}$$

Therefore, $\eta = 0$ and hence the regularity of u can be achieved by generic J.

This proves the regularity of elements in $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)_{\text{pre}}$. The transversality between the evaluation map $\mathcal{R}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)_{\text{pre}} \to (\mathrm{Hilb}^n(E))^h$ and a pseudocycle representing the inclusion $(D_{HC})^h \to (\mathrm{Hilb}^n(E))^h$ can be addressed by the same reasoning. This finishes the proof. ■

By Lemma 4.3, we can define the A_∞ operations

$$CF(\underline{T}^{I_{d-1}}, \underline{T}^{I_d}) \times \cdots \times CF(\underline{T}^{I_0}, \underline{T}^{I_1}) \to CF(\underline{T}^{I_0}, \underline{T}^{I_d}) \tag{4.15}$$

without introducing a Hamiltonian term K, whenever $I_0 < I_1 < \cdots < I_d$. In a standard way, we can extend the A_∞ operations by adding idempotents.

Corollary 4.4. *Let $R = \bigoplus_{I \in \mathcal{I}} \mathbb{K} e_I$ and $e_I^2 = e_I$. We have a strictly unital A_∞ algebra*

$$R \oplus \bigoplus_{I_0 < I_1} CF(\underline{T}^{I_0}, \underline{T}^{I_1}) \tag{4.16}$$

with unit $\sum_{I \in \mathcal{I}} e_I$.

Proof. See [42, Section 7] for how the A_∞ structure is defined after adjoining the strict units e_I of the idempotents. ∎

Corollary 4.5. *The A_∞ algebra (4.16) is quasi-isomorphic to* $\mathrm{End}_{\mathcal{FS}^{\mathrm{cyl},n}}(\bigoplus_{I\in\mathcal{I}} \underline{T}^I)$.

Proof. This follows from Lemmas 4.2 and 4.3, Corollary 4.4 and homological perturbation. ∎

4.2. Fukaya–Seidel embeds into cylindrical Fukaya–Seidel

In this section, we show that

$$\pi_{\mathcal{Y}_E} := \pi_E^{[n]}|_{\mathcal{Y}_E} : \mathcal{Y}_E := \mathrm{Hilb}^n(E) \setminus D_r \to \mathbb{C} \tag{4.17}$$

is a Lefschetz fibration, and identify the subcategory of the associated Fukaya–Seidel category $D^\pi \mathcal{FS}(\pi_{\mathcal{Y}_E})$ generated by thimbles with the category of perfect modules over the A_∞-algebra (4.16).

Lemma 4.6. *If $f : \mathbb{C}^2 \to \mathbb{C}$ is a holomorphic map without critical points, then so is the induced map $f^{[n]} : \mathrm{Hilb}^n(\mathbb{C}^2) \to \mathbb{C}$.*

Proof. Without loss of generality, it suffices to show that $f^{[n]}$ is regular at a 0-dimensional length n subscheme z supported at 0. We can also assume that $f(x, y) = x$.

Let z_t be a family of 0-dimensional length n subschemes such that $z_0 = z$ and z_t is a length n subscheme supported at $(t, 0)$. Then $f^{[n]}(z_t) = nt$ so $f^{[n]}$ is regular at 0. ∎

Corollary 4.7. *Every critical point of $\pi_E^{[n]} : \mathrm{Hilb}^n(E) \to \mathbb{H}^\circ$ has support lying inside the union of critical points of π_E.*

Proof. If the support of $z \in \mathrm{Hilb}^n(E)$ contains a point p that is not a critical point of π_E, then we can apply Lemma 4.6 locally near p to show that z is not a critical point of $\pi_E^{[n]}$. ∎

If z is a critical point of $\pi_E^{[n]}$, consider whether the support is a disjoint union of n points or not. We consider the former case first. For $I \in \mathcal{I}$, we use $z_I \in \mathrm{Conf}^n(E) \subset \mathrm{Hilb}^n(E)$ to denote the subscheme with support being the union of the critical points lying above $\{\boldsymbol{c}_i \mid i \in I\}$.

Lemma 4.8 ([5, Proposition 2.1 and Lemma 2.2]). *For each $I \in \mathcal{I}$, the point z_I is a Lefschetz critical point of $\pi_E^{[n]}$ and $\mathrm{Sym}(\underline{T}^I)$ is a Lefschetz thimble.*

Proof. Since z_I consists of pairwise distinct points, near z_I, $\pi_E^{[n]}$ is locally given by $(u_1, v_1, \dots, u_n, v_n) \mapsto u_1^2 + v_1^2 + \cdots + u_n^2 + v_n^2$. Therefore, it admits a Lefschetz critical point at z_I.

Since $\{\pi_E(T^{I,k})\}_{k=1}^n$ are parallel lines, $\pi_E^{[n]}(\mathrm{Sym}(\underline{T}^I))$ is also a straight line so it is a Lefschetz thimble. ∎

For the other case, we have a local lemma.

Lemma 4.9. *Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be $\pi(u, v) = u^2 + v^2$. For $n \geq 2$ and $z \in \mathrm{Hilb}^n(\mathbb{C}^2)$ supported at the origin of $\mathbb{C}^2$, the length of the projection of z by π is strictly less than n.*

Proof. Let $\mathbb{C} = \mathrm{Spec}(\mathbb{C}[w])$ and $f : \mathbb{C}[w] \to \mathbb{C}[u, v]$ be the algebra homomorphism sending w to $u^2 + v^2$. Let I be a length n ideal of $\mathbb{C}[u, v]$ supported at the origin, which corresponds to the point z in the statement. We want to show that the ideal $f^{-1}(I)$ in $\mathbb{C}[w]$ has length strictly less than n.

Since $\dim_{\mathbb{C}}(\mathbb{C}[u, v]/I) = n$, we know that $u^a v^b \in I$ if $a, b, \geq 0$ and $a + b \geq n$. It means that when n is even (resp. odd), we have $(u^2 + v^2)^{n/2} \in I$ (resp. $(u^2 + v^2)^{(n+1)/2} \in I$). Therefore, $z^{n/2} \in f^{-1}(I)$ when n is even (resp. $z^{(n+1)/2} \in f^{-1}(I)$ when n is odd), which in turn implies that $f^{-1}(I)$ has length no greater than $(n + 1)/2$. ∎

Corollary 4.10. *The critical points of* (4.17) *are precisely $\{z_I\}_{I \in \mathcal{I}}$ and they are Lefschetz.*

Proof. By Lemma 4.9, if the support of z has multiplicity 2 at some critical point of π_E, then $z \in D_r$ which is not in $\mathcal{Y}_E = \mathrm{Hilb}^n(E) \setminus D_r$. The result then follows from Corollary 4.7 and Lemma 4.8. ∎

Remark 4.11. Even though (4.17) has only Lefschetz critical points, it is in general not symplectically locally trivial near the horizontal boundary, because there are critical points of $\pi_E^{[n]} : \mathrm{Hilb}^n(E) \to \mathbb{C}$ lying in D_r (when $n > 1$ so $D_r \neq \emptyset$).

In light of Remark 4.11, we should clarify what we mean by $\mathcal{FS}(\pi_{\mathcal{Y}})$. Objects of the Fukaya–Seidel category $\mathcal{FS}(\pi_{\mathcal{Y}})$ are restricted to be Lefschetz thimbles $\mathrm{Sym}(\underline{T}^I)$ for $I \in \mathcal{I}$. As in Corollary 4.5 (see [42, Section 7]), the A_∞ endomorphism algebra of the direct sum of the objects is defined to be

$$R \oplus \bigoplus_{I_0 < I_1} CF(\mathrm{Sym}(\underline{T}^{I_0}), \mathrm{Sym}(\underline{T}^{I_1})) \tag{4.18}$$

where the Floer cochains are taken in $\mathrm{Hilb}^n(E) \setminus D_r$ and no Hamiltonian perturbation is put on the Floer equations defining the A_∞ structure. As in [42], this definition is quasi-isomorphic to the directed subcategory of the vanishing cycles in the distinguished fibre.

Proposition 4.12. *There is a quasi-isomorphism between* (4.16) *and* (4.18)*. As a result, there is a cohomologically full and faithful embedding $D^\pi \mathcal{FS}(\pi_{\mathcal{Y}}) \to D^\pi \mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$.*

Proof. There is an obvious bijective correspondence of objects and generators between (4.16) and (4.18). Note that all pseudo-holomorphic maps involved in defining the μ^d operations in (4.16) are contained in $\mathrm{Hilb}^n(E) \setminus D_r$.

For the definition of the μ^d operations in (4.18), we can use moduli of pseudo-holomorphic maps with no Hamiltonian perturbation term as in (4.16) (see Lemma 4.3) but we need to use the domain moduli $\mathcal{R}^{d+1}$ instead of $\mathcal{R}^{d+1,h}$. It means that we equip $\bigcup_d \overline{\mathcal{R}}^{d+1}$ with a consistent choice of domain-dependent almost complex structures that are equal to $J_E^{[n]}$ outside a compact subset of $\mathrm{Hilb}^n(E) \setminus D_r$ and generic inside the compact subset, and we count the corresponding moduli of maps to define the μ^d operations in (4.18).

Given the difference of domain moduli for (4.18) and in (4.16), we cannot directly compare the μ^d operations. The standard method to circumvent this is to use the 'total Fukaya category' trick from [40, Section 10a].

Briefly, one can show that μ^d operations in (4.16) yield a category quasi-isomorphic to another one in which the operations $\tilde{\mu}^d$ are defined by the domain moduli $\bigcup_{d,h} \overline{\mathcal{R}}^{d+1,h}$ but with the domain dependent J chosen *independent* of the interior marked points, equal to $J_E^{[n]}$ outside a compact subset of $\mathrm{Hilb}^n(E) \setminus (D_{HC} \cup D_r)$ and generic inside a compact subset. Moreover, for generic J, we can assume that the universal evaluation map to $(\mathrm{Hilb}^n(E))^h$ is transverse to D_{HC}^h. With these data, $\tilde{\mu}^d$ is defined by counting the corresponding moduli of maps u such that $u(z) \in D_{HC}$ for $z \in \mathrm{mk}(S)$. In this modification, the interior marked points are merely decorative, and one can canonically identify the moduli spaces with the ones defining the μ^d operations in (4.18). ■

It is natural to ask when the thimbles $\{\underline{T}^I\}_{I \in \mathcal{I}}$ split-generate $D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$. We next show that this holds when π_E is the standard Lefschetz fibration on the type A Milnor fibre.

5. Type A geometry

In this section, we apply the results in Sections 2–4 to the case in which $E = A_{m-1}$ is a type A Milnor fibre. When $2n \leq m$, the corresponding $\mathcal{Y}_E = \mathrm{Hilb}^n(E) \setminus D_r$ is the generic fibre of the adjoint quotient map restricted to a nilpotent (or Slodowy) slice associated to a nilpotent with two Jordan blocks, as studied in [32, 47] in the context of symplectic Khovanov cohomology. At the end of this section, we show that if Claim 3.6 holds, then the embedding in Proposition 4.12 is essentially surjective in this case.

5.1. A_{m-1} *Milnor fibres*

We recall the symplectic geometry of type A Milnor fibres, with an emphasis on verifying the assumptions made in the set-up of Section 2. The following model is a slight modification of the one in [19, Section 7]. Let $m > 1$ be an integer. Let

$$X := \mathbb{C} \times \mathbb{CP}^1 \times \prod_{i=1}^{m} \mathbb{CP}^1_i \tag{5.1}$$

with coordinates $(x, [Y_0 : Y_1], [a_1, b_1], \ldots, [a_m, b_m])$. Let M be the subvariety given by the equations

$$a_i Y_0 = b_i Y_1 (x - i) \quad \text{for } i = 1, \ldots, m. \tag{5.2}$$

Let $\pi_M : M \to \mathbb{C}$ be the projection to the x coordinate. For $x \neq 1, \ldots, m$, the fibre at x is given by

$$P_x := \pi_E^{-1}(x) = \{(x, [Y_0 : Y_1], [Y_1(x-1) : Y_0], \ldots, [Y_1(x-m) : Y_0])\}, \tag{5.3}$$

which is a smooth rational curve. For $x = i \in \{1, \dots, m\}$,

$$P_i := \pi_E^{-1}(i) = \{(i, [Y_0 : Y_1], [Y_1(i-1) : Y_0], \dots, [Y_1(i-m) : Y_0]) \mid Y_0 \neq 0\} \\ \cup \{(i, [0:1], [1:0], \dots, [a_i : b_i], \dots, [1:0]) \mid [a_i : b_i] \in \mathbb{CP}^1_i\}$$

is a union of two irreducible smooth rational curves. One can check that π_M is a Lefschetz fibration. Consider the following sections

$$D_\infty := \{(x, [1:0], [0:1], \dots, [0:1]) \mid x \in \mathbb{C}\}, \tag{5.4}$$

$$D_0 := \{(x, [0:1], [1:0], \dots, [1:0]) \mid x \in \mathbb{C}\}, \tag{5.5}$$

and define $D := D_0 \cup D_\infty$.

Lemma 5.1 (cf. [19, Lemma 7.1]). *$M \setminus D$ is biholomorphic to $\{a^2 + b^2 + (c-1) \cdots \cdot (c-m) = 0\}$.*

Proof. For a dense subset, the identification is given by

$$(a, b, c) \mapsto (c, [a + \sqrt{-1}\, b : 1], [c-1 : a + \sqrt{-1}\, b], \dots, [c-m : a + \sqrt{-1}\, b]). \tag{5.6}$$

We leave the rest to the reader. ∎

We call a smooth affine variety of the form $\{a^2 + b^2 + (c-c_1) \cdots (c-c_m) = 0\}$ with pairwise distinct $c_1, \dots, c_m \in \mathbb{C}$ an A_{m-1} *Milnor fibre*. In particular, $M \setminus D$ is an A_{m-1} Milnor fibre by Lemma 5.1.

Remark 5.2. By projecting to the c coordinate, $\{a^2 + b^2 + (c-c_1) \cdots (c-c_m) = 0\} \subset \mathbb{C}^3$ is the total space of a Lefschetz fibration with general fibre $\mathbb{C}^*$ and with m nodal fibres (over the c_i). Since we are primarily interested in the symplectic geometry and not complex geometry of $M \setminus D$, we will also refer to any such Lefschetz fibration as an A_{m-1} Milnor fibre; in particular, the restriction of this Lefschetz fibration of $M \setminus D$ to a disc in the c-plane containing all the critical values is a Milnor fibre.

Lemma 5.3 (cf. [19, Lemma 7.2]). *Let ω_X be a product symplectic form on X that tames the complex structure. Then $\omega_M := \omega_X|_M$ is a symplectic form and P_x is ω_M-orthogonal to D for all x.*

Proof. Both assertions are clear. For the second one, observe that TD and TP_x lie in the first and second factor of $T\mathbb{C} \oplus (T(\mathbb{CP}^1)^{m+1}) = TX$, respectively. ∎

Let ω_X be the standard symplectic form on X and define $\omega_M := \omega_X|_M$. For $R > 0$, let $B_R \subset \mathbb{C}$ be the open disc of radius R and $M_R = \pi_M^{-1}(B_R)$. Let $\pi_{M_R} : \frac{1}{R}\pi_M|_{M_R} : M_R \to B_1$, which is still a holomorphic Lefschetz fibration, and a symplectic Lefschetz fibration with respect to $\omega_{M_R} := \omega_M|_{M_R}$. We equip the base unit disc B_1 with the hyperbolic area form ω_{hyp}.

Lemma 5.4. *Let $R > m$. There exists a symplectic form ω_R on M_R and a compact subset $C_B \subset B_1$ such that*

$$\omega_R \text{ tames the complex structure}, \tag{5.7}$$

$$\omega_R \text{ is symplectically locally trivial in } \pi_{M_R}^{-1}(B_1 \setminus C_B), \tag{5.8}$$

$$D \cap M_R \text{ is } \omega_R\text{-symplectic-orthogonal to } \pi_{M_R}^{-1}(x) \text{ for all } x \in B_1 \setminus C_B. \tag{5.9}$$

Proof. For $\epsilon > 0$ small, we have a holomorphic embedding

$$\Phi : (B_1 \setminus B_{1-\epsilon}) \times \mathbb{CP}^1 \to M_R,$$
$$(x, [Y_0 : Y_1]) \mapsto (Rx, [Y_0 : Y_1], [Y_1(x-1) : Y_0], \ldots, [Y_1(x-m) : Y_0]),$$

which is compatible with the projection to $B_1 \setminus B_{1-\epsilon}$.

Let $g : (1-\epsilon, 1] \to (1-\epsilon, 1]$ be an increasing function such that there exists $\delta > 0$ such that $g(r) = r$ for r near $1-\epsilon$, g is strictly increasing when $r \in (1-\epsilon, 1-\delta)$, and $g(r) = 1$ for $r \geq 1-\delta$. It induces a smooth map $\phi_g : M_R \to M_R$ such that

$$\phi_g(z) := z \quad \text{for } z \notin \mathrm{Im}(\Phi), \tag{5.10}$$

$$\phi_g(\Phi(re^{\sqrt{-1}\theta}, y)) := \Phi(g(r)e^{\sqrt{-1}\theta}, y), \text{ where } (x, [Y_0 : Y_1]) = (re^{\sqrt{-1}\theta}, y). \tag{5.11}$$

Then $\phi_g^* \omega_{M_R}$ is a closed 2-form, non-degenerate in $\pi_{M_R}^{-1}(B_{1-\epsilon})$ and fibrewise symplectic.

A direct calculation shows that there is $A > 0$ such that for all $a > A$, $\frac{1}{a}\phi_g^*\omega_{M_R} + \pi_{M_R}^* \omega_{\mathrm{hyp}}$ is a symplectic form that satisfies (5.7)–(5.9) for $C_B := \overline{B_{1-\delta}}$. ■

Now, we give a dictionary to the set-up in Section 2.2.4. Let $f : \overline{B}_1 \to \mathbb{H}$ be a hyperbolic isometry. Let

$$\begin{aligned} E^{\urcorner} &:= \overline{M}_R, \quad \pi_{E^{\urcorner}} := f \circ \frac{1}{R}\pi_M|_{\overline{M}_R}, \quad D_E := D \cap \overline{M}_R, \\ \overline{E} &:= M_R, \quad \pi_{\overline{E}} := f \circ \pi_{M_R}, \quad \omega_{\overline{E}} := \omega_R, \quad C_{\mathbb{H}} := f(C_B). \end{aligned} \tag{5.12}$$

It is straightforward to check that the assumptions made in Section 2.2.4 are satisfied. Most notably, $\omega_{\overline{E}}$ tames the complex structure, $\pi_{\overline{E}}|_{\pi_{\overline{E}}^{-1}(\mathbb{H}^\circ \setminus C_{\mathbb{H}})}$ is symplectically locally trivial and every holomorphic map $\mathbb{CP}^1 \to E^{\urcorner}$ has positive algebraic intersection number with D_E.

Remark 5.5. There is an additional feature in this setting that is not assumed in Section 2.2.4, namely, for all $x \in \pi_{\overline{E}}|_{\pi_{\overline{E}}^{-1}(\mathbb{H}^\circ \setminus C_{\mathbb{H}})}$, the complex structure at x respects the symplectic decomposition $T_x\overline{E} = T_x^v\overline{E} \oplus T_x^h\overline{E}$.

5.2. Nilpotent slices

Let $m, n \in \mathbb{N}$ be such that $2n \leq m$. Let $G = \mathrm{GL}_m(\mathbb{C})$ and $\mathfrak{g}$ be its Lie algebra. The adjoint quotient map $\chi : \mathfrak{g} \to \mathfrak{h}/W = \mathrm{Sym}^m(\mathbb{C})$ take an element $A \in \mathfrak{g}$ to the coefficients of its characteristic polynomial, i.e. the symmetric functions of the eigenvalues (with multiplicities) of A. The set of regular values is given by $\mathrm{Conf}^m(\mathbb{C}) \subset \mathrm{Sym}^m(\mathbb{C}) = \mathbb{C}^m$. Let $S_{n,m} \subset \mathfrak{g}$ be the affine subspace consisting of the matrices

$$A = \begin{pmatrix} a_1 & 1 & & & b_1 & & & & \\ a_2 & 0 & 1 & & b_2 & & & & \\ \dots & & \dots & & \dots & & & & \\ a_{n-1} & & & 1 & b_{n-1} & & & & \\ a_n & & & 0 & b_n & & & & \\ 0 & & & & d_1 & 1 & & & \\ \dots & & & & d_2 & 0 & 1 & & \\ 0 & & & & \dots & & & \dots & \\ c_1 & & & & \dots & & & \dots & \\ \dots & & & & d_{m-n-1} & & & & 1 \\ c_n & & & & d_{m-n} & & & & 0 \end{pmatrix}$$

such that $a_i, b_i, c_i, d_i \in \mathbb{C}$ for all i. Note that, for the first column of A to be well-defined, we used $2n \leq m$. The affine subspace $S_{n,m}$ is a nilpotent (Slodowy) slice of the nilpotent element of Jordan type $(n, m-n)$. The restriction $\chi|_{S_{n,m}} : S_{n,m} \to \mathrm{Sym}^m(\mathbb{C})$ admits a simultaneous resolution by Grothendieck (see [50], [14]), and $\chi|_{S_{n,m}}^{-1}(\mathrm{Conf}^m(\mathbb{C})) \to \mathrm{Conf}^m(\mathbb{C})$ is a differentiable fibre bundle. For $\tau \in \mathrm{Conf}^m(\mathbb{C})$, we define

$$\mathcal{Y}_{n,\tau} := \chi|_{S_{n,m}}^{-1}(\tau). \tag{5.13}$$

Example 5.6. When $n = 1$,

$$\mathcal{Y}_{1,\tau} = \{(b, c, z) \in \mathbb{C}^3 \mid bc + P_\tau(z) = 0\} \tag{5.14}$$

where $P_\tau(t)$ is the degree m monic polynomial with roots given by elements in τ. Note that if $\tau = \{1, \dots, m\}$, then $\mathcal{Y}_{1,\tau}$ is biholomorphic to $\{a^2 + b^2 + (c-1)\dots(c-m) = 0\}$ (cf. Lemma 5.1).

We denote the projection to the z coordinate by $\pi_{1,\tau} : \mathcal{Y}_{1,\tau} \to \mathbb{C}$.

Lemma 5.7 ([31]). *When $2n \leq m$, there is a holomorphic open embedding $j : \mathcal{Y}_{n,\tau} \to \mathrm{Hilb}^n(\mathcal{Y}_{1,\tau})$. The complement of the image is the relative Hilbert scheme of the projection $\pi_{1,\tau}$, i.e. the subschemes whose projection to $\mathbb{C}_z$ does not have length n.*

Proof. Although Manolescu [31] only considers the case m even, the same proof goes through in general. More precisely, for $A \in S_{n,m}$, it is easy to check that

$$\det(tI - A) = A(t)D(t) - B(t)C(t) \tag{5.15}$$

where

$$\begin{aligned} A(t) &= t^n - a_1 t^{n-1} + \dots + (-1)^n a_n, \\ B(t) &= b_1 t^{n-1} - b_2 t^{n-2} + \dots + (-1)^{n-1} b_n, \\ C(t) &= c_1 t^{n-1} - c_2 t^{n-2} + \dots + (-1)^{n-1} c_n, \\ D(t) &= t^{m-n} - d_1 t^{m-n-1} + \dots + (-1)^{m-n} d_{m-n}. \end{aligned}$$

Points in $\mathcal{Y}_{n,\tau}$ are identified with 4-tuples of polynomials $(A(t), B(t), C(t), D(t))$ in $\mathbb{C}[t]$ such that $A(t)D(t) - B(t)C(t) = P_\tau(t)$. Points in $\mathrm{Hilb}^n(\mathcal{Y}_{1,\tau})$ are identified with ideals $\mathcal{I}$ in $\mathcal{O} := \mathbb{C}[b,c,z]/(bc + P_\tau(z))$ such that $\dim \mathcal{O}/\mathcal{I} = n$. The holomorphic embedding $j : \mathcal{Y}_{n,\tau} \to \mathrm{Hilb}^n(\mathcal{Y}_{1,\tau})$ is given by

$$j(A(t), B(t), C(t), D(t)) = \{Q(b,c,z) \mid A(t) \text{ divides } Q(B(t), C(t), t)\}. \qquad \blacksquare$$

Let $\tau = \{1, \dots, m\}$ and identify $\mathcal{Y}_{1,\tau}$ and $\pi_{1,\tau}$ with M and π_M from Section 5.1, respectively. For $E^\rceil = \overline{M}_R$ as in the dictionary (5.12), we know that, by Lemma 5.7, $\mathcal{Y}_{n,m} := \mathrm{Hilb}^n(E) \setminus D_r$ is an open subset of $\mathcal{Y}_{n,\tau}$ when $2n \le m$. Moreover, $\mathcal{Y}_{n,m}$ exhausts $\mathcal{Y}_{n,\tau}$ as R goes to infinity.

Lemma 5.8. *When $2n \le m$, the map $\pi_{1,\tau}^{[n]}|_{j(\mathcal{Y}_{n,\tau})}$ is given by sending a matrix $A \in \mathcal{Y}_{n,\tau}$ to the top left entry a_1.*

Proof. By [31, Remark 2.8], the composition

$$\mathcal{Y}_{n,\tau} \hookrightarrow \mathrm{Hilb}^n(\mathcal{Y}_{1,\tau}) \xrightarrow{\pi_{HC}} \mathrm{Sym}^n(\mathcal{Y}_{1,\tau}) \xrightarrow{\mathrm{Sym}^n(\pi_E)} \mathrm{Sym}^n(\mathbb{C}) \tag{5.16}$$

is given by the roots of $A(t)$ (in the notation of the proof of Lemma 5.7). Therefore the map $\pi_{1,\tau}^{[n]} \circ j : \mathcal{Y}_{n,\tau} \to \mathbb{C}$ is the sum of the roots of $A(t)$, which is a_1. $\blacksquare$

5.3. Sliding invariance

If $\underline{L}_0$ and $\underline{L}_1$ are not connected by a path $\underline{L}_t \in \mathcal{L}^{\mathrm{cyl},n}$, then the quasi-isomorphism types of $\underline{L}_0$ and $\underline{L}_1$ in $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ are in general different. However, we present three special cases where the quasi-isomorphism type is unchanged by a move not arising from an isotopy through admissible tuples (recall Definition 4.1).

The first case is the analogue of [47, Lemma 49].

Lemma 5.9. *Let $\Gamma := \{\gamma_1, \dots, \gamma_n\}$ be an admissible tuple. Suppose γ_i and γ_j are matching paths and $i \neq j$. Let γ' be the matching path obtained by sliding γ_i across γ_j. If Γ' is the admissible tuple obtained by replacing γ_i by γ', then $\underline{L}_\Gamma$ is quasi-isomorphic to $\underline{L}_{\Gamma'}$.*

Proof. First consider the case that $n = 2$, $i = 1$, $j = 2$ and $m \ge 2n$. In this case, it is proved in [47, Lemma 49] that $\mathrm{Sym}(\underline{L}_\Gamma)$ and $\mathrm{Sym}(\underline{L}_{\Gamma'})$ are Hamiltonian isotopic in $\mathcal{Y}_{n,m}$ (they treat the case $m = 2n$ but the proof works for all $m \ge 2n$). In particular, we can find $a \in HF(\mathrm{Sym}(\underline{L}_\Gamma), \mathrm{Sym}(\underline{L}_{\Gamma'}))$ and $b \in HF(\mathrm{Sym}(\underline{L}_{\Gamma'}), \mathrm{Sym}(\underline{L}_\Gamma))$ such that

$$\mu^2(b,a) = 1_{HF(\mathrm{Sym}(\underline{L}_\Gamma),\mathrm{Sym}(\underline{L}_\Gamma))} \quad \text{and} \quad \mu^2(a,b) = 1_{HF(\mathrm{Sym}(\underline{L}_{\Gamma'}),\mathrm{Sym}(\underline{L}_{\Gamma'}))}. \tag{5.17}$$

As in the proof of Proposition 4.12, when we turn off the Hamiltonian perturbation, there is a bijective correspondence between the J-holomorphic curves in $\mathcal{Y}_{n,m}$ and the corresponding curves in E. In particular, we can find $a' \in HF(\underline{L}_\Gamma, \underline{L}_{\Gamma'})$ and $b' \in HF(\underline{L}_{\Gamma'}, \underline{L}_\Gamma)$ such that

$$\mu^2(b',a') = 1_{HF(\underline{L}_\Gamma,\underline{L}_\Gamma)} \quad \text{and} \quad \mu^2(a',b') = 1_{HF(\underline{L}_{\Gamma'},\underline{L}_{\Gamma'})}, \tag{5.18}$$

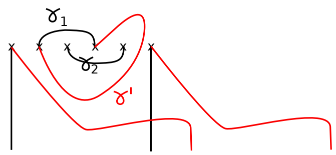


Fig. 5.1. Floer cochain $CF(\underline{L}_{\Gamma'}, \underline{L}_{\Gamma})$, where $\underline{L}_{\Gamma'}$ is in red and $\underline{L}_{\Gamma}$ is in black.

which implies that $\underline{L}_{\Gamma}$ and $\underline{L}_{\Gamma'}$ are quasi-isomorphic objects in $\mathcal{FS}^{\mathrm{cyl},n}(E)$. Moreover, by the open mapping theorem, the projection to $\mathbb{H}^\circ$ of the curves contributing to μ^2 in (5.18) is contained in the disc bounded by γ_1 and γ' (see Figure 5.1).

Now, we consider the general n with $2n \le m$. Without loss of generality, we continue to assume $i = 1$ and $j = 2$. If Γ has no thimble path, then we do not need to impose Hamiltonian perturbations in the Floer equation when we compute the differential and product. In this case, the Floer solutions contributing to μ^2 in (5.18) persist because they lie above the disc bounded by γ_1 and γ', which is not altered when adding the other Lagrangian components. Together with the constant triangles on the other Lagrangian components representing $\mu^2(e, e) = e$, we conclude that there exist $a' \in HF(\underline{L}_{\Gamma}, \underline{L}_{\Gamma'})$ and $b' \in HF(\underline{L}_{\Gamma'}, \underline{L}_{\Gamma})$ such that (5.18) holds, and hence $\underline{L}_{\Gamma}$ is quasi-isomorphic to $\underline{L}_{\Gamma'}$ (see Figure 5.1).

If Γ contains some thimble paths, then Hamiltonian perturbation terms in the Floer equation are necessary. However, the Hamiltonian terms can be taken to be zero over the disc bound by γ_1 and γ' and hence the Floer solutions contributing to μ^2 in (5.18) persist. Again, together with the Floer triangles on the other Lagrangian components representing $\mu^2(e, e) = e$ (all of which are constant for an appropriate choice of Hamiltonian perturbation, cf. the proof of Lemma 3.2), we conclude that $\underline{L}_{\Gamma}$ is quasi-isomorphic to $\underline{L}_{\Gamma'}$.

Finally, we explain the remaining case where $2n > m$. To deal with this case, we want to compare $\mathcal{Y}_{n,m}$ and $\mathcal{Y}_{n,2n}$. Let E be as above so that $\mathcal{Y}_{n,m} = \mathrm{Hilb}^n(E) \setminus D_r$. We denote the corresponding E for $\mathcal{Y}_{n,2n}$ by E_+ (i.e. $\mathcal{Y}_{n,2n} = \mathrm{Hilb}^n(E_+) \setminus D_r$). Similarly, we denote the Lefschetz fibration $E_+ \to \mathbb{H}^\circ$ by π_{E_+}. Let $W \subset \mathbb{H}$ be $\{\mathrm{re}(z) > 2n - m + 1/2\}$ so W contains exactly m of the $2n$ critical values of π_{E_+}. Pick a diffeomorphism from W to $\mathbb{H}$ which sends $\{2n - m + 1, \dots, 2n\}$ to $\{1, \dots, m\}$. The pre-images of Γ and Γ' under this diffeomorphism define two admissible tuples, denoted by Γ_+ and Γ'_+, respectively. From the discussion above, we know that $\underline{L}_{\Gamma_+}$ is quasi-isomorphic to $\underline{L}_{\Gamma'_+}$ in $\mathcal{FS}^{\mathrm{cyl},n}_W(\pi_{E_+})$ because we can find a', b' such that the corresponding (5.18) holds for $\underline{L}_{\Gamma_+}$ and $\underline{L}_{\Gamma'_+}$. The pairs $\underline{L}_{\Gamma}, \underline{L}_{\Gamma'}$ in E and $\underline{L}_{\Gamma_+}, \underline{L}_{\Gamma'_+}$ in $\pi^{-1}_{E_+}(W)$ are isotopic through a family of Lagrangians associated to a family of admissible tuples in Lefschetz fibrations over the disc with varying symplectic form and almost complex structure, and where the isotopy does not create or cancel intersection points. There is no bifurcation in the moduli spaces of constant holomorphic triangles in such a deformation, so one can find the corresponding a', b' for $\underline{L}_{\Gamma}$ and $\underline{L}_{\Gamma'}$ such that (5.18) holds. Therefore, the result follows. ∎

Lemma 5.10. *Let $\Gamma := \{\gamma_1, \ldots, \gamma_n\}$ be an admissible tuple. Suppose γ_i is a thimble path and γ_j is a matching path. Let γ' be the thimble path obtained by sliding γ_i across γ_j. If Γ' is the admissible tuple obtained by replacing γ_i with γ', then $\underline{L}_\Gamma$ is quasi-isomorphic to $\underline{L}_{\Gamma'}$.*

Proof. We first consider the case $n = 2$. Let γ_1 be the thimble path and γ_2 be the matching path. We can assume that the thimble path γ' only intersects γ_1 at the starting point, which we denote by c. Along with γ', we consider two more auxiliary thimble paths $\gamma_{1,w}$ and γ'_w, which are obtained by positively wrapping γ_1 and γ' along the real line, respectively. Without loss of generality (by possibly switching the roles of γ_1 and γ'), we can assume that $\gamma_{1,w}$ intersects γ' in two points c and c', and any other pair of thimble paths amongst $\gamma_1, \gamma', \gamma_{1,w}, \gamma'_w$ intersect only in c (see Figure 5.2).

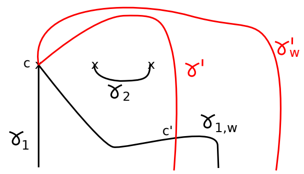


Fig. 5.2

Let $\Gamma_w := \{\gamma_{1,w}, \gamma_2\}$ and $\Gamma'_w := \{\gamma'_w, \gamma_2\}$. It is clear that $\underline{L}_\Gamma$ is quasi-isomorphic to $\underline{L}_{\Gamma_w}$ and $\underline{L}_{\Gamma'}$ is quasi-isomorphic to $\underline{L}_{\Gamma'_w}$. To show that $\underline{L}_\Gamma$ is quasi-isomorphic to $\underline{L}_{\Gamma'}$, it suffices to find $a \in HF(\underline{L}_{\Gamma'}, \underline{L}_\Gamma)$, $b \in HF(\underline{L}_{\Gamma_w}, \underline{L}_{\Gamma'})$, and $a' \in HF(\underline{L}_{\Gamma'_w}, \underline{L}_{\Gamma_w})$ such that

$$\mu^2(a, b) = 1_{HF(\underline{L}_{\Gamma_w}, \underline{L}_\Gamma)} \quad \text{and} \quad \mu^2(b, a') = 1_{HF(\underline{L}_{\Gamma'_w}, \underline{L}_{\Gamma'})} \tag{5.19}$$

and such that a is identified with a' under the continuation map (with respect to positive wrapping along the real line). In the given positions of the Lagrangians, we do not need to use Hamiltonian terms on the Floer multiplication maps

$$HF(\underline{L}_{\Gamma'}, \underline{L}_\Gamma) \times HF(\underline{L}_{\Gamma_w}, \underline{L}_{\Gamma'}) \to HF(\underline{L}_{\Gamma_w}, \underline{L}_\Gamma), \tag{5.20}$$

$$HF(\underline{L}_{\Gamma_w}, \underline{L}_{\Gamma'}) \times HF(\underline{L}_{\Gamma'_w}, \underline{L}_{\Gamma_w}) \to HF(\underline{L}_{\Gamma'_w}, \underline{L}_{\Gamma'}). \tag{5.21}$$

To compute these maps and hence verify (5.19), we rely on (5.18) and a restriction argument as follows.

We add a critical value $\hat{c}$ and extend the thimble paths $\gamma_1, \gamma', \gamma_{1,w}, \gamma'_w$ to run into $\hat{c}$. The thimble paths become matching paths $\hat{\gamma}_1, \hat{\gamma}', \hat{\gamma}_{1,w}, \hat{\gamma}'_w$ and we define $\hat{\Gamma} := \{\hat{\gamma}_1, \gamma_2\}$, $\hat{\Gamma}' := \{\hat{\gamma}', \gamma_2\}$, $\hat{\Gamma}_{1,w} = \{\hat{\gamma}_{1,w}, \gamma_2\}$ and $\hat{\Gamma}'_w := \{\hat{\gamma}'_w, \gamma_2\}$ (see Figure 5.3). By Lemma 5.9, $\underline{L}_{\hat{\Gamma}}$, $\underline{L}_{\hat{\Gamma}'}$, $\underline{L}_{\hat{\Gamma}_{1,w}}$ and $\underline{L}_{\hat{\Gamma}'_w}$ are all quasi-isomorphic objects. Therefore, there are $\hat{a} \in HF(\underline{L}_{\hat{\Gamma}'}, \underline{L}_{\hat{\Gamma}})$, $\hat{b} \in HF(\underline{L}_{\hat{\Gamma}_w}, \underline{L}_{\hat{\Gamma}'})$ and $\hat{a}' \in HF(\underline{L}_{\hat{\Gamma}'_w}, \underline{L}_{\hat{\Gamma}_w})$ such that

$$\mu^2(\hat{a}, \hat{b}) = 1_{HF(\underline{L}_{\hat{\Gamma}_w}, \underline{L}_{\hat{\Gamma}})} \quad \text{and} \quad \mu^2(\hat{b}, \hat{a}') = 1_{HF(\underline{L}_{\hat{\Gamma}'_w}, \underline{L}_{\hat{\Gamma}'})}. \tag{5.22}$$

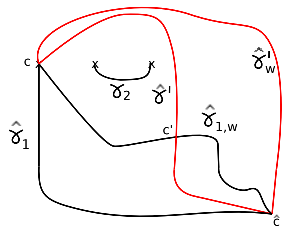


Fig. 5.3

The degree 0 generator for each of the Floer cochain groups above is given by an intersection point lying above either c or c', tensored with the degree 0 generator of $CF(L_{\gamma_2}, L_{\gamma_2})$. For example, the cochain complex $CF(L_{\hat{\gamma}_{1,w}}, L_{\hat{\gamma}'})$ is generated by one degree 0 and one degree 1 generator lying over c' (arising from morsifying the clean S^1-intersection locus [38]), and transverse degree 2 intersection points lying over each of c and $\hat{c}$. Since each of the CF^0 groups involved has rank one, $\hat{a}$, $\hat{b}$ and $\hat{a}'$ are the unique degree 0 generators (up to sign). The curves contributing to (5.22) cannot hit the generator lying above $\hat{c}$, so all the curves lie away from the fibre above $\hat{c}$. By the open mapping theorem, the projections of these curves are contained in the disc bounded by γ' and $\gamma_{1,w}$, which shows that (5.19) holds before adding the critical value $\hat{c}$. This verifies the case in which $n = 2$.

Since the previous computation is local, the general case where $n > 2$ can be treated as in the proof of Lemma 5.9. ∎

The last case involves sliding a thimble path across another thimble path. The proof is again an adaption of Lemma 5.9, extending the thimble paths in order to apply the same kind of restriction argument as in the proof of Lemma 5.10, so we omit it.

Lemma 5.11. *Let $\Gamma := \{\gamma_1, \dots, \gamma_n\}$ be an admissible tuple. Suppose γ_i and γ_j are thimble paths and $i \neq j$. Let γ' be the thimble path obtained by sliding γ_i across γ_j. If Γ' is the admissible tuple obtained by replacing γ_i with γ', then $\underline{L}_\Gamma$ is quasi-isomorphic to $\underline{L}_{\Gamma'}$.*

5.4. *Generation*

We now show that when E is the A_{m-1} Milnor fibre, the embedding in Proposition 4.12 is essentially surjective. In other words, we want to show that the split-closure $\mathcal{A}$ of the thimbles $\underline{T}^I$ is the entire $D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$. We first recall some general facts for A_∞/triangulated categories.

Lemma 5.12. *The full subcategory generated by an exceptional collection is admissible* (*i.e. admits right and left adjoints*). *In particular, $\mathcal{A}$ is an admissible subcategory of $D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$.*

Proof. Recall that a category admitting a full exceptional collection is already split-closed [40, Remark 5.14]. The result then follows from e.g. [23, Lemma 1.58]. ∎

For a subcategory $\mathcal{A}$ of a triangulated category $\mathcal{C}$, the *right orthogonal* $\mathcal{A}^\perp$ of $\mathcal{A}$ is the full subcategory of objects $\{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(A, X) = 0 \ \forall A \in \mathcal{A}\}$. The left orthogonal is defined similarly.

Lemma 5.13. *Let $\mathcal{S}$ be the Serre functor of $D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$. If $\mathcal{S}(X), \mathcal{S}^{-1}(X) \in \mathcal{A}$ for all $X \in \mathcal{A}$, then $D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E) = \mathcal{A} \oplus \mathcal{A}^\perp$.*

Proof. If $Y \in D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ satisfies $\mathrm{Hom}(Y, X) = 0$ for all $X \in \mathcal{A}$, then $\mathrm{Hom}(X, \mathcal{S}(Y))^\vee = 0$ for all $X \in \mathcal{A}$. It means that $\mathrm{Hom}(\mathcal{S}^{-1}(X), Y) = 0$ for all $X \in \mathcal{A}$. By assumption, this is equivalent to $\mathrm{Hom}(X, Y) = 0$ for all $X \in \mathcal{A}$. Similarly, $\mathrm{Hom}(X, Y) = 0$ for all $X \in \mathcal{A}$ implies that $\mathrm{Hom}(Y, X) = 0$ for all $X \in \mathcal{A}$. As a result, the left orthogonal of $\mathcal{A}$ coincides with its right orthogonal, so $D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ splits as a direct sum $\mathcal{A} \oplus \mathcal{A}^\perp$. ■

Lemma 5.14. *Suppose $\underline{T} = \{T_1, \ldots, T_n\} \in \mathcal{L}^{\mathrm{cyl},n}$ is a Lagrangian tuple such that each T_i is a thimble of π_E. Then $\underline{T}$ is generated by $\{\underline{T}^I\}_{I \in \mathcal{I}}$.*

Proof. First note that the braid group acts transitively (up to isotopy) on all the $\underline{T} = \{T_1, \ldots, T_n\}$ such that T_i is a thimble for each i. Therefore, it suffices to show that for each simple braid σ and the associated symplectomorphism ϕ_σ, the images $\phi_\sigma(\underline{T}^I)$ and $\phi_\sigma^{-1}(\underline{T}^I)$ are generated by $\{\underline{T}^I\}_{I \in \mathcal{I}}$.

Let σ be the positive half-twist swapping $\boldsymbol{c}_j$ and $\boldsymbol{c}_{j+1}$. There are four cases of $\underline{T}^I$ to consider, namely, whether j and/or $j+1$ is contained in I or not (recall that I is a cardinality n subset of $\{1, \ldots, m\}$ and $\boldsymbol{c}_i = i + \sqrt{-1}$ for all $i = 1, \ldots, m$). For each of these four cases, one can apply the exact triangles from Corollary 3.5 and the sliding invariance property in Lemma 5.11 to show that for all $I \in \mathcal{I}$, $\phi_\sigma(\underline{T}^I)$ and $\phi_\sigma^{-1}(\underline{T}^I)$ are generated by $\{\underline{T}^I\}_{I \in \mathcal{I}}$.

More precisely, if $j, j+1 \notin I$, then $\phi_\sigma(\underline{T}^I) = \phi_\sigma^{-1}(\underline{T}^I) = \underline{T}^I$. If $j+1 \in I$ and $j \notin I$, then $\phi_\sigma^{-1}(\underline{T}^I) = \underline{T}^{I'}$, where $I' = (I \setminus \{j+1\}) \cup \{j\}$. On the other hand, $\phi_\sigma(\underline{T}^I)$ can be obtained from applying iterated exact triangles to $\underline{T}^I$ and $\underline{T}^{I'}$ (see the first row of Figure 5.4 where $n = 4$, $j = 2$ and $\phi_\sigma(\underline{T}^I)$ corresponds to the third term in the exact triangle). If $j \in I$ and $j+1 \notin I$, it is similar to the previous case. The last case is $j, j+1 \in I$. In this cae, both $\phi_\sigma(\underline{T}^I)$ and $\phi_\sigma^{-1}(\underline{T}^I)$ can be obtained from applying the sliding invariance property to $\underline{T}^I$ (see the second row of Fgure 5.4). ■

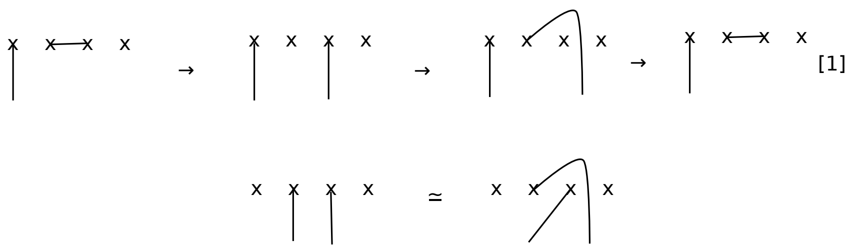


Fig. 5.4. The top row is an exact triangle so the third term is generated by the first and second terms, which are in turn generated by $\{\underline{T}^I\}_{I \in \mathcal{I}}$. The second row represents two quasi-isomorphic objects.

Corollary 5.15. *Assume that Claim 3.6 holds. Then the assumption in Lemma 5.13 holds, i.e. $\mathcal{S}(X), \mathcal{S}^{-1}(X) \in \mathcal{A}$ for all $X \in \mathcal{A}$.*

Proof. By Claim 3.6, the Serre functor of $D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ is given by the global monodromy τ up to grading shift. Therefore, it suffices to prove that for each thimble $\underline{T}^I$, the images $\tau(\underline{T}^I)$ and $\tau^{-1}(\underline{T}^I)$ are split-generated by the collection of thimbles $\{\underline{T}^I\}_{I\in\mathcal{I}}$. This is the content of Lemma 5.14. ■

Proposition 5.16. *If Claim 3.6 holds, then $\mathcal{A}^\perp = 0$, so $D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E) = \mathcal{A}$.*

Proof. It suffices to show that for each $\underline{L} \in \mathcal{L}^{\mathrm{cyl},n}$, there is an object $\underline{T}$ in $\mathcal{A}$ such that $\mathrm{Hom}(\underline{T}, \underline{L}) \neq 0$. By definition, we have $\underline{L} = \{L_1, \dots, L_n\}$ and $\pi_E(L_i) \subset U_i$ for some contractible U_i. We can assume that $\overline{U_i} \cap \partial\mathbb{H}$ is connected and non-empty so that $\mathbb{H}^\circ \setminus U_i$ is also contractible.

For each i, there is a thimble T_i of π_E such that $\pi_E(T_i) \subset U_i$ and $HF(T_i, L_i) \neq 0$. This follows from the fact that thimbles generate $\mathcal{F}\mathcal{S}^{\mathrm{cyl},1}_{U_i}(\pi_E) = \mathcal{F}\mathcal{S}(\pi_E|_{\pi_E^{-1}(U_i)})$, the usual Fukaya–Seidel category of the Milnor fibre [40] (note that when $n = 1$ there are no critical points at infinity). Let $\underline{T} = \{T_1, \dots, T_n\}$. It is clear that there is a cochain isomorphism

$$CF(\underline{T}, \underline{L}) \simeq \bigotimes_{i=1}^{n} CF(T_i, L_i), \tag{5.23}$$

which implies that $HF(\underline{T}, \underline{L}) \neq 0$. By Lemma 5.14, we have $\underline{T} \in \mathcal{A}$, concluding the proof. ■

Corollary 5.17. *Given Claim 3.6, when $n = m$ we have $\mathcal{A} = \mathbb{K}$ so $D^\pi \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E) = D^b(\mathbb{K})$.*

Proof. When $n = m$, there is only one cardinality n subset in $\{1, \dots, m\}$ so there is only one object in $\mathcal{A}$ up to quasi-isomorphism (recall that $\mathcal{A}$ is defined to be the full subcategory of $\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ with objects $\underline{T}^I$). Moreover, this object is an exceptional object. Therefore, the result follows from Proposition 5.16. ■

This special case is not very important at this point but we will come back to it in Section 11 when we define the symplectic annular Khovanov homology and compare it to the algebraically defined annular Khovanov homology.

Remark 5.18. Even though Corollary 5.17 depends on Claim 3.6, the fact that $D^\pi \mathcal{F}\mathcal{S}(\pi_{\mathcal{Y}}) = D^b(\mathbb{K})$ when $n = m$ follows from the definitions.

6. The extended symplectic arc algebra

In this section, we will introduce a particular collection of admissible tuples, and hence the corresponding collection of objects in $\mathcal{L}^{\mathrm{cyl},n}$; these objects are motivated by the diagrammatics in [13, 51]. We will prove that the cohomological Floer endomorphism algebra of these objects recovers the algebraic extended arc algebra as a graded vector space (cf.

Lemma 6.5). The A_∞ endomorphism algebra of this collection of objects will be the 'extended symplectic arc algebra', and will contain the symplectic arc algebra from [1] as a subalgebra. The corresponding Lagrangian products in $\mathrm{Hilb}^n(A_{m-1})\backslash D_r$ would not be cones over Legendrian submanifolds at infinity, which is why it is important to be able to study these Lagrangians in the cylindrical model. The later parts of the section begin the study of the algebra structure on, and formality of, the extended symplectic arc algebra; these studies continue in Sections 7 and 8 respectively.

6.1. Weights and projective Lagrangians

To introduce the collection of admissible tuples, we start with some terminology. Without loss of generality, we assume that the critical values are $\boldsymbol{c}_k := k + \sqrt{-1}$ for $k = 1, \ldots, m$. A weight of type (n, m) is a function $\lambda : \{1, \ldots, m\} \to \{\wedge, \vee\}$ such that $|\lambda^{-1}(\vee)| = n$. Let $\Lambda_{n,m}$ be the set of all weights of type (n, m). For $\lambda \in \Lambda_{n,m}$, let $c_{\lambda,1} < \cdots < c_{\lambda,n}$ be the integers such that $\lambda(c_{\lambda,j}) = \vee$. For each $c_{\lambda,j}$, if there exists $c' \in \lambda^{-1}(\wedge)$ with $c' > c_{\lambda,j}$ and such that

$$|\{c \in \{1, \ldots, m\} \mid \lambda(c) = \vee,\ c_{\lambda,j} < c < c'\}| = |\{c \in \{1, \ldots, m\} \mid \lambda(c) = \wedge,\ c_{\lambda,j} < c < c'\}| \quad (6.1)$$

then we call $c_{\lambda,j}$ a *good point* of λ; the minimum of all c' satisfying (6.1) is denoted by $c^{\wedge}_{\lambda,j}$. If $c_{\lambda,j}$ is not a good point of λ, then we call it a *bad point* (see Figure 6.1 for an example).

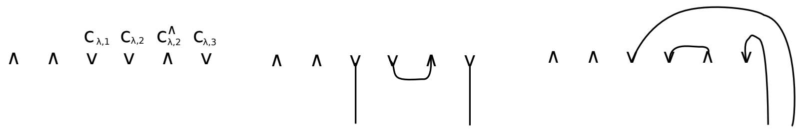


Fig. 6.1. The left figure represents a weight $\lambda \in \Lambda_{3,6}$ with a good point $c_{\lambda,2}$ and two bad points $c_{\lambda,1}, c_{\lambda,3}$. The middle figure is $\underline{\lambda}$ and the right figure is $\overline{\lambda}$.

For $\lambda \in \Lambda_{n,m}$, we choose n pairwise disjoint embedded curves $\underline{\gamma}_{\lambda,1}, \ldots, \underline{\gamma}_{\lambda,n}$ in $\{z \in \mathbb{H}^\circ \mid \mathrm{im}(z) \le 1,\ \mathrm{re}(z) < 2m\}$ such that

if $c_{\lambda,j}$ is a good point, then $\underline{\gamma}_{\lambda,j}$ is a matching path joining $c_{\lambda,j} + \sqrt{-1}$ and $c^{\wedge}_{\lambda,j} + \sqrt{-1}$, (6.2)

if $c_{\lambda,j}$ is a bad point, then $\underline{\gamma}_{\lambda,j}$ is a thimble path from $c_{\lambda,j} + \sqrt{-1}$ to $c_{\lambda,j}$. (6.3)

We define $\underline{\lambda} := \{\underline{\gamma}_{\lambda,1}, \ldots, \underline{\gamma}_{\lambda,n}\}$ and

$$\underline{L_\lambda} := \{L_{\underline{\gamma}_{\lambda,1}}, \ldots, L_{\underline{\gamma}_{\lambda,n}}\} \in \mathcal{L}^{\mathrm{cyl},n}. \quad (6.4)$$

The quasi-isomorphism type of $\underline{L_\lambda}$ is independent of the choice of $\underline{\lambda}$ (i.e. of the particular choice of paths) by Lemma 3.3.

Similarly, for $\lambda \in \Lambda_{n,m}$, we choose n pairwise disjoint embedded curves $\overline{\gamma}_{\lambda,1}, \ldots, \overline{\gamma}_{\lambda,n}$ in $\{z \in \mathbb{H}^\circ \mid \mathrm{im}(z) \geq 1 \text{ or } \mathrm{re}(z) > 2m\}$ such that

$$\text{if } c_{\lambda,j} \text{ is a good point, then } \overline{\gamma}_{\lambda,j} \text{ is a matching path joining } c_{\lambda,j} + \sqrt{-1} \text{ and } c^{\wedge}_{\lambda,j} + \sqrt{-1}, \tag{6.5}$$

$$\text{if } c_{\lambda,j} \text{ is a bad point, then } \overline{\gamma}_{\lambda,j} \text{ is a thimble path from } c_{\lambda,j} + \sqrt{-1} \text{ to } 6m - c_{\lambda,j}. \tag{6.6}$$

We define $\overline{\lambda} := \{\overline{\gamma}_{\lambda,1}, \ldots, \overline{\gamma}_{\lambda,n}\}$ and

$$\underline{L}_{\overline{\lambda}} := \{L_{\overline{\gamma}_{\lambda,1}}, \ldots, L_{\overline{\gamma}_{\lambda,n}}\} \in \mathcal{L}^{\mathrm{cyl},n}. \tag{6.7}$$

The quasi-isomorphism type of $\underline{L}_{\overline{\lambda}}$ is again independent of the choices of paths made in defining $\overline{\lambda}$, by Lemma 3.3.

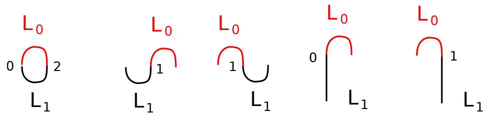


Fig. 6.2. The integer near an intersection point labels its degree as an element of $CF(L_0, L_1)$.

We want to choose a grading function on each $L_{\underline{\gamma},j}$ and $L_{\overline{\gamma},k}$ to induce a grading on $\underline{L}_{\underline{\lambda}}$ and $\underline{L}_{\overline{\lambda}}$. These grading functions are chosen so that (see Figure 6.2)

$$x \in CF(L_{\overline{\gamma},k}, L_{\underline{\gamma},j}) \text{ has degree } a \in \mathbb{N} \text{ if } x \text{ is the right end point of } a \text{ matching spheres;} \tag{6.8}$$

in particular, $\deg(x) \in \{0, 1, 2\}$ for all $x \in CF(L_{\overline{\gamma},k}, L_{\underline{\gamma},j})$.

By iteratively applying Lemmas 5.9, 5.10 and 5.11, we obtain the following (see Figures 6.3, 6.4, 6.5):

Proposition 6.1. *$\underline{L}_{\underline{\lambda}}$ is quasi-isomorphic to $\underline{L}_{\overline{\lambda}}$ in $\mathcal{FS}^{\mathrm{cyl},n}$.*

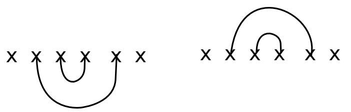


Fig. 6.3. Quasi-isomorphic compact objects.

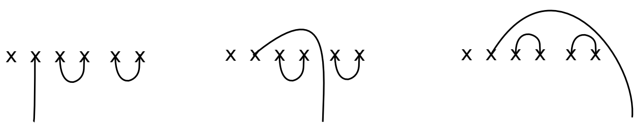


Fig. 6.4. Quasi-isomorphic 'mixed' objects.

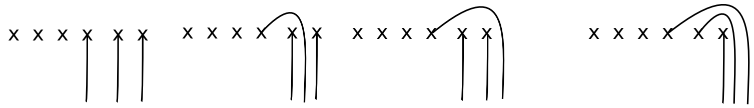


Fig. 6.5. Quasi-isomorphic thimble objects.

When only the quasi-isomorphism type is important, we denote either of $\underline{L}_\lambda$ and $\underline{L}_{\overline{\lambda}}$ by $\underline{L}_\lambda$.

Definition 6.2. The *extended symplectic arc algebra* is the A_∞ algebra

$$\mathcal{K}_{n,m}^{\text{symp}} := \bigoplus_{\lambda,\lambda' \in \Lambda_{n,m}} CF(\underline{L}_\lambda, \underline{L}_{\lambda'}), \tag{6.9}$$

which is well-defined up to quasi-isomorphism.

We want to choose a basis for the cohomology of $\mathcal{K}_{n,m}^{\text{symp}}$ as follows. Let $\overline{\lambda} \cup \underline{\lambda}'$ be the union of all the paths in $\overline{\lambda}$ and $\underline{\lambda}'$. By definition, $\overline{\lambda} \cup \underline{\lambda}'$ is a union of embedded circles and arcs; some circles might be nested inside one another. It will be helpful to consider alternative admissible tuples which avoid such nesting (but for which the quasi-isomorphism type of the associated Lagrangian tuple is unchanged).

Lemma 6.3 (cf. [1, Lemma 5.15]). *There is an admissible tuple $\tilde{\lambda}'$ such that*

- *if $\gamma \in \underline{\lambda}'$ is not contained in a circle of $\overline{\lambda} \cup \underline{\lambda}'$ (for example, when γ is a thimble path), then $\gamma \in \tilde{\lambda}'$;*
- *if $\gamma \in \underline{\lambda}'$ is contained in a circle C of $\overline{\lambda} \cup \underline{\lambda}'$, then there is a matching path $\tilde{\gamma} \in \tilde{\lambda}'$ with the same end points as γ such that $\tilde{\gamma}$ is enclosed in C;*
- *$\overline{\lambda} \cup \tilde{\lambda}'$ is a union of embedded circles and arcs such that none are nested.*

Proof. The proof is analogous to the proof of Lemma 5.15 of [1]. The only difference for our case is that we could have some thimble paths in admissible tuples.

More precisely, if γ is contained in a circle C, then each critical value enclosed in C is an end point of a matching path of $\underline{\lambda}'$ (and also a matching path of $\overline{\lambda}$), directly from the definitions (6.2), (6.3) (resp. (6.5), (6.6)). Thus, a suitable $\tilde{\lambda}'$ can be obtained by iteratively applying Lemma 5.9 to $\underline{\lambda}'$, ensuring that $\underline{L}_{\lambda'}$ is quasi-isomorphic to $\underline{L}_{\tilde{\lambda}'}$ (see Figure 6.6). Note that thimble paths are never contained in a circle, so do not need to be changed. ∎

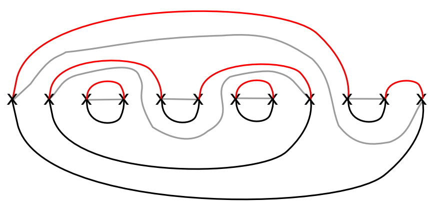


Fig. 6.6. Here $\overline{\lambda}$ is red, $\tilde{\lambda}'$ is grey, and $\underline{\lambda}'$ is black.

Note that the definition of $\tilde{\lambda}'$ depends on the pair (λ, λ'), and not just on λ'.

Lemma 6.4. *The cohomology of $\mathcal{K}_{n,m}^{\text{symp}}$, denoted by $K_{n,m}^{\text{symp}}$, is given by*

$$K_{n,m}^{\text{symp}} = \bigoplus_{\lambda,\lambda' \in \Lambda_{n,m}} HF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\tilde{\lambda}'}) = \bigoplus_{\lambda,\lambda' \in \Lambda_{n,m}} CF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\tilde{\lambda}'}) \tag{6.10}$$

as a graded vector space.

Proof. With the grading conventions of Figure 6.2, the pure degree elements in $CF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\tilde{\lambda}'})$ are concentrated either all in odd degrees or all in even degrees, so the Floer differential vanishes (compare to [1, Proposition 5.12]). ∎

We call a basis $\mathcal{B}^{\text{symp}}$ for $K_{n,m}^{\text{symp}}$ *geometric* if each basis element in $\mathcal{B}^{\text{symp}}$ is concentrated at a single intersection point in $\underline{L}_{\overline{\lambda}} \cap \underline{L}_{\tilde{\lambda}'}$ under the isomorphism (6.10). Once a $\tilde{\lambda}'$ has been chosen for each pair (λ, λ'), two different geometric bases can differ only by signs.

6.2. *Extended arc algebra*

We briefly recall the diagrammatic extended arc algebra $K_{n,m}^{\text{alg}}$. Details can be found in [13, 51], to which we refer for many details. For each weight $\lambda \in \Lambda_{n,m}$, there is an associated cup diagram $\underline{\lambda}^{\text{alg}}$ and a cap diagram $\overline{\lambda}^{\text{alg}}$ as follows. The cup diagram $\underline{\lambda}^{\text{alg}}$ can be obtained by adding a thimble path to $\underline{\lambda}$ from $a + \sqrt{-1}$ to a, for each $a \in \{1, \dots, m\}$, such that $a + \sqrt{-1}$ is not contained in any of the paths in $\underline{\lambda}$ (see Figure 6.7). For the cap diagram $\overline{\lambda}^{\text{alg}}$, we need to replace the thimble paths from $c + \sqrt{-1}$ to $6m - c$ in $\overline{\lambda}$ by vertical rays from $c + \sqrt{-1}$ to $c + \sqrt{-1}\,\infty$, and in addition, for each $a \in \{1, \dots, m\}$ such that $a + \sqrt{-1}$ is not contained in any of the paths in $\overline{\lambda}$, add a thimble path from $a + \sqrt{-1}$ to $a + \sqrt{-1}\,\infty$ (see Figure 6.7). In this paper, the only cup and cap diagrams we will encounter are given by $\underline{\lambda}^{\text{alg}}$ or $\overline{\lambda}^{\text{alg}}$ for some $\lambda \in \Lambda_{n,m}$.

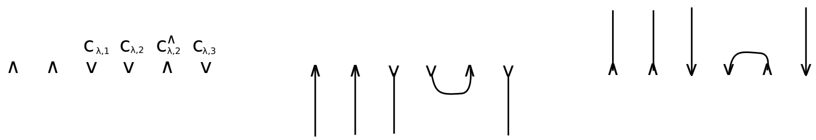


Fig. 6.7. From left to right: a weight λ, the cup diagram $\underline{\lambda}^{\text{alg}}$ and the cap diagram $\overline{\lambda}^{\text{alg}}$.

If β is a cup diagram and λ is a weight, then we say that $\beta\lambda$ is an *oriented cup diagram* if

the λ-values of the two ends of every matching path in β are different, (6.11)

if γ_a and γ_b are thimble paths in β containing $a + \sqrt{-1}$ and $b + \sqrt{-1}$, respectively, such that $a < b$ and $\lambda(a) = \vee$, then $\lambda(b) = \vee$. (6.12)

A *cap diagram* is defined to be the reflection β^r of a cup diagram β along the line $\{\mathrm{im}(z) = 1\}$. If α is a cap diagram and λ is a weight, then we say that $\lambda\alpha$ is an *oriented cap diagram* if $\alpha^r\lambda$ is an oriented cup diagram.

The union of a cup diagram β and a cap diagram α is denoted by $\beta \cup \alpha$ and called a *circle diagram*. This is a union of embedded circles and arcs in the upper half-plane. An *orientation* of a circle diagram $\beta \cup \alpha$ is a weight λ such that $\lambda\alpha$ and $\beta\lambda$ are an oriented cap diagram and an oriented cup diagram, respectively. Given such a λ, we denote the resulting oriented circle diagram by $\beta\lambda\alpha$.

A *clockwise cap* (resp. *cup*) of an oriented cap (resp. cup) diagram $\lambda\alpha$ (resp. $\alpha\lambda$) is a matching path $\gamma \in \alpha$ such that the λ-value of the left end point is $\wedge$, and hence the λ-value of the right end point is $\vee$. The *degree* (or *grading*) of an oriented cap/cup/circle diagram is defined to be the number of clockwise cups and caps in it. As a result, we have

$$\deg(\beta\lambda\alpha) = \deg(\lambda\alpha) + \deg(\beta\lambda). \tag{6.13}$$

As a graded vector space, $K^{\mathrm{alg}}_{n,m}$ is generated by oriented circle diagrams of the form $\underline{\lambda}^{\mathrm{alg}}_b \lambda \overline{\lambda}^{\mathrm{alg}}_a$, for $\lambda, \lambda_a, \lambda_b \in \Lambda_{n,m}$, and the grading of an oriented circle diagram is given by its degree.

Lemma 6.5. *There is a graded vector space isomorphism $\Phi : K^{\mathrm{symp}}_{n,m} \to K^{\mathrm{alg}}_{n,m}$ of the cohomological symplectic extended arc algebra and its algebraic counterpart.*

Proof. Let $\lambda_0, \lambda_1 \in \Lambda_{n,m}$. On the symplectic side, we consider the Floer cochains $CF(\underline{L}_{\overline{\lambda}_0}, \underline{L}_{\tilde{\lambda}_1})$. On the diagrammatic side, we consider the graded vector space $S(\lambda_0, \lambda_1)$ generated by the orientations of the circle diagram $\underline{\lambda}^{\mathrm{alg}}_1 \cup \overline{\lambda}^{\mathrm{alg}}_0$. By (6.10), $K^{\mathrm{symp}}_{n,m} = \bigoplus_{\lambda_0,\lambda_1 \in \Lambda_{n,m}} CF(\underline{L}_{\overline{\lambda}_0}, \underline{L}_{\tilde{\lambda}_1})$, so it suffices to find a graded vector space isomorphism between $CF(\underline{L}_{\overline{\lambda}_0}, \underline{L}_{\tilde{\lambda}_1})$ and $S(\lambda_0, \lambda_1)$ for all λ_0, λ_1.

Each generator $\underline{x} = \{x_1, \ldots, x_n\}$ of $CF(\underline{L}_{\overline{\lambda}_0}, \underline{L}_{\tilde{\lambda}_1})$ projects to an n-tuple of pairwise distinct points $\pi_E(\underline{x})$ in $\{1, \ldots, m\} + \sqrt{-1}$, such that each Lagrangian component in $\overline{\lambda}_0$ and $\tilde{\lambda}_1$ contains exactly one $\pi_E(x_i)$. Conversely, every n-tuple of pairwise distinct points in $\{1, \ldots, m\} + \sqrt{-1}$ satisfying this property uniquely determines a generator of $CF(\underline{L}_{\overline{\lambda}_0}, \underline{L}_{\tilde{\lambda}_1})$. Let $\lambda_{\underline{x}}$ be the weight given by $\lambda_{\underline{x}}(a) = \vee$ if and only if $a + \sqrt{-1} \in \pi_E(\underline{x})$. We claim that the linear map $\Phi_{\lambda_0,\lambda_1} : CF(\underline{L}_{\overline{\lambda}_0}, \underline{L}_{\tilde{\lambda}_1}) \to S(\lambda_0, \lambda_1)$ given by

$$\underline{x} = \{x_1, \ldots, x_n\} \mapsto \lambda_{\underline{x}} \tag{6.14}$$

is a graded vector space isomorphism.

To see that $\Phi_{\lambda_0,\lambda_1}$ is well-defined, we observe that all the thimble paths contained in $\underline{\lambda}^{\mathrm{alg}}$ but not in $\underline{\lambda}$ are on the left of the thimble paths (if any) in $\underline{\lambda}$, and the same is true for $\overline{\lambda}^{\mathrm{alg}}$ and $\overline{\lambda}$. Therefore, $\lambda_{\underline{x}}$ satisfies (6.12). On the other hand, since each Lagrangian component contains exactly one of the x_i, it means that $\lambda_{\underline{x}}$ also satisfies (6.11).

It is routine to check that $\Phi_{\lambda_0,\lambda_1}$ is bijective and preserves the grading, using (6.8) and (6.13). ∎

The algebra structure on $K^{\mathrm{alg}}_{n,m}$ is defined by applying an appropriate diagrammatic TQFT. We briefly recall one possible definition of this algebra, and some crucial properties that we will use later, in Section 6.4 below, and refer the readers to [13, Sections 3 & 4] for a detailed exposition.

6.3. Compact subalgebra

We call a weight $\lambda \in \Lambda_{n,m}$ a *compact weight* if $\overline{\lambda}$ (and hence $\underline{\lambda}$) consists only of matching paths. Let $\Lambda^c_{n,m} \subset \Lambda_{n,m}$ be the subset of compact weights. We define

$$\mathcal{H}^{\mathrm{symp}}_{n,m} := \bigoplus_{\lambda,\lambda' \in \Lambda^c_{n,m}} CF(\underline{L}_{\lambda}, \underline{L}_{\lambda'}), \tag{6.15}$$

which is well-defined up to quasi-isomorphism. As in (6.10), its cohomology is given by

$$H^{\mathrm{symp}}_{n,m} = \bigoplus_{\lambda,\lambda' \in \Lambda^c_{n,m}} HF(\underline{L}_{\lambda}, \underline{L}_{\lambda'}) = \bigoplus_{\lambda,\lambda' \in \Lambda^c_{n,m}} CF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\tilde{\lambda}'}). \tag{6.16}$$

A basis of $H^{\mathrm{symp}}_{n,m}$ is called *geometric* if it is given by the geometric intersection points in $CF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\tilde{\lambda}'})$ under the isomorphism (6.16) (again such bases are well-defined up to sign).

On the diagrammatic side, we can define the corresponding subalgebra $H^{\mathrm{alg}}_{n,m}$ of $K^{\mathrm{alg}}_{n,m}$, which is generated by oriented circle diagrams such that the underlying cap and cup diagrams are given by $\overline{\lambda}^{\mathrm{alg}}$ and $\underline{\lambda}'^{\mathrm{alg}}$ for some $\lambda, \lambda' \in \Lambda^c_{n,m}$. It is clear that the graded vector space isomorphism in Lemma 6.5 induces a graded vector space isomorphism between $H^{\mathrm{symp}}_{n,m}$ and $H^{\mathrm{alg}}_{n,m}$.

Previous study has focussed on the case $m = 2n$. In this case, $H^{\mathrm{alg}}_{n,2n}$ is generated by oriented circle diagrams whose underlying diagram only contains circles (and the corresponding Lagrangian submanifolds of $\mathrm{Hilb}^n(A_{2n-1})$ are compact, being products of spheres rather than products of spheres and thimbles). The algebra $H^{\mathrm{alg}}_{n,2n}$ is also known as Khovanov's *arc algebra*.

Theorem 6.6 ([1, 2]). *The A_∞ algebra $H^{\mathrm{symp}}_{n,2n}$ is formal. Moreover, there is an isomorphism between $H^{\mathrm{alg}}_{n,2n}$ and $H^{\mathrm{symp}}_{n,2n}$, sending the oriented circle diagram basis of $H^{\mathrm{alg}}_{n,2n}$ to a geometric basis of $H^{\mathrm{symp}}_{n,2n}$.*

Sketch of proof. Formality of $H^{\mathrm{symp}}_{n,2n}$ is the main result of [1], whilst a basis-preserving algebra isomorphism between $H^{\mathrm{alg}}_{n,2n}$ and $H^{\mathrm{symp}}_{n,2n}$ is one of the main results of [2]. We now recall the basis for $H^{\mathrm{symp}}_{n,2n}$ chosen in [2], and explain why it is a geometric basis in our sense. The essential point [2, Corollary 5.5] is the compatibility of the basis with various Künneth-type functors and decompositions (of the cohomology of products of spheres and thimbles with the cohomologies of the constituent factors).

The natural action of the braid group Br_{2n} on A_{2n-1} and on $\mathcal{Y}_{n,2n}$ factors through an action of the symmetric group Sym_{2n} on cohomology. Furthermore, there are inclusions

$$A_{2n-1} = \mathcal{Y}_{1,2n} \subset \mathcal{Y}_{n,2n} \subset (\mathbb{P}^1)^{2n}$$

which induce Sym_{2n}-equivariant cohomology isomorphisms

$$\begin{aligned} H^2(A_{2n-1}) = H^2(\mathcal{Y}_{1,2n}) \simeq H^2(\mathcal{Y}_{n,2n}) \simeq \mathbb{Z}\langle e_1, \ldots, e_{2n}\rangle / \left\langle \sum e_i \right\rangle \leftarrow \mathbb{Z}\langle e_1, \ldots, e_{2n}\rangle \\ = H^2((\mathbb{P}^1)^{2n}). \qquad (6.17) \end{aligned}$$

In particular, the image of $\{e_i \mid 1 \leq i \leq 2n-1\}$ in $H^2(\mathcal{Y}_{n,2n})$ forms a basis. The restriction map $H^2(\mathcal{Y}_{n,2n}) \to H^2(\mathrm{Sym}(\underline{L}_\lambda))$ is surjective for each $\lambda \in \Lambda^c_{n,2n}$, and by identifying the range with the cohomology of suitable multi-diagonals in $(\mathbb{P}^1)^{2n}$, it is proved in [2] that the chosen basis has the property that it induces a well-defined basis for $H^2(\mathrm{Sym}(\underline{L}_\lambda))$, i.e. there is a basis of the latter such that each e_i maps to a basis element or to zero.

Now, we translate the basis of $H^2(\mathcal{Y}_{n,2n})$ to a basis of $H^2(A_{2n-1})$. Denote by $s_1, \ldots, s_{2n-1}$ the homology classes of the $2n-1$ standard matching spheres in A_{2n-1}, i.e. the ones that lie above $\{\mathrm{im}(z) = 1\}$; here they are labelled from left to right, and the spheres are oriented as the complex curves in the resolution of the A_{2n-1} surface singularity. We denote the image of e_i under (6.17) by $v_i \in H^2(A_{2n-1})$ for $1 \leq i \leq 2n-1$. They satisfy

$$v_1 = s_1^* \quad \text{and} \quad v_j = (-1)^j (s_{j-1}^* - s_j^*) \quad \text{for } 1 < j \leq 2n-1 \qquad (6.18)$$

where $*$ stands for linear dual. The set $\{v_j\}_{j=1}^{2n-1}$ is the corresponding basis for $H^2(A_{2n-1})$. It induces a well-defined basis for $HF^2(\underline{L}_\lambda, \underline{L}_\lambda)$ via restriction

$$H^2(A_{2n-1}) \to H^2(\underline{L}_\lambda) = HF^2(\underline{L}_\lambda, \underline{L}_\lambda) \qquad (6.19)$$

where the first map is defined by regarding $\underline{L}_\lambda$ as the disjoint union of the matching spheres it contains, and hence as a submanifold of A_{2n-1}. This basis for $H^2(\underline{L}_\lambda)$ naturally corresponds to the basis of $H^2(\mathrm{Sym}(\underline{L}_\lambda))$. Monomial products of the resulting elements give a basis for $HF^*(\underline{L}_\lambda, \underline{L}_\lambda)$.

When $HF(\underline{L}_\lambda, \underline{L}_{\lambda'}) \neq \{0\}$, it has rank 1 in minimal degree. Let $a_{\min}$ be a minimal degree generator of $HF(\underline{L}_\lambda, \underline{L}_{\lambda'})$ over the integers $\mathbb{Z}$. Then $HF(\underline{L}_\lambda, \underline{L}_{\lambda'})$ is generated by $a_{\min}$ as a module over each of $HF(\underline{L}_\lambda, \underline{L}_\lambda)$ and $HF(\underline{L}_{\lambda'}, \underline{L}_{\lambda'})$. A basis in $HF(\underline{L}_\lambda, \underline{L}_{\lambda'})$ can therefore be defined by taking products of $a_{\min}$ with the bases for either $HF(\underline{L}_\lambda, \underline{L}_\lambda)$ or $HF(\underline{L}_{\lambda'}, \underline{L}_{\lambda'})$; these act via the action of $H^*(\mathcal{Y}_{n,2n})$, which acts centrally, so there is no ambiguity (aside from a choice of sign of $a_{\min}$). This finishes recalling the basis for $H^{\mathrm{symp}}_{n,2n}$, which we call a *convenient* basis.

We claim this convenient basis is geometric. Let $S_{\lambda,1}, \ldots, S_{\lambda,n}$ be the matching spheres that $\underline{L}_\lambda$ contains. The first observation is that $v_j(S_{\lambda,k}) \in \{0, \pm 1\}$ for each j, and furthermore, for each j there is exactly one k for which $v_j(S_{\lambda,k}) \neq 0$. This means that the convenient basis of $H^2(\mathrm{Sym}(\underline{L}_\lambda))$ induced from $H^2(\mathcal{Y}_{n,2n})$ coincides with the product basis of the cohomology of $\mathrm{Sym}(\underline{L}_\lambda)$ as a product of matching spheres. The geometric basis in $CF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\tilde{\lambda}})$ also coincides with the product basis, because $\tilde{\lambda}$ is a Hamiltonian push-off of $\overline{\lambda}$, and $\overline{\lambda} \cup \tilde{\lambda}$ consists of n embedded pairwise non-nested circles. That suffices to show that the convenient basis on $HF(\underline{L}_{\lambda}, \underline{L}_{\lambda})$ is geometric.

On the other hand, the minimal degree generator of $HF(\underline{L}_{\lambda}, \underline{L}_{\lambda'})$ is forced to be geometric, because the minimal degree subspace has rank 1. Since $\overline{\lambda} \cup \tilde{\lambda}'$ is a union of embedded and non-nested circles, the product map

$$HF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\tilde{\lambda}'}) \otimes HF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\overline{\lambda}}) \to HF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\tilde{\lambda}'}) \tag{6.20}$$

decomposes into pieces, one for each circle C in $\overline{\lambda} \cup \tilde{\lambda}'$. More precisely, we have isomorphisms

$$HF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\tilde{\lambda}'}) = \bigotimes_C H^*(S^2), \tag{6.21}$$

$$HF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\overline{\lambda}}) = \bigotimes_C (H^*(S^2))^{\otimes m_C}, \tag{6.22}$$

where the tensor product is over all circles C in $\overline{\lambda} \cup \tilde{\lambda}'$, and m_C is the number of paths of $\overline{\lambda}$ that lie in C. The product map (6.20) decomposes into the tensor product over all circles C of maps

$$H^*(S^2) \otimes (H^*(S^2))^{\otimes m_C} \to H^*(S^2) \tag{6.23}$$

for which a typical local model is given by Figure 6.8. The product of the degree 0 geometric generator in $H^*(S^2)$ and a degree 2 geometric generator in $(H^*(S^2))^{\otimes m_C}$ is, up to sign, the degree 2 geometric generator in $H^*(S^2)$, because $H^2(S^2)$ has rank 1, the product is (by definition) a basis element of the convenient basis, and [2, Section 5] proves that the convenient basis is a basis for the cohomology groups over $\mathbb{Z}$. We conclude that the convenient basis is geometric. ∎

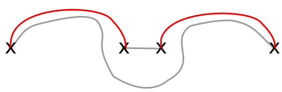


Fig. 6.8. A circle C in $\overline{\lambda} \cup \tilde{\lambda}'$ with $m_C = 2$ (see Figure 6.6).

Corollary 6.7. *After possibly changing Φ by sign on certain basis elements, the resulting graded vector space isomorphism $\Phi|_{H^{\mathrm{symp}}_{n,2n}}$ is an algebra isomorphism between $H^{\mathrm{symp}}_{n,2n}$ and $H^{\mathrm{alg}}_{n,2n}$.*

From now on, we assume $\Phi|_{H^{\mathrm{symp}}_{n,2n}}$ has been chosen to be an algebra isomorphism.

6.4. From the compact arc algebra to the extended arc algebra

In this section, we summarize the multiplication rule for $K^{\mathrm{alg}}_{n,m}$ and $H^{\mathrm{alg}}_{n,m}$ from [13, Sections 3 & 4]. Our approach is dictated by two considerations. First, the easiest (though not the original) description of the algebra structure on $K^{\mathrm{alg}}_{n,m}$ is to realise $K^{\mathrm{alg}}_{n,m}$ as a quotient algebra of $H^{\mathrm{alg}}_{m,2m}$. Second, in the next section, we precisely compute $K^{\mathrm{symp}}_{n,m}$ by realising it as a quotient of $H^{\mathrm{symp}}_{m,2m}$ by an ideal which, under the isomorphism $H^{\mathrm{alg}}_{m,2m} = H^{\mathrm{symp}}_{m,2m}$ (Theorem 6.6), we identify with the corresponding ideal on the algebraic side. The main purpose of this section is therefore to recall this quotient description of $K^{\mathrm{alg}}_{n,m}$.

For completeness, we briefly recall the multiplication rule for $H^{\mathrm{alg}}_{m,2m}$ from [13, Section 3, multiplication]. Let $\underline{d}\lambda\overline{c}$ and $\underline{b}\mu\overline{a}$ be two oriented circle diagrams with $a, b, c, d \in \Lambda^c_{n,m}$. If $b \neq c$, then the product is defined to be zero. If $b = c$, then we put the diagram $\underline{b}\mu\overline{a}$ above the diagram $\underline{d}\lambda\overline{c}$ and apply the TQFT surgery procedure – based on the Frobenius algebra underlying $H^*(S^2; \mathbb{Z})$, and described e.g. in [13, Section 3, the surgery procedure] iteratively to convert it into a disjoint union of diagrams each of which has no cups/caps in the 'middle section'. After this, the middle sections are unions of line segments; we shrink each such line segment to a point to obtain a disjoint union of some new oriented circle diagrams. The product $(\underline{d}\lambda\overline{c})(\underline{b}\mu\overline{a})$ is then defined to be the sum of the corresponding basis vectors of $H^{\mathrm{alg}}_{m,2m}$.

One distinguished feature of this TQFT surgery procedure is that the output circle (resp. disjoint union of two circles) is (resp. are) oriented according to the following rules, where $1 \Leftrightarrow$ counterclockwise orientation, and $x \Leftrightarrow$ clockwise orientation:

$$1 \otimes 1 \mapsto 1, \quad 1 \otimes x \mapsto x, \quad x \otimes 1 \mapsto x, \quad x \otimes x \mapsto 0, \tag{6.24}$$

$$1 \mapsto 1 \otimes x + x \otimes 1, \quad x \mapsto x \otimes x. \tag{6.25}$$

Each TQFT operation yields a disjoint union of zero, one or two new oriented diagrams replacing the old diagram (the iterative application may yield larger linear combinations of diagrams). This completes our résumé of $H^{\mathrm{alg}}_{m,2m}$ as an algebra.

We now recall how to realise $K^{\mathrm{alg}}_{n,m}$ as a quotient algebra of $H^{\mathrm{alg}}_{m,2m}$; for an independent definition of $K^{\mathrm{alg}}_{n,m}$, see [13]. For $\lambda \in \Lambda_{n,m}$, we define its *closure* $\mathrm{cl}(\lambda) \in \Lambda^c_{m,2m}$ by

$$\mathrm{cl}(\lambda) = \begin{cases} \vee & \text{if } a \leq m-n, \\ \lambda(a-(m-n)) & \text{if } m-n < a \leq 2m-n, \\ \wedge & \text{if } a > 2m-n. \end{cases} \tag{6.26}$$

Intuitively, $\mathrm{cl}(\lambda)$ can be regarded as putting $m-n$ many $\vee$'s and n many $\wedge$'s to the left and right of λ, respectively, to make $\mathrm{cl}(\lambda)$ a compact weight. For any oriented circle diagram $\underline{b}\lambda\overline{a} \in K^{\mathrm{alg}}_{n,m}$, we define

$$\mathrm{cl}(\underline{b}\lambda\overline{a}) := \underline{\mathrm{cl}(b)}\,\mathrm{cl}(\lambda)\,\overline{\mathrm{cl}(a)}, \tag{6.27}$$

which is an oriented circle diagram of $H^{\mathrm{alg}}_{m,2m}$.

Lemma 6.8 ([13, Lemma 4.2]). *The map $\underline{b}\lambda\overline{a} \mapsto \mathrm{cl}(\underline{b}\lambda\overline{a})$ is a degree preserving bijection between the set of oriented circle diagrams in $K^{\mathrm{alg}}_{n,m}$ with underlying weight $\lambda \in \Lambda_{n,m}$ and the set of oriented circle diagrams in $H^{\mathrm{alg}}_{m,2m}$ with underlying weight* $\mathrm{cl}(\lambda)$.

Let $I_{\Lambda_{n,m}}$ be the subspace of $H^{\text{alg}}_{m,2m}$ spanned by the vectors

$$\{\underline{b}\lambda\overline{a} \in H^{\text{alg}}_{m,2m} \mid \text{oriented circle diagram } \underline{b}\lambda\overline{a} \text{ with } \lambda \in \Lambda_{m,2m} \setminus \text{cl}(\Lambda_{n,m})\}, \tag{6.28}$$

which is a two-sided ideal. The condition $\lambda \in \Lambda_{m,2m} \setminus \text{cl}(\Lambda_{n,m})$ is equivalent to

$$\{1, \dots, m-n\} \not\subseteq \lambda^{-1}(\vee) \text{ or } \{2m-n+1, \dots, 2m\} \not\subseteq \lambda^{-1}(\wedge). \tag{6.29}$$

In view of Lemma 6.8, the vectors

$$\{\text{cl}(\underline{b}\lambda\overline{a}) + I_{\Lambda_{n,m}} \mid \text{oriented circle diagrams } \underline{b}\lambda\overline{a} \text{ with } \lambda \in \Lambda_{n,m}\} \tag{6.30}$$

give a basis for the quotient algebra $H^{\text{alg}}_{m,2m}/I_{\Lambda_{n,m}}$. We deduce that the map

$$\text{cl} : K^{\text{alg}}_{n,m} \to H^{\text{alg}}_{m,2m}/I_{\Lambda_{n,m}}, \qquad \underline{b}\lambda\overline{a} \mapsto \text{cl}(\underline{b}\lambda\overline{a}) + I_{\Lambda_{n,m}}, \tag{6.31}$$

is an isomorphism of graded vector spaces. We use this to transport the algebra structure on $H^{\text{alg}}_{m,2m}/I_{\Lambda_{n,m}}$ to $K^{\text{alg}}_{n,m}$.

We want to extract from (6.31) two algebra isomorphisms

$$H^{\text{alg}}_{n,m} = H^{\text{alg}}_{m-n,2(m-n)}/I \qquad \text{for } 2n \le m, \tag{6.32}$$

$$K^{\text{alg}}_{n,m} = H^{\text{alg}}_{n,m+n}/J \qquad \text{for all } n \le m, \tag{6.33}$$

where I and J are certain ideals to be specified. Note that (6.32) is an empty statement when $2n > m$ because, in this case, $H^{\text{alg}}_{n,m} = 0$, so one can simply take I to be $H^{\text{alg}}_{m-n,2(m-n)}$.

By definition, an oriented circle diagram $\underline{b}\lambda\overline{a} \in K^{\text{alg}}_{n,m}$ lies in $H^{\text{alg}}_{n,m}$ if and only if a and b have the property that all the $c_{a,j}$ and $c_{b,j}$ are good points (see (6.1)). That is equivalent to asking

$$c^{\wedge}_{\text{cl}(a),k} = c^{\wedge}_{\text{cl}(b),k} = 2m+1-k \qquad \text{for all } k = 1, \dots, n. \tag{6.34}$$

It means that for $\underline{b}\lambda\overline{a} \in H^{\text{alg}}_{n,m}$, $\text{cl}(\underline{b}\lambda\overline{a})$ has n counterclockwise circles enclosing an oriented circle diagram, which we call $\boldsymbol{c}(\underline{b}\lambda\overline{a})$, and which defines an element in $H^{\text{alg}}_{m-n,2(m-n)}$. Let $f : H^{\text{alg}}_{m-n,2(m-n)} \to H^{\text{alg}}_{m,2m}$ be the algebra embedding given by adding to an oriented circle diagram in $H^{\text{alg}}_{m-n,2(m-n)}$ precisely n counterclockwise circles enclosing it, so we have $f \circ \boldsymbol{c} = \text{cl}$.

Lemma 6.9. *We have a commutative diagram of algebra maps*

$$\begin{array}{ccc} H^{\text{alg}}_{n,m} & \xhookrightarrow{\quad\text{cl}\quad} & H^{\text{alg}}_{m,2m}/I_{\Lambda_{n,m}} \\ & {\scriptstyle \boldsymbol{c}}\searrow \qquad \nearrow{\scriptstyle f} & \\ & H^{\text{alg}}_{m-n,2(m-n)}/f^{-1}(I_{\Lambda_{n,m}}) & \end{array}$$

Proof. The diagram is commutative by construction, and we know that f and cl are algebra maps. It remains to check that $\boldsymbol{c}$ is also an algebra map, which follows from the injectivity of f and the fact that f and cl are algebra maps. ■

By (6.28), (6.29) and the description of f, one checks that the ideal $I := f^{-1}(I_{\Lambda_{n,m}})$ is the subspace of $H^{\mathrm{alg}}_{m-n,2(m-n)}$ spanned by

$$\big\{\underline{b}\lambda\overline{a} \in H^{\mathrm{alg}}_{m-n,2(m-n)} \;\big|\; \text{oriented circle diagrams } \underline{b}\lambda\overline{a} \text{ such that } \{1,\dots,m-2n\} \not\subseteq \lambda^{-1}(\vee)\big\}. \tag{6.35}$$

From (6.35), it is obvious that $\boldsymbol{c}$ is an isomorphism (of vector spaces, and hence algebras)

$$\boldsymbol{c} : H^{\mathrm{alg}}_{n,m} \simeq H^{\mathrm{alg}}_{m-n,2(m-n)}/I. \tag{6.36}$$

We given an equivalent reformulation of (6.36). For $2n \le m$, define $\boldsymbol{c} : \Lambda_{n,m} \to \Lambda_{m-n,2(m-n)}$ by

$$\boldsymbol{c}(\lambda)(a) := \begin{cases} \vee & \text{if } a \le m-2n, \\ \lambda(a-(m-2n)) & \text{if } a > m-2n. \end{cases} \tag{6.37}$$

This has the property that $\boldsymbol{c}(\Lambda^c_{n,m}) \subset \Lambda^c_{m-n,2(m-n)}$. For $\underline{b}\lambda\overline{a} \in H^{\mathrm{alg}}_{n,m}$, we define

$$\boldsymbol{c}(\underline{b}\lambda\overline{a}) = \underline{\boldsymbol{c}(b)}\boldsymbol{c}(\lambda)\overline{\boldsymbol{c}(a)} \in H^{\mathrm{alg}}_{m-n,2(m-n)}. \tag{6.38}$$

Let I be as in (6.35). Then

$$\boldsymbol{c} : H^{\mathrm{alg}}_{n,m} \to H^{\mathrm{alg}}_{m-n,2(m-n)}/I, \quad \underline{b}\lambda\overline{a} \mapsto \boldsymbol{c}(\underline{b}\lambda\overline{a}) + I, \tag{6.39}$$

is an algebra isomorphism.

Now we explain the definition of the ideal J appearing in (6.33). Let $I^{(1)}_{n,m}$ and $I^{(2)}_{n,m}$ be the ideals of $H^{\mathrm{alg}}_{m,2m}$ spanned by

$$\big\{\underline{b}\lambda\overline{a} \in H^{\mathrm{alg}}_{m,2m} \;\big|\; \{1,\dots,m-n\} \not\subseteq \lambda^{-1}(\vee))\big\} \quad \text{and} \tag{6.40}$$

$$\big\{\underline{b}\lambda\overline{a} \in H^{\mathrm{alg}}_{m,2m} \;\big|\; \{2m-n+1,\dots,2m\} \not\subseteq \lambda^{-1}(\wedge)\big\} \tag{6.41}$$

respectively (see Figure 6.9). In view of (6.28) and (6.29), we have $I_{n,m} = I^{(1)}_{n,m} + I^{(2)}_{n,m}$.

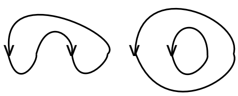


Fig. 6.9. When $m = 4$ and $n = 2$, the oriented circle diagram above represents an element in $I^{(1)}_{n,m}$ but not in $I^{(2)}_{n,m}$.

This implies that

$$K^{\mathrm{alg}}_{n,m} = H^{\mathrm{alg}}_{m,2m}/I_{\Lambda_{n,m}} = (H^{\mathrm{alg}}_{m,2m}/I^{(1)}_{n,m})/(I^{(1)}_{n,m} + I^{(2)}_{n,m}/I^{(1)}_{n,m}). \tag{6.42}$$

From (6.39), we have $H^{\mathrm{alg}}_{m,2m}/I^{(1)}_{n,m} = H^{\mathrm{alg}}_{n,m+n}$ so (6.42) becomes

$$K^{\mathrm{alg}}_{n,m} = H^{\mathrm{alg}}_{n,m+n}/J \tag{6.43}$$

where J is spanned by

$$\bigl\{\underline{b}\lambda\overline{a} \in H^{\mathrm{alg}}_{n,m+n} \bigm| \underline{b}\lambda\overline{a} \text{ is an oriented circle diagram and } \{m+1,\dots,m+n\} \not\subseteq \lambda^{-1}(\wedge)\bigr\} \tag{6.44}$$

We give an equivalent reformulation of (6.43). Let $\boldsymbol{e} : \Lambda_{n,m} \to \Lambda^{c}_{n,m+n}$ be the inclusion

$$\boldsymbol{e}(\lambda)(a) := \begin{cases} \lambda(a) & \text{if } a \le m, \\ \wedge & \text{if } a > m. \end{cases} \tag{6.45}$$

For $\underline{b}\lambda\overline{a} \in K^{\mathrm{alg}}_{n,m}$, we define

$$\boldsymbol{e}(\underline{b}\lambda\overline{a}) = \underline{\boldsymbol{e}(b)}\boldsymbol{e}(\lambda)\overline{\boldsymbol{e}(a)} \in H^{\mathrm{alg}}_{n,m+n}. \tag{6.46}$$

Let J be as in (6.44). Then

$$\boldsymbol{e} : K^{\mathrm{alg}}_{n,m} \to H^{\mathrm{alg}}_{n,m+n}/J, \quad \underline{b}\lambda\overline{a} \mapsto \boldsymbol{e}(\underline{b}\lambda\overline{a}) + J, \tag{6.47}$$

is an algebra isomorphism.

7. More algebra isomorphisms

We will next identify $K^{\mathrm{symp}}_{n,m}$ and $K^{\mathrm{alg}}_{n,m}$ as algebras for all n, m. The proof of Theorem 6.6 in [2] relied in an essential way on the fact that the Floer product for a triple of Lagrangians meeting pairwise cleanly can be understood, via 'plumbing models', as a possibly sign-twisted convolution product, and that for every triple of Lagrangians associated to compact weights, one could find non-vanishing Floer products which factored through products associated to triples with plumbing models. By contrast, for the extended algebra, there are triples of weights and corresponding Lagrangians for which the product of minimal degree generators cannot be written as a product of minimal degree generators between interpolating Lagrangians with plumbing models. Instead of mimicking the strategy of [2] in proving $H^{\mathrm{symp}}_{n,2n} = H^{\mathrm{alg}}_{n,2n}$, we will instead reduce the isomorphism of extended arc algebras (in stages) to Theorem 6.6 by restriction-type arguments similar to those appearing in the proof of Lemma 5.10. The crucial point, as explained in Section 6.4, is that all the extended arc algebras can be understood as subquotients of the algebras $H^{\mathrm{alg}}_{n,2n}$.

7.1. The compact cases

We first explain how to compute the algebra structure on $H^{\mathrm{symp}}_{n,m}$ for all n, m. We assume $2n \leq m$ because $H^{\mathrm{symp}}_{n,m} = 0$ otherwise. Let $\boldsymbol{c} : \Lambda^c_{n,m} \to \Lambda^c_{m-n,2(m-n)}$ be the injection given by (6.37). Let

$$H^{\boldsymbol{c}}_{n,m} := \bigoplus_{\lambda_0,\lambda_1 \in \Lambda^c_{n,m}} HF(\underline{L}_{\boldsymbol{c}(\lambda_0)}, \underline{L}_{\boldsymbol{c}(\lambda_1)}), \tag{7.1}$$

which is a subalgebra of $H^{\mathrm{symp}}_{m-n,2(m-n)} = H^{\mathrm{alg}}_{m-n,2(m-n)}$, so in particular $H^{\boldsymbol{c}}_{n,m}$ is formal. We want to compare $H^{\mathrm{symp}}_{n,m}$ with $H^{\boldsymbol{c}}_{n,m}$. Note that the Lagrangians underlying $H^{\mathrm{symp}}_{n,m}$ are diffeomorphic to $(S^2)^n$, whilst those relevant to $H^{\boldsymbol{c}}_{n,m}$ are $(S^2)^{m-n}$, where $m - n \geq n$.

For each ordered pair (λ_0, λ_1) such that $\lambda_0, \lambda_1 \in \Lambda^c_{n,m}$, we choose upper matchings $\overline{\boldsymbol{c}(\lambda_0)}$ and $\overline{\boldsymbol{c}(\lambda_1)}$ such that

for $a = 1, \dots, m - 2n$, the slope at $a + \sqrt{-1}$ of the matching path in $\overline{\boldsymbol{c}(\lambda_0)}$ starting from $a + \sqrt{-1}$ is larger than that of the corresponding path in $\overline{\boldsymbol{c}(\lambda_1)}$. (7.2)

Remark 7.1. Condition (7.2) is used to eliminate the existence of certain pseudo-holomorphic maps. For example, let l_0, l_1, l_2 be upper matching paths from $1 + \sqrt{-1}$ to $2 + \sqrt{-1}$ such that the slopes at $1 + \sqrt{-1}$ are in decreasing order (Figure 7.1). Let L_i be the corresponding matching spheres. Let u be a solution contributing to the multiplication map

$$CF(L_1, L_2) \times CF(L_0, L_1) \to CF(L_0, L_2) \tag{7.3}$$

such that $\pi_E \circ v$ is holomorphic. We claim that if the output of u maps to $1 + \sqrt{-1}$ under π_E, so do all the inputs of v. This is because if $\pi_E \circ v$ restricted to the boundary labelled by L_0 is not a constant, then the image of v must have non-empty intersection with the unbounded region of $\mathbb{H} \setminus (l_0 \cup l_1 \cup l_2)$ by the holomorphicity of $\pi_E \circ v$. This in turn implies that $\pi_E \circ v$ is not relatively compact by the open mapping theorem, a contradiction. As a result, one can show that $\pi_E \circ v$ restricted to the boundary labelled by L_0 is a constant. Inductively applying this argument, one can show that the restriction of $\pi_E \circ v$ to the whole boundary is constant, and hence u itself is a constant map.

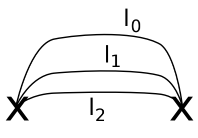


Fig. 7.1

The choices of matchings depend on the ordered pair we start with (cf. the discussion around Figure 6.6); to emphasize this dependence, we denote the matchings by $\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}$

and $\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}$ respectively. Let $\mathcal{I}_{\lambda_0,\lambda_1}$ be the subspace of $CF(\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}})$ generated by

$$\mathcal{B}_{\lambda_0,\lambda_1} := \left\{\underline{x} \in \mathcal{X}(\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}}) \;\middle|\; \{1,\dots,m-2n\} + \sqrt{-1} \not\subset \pi_E(\underline{x})\right\}$$

Let $\mathcal{G}_{\lambda_0,\lambda_1} := \mathcal{X}(\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}}) \setminus \mathcal{B}_{\lambda_0,\lambda_1}$. We define $\mathcal{I} = \bigoplus \mathcal{I}_{\lambda_0,\lambda_1}$, $\mathcal{B} := \bigoplus \mathcal{B}_{\lambda_0,\lambda_1}$ and $\mathcal{G} = \bigoplus \mathcal{G}_{\lambda_0,\lambda_1}$. We use the symbol $\mathcal{I}$ because, as we will see in Lemma 7.6, $\mathcal{I}$ is actually an ideal.

Lemma 7.2. *$\mu^1(\mathcal{I}_{\lambda_0,\lambda_1}) \subset \mathcal{I}_{\lambda_0,\lambda_1}$, so $\mathcal{I}_{\lambda_0,\lambda_1}$ descends to a vector subspace of $H^{\boldsymbol{c}}_{n,m}$, which we denote by I_{λ_0,λ_1}.*

Proof. It suffices to show that if the $\underline{x}_0$-coefficient of $\mu^1(\underline{x}_1)$ is non-zero for some $\underline{x}_0 \in \mathcal{G}$, then $\underline{x}_1 \in \mathcal{G}$.

By definition, $\underline{x}_0 \in \mathcal{G}$ implies that $\{1,\dots,m-2n\} + \sqrt{-1} \subset \pi_E(\underline{x}_0)$. Let $v : \Sigma \to E$ be a J-holomorphic map contributing to the $\underline{x}_0$-coefficient of $\mu^1(\underline{x}_1)$ such that near an output puncture ξ^0, v is asymptotic to the output lying above $1 + \sqrt{-1}$. We project v to $\mathbb{H}^\circ$ by π_E and apply the open mapping theorem. The condition (7.2) when $a = 1$ forces at least one of the boundary components of Σ adjacent to ξ^0 to be mapped constantly to $1 + \sqrt{-1}$ by $\pi_E \circ v$. Let the two boundary components of Σ adjacent to ξ^0 be $\partial^i\Sigma$ and $\partial^j\Sigma$, respectively, and suppose $\pi_E \circ v|_{\partial^j\Sigma} = 1 + \sqrt{-1}$. As a result, the other puncture ξ' that is adjacent to $\partial^j\Sigma$ is also mapped to $1 + \sqrt{-1}$ by $\pi_E \circ v$. Let the other boundary component of Σ that is adjacent to ξ' be $\partial^{j'}\Sigma$.

By the Lagrangian boundary conditions, $v(\partial^i\Sigma)$ and $v(\partial^{j'}\Sigma)$ are both contained in the Lagrangian components of $\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}}$ (or $\underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}}$). However, each of $\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}}$ and $\underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}}$ has only one component whose projection to $\mathbb{H}^\circ$ contains $1 + \sqrt{-1}$. Therefore $\partial^i\Sigma = \partial^{j'}\Sigma$, and the two punctures adjacent to $\partial^i\Sigma$ are both mapped to $1 + \sqrt{-1}$ under $\pi_E \circ v$. By the open mapping theorem, the restriction of $\pi_E \circ v$ to the connected component of Σ that contains ξ^0 is a constant map, and that connected component of Σ is a bigon.

The matching paths starting from $1 + \sqrt{-1}$ and the Lagrangian matching spheres lying above those paths are not Lagrangian boundary conditions of the restriction of v to the other connected components of Σ. We can therefore apply the previous reasoning inductively to $a = 1,\dots,m-2n$. The conclusion is that any $v : \Sigma \to E$ contributing to the $\underline{x}_0$-coefficient of $\mu^1(\underline{x}_1)$ contains $m-2n$ bigon components, each of which maps by a constant map to $a + \sqrt{-1}$ under $\pi_E \circ v$, for $a = 1,\dots,m-2n$, respectively. This implies that $\{1,\dots,m-2n\} + \sqrt{-1} \subset \pi_E(\underline{x}_1)$, and hence $\underline{x}_1 \in \mathcal{G}$. ■

Corollary 7.3. *$HF(\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}})/I_{\lambda_0,\lambda_1} = HF(\underline{L}_{\lambda_0}, \underline{L}_{\lambda_1})$ as graded vector spaces.*

Proof. For $j = 0, 1$, by removing the matching paths of $\overline{\boldsymbol{c}(\lambda_j)}^{\lambda_0,\lambda_1}$ that contain $a + \sqrt{-1}$ for some $a = 1,\dots,m-2n$ and translating the remaining matching paths by $-(m-2n)$, we get an upper matching $\overline{\lambda_j}$ of λ_j. Therefore, there is an obvious bijective correspondence

$$\mathcal{G}_{\lambda_0,\lambda_1} \simeq \mathcal{X}(\underline{L}_{\overline{\lambda_0}}, \underline{L}_{\overline{\lambda_1}}) \tag{7.4}$$

given by forgetting the elements in a tuple that lie above $\{1,\dots,m-2n\}+\sqrt{-1}$, and then translating the rest by $-(m-2n)$.

By the last paragraph of the proof of Lemma 7.2, there is a canonical isomorphism between the chain complexes

$$CF(\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}})/\mathcal{I}_{\lambda_0,\lambda_1} \cong CF(\underline{L}_{\overline{\lambda_0}}, \underline{L}_{\overline{\lambda_1}}) \tag{7.5}$$

which on generators is given by (7.4), and on differentials by incorporating or removing the $m-2n$ constant bigon components from the proof of Lemma 7.2. ∎

Lemma 7.4. *The subspace I_{λ_0,λ_1} is independent of the choices of $\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}$ and $\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}$, provided (7.2) is satisfied.*

Proof. Let $\boldsymbol{c}(\lambda_0)^a$ and $\boldsymbol{c}(\lambda_0)^b$ be two different choices of $\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}$ such that (7.2) is satisfied for the pairs $(\boldsymbol{c}(\lambda_0)^a, \overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1})$ and $(\boldsymbol{c}(\lambda_0)^b, \overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1})$.

By interpolating slopes, there exists another choice $\boldsymbol{c}(\lambda_0)^c$ of $\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}$ such that (7.2) is satisfied for the pairs $(\boldsymbol{c}(\lambda_0)^c, \boldsymbol{c}(\lambda_0)^a)$ and $(\boldsymbol{c}(\lambda_0)^c, \boldsymbol{c}(\lambda_0)^b)$. Let θ be a continuation element (i.e the image of the identity element under a continuation map) of $HF(\underline{L}_{\boldsymbol{c}(\lambda_0)^c}, \underline{L}_{\boldsymbol{c}(\lambda_0)^a})$.

Let I^a and I^c be the respective I_{λ_0,λ_1} for the spaces $HF(\underline{L}_{\boldsymbol{c}(\lambda_0)^a}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}})$ and $HF(\underline{L}_{\boldsymbol{c}(\lambda_0)^c}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}})$. We need to show that the isomorphism

$$\mu^2(-,\theta): HF(\underline{L}_{\boldsymbol{c}(\lambda_0)^a}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}}) \to HF(\underline{L}_{\boldsymbol{c}(\lambda_0)^c}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}}) \tag{7.6}$$

sends I^a to I^c. By Corollary 7.3 and for dimension reasons, it suffices to show that the image of I^a under $\mu^2(-,\theta)$ is contained in I^c. The same will then be true when we replace $\boldsymbol{c}(\lambda_0)^a$ by $\boldsymbol{c}(\lambda_0)^b$, so the result will follow.

The proof that $\mu^2(I^a,\theta) \subset I^c$ is similar to the proof of Lemma 7.2. Let $\mathcal{G}^a$ and $\mathcal{G}^c$ be the respective $\mathcal{G}_{\lambda_0,\lambda_1}$ for $CF(\underline{L}_{\boldsymbol{c}(\lambda_0)^a}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}})$ and $CF(\underline{L}_{\boldsymbol{c}(\lambda_0)^c}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}})$. By slight abuse of notation, we denote a chain level lift of θ to $CF(\underline{L}_{\boldsymbol{c}(\lambda_0)^c}, \underline{L}_{\boldsymbol{c}(\lambda_0)^a})$ by θ. It suffices to show that if the $\underline{x}_0$-coefficient of $\mu^2(\underline{x}_1,\theta)$ is non-zero for some $\underline{x}_0 \in \mathcal{G}^c$, then $\underline{x}_1 \in \mathcal{G}^a$.

By definition, $\underline{x}_0 \in \mathcal{G}^c$ implies that $\{1,\dots,m-2n\}+\sqrt{-1} \subset \pi_E(\underline{x}_0)$. Let $v:\Sigma \to E$ be a J-holomorphic map contributing to the $\underline{x}_0$-coefficient of $\mu^2(\underline{x}_1,\theta)$ such that near an output puncture ξ^0, v is asymptotic to the output lying above $1+\sqrt{-1}$. We project v to $\mathbb{H}^\circ$ by π_E and apply the open mapping theorem. Note that both $(\boldsymbol{c}(\lambda_0)^a, \overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1})$ and $(\boldsymbol{c}(\lambda_0)^c, \boldsymbol{c}(\lambda_0)^a)$ satisfying (7.2) implies that $(\boldsymbol{c}(\lambda_0)^c, \overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1})$ also satisfies (7.2). This forces at least one of the boundary components of Σ adjacent to ξ^0 to be mapped constantly to $1+\sqrt{-1}$ by $\pi_E \circ v$. Let the two boundary components of Σ adjacent to ξ^0 be $\partial^i \Sigma$ and $\partial^j \Sigma$, respectively, and suppose $\pi_E \circ v|_{\partial^j \Sigma} = 1+\sqrt{-1}$. As before, the other puncture ξ' that is adjacent to $\partial^j \Sigma$ is also mapped to $1+\sqrt{-1}$ by $\pi_E \circ v$. Let the other boundary component of Σ that is adjacent to ξ' be $\partial^{j'} \Sigma$.

By examining the slopes of the matching paths at $1+\sqrt{-1}$ again, either $\partial^i \Sigma$ or $\partial^{j'} \Sigma$ is mapped to the constant value $1+\sqrt{-1}$ by $\pi_E \circ v$. Say $\partial^{j'} \Sigma$ is mapped this way; then the next boundary components of Σ and $\partial^i \Sigma$ are mapped to components of the same Lagrangian tuple (i.e. $\underline{L}_{\boldsymbol{c}(\lambda_0)^c}$, $\underline{L}_{\boldsymbol{c}(\lambda_0)^a}$ or $\underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}}$) under v. As a result, these two boundary components coincide, $\pi_E \circ v$ restricted to the component of Σ containing ξ^0 is a constant map, and this component is a triangle.

Applying this argument to all $a+\sqrt{-1}$ for $a = 1,\dots,m-2n$, we conclude that $\{1,\dots,m-2n\}+\sqrt{-1} \subset \pi_E(\underline{x}_1)$ and hence $\underline{x}_1 \in \mathcal{G}^a$. ■

Remark 7.5. Lemma 7.4 also proves that the continuation element θ lies in the vector subspace spanned by the corresponding $\mathcal{G} \subset CF(\underline{L}_{\boldsymbol{c}(\lambda_0)^c}, \underline{L}_{\boldsymbol{c}(\lambda_0)^a})$.

Lemma 7.6. *$I \subset H^{\boldsymbol{c}}_{n,m}$ is an ideal. Moreover, there is an algebra isomorphism $H^{\boldsymbol{c}}_{n,m}/I \simeq H^{\mathrm{symp}}_{n,m}$.*

Proof. By Lemma 7.4, for $\lambda_0,\lambda_1,\lambda_2 \in \Lambda^c_{n,m}$, we can choose the upper matchings $\overline{\boldsymbol{c}(\lambda_j)}$ such that condition (7.2) is satisfied for both $(\overline{\boldsymbol{c}(\lambda_0)}, \overline{\boldsymbol{c}(\lambda_1)})$ and $(\overline{\boldsymbol{c}(\lambda_1)}, \overline{\boldsymbol{c}(\lambda_2)})$.

In this case, the same argument as in the proof of Lemma 7.4 shows that if the $\underline{x}_0$-coefficient of $\mu^2(\underline{x}_2,\underline{x}_1)$ is non-zero for some $\underline{x}_0 \in \mathcal{G}^{0,2}$, then $\underline{x}_1 \in \mathcal{G}^{0,1}$ and $\underline{x}_2 \in \mathcal{G}^{1,2}$, where $\mathcal{G}^{i,j}$ is the respective $\mathcal{G}$ for $CF(\underline{L}_{\overline{\boldsymbol{c}(\lambda_i)}}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_j)}})$. We define $\mathcal{B}^{i,j}$ similarly. This implies that if $\underline{x}_1 \in \mathcal{B}^{0,1}$, $\underline{x}_2 \in \mathcal{B}^{1,2} \cup \mathcal{G}^{1,2}$ (i.e. $\underline{x}_2$ is a basis element in $CF(\underline{L}_{\overline{\boldsymbol{c}(\lambda_1)}}, \underline{L}_{\overline{\boldsymbol{c}(\lambda_2)}})$), $\underline{x}_0 \in \mathcal{B}^{0,2} \cup \mathcal{G}^{0,2}$ and the $\underline{x}_0$-coefficient of $\mu^2(\underline{x}_2,\underline{x}_1)$ is non-zero, then $\underline{x}_0 \in \mathcal{B}^{0,2}$. Similarly, if $\underline{x}_1 \in \mathcal{B}^{0,1} \cup \mathcal{G}^{0,1}$, $\underline{x}_2 \in \mathcal{B}^{1,2}$, $\underline{x}_0 \in \mathcal{B}^{0,2} \cup \mathcal{G}^{0,2}$ and the $\underline{x}_0$-coefficient of $\mu^2(\underline{x}_2,\underline{x}_1)$ is non-zero, then $\underline{x}_0 \in \mathcal{B}^{0,2}$. This precisely says that I is an ideal in $H^{\boldsymbol{c}}_{n,m}$. (We are working in the cohomological algebra, so we only need to check closure under μ^2.)

Moreover, we also know that any u contributing to the $\underline{x}_0$-coefficient of $\mu^2(\underline{x}_2,\underline{x}_1)$ has $m-2n$ components of constant triangles. Therefore, under the identification of generators (7.4) in Corollary 7.3, the moduli spaces of holomorphic triangles defining the products μ^2 on the two sides are also canonically identified. The result follows. ■

Next, we want to understand the ideal I in terms of the geometric basis of $H^{\boldsymbol{c}}_{n,m}$. Consider the cochain model (cf. (6.16), (7.1))

$$H^{\boldsymbol{c}}_{n,m} = \bigoplus_{\lambda_0,\lambda_1 \in \Lambda^c_{n,m}} HF(\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}}, \underline{L}_{\widetilde{\boldsymbol{c}(\lambda_1)}}) = \bigoplus_{\lambda_0,\lambda_1 \in \Lambda^c_{n,m}} CF(\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}}, \underline{L}_{\widetilde{\boldsymbol{c}(\lambda_1)}}). \tag{7.7}$$

Define $\mathcal{I}^{\mathrm{arc}}_{\lambda_0,\lambda_1}$ to be the vector subspace of $CF(\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}}, \underline{L}_{\widetilde{\boldsymbol{c}(\lambda_1)}})$ generated by

$$\mathcal{B}^{\mathrm{arc}}_{\lambda_0,\lambda_1} := \left\{\underline{x} \in \mathcal{X}(\underline{L}_{\overline{\boldsymbol{c}(\lambda_0)}}, \underline{L}_{\widetilde{\boldsymbol{c}(\lambda_1)}}) \;\middle|\; \{1,\dots,m-2n\}+\sqrt{-1} \not\subset \pi_E(\underline{x})\right\}. \tag{7.8}$$

Let $\mathcal{G}^{\mathrm{arc}}_{\lambda_0,\lambda_1} := \mathcal{X}(L_{\overline{\boldsymbol{c}(\lambda_0)}}, L_{\widetilde{\boldsymbol{c}(\lambda_1)}}) \setminus \mathcal{B}^{\mathrm{arc}}_{\lambda_0,\lambda_1}$ and define $\mathcal{J}^{\mathrm{arc}}$, $\mathcal{B}^{\mathrm{arc}}$ and $\mathcal{G}^{\mathrm{arc}}$ accordingly.

Lemma 7.7. *There is an algebra isomorphism from* $\bigoplus HF(L_{\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}}, L_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}})$ *to* $\bigoplus CF(L_{\overline{\boldsymbol{c}(\lambda_0)}}, L_{\widetilde{\boldsymbol{c}(\lambda_1)}})$ *sending* I *to* $\mathcal{J}^{\mathrm{arc}}$.

Proof. Let $\theta_{\lambda_0,\lambda_1} \in HF^0(L_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}}, L_{\widetilde{\boldsymbol{c}(\lambda_1)}})$ be a generator over $\mathbb{Z}$. Then, for a sign-consistent choice of $\theta_{\lambda_0,\lambda_1}$, the direct sum of

$$\mu^2(\theta_{\lambda_0,\lambda_1}, -) : HF(L_{\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}}, L_{\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}}) \to HF(L_{\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1}}, L_{\widetilde{\boldsymbol{c}(\lambda_1)}}) \tag{7.9}$$

over all (λ_0, λ_1) is an algebra isomorphism.

Without loss of generality, we can assume that $\overline{\boldsymbol{c}(\lambda_0)}^{\lambda_0,\lambda_1} = \overline{\boldsymbol{c}(\lambda_0)}$. Note that we can choose $\widetilde{\boldsymbol{c}(\lambda_1)}$ such that the pair $(\overline{\boldsymbol{c}(\lambda_1)}^{\lambda_0,\lambda_1}, \widetilde{\boldsymbol{c}(\lambda_1)})$ also satisfies (7.2). By the same reasoning as in Lemmas 7.4 and 7.6, the image of $\mathcal{J}$ under (7.9) is contained in $\mathcal{J}^{\mathrm{arc}}$. Since $\mu^2(\theta_{\lambda_0,\lambda_1}, -)$ is an isomorphism on cohomology and the dimensions spanned by I and $\mathcal{J}^{\mathrm{arc}}$ in cohomology are the same, the image of I is precisely $\mathcal{J}^{\mathrm{arc}}$. ∎

Lemmas 7.6 and 7.7 yield an algebra isomorphism $\bigoplus CF(L_{\overline{\boldsymbol{c}(\lambda_0)}}, L_{\widetilde{\boldsymbol{c}(\lambda_1)}})/\mathcal{J}^{\mathrm{arc}} \simeq H^{\mathrm{symp}}_{n,m}$. We want to show that this isomorphism respects the geometric basis.

For $\lambda_0, \lambda_1 \in \Lambda^c_{n,m}$, we consider the composition of quasi-isomorphisms

$$\begin{aligned} CF(L_{\bar{\lambda}_0}, L_{\tilde{\lambda}_1}) \to CF(L_{\bar{\lambda}_0}, L_{\bar{\lambda}_1}) \to CF(L_{\overline{\boldsymbol{c}(\lambda_0)}}, L_{\overline{\boldsymbol{c}(\lambda_1)}})/\mathcal{J}_{\lambda_0,\lambda_1} \\ \to CF(L_{\overline{\boldsymbol{c}(\lambda_0)}}, L_{\widetilde{\boldsymbol{c}(\lambda_1)}})/\mathcal{J}^{\mathrm{alg}}_{\lambda_0,\lambda_1} \end{aligned} \tag{7.10}$$

The first arrow is given by $\mu^2(\theta, -)$ for a continuation element $\theta \in CF(L_{\tilde{\lambda}_1}, L_{\bar{\lambda}_1})$. The second arrow is the inverse of the chain isomorphism (7.5). The last arrow is (7.9), which is given by $\mu^2(\theta', -)$ for a continuation element $\theta' \in CF^0(L_{\overline{\boldsymbol{c}(\lambda_1)}}, L_{\widetilde{\boldsymbol{c}(\lambda_1)}})$.

We can assume that $\bar{\lambda}_0$, $\tilde{\lambda}_1$ and $\bar{\lambda}_1$ are obtained, respectively, by removing the left-most $m-2n$ matching paths of $\overline{\boldsymbol{c}(\lambda_0)}$, $\widetilde{\boldsymbol{c}(\lambda_1)}$ and $\overline{\boldsymbol{c}(\lambda_1)}$, and translating by $-(m-2n)$. Let θ^{-1} be a quasi-inverse of θ. We can choose θ' to be the image of θ^{-1} under the canonical inclusion $CF(L_{\bar{\lambda}_1}, L_{\tilde{\lambda}_1}) \to CF(L_{\overline{\boldsymbol{c}(\lambda_1)}}, L_{\widetilde{\boldsymbol{c}(\lambda_1)}})$ In this case, the composition (7.10) coincides with the canonical cochain isomorphism

$$CF(L_{\bar{\lambda}_0}, L_{\tilde{\lambda}_1}) \to CF(L_{\overline{\boldsymbol{c}(\lambda_0)}}, L_{\widetilde{\boldsymbol{c}(\lambda_1)}})/\mathcal{J}^{\mathrm{alg}}_{\lambda_0,\lambda_1}, \quad \underline{x} \mapsto \underline{y}, \tag{7.11}$$

characterized by $\pi_E(\underline{y}) = \{1, \dots, m-2n\} \cup (\pi_E(\underline{x}) + m - 2n)$ (i.e. essentially the same map as in (7.4), (7.5)). As a consequence, there is an algebra isomorphism from $H^{\mathrm{symp}}_{n,m}$ to $H^{\boldsymbol{c}}_{n,m}/I$ respecting the geometric basis.

For the combinatorial arc algebra, we have exactly the same quotient description for $H^{\mathrm{alg}}_{n,m}$ (see (6.39)). To conclude, we have

Proposition 7.8. *The isomorphism* Φ *of Lemma* 6.5 *restricts to an algebra isomorphism from* $H^{\mathrm{symp}}_{n,m}$ *to* $H^{\mathrm{alg}}_{n,m}$.

7.2. All cases

In this section we show that Φ is an algebra isomorphism from $K^{\mathrm{symp}}_{n,m}$ to $K^{\mathrm{alg}}_{n,m}$ for all n, m. The strategy is similar to the previous section.

Let $\boldsymbol{e} : \Lambda_{n,m} \to \Lambda^c_{n,m+n}$ be the inclusion given by (6.45). Let

$$H^{\boldsymbol{e}}_{n,m} := \bigoplus_{\lambda_0,\lambda_1 \in \Lambda_{n,m}} HF(\underline{L}_{\boldsymbol{e}(\lambda_0)}, \underline{L}_{\boldsymbol{e}(\lambda_1)}), \tag{7.12}$$

which is a subalgebra of $H^{\mathrm{symp}}_{n,m+n} = H^{\mathrm{alg}}_{n,m+n}$, so in particular $H^{\boldsymbol{e}}_{n,m}$ is formal. We want to compare $K^{\mathrm{symp}}_{n,m}$ with $H^{\boldsymbol{e}}_{n,m}$.

For each ordered pair (λ_0, λ_1) such that $\lambda_0, \lambda_1 \in \Lambda_{n,m}$, we choose upper matchings $\overline{\boldsymbol{e}(\lambda_0)}$ and $\overline{\boldsymbol{e}(\lambda_1)}$ such that

for $a = 1, \dots, n$, the slope at $m + a + \sqrt{-1}$ of the matching paths in $\overline{\boldsymbol{e}(\lambda_0)}$ that ends at $m + a + \sqrt{-1}$ is more negative than that of $\overline{\boldsymbol{e}(\lambda_1)}$. (7.13)

Since the choices of matching depend on the ordered pair of weights, we denote the matchings by $\overline{\boldsymbol{e}(\lambda_0)}^{\lambda_0,\lambda_1}$ and $\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}$ respectively. Let J_{λ_0,λ_1} be the subspace of $CF(\underline{L}_{\overline{\boldsymbol{e}(\lambda_0)}^{\lambda_0,\lambda_1}}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}})$ generated by

$$\mathcal{B}_{\lambda_0,\lambda_1} := \{\underline{x} \in \mathcal{X}(\underline{L}_{\overline{\boldsymbol{e}(\lambda_0)}^{\lambda_0,\lambda_1}}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}}) \mid m + a + \sqrt{-1} \in \pi_E(\underline{x}) \text{ for some } a = 1, \dots, n\}. \tag{7.14}$$

Let

$$\mathcal{G}_{\lambda_0,\lambda_1} := \mathcal{X}(\underline{L}_{\overline{\boldsymbol{e}(\lambda_0)}^{\lambda_0,\lambda_1}}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}}) \setminus \mathcal{B}_{\lambda_0,\lambda_1}$$

and define $\mathcal{J} := \bigoplus \mathcal{J}_{\lambda_0,\lambda_1}$, $\mathcal{B} := \bigoplus \mathcal{B}_{\lambda_0,\lambda_1}$ and $\mathcal{G} := \bigoplus \mathcal{G}_{\lambda_0,\lambda_1}$.

Lemma 7.9. *$\mu^1(\mathcal{J}_{\lambda_0,\lambda_1}) \subset \mathcal{J}_{\lambda_0,\lambda_1}$ so $\mathcal{J}_{\lambda_0,\lambda_1}$ descends to a vector subspace of $H^{\boldsymbol{e}}_{n,m}$, which we denote by J_{λ_0,λ_1}.*

Proof. Let $\underline{x}_1 \in \mathcal{B}$. We want to show that if the $\underline{x}_0$-coefficient of $\mu^1(\underline{x}_1)$ is non-zero then $\underline{x}_0 \in \mathcal{B}$.

By definition, $\underline{x}_1 \in \mathcal{G}$ implies that $m + a + \sqrt{-1} \subset \pi_E(\underline{x}_1)$ for some $a = 1, \dots, n$. Let $v : \Sigma \to E$ be a J-holomorphic map contributing to the $\underline{x}_0$-coefficient of $\mu^1(\underline{x}_1)$ such that near an input puncture ξ, v is asymptotic to the element of $\underline{x}_1$ lying above $m + a + \sqrt{-1}$. We project v to $\mathbb{H}^\circ$ by π_E and apply the open mapping theorem. The condition (7.13) at $m + a + \sqrt{-1}$ forces at least one of the boundary components of Σ adjacent to ξ to be mapped constantly to $m + a + \sqrt{-1}$ by $\pi_E \circ v$. It immediately implies that $m + a + \sqrt{-1} \in \pi_E(\underline{x}_0)$. ∎

Corollary 7.10. *$HF(\underline{L}_{\boldsymbol{e}(\lambda_0)}, \underline{L}_{\boldsymbol{e}(\lambda_1)})/J_{\lambda_0,\lambda_1} = HF(\underline{L}_{\lambda_0}, \underline{L}_{\lambda_1})$ as graded vector spaces.*

Proof. For $j = 0, 1$, by forgetting the points $\{1, \ldots, n\} + m + \sqrt{-1}$ and 'extending' the matching paths of $\boldsymbol{e}(\lambda_j)^{\lambda_0,\lambda_1}$ that contain $m + a + \sqrt{-1}$ for some $a = 1, \ldots, n$ to thimble paths, we get an upper matching $\overline{\lambda_j}$ of λ_j. There is then an obvious bijective correspondence

$$\mathcal{G}_{\lambda_0,\lambda_1} \simeq \mathcal{X}(\underline{L}_{\overline{\lambda_0}}, \underline{L}_{\overline{\lambda_1}}) \tag{7.15}$$

Moreover, there is a canonical isomorphism between the chain complexes

$$CF(\underline{L}_{\overline{\boldsymbol{e}(\lambda_0)}^{\lambda_0,\lambda_1}}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}})/\mathcal{J}_{\lambda_0,\lambda_1} \cong CF(\underline{L}_{\overline{\lambda_0}}, \underline{L}_{\overline{\lambda_1}}) \tag{7.16}$$

which on generators is given by (7.15); the differentials on the left and right agree by the proof of Lemma 7.9. ∎

Lemma 7.11. *The subspace J_{λ_0,λ_1} is independent of the choices of $\overline{\boldsymbol{e}(\lambda_0)}^{\lambda_0,\lambda_1}$ and $\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}$, provided (7.13) is satisfied.*

Proof. Let $\boldsymbol{e}(\lambda_0)^a$ and $\boldsymbol{e}(\lambda_0)^b$ be two different choices of $\overline{\boldsymbol{e}(\lambda_0)}^{\lambda_0,\lambda_1}$ such that (7.13) is satisfied for the pairs $(\boldsymbol{e}(\lambda_0)^a, \overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1})$ and $(\boldsymbol{e}(\lambda_0)^b, \overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1})$.

There exists another choice $\boldsymbol{e}(\lambda_0)^c$ of $\overline{\boldsymbol{e}(\lambda_0)}^{\lambda_0,\lambda_1}$ such that (7.2) is satisfied for the pairs $(\boldsymbol{e}(\lambda_0)^c, \boldsymbol{e}(\lambda_0)^a)$ and $(\boldsymbol{e}(\lambda_0)^c, \boldsymbol{e}(\lambda_0)^b)$. Let θ be a continuation element of $HF(\underline{L}_{\boldsymbol{e}(\lambda_0)^c}, \underline{L}_{\boldsymbol{e}(\lambda_0)^a})$.

Let J^a and J^c be the respective J_{λ_0,λ_1} for the spaces $HF(\underline{L}_{\boldsymbol{e}(\lambda_0)^a}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}})$ and $HF(\underline{L}_{\boldsymbol{e}(\lambda_0)^c}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}})$. We need to show that the isomorphism

$$\mu^2(-, \theta) : HF(\underline{L}_{\boldsymbol{e}(\lambda_0)^a}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}}) \to HF(\underline{L}_{\boldsymbol{e}(\lambda_0)^c}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}}) \tag{7.17}$$

sends J^a to J^c. By Corollary 7.10 and a dimension count, it suffices to show that the image of J^a under $\mu^2(-, \theta)$ is contained in J^c.

The proof that $\mu^2(J^a, \theta) \subset J^c$ is similar to the proof of Lemma 7.9. Let $\mathcal{B}^a$ and $\mathcal{B}^c$ be the respective $\mathcal{B}_{\lambda_0,\lambda_1}$ for $CF(\underline{L}_{\boldsymbol{e}(\lambda_0)^a}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}})$ and $CF(\underline{L}_{\boldsymbol{e}(\lambda_0)^c}, \underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}})$. By slight abuse of notation, we denote a chain level lift of θ to $CF(\underline{L}_{\boldsymbol{e}(\lambda_0)^c}, \underline{L}_{\boldsymbol{e}(\lambda_0)^a})$ by θ. It suffices to show that if $\underline{x}_1 \in \mathcal{B}^a$ and the $\underline{x}_0$-coefficient of $\mu^2(\underline{x}_1, \theta)$ is non-zero, then $\underline{x}_0 \in \mathcal{B}^c$.

Let $v : \Sigma \to E$ be a J-holomorphic map contributing to the $\underline{x}_0$-coefficient of $\mu^2(\underline{x}_1, \theta)$ such that near an input puncture ξ, v is asymptotic to the element of $\underline{x}_1$ lying above $m + a + \sqrt{-1}$. We project v to $\mathbb{H}^\circ$ by π_E and apply the open mapping theorem. Since $(\boldsymbol{e}(\lambda_0)^a, \overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1})$ satisfies (7.13), it forces at least one of the boundary components of Σ adjacent to ξ to be mapped constantly to $m + a + \sqrt{-1}$ by $\pi_E \circ v$. Let the two boundary components of Σ adjacent to ξ be $\partial^i \Sigma$ and $\partial^j \Sigma$, with Lagrangian labels $\underline{L}_{\boldsymbol{e}(\lambda_0)^a}$ and $\underline{L}_{\overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1}}$, respectively.

If $\pi_E \circ v|_{\partial^j \Sigma} = m + a + \sqrt{-1}$, then we are done because we must have $m + a + \sqrt{-1} \in \pi_E(\underline{x}_0)$.

If $\pi_E \circ v|_{\partial^i \Sigma} = m + a + \sqrt{-1}$, then we denote the next boundary component of Σ adjacent to $\partial^i \Sigma$ by $\partial^{i'} \Sigma$, which is equipped with the Lagrangian label $\underline{L}_{\boldsymbol{e}(\lambda_0)^c}$. Using the fact that $(\boldsymbol{e}(\lambda_0)^c, \overline{\boldsymbol{e}(\lambda_1)}^{\lambda_0,\lambda_1})$ satisfies (7.13), we conclude that either $\partial^j \Sigma$ or $\partial^{i'} \Sigma$ is mapped to $m + a + \sqrt{-1}$ under $\pi_E \circ v$. In either case, $m + a + \sqrt{-1} \in \pi(\underline{x}_0)$. ∎

By very similar arguments, the analogues of Lemmas 7.6 and 7.7 hold. In particular, we obtain algebra isomorphisms

$$K_{n,m}^{\mathrm{symp}} \simeq H_{n,m}^{\boldsymbol{e}}/J \simeq \bigoplus CF(\underline{L}_{\overline{\boldsymbol{e}(\lambda_0)}}, \underline{L}_{\boldsymbol{e}(\tilde{\lambda}_1)})/\mathcal{J}^{\mathrm{arc}} \tag{7.18}$$

where $\mathcal{J}^{\mathrm{arc}}$ is generated by (7.14). There is also exactly the same quotient description for $K_{n,m}^{\mathrm{alg}}$ (see (6.47)). As a result, the analogue of Proposition 7.8 is also true.

Proposition 7.12. *The map Φ of Lemma 6.5 gives an algebra isomorphism from $K_{n,m}^{\mathrm{symp}}$ to $K_{n,m}^{\mathrm{alg}}$.*

8. An nc-vector field

The algebra $K_{n,m}^{\mathrm{symp}}$ in principle carries a non-trivial A_∞ structure. Seidel gave a necessary and sufficient criterion for formality of an A_∞ algebra: it should admit a degree 1 Hochschild cohomology class which acts by the Euler field (see Theorem 8.8; a proof is given in [1]). Following the language of non-commutative geometry, we call a degree 1 Hochschild cocycle a *non-commutative vector field* or nc-vector field; the motivating example is a global vector field $V \in H^0(\mathcal{T}_Z) \subset HH^1(D^b(\mathrm{Coh}(Z))$ on an algebraic variety Z. In this section, we apply the method from [1] to construct an nc-vector field by counting holomorphic discs with prescribed conormal-type conditions at infinity. This will be well-defined even if $\underline{L}$ has some non-compact Lagrangian components (see Remark 8.2).

8.1. *Moduli spaces of maps revisited*

We first explain how to modify the moduli spaces of maps used in [1] to construct an nc-vector field in our setting. Let $\mathcal{R}_{(0,1)}^{d+1,h}$ be the moduli space of unit discs S with $d+1$ boundary punctures $\{\xi_i\}_{i=0}^d$, h ordered interior marked points $\mathrm{mk}(S)$ and two more **distinguished** ordered interior marked points s_0, s_1 such that s_1 lies on the hyperbolic geodesic arc between s_0 and ξ_0 (but $s_1 \neq s_0$ and $s_1 \neq \xi_0$). We have $\dim(\mathcal{R}_{(0,1)}^{d+1,h}) = \dim(\mathcal{R}^{d+1,h}) + 3$. We call the h ordered interior marked points *type*-1 interior marked points, and the two distinguished ordered interior marked points *type*-2 interior marked points. We denote $\mathrm{mk}(S) \cup \{s_0, s_1\}$ by $\mathrm{mk}(S)^+$.

We can compactify $\mathcal{R}_{(0,1)}^{d+1,h}$ to $\overline{\mathcal{R}}_{(0,1)}^{d+1,h}$ as in [1, Section 3.6] (which treats the case $h = 0$). Informally, the compactification includes stable broken configurations as considered in [1, Section 3.6], with an extra condition that nodal sphere components arise when some of the interior marked points collide. In particular, the codimension 1 boundary facets of $\overline{\mathcal{R}}^{d+1,h}_{(0,1)}$ are given by (in all the moduli below, the subset $P \subset \{1, \dots, h\}$ remembers the type-1 ordered marked points that go to the same component):

(1) ($\{\xi_{i+1}, \dots, \xi_{i+j}\}$ move together) A nodal domain in which a collection of input boundary punctures bubble off:

$$\coprod_{1 \le j \le d,\, 0 \le i \le d-j,\, h_1+h_2=h,\, P \subset \{1,\dots,h\},\, |P|=h_1} \mathcal{R}^{d-j+1,h_1}_{(0,1)} \times \mathcal{R}^{j+1,h_2}. \tag{8.1}$$

When $j = 1$, we need $h_2 > 0$ so that the domain is stable.

(2) ($s_1 \to s_0$) A domain with a sphere bubble carrying the two type-2 interior marked points and some type-1 interior marked points attached to a disc carrying the remaining type-1 interior marked points and the $d+1$ boundary punctures. Letting $\mathcal{M}^h_{0,3}$ denote the moduli space of spheres with one interior node, two type-2 marked points and h unordered type-1 interior marked points, and $\mathcal{R}^{d+1,h}_1$ denote the moduli space of unit discs with one interior node, $d+1$ boundary punctures and h type-1 interior marked points, this component is

$$\coprod_{h_1+h_2=h,\, P \subset \{1,\dots,h\},\, |P|=h_1} \mathcal{R}^{d+1,h_1}_1 \times \mathcal{M}^{h_2}_{0,3}. \tag{8.2}$$

The attaching point is understood to be the node on the disc and on the sphere respectively.

(3) ($\{\xi_{d-l+1}, \dots, \xi_d, \xi_1, \dots, \xi_i\} \to \xi_0$) A nodal domain with two discs, one carrying both type-2 interior marked points, some type-1 interior marked points and the boundary punctures $\{\xi_{i+1}, \dots, \xi_{d-l}\}$, the other carrying the remaining type-1 interior marked points and boundary punctures

$$\coprod_{0 \le i+l \le d,\, h_1+h_2=h,\, P \subset \{1,\dots,h\},\, |P|=h_1} \mathcal{R}^{i+l+1,h_1} \times \mathcal{R}^{d-i-l+1,h_2}_{(0,1)}. \tag{8.3}$$

When $i + l = 0$, we need $h_1 > 0$ so that the domain is stable.

(4) ($\{s_1\} \cup \{\xi_1, \dots, \xi_i, \xi_{d-l+1}, \dots, \xi_d\} \to \xi_0$) A nodal domain with two discs, one carrying $d-i-l$ input boundary punctures, one type-2 interior marked point and some type-1 interior marked points, the bubble carrying the second type-2 interior marked point, and the remaining type-1 interior marked points and boundary punctures:

$$\coprod_{0 \le i+l \le d,\, h_1+h_2=h,\, P \subset \{1,\dots,h\},\, |P|=h_1} \mathcal{R}^{(i+l+1)+1,h_1}_1 \times \mathcal{R}^{d-i-l+1,h_2}_1. \tag{8.4}$$

We pick a consistent choice of strip-like ends and marked-points neighbourhoods for elements in $\overline{\mathcal{R}}^{d+1,h}_{(0,1)}$ as in Section 2.2.2. This time, we require each marked-points neighbourhood to contain s_0 and s_1, and denote it by $\nu(\mathrm{mk}(S)^+)$. For each $S \in \mathcal{R}^{d+1,h}_{(0,1)}$, we can define $\mathcal{G}_{\mathrm{aff}}(S)$ by (2.42) but with $\nu(\mathrm{mk}(S))$ being replaced by $\nu(\mathrm{mk}(S)^+)$. For a cylindrical Lagrangian label associating $\partial_j S$ to $\underline{L}_j$ for $j = 0, \dots, d$ and a choice $A_S \in \mathcal{P}_{\mathrm{aff}}(S, \{(A_j, \lambda_{\underline{L}_{j-1}}, \lambda_{\underline{L}_j})\}_{j=0}^d)$, we equip S with the additional data (J, K) as in (2.48) and (2.49), again with $\nu(\mathrm{mk}(S))$ replaced by $\nu(\mathrm{mk}(S)^+)$.

Let $D_0 \subset \mathrm{Hilb}^n(\overline{E})$ be the divisor of ideals whose support meets D_E. We now assume that

$$D_0 \text{ is moveable, and the base locus of its linear system contains no rational curve,} \tag{8.5}$$

$$\text{there is a holomorphic volume form with simple poles on } D_0 \ (\text{so } c_1(\mathrm{Hilb}^n(\overline{E})) = PD(D_0)). \tag{8.6}$$

Let D_0' be a divisor in $\mathrm{Hilb}^n(\overline{E})$ linearly equivalent to, but sharing no common irreducible component with, D_0. Let $B_0 = D_0 \cap D_0'$ be the base locus of the corresponding pencil; we assume that it contains no rational curve. Note that these conditions will hold in the case of type A Milnor fibres (see [1, Section 6]), and more generally (8.5) holds when $D_E \subset \overline{E}$ is moveable and (8.6) holds when there is a holomorphic volume form on $\overline{E}$ with simple poles on D_E (see [1, Lemma 6.3]).

Given $\underline{x}_0 \in \mathcal{X}(H_0, \underline{L}_0, \underline{L}_d)$ and $\underline{x}_j \in \mathcal{X}(H_j, \underline{L}_{j-1}, \underline{L}_j)$ for $j = 1, \dots, d$, we define

$$\mathcal{R}^{d+1,h}_{(0,1)}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1) \tag{8.7}$$

to be the moduli space of all maps $u : S \to \mathrm{Hilb}^n(\overline{E})$ such that

$$\begin{cases} [u] \cdot [D_r] = 0 \text{ and } [u] \cdot [D_0] = 1, \\ u(\mathrm{mk}(S)) \subset D_{HC},\ u(s_0) \in D_0,\ u(s_1) \in D_0', \\ (Du|_z - X_K|_z)^{0,1} = 0 \text{ with respect to } (J_z)_{u(z)} \text{ for } z \in S, \\ u(z) \in \mathrm{Sym}(\underline{L}_j) \text{ for } z \in \partial_j S, \\ \lim_{s \to \pm\infty} u(\epsilon^j(s, \cdot)) = \underline{x}_j(\cdot) \text{ uniformly}, \end{cases} \tag{8.8}$$

where J, K should be understood as their extension to $\mathrm{Conf}^n(\overline{E})$ (see Remark 2.14).

Note that Lemmas 2.20 and 2.21 remain true for $u \in \mathcal{R}^{d+1,h}_{(0,1)}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ so when $h = I_{\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1}$, (8.8) implies that $\mathrm{Im}(u) \cap D_r = \emptyset$ and u intersects D_{HC} transversally. On the other hand, since $J = J_E^{[n]}$ near s_0, s_1, $[u] \cdot [D_0] = 1$ implies that the intersection multiplicity of u with D_0 at s_0 and with D_0' at s_1 are both 1. (We will see later, in the proof of Lemma 8.7, that every intersection between u and D_0 contributes positively to their intersection number.)

We want to discuss the regularity and compactification of $\mathcal{R}^{d+1,h}_{(0,1)}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ following Sections 2.2.7, 2.2.9, 2.2.10 and [1].

First, we pick a consistent choice of (A_S, J, K) for all elements in $\overline{\mathcal{R}}^{d+1,h}_{(0,1)}$ and all possible cylindrical Lagrangian labels and in/outputs. In particular, this implies that if $S \in \overline{\mathcal{R}}^{d+1,h}_{(0,1)}$ has a sphere component, then over that component $K \equiv 0$ and the complex structure is $J_E^{[n]}$. We can then introduce the following moduli spaces of maps that are relevant for the compactification of $\mathcal{R}_{(0,1)}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$:

$$\mathcal{R}_1^{d+1,h}(\mathrm{Hilb}^n(\overline{E}), (d_0, d_r) \mid \underline{x}_0; \underline{x}_d, \dots, \underline{x}_1), \tag{8.9}$$

$$\mathcal{M}^h(\mathrm{Hilb}^n(\overline{E}) \setminus D_r \mid 1). \tag{8.10}$$

The moduli space (8.9) consists of maps $u : S \to \mathrm{Hilb}^n(\overline{E})$ such that $S \in \mathcal{R}_1^{d+1,h}$, and

$$\begin{cases} [u] \cdot [D_0] = d_0, \ [u] \cdot [D_r] = d_r, \\ u(\mathrm{mk}(S)) \subset D_{HC}, \\ (Du|_z - X_K|_z)^{0,1} = 0 \text{ with respect to } (J_z)_{u(z)} \text{ for } z \in S, \\ u(z) \in \mathrm{Sym}(\underline{L}_j) \text{ for } z \in \partial_j S, \\ \lim_{s \to \pm\infty} u(\epsilon^j(s, \cdot)) = \underline{x}_j(\cdot) \text{ uniformly.} \end{cases} \tag{8.11}$$

It has an evaluation map using the node to $\mathrm{Hilb}^n(\overline{E})$. The virtual dimension of (8.9) is

$$d + 2d_0 + |\underline{x}_0| - \sum_{j=1}^{d} |\underline{x}_j| \tag{8.12}$$

where $2d_0$ comes from $c_1(\mathrm{Hilb}^n(\overline{E})) = PD(D_0)$ and d comes from $\dim(\mathcal{R}_1^{d+1})$,

For $S \in \mathcal{M}_{0,3}^h$, we can rigidify the domain and assume the type-2 marked points are $0, 1$ and the node is ∞. The moduli space (8.10) consists of maps $u : S \to \mathrm{Hilb}^n(\overline{E}) \setminus D_r$ such that $S \in \mathcal{M}_{0,3}^h$,

$$\begin{cases} [u] \cdot [D_0] = 1, \\ u(\mathrm{mk}(S)) \subset D_{HC}, u(0) \in D_0, u(1) \in D_0', \\ u \text{ is } J_E^{[n]}\text{-holomorphic.} \end{cases} \tag{8.13}$$

It also has an evaluation map to $\mathrm{Hilb}^n(\overline{E})$ by evaluating at ∞. The virtual dimension of (8.10) is

$$4n + 2c_1([u]) + 2h - (4 + 2h) = 4n - 2 \tag{8.14}$$

where $4n$ is the dimension of $\mathrm{Hilb}^n(\overline{E})$, $c_1([u]) = [u] \cdot [D_0] = 1$, $2h$ is the dimension of $\mathcal{M}_{0,3}^h$ and $4 + 2h$ comes from the incidence conditions.

Proposition 8.1 (cf. [1, Lemmas 3.18, 3.19]). *Let $h = I_{\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1}$. For generic consistent choice of J, K satisfying* (2.48) *and* (2.49)*, every $u \in \mathcal{R}_{(0,1)}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ is regular. Moreover, when $\mathcal{R}_{(0,1)}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ is 0-dimensional, it is compact. When $\mathcal{R}_{(0,1)}^{d+1,h}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ is 1-dimensional, it can be compactified by adding the following boundary strata, which are themselves regular.*

(1) (*corresponding to* (8.1), (8.3))

$$\coprod_{\underline{x},\, h_1+h_2=h,\, P\subset\{1,\dots,h\},\, |P|=h_1} \mathcal{R}^{d-j+1,h_1}_{(0,1)}(\underline{x}_0;\underline{x}_d,\dots,\underline{x}_{i+j+1},\underline{x},\underline{x}_i,\dots,\underline{x}_1) \times \mathcal{R}^{j+1,h_2}(\underline{x};\underline{x}_{i+j},\dots,\underline{x}_{i+1}), \tag{8.15}$$

$$\coprod_{\underline{x},\, h_1+h_2=h,\, P\subset\{1,\dots,h\},\, |P|=h_1} \mathcal{R}^{i+l+1,h_1}(\underline{x}_0;\underline{x}_d,\dots,\underline{x}_{d-l+1},\underline{x},\underline{x}_i,\dots,\underline{x}_1) \times \mathcal{R}^{d-i-l+1,h_2}_{(0,1)}(\underline{x};\underline{x}_{d-j},\dots,\underline{x}_{i+1}), \tag{8.16}$$

(2) (*corresponding to* (8.2))

$$\mathcal{R}^{d+1,h}_1(\mathrm{Hilb}^n(\overline{E}),(0,0)\mid \underline{x}_0;\underline{x}_d,\dots,\underline{x}_1) \times_{\mathrm{Hilb}^n(\overline{E})} \mathcal{M}^0(\mathrm{Hilb}^n(\overline{E})\setminus D_r\mid 1), \tag{8.17}$$

$$\mathcal{R}^{d+1,h}_1(\mathrm{Hilb}^n(\overline{E}),(1,0)\mid \underline{x}_0;\underline{x}_d,\dots,\underline{x}_1) \times_{\mathrm{Hilb}^n(\overline{E})} B_0, \tag{8.18}$$

where in (8.17) *and* (8.18) *the fibre products are taken with respect to the evaluation maps, which are transverse.*

We will see that $\mathcal{M}^h(\mathrm{Hilb}^n(\overline{E})\setminus D_r\mid 1)$ is empty when $h>0$ (see Lemma 8.3 and Corollary 8.4), which explains why it does not appear in (8.17). In the rest of this section, we establish some regularity and compactness statements and prove Proposition 8.1.

Remark 8.2. We will establish the compactness result below following the method in Section 2.2.10. The reason one should expect compactness to hold is because we are counting discs with interior marked points going to D_0 which corresponds to the vertical infinity of E, while non-compact Lagrangian components in $\underline{L}$ are only non-compact with respect to the horizontal infinity of E, so the method in Section 2.2.10 applies without substantial changes.

8.1.1. Moduli of Chern 1 spheres. We first discuss $\mathcal{M}^h(\mathrm{Hilb}^n(\overline{E})\setminus D_r\mid 1)$. Let C be a Chern number 1 rational curve in $\overline{E}$ and $q_2,\dots,q_n\in\overline{E}$ be $n-1$ pairwise distinct points such that at most one of $q_2,\dots,q_n$ lies in C. The product determines a Chern number 1 rational curve in $\mathrm{Sym}^n(\overline{E})$ which meets $\Delta_{\overline{E}}$ in at most one point. Therefore, it uniquely lifts to an irreducible Chern number 1 rational curve $\tilde{C}$ in $\mathrm{Hilb}^n(\overline{E})$. We call $\tilde{C}$ a Chern number 1 rational curve of product type. In fact, these are essentially all the irreducible Chern number 1 rational curves in $\mathrm{Hilb}^n(\overline{E})$.

Lemma 8.3 ([1, Lemma 6.4]). *Let C be an irreducible rational curve in* $\mathrm{Hilb}^n(\overline{E})$ *with Chern number* 1 *and not contained in D_{HC}. Then C is of product type.*

Proof. By assumption, $\pi_{HC}(C)$ is an irreducible Chern number 1 rational curve in $\mathrm{Sym}^n(\overline{E})$. By the tautological correspondence, we get a (possibly disconnected) closed complex curve C' in $\overline{E}$ of Chern number 1. It means that C' lies inside a finite union of fibres of $\pi_{\overline{E}}$. Having Chern number 1 means that $[C']\cdot[D_E]=1$, so C' is a union of a Chern number 1 rational curve C'' and $n-1$ points in $\overline{E}$. By the assumption that C is not contained in D_{HC} (or equivalently $\pi_{HC}(C)$ is not contained in $\Delta_{\overline{E}}$), we know that the points are pairwise disjoint and at most one of them lies inside C''. ■

Corollary 8.4 (cf. [1, Lemma 6.4]). *The moduli space* $\mathcal{M}^h(\mathrm{Hilb}^n(\overline{E}) \setminus D_r \mid 1)$ *is regular and the evaluation map* $\mathcal{M}^h(\mathrm{Hilb}^n(\overline{E}) \setminus D_r \mid 1) \to \mathrm{Hilb}^n(\overline{E})$ *defines a pseudocycle. In other words, the image of the evaluation map can be compactified by adding a real codimension* 2 (*with respect to the image*) *subset.*

Proof. When $h > 0$, by Lemma 8.3, $\mathcal{M}^h(\mathrm{Hilb}^n(\overline{E}) \setminus D_r \mid 1)$ is actually empty so there is nothing to prove.

When $h = 0$, the regularity of $\mathcal{M}^h(\mathrm{Hilb}^n(\overline{E}) \setminus D_r \mid 1)$ follows from the explicit description of Lemma 8.3 and the fact that every direct summand of the normal bundle of a product-type Chern 1 rational curve in $\mathrm{Hilb}^n(\overline{E}) \setminus D_r$ has Chern number ≥ -1. Therefore, automatic regularity for these somewhere injective curves applies.

To define a pseudocycle, the codimension 2 subset to be added is the union of the images of *stable* Chern number 1 rational curves in $\mathrm{Hilb}^n(\overline{E})$ that meet D_r non-trivially, which is denoted by B_r in [1]. It is of codimension 2 because every such stable Chern number 1 rational curve maps to a point in $\mathrm{Sym}^n(\mathbb{H}^\circ)$ that lies inside $\Delta_{\mathbb{H}^\circ} \cap D'$ where D' is the divisor in $\mathrm{Sym}^n(\mathbb{H}^\circ)$ consisting of subschemes of $\mathbb{H}^\circ$ whose support meets critical values of π_E. ■

We denote the pseudocycle by GW_1.

8.1.2. Moduli of discs. The following regularity statement follows as in Lemma 2.15.

Lemma 8.5 (Regularity). *For generic consistent choice of* (J, K) *satisfying* (2.48) *and* (2.49)*, every element in the moduli spaces* (8.7) *and* (8.9) *is regular.*

For an element u in the moduli space (8.7) or (8.9), we can define $E(u)$ by (2.65). Even though $\overline{E}$ is not exact, we can still define ω_K^{geom} and ω_K^{top} on $\mathrm{Conf}^n(\overline{E}) \times S$. They differ by R_K which, by the same reasoning as in Lemma 2.28, is uniformly bounded independent of u. Moreover, the integration of ω_K^{top} over the graph of u is bounded *a priori* by the Lagrangian boundary conditions and the intersection numbers with D_0, D_r, because $\mathrm{Hilb}^n(\overline{E}) \setminus (D_0 \cup D_r)$ is exact. To conclude, we have

Lemma 8.6 (Energy). *For a fixed choice of* $\{\underline{x}_j\}_{j=1}^d$ *and a moduli space of maps* (8.7) *or* (8.9)*, there exists* $T > 0$ *such that for all elements* u *in the moduli space, we have* $E(u) < T$.

We also have the corresponding statement for positivity of intersections.

Lemma 8.7 (Positivity of intersection). *Let* u *be an element in* (8.7) *or* (8.9)*. If* $u(z) \in D_0$ (*resp.* $u(z) \in D_r$)*, then the intersection* $u(z)$ *between* $\mathrm{Im}(u)$ *and* D_0 (*resp.* D_r) *contribute positively to the algebraic intersection number.*

Proof. For both cases (D_0 and D_r), this follows from the fact that the Hamiltonian vector field in the perturbation term is tangent to (every stratum of) the divisor. For D_0, this is exactly Lemma 2.34. For D_r, the argument is similar to that of Lemmas 2.21 and 2.36. ■

Proof of Proposition 8.1. Let $u_k : S_k \to \mathrm{Hilb}^n(\overline{E})$ be a sequence of maps in (8.7). First consider the case that there is a subsequence of S_k converging to $S \in \mathcal{R}^{d+1,h}_{(0,1)}$. Without loss of generality, we can assume $S_k = S$ for all k.

The analogue of Lemma 2.32 holds. More precisely, if there is energy concentration at a point in S but outside $\nu(\mathrm{mk}(S)^+)$, then it will produce a sphere bubble or a disc bubble in $E^{\rceil}$ which intersects D_E. By Lemma 8.7, this will imply that for large k, the algebraic intersection number between u_k and D_0 is greater than 1, a contradiction.

Energy concentration cannot happen at a point outside $\mathrm{mk}(S)^+$: If energy concentration happens at $\mathrm{mk}(S)^+$, then it is either at $\{s_0, s_1\}$ or not. If not, the resulting sphere bubble (tree) has Chern number 0, so it is contained in $\mathrm{Hilb}^n(E)$ (recall that $c_1(\mathrm{Hilb}^n(\overline{E})) = PD(D_0)$). Therefore, it intersects D_r positively by Lemma 2.5 and gives a contradiction. If energy concentration happens at s_0 or s_1, say s_0, then the bubble tree has either Chern number 0 or Chern number 1. We get a contradiction in the former case as above. For the latter case, the Chern number 1 bubble intersects D_0'. Note that s_1 is another point in the stable domain that is mapped to D_0'. Since D_0 and D_0' are homologous, that implies that the algebraic intersection number between u and D_0 is at least 2 in total. This again gives a contradiction.

Energy concentration at strip-like ends give parts of (8.15) and (8.16). This finishes the bubbling analysis when S_k has a subsequence converging to $S \in \mathcal{R}^{d+1,h}_{(0,1)}$.

Now, if S_k has a subsequence converging to the boundary strata (8.1), (8.3), we get the remaining terms in (8.15) and (8.16).

If S_k has a subsequence converging to the boundary strata (8.4), then for k large, u has algebraic intersection at least 2 with D_0 because each disc component in the limit contributes at least 1.

Next, we consider the case that S_k has a subsequence converging to the boundary strata (8.2). Without loss of generality, we can assume $S_k = S$ lies in the boundary strata (8.2) for all k. The analogue of Lemma 2.32 still holds. That is, there is no energy concentration at a point in $S \setminus \nu(\mathrm{mk}(S)^+)$, and hence no energy concentration at a point in $S \setminus \mathrm{mk}(S)^+$. If energy concentration happens at $\mathrm{mk}(S)$, we get a Chern 0 bubble and the usual contradiction. If energy concentration happens at s_0 or s_1, say s_0, then at least one of the spheres (including the original one in the domain) has vanishing Chern number, so meets D_r. Therefore, the tree of spheres define a stable Chern number 1 rational curve in $\mathrm{Hilb}^n(\overline{E})$ which lies in a real codimension 4 subset of $\mathrm{Hilb}^n(\overline{E})$ (see Corollary 8.4). As a result, for a generic choice of data, there is no such stable Chern number 1 rational curve in the codimension 1 boundary strata of $\mathcal{R}^{d+1,h}_{(0,1)}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ when $\mathcal{R}^{d+1,h}_{(0,1)}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ is 1-dimensional. Energy concentration of solutions in (8.2) at strip-like ends is a higher codimension phenomenon.

When there is no energy concentration, the boundary strata (8.2) correspond precisely to (8.17) and (8.18).

Finally, when S_k converges to some element in higher codimension strata, regularity reasoning as in Corollary 2.39 shows that it cannot exist for generic (J, K). This completes the proof. ∎

8.2. Pure Lagrangians

An nc-vector field of an A_∞ category $\mathcal{A}$ is a cocycle $b \in CC^1(\mathcal{A}, \mathcal{A})$. Given an nc-vector field b, a b-equivariant object is a pair (L, c_L) such that $L \in \mathrm{Ob}(\mathcal{A})$, $c_L \in \hom^0_{\mathcal{A}}(L, L)$ and $b^0|_L = \mu^1(c_L)$. In this case, (L, c_L) is called a *b-equivariant* lift of L. Given two b-equivariant objects (L, c_L) and $(L', c_{L'})$, the map

$$a \mapsto b^1(a) - \mu^2(c_{L'}, a) + \mu^2(a, c_L) \tag{8.19}$$

is a chain map from $\hom_{\mathcal{A}}(L, L')$ to itself. An nc-vector field is called *pure* if every object $L \in \mathrm{Ob}(\mathcal{A})$ admits a b-equivariant lift (L, c_L) and for every pair $(L, c_L), (L', c_{L'})$ of b-equivariant lifts, the endomorphism on $H(\hom_{\mathcal{A}}(L, L'))$ induced by (8.19) is given by

$$a \mapsto |a|a \tag{8.20}$$

for all pure degree elements $a \in H(\hom_{\mathcal{A}}(L, L'))$, where $|\cdot|$ denotes degree. In other words, purity asserts that (8.19) agrees with the Euler vector field.

Theorem 8.8 (Seidel). *If an A_∞ category $\mathcal{A}$ over a field of characteristic* 0 *admits a pure nc-vector field, then $\mathcal{A}$ is formal.*

A proof is given in [1].

8.2.1. Construction. By Proposition 8.1, we can define a Hochschild cochain

$$\tilde{b} \in CC^*(\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E), \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)) \tag{8.21}$$

by counting rigid elements in $\mathcal{R}^{d+1,h}_{(0,1)}(\underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ and then divide it by $h!$. The cochain $\tilde{b}$ is not closed due to the existence of the boundary strata (8.17) and (8.18).

To compensate for these strata, we consider the closed-open maps

$$CO : C^*(\mathrm{Hilb}^n(E) \setminus D_r) \to CC^*(\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E), \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)), \tag{8.22}$$

$$co : C^{\mathrm{lf}}_{2n-2-j}(D_0, (D_0 \cap D_r) \cup D_0^{\mathrm{sing}}) \to CC^j(\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E), \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)) \quad \text{for } j \leq 3, \tag{8.23}$$

which are the direct analogues of the ones in [1, Section 3.5]. Here, lf refers to locally finite chains. The only difference is that we added the type-1 interior marked points and hence use $\mathcal{R}^{d+1,h}_1(\mathrm{Hilb}^n(\overline{E}), (d_0, d_r) \mid \underline{x}_0; \underline{x}_d, \dots, \underline{x}_1)$ for $(d_0, d_r) = (0, 0)$ and $(1, 0)$, respectively, to define CO and co.

The boundary strata (8.17) and (8.18) correspond to

$$CO(GW_1) \quad \text{and} \quad co([B_0]) \tag{8.24}$$

respectively, where we recall $B_0 = D_0 \cap D_0'$. Using the fact that both CO and co are chain maps, we get

Proposition 8.9 (cf. [1, Proposition 3.20]). *If both GW_1 and $[B_0]$ are null-homologous with primitives gw_1 and β_0, then the sum*

$$b := \tilde{b} + CO(gw_1) + co(\beta_0) \tag{8.25}$$

defines an nc-vector field for $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$.

We now specialize to the case where $E = A_{m-1}$ is the type A Milnor fibre. It is shown in [1, Lemmas 6.7, 6.8] that both GW_1 and $[B_0]$ are null-homologous, so Proposition 8.9 applies. The Lagrangian spheres and thimbles in E have trivial first cohomology and so (by exactness) trivial first Floer cohomology, so admit equivariant structures.

Lemma 8.10. *For each $\lambda \in \Lambda_{n,m}$, $\underline{L}_\lambda$ is pure.*

Proof. The argument of [1, Lemma 6.9] generalizes to our situation: we briefly recall the key points, adapted to the current set-up. By Lemma 2.24, the rigid count of $\mathcal{R}^{d+1,h}_{(0,1)}(\underline{x}_0)$ is 0 when $h > 0$. When $h = 0$, the domain is a disjoint union of n discs which are mapped to different Lagrangian components of $\underline{L}_\lambda$. Since the intersection with D_0 is 1, exactly one of these n discs hits D_E transversally once, and the others are disjoint from D_E and are hence constant maps by exactness. The key point in the argument of [1, Lemma 6.9] is that, over each point $t \in \gamma$ of a fibred Lagrangian $L_\gamma \subset \overline{E}$, there are exactly two holomorphic discs with boundary on L_γ (the hemispheres of a $\mathbb{P}^1$-fibre of $\overline{E}$). When γ is a matching path, this algebraic count gives $a \mapsto 2a$ under the map (8.19) for the degree 2 generator $a \in HF(L_\gamma)$. Moreover, by [1, Lemma 2.12] and the paragraph thereafter, (8.19) necessarily vanishes on degree 0 generators of $HF(L_\gamma)$ whenever $HF^0(L_\gamma)$ has rank 1 (which holds when γ is a matching path or a thimble path). The result follows. ∎

9. Formality

By Proposition 7.12, we know that on the cohomological level, the extended symplectic arc algebra $K^{\mathrm{symp}}_{n,m}$ agrees with the extended arc algebra $K^{\mathrm{alg}}_{n,m}$. In this section, we prove that the extended symplectic arc algebra is formal when working over a field of characteristic 0, by proving that the nc-vector field constructed in 8.2.1 is pure. To simplify notation, in this section, given a weight λ, we use $\overline{\lambda}$ and $\underline{\lambda}$ to denote $\overline{\lambda}^{\mathrm{alg}}$ and $\underline{\lambda}^{\mathrm{alg}}$, respectively.

9.1. Some lemmas

Let $\lambda_0, \lambda_1 \in \Lambda_{n,m}$.

Lemma 9.1 (Cyclic module structure). *Let $x_{\min}$ be an oriented circle diagram $\underline{\lambda}_1\eta_{\min}\overline{\lambda}_0$ which corresponds to the minimal degree generator of $HF^*(\underline{L}_{\lambda_0}, \underline{L}_{\lambda_1})$. Let x be another oriented circle diagram $\underline{\lambda}_1\eta\overline{\lambda}_0$. Then there are two oriented circle diagrams $\underline{\lambda}_0\eta_0\overline{\lambda}_0$ and $\underline{\lambda}_1\eta_1\overline{\lambda}_1$ such that*

$$\mu^2(\underline{\lambda}_1\eta_1\overline{\lambda}_1, x_{\min}) = x = \mu^2(x_{\min}, \underline{\lambda}_0\eta_0\overline{\lambda}_0). \tag{9.1}$$

In particular, $HF^*(\underline{L}_{\lambda_0}, \underline{L}_{\lambda_1})$ is a cyclic module over $HF^*(\underline{L}_{\lambda_i}, \underline{L}_{\lambda_i})$ for both $i = 0, 1$.

Proof. We give a proof when $i = 0$; the case $i = 1$ can be dealt with similarly.

Recall that, schematically, the product in the combinatorial arc algebra is computed by stacking an oriented $\underline{\lambda}_0 \cup \overline{\lambda}_0$ on top of an oriented $\underline{\lambda}_1 \cup \overline{\lambda}_0$, and resolving / removing the middle levels by a sequence of elementary cobordisms, which induce maps by the action of an underlying TQFT.

The diagram $\underline{\lambda}_1 \cup \overline{\lambda}_0$ consists of circles and lines. Let $C_1, \ldots, C_k$ be the circles, and define, for $j = 1, \ldots, k$,

$$\Gamma_{o,j} := \{\gamma \in \overline{\lambda}_0 \mid \gamma \text{ is contained in the circle } C_j\}. \tag{9.2}$$

It is clear that $\Gamma_{o,j} \neq \emptyset$ for all j. Let $\Gamma_o := \bigcup_j \Gamma_{o,j}$ and $\Gamma_l := \overline{\lambda}_0 \setminus \Gamma_o$. In particular, if $\gamma \in \overline{\lambda}_0$ is a line, then $\gamma \in \Gamma_l$.

The minimal degree generator $x_{\min}$ corresponds to orienting all the circles in $\underline{\lambda}_1 \cup \overline{\lambda}_0$ counterclockwise (note that the orientations of the arcs in $\underline{\lambda}_1 \cup \overline{\lambda}_0$ are uniquely determined by the rule (6.12), namely, the remaining $\vee$ *must* occupy the rightmost available positions). On the other hand, let $C_{s_1}, \ldots, C_{s_t}$ be the circles in $\underline{\lambda}_1 \cup \overline{\lambda}_0$ that are oriented clockwise by x. For each $j \in \{s_1, \ldots, s_t\}$, we pick a single $\gamma_j \in \Gamma_{o,j}$. Now we give $\underline{\lambda}_0 \cup \overline{\lambda}_0$ the orientation y that is uniquely determined by the property that a circle C in $\underline{\lambda}_0 \cup \overline{\lambda}_0$ is oriented clockwise if and only if C contains γ_j for some $j \in \{s_1, \ldots, s_t\}$.

One feature of the Khovanov TQFT ([24], [25], which is recalled in (6.24) and (6.25)) is that a counterclockwise circle times a counterclockwise (resp. clockwise) circle is a counterclockwise (resp. clockwise) circle. The analogue is true for the Khovanov-type TQFT from [13, 51]. One can check, by applying the rules underlying the Khovanov-type TQFT from [13, 51], that our prescription ensures that $\mu^2(x_{\min}, y) = x$ (see Figure 9.1 for an example). ∎

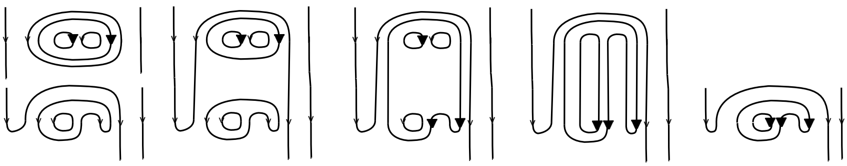


Fig. 9.1. From left to right: y (top) and $x_{\min}$ (bottom) before taking product; successively applying two TQFT operations to the first picture; the product x. Clockwise orientations on circles are indicated by ▼. We choose γ_j to be the outermost upper arcs in the 2-circle components of x; this forces the indicated ▼s on y.

Lemma 9.2. *Let λa be an oriented cap diagram and $b\lambda$ be an oriented cup diagram. Then*

$$\mu^2(b\lambda\overline{\lambda}, \underline{\lambda}\lambda a) = b\lambda a. \tag{9.3}$$

In particular, the product map

$$HF(L_\lambda, L_b) \times HF(L_a, L_\lambda) \to HF(L_a, L_b) \tag{9.4}$$

is non-zero.

Proof. Note that $\underline{\lambda}\lambda a$, $b\lambda\overline{\lambda}$ and $b\lambda a$ are oriented circle diagrams so the LHS and RHS of (9.3) are well-defined.

Now we need to justify (9.3). If $\gamma \in \underline{\lambda}$ is a half-circle and the component in $\underline{\lambda}\lambda a$ containing γ is a circle C, then by the definition of $\underline{\lambda}$, we know that C is oriented counterclockwise. Similarly, if $\gamma^* \in \overline{\lambda}$ is a half-circle and the component in $b\lambda\overline{\lambda}$ containing γ^* is a circle C', then C' is oriented counterclockwise. In this case, when we do the resolution between γ and γ^* involved in the product, the outcome is also oriented counterclockwise. In other words, the resolution cobordism move does not change the orientation (see Figure 9.2).

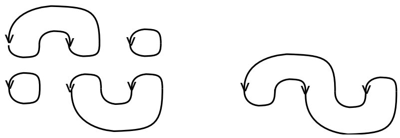


Fig. 9.2. Orientation unchanged.

On the other hand, if the component of $\underline{\lambda}\lambda a$ containing γ is a line, then we need to first complete it to a circle before computing the product. Moreover, the circle to which it is completed is oriented counterclockwise (see Figure 9.3). The same holds for $b\lambda\overline{\lambda}$.

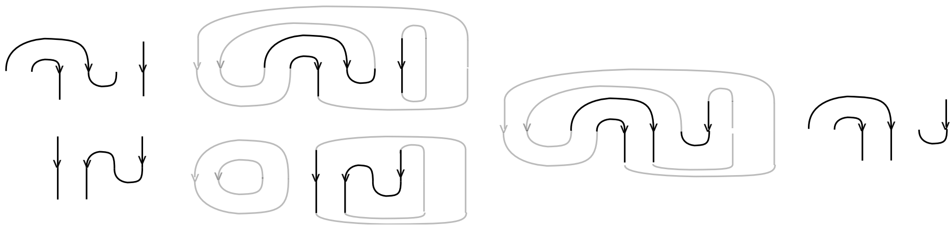


Fig. 9.3. From left to right: 1) two oriented circle diagrams $\underline{\lambda}\lambda a$, $b\lambda\overline{\lambda}$, 2) closure of the oriented circle diagrams, all circles are oriented counterclockwise (see Section 6.4), 3) taking product, 4) removing the completion (grey).

Therefore, each type of resolution, and hence the product obtained from a composition of such, does not change the orientations. This implies $\mu^2(b\lambda\overline{\lambda}, \underline{\lambda}\lambda a) = b\lambda a$.

By Proposition 7.12, we have the algebra isomorphism $\Phi : K^{\mathrm{symp}}_{n,m} \to K^{\mathrm{alg}}_{n,m}$ so we have

$$\mu^2(\Phi^{-1}(b\lambda\overline{\lambda}), \Phi^{-1}(\underline{\lambda}\lambda a)) = \Phi^{-1}(b\lambda a)$$

and (9.4) is non-zero. ■

The set $\Lambda_{n,m}$ of weights has a natural 'Bruhat' partial ordering (see Section 4.1). Intuitively, it is given by exchanging pairs $\vee$ and $\wedge$; one increases in the Bruhat order by moving $\vee$'s to the right [13, Section 2]. Given $\lambda \in \Lambda_{n,m}$, there is a unique maximal element w_λ in the set

$$\{w \in \Lambda_{n,m} \mid \underline{\lambda} w \overline{\lambda} \text{ is an oriented circle diagram}\}. \tag{9.5}$$

Explicitly, the maximal element is given by putting a $\vee$ on the right end points of all the half-circles and lines in $\overline{\lambda}$ (or $\underline{\lambda}$) and a $\wedge$ otherwise. In particular, it implies that $\lambda = w_\lambda$ only when $\overline{\lambda}$ has no half-circle component, that is, when λ is the maximal weight in $\Lambda_{n,m}$.

Lemma 9.3. *Let $\lambda_0, \lambda_1 \in \Lambda_{n,m}$. At least one of the following statements holds.*

(1) *There is no $\eta \in \Lambda_{n,m}$ such that $\underline{\lambda_1}\eta\overline{\lambda_0}$ is an oriented circle diagram.*

(2) *There is some $\eta \in \Lambda_{n,m} \backslash \{\lambda_0, \lambda_1\}$ such that $\underline{\lambda_1}\eta\overline{\lambda_0}$ is an oriented circle diagram.*

(3) $\lambda_0 = w_{\lambda_1}$ *or* $\lambda_1 = w_{\lambda_0}$.

Proof. Suppose neither (1) nor (2) hold. We want to prove (3).

Since (1) and (2) fail, there exists η such that $\underline{\lambda_1}\eta\overline{\lambda_0}$ is an oriented circle diagram, but every such η is given by λ_0 or λ_1. Without loss of generality, we assume that $\eta = \lambda_1$, so in particular $\lambda_0 \leq \lambda_1$, since whenever $\alpha\lambda\beta$ is an oriented circle diagram, one has $\alpha \leq \lambda$ and $\beta \leq \lambda$ in the Bruhat order [13, Lemma 2.3]. We want to prove that $\lambda_1 = w_{\lambda_0}$.

Notice that if $\underline{\lambda_1} \cup \overline{\lambda_0}$ has a circle component, then there is $\eta' \in \Lambda_{n,m}$ such that $\eta' \neq \eta$ and $\underline{\lambda_1}\eta'\overline{\lambda_0}$ is an oriented circle diagram. It implies that $\eta = \lambda_1 < \eta'$ but we also have $\lambda_0 \leq \eta$ so $\eta' \in \Lambda_{n,m} \setminus \{\lambda_0, \lambda_1\}$. Since we have assumed that (2) fails, we get a contradiction and hence $\underline{\lambda_1} \cup \overline{\lambda_0}$ is a diagram with only line components (and there is only one orientation given by $\eta = \lambda_1$).

Let $s_1 < \cdots < s_k$ be the integers which are the left end points of *half-circles* of $\overline{\lambda_0}$. By definition, $\lambda_0(s_i) = \vee$ for all i. If $\lambda_1(s_i) = \wedge$ for all i, then we are done, because then λ_1 has $\vee$ on all the right end points of half-circles and lines of $\overline{\lambda_0}$, and hence $\lambda_1 = w_{\lambda_0}$.

If this is not the case, then let s_j be the largest among $s_1, \ldots, s_k$ such that $\lambda_1(s_j) = \vee$. Let $s_j + t$ be the right end point of the half-circle of $\overline{\lambda_0}$ with left end point s_j. By definition of $\overline{\lambda_0}$, exactly half of $\{s_j + 1, \ldots, s_j + t - 1\}$ are labelled by $\wedge$ and half by $\vee$ with respect to λ_0. Moreover, all of them lie in a half-circle of $\overline{\lambda_0}$.

By the definition of s_j, for $l = 1, \ldots, t - 1$, if $\lambda_0(s_j + l) = \vee$, then $s_j + j \in \{s_{j+1}, \ldots, s_k\}$ so we have $\lambda_1(s_j + l) = \wedge$. Moreover, for $\underline{\lambda_1}\eta\overline{\lambda_0}$ to have at least one orientation, we need that $\lambda_1(s_j + t) = \wedge$ and for $l = 1, \ldots, t - 1$, if $\lambda_0(s_j + l) = \wedge$, then $\lambda_1(s_j + l) = \vee$ (see Figure 9.4). Now, by the definition of $\underline{\lambda_1}$, we see that $\underline{\lambda_1}\eta\overline{\lambda_0}$ has a circle passing through s_j and $s_j + t$. This contradicts the fact that $\underline{\lambda_1}\eta\overline{\lambda_0}$ does not have circle components. ■

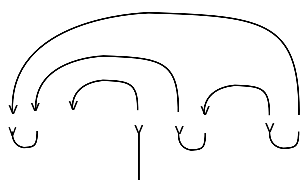


Fig. 9.4. The weight and the cap diagram on top are λ_0 and $\overline{\lambda_0}$; the weight and the cup diagram on the bottom are λ_1 and $\underline{\lambda_1}$.

9.2. Consistent choice of equivariant structures

Let b be the nc vector field constructed in Proposition 8.9.

Proposition 9.4. *There is a choice of b-equivariant structures on $\{\underline{L}_\lambda\}_{\lambda\in\Lambda_{n,m}}$ such that b is pure.*

Proof. By Lemma 8.10, $\underline{L}_\lambda$ is pure for each $\lambda \in \Lambda_{n,m}$.

Any partial order admits a total order refinement. We choose a total ordering $\leq^T$ on $\Lambda_{n,m}$ that is compatible with the Bruhat partial ordering $\leq$. We make choices of b-equivariant structure on the $\underline{L}_\lambda$ with decreasing order of λ.

Let $\eta \in \Lambda_{n,m}$ and suppose that a b-equivariant structure on $\underline{L}_\lambda$ is chosen for all $\lambda >^T \eta$, such that b is pure. We want to choose a b-equivariant structure on $\underline{L}_\eta$ such that b is pure on $\{L_\lambda\}_{\lambda\geq^T\eta}$.

We choose a b-equivariant structure on $\underline{L}_\eta$ such that the minimal degree element of $HF(\underline{L}_\eta, \underline{L}_{w_\eta})$ has weight equal to the degree. By Lemma 9.1, $HF(\underline{L}_\eta, \underline{L}_{w_\eta})$ is cyclic as a module over $HF(\underline{L}_{w_\eta}, \underline{L}_{w_\eta})$ so the weight equals the degree for all pure degree elements in $HF(\underline{L}_\eta, \underline{L}_{w_\eta})$.

By Lemma 9.2, we have

$$\mu^2(\underline{\eta}w_\eta\overline{w_\eta}, \underline{w}_\eta w_\eta\overline{\eta}) = \underline{\eta}w_\eta\overline{\eta}. \tag{9.6}$$

This implies that $x := \underline{\eta}w_\eta\overline{w}_\eta$ in $HF(\underline{L}_{w_\eta}, \underline{L}_\eta)$ has weight equal to its degree.

Let $x_{\min}$ be the minimal degree generator of $HF(\underline{L}_{w_\eta}, \underline{L}_\eta)$. By Lemma 9.1, we know that $x = \mu^2(x_{\min}, y)$ for some $y \in HF(\underline{L}_{w_\eta}, \underline{L}_{w_\eta})$. It means that $x_{\min}$ also has weight equal to its degree, and hence all elements of $HF(\underline{L}_{w_\eta}, \underline{L}_\eta)$ have weights equal to their degrees.

Now, suppose that $\lambda >^T \eta$ but $\lambda \neq w_\eta$, and purity holds for all of $HF(L_{\lambda'}, L_\eta)$, $HF(L_\eta, L_{\lambda'})$ such that $\lambda' >^T \lambda$. By Lemma 9.3, λ, η satisfies either (1) or (2) of Lemma 9.3. If (1) holds, then $HF(L_\lambda, L_\eta) = HF(L_\eta, L_\lambda) = 0$ so purity is automatic. If (2) holds, let $\lambda' > \lambda, \eta$ be such that $\overline{\eta}\lambda'\underline{\lambda}$ is an oriented circle diagram.

By Lemma 9.2, we have

$$\mu^2(\underline{\lambda}\lambda'\overline{\lambda'}, \underline{\lambda}'\lambda'\overline{\eta}) = \underline{\lambda}\lambda'\overline{\eta}. \tag{9.7}$$

This means that $\underline{\lambda}\lambda'\overline{\eta}$ in $HF(\underline{L}_\eta, \underline{L}_\lambda)$ has weight equal to its degree. By Lemma 9.1 again, we know the minimal degree elements in $HF(\underline{L}_\eta, \underline{L}_\lambda)$ have weight equal to degree, hence so do all elements in $HF(\underline{L}_\eta, \underline{L}_\lambda)$. The same is true for $HF(\underline{L}_\lambda, \underline{L}_\eta)$ because Lemma 9.2 also implies that

$$\mu^2(\underline{\eta}\lambda'\overline{\lambda'}, \underline{\lambda}'\lambda'\overline{\lambda}) = \underline{\eta}\lambda'\overline{\lambda}. \tag{9.8}$$

By induction over the total order, the result follows. ∎

Corollary 9.5. *Over a field of characteristic 0, the A_∞ endomorphism algebra of the direct sum of objects in the collection $\{\underline{L}_\lambda\}_{\lambda\in\Lambda_{n,m}}$ is formal.*

Proof. This follows from Theorem 8.8 and Proposition 9.4. ∎

Proof of Theorem 1.1. Recall the notation for the tuple of Lagrangian thimbles $\underline{T}^I$ from Section 4. One can argue inductively to show that $\underline{T}^{(\lambda)^{-1}(\vee)}$ can be generated by $\{\underline{L}_\lambda \mid \lambda \in \Lambda_{n,m}\}$ using the exact triangles constructed in Section 3.2 (compare with Proposition 10.1).

We now give the details. When $n = 1$, $\{\underline{L}_\lambda \mid \lambda \in \Lambda_{n,m}\}$ consists of one Lefschetz thimble at the rightmost critical point, and all the matching spheres over matching paths whose end points are consecutive critical values. This collection clearly generates $\underline{T}^{(\lambda)^{-1}(\vee)}$, by applying Dehn twists in the matching spheres to the Lefschetz thimble. We now argue inductively. Let $k \in \mathbb{N}_+$. We assume that for all $n < k$, all $m \geq n$ and all $\lambda_0 \in \Lambda_{n,m}$, the Lagrangian tuple $\underline{T}^{(\lambda_0)^{-1}(\vee)}$ can be generated by the collection $\{\underline{L}_\lambda \mid \lambda \in \Lambda_{n,m}\}$. Now we consider the case $n = k$, $m \geq n$ and let $\lambda \in \Lambda_{n,m}$.

When $\lambda^{-1}(\vee) = \{m - n + 1, \dots, m\}$, λ is the maximal weight and we have $\underline{L}_\lambda = \underline{T}^{\lambda^{-1}(\vee)}$.

If λ is not the maximal weight, then there is $k \in \{1, \dots, m\}$ such that $\lambda(k) = \vee$ and $\lambda(k+1) = \wedge$. Let λ'' be the weight given by 'swapping positions k and $k + 1$', that is,

$$\lambda''(i) = \begin{cases} \lambda(i) & \text{if } i \neq k, k+1, \\ \wedge & \text{if } i = k, \\ \vee & \text{if } i = k+1. \end{cases} \tag{9.9}$$

Let $\lambda' \in \Lambda_{n-1,m-2}$ be the weight given by forgetting the positions $k, k + 1$, that is,

$$\lambda'(i) = \begin{cases} \lambda(i) & \text{if } i < k, \\ \lambda(i+2) & \text{if } i \geq k. \end{cases} \tag{9.10}$$

We define $\bigcup_{k,k+1} \lambda'$ to be the admissible tuple in $\{\mathrm{im}(z) \leq 1\}$ consisting of one matching path connecting $k + \sqrt{-1}$ and $k + 1 + \sqrt{-1}$ and $n - 1$ thimble paths from $i + \sqrt{-1}$ to i for each $i \in \lambda^{-1}(\vee) \setminus \{k\}$. The notation suggests that it is obtained from applying a 'cup functor' to λ' in position $k, k + 1$ in the sense of [2].

By Corollary 3.5, we have the exact triangle

$$\underline{T}^{(\lambda'')^{-1}(\vee)}[-1] \to \underline{L}_{\bigcup_{k,k+1}\lambda'} \to \underline{T}^{(\lambda)^{-1}(\vee)} \to \underline{T}^{(\lambda'')^{-1}(\vee)}. \tag{9.11}$$

Note that the second term $\underline{L}_{\bigcup_{k,k+1}\lambda'}$ can be generated by $\{\underline{L}_\lambda \mid \lambda \in \Lambda_{n,m}\}$ by applying the 'cup functor' $\bigcup_{k,k+1}$ to the induction hypothesis. More precisely, by induction hypothesis, the collection

$$\{\underline{L}_\lambda \mid \lambda \in \Lambda_{n-1,m-2}\}$$

is sufficient to generate $\underline{L}_\Gamma$ for all admissible tuples $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ such that for all i, γ_i is a thimble path from $q_i + \sqrt{-1}$ to q_i for some $q_i \in \{1, \dots, m\}$. Together with Corollary 3.5 (which implies that the 'cup functor' takes exact triangles to exact triangles), we know that the collection

$$\{\underline{L}_\lambda \mid \lambda \in \Lambda_{n,m},\ \lambda(k) = \vee,\ \lambda(k+1) = \wedge\}$$

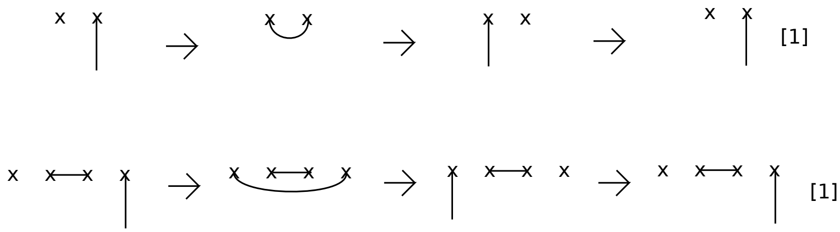


Fig. 9.5. The first row represents an exact triangle in the induction hypothesis. The second row is the exact triangle after applying the 'cup functor' $\bigcup_{2,3}$ to it.

is sufficient to generate $\underline{L}_\Gamma$ for all admissible tuples $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ such that (see Figure 9.5)

(1) $\gamma_i \subset \{\mathrm{im}(z) \le 1\}$ for all i,
(2) γ_1 is a matching path joining $k + \sqrt{-1}$ and $k + 1 + \sqrt{-1}$,
(3) for $i > 1$, γ_i is a thimble path from $q_i + \sqrt{-1}$ to q_i for some $q_i \in \{1, \ldots, m\}$.

On the other hand, the first and third terms of (9.11) have the property that $\lambda'' > \lambda$. Since $\underline{L}_\lambda = \underline{T}^{\lambda^{-1}(\vee)}$ when λ is maximal, by (9.11) and a second induction with respect to the Bruhat ordering starting from the maximal weight, we know that $\underline{T}^{(\lambda)^{-1}(\vee)}$ can be generated by $\{\underline{L}_\lambda \mid \lambda \in \Lambda_{n,m}\}$.

Conversely, $\underline{L}_\lambda$ can be generated by $\{\underline{T}^{(\lambda)^{-1}(\vee)} \mid \lambda \in \Lambda_{n,m}\}$, either again by directly applying the exact triangles, or by appeal to Proposition 5.16. Therefore, by Proposition 4.5, we have quasi-equivalences

$$D^\pi \mathcal{F}\mathcal{S}(\pi_{n,m}) \simeq \text{perf-End}\Big(\bigoplus \underline{T}^{(\lambda)^{-1}(\vee)}\Big) \simeq \text{perf-}K^{\text{symp}}_{n,m} = \text{perf-}K^{\text{alg}}_{n,m}. \tag{9.12}$$

■

Remark 9.6. Theorem 1.1 and Claim 3.6 together give a new proof of a result of [7], namely that the Serre functor of the constructible derived category of the Grassmannian is given by the action of the centre of the braid group.

There is an algebra isomorphism $K^{\text{alg}}_{n,m} \simeq (K^{\text{alg}}_{n,m})^{\text{op}}$, given on basis elements by sending an oriented circle diagram $\beta\lambda\alpha$ to its reflection $\alpha^r\lambda\beta^r$ (see [13, (3.9)]). Together with Theorem 1.1, this implies

Corollary 9.7. *For all $\lambda, \lambda' \in \Lambda_{n,m}$, we have*

$$HF(\underline{L}_\lambda, \underline{L}_{\lambda'}) = HF(\underline{L}_{\lambda'}, \underline{L}_\lambda) \tag{9.13}$$

as graded vector spaces.

This is already non-trivial, because $\underline{L}_\lambda, \underline{L}_{\lambda'}$ are not compact Lagrangians, so it is not *a priori* clear why they should satisfy this Poincaré duality type of equality (because of the non-trivial wrapping in constructing the category, the underlying cochain groups are typically *not* equal).

The algebra isomorphism $K_{n,m}^{\text{alg}} \simeq (K_{n,m}^{\text{alg}})^{\text{op}}$ can be understood geometrically: we give a sketch of the argument. By a hyperbolic isometry between the upper half-plane and the unit disc, we can assume the target of π_E is the unit disc B. We also assume that the critical values lie on the real line, and that infinity in the upper half-plane is mapped to $\sqrt{-1} \in \partial B$. For example, it can be achieved by applying the inverse of the hyperbolic isometry $f : \overline{B}_1 \to \mathbb{H}$ of (5.12). Let $\iota_B : B \to B$ be the reflection along the real line, which is an anti-symplectic involution of B. We recall that for the construction of (E, ω_E) in Section 5.1, ω_M is restricted from a product symplectic form ω_X. Therefore, by appropriately choosing ω_R in Lemma 5.4, we can assume that there is an anti-symplectic involution $\iota_E : E \to E$ covering ι_B.

Let $u : S \to \mathcal{Y}_E$ be a solution that contributes to the A_∞ structure of $\mathcal{FS}^{\text{cyl},n}(\pi_E)$. Let $\iota_S : S \to \overline{S}$ be reflection in the real diameter of the disc, and $\overline{S}$ the image of S (equipped with the push-forward data by ι_S). Let $\iota_{\mathcal{Y}_E} : \mathcal{Y}_E \to \mathcal{Y}_E$ be the anti-symplectic involution induced by ι_E. In this case, $\iota_{\mathcal{Y}_E} \circ u \circ \iota_S : \overline{S} \to \mathcal{Y}_E$ is tautologically a solution to the push-forward equation, which is itself a perturbed pseudo-holomorphic equation. Note, however, that the ordering of the Lagrangian boundary conditions is reversed on $\partial\overline{S}$.

This illustrates that, for appropriate choices of Floer data, the involution ι_E induces an A_∞-equivalence

$$\mathcal{FS}^{\text{cyl},n}(\pi_E) \simeq (\mathcal{FS}^{\text{cyl},n}(\pi_E)^\dagger)^{\text{op}}, \quad \underline{L} \mapsto \iota_E(\underline{L}), \tag{9.14}$$

where $\mathcal{FS}^{\text{cyl},n}(\pi_E)^\dagger$ is quasi-equivalent to $\mathcal{FS}^{\text{cyl},n}(\pi_E)$ but the wrapping at infinity for $CF(?, ??)$ in $\mathcal{FS}^{\text{cyl},n}(\pi_E)^\dagger$ is defined to be wrapping the latter entry clockwise, with stop at $-\sqrt{-1}$, whilst the wrapping in $\mathcal{FS}^{\text{cyl},n}(\pi_E)$ is wrapping the former entry counterclockwise with stop at $\sqrt{-1}$ (see Figure 9.6). For more details of (quasi-)equivalences of the kind of (9.14), see [48, Appendix].

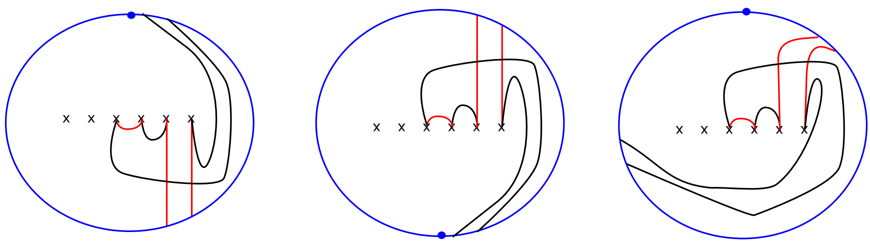


Fig. 9.6. $CF(\underline{L}_\lambda, \underline{L}_{\lambda'})$ (left), $CF(\iota_E(\underline{L}_{\lambda'}), \iota_E(\underline{L}_\lambda))$ in $\mathcal{FS}^{\text{cyl},n}(\pi_E)^\dagger$ (middle) and $CF(\Psi \circ \iota_E(\underline{L}_{\lambda'}), \Psi \circ \iota_E(\underline{L}_\lambda))$ (right). The dot on the boundary indicates the stop.

To identify $\mathcal{FS}^{\text{cyl},n}(\pi_E)^\dagger$ with $\mathcal{FS}^{\text{cyl},n}(\pi_E)$, we need to pick a quasi-equivalence Ψ between them. We choose the one that corresponds to moving the 'stop' $-\sqrt{-1}$ for $\mathcal{FS}^{\text{cyl},n}(\pi_E)^\dagger$ to $\sqrt{-1}$ clockwise along the boundary of B and keeping a compact region containing the critical points unchanged (see Figure 9.6). This choice of quasi-equivalence has the property that

$$\Psi \circ \iota_E(\underline{L}_{\lambda}) = \underline{L}_{\overline{\lambda}} \tag{9.15}$$

for all $\lambda \in \Lambda_{n,m}$.

Applying $\Psi \circ \iota_E$ to the hom space between the Lagrangians associated to weights in $\Lambda_{n,m}$, we get Corollary 9.7:

$$HF(\underline{L}_{\lambda}, \underline{L}_{\lambda'}) = HF(\Psi \circ \iota_E(\underline{L}_{\lambda'}), \Psi \circ \iota_E(\underline{L}_{\lambda})) = HF(\underline{L}_{\lambda'}, \underline{L}_{\lambda}) \tag{9.16}$$

where the first equality comes from (9.14) and the second one comes from (9.15). Similarly, applying $\Psi \circ \iota_E$ to the multiplication maps, we can deduce $K^{\mathrm{symp}}_{n,m} \simeq (K^{\mathrm{symp}}_{n,m})^{\mathrm{op}}$, so (9.14) can be viewed as a geometric interpretation of the isomorphism $K^{\mathrm{alg}}_{n,m} \simeq (K^{\mathrm{alg}}_{n,m})^{\mathrm{op}}$.

Proof of Corollary 1.5. The statement follows from Theorem 1.1 and the fact that $K^{\mathrm{alg}}_{n,m} = (K^{\mathrm{alg}}_{m-n,m})^{\mathrm{op}}$. The geometric origin of the latter equality is the Schubert-cells compatible identification of the Grassmannians $\mathrm{Gr}(n,m) = \mathrm{Gr}(m-n,m)$ but, following [13, (3.10)], it is not hard to give a concrete isomorphism in diagrammatic terms. We present their isomorphism here using our notations.

Let $PD : \Lambda_{n,m} \to \Lambda_{m-n,m}$ be the bijection

$$PD(\lambda)(a) = \begin{cases} \wedge & \text{if } \lambda(m+1-a) = \vee, \\ \vee & \text{if } \lambda(m+1-a) = \wedge. \end{cases} \tag{9.17}$$

In words, PD is obtained by rotating λ by π (see Figure 9.7).

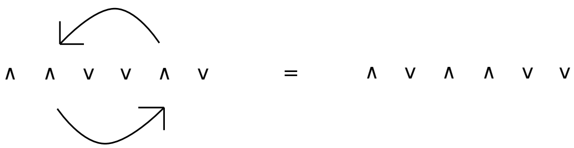


Fig. 9.7. Left: a weight λ. Right: the weight $PD(\lambda)$.

Each basis element of $K^{\mathrm{alg}}_{n,m}$ is an oriented circle diagram, for which we can perform a π-rotation to obtain another oriented circle diagram, and hence a basis element of $(K^{\mathrm{alg}}_{m-n,m})^{\mathrm{op}}$. This gives a bijection between basis elements of $K^{\mathrm{alg}}_{n,m}$ and $(K^{\mathrm{alg}}_{m-n,m})^{\mathrm{op}}$. Since clockwise cup becomes clockwise cap, and vice versa, this bijection preserves grading. Because of the diagrammatic nature of the defining TQFT, it is immediate that this induces an algebra isomorphism, as observed in [13, (3.10)]. ∎

10. Dictionary between Lagrangians and modules

In this section, we summarize the dictionary between certain objects in $\mathcal{FS}^{\mathrm{cyl},n}(A_{m-1})$ (or their Yoneda images as modules over $K^{\mathrm{symp}}_{n,m}$) and modules over $K^{\mathrm{alg}}_{n,m}$. None of our results rely on this dictionary: it is simply meant to help orient the interested reader. Given that, we make free use of Claim 3.6 in this section.

Modules in this section are right modules, which differs from the convention used in [13]. However, as remarked previously, there is a canonical algebra isomorphism $K_{n,m}^{\mathrm{alg}} \simeq (K_{n,m}^{\mathrm{alg}})^{\mathrm{op}}$, which identifies the right $K_{n,m}^{\mathrm{alg}}$-modules used below with the left $K_{n,m}^{\mathrm{alg}}$-modules used in [13].

For an A_∞ or dg category/algebra $\mathcal{C}$, we use perf-$\mathcal{C}$ and $\mathcal{C}$-perf to denote the dg category of perfect right, and respectively left, $\mathcal{C}$-modules. We use $[\mathcal{C}, \mathcal{D}]$ denote the dg category of $\mathcal{C}$-$\mathcal{D}$-bimodules. Since $K_{n,m}^{\mathrm{alg}}$ is homologically smooth, every proper (cohomologically finite) module or bimodule is perfect.

10.1. Indecomposable projectives

For each $\lambda \in \Lambda_{n,m}$, we define $P(\lambda)$ to be the submodule of $K_{n,m}^{\mathrm{alg}}$ generated by the oriented circle diagrams of the form

$$P(\lambda) := \bigoplus \mathbb{K}\underline{\lambda}^{\mathrm{alg}}\mu\alpha. \tag{10.1}$$

The collection of all $P(\lambda)$ (together with their grading shifts) is the set of all indecomposable projective modules of $K_{n,m}^{\mathrm{alg}}$. Under the isomorphism $K_{n,m}^{\mathrm{symp}} \simeq K_{n,m}^{\mathrm{alg}}$, $P(\lambda)$ is the same as the Yoneda embedding of $\underline{L_\lambda}$ as a right $K_{n,m}^{\mathrm{symp}}$-module. Thus, *the indecomposable projectives are the Lagrangians associated to weights.*

By [13, Theorem 5.3], the set of all indecomposable injective modules of $K_{n,m}^{\mathrm{alg}}$ is given by

$$P(\lambda)^* := \mathrm{Hom}(P(\lambda), \mathbb{K}) \tag{10.2}$$

where the right module structure is given by $fa(m) := f(ma^*)$ for $a \in K_{n,m}^{\mathrm{alg}}$, $m \in P(\lambda)$ and $f \in P(\lambda)^*$. In other words, $P(\lambda)^*$ is obtained by pulling back the left module $\mathrm{Hom}(P(\lambda), \mathbb{K})$ via the algebra isomorphism $K_{n,m}^{\mathrm{alg}} = (K_{n,m}^{\mathrm{alg}})^{\mathrm{op}}$. On the symplectic side, up to a grading shift, $\mathrm{Hom}(P(\lambda), \mathbb{K})$ is given by the Yoneda embedding of $\tau^{-1}(\underline{L_\lambda})$ as a left $K_{n,m}^{\mathrm{symp}}$-module, where τ is the global monodromy (see Claim 3.6), so $P(\lambda)^*$ corresponds to the pull-back of $\tau^{-1}(\underline{L_\lambda})$ via the algebra isomorphism $K_{n,m}^{\mathrm{symp}} = (K_{n,m}^{\mathrm{symp}})^{\mathrm{op}}$.

In view of the geometric explanation of the equivalence $K_{n,m}^{\mathrm{symp}} = (K_{n,m}^{\mathrm{symp}})^{\mathrm{op}}$ given in (9.14), $P(\lambda)^*$ is given by the right Yoneda embedding of $\Psi \circ \iota_E \circ \tau^{-1}(\underline{L_\lambda})$.

10.2. Standard modules

For each $\lambda \in \Lambda_{n,m}$, there is a standard module $V(\lambda)$ defined in [13, (5.11)]. As a graded vector space, $V(\lambda)$ is generated by oriented cap diagrams of the form $\lambda\alpha$. The collection of all $V(\lambda)$ gives a full exceptional collection for the dg category of right $K_{n,m}^{\mathrm{alg}}$-modules and $\mathrm{Ext}^i(V(\lambda_0), V(\lambda_1)) \neq 0$ for some i only if $\lambda_0 < \lambda_1$, where $<$ is with respect to the Bruhat partial ordering.

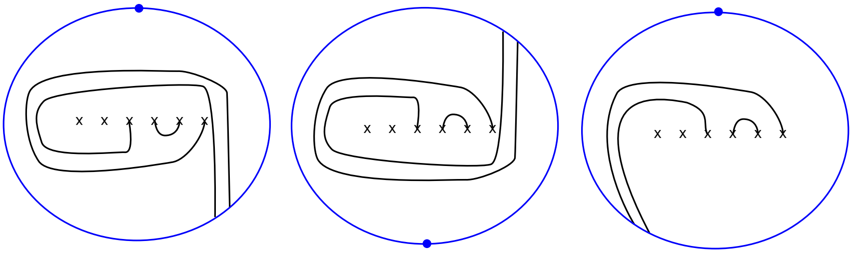


Fig. 10.1. An example of $\tau^{-1}(\underline{L_\lambda})$ (left), $\iota_E \circ \tau^{-1}(\underline{L_\lambda})$ (middle) and $\Psi \circ \iota_E \circ \tau^{-1}(\underline{L_\lambda})$ (right).

An inductive construction of a projective resolution of $V(\lambda)$ is described in [12, Theorem 5.3]. By identifying $P(\lambda)$ with the Yoneda image of $\underline{L_\lambda}$ and imitating that inductive construction, we have

Proposition 10.1. *The standard module $V(\lambda)$, as an A_∞ module over $K^{\mathrm{alg}}_{n,m} = K^{\mathrm{symp}}_{n,m}$, is quasi-isomorphic to the Yoneda image of $\underline{T}^{\lambda^{-1}(\vee)}$ up to grading shift.*

Sketch of proof. We want to compare the projective resolution of the standard module $V(\lambda)$ and the iterated mapping cone decomposition of $\underline{T}^{\lambda^{-1}(\vee)}$ (cf. the proof of Theorem 1.1).

When $\lambda^{-1}(\vee) = \{m-n+1,\dots,m\}$, λ is the maximal weight and $P(\lambda) = V(\lambda)$. It corresponds to $\underline{L_\lambda} = \underline{T}^{\lambda^{-1}(\vee)}$.

If λ is not the maximal weight, then there is $k \in \{1,\dots,m\}$ such that $\lambda(k) = \vee$ and $\lambda(k+1) = \wedge$. Let λ'', λ' and $\bigcup_{k,k+1} \lambda'$ be as in the proof of Theorem 1.1.

The construction of the projective resolution of $V(\lambda)$ comes from iteratively applying the exact sequence of modules in [12, (5.8)]. On the other hand, by Corollary 3.5, we have the exact triangle of Yoneda modules

$$\underline{T}^{(\lambda'')^{-1}(\vee)}[-1] \to \underline{L}_{\bigcup_{k,k+1}\lambda'} \to \underline{T}^{(\lambda)^{-1}(\vee)} \to \underline{T}^{(\lambda'')^{-1}(\vee)}. \tag{10.3}$$

Therefore, it is sufficient to prove that the exact sequence of modules in [12, (5.8)] corresponds to (10.3). This correspondence can be proved by induction on m, induction on the partial ordering of weights (from high to low; note that $\lambda'' > \lambda$) and the fact that $HF(\underline{T}^{(\lambda'')^{-1}(\vee)}[-1], \underline{L}_{\bigcup_{k,k+1}\lambda'})$ has rank 1, so up to quasi-isomorphism there is only one non-trivial mapping cone. We leave the details to the readers. ∎

Thus, *the standard modules are the Lefschetz thimbles.*

Remark 10.2. In contrast to the collection of indecomposable projective modules, the collection of standard modules does not have formal endomorphism algebra when $n > 1$. A minimal model for the algebra when $n = 2$ is given in [27], which conjectured that this minimal model is not formal; we give a proof of that conjecture in Appendix A.

By [13, Theorem 5.3], the set of all costandard modules of $K^{\text{alg}}_{n,m}$ are given by

$$V(\lambda)^* := \text{Hom}(V(\lambda), \mathbb{K}) \tag{10.4}$$

where the right module structure is given by $fa(m) := f(ma^*)$ for $a \in K^{\text{alg}}_{n,m}$, $m \in V(\lambda)^*$ and $f \in V(\lambda)^*$. In other words, up to grading shift, it is given by the Yoneda embedding of $\tau^{-1}(\underline{T}^{\lambda^{-1}(\vee)})$ as a right $(K^{\text{symp}}_{n,m})^{\text{op}}$-module after pull-back via the algebra isomorphism $K^{\text{symp}}_{n,m} = (K^{\text{symp}}_{n,m})^{\text{op}}$.

In view of (9.14) again, $V(\lambda)^*$ is given by the right Yoneda embedding of $\Psi \circ \iota_E \circ \tau^{-1}(\underline{T}^{\lambda^{-1}(\vee)})$.

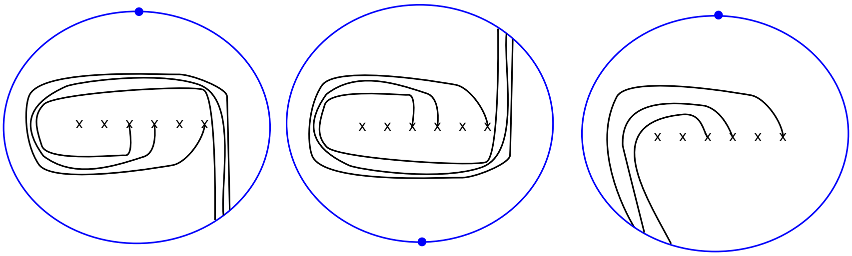


Fig. 10.2. An example of $\tau^{-1}(\underline{T}^{\lambda^{-1}(\vee)})$ (left), $\iota_E \circ \tau^{-1}(\underline{T}^{\lambda^{-1}(\vee)})$ (middle) and $\Psi \circ \iota_E \circ \tau^{-1}(\underline{T}^{\lambda^{-1}(\vee)})$ (right).

In fact, the $\Psi \circ \iota_E \circ \tau^{-1}(\underline{T}^{\lambda^{-1}(\vee)})$ form the right Koszul duals of the thimbles (i.e. $HF(\underline{T}^{\lambda_0^{-1}(\vee)}, \Psi \circ \iota_E \circ \tau^{-1}(\underline{T}^{\lambda_1^{-1}(\vee)})) = \mathbb{K}$ if and only if $\lambda_0 = \lambda_1$, and equal to 0 otherwise). It is well-known that costandard modules are right Koszul dual to standard modules (see for example [9]).

10.3. Irreducible modules

For each $\lambda \in \Lambda_{n,m}$, we define $L(\lambda)$ to be the submodule of $K^{\text{alg}}_{n,m}$ generated by the single oriented circle diagram

$$L(\lambda) := \mathbb{K}\underline{\lambda}\lambda\overline{\lambda}. \tag{10.5}$$

The collection of all $L(\lambda)$ is the set of all irreducible modules of $K^{\text{alg}}_{n,m}$.

It is not immediately clear in our context which Lagrangians correspond to the irreducible modules. In view of the work of [18], and the ‘topological model’ for the compact core of the space $\mathcal{Y}_{n,m}$ arising from its description as a quiver variety (see e.g. [2]), it is reasonable to believe that one can enlarge the category $\mathcal{FS}^{\text{cyl},n}(E)$ to allow Lagrangian discs that intersect exactly one of the $\{\underline{L}_\lambda\}_\lambda$ once and are disjoint from the others. These would be the analogues of ‘cocore discs’ over the components of the skeleton of a plumbing of cotangent bundles, and the Yoneda images of these Lagrangian discs are natural candidates for the irreducible modules.

11. Symplectic annular Khovanov homology

This section gives an application of our main results to link homology theories in the sense of Khovanov. Suppose $m = 2n$. A braid $\beta \in \mathrm{Br}_n$ in the n-string braid group defines bimodules $P_\beta^{(j)}$ over each of the extended arc algebras $K_{j,n}^{\mathrm{alg}}$ with $0 \le j \le n$. One can combine a theorem of Roberts [39] with deep recent work of Beliakova, Putyra and Wehrli [8] to infer that there is a spectral sequence $\bigoplus_j HH_*(P_\beta^{(j)}) \Rightarrow \mathrm{Kh}(\kappa(\beta))$, where $\kappa(\beta)$ is the link closure of β, and where $\mathrm{Kh}(\kappa)$ is the Khovanov homology [24] of the link $\kappa \subset S^3$, constructed as a categorification of the Jones polynomial. It is by now classical that Khovanov homology can be understood as a certain morphism group $\mathrm{Ext}_{\text{perf-}H_{n,2n}^{\mathrm{alg}}}(P, (\beta \times \mathrm{id})(P))$ in the derived category of modules over the *compact* arc algebra $H_{n,2n}^{\mathrm{alg}}$ (see [25]), where $\beta \times \mathrm{id} \in \mathrm{Br}_{2n}$ belongs to the $2n$-string braid group and P is a particular projective. From the viewpoint of the (extended) arc algebras, the existence of such a spectral sequence is rather mysterious. The purpose of this section is to give a transparent account of why it should exist, starting from a new semi-orthogonal decomposition of perf-$K_{n,2n}^{\mathrm{alg}}$ which we shall derive from the geometric viewpoint afforded by Theorem 1.1.

We assume $m = 2n$ in Sections 11 and 12 unless stated otherwise.

11.1. The annular symplectic theory

Let $\mathrm{hs} \in \Lambda_{n,m}$ be the weight such that $\mathrm{hs}^{-1}(\vee) = \{1, \dots, n\}$. The corresponding tuple $\underline{L}_{\mathrm{hs}}$ is called the 'horseshoe' Lagrangian tuple and the Lagrangian $\mathrm{Sym}(\underline{L}_{\mathrm{hs}}) \subset \mathcal{Y}_{n,m}$ is the horseshoe Lagrangian from [1,47]. Let Br_n be the braid group on n strands. The paper [47] associates to each element $\beta \in \mathrm{Br}_n$ a symplectomorphism $\phi_\beta^{(n)} : \mathcal{Y}_{n,m} \to \mathcal{Y}_{n,m}$, and defines the *symplectic Khovanov cohomology* of the braid closure $\kappa = \kappa(\beta)$ of β to be

$$\mathrm{Kh}^{\mathrm{symp}}(\kappa(\beta)) := HF(\mathrm{Sym}(\underline{L}_{\mathrm{hs}}), \phi_\beta^{(n)}(\mathrm{Sym}(\underline{L}_{\mathrm{hs}}))) \tag{11.1}$$

The main result of [47] is that $\mathrm{Kh}^{\mathrm{symp}}(\kappa(\beta))$ is a link invariant, i.e. it is independent of the representation of κ as a braid closure.

One can give an equivalent definition of symplectic Khovanov cohomology using $\underline{L}_{\mathrm{hs}}$ instead of its product $\mathrm{Sym}(\underline{L}_{\mathrm{hs}})$. Let $E := A_{m-1}$ denote the Milnor fibre. To each simple braid element σ_i we associate the symplectomorphism $\phi_{\sigma_i} : E \to E$ given by the Dehn twist along the i-th matching sphere (lying above the line joining $i + \sqrt{-1}$ and $i + 1 + \sqrt{-1}$). This defines a representation $\mathrm{Br}_m \to \pi_0 \mathrm{Symp}_{ct}(E)$. Let $\mathrm{Br}_n \hookrightarrow \mathrm{Br}_m$ be the embedding of the left n strands. We have the restricted representation $\mathrm{Br}_n \to \pi_0 \mathrm{Symp}_{ct}(E)$ so each braid $\beta \in \mathrm{Br}_n$ determines a symplectomorphism ϕ_β of E that acts as the identity on the 'right half'.

Then (see [31])

$$\mathrm{Kh}^{\mathrm{symp}}(\kappa(\beta)) = HF(\underline{L}_{\mathrm{hs}}, \phi_\beta(\underline{L}_{\mathrm{hs}})). \tag{11.2}$$

Let $E_{1/2}$ be the A_{n-1} Milnor fibre with its standard Lefschetz fibration $\pi_{E_{1/2}}$. We can define the symplectomorphism ϕ_β of $E_{1/2}$ for $\beta \in \mathrm{Br}_n$ accordingly. Then ϕ_β

induces an A_∞ endofunctor of $\mathcal{F}\mathcal{S}^{\mathrm{cyl},j}(\pi_{E_{1/2}})$, and hence an A_∞-bimodule $P_\beta^{(j)}$ over $\mathcal{F}\mathcal{S}^{\mathrm{cyl},j}(\pi_{E_{1/2}})$.

Definition 11.1. The *symplectic annular Khovanov homology* of the braid $\beta \in \mathrm{Br}_n$ is the direct sum of Hochschild homology groups

$$\mathrm{AKh}^{\mathrm{symp}}(\beta) := \bigoplus_{j=0}^{n} HH_*(\mathcal{F}\mathcal{S}^{\mathrm{cyl},j}(\pi_{E_{1/2}}), P_\beta^{(j)}) \tag{11.3}$$

where, by convention, $\mathcal{F}\mathcal{S}^{\mathrm{cyl},j}(\pi_{E_{1/2}}) = \mathbb{K}$ and $P_\beta^{(j)} = \mathbb{K}$ is the diagonal bimodule when $j = 0$ (cf. Remark 1.2).

Symplectic annular Khovanov homology is clearly a braid invariant. It is not an invariant of the link closure of the braid. Its definition is motivated by the fact that combinatorial annular Khovanov homology, originally defined by a diagrammatic calculus for links in a solid torus [4], is itself isomorphic to a direct sum of Hochschild homologies of braid bimodules over the extended arc algebras, a recent theorem of [8] establishing a conjecture due to [6]. Indeed, Theorem 1.1 and the same formality arguments for bimodules as in [2] would prove that

Proposition 11.2. $\mathrm{AKh}^{\mathrm{symp}}(\beta)$ *is isomorphic to annular Khovanov homology when* $\mathbb{K}$ *has characteristic* 0.

We will prove Theorem 1.9 by constructing a spectral sequence $\mathrm{AKh}^{\mathrm{symp}}(\beta) \Rightarrow \mathrm{Kh}^{\mathrm{symp}}(\kappa(\beta))$.

Remark 11.3. One could also define a braid invariant as the direct sum of fixed point Floer homologies of ϕ_β on $\mathcal{Y}_{j,n}$ over $j \in \{0, \ldots, n\}$ (where 'partial wrapping' would remove fixed points at infinity). It seems likely that this geometric definition would recover that of Definition 11.1, but establishing such an isomorphism is beyond the scope of this paper. See [43] for closely related results.

Remark 11.4. There is an $\mathfrak{sl}_2$ action on $\mathrm{AKh}^{\mathrm{symp}}(\beta)$, by combining [21] and Proposition 11.2, which implies a rank inequality

$$\mathrm{rank}(HH_*(\mathcal{F}\mathcal{S}^{\mathrm{cyl},j}(\pi_{E_{1/2}}), P_\beta^{(j)})) \leq \mathrm{rank}(HH_*(\mathcal{F}\mathcal{S}^{\mathrm{cyl},j+1}(\pi_{E_{1/2}}), P_\beta^{(j+1)}))$$

for all $j < n/2$. Coupled with Remark 11.3, this predicts a non-trivial existence result for periodic points of the symplectomorphisms of $\mathcal{Y}_{j,n}$ associated to braids. It would be interesting to construct the $\mathfrak{sl}_2$ action symplectic-geometrically.

11.2. Embedding of algebras

Let $E := A_{m-1}$ and π_E be the standard Lefschetz fibration. Let $W_1 := \{\mathrm{re}(z) < n + 1/2\}$ and $W_2 := \{\mathrm{re}(z) > n + 1/2\}$. Note that $\mathcal{F}\mathcal{S}^{\mathrm{cyl},j}_{W_1}(\pi_E) = \mathcal{F}\mathcal{S}^{\mathrm{cyl},j}(\pi_{E_{1/2}})$. As explained in Section 3.2, there is a faithful A_∞ functor $\sqcup \underline{K} : \mathcal{F}\mathcal{S}^{\mathrm{cyl},j}_{W_1}(\pi_E) \to \mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ for each object $\underline{K}$ of $\mathcal{F}\mathcal{S}^{\mathrm{cyl},n-j}_{W_2}(\pi_E)$.

By essentially the same argument, we can prove that

$$HF(\underline{L}_0 \sqcup \underline{K}_0, \underline{L}_1 \sqcup \underline{K}_1) = HF(\underline{L}_0, \underline{L}_1) \otimes HF(\underline{K}_0, \underline{K}_1) \tag{11.4}$$

for $\underline{L}_i \in \mathcal{FS}^{\mathrm{cyl},j}_{W_1}(\pi_E)$ and $\underline{K}_i \in \mathcal{FS}^{\mathrm{cyl},j}_{W_2}(\pi_E)$. Varying either the $\underline{L}_i$ or $\underline{K}_i$, Floer multiplication amongst the corresponding groups also respects the tensor product structure. Let $0 \le j \le n$ be an integer. For each $\lambda \in \Lambda_{j,n}$, one can define an object $\underline{L}_\lambda$ in $\mathcal{FS}^{\mathrm{cyl},j}_{W_1}(\pi_E)$ by forgetting the right n critical values (see Figure 11.1). Our convention is $\mathcal{FS}^{\mathrm{cyl},j}_{W_1}(\pi_E) = \mathbb{K}$ when $j = 0$, i.e. the category contains a unique object whose endomorphism algebra is the ground field concentrated in degree zero; the object $\underline{L}_\lambda$ is by definition the unique non-zero object in the category in this case. For the other extreme, when $j = n$, the object $\underline{L}_\lambda$ is the tuple of Lagrangian thimbles (cf. Corollary 5.17). Similarly, for each $\mu \in \Lambda_{n-j,m-n}$, one can define an object $\underline{K}_\mu$ in $\mathcal{FS}^{\mathrm{cyl},n-j}_{W_2}(\pi_E)$ by forgetting the left n critical values. We have an algebra isomorphism

$$\bigoplus HF(\underline{L}_{\lambda_0} \sqcup \underline{K}_{\mu_0}, \underline{L}_{\lambda_1} \sqcup \underline{K}_{\mu_1}) = K^{\mathrm{symp}}_{j,n} \times K^{\mathrm{symp}}_{n-j,m-n} \tag{11.5}$$

where the sum is over all $\lambda_0, \lambda_1 \in \Lambda_{j,n}$ and $\mu_0, \mu_1 \in \Lambda_{n-j,m-n}$.

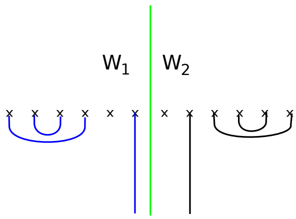


Fig. 11.1. Lagrangian tuples $\underline{L}_\lambda$ (blue, left) and $\underline{K}_\mu$ (black, right).

Let A be the A_∞ full subcategory of $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ consisting of objects $\underline{L}_\lambda \sqcup \underline{K}_\mu$ over all $\lambda \in \Lambda_{j,n}$, $\mu \in \Lambda_{n-j,m-n}$ and $j = 0, \dots, n$. For each integer $0 \le j \le n$, we have the full subcategory A_j of A given by the objects $\underline{L}_\lambda \sqcup \underline{K}_\mu$ over all $\lambda \in \Lambda_{j,n}$ and $\mu \in \Lambda_{n-j,m-n}$.

Lemma 11.5. *For each integer $0 \le j \le n$, the endomorphism algebra of the objects in A_j is formal, and hence quasi-isomorphic to the right hand side of* (11.5).

Proof. This can be seen as a version of the Künneth theorem: the A_∞ structure on a product Lagrangian is quasi-isomorphic to the tensor product of a dg model for each factor (see [3] for the relevant theory).

Alternatively, and more concretely, in our case we observe that each $\underline{L}_\lambda \sqcup \underline{K}_\mu$ is pure with respect to the nc vector field for the same reason as in Lemma 8.10. Moreover, the cohomological algebra of the collection of $\underline{L}_\lambda \sqcup \underline{K}_\mu$ splits as a tensor product, so we can repeat the strategy from Section 9. We next give the details of this second approach.

We choose total orderings $\leq_1^T$ and $\leq_2^T$ for $\Lambda_{j,n}$ and $\Lambda_{n-j,m-n}$, respectively, both of which refine the Bruhat partial ordering. We can define a total order $\leq^T$ on the pairs $(\lambda, \mu) \in \Lambda_{j,n} \times \Lambda_{n-j,m-n}$ by declaring that

$$(\lambda_0, \mu_0) <^T (\lambda_1, \mu_1) \quad \text{if and only if} \quad (\lambda_0 <_1^T \lambda_1) \text{ or } (\lambda_0 = \lambda_1 \text{ and } \mu_0 <_2^T \mu_1). \tag{11.6}$$

Now, we can run the induction in the proof of Proposition 9.4.

For the rest of the proof, we use $L_{\lambda,\mu}$ to denote $\underline{L}_\lambda \sqcup \underline{K}_\mu$. Let $(\eta_1, \eta_2) \in \Lambda_{j,n} \times \Lambda_{n-j,m-n}$ and suppose that a b-equivariant structure on $\underline{L}_{\lambda,\mu}$ has been chosen for all $(\lambda, \mu) >^T (\eta_1, \eta_2)$ such that b is moreover pure. We want to choose a b-equivariant structure on $\underline{L}_{\eta_1,\eta_2}$ such that b is pure on $\{\underline{L}_{\lambda,\mu}\}_{(\lambda,\mu)\geq^T(\eta_1,\eta_2)}$.

Recall the notation for the maximal element w_λ associated to $\lambda \in \Lambda_{j,n}$ from (9.5). We choose a b-equivariant structure on $\underline{L}_{\eta_1,\eta_2}$ such that the minimal degree element of $HF(\underline{L}_{\eta_1,\eta_2}, \underline{L}_{w_{\eta_1},w_{\eta_2}})$ has weight equal to the degree. By Lemma 9.1, $HF(\underline{L}_{\eta_1,\eta_2}, \underline{L}_{w_{\eta_1},w_{\eta_2}})$ is cyclic as a module over $HF(\underline{L}_{w_{\eta_1},w_{\eta_2}}, \underline{L}_{w_{\eta_1},w_{\eta_2}})$ so weight equals degree for all pure degree elements in $HF(\underline{L}_{\eta_1,\eta_2}, \underline{L}_{w_{\eta_1},w_{\eta_2}})$. By Lemmas 9.1 and 9.2, and exactly the same argument as in Proposition 9.4, we know that all elements of $HF(\underline{L}_{w_{\eta_1},w_{\eta_2}}, \underline{L}_{\eta_1,\eta_2})$ have weights equal to their degrees.

Now, suppose that $(\lambda, \mu) >^T (\eta_1, \eta_2)$ but $(\lambda, \mu) \neq (w_{\eta_1}, w_{\eta_2})$, and purity holds for all of $HF(L_{\lambda',\mu'}, L_{\eta_1,\eta_2})$, $HF(L_{\eta_1,\eta_2}, L_{\lambda',\mu'})$ such that $(\lambda', \mu') >^T (\lambda, \mu)$.

Claim 11.6. *If $HF(L_{\lambda,\mu}, L_{\eta_1,\eta_2}) \neq 0$, then there exists $(\lambda', \mu') >^T (\lambda, \mu)$ such that both $\underline{\lambda}\lambda'\overline{\eta_1}$ and $\underline{\mu}\mu'\overline{\eta_2}$ are oriented circle diagrams.*

Assuming Claim 11.6, one can show that purity also holds for $HF(L_{\lambda,\mu}, L_{\eta_1,\eta_2})$, $HF(L_{\eta_1,\eta_2}, L_{\lambda,\mu})$ exactly as in the last two paragraphs of the proof of Proposition 9.4. Therefore, the result follows by induction. ∎

Proof of Claim 11.6. By assumption, we have

$$(w_{\eta_1}, w_{\eta_2}) >^T (\lambda, \mu) >^T (\eta_1, \eta_2). \tag{11.7}$$

By the definition of $>^T$, we have $\lambda = \eta_1$ or $\lambda >_1^T \eta_1$.

If $\lambda = \eta_1$, then $\mu >_2^T \eta_2$. If $\mu \neq w_{\eta_2}$, then by (2) of Lemma 9.3, there exists $\mu' >_2^T \mu, \eta_2$ such that $\underline{\mu}\mu'\overline{\eta_2}$ is an oriented circle diagram. We can take $\lambda' = \eta_1$ so that we have $(\lambda', \mu') >^T (\lambda, \mu)$ and both $\underline{\lambda}\lambda'\overline{\eta_1}$ and $\underline{\mu}\mu'\overline{\eta_2}$ are oriented circle diagrams.

If $\lambda = \eta_1$ and $\mu = w_{\eta_2}$, then by (11.7), we have $\eta_1 \neq w_{\eta_1}$ (in fact, $\nu = w_\nu$ only when ν is the maximal weight, see (9.5)). In this case, we can take $(\lambda', \mu') = (w_{\eta_1}, w_{\eta_2})$.

If $\lambda >_1^T \eta_1$, then either $\lambda = w_{\eta_1}$ or $\lambda \neq w_{\eta_1}$. If $\lambda \neq w_{\eta_1}$, then by (2) of Lemma 9.3, there exists $\lambda' >_1^T \lambda, \eta_1$ such that $\underline{\lambda}\lambda'\overline{\eta_1}$ is an oriented circle diagram. As $HF(L_{\lambda,\mu}, L_{\eta_1,\eta_2}) \neq 0$, there exists μ' such that $\underline{\mu}\mu'\overline{\eta_2}$ is an oriented circle diagram. We can pick any such μ' because $(\lambda', \mu') >^T (\lambda, \mu)$ no matter which μ' we pick.

If $\lambda >_1^T \eta_1$ but $\lambda = w_{\eta_1}$, then we pick $\lambda' = \lambda = w_{\eta_1}$. Since $(w_{\eta_1}, w_{\eta_2}) >^T (\lambda, \mu)$, we have $\mu <_2^T w_{\eta_2}$. If $\eta_2 \neq w_\mu$, then by (2) of Lemma 9.3, there exists $\mu' >_2^T \mu, \eta_2$ such that $\underline{\mu}\mu'\overline{\eta_2}$ is an oriented circle diagram so we are done. If $\eta_2 = w_\mu$, then $\mu <_2^T \eta_2$ so we can take $\mu' = \eta_2$ because it implies that $(\lambda', \mu') = (w_{\eta_1}, \eta_2) >^T (w_{\eta_1}, \mu) = (\lambda, \mu)$ and both $\underline{\lambda}\lambda'\overline{\eta_1}$ and $\underline{\mu}\mu'\overline{\eta_2}$ are oriented circle diagrams.

This completes the proof. ■

It is not clear whether the endomorphism algebra of the direct sum of all the objects in A is formal or not. However, the dg category perf-A of perfect modules is formal in the following weaker sense.

Lemma 11.7. perf-A *is quasi-equivalent to* perf-$K_{n,m}^{\mathrm{alg}}$.

Proof. By iteratively applying exact triangles, we see that the objects in A_j can generate thimbles $\underline{T} = \{T_1, \dots, T_n\}$ in $\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ such that j of the T_i's are contained in $\pi_E^{-1}(W_1)$, and the remaining $n-j$ are contained in $\pi_E^{-1}(W_2)$. Therefore, every thimble of $\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ can be generated by objects in A, so the result follows. ■

Lemma 11.8. *Let* $X_0 \in A_{j_0}$ *and* $X_1 \in A_{j_1}$. *If* $HF(X_0, X_1) \neq 0$, *then* $j_0 \geq j_1$. *As a result,* $\langle A_n, \dots, A_0\rangle$ *is a semi-orthogonal decomposition of* A.

Proof. It is clear that $CF(X_0, X_1) = 0$ if $j_0 < j_1$, because of the definition of Floer cochains via positive isotopies along $\mathbb{R} = \partial\mathbb{H}$. ■

Theorem 11.9. perf-$K_{n,m}^{\mathrm{alg}}$ *admits a semi-orthogonal decomposition* $\langle$perf-$A_n, \dots,$ perf-$A_0\rangle$, *where for each* j, perf-A_j *is quasi-equivalent to* perf-$(K_{j,n}^{\mathrm{alg}} \times K_{n-j,m-n}^{\mathrm{alg}})$.

Proof. This immediately follows from Lemmas 11.5, 11.7 and 11.8. ■

Theorem 11.9 categorifies the identity $\binom{m}{n} = \sum_{j=0}^{n}\binom{n}{j}\binom{m-n}{n-j}$ for ranks of Grothendieck groups.

Remark 11.10. Even though we are primarily interested in the case $m = 2n$ in this section, Theorem 11.9 holds for all $n < m$. This algebraic result seems to be new and may be of independent interest.

We need another full subcategory of $\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ that is analogous to A. Let

$$\begin{aligned} W_2^! &:= \{n + 1/2 < \mathrm{re}(z) < m + 1/2 \text{ and } \mathrm{im}(z) < 2\} \\ &\quad \cup \{\mathrm{re}(z) < m + 1/2 \text{ and } \mathrm{im}(z) < 1/2\} \end{aligned} \tag{11.8}$$

and $W_1^!$ be the complement of $W_2^!$. For $\lambda \in \Lambda_{j,n}$ and $\mu \in \Lambda_{n-j,m-n}$, we can consider the corresponding upper Lagrangian tuples $\underline{L}_{\overline{\lambda}}^! \in \mathcal{F}\mathcal{S}_{W_1^!}^{\mathrm{cyl},j}(\pi_E)$ and $\underline{K}_{\overline{\mu}}^! \in \mathcal{F}\mathcal{S}_{W_2^!}^{\mathrm{cyl},n-j}(\pi_E)$ (see Figure 11.2).

Let $A^!$ be the A_∞ full subcategory of $\mathcal{F}\mathcal{S}^{\mathrm{cyl},n}(\pi_E)$ consisting of objects $\underline{L}_\lambda^! \sqcup \underline{K}_\mu^!$ over all $\lambda \in \Lambda_{j,n}$, $\mu \in \Lambda_{n-j,m-n}$ and $j = 0, \dots, n$. For each integer $0 \leq j \leq n$, we have the full subcategory $A_j^!$ of $A^!$ given by the objects $\underline{L}_\lambda^! \sqcup \underline{K}_\mu^!$ over all $\lambda \in \Lambda_{j,n}$ and $\mu \in \Lambda_{n-j,m-n}$. The following Koszulness property justifies the notation $(-)^!$.

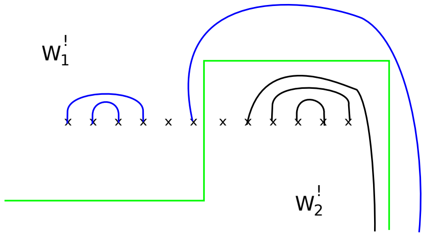


Fig. 11.2. Lagrangian tuples $\underline{L}^!_{\lambda}$ (blue) and $\underline{K}^!_{\mu}$ (black).

Lemma 11.11. *Let $X^! \in A^!_i$ and $X \in A_j$. If $HF(X^!, X) \neq 0$, then $i = j$.*

Proof. The direction of wrapping shows that $CF(X^!, X)$ is non-zero only if $i = j$. ■

On the other hand, $A^!$ shares many features with A. By the same arguments as in Lemmas 11.5 and 11.7, we know that perf-$A^!$ is quasi-equivalent to perf-$K^{\text{alg}}_{n,m}$ and for $j = 0, \dots, n$, the category $A^!_j$ is formal with endomorphism algebra

$$\bigoplus HF(\underline{L}^!_{\lambda_0} \sqcup \underline{K}^!_{\mu_0}, \underline{L}^!_{\lambda_1} \sqcup \underline{K}^!_{\mu_1}) = K^{\text{symp}}_{j,n} \times K^{\text{symp}}_{n-j,m-n} \tag{11.9}$$

The analogues of Lemma 11.12 and Theorem 11.9 are

Lemma 11.12. *Let $X_0 \in A^!_{j_0}$ and $X_1 \in A^!_{j_1}$. If $HF(X_0, X_1) \neq 0$, then $j_0 \leq j_1$. As a result, $\langle A^!_0, \dots, A^!_n \rangle$ is a semi-orthogonal decomposition of $A^!$.*

Theorem 11.13. perf-$K^{\text{alg}}_{n,m}$ *admits a semi-orthogonal decomposition* $\langle$perf-$A^!_0, \dots,$ perf-$A^!_n\rangle$, *where for each j,* perf-$A^!_j$ *is quasi-equivalent to* perf-$(K^{\text{alg}}_{j,n} \times K^{\text{alg}}_{n-j,m-n})$.

In view of Theorems 11.9 and 11.13, when $m = 2n$, perf-$(K_{j,n} \times K_{n-j,m-n})$ is the same as $[(K_{n-j,n})^{\text{op}}, K_{j,n}]$. By the isomorphism $(K_{n-j,n})^{\text{op}} = K_{j,n}$ (see Corollary 1.5), this in turn is isomorphic to $[K_{j,n}, K_{j,n}]$. Similarly, $(K_{j,n} \times K_{n-j,n})$-perf is the same as $[K_{j,n}, (K_{n-j,n})^{\text{op}}]$, which in turn is isomorphic to $[K_{j,n}, K_{j,n}]$.

11.3. A Beilinson-type spectral sequence

We next explain why a semi-orthogonal decomposition of an A_∞ category $\mathcal{C}$ induces a spectral sequence with target a given morphism group in $\mathcal{C}$. From now on, for an object X, we use X^r and X^l to denote its right and left Yoneda modules, respectively.

Let $\mathcal{C}$ be a split-closed triangulated A_∞ category with a semi-orthogonal decomposition

$$\mathcal{C} = \langle \mathcal{C}_n, \dots, \mathcal{C}_0 \rangle. \tag{11.10}$$

A *Koszul dual* semi-orthogonal decomposition is a semi-orthogonal decomposition

$$\mathcal{C} = \langle \mathcal{C}^!_0, \dots, \mathcal{C}^!_n \rangle \tag{11.11}$$

such that for $X \in \mathcal{C}_i^!$ and $Y \in \mathcal{C}_k$

$$H(\hom_{\mathcal{C}}(X, Y)) \neq 0 \quad \text{only if } i = k. \tag{11.12}$$

Without loss of generality, we can assume each summand in the semi-orthogonal decomposition is split-closed triangulated. Note that Koszul dual semi-orthogonal decompositions always exist [28].

Let $\iota_j : \mathcal{C}_j \to \mathcal{C}$ and $\iota_j^! : \mathcal{C}_j^! \to \mathcal{C}$ be the embeddings. Let $\pi_j : \mathcal{C} \to \mathcal{C}_j$ and $\pi_j^! : \mathcal{C} \to \mathcal{C}_j^!$ be the projection functors (i.e. the unique functors such that $\pi_j \circ \iota_j = \mathrm{id}_{\mathcal{C}_j}$ and $\pi_j \circ \iota_i = 0$ for $i \neq j$). These functors induce pull-back functors on modules.

Lemma 11.14. *Up to quasi-isomorphism, the left Yoneda embedding $\mathcal{C}_j^! \to \mathcal{C}$-perf factors through $(\pi_j)^* : \mathcal{C}_j$-perf $\to \mathcal{C}$-perf.*

Proof. Let $X \in \mathcal{C}_j^!$. By (11.12), we have $H(\hom_{\mathcal{C}}(X, X')) = 0$ for all $X' \in \mathcal{C}_i$ such that $i \neq j$. It implies that, up to quasi-isomorphism, we have

$$X^l = (\pi_j)^* \circ (\iota_j)^*(X^l), \tag{11.13}$$

so the result follows. ■

Since Yoneda embedding and $(\pi_j)^*$ are both cohomologically full and faithful, so is the functor

$$(\iota_j)^*((-)^l) : \mathcal{C}_j^! \to \mathcal{C}_j\text{-perf}. \tag{11.14}$$

By (11.12) again, we can see that

$$(\iota_j)^*((-)^l) : \mathcal{C}_i^! \to \mathcal{C}_j\text{-perf} \tag{11.15}$$

is the 0 functor when $i \neq j$.

Lemma 11.15. (11.14) *is essentially surjective, so induces a quasi-equivalence $\mathcal{C}_j^! \to \mathcal{C}_j$-perf.*

Proof. Combining the fact that (11.15) is the 0 functor and that both Yoneda embedding and $(\iota_j)^*$ are essentially surjective, we find that (11.14) is essentially surjective, so the result follows. ■

Let $\psi_j : \mathcal{C}_j\text{-perf} \to \mathcal{C}_j^!$ be a quasi-inverse.

Lemma 11.16. *Let $X \in \mathcal{C}$ be quasi-isomorphic to an iterated mapping cone of the form*

$$X = \mathrm{Cone}(\dots \mathrm{Cone}(X_0 \to X_1) \dots X_n) \tag{11.16}$$

where $X_j \in \mathcal{C}_j^!$ for $j = 0, \dots, n$. Then $\psi_j \circ (\iota_j)^*(X^l)$ is quasi-isomorphic to X_j.

Proof. By applying the functor $\psi_j \circ (\iota_j)^*((-)^l)$ to the iterated mapping cone (11.16), and using the fact that (11.15) is the 0 functor, we see that only the object X_j contributes and hence

$$\psi_j \circ (\iota_j)^*(X^l) \simeq \psi_j \circ (\iota_j)^*(X_j^l). \tag{11.17}$$

By definition, $\psi_j \circ (\iota_j)^*((-)^l)$ is quasi-isomorphic to the identity functor on $\mathcal{C}_j$ so the right hand side of (11.17) is in turn quasi-isomorphic to X_j. ■

Proposition 11.17. *Let $X, Y \in \mathcal{C}$. There is a spectral sequence to $H(\hom_{\mathcal{C}}(X, Y))$ from*

$$\bigoplus_{j=0}^{n} H((\iota_j^!)^*(Y^r) \otimes_{\mathcal{C}_j^!} \psi_j \circ (\iota_j)^*(X^l)) \tag{11.18}$$

where $\psi_j \circ (\iota_j)^*(X^l)$, as an object in $\mathcal{C}_j^!$, is regarded as a left $\mathcal{C}_j^!$-module via the Yoneda embedding.

Proof. By (11.10), X is quasi-isomorphic to an iterated mapping cone of the form (11.16). An expression of X as such a mapping cone induces a filtration on $Y^r(X)$, with direct sum of graded pieces being

$$\bigoplus_{j=0}^{n} Y^r(X_j). \tag{11.19}$$

Thus we have a spectral sequence from the cohomology of (11.19) to $H(\hom_{\mathcal{C}}(X, Y))$.

On the other hand, we have quasi-isomorphisms of cochain complexes

$$\begin{aligned} Y^r(X_j) &= (\iota_j^!)^*(Y^r)(X_j) = (\iota_j^!)^*(Y^r)(\psi_j \circ (\iota_j)^*(X^l)) \\ &= (\iota_j^!)^*(Y^r) \otimes_{\mathcal{C}_j^!} \psi_j \circ (\iota_j)^*(X^l) \end{aligned}$$

where the second quasi-isomorphism comes from Lemma 11.16, and the other quasi-isomorphisms come from the definitions of the objects involved. ■

Remark 11.18. In the extreme case where each $\mathcal{C}_j$ (and $\mathcal{C}_j^!$) is an exceptional object, the semi-orthogonal decomposition is a full exceptional collection and Proposition 11.17 reduces to the well-known Beilinson-type spectral sequence (cf. [40, Section (5l)]).

12. From annular to ordinary Khovanov homology

In this section, we explain how to apply Proposition 11.17 to the semi-orthogonal decompositions in Theorem 11.9 and 11.13 to obtain a spectral sequence from annular to ordinary (symplectic) Khovanov homology.

Recall that for an A_∞ or dg category/algebra $\mathcal{C}$, the Hochschild homology of a $\mathcal{C}$-$\mathcal{C}$-bimodule P is defined to be

$$HH_*(\mathcal{C}, P) := H(\Delta_{\mathcal{C}} \otimes_{\mathcal{C}\text{-}\mathcal{C}} P) \tag{12.1}$$

where $\Delta_{\mathcal{C}}$ is the diagonal bimodule and $\otimes_{\mathcal{C}\text{-}\mathcal{C}}$ is the derived tensor product over $[\mathcal{C}, \mathcal{C}]$. Equivalently, $\Delta_{\mathcal{C}}$ defines a right $\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}$-module and P defines a left $\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}$-module, and we have

$$HH_*(\mathcal{C}, P) := H(\Delta_{\mathcal{C}} \otimes_{\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}} P) \tag{12.2}$$

The fact that Hochschild homology groups appear in the first page of a spectral sequence to symplectic Khovanov cohomology arises from a relation between the horseshoe Lagrangian and the diagonal bimodule, which we explain next. More precisely, in Section 12.1, we compute $(\iota_j)^*(\underline{L}^l_{\mathrm{hs}})$ and $(\iota^!_j)^*(\phi_\beta(\underline{L}_{\mathrm{hs}})^r)$ for the embeddings $\iota_j : A_j \to \mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ and $\iota^!_j : A^!_j \to \mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$. Then, we describe the quasi-inverse ψ_j and complete the proof of Theorem 1.9 in Section 12.2.

12.1. Horseshoe Lagrangian tuple and diagonal bimodule

Recall that we have the pull-back functor

$$(\iota_j)^* : \mathcal{FS}^{\mathrm{cyl},n}(\pi_E)\text{-perf} \to A_j\text{-perf} \simeq [K_{j,n}, K_{j,n}]. \tag{12.3}$$

The following is the key technical result.

Proposition 12.1. *Up to grading shift, $(\iota_j)^*(\underline{L}^l_{\mathrm{hs}})$ is quasi-isomorphic to the diagonal $K_{j,n}$-bimodule.*

We divide the proof into two steps. First, we will show that $(\iota_j)^*(\underline{L}^l_{\mathrm{hs}})$ coincides with the diagonal bimodule (up to grading shift) on the cohomological level. Then, we will explain how to adapt the strategy in Section 9 to prove that this bimodule is formal. For all the Lagrangian tuples in the proof, we continue to use our grading convention (6.8) and the cost is that we will see eventually that $(\iota_j)^*(\underline{L}^l_{\mathrm{hs}})$ corresponds to the diagonal bimodule shifted by $n-j$ instead of the diagonal bimodule. This arises from the fact that there are $n-j$ matching paths of $(\iota_j)^*(\underline{L}^l_{\mathrm{hs}})$ that are oriented clockwise in the sense of the algebraic extended arc algebra (see Remark 12.5).

Let $\mu \in \Lambda_{n-j,n}$ and define $1 \le c_{\mu,1} < \cdots < c_{\mu,n-j} \le n$ as at the beginning of Section 6.1. Let $c_{\mu,i_1} < \cdots < c_{\mu,i_s}$ be all the good points. Let

$$T := \{n+1-c_{\mu,i_l} \mid l = 1,\dots,s\}, \tag{12.4}$$
$$S := \{1,\dots,n\} \cup (T+1/3) \cup (T+2/3). \tag{12.5}$$

We define $(\mu^+)', (\mu^-)' : S \to \{\vee,\wedge\}$ by

$$(\mu^+)'(a) = \begin{cases} PD(\mu)(a) & \text{if } a \in \{1,\dots,n\}\setminus T, \\ \vee & \text{if } a \in T, \\ \wedge & \text{if } a \in (T+1/3)\cup(T+2/3), \end{cases} \tag{12.6}$$

and

$$(\mu^-)'(a) = \begin{cases} \vee & \text{if } a \in T+1/3, \\ \wedge & \text{otherwise.} \end{cases} \tag{12.7}$$

There is a unique order preserving bijective map (with the order induced from $\mathbb{R}$) $f : \{1,\dots,n+2s\} \to S$. Let $\mu^+ := (\mu^+)' \circ f \in \Lambda_{j+s,n+2s}$ and $\mu^- := (\mu^-)' \circ f \in \Lambda_{s,n+2s}$.

For $\lambda \in \Lambda_{j,n}$, we define $(\lambda \sqcup \mu^-)' : S \to \{\vee, \wedge\}$ by

$$(\lambda \sqcup \mu^-)'(a) = \begin{cases} \lambda(a) & \text{if } a \in \{1, \dots, n\}, \\ (\mu^-)'(a) & \text{if } a \in (T + 1/3) \cup (T + 2/3), \end{cases} \tag{12.8}$$

and let $\lambda \sqcup \mu^- := (\lambda \sqcup \mu^-)' \circ f \in \Lambda_{j+s,n+2s}$. Note that both μ^+ and $\lambda \sqcup \mu^-$ lie in $\Lambda_{j+s,n+2s}$.

Lemma 12.2. *For $\lambda \in \Lambda_{j,n}$ and $\mu \in \Lambda_{n-j,n}$, we have vector space isomorphisms*

$$HF^{k+n-j-s}(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_\mu)) = HF^k(\underline{L}_{\mu^+}, \underline{L}_{\lambda \sqcup \mu^-}) \tag{12.9}$$

for all k. (Here $n - j - s$ is the number of thimble paths in $\underline{K}_\mu$.)

The importance of Lemma 12.2 is that the right hand side of (12.9) is part of the symplectic extended arc algebra. We will see later how to use (12.9), and the identification of the symplectic and algebraic extended arc algebras (Proposition 7.12), to compute the bimodule multiplication map of $(\iota_j)^*(\underline{L}^l_{\mathrm{hs}})$.

Proof of Lemma 12.2. We use an upper matching to represent $\underline{L}_{\mathrm{hs}}$ and a lower matching to represent $\underline{K}_\mu$ (see the first picture of Figure 12.1). Now, we apply a symplectomorphism to 'move the right half to the top' and make all the matching paths of $\underline{L}_{\mathrm{hs}}$ be straight line segments (see the second picture of Figure 12.1). For each matching path of $\underline{K}_\mu$, we apply a further symplectomorphism to 'bend it to the right' and make the two end points become $n + 1 - c_{\mu,i_l} + 1/3 + \sqrt{-1}$ and $n + 1 - c_{\mu,i_l} + 2/3 + \sqrt{-1}$ (see the third picture of Figure 12.1). Then, we remove all critical values contained in the thimble paths of $\underline{K}_\mu$ and all the paths that contain these critical values (see the fourth picture of Figure 12.1). Finally, for those matching paths of $\underline{L}_{\mathrm{hs}}$ that do not intersect with a matching path of $\underline{K}_\mu$, we bend them to the right and turn them into thimble paths (see the fifth picture of Figure 12.1).

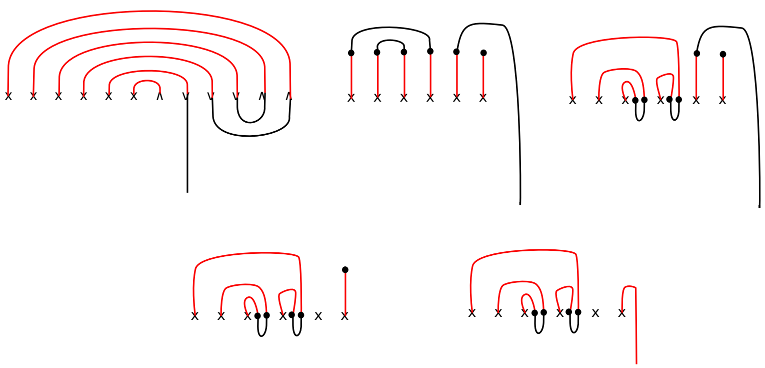


Fig. 12.1. Turning $\underline{L}_{\mathrm{hs}}$ (red) and $\underline{K}_\mu$ (black) to $\underline{L}_{\mu^+}$ (red) and $\underline{L}_{\mu^-}$ (black), respectively.

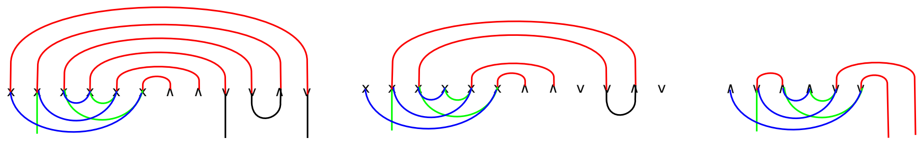


Fig. 12.2. $CF(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_\mu))$ (left) and $CF(\underline{L}_{\mu^+}, \underline{L}_{\lambda\sqcup\mu^-})$ (right).

After this procedure, the set of critical values (the crosses and black dots in Figure 12.1) is exactly S, and we identify it with $\{1, \dots, n+2s\}$ by f. The Lagrangian tuples $\underline{L}_{\mathrm{hs}}$ and $\underline{K}_\mu$ become $\underline{L}_{\mu^+}$ and $\underline{L}_{\mu^-}$ respectively.

If we now add in a lower matching of $\underline{L}_\lambda$ that does not intersect with $\underline{L}_{\mu^-}$, then together with $\underline{L}_{\mu^-}$ we obtain a representative of $\underline{L}_{\lambda\sqcup\mu^-}$ (see the second picture of Figure 12.2). Moreover, for degree reasons (either all generators have odd degree, or all have even degree), both the cochain models $CF(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_\mu))$ and $CF(\underline{L}_{\mu^+}, \underline{L}_{\lambda\sqcup\mu^-})$ have vanishing differentials. Furthermore, there is an identification of the generators by sending a generator $\underline{x}$ of $CF(\underline{L}_{\mu^+}, \underline{L}_{\lambda\sqcup\mu^-})$ to the generator $\underline{y}$ of $CF(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_\mu))$ such that $\pi_E(\underline{y})$ is the union of $\pi_E(\underline{x})$ and the end points of the thimble paths of $\underline{K}_\mu$ (Figure 12.2). This identification increases the grading by the number of points added, which is $n - j - s$. This completes the proof. ∎

To prove Lemma 12.2, it suffices to consider a lower matching of $\underline{L}_\lambda$ to give a representative of $\underline{L}_{\lambda\sqcup\mu^-}$. However, for this representative, there may be nested circles in the union of matching and thimble paths of $\underline{L}_{\mu^+}$ and $\underline{L}_{\lambda\sqcup\mu^-}$, which will be inconvenient later on. By possibly sliding the lower matching of $\underline{L}_\lambda$ across some matchings of $\underline{L}_{\mu^-}$ (and $\underline{L}_\lambda$ itself), we can choose a representative $\underline{L}_{\widetilde{\lambda\sqcup\mu^-}}$ of $\underline{L}_{\lambda\sqcup\mu^-}$ with no nested circles (see the second picture of Figure 12.3). This corresponds to choosing another representative $\underline{L}_{\widetilde{\lambda\sqcup\mu}}$ of $\iota_j(\underline{L}_\lambda, \underline{K}_\mu)$, by sliding across some matching paths of $\underline{K}_\mu$ (see the first picture of Figure 12.3). For these representatives, we still have the (grading shifting) cochain isomorphism with 0 differential,

$$CF^{*+n-j-s}(\underline{L}_{\mathrm{hs}}, \underline{L}_{\widetilde{\lambda\sqcup\mu}}) = CF^*(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda\sqcup\mu^-}}), \tag{12.10}$$

which is again given by sending a generator $\underline{x}$ of $CF(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda\sqcup\mu^-}})$ to the generator $\underline{y}$ of $CF(\underline{L}_{\mathrm{hs}}, \underline{L}_{\widetilde{\lambda\sqcup\mu}})$ such that $\pi_E(\underline{y})$ is the union of $\pi_E(\underline{x})$ and the end points of the thimble

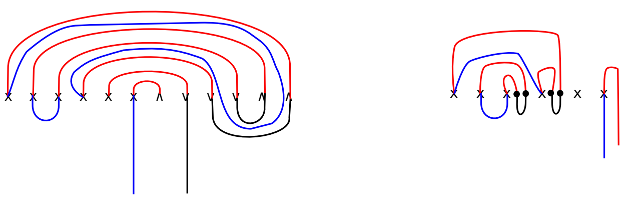


Fig. 12.3. $CF(\underline{L}_{\mathrm{hs}}, \underline{L}_{\widetilde{\lambda\sqcup\mu}})$ (left) and $CF(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda\sqcup\mu^-}})$ (right).

paths of $\underline{K}_\mu$. The geometric generators on the right hand side of (12.10) are geometric basis elements of $K^{\mathrm{symp}}_{j+s,n+2s}$, which are identified with the corresponding diagrammatic basis elements of $K^{\mathrm{alg}}_{j+s,n+2s}$ under the map Φ of Proposition 7.12.

Next, we want to compare the multiplication maps

$$CF(\underline{L}_{\widetilde{\lambda_0 \sqcup \mu}}, \underline{L}_{\widetilde{\lambda_1 \sqcup \mu}}) \times CF(\underline{L}_{\mathrm{hs}}, \underline{L}_{\widetilde{\lambda_0 \sqcup \mu}}) \to CF(\underline{L}_{\mathrm{hs}}, \underline{L}_{\widetilde{\lambda_1 \sqcup \mu}}) \tag{12.11}$$

and

$$CF(\underline{L}_{\widetilde{\lambda_0 \sqcup \mu^-}}, \underline{L}_{\widetilde{\lambda_1 \sqcup \mu^-}}) \times CF(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda_0 \sqcup \mu^-}}) \to CF(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda_1 \sqcup \mu^-}}). \tag{12.12}$$

The second and third terms of (12.11) and (12.12) are identified via (12.10), and the identification of the first terms is defined similarly (i.e. adding to a generator $\underline{x}$ of $CF(\underline{L}_{\widetilde{\lambda_0 \sqcup \mu^-}}, \underline{L}_{\widetilde{\lambda_1 \sqcup \mu^-}})$ the intersection points of $CF(K_\mu, K_\mu)$ that lie above those critical values that are contained in the thimble paths of K_μ – see Figure 12.4; note that this identification is degree preserving, because the points being added now have degree 0).

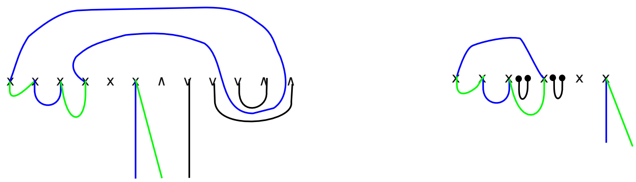


Fig. 12.4. $CF(\underline{L}_{\widetilde{\lambda_0 \sqcup \mu}}, \underline{L}_{\widetilde{\lambda_1 \sqcup \mu}})$ (left) and $CF(\underline{L}_{\widetilde{\lambda_0 \sqcup \mu^-}}, \underline{L}_{\widetilde{\lambda_1 \sqcup \mu^-}})$ (right).

By the open mapping theorem, all solutions u contributing to the map (12.11) must include $n-j-s$ constant triangles which, under π_E, map to the $n-j-s$ critical values that are contained in the thimble paths of K_μ. Removing these constant triangles (and removing the thimble paths of K_μ), we can identify the moduli spaces defining the maps (12.11) and (12.12), so the maps (12.11) and (12.12) agree under the identification above.

Let $CF(\underline{L}^\dagger_{\lambda_0 \sqcup \mu^-}, \underline{L}^\dagger_{\lambda_1 \sqcup \mu^-})$ be a cochain model for the pair $(\underline{L}_{\lambda_0 \sqcup \mu^-}, \underline{L}_{\lambda_1 \sqcup \mu^-})$ which contains the matching paths of μ^- used above, and such that the union of the matching and thimble paths has no nested circles (see the first picture of Figure 12.5). In particular, as the matching paths in μ^- are not nested, the Floer cochain complex canonically splits,

$$CF(\underline{L}^\dagger_{\lambda_0 \sqcup \mu^-}, \underline{L}^\dagger_{\lambda_1 \sqcup \mu^-}) = CF(\underline{L}^\dagger_{\lambda_0}, \underline{L}^\dagger_{\lambda_1}) \otimes CF(\underline{L}_{\mu^-}, \underline{L}_{\mu^-}) \tag{12.13}$$

where, for $j = 0, 1$, $\underline{L}^\dagger_{\lambda_j}$ is obtained by removing the matchings corresponding to μ^- (see the second picture of Figure 12.5). Moreover, as there is no nesting, the geometric generators are geometric basis elements of $K^{\mathrm{symp}}_{j,n} \otimes K^{\mathrm{symp}}_{s,2s}$. By Proposition 7.12, these geometric generators can in turn be identified with diagrammatic basis elements in $K^{\mathrm{alg}}_{j,n} \otimes K^{\mathrm{alg}}_{s,2s}$. Let

$$\kappa : CF(\underline{L}^\dagger_{\lambda_0 \sqcup \mu^-}, \underline{L}^\dagger_{\lambda_1 \sqcup \mu^-}) \to CF(\underline{L}_{\widetilde{\lambda_0 \sqcup \mu^-}}, \underline{L}_{\widetilde{\lambda_1 \sqcup \mu^-}}) \tag{12.14}$$

Fig. 12.5. $CF(\underline{L}^{\dagger}_{\lambda_0\sqcup\mu^-}, \underline{L}^{\dagger}_{\lambda_1\sqcup\mu^-})$ (left) and $CF(\underline{L}^{\dagger}_{\lambda_0}, \underline{L}^{\dagger}_{\lambda_1})$ (right).

be a continuation map (the induced map on cohomology is independent of choices). By Proposition 7.12, we can completely describe

$$\mu^2(\kappa(-),-) : CF(\underline{L}^{\dagger}_{\lambda_0\sqcup\mu^-}, \underline{L}^{\dagger}_{\lambda_1\sqcup\mu^-}) \times CF(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda_0\sqcup\mu^-}}) \to CF(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda_1\sqcup\mu^-}}) \tag{12.15}$$

using the algebraic extended arc algebra. In particular, by applying (12.13) to the first term of (12.15), we completely understand the map

$$\mu^2(\kappa(-\otimes e_{\mu^-}),-) : CF(\underline{L}^{\dagger}_{\lambda_0}, \underline{L}^{\dagger}_{\lambda_1}) \times CF(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda_0\sqcup\mu^-}}) \to CF(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda_1\sqcup\mu^-}}) \tag{12.16}$$

where $e_{\mu^-} \in CF(\underline{L}_{\mu^-}, \underline{L}_{\mu^-})$ is the unit (cf. Lemma 3.2).

On the other hand, we now consider

$$CF(\underline{L}_{\tilde{\lambda}_0}, \underline{L}_{\tilde{\lambda}_1}) \times CF(\underline{L}_{\overline{PD(\mu)}}, \underline{L}_{\tilde{\lambda}_0}) \to CF(\underline{L}_{\overline{PD(\mu)}}, \underline{L}_{\tilde{\lambda}_1}) \tag{12.17}$$

where, for $j = 0, 1$, $\underline{L}_{\tilde{\lambda}_j}$ is defined by removing the matching paths (and their end points) of $\underline{L}_{\lambda_j \tilde{\sqcup}\mu^-}$ that correspond to μ^-, and $\underline{L}_{\overline{PD(\mu)}}$ is an upper matching such that there are no nested circles in the union of the matching paths of $\underline{L}_{\overline{PD(\mu)}}$ and $\underline{L}_{\tilde{\lambda}_j}$ (see Figure 12.6).

Fig. 12.6. $CF(\underline{L}_{\mu^+}, \underline{L}_{\widetilde{\lambda\sqcup\mu^-}})$ (left) and $CF(\underline{L}_{\overline{PD(\mu)}}, \underline{L}_{\tilde{\lambda}_0})$ (right).

Analogously, we have a continuation map

$$\kappa' : CF(\underline{L}^{\dagger}_{\lambda_0}, \underline{L}^{\dagger}_{\lambda_1}) \to CF(\underline{L}_{\tilde{\lambda}_0}, \underline{L}_{\tilde{\lambda}_1}) \tag{12.18}$$

and hence we can define

$$\mu^2(\kappa'(-),-) : CF(\underline{L}^{\dagger}_{\lambda_0}, \underline{L}^{\dagger}_{\lambda_1}) \times CF(\underline{L}_{\overline{PD(\mu)}}, \underline{L}_{\tilde{\lambda}_0}) \to CF(\underline{L}_{\overline{PD(\mu)}}, \underline{L}_{\tilde{\lambda}_1}), \tag{12.19}$$

which we also know completely because the geometric generators are all geometric basis elements and we can apply Proposition 7.12.

For $j = 0, 1$, we have a (grading shifting) cochain isomorphism (both sides have 0 differential as usual)

$$CF^{*+s}(\underline{L}_{\mu^+}, \underline{L}_{\lambda_j \tilde{\sqcup} \mu^-}) \to CF^*(\underline{L}_{\overline{PD(\mu)}}, \underline{L}_{\tilde{\lambda}_j}) \tag{12.20}$$

given by forgetting the points of a generator $\underline{x}$ that lie above $(T + 1/3 + \sqrt{-1}) \cup (T + 2/3 + \sqrt{-1})$. The grading shift arises from removing the grading contribution of the points lying above $(T + 1/3 + \sqrt{-1}) \cup (T + 2/3 + \sqrt{-1})$. There are two cases. If a point lying above $T + 1/3 + \sqrt{-1}$ is removed, then its grading contribution, which is 1, is removed and the rest of the gradings are unchanged. If a point lying above $(T + 2/3 + \sqrt{-1})$ is removed, then its grading contribution, which is 2, is removed but there is grading change from 1 to 2 for the point lying above $T + \sqrt{-1}$. Indeed, since there is a matching path from $T + \sqrt{-1}$ to $T + 1/3 + \sqrt{-1}$ and another matching path from $T + 1/3 + \sqrt{-1}$ to $T + 2/3 + \sqrt{-1}$, a generator has a point lying above $T + 2/3 + \sqrt{-1}$ if and only if it has a point lying above $T + \sqrt{-1}$. Therefore, in both cases, each point removal results in a decrease of overall grading by 1.

The outcome of this discussion is the following:

Lemma 12.3. *Under the cochain identifications* (12.20)*, the maps* (12.16) *and* (12.19) *are identified.*

Proof. After applying Proposition 7.12 to the geometric basis elements for all the Floer cochain groups involved, the maps (12.16) and (12.19) can be computed by the algebraic extended arc algebra in $K^{\mathrm{alg}}_{j+s,n+2s}$ and in $K^{\mathrm{alg}}_{j,n}$, respectively.

The upshot is that, for (12.16), elements of the form $- \otimes e_{\mu^-}$ correspond to orienting the s circles in $\underline{\lambda_1 \sqcup \mu^{-}}^{\mathrm{alg}} \cup \overline{\lambda_0 \sqcup \mu^-}^{\mathrm{alg}}$ associated to μ^- counterclockwise. Applying TQFT type multiplication to these circles does nothing [13, (3.4)], and the rest of the diagrammatic calculus operations (for (12.16) and (12.19)) are obviously identified. ∎

As a consequence of Lemma 12.2, the identification between (12.11) and (12.12), and Lemma 12.3, we get

Corollary 12.4. *We have canonical vector space isomorphisms*

$$HF^{*+n-j}(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_\mu)) = HF^*(\underline{L}_{PD(\mu)}, \underline{L}_\lambda), \tag{12.21}$$

and under these isomorphisms, the multiplication maps

$$\mu^2(- \otimes e_K, -) : HF(\underline{L}_{\lambda_0}, \underline{L}_{\lambda_1}) \times HF(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_{\lambda_0}, \underline{K}_\mu)) \to HF(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_{\lambda_1}, \underline{K}_\mu)) \tag{12.22}$$

and

$$\mu^2(-,-) : HF(\underline{L}_{\lambda_0}, \underline{L}_{\lambda_1}) \times HF(\underline{L}_{PD(\mu)}, \underline{L}_{\lambda_0}) \to HF(\underline{L}_{PD(\mu)}, \underline{L}_{\lambda_1}) \tag{12.23}$$

are identified. Here, for (12.22)*,*

$$-\otimes e_K : HF(\underline{L}_{\lambda_0}, \underline{L}_{\lambda_1}) \to HF(\underline{L}_{\lambda_0}, \underline{L}_{\lambda_1}) \otimes HF(\underline{K}_{\mu}, \underline{K}_{\mu}) \simeq HF(\iota_j(\underline{L}_{\lambda_0}, \underline{K}_{\mu}), \iota_j(\underline{L}_{\lambda_1}, \underline{K}_{\mu})) \tag{12.24}$$

is the obvious map where e_K is the cohomological unit of $HF(\underline{K}_{\mu}, \underline{K}_{\mu})$.

Remark 12.5. Another way to interpret the grading shift is as follows. We have seen from Lemma 6.5 (or Proposition 7.12) that the map from the symplectic extended arc algebra to the algebraic extended arc algebra is given by putting a $\vee$ to the points $\pi_E(\underline{x})$, where $\underline{x}$ is a geometric basis element. Each generator $\underline{x}$ of $CF(\underline{L}_{\mathrm{hs}}, \underline{L}_{\widetilde{\lambda \sqcup \mu}})$ (see the first picture of Figure 12.3) must have j points lying above $\{1, \dots, n\}$ and $n-j$ points lying above $\{n+1, \dots, 2n\}$. That corresponds to making j counterclockwise caps and $n-j$ clockwise caps for the oriented cap diagram associated to $\underline{L}_{\mathrm{hs}}$. This is where the extra $n-j$ in the grading comes from.

By a completely analogous reasoning, we get

Corollary 12.6. *We have canonical vector space isomorphisms*

$$HF^{*+n-j}(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_{\lambda}, \underline{K}_{\mu})) = HF^*(\underline{L}_{PD(\lambda)}, \underline{K}_{\mu}), \tag{12.25}$$

and under these isomorphisms, the multiplication maps

$$\mu^2(e_L \otimes -, -) : HF(\underline{K}_{\mu_0}, \underline{K}_{\mu_1}) \times HF(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_{\lambda}, \underline{K}_{\mu_0})) \to HF(\underline{L}_{\mathrm{hs}}, \iota_j(\underline{L}_{\lambda}, \underline{K}_{\mu_1})) \tag{12.26}$$

and

$$\mu^2(-,-) : HF(\underline{K}_{\mu_0}, \underline{K}_{\mu_1}) \times HF(\underline{L}_{PD(\lambda)}, \underline{K}_{\mu_0}) \to HF(\underline{L}_{PD(\lambda)}, \underline{K}_{\mu_1}) \tag{12.27}$$

are identified. Here, for (12.26)*,*

$$e_L \otimes - : HF(\underline{K}_{\mu_0}, \underline{K}_{\mu_1}) \to HF(\underline{L}_{\lambda}, \underline{L}_{\lambda}) \otimes HF(\underline{K}_{\mu_0}, \underline{K}_{\mu_1}) \simeq HF(\iota_j(\underline{L}_{\lambda}, \underline{K}_{\mu_0}), \iota_j(\underline{L}_{\lambda}, \underline{K}_{\mu_1})) \tag{12.28}$$

is the obvious map where e_L is the cohomological unit of $HF(\underline{L}_{\lambda}, \underline{L}_{\lambda})$.

Note that, after applying PD (which is grading preserving), (12.27) would become

$$\mu^2(-,-) : HF(\underline{L}_{PD(\mu_0)}, \underline{L}_{\lambda}) \times HF(\underline{L}_{PD(\mu_1)}, \underline{L}_{PD(\mu_0)}) \to HF(\underline{L}_{PD(\mu_1)}, \underline{L}_{\lambda}) \tag{12.29}$$

On account of Corollaries 12.4 and 12.6, and the fact that every basis element of $HF(\iota_j(\underline{L}_{\lambda_0}, \underline{K}_{\mu_0}), \iota_j(\underline{L}_{\lambda_1}, \underline{K}_{\mu_1}))$ can be written as the product of

$$x \otimes e_K \in HF(\iota_j(\underline{L}_{\lambda_0}, \underline{K}_{\mu_0}), \iota_j(\underline{L}_{\lambda_1}, \underline{K}_{\mu_0})) \quad \text{and}$$
$$e_L \otimes y \in HF(\iota_j(\underline{L}_{\lambda_1}, \underline{K}_{\mu_0}), \iota_j(\underline{L}_{\lambda_1}, \underline{K}_{\mu_1}))$$

for some x, y, we now know that the multiplication map

$$HF(\iota_j(\underline{L}_{\lambda_0}, \underline{K}_{\mu_0}), \iota_j(\underline{L}_{\lambda_1}, \underline{K}_{\mu_1})) \times HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_{\lambda_0}, \underline{K}_{\mu_0})) \to HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_{\lambda_1}, \underline{K}_{\mu_1})) \quad (12.30)$$

can be identified with the left and right multiplication maps

$$\mu^2(-, \mu^2(-,-)) : HF(\underline{L}_{\lambda_0}, \underline{L}_{\lambda_1}) \times HF(\underline{L}_{PD(\mu_0)}, \underline{L}_{\lambda_0}) \times HF(\underline{L}_{PD(\mu_1)}, \underline{L}_{PD(\mu_0)}) \to HF(\underline{L}_{PD(\mu_1)}, \underline{L}_{\lambda_1}). \quad (12.31)$$

In other words, we have now shown that $(\iota_j)^*(\underline{L}^l_{\text{hs}})$ coincides with the diagonal bimodule (shifted by $n-j$) on the cohomological level. Having this, we can now finish the proof of Proposition 12.1

Completion of the proof of Proposition 12.1. In the previous paragraph, we have shown that $(\iota_j)^*(\underline{L}^l_{\text{hs}})$ coincides with the diagonal $K^{\text{symp}}_{j,n}$-$K^{\text{symp}}_{j,n}$-bimodule on the cohomological level. It remains to show that $(\iota_j)^*(\underline{L}^l_{\text{hs}})$ is formal as a bimodule over $K^{\text{symp}}_{j,n}$, or formal as a left module over A_j.

As in the proof of formality in Section 9, it suffices to show that we can fix a consistent choice of b-equivariant structures for the collection of Lagrangians in A_j together with the single additional Lagrangian $\underline{L}_{\text{hs}}$. As explained in Lemma 11.5, the existence of a consistent choice of b-equivariant structures for the collection of Lagrangians in A_j follows from Section 9 so we now only need to show the consistency for the pairs $(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_\mu))$.

The strategy is as before. First observe that, by the identification between (12.30) and (12.31) and Lemma 9.1, we know that $HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_\mu))$ is a cyclic module over $HF(\iota_j(\underline{L}_\lambda, \underline{K}_\mu), \iota_j(\underline{L}_\lambda, \underline{K}_\mu))$.

Then, we can run the proof of Proposition 9.4. We choose a b-equivariant structure on $\underline{L}_{\text{hs}}$ such that the rank 1 vector space $HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_{\lambda_{\max}}, \underline{K}_{\mu_{\max}}))$ is pure, where $\lambda_{\max}$ and $\mu_{\max}$ are the unique maximal weights in $\Lambda_{j,n}$ and $\Lambda_{n-j,n}$, respectively. Fixing $\underline{K}_{\mu_{\max}}$, we want to inductively argue that $HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_{\mu_{\max}}))$ is pure for all λ. Let λ be such that the purity of $HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_{\lambda'}, \underline{K}_{\mu_{\max}}))$ has been verified for all $\lambda' >^T \lambda$, where $>^T$ is the total ordering of weights in Proposition 9.4; then we need to verify purity for $HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_{\mu_{\max}}))$. By Lemma 9.2, and the identification between (12.30) and (12.31), there exists a weight $\lambda' >^T \lambda$ such that we have a non-trivial product (this is the analogue of (9.7))

$$HF(\iota_j(\underline{L}_{\lambda'}, \underline{K}_{\mu_{\max}}), \iota_j(\underline{L}_\lambda, \underline{K}_{\mu_{\max}})) \times HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_{\lambda'}, \underline{K}_{\mu_{\max}})) \to HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_{\mu_{\max}})).$$

Together with cyclicity, it implies that $HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_{\mu_{\max}}))$ is pure. Thus $HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_{\mu_{\max}}))$ is pure for all λ.

Similarly, we can fix $\underline{L}_\lambda$ and apply the inductive argument to $HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_\mu))$ with varying μ. Altogether, we conclude that $HF(\underline{L}_{\text{hs}}, \iota_j(\underline{L}_\lambda, \underline{K}_\mu))$ is pure for all λ, μ and hence $(\iota_j)^*(\underline{L}^l_{\text{hs}})$ is formal as a left module over A_j. ∎

Lemma 12.7. *Up to grading shift, $(\iota_j)^*(\phi_\beta(\underline{L}_{\text{hs}})^l)$ is quasi-isomorphic to $P^{(j)}_\beta$.*

Proof. We can apply the same method as in the proof of Proposition 12.1 to $(\phi_\beta)^*(\iota_j)^*(\phi_\beta(\underline{L}_{\mathrm{hs}}))^l$, where $(\phi_\beta)^* : \text{perf-}A_j \to \text{perf-}A_j$ is the pull-back functor under ϕ_β. This is because, geometrically, $CF(\phi_\beta(\underline{L}_{\mathrm{hs}}), \phi_\beta(\underline{L}) \sqcup \underline{K})$ can be canonically identified with $CF(\underline{L}_{\mathrm{hs}}, \underline{L} \sqcup \underline{K})$, as can all holomorphic polygons contributing to the A_∞ structure amongst tuples of such Floer groups. It may be worth emphasising that at this point in the argument we use the fact that, working with pullback data for Hamiltonians, almost complex structures, etc., the symplectomorphism ϕ_β gives a canonical identification of moduli spaces of polygons; the corresponding isomorphism, from the algebraic viewpoint, is much less transparent.

In consequence, $(\phi_\beta)^*(\iota_j)^*(\phi_\beta(\underline{L}_{\mathrm{hs}}))^l$ is also quasi-isomorphic to the diagonal bimodule, and hence $(\iota_j)^*(\phi_\beta(\underline{L}_{\mathrm{hs}}))^l$ is quasi-isomorphic to $(\phi_\beta^{-1})^*\Delta = P_\beta^{(j)}$. ∎

We have the analogous results for the embedding $\iota_j^! : A_j^! \to \mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ and the induced pull-back

$$(\iota_j^!)^* : \text{perf-}\mathcal{FS}^{\mathrm{cyl},n}(\pi_E) \to \text{perf-}A_j^! \simeq [K_{j,n}, K_{j,n}]. \tag{12.32}$$

Proposition 12.8. *Up to grading shift, $(\iota_j^!)^*(\underline{L}_{\mathrm{hs}}^r)$ is quasi-isomorphic to the diagonal bimodule.*

Sketch of proof. A completely parallel argument applies (see Figures 12.7 and 12.8 for illustration). ∎

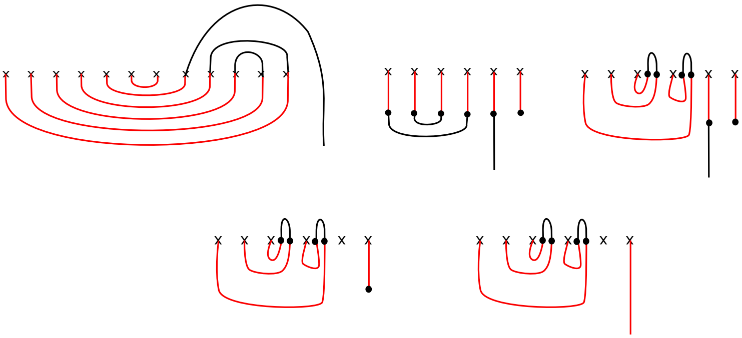


Fig. 12.7. The counterpart of Figure 12.1.

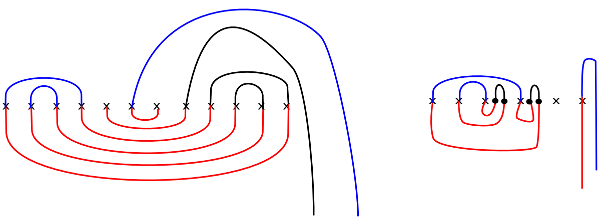


Fig. 12.8. The counterpart of Figure 12.2.

If one keeps track of the grading and follows our grading convention, then $(\iota_j^!)^*(\underline{L}_{\text{hs}}^r)$ is quasi-isomorphic to the diagonal bimodule shifted by $n-j$ (cf. Remark 12.5).

Lemma 12.9. *Up to grading shift,* $(\iota_j)^*(\phi_\beta(\underline{L}_{\text{hs}})^r)$ *is quasi-isomorphic to* $P_\beta^{(j)}$.

Proof. The proof repeats that of Lemma 12.7, but replacing Proposition 12.1 by Proposition 12.8. ■

12.2. Spectral sequence to Khovanov homology

Having Proposition 12.1 and Lemma 12.9, the last piece of information we need to understand the E_1-page of the spectral sequence in Proposition 11.17 is a description of a quasi-inverse

$$\psi_j : A_j\text{-perf} \to A_j^! \tag{12.33}$$

of the functor $(\iota_j)^*(-)^l : A_j^! \to A_j$-perf. We are going to describe ψ_j geometrically.

For $\lambda \in \Lambda_{j,n}$ and $\mu \in \Lambda_{n-j,m-n}$, we consider the upper Lagrangian tuples $\underline{L}_{\overline{\lambda}} \in \mathcal{FS}^{\text{cyl},j}_{W_1}(\pi_E)$ and $\underline{K}_{\overline{\mu}} \in \mathcal{FS}^{\text{cyl},n-j}_{W_2}(\pi_E)$ (see Figure 12.9). The disjoint union $\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}}$ gives an object in $\mathcal{FS}^{\text{cyl},n}(\pi_E)$. On the other hand, we have $\underline{L}^!_{\overline{\lambda}} \sqcup \underline{K}^!_{\overline{\mu}} \in A_j^!$, which is in particular an object in $\mathcal{FS}^{\text{cyl},n}(\pi_E)$.

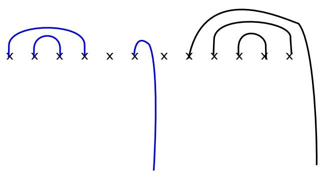


Fig. 12.9. Lagrangian tuples $\underline{L}_{\overline{\lambda}}$ (blue) and $\underline{K}_{\overline{\mu}}$ (black).

Lemma 12.10. *For* $\lambda \in \Lambda_{j,n}$ *and* $\mu \in \Lambda_{n-j,m-n}$*, we have a quasi-isomorphism*

$$(\iota_j)^*(\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}})^l \simeq (\iota_j)^*(\underline{L}^!_{\overline{\lambda}} \sqcup \underline{K}^!_{\overline{\mu}})^l \tag{12.34}$$

of objects in A_j-perf.

Proof. For any object $\underline{L}_{\lambda'} \sqcup \underline{K}_{\mu'} \in A_j$, there is an obvious isomorphism (see Figure 12.10)

$$\begin{aligned} CF(\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}}, \underline{L}_{\lambda'} \sqcup \underline{K}_{\mu'}) &= CF(\underline{L}_{\overline{\lambda}}, \underline{L}_{\lambda'}) \otimes CF(\underline{K}_{\overline{\mu}}, \underline{K}_{\mu'}) \\ &= CF(\underline{L}^!_{\overline{\lambda}}, \underline{L}_{\lambda'}) \otimes CF(\underline{K}^!_{\overline{\mu}}, \underline{K}_{\mu'}) \\ &= CF(\underline{L}^!_{\overline{\lambda}} \sqcup \underline{K}^!_{\overline{\mu}}, \underline{L}_{\lambda'} \sqcup \underline{K}_{\mu'}). \end{aligned}$$

Moreover, when $X_0, \dots, X_d \in A_j$, for a careful choice of Floer data, one can actually make the A_∞ structural maps

$$CF(X_{d-1}, X_d) \times \dots \times CF(X_0, X_1) \times CF(\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}}, X_0) \to CF(\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}}, X_d),$$

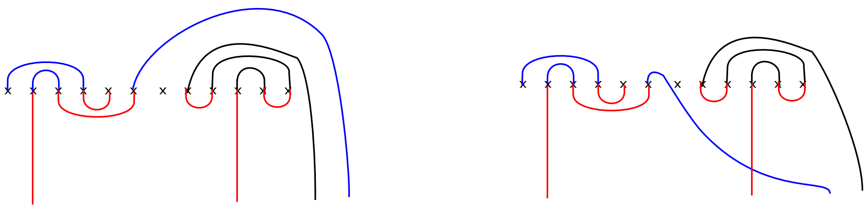


Fig. 12.10. Floer cochain $CF(\underline{L}^!_{\overline{\lambda}} \sqcup \underline{K}^!_{\overline{\mu}}, \underline{L}_{\lambda'} \sqcup \underline{K}_{\mu'})$ (left) and $CF(\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}}, \underline{L}_{\lambda'} \sqcup \underline{K}_{\mu'})$ (right). The intersection between the positive wrapping of $\underline{L}_{\overline{\lambda}}$ and $\underline{K}_{\mu'}$ cannot contribute a generator to $CF(\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}}, \underline{L}_{\lambda'} \sqcup \underline{K}_{\mu'})$ so the Floer cochain splits.

$$CF(X_{d-1}, X_d) \times \cdots \times CF(X_0, X_1) \times CF(\underline{L}^!_{\overline{\lambda}} \sqcup \underline{K}^!_{\overline{\mu}}, X_0) \to CF(\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}}, X_d)$$

completely coincide. This implies the result. ∎

Remark 12.11. By taking morphisms with an appropriate object in A_i for $i \neq j$, one can see that $(\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}})^l$ is not quasi-isomorphic to $(\underline{L}^!_{\overline{\lambda}} \sqcup \underline{K}^!_{\overline{\mu}})^l$.

By Lemma 12.10, we know that the quasi-inverse ψ_j is given, on the object level, by

$$\psi_j : (\iota_j)^*(\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}})^l \mapsto \underline{L}^!_{\overline{\lambda}} \sqcup \underline{K}^!_{\overline{\mu}}. \tag{12.35}$$

A further inspection of the proof of Lemma 12.10 shows that the quasi-isomorphism (12.34) is functorial when we vary $\lambda \in \Lambda_{j,n}$ and $\mu \in \Lambda_{n-j,m-n}$. Together with the fact that $\underline{L}_{\overline{\lambda}} \sqcup \underline{K}_{\overline{\mu}}$ is quasi-isomorphic to $\underline{L}_\lambda \sqcup \underline{K}_\mu$ in $\mathcal{FS}^{\mathrm{cyl},n}(\pi_E)$ (so the former can be regarded as an object in A_j), we have a commutative diagram

$$\begin{array}{ccccc} A_j & \xrightarrow{\simeq} & K_{j,n} \times K_{n-j,n} & \xrightarrow{\simeq} & A^!_j \\ \downarrow (-)^l & & & \swarrow (\iota_j)^*(-)^l & \\ A_j\text{-perf} & & & & \end{array}$$

where the horizontal isomorphisms are the ones in (11.5), (11.9), which in particular send $\underline{L}_\lambda \sqcup \underline{K}_\mu \in A_j$ to $\underline{L}^!_\lambda \sqcup \underline{K}^!_\mu \in A^!_j$. This implies that ψ_j sits inside the commutative diagram

$$\begin{array}{ccccc} A_j & \xrightarrow{\simeq} & K_{j,n} \times K_{n-j,n} & \xrightarrow{\simeq} & A^!_j \\ \downarrow (-)^l & & & \nearrow \psi_j & \\ A_j\text{-perf} & & & & \end{array}$$

Proof of Theorem 1.9. By Proposition 11.17, we have a spectral sequence to $\mathrm{Kh}(\kappa(\beta))$ from

$$\bigoplus_{j=0}^{n} H\big((\iota^!_j)^*(\phi_\beta(\underline{L}_{\mathrm{hs}})^r) \otimes_{A^!_j} \psi_j \circ (\iota_j)^*(\underline{L}^l_{\mathrm{hs}})\big). \tag{12.36}$$

By Proposition 12.1, Lemma 12.9 and the commutative diagram above, we have, up to grading shift,

$$H\big((\iota_j^!)^*(\phi_\beta(\underline{L}_{\mathrm{hs}})^r)\otimes_{A_j^!}\psi_j\circ(\iota_j)^*(\underline{L}_{\mathrm{hs}}^l)\big)=H(P_\beta^{(j)}\otimes_{K_{j,n}-K_{j,n}}\Delta_{K_{j,n}})$$
$$=HH_*(K_{j,n},P_\beta^{(j)}),$$

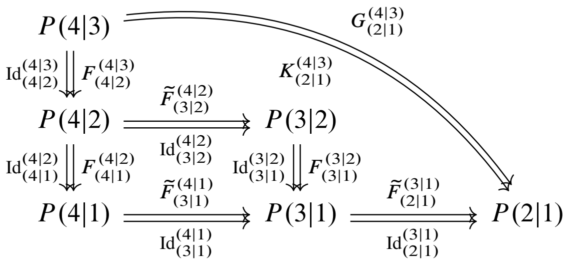
which is what we want. ∎

Appendix A. Non-formality of standard modules/thimbles

Here we explain that the endomorphism algebra of the Lefschetz thimbles in $\mathcal{FS}(\pi_{n,m})$ may be non-formal when $n>1$. The non-formality follows rather directly from the minimal model computed in Klamt's thesis [27]. Since this is not required for our main results, our treatment is somewhat brief.

When $(n,m)=(1,m)$ the algebra is known to be formal; by duality, the same holds when $n=m-1$. On the other hand, if $n>1$ and $(n,m)\neq(m-1,m)$, the algebra associated to $(n,m)=(2,4)$ occurs as an A_∞-subalgebra of the algebra associated to (n,m). It therefore suffices to consider the case $(n,m)=(2,4)$. In that case, there are six Lefschetz thimbles (up to grading shift). We denote them by $P(4|3)$, $P(4|2)$, $P(4|1)$, $P(3|2)$, $P(3|1)$ and $P(2|1)$, where $P(n|m)$ corresponds to $\underline{T}^{\{1,2,3,4\}\backslash\{n,m\}}$.

The cohomological algebra is given by the following quiver with relations, by Proposition 10.1 and [27, Section 5.2.8, 5.2.9]:



The subscript and superscript of the name of the arrow indicate the target and the source of the arrow, respectively. The definition of $\mathrm{Id}^{(k|l)}_{(a|b)}$, $F^{(k|l)}_{(a|b)}$, $\widetilde{F}^{(k|l)}_{(a|b)}$, $G^{(k|l)}_{(a|b)}$, $K^{(k|l)}_{(a|b)}$ for various k,l,a,b can be found in [27, Theorems 5.16–5.19]. The complete list of relations of the product structure can be found in [27, Table 5.25]. For example, $\widetilde{F}^{(4|1)}_{(2|1)}=\mathrm{Id}^{(3|1)}_{(2|1)}\widetilde{F}^{(4|1)}_{(3|1)}$ and $F^{(4|2)}_{(4|1)}F^{(4|3)}_{(4|2)}=0$. We remark that $J^{(k|l)}_{(a|b)}$ in the table is defined in Theorem 5.26 and $J^{(4|2)}_{(3|1)}$ equals to $F^{(3|2)}_{(3|1)}\widetilde{F}^{(4|2)}_{(3|2)}$ and $\widetilde{F}^{(4|1)}_{(3|1)}F^{(4|2)}_{(4|1)}$ up to sign. Moreover, the dimension of $\mathrm{Ext}(P(k|l),P(a|b))$ can be found in [27, Section 5.2.6].

For the A_∞ structure, Klamt computed a minimal model with $\mu^3\neq 0$ but $\mu^d=0$ for $d>3$. The list of non-zero μ^3 is given in [27, Table 8.4].

Lemma A.1. *When* $(n,m) = (2,4)$*, the endomorphism algebra of Lefschetz thimbles is not formal.*

Proof. Let $\mathcal{B}$ be the A_∞ endomorphism algebra of Lefschetz thimbles and $B := H(\mathcal{B})$ be the underlying cohomological algebra. If $\mathcal{B}$ were formal, then the μ^3 of the minimal model Klamt found would define the zero class in $HH^3(B, B[-1])$. This means that $\mu^3 = d_{CC}\tau$ for some $\tau \in CC^2(B, B[-1])$, where d_{CC} is the Hochschild differential.

From [27, Table 8.4], we have

$$\mu^3(\widetilde{F}^{(4|1)}_{(2|1)}, F^{(4|2)}_{(4|1)}, F^{(4|3)}_{(4|2)}) = \pm K^{(4|3)}_{(2|1)}. \tag{A.1}$$

On the other hand, up to sign,

$$\begin{aligned} d_{CC}\tau(\widetilde{F}^{(4|1)}_{(2|1)}, F^{(4|2)}_{(4|1)}, F^{(4|3)}_{(4|2)}) &= \widetilde{F}^{(4|1)}_{(2|1)}\tau(F^{(4|2)}_{(4|1)}, F^{(4|3)}_{(4|2)}) + \tau(\widetilde{F}^{(4|1)}_{(2|1)}, F^{(4|2)}_{(4|1)})F^{(4|3)}_{(4|2)} \\ &\quad + \tau(\widetilde{F}^{(4|1)}_{(2|1)}F^{(4|2)}_{(4|1)}, F^{(4|3)}_{(4|2)}) + \tau(\widetilde{F}^{(4|1)}_{(2|1)}, F^{(4|2)}_{(4|1)}F^{(4|3)}_{(4|2)}). \end{aligned} \tag{A.2}$$

By [27, Table 5.25], we have $\widetilde{F}^{(4|1)}_{(2|1)}F^{(4|2)}_{(4|1)} = F^{(4|2)}_{(4|1)}F^{(4|3)}_{(4|2)} = 0$ so the last two terms of (A.2) vanish.

One can check from [27, Section 5.2.6] that the dimension of $\mathrm{Ext}(P(2|1), P(4|3))$ is 4 and it is generated by $\mathrm{Id}^{(4|3)}_{(2|1)}$, $F^{(4|3)}_{(2|1)} = \widetilde{F}^{(4|3)}_{(2|1)}$, $G^{(4|3)}_{(2|1)}$ and $K^{(4|3)}_{(2|1)}$. On the other hand, one can check from [27, Table 5.25] that for any $a \in \mathrm{Ext}(P(4|1), P(4|3))$ and $b \in \mathrm{Ext}(P(2|1), P(4|2))$, the sum $\widetilde{F}^{(4|1)}_{(2|1)}a + bF^{(4|3)}_{(4|2)}$ lie in the subspace spanned by $F^{(4|3)}_{(2|1)}$. Therefore, $K^{(4|3)}_{(2|1)}$ cannot be written as the sum of the first two terms of (A.2) so it is not equal to $d_{CC}\tau(\widetilde{F}^{(4|1)}_{(2|1)}, F^{(4|2)}_{(4|1)}, F^{(4|3)}_{(4|2)})$ for any τ in $CC^2(B, B[-1])$, a contradiction, and the result follows. ∎

Remark A.2. Lemma A.1 shows that the strategy from Section 9 cannot be used to prove that the collection of thimbles is formal. In other words, the cohomological algebra of the thimbles is not compatible with the existence of such a consistent choice of equivariant structure. It seems hard to pinpoint why one would expect the collection of Lagrangians $\{\underline{L}_\lambda\}_{\lambda\in\Lambda_{n,m}}$ to be better in this respect.

Acknowledgements. The authors are grateful to Mohammed Abouzaid, Robert Lipshitz, Jacob Rasmussen, Catharina Stroppel and Filip Živanović for many helpful conversations. We thank the anonymous referee for their numerous helpful comments and corrections. We also thank Zhengfang Wang for pointing out a typo in the previous version.

Funding. Both authors are supported by EPSRC Fellowship EP/N01815X/1.

References

[1] Abouzaid, M., Smith, I.: The symplectic arc algebra is formal. Duke Math. J. **165**, 985–1060 (2016) Zbl 1346.53073 MR 3486414

[2] Abouzaid, M., Smith, I.: Khovanov homology from Floer cohomology. J. Amer. Math. Soc. **32**, 1–79 (2019) Zbl 1401.57005 MR 3867999

[3] Amorim, L.: Tensor product of filtered A_∞-algebras. J. Pure Appl. Algebra **220**, 3984–4016 (2016) Zbl 1344.18012 MR 3517566

[4] Asaeda, M. M., Przytycki, J. H., Sikora, A. S.: Categorification of the Kauffman bracket skein module of I-bundles over surfaces. Algebr. Geom. Topol. **4**, 1177–1210 (2004) Zbl 1070.57008 MR 2113902

[5] Auroux, D.: Fukaya categories of symmetric products and bordered Heegaard–Floer homology. J. Gökova Geom. Topol. **4**, 1–54 (2010) Zbl 1285.53077 MR 2755992

[6] Auroux, D., Grigsby, J. E., Wehrli, S. M.: Sutured Khovanov homology, Hochschild homology, and the Ozsváth–Szabó spectral sequence. Trans. Amer. Math. Soc. **367**, 7103–7131 (2015) Zbl 1365.57011 MR 3378825

[7] Beilinson, A., Bezrukavnikov, R., Mirković, I.: Tilting exercises. Moscow Math. J. **4**, 547–557, 782 (2004) Zbl 1075.14015 MR 2119139

[8] Beliakova, A., Putyra, K. K., Wehrli, S. M.: Quantum link homology via trace functor I. Invent. Math. **215**, 383–492 (2019) Zbl 1411.57021 MR 3910068

[9] Bodzenta, A., Külshammer, J.: Ringel duality as an instance of Koszul duality. J. Algebra **506**, 129–187 (2018) Zbl 1420.16005 MR 3800074

[10] Braden, T.: Perverse sheaves on Grassmannians. Canad. J. Math. **54**, 493–532 (2002) Zbl 1009.32019 MR 1900761

[11] Braden, T., Licata, A., Proudfoot, N., Webster, B.: Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality (with an appendix by I. Losev). Astérisque **384**, 75–179 (2016) Zbl 1360.53001 MR 3594665

[12] Brundan, J., Stroppel, C.: Highest weight categories arising from Khovanov's diagram algebra II: Koszulity. Transform. Groups **15**, 1–45 (2010) Zbl 1205.17010 MR 2600694

[13] Brundan, J., Stroppel, C.: Highest weight categories arising from Khovanov's diagram algebra I: cellularity. Moscow Math. J. **11**, 685–722, 821–822 (2011) Zbl 1275.17012 MR 2918294

[14] Chriss, N., Ginzburg, V.: Representation Theory and Complex Geometry. Birkhäuser Boston, Boston, MA (1997) Zbl 0879.22001 MR 1433132

[15] Cieliebak, K., Mohnke, K.: Symplectic hypersurfaces and transversality in Gromov–Witten theory. J. Symplectic Geom. **5**, 281–356 (2007) Zbl 1149.53052 MR 2399678

[16] Costello, K.: Higher genus Gromov–Witten invariants as genus zero invariants of symmetric products. Ann. of Math. (2) **164**, 561–601 (2006) Zbl 1209.14046 MR 2247968

[17] Donaldson, S., Smith, I.: Lefschetz pencils and the canonical class for symplectic four-manifolds. Topology **42**, 743–785 (2003) Zbl 1012.57040 MR 1958528

[18] Etgü, T., Lekili, Y.: Koszul duality patterns in Floer theory. Geom. Topol. **21**, 3313–3389 (2017) Zbl 1378.57041 MR 3692968

[19] Evans, J. D.: Symplectic mapping class groups of some Stein and rational surfaces. J. Symplectic Geom. **9**, 45–82 (2011) Zbl 1242.58004 MR 2787361

[20] Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K.: Lagrangian Intersection Floer Theory: Anomaly and Obstruction. Part II. AMS/IP Stud. Adv. Math. 46, Amer. Math. Soc., Providence, RI, and Int. Press, Somerville, MA (2009) Zbl 1181.53003 MR 2548482

[21] Grigsby, J. E., Licata, A. M., Wehrli, S. M.: Annular Khovanov homology and knotted Schur–Weyl representations. Compos. Math. **154**, 459–502 (2018) Zbl 1422.57036 MR 3731256

[22] Hanlon, A.: Monodromy of monomially admissible Fukaya–Seidel categories mirror to toric varieties. Adv. Math. **350**, 662–746 (2019) Zbl 1423.53109 MR 3948684

[23] Huybrechts, D.: Fourier–Mukai Transforms in Algebraic Geometry. Oxford Math. Monogr., Clarendon Press, Oxford Univ. Press, Oxford (2006) Zbl 1095.14002 MR 2244106

[24] Khovanov, M.: A categorification of the Jones polynomial. Duke Math. J. **101**, 359–426 (2000) Zbl 0960.57005 MR 1740682

[25] Khovanov, M.: A functor-valued invariant of tangles. Algebr. Geom. Topol. **2**, 665–741 (2002) Zbl 1002.57006 MR 1928174

[26] Khovanov, M., Seidel, P.: Quivers, Floer cohomology, and braid group actions. J. Amer. Math. Soc. **15**, 203–271 (2002) Zbl 1035.53122 MR 1862802

[27] Klamt, A.: A_∞-structures on the algebra of extensions of Verma modules in the parabolic category $\mathcal{O}$. arXiv:1104.0102 (2011)

[28] Kuznetsov, A.: Hochschild homology and semiorthogonal decompositions. arXiv:0904.4330 (2009)

[29] Lipshitz, R.: A cylindrical reformulation of Heegaard Floer homology. Geom. Topol. **10**, 955–1096 (2006) Zbl 1130.57035 MR 2240908

[30] Lipshitz, R., Ozsváth, P. S., Thurston, D. P.: Bimodules in bordered Heegaard Floer homology. Geom. Topol. **19**, 525–724 (2015) Zbl 1315.57036 MR 3336273

[31] Manolescu, C.: Nilpotent slices, Hilbert schemes, and the Jones polynomial. Duke Math. J. **132**, 311–369 (2006) Zbl 1110.57010 MR 2219260

[32] Manolescu, C.: Link homology theories from symplectic geometry. Adv. Math. **211**, 363–416 (2007) Zbl 1117.57012 MR 2313538

[33] Maydanskiy, M., Seidel, P.: Lefschetz fibrations and exotic symplectic structures on cotangent bundles of spheres. J. Topol. **3**, 157–180 (2010) Zbl 1235.53088 MR 2608480

[34] McDuff, D., Salamon, D.: J-holomorphic Curves and Symplectic Topology. Amer. Math. Soc. Colloq. Publ. 52, Amer. Math. Soc., Providence, RI (2004) Zbl 1064.53051 MR 2045629

[35] Ozsváth, P., Szabó, Z.: Holomorphic disks and topological invariants for closed three-manifolds. Ann. of Math. (2) **159**, 1027–1158 (2004) Zbl 1073.57009 MR 2113019

[36] Perutz, T.: Lagrangian matching invariants for fibred four-manifolds. I. Geom. Topol. **11**, 759–828 (2007) Zbl 1143.53079 MR 2302502

[37] Perutz, T.: Lagrangian matching invariants for fibred four-manifolds. II. Geom. Topol. **12**, 1461–1542 (2008) Zbl 1144.53104 MR 2421133

[38] Poźniak, M.: Floer homology, Novikov rings and clean intersections. In: Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. (2) 196, Amer. Math. Soc., Providence, RI, 119–181 (1999) Zbl 0948.57025 MR 1736217

[39] Roberts, L. P.: On knot Floer homology in double branched covers. Geom. Topol. **17**, 413–467 (2013) Zbl 1415.57009 MR 3035332

[40] Seidel, P.: Fukaya Categories and Picard–Lefschetz Theory. Zurich Lectures in Adv. Math., Eur. Math. Soc., Zürich (2008) Zbl 1159.53001 MR 2441780

[41] Seidel, P.: Symplectic homology as Hochschild homology. In: Algebraic Geometry—Seattle 2005, Part 1, Proc. Sympos. Pure Math. 80, Amer. Math. Soc., Providence, RI, 415–434 (2009) Zbl 1179.53085 MR 2483942

[42] Seidel, P.: Fukaya A_∞-structures associated to Lefschetz fibrations. I. J. Symplectic Geom. **10**, 325–388 (2012) Zbl 1267.53094 MR 2983434

[43] Seidel, P.: Fukaya A_∞-structures associated to Lefschetz fibrations. II 1/2. Adv. Theor. Math. Phys. **20**, 883–944 (2016) Zbl 1359.53062 MR 3573028

[44] Seidel, P.: Fukaya A_∞-structures associated to Lefschetz fibrations. II. In: Algebra, Geometry, and Physics in the 21st Century, Progr. Math. 324, Birkhäuser/Springer, Cham, 295–364 (2017) Zbl 1390.53098 MR 3727564

[45] Seidel, P.: Fukaya A_∞-structures associated to Lefschetz fibrations. IV. arXiv:1709.06018 (2017)

[46] Seidel, P.: Fukaya A_∞-structures associated to Lefschetz fibrations. IV 1/2. arXiv:1802.09461 (2018)

[47] Seidel, P., Smith, I.: A link invariant from the symplectic geometry of nilpotent slices. Duke Math. J. **134**, 453–514 (2006) Zbl 1108.57011 MR 2254624

[48] Sheridan, N.: On the homological mirror symmetry conjecture for pairs of pants. J. Differential Geom. **89**, 271–367 (2011) Zbl 1255.53065 MR 2863919

[49] Siegel, K.: Squared Dehn twists and deformed symplectic invariants. arXiv:1609.08545 (2016)

[50] Slodowy, P.: Simple Singularities and Simple Algebraic Groups. Lecture Notes in Math. 815, Springer, Berlin (1980) Zbl 0441.14002 MR 584445

[51] Stroppel, C.: Parabolic category $\mathcal{O}$, perverse sheaves on Grassmannians, Springer fibres and Khovanov homology. Compos. Math. **145**, 954–992 (2009) Zbl 1187.17004 MR 2521250

[52] Voisin, C.: On the punctual Hilbert scheme of a symplectic fourfold. In: Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), Contemp. Math. 312, Amer. Math. Soc., Providence, RI, 265–289 (2002) Zbl 1103.14006 MR 1941585

[53] Vybornov, M.: Perverse sheaves, Koszul IC-modules, and the quiver for the category $\mathcal{O}$. Invent. Math. **167**, 19–46 (2007) Zbl 1114.14012 MR 2264803