J. Eur. Math. Soc. 24, 3439–3503 (2022) DOI 10.4171/JEMS/1165

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p-adic L-functions of Hilbert cusp forms and the trivial zero conjecture

Received July 5, 2019

Abstract. We prove a strong form of the trivial zero conjecture at the central point for the p-adic L-function of a non-critically refined self-dual cohomological cuspidal automorphic representation of GL_2 over a totally real field, which is Iwahori spherical at places above p.

In the case of a simple zero we adapt the approach of Greenberg and Stevens, based on the functional equation for the p-adic L-function of a nearly finite slope family and on improved p-adic L-functions that we construct using automorphic symbols and overconvergent cohomology.

For higher order zeros we develop a conceptually new approach studying the variation of the root number in partial families and establishing the vanishing of many Taylor coefficients of the p-adic L-function of the family.

Keywords. Hilbert cusp forms, p-adic L-functions, trivial zero conjecture, overconvergent cohomology, automorphic symbols, partial eigenvarieties

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Mathematics Subject Classification (2020): Primary 11F67; Secondary 11F41, 11G40

Introduction

The complex analytic L-function of an algebraic cuspidal automorphic representation π on a reductive group over a number field F lies at the heart of the Langlands program, and the relationship between its analytic properties, namely the order of vanishing at critical points, and the arithmetic of the conjecturally attached p-adic representation V_π of the absolute Galois group G_F is the content of the famous Bloch–Kato conjectures. Iwasawa theory, in turn, seeks to relate the arithmetic of the restriction of V_π to the p-adic cyclotomic extension of F, and the behavior of the p-adic analytic L-function $L_p(\pi, s)$ of π. The existence of p-adic L-functions for automorphic representations and families thereof is a challenging problem in itself, but even when they have been constructed, properties such as the location of their zeros and orders of vanishing have remained poorly understood.

To ensure good analytic properties in the cyclotomic variable s, $L_p(\pi, s)$ contains extra interpolation factors which can possibly vanish at a critical integer. Such zeros, called *trivial*, were first considered for an elliptic curve E over $\mathbb{Q}$ in the seminal work of Mazur, Tate and Teitelbaum [32]. If E has split multiplicative reduction at p, the p-adic L-function $L_p(E, s)$ has a trivial zero at $s = 1$ and it was conjectured, and later proven by Greenberg and Stevens [21], that

$$L_p'(E, 1) = \mathscr{L}(E) \cdot \frac{L(E, 1)}{\Omega_E},$$

where Ω_E is the real period of E and $\mathscr{L}(E) = \frac{\log_p q_E}{\mathrm{ord}_p\, q_E}$ is the so-called $\mathscr{L}$-invariant, q_E being the Tate period of E. While trivial zeros of p-adic L-functions and their $\mathscr{L}$-invariants were considered by Mazur, Tate and Teitelbaum in their quest to formulate a p-adic analogue of the Birch and Swinnerton-Dyer conjecture, various recent works on the Bloch–Kato conjecture rely crucially on p-adic L-functions and the Iwasawa main conjecture. In the context of geometric Galois representations the following more general, albeit somewhat vague, trivial zero conjecture springs from various places in the literature and is part of the 'folklore'.

Let V be a p-adic representation of $\mathrm{G}_{\mathbb{Q}}$, critical in the sense of Deligne, such that $V_p = V|_{\mathrm{G}_{\mathbb{Q}_p}}$ is semistable. Let $D \subset \mathscr{D}_{\mathrm{st}}(V_p)$ be a regular submodule in the sense of Perrin-Riou [38]. The works of Coates and Perrin-Riou posit the existence of a p-adic L-function $L_p(V, D, s)$ satisfying an interpolation formula of the form $L_p(V, D, 0) = \Omega_V^{-1} L(V, 0)\mathscr{E}(V_p, D)$, where Ω_V is a Deligne period, $L(V, s)$ is the complex L-function and $\mathscr{E}(V_p, D)$ is a product of linear Euler factors.

Trivial Zero Conjecture. *Letting e denote the number of vanishing Euler factors in $\mathcal{E}(V_p, D)$ and $\mathcal{E}^+(V_p, D)$ the product of the remaining non-vanishing ones, the p-adic L-function $L_p(V, D, s)$ vanishes to order at least e at $s = 0$ and*

$$L_p^{(e)}(V, D, 0) = e! \cdot \mathscr{L}(V, D) \cdot \mathcal{E}^+(V_p, D) \cdot \frac{L(V, 0)}{\Omega_V},$$

where $\mathscr{L}(V, D)$ is an arithmetic $\mathscr{L}$-invariant. More generally, by the Trivial Zero Conjecture for a geometric representation V of G_F and a collection of regular submodules $D_v \subset \mathcal{D}_{\mathrm{st}}(V_{|\mathrm{G}_{F_v}})$ for $v \mid p$ we mean the one for $\mathrm{Ind}_{\mathrm{G}_F}^{\mathrm{G}_\mathbb{Q}} V$ and the regular submodule $\bigoplus_{v|p} \mathrm{Ind}_{\mathrm{G}_{F_v}}^{G_{\mathbb{Q}_p}} D_v$.

Precise formulations of the Trivial Zero Conjecture exist in a number of restricted settings. In the case when V is crystalline at p, Greenberg and Benois made explicit the conjectural interpolation factor $\mathcal{E}(V_p, D)$. Moreover, Greenberg in the case of ordinary representations, and Benois in the semistable case, have defined arithmetic $\mathscr{L}$-invariants Galois cohomologically, when V satisfies a number of technical hypotheses (S, U, T of [20] and (C1)–(C5) of [7]).

This article is devoted to proving the Trivial Zero Conjecture at the central point $s = 0$, with precise interpolation factors and with the Greenberg–Benois arithmetic $\mathscr{L}$-invariant, for the Galois representation $V_\pi(1)$ attached to a unitary self-dual cuspidal automorphic representation π of GL_2 over a totally real field F having an arbitrary cohomological weight.

The construction of a p-adic L-function for π requires the choice of a regular p-refinement $\tilde{\pi}$, i.e., the choice for each v dividing p of a character ν_v of $F_v^\times$ which can be realized uniquely as a subrepresentation of the Weil–Deligne representation attached to π_v via the local Langlands correspondence. Assuming that $\tilde{\pi}$ is *non-critical* (see Definition 2.12), there exists a p-adic L-function $L_p(\tilde{\pi}, s)$.

Let S_p be the set of places of F above p and St_p the subset of places at which π is a twist of the Steinberg representation. The set E of places for which the local interpolation factor of $L_p(\tilde{\pi}, s)$ vanishes at $s = 1$ consists of $v \in \mathrm{St}_p$ such that π_v is the Steinberg representation.

Main Theorem (Theorem 7.1). *Let $\tilde{\pi}$ be a non-critically refined cohomological self-dual cuspidal automorphic representations of GL_2 over F, which is Iwahori spherical at places above p. Then $L_p(\tilde{\pi}, s)$ has order of vanishing at least $e = |E|$ at $s = 1$ and*

$$L_p^{(e)}(\tilde{\pi}, 1) = e! \mathscr{L}(\tilde{\pi}) \cdot \frac{L(\pi, 1/2)}{\Omega_{\tilde{\pi}}} \cdot 2^{|\mathrm{St}_p \backslash E|} \prod_{v \in S_p \backslash \mathrm{St}_p} (1 - \nu_v^{-1}(\varpi_v))^2, \qquad (0.1)$$

where ϖ_v is a uniformizer at v, $\mathscr{L}(\tilde{\pi})$ is the Fontaine–Mazur $\mathscr{L}$-invariant of Definition 5.3, and $\Omega_{\tilde{\pi}}$ is a Betti–Whittaker period defined in §1.7. Moreover, if the Greenberg–Benois arithmetic $\mathscr{L}$-invariant is defined, then the Trivial Zero Conjecture holds for the Galois representation $V_\pi(1)$ with the choice of regular submodule as in §5.3.

The conjectural non-vanishing of the $\mathscr{L}$-invariant is currently only known for elliptic curves over $\mathbb{Q}$ (see [1]). Previously, the Trivial Zero Conjecture at the central point was proved for modular forms over $\mathbb{Q}$ in [21, 48], and for parallel weight 2 ordinary Hilbert cusp forms in [33] using a Rankin–Selberg construction for a single trivial zero and in general in [47], building on ideas of [16, 36]. The non-criticality condition is implied by the assumption of $\tilde{\pi}$ having non-critical slope (see Corollary 2.13) and is expected to be true for most regular $\tilde{\pi}$ (see Bellaïche [4], Pollack–Stevens [39], and Bellaïche–Dimitrov [5] when $F = \mathbb{Q}$). After the completion of this article we were made aware of a preprint of Bergdall and Hansen [8] who construct p-adic L-functions under the weaker condition of $\tilde{\pi}$ being *decent* (see Remark 4.9).

In the case of a single trivial zero our approach to the Main Theorem is inspired by the work of Greenberg–Stevens [21], and crucially uses the p-adic L-function of the unique p-adic family containing $\tilde{\pi}$, constructed and studied in the first part of the paper.

However, in the case of a multiple trivial zero, the computation of higher order derivatives has long been known to lie outside the reach of the Greenberg–Stevens method, thus requiring some genuinely new ideas. Indeed, the use of the functional equation for the p-adic L-function of the family of maximal dimension containing $\tilde{\pi}$, as suggested by Hida and Mazur in the nearly ordinary case (see [33, §1]), does not suffice alone to compute higher order derivatives. Our innovation consists in making use of partial p-adic families to flip the sign of the root number and deduce the vanishing of many Taylor coefficients of a certain p-adic analytic function $\mathbb{L}_p(x_1, \ldots, x_e; u)$. We deduce the Main Theorem from the following properties:

(i) (Specialization) $\mathbb{L}_p(0, \ldots, 0; u) = \langle \mathfrak{n} \rangle_p^{u/4} L_p\big(\tilde{\pi}, \frac{2-u}{2}\big)$, where π has tame conductor $\mathfrak{n}$.

(ii) (Functional equation) $\mathbb{L}_p(x_1, \ldots, x_e; -u) = \tilde{\varepsilon} \cdot \mathbb{L}_p(x_1, \ldots, x_e; u)$ with $\tilde{\varepsilon} \in \{\pm 1\}$.

(iii) (Retrieved L-value) $(-2)^e \frac{d^e}{du^e} \mathbb{L}_p(u, \ldots, u; u)\big|_{u=0} =$ R.H.S. of (0.1).

(iv) (Taylor coefficients) $\mathbb{L}_p(x_1, \ldots, x_e; u)$ contains only multinomials of total degrees $\geq e$, and $\frac{d^e}{du^e} \mathbb{L}_p(0, \ldots, 0; u)\big|_{u=0} = \frac{d^e}{du^e} \mathbb{L}_p(u, \ldots, u; u)|_{u=0}$.

Our construction of p-adic L-functions is geometric, based on the theory of automorphic symbols introduced in [18] and on the construction of eigenvarieties using overconvergent cohomology as in [50] and [22]. For a Hilbert modular variety Y_K this was initiated in [2] and is fully developed in the present work. To an admissible affinoid neigborhood $\mathcal{U}$ of the weight of π in the weight space, and a non-zero U_p-eigenclass Φ in the compactly supported overconvergent cohomology $\mathrm{H}_c^d(Y_K, \mathcal{D}_{\mathcal{U}})$, we attach in §3 a canonical $\mathcal{O}(\mathcal{U})$-valued distribution $\mathrm{ev}(\Phi)$ having controlled growth on the Galois group of the maximal abelian extension of F which is unramified outside $p\infty$. Specializing to the case where Φ corresponds to the p-adic family passing through $\tilde{\pi}$ (see Theorem 2.14) we define $L_p(\lambda, s)$ as the p-adic Mellin transform of $\mathrm{ev}(\Phi)$ (see §4.2 and (6.4)). A specific feature of our treatment is that, thanks to a precise choice of a p-refined automorphic newform in π, the interpolation formula for $L_p(\lambda, s)$ has no superfluous factors, allowing us to establish the concise functional equation (ii) (see Theorem 6.4). While the proof of (iii) uses the improved p-adic L-functions constructed in §4.3, reinterpreting (iii) in terms of arithmetic $\mathscr{L}$-invariants requires an extra input from p-adic Hodge theory, namely the existence of rigid analytic triangulations in the category of (φ, Γ)-modules.

In the case of a simple zero ($e = 1$), property (iv) is an immediate consequence of the functional equation (ii) as in the Greenberg–Stevens method. Establishing (iv) in the case of a zero of higher order ($e > 1$), which is the keystone in our approach, demands to go beyond the Greenberg–Stevens method and use partially improved p-adic L-functions as well as study the behavior of $\pi_{\lambda,v}$ in certain 'partial' p-adic families defined in §2.6. This allows us to establish a number of relations between the Taylor coefficients of $\mathbb{L}_p(x_1, \dots, x_e; u)$ which are not all predicted by the Trivial Zero Conjecture and which we believe are of independent interest. Our results do not rely on the Leopoldt or the Bloch–Kato conjectures. The formula that we show is true even when the archimedean L-function vanishes at the central point, implying then that the order of vanishing of the p-adic L-function is at least $e + 1$.

0. Notations and conventions

Throughout this paper, F will be a totally real number field of degree d and with ring of integers $\mathcal{O}_F$. Let $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F = \mathbb{A}_{F,f} \times F_\infty$ be the ring of adeles of F and denote $\widehat{\mathcal{O}}_F = \mathcal{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

We choose a generator $\varpi_{\mathfrak{f}} \in \mathbb{A}_{F,f}^\times$ of each fractional ideal $\mathfrak{f}$ of F such that for any finite place v of F one has $\varpi_{v\mathfrak{f}} = \varpi_v \cdot \varpi_{\mathfrak{f}}$, where ϖ_v is a uniformizer of the ring of integers $\mathcal{O}_v$ of F_v.

Moreover, we define the adele $1_{\mathfrak{f}} \in \mathbb{A}_F$ by

$$(1_{\mathfrak{f}})_v = \begin{cases} 1 & \text{if } n_v \neq 0, \\ 0 & \text{if } n_v = 0. \end{cases}$$

When $\mathfrak{f} \subset \mathcal{O}_F$ we consider the strict idele class group

$$\mathscr{C}\ell_F^+(\mathfrak{f}) = F^\times \backslash \mathbb{A}_F^\times / U(\mathfrak{f}) F_\infty^{\times +},$$

where $(\cdot)^+$ denotes the connected component of identity in a real Lie group and $U(\mathfrak{f})$ denotes the principal congruence subgroup of level $\mathfrak{f}$ of $\widehat{\mathcal{O}}_F^\times$. Moreover we denote by $E(\mathfrak{f}) \subseteq \mathcal{O}_F^\times$ the group of totally positive units which are congruent to 1 modulo $\mathfrak{f}$.

Let Σ be the set of infinite places of F which are all real and can also be seen as embeddings of F in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ inside $\mathbb{C}$. The choice, for a given prime number p, of an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ induces a partition $\Sigma = \bigsqcup_{v \in S_p} \Sigma_v$, where $\sigma \in \Sigma_v$ if and only if v is the kernel of the composite map $\mathcal{O}_F \xrightarrow{\iota_p \circ \sigma} \overline{\mathbb{Z}}_p \twoheadrightarrow \overline{\mathbb{F}}_p$.

We let $(\cdot)^\epsilon$ denote the eigenspace corresponding to a character ϵ of $F_\infty^\times / F_\infty^{\times +} = \{\pm 1\}^\Sigma$.

We denote by G_E the absolute Galois group of a perfect field E. For S a finite set of places of F we let Gal_S denote the Galois group of the maximal abelian extension of F which is unramified outside S. We let $\mathrm{Gal}_{S\infty} = \mathrm{Gal}_{S\cup\Sigma}$ and $\mathrm{Gal}_{p\infty} = \mathrm{Gal}_{S_p\cup\Sigma}$, where S_p denotes the set of places above p. For $S \subset S_p$, we let $\Sigma_S = \bigsqcup_{v\in S} \Sigma_v$ and $\mathcal{O}_{F,S} = \prod_{v\in S} \mathcal{O}_v$.

The cyclotomic character $\chi_{\mathrm{cyc}} : \mathrm{Gal}_{p\infty} \twoheadrightarrow \mathrm{Gal}(F(\mu_{p^\infty})/F) \to \mathbb{Z}_p^\times$ corresponds, via global class field theory, to the idele class character $\chi_{\mathrm{cyc}} : F_+^\times\backslash\mathbb{A}_{F,f}^\times \to \mathbb{Z}_p^\times$ sending y to $\prod_{v\in S_p} \mathrm{N}_{F_v/\mathbb{Q}_p}(y_v)|y_f|_F$. One has $\chi_{\mathrm{cyc}}(\varpi_v) = |\varpi_v|_v = q_v^{-1}$ if $v \notin S_p \cup \Sigma$ and $\chi_{\mathrm{cyc}}(\varpi_v) = \mathrm{N}_{F_v/\mathbb{Q}_p}(\varpi_v)q_v^{-1}$ if $v \in S_p$. Define

$$\langle\cdot\rangle_p = \chi_{\mathrm{cyc}}\omega_p^{-1} : \mathrm{Gal}_{p\infty} \to 1 + 2p\mathbb{Z}_p,$$

where ω_p is the Teichmüller lift of χ_{cyc} mod p if p is odd (resp. of χ_{cyc} mod 4 if $p = 2$). Note that the character $\langle\cdot\rangle_p$ factors through the Galois group $\mathrm{Gal}_{\mathrm{cyc}} = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ of the cyclotomic $\mathbb{Z}_p$-extension $F_{\mathrm{cyc}} \subset F(\mu_{p^\infty})$ of F, hence can be raised to power any $s \in \mathcal{O}_{\mathbb{C}_p}$.

We normalize the Artin reciprocity map so that a uniformizer ϖ_v is sent to a geometric Frobenius Frob_v, and p-adic Hodge theory so that the cyclotomic character χ_{cyc} has Hodge–Tate weight -1. We consider the following non-trivial additive unitary character of $\mathbb{A}_F/F$:

$$\psi : \mathbb{A}_F/F \to \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times,$$

where the first map is the trace, and the second is the usual additive character ψ_0 on $\mathbb{A}/\mathbb{Q}$ characterized by $\ker(\psi_{0|\mathbb{Q}_\ell}) = \mathbb{Z}_\ell$ for every prime number ℓ and $\psi_{0|\mathbb{R}} = \exp(2i\pi\,\cdot)$. We remark that the largest fractional ideal contained in $\ker(\psi_v)$ equals $(\varpi_v^{-\delta_v})$, where δ_v is the valuation at v of the different $\mathfrak{d}$ of F. With this notation the discriminant of F is $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d})$.

Let $dx = \otimes_v dx_v$ be the (self-dual) Haar measure on $\mathbb{A}_F$ which induces the discrete measure on $F \subset \mathbb{A}_F$ and the Haar measure with volume 1 on $\mathbb{A}_F/F$. It has the property that dx_σ is the usual Lebesgue measure for $\sigma \in \Sigma$ and when v is a finite place, $\int_{\mathcal{O}_v} dx_v = q_v^{-\delta_v/2}$. We also let $d^\times x = \otimes d^\times x_v$ be the Haar measure on $\mathbb{A}_F^\times/F^\times$ such that $d^\times x_\sigma = |x_\sigma|_\sigma^{-1}dx_\sigma$ for $\sigma \in \Sigma$ and $\int_{\mathcal{O}_v^\times} d^\times x_v = 1$ for v finite.

Given a finite place v and a character χ_v of $F_v^\times$ of conductor c_v, one can define the local Gauss sum, which is independent of the choice of uniformizers, by

$$\tau(\chi_v, \psi_v, d^\times) = \int_{F_v^\times} \chi_v(y)\psi_v(y)\, d^\times y = \int_{\mathcal{O}_v^\times} \chi_v(u\varpi_v^{-c_v-\delta_v})\psi_v(u\varpi_v^{-c_v-\delta_v})\, du. \tag{0.1}$$

For $\chi : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ an idele class character of conductor $\mathfrak{c}_\chi$ we define the global Gauss sum

$$\tau(\chi) = \prod_{v\nmid\infty} \tau(\chi_v, \psi_v, d_{\chi_v}) = \prod_{v|\mathfrak{c}_\chi} \tau(\chi_v, \psi_v, d^\times) \prod_{v\nmid\mathfrak{c}_\chi\infty} \chi_v(\varpi_v^{-\delta_v}), \tag{0.2}$$

where the Haar measure d_{χ_v} on $F_v^\times$ gives $\mathcal{O}_v^\times$ volume 1 (resp. $1 - q_v^{-1}$) when χ_v is unramified (resp. ramified).

Part I
p-adic L-functions for families of nearly finite slope Hilbert cusp forms

We develop a natural framework yielding simultaneous constructions of p-adic L-functions and their improved counterparts for nearly finite slope families of Hilbert cusp forms.

1. Automorphic theory of Hilbert cusp forms

We recall the representation theory of Hilbert automorphic cusp forms and construct normalized p-refined nearly finite slope newforms allowing us to define canonical periods.

1.1. Hilbert modular varieties

We consider the reductive group scheme $G = \mathrm{Res}_{\mathbb{Z}}^{\mathcal{O}_F} \mathrm{GL}_2$ over $\mathbb{Z}$. We let C_∞ be the standard maximal compact subgroup of G_∞ and $K_\infty = C_\infty F_\infty^\times$.

The Hilbert modular variety of level K, an open compact subgroup of $G(\mathbb{A}_f)$, is defined as the locally symmetric space

$$Y_K = G(\mathbb{Q})\backslash G(\mathbb{A})/KK_\infty^+.$$

By the Strong Approximation Theorem for $\mathrm{SL}_2(\mathbb{A}_F)$, the fibers of the map

$$\det_K : Y_K \to F^\times\backslash\mathbb{A}_F^\times/\det(K)F_\infty^{\times+}$$

are connected. For each $[\eta] \in \mathbb{A}_F^\times/(F^\times \det(K)F_\infty^{\times+}) = \pi_0(Y_K)$ the connected component $Y_K[\eta] = \det_K^{-1}([\eta])$ can be described as a quotient of the unbounded hermitian symmetric domain G_∞^+/K_∞^+ by a congruence subgroup as follows. Choosing a representative $\eta \in \mathbb{A}_{F,f}^\times$ of $[\eta]$ (one can take it to be a uniformizer at some finite place), there is an isomorphism

$$\Gamma_\eta\backslash G_\infty^+/K_\infty^+ \simeq Y_K[\eta],\, g_\infty \mapsto g_\infty\left(\begin{smallmatrix}\eta & 0\\ 0 & 1\end{smallmatrix}\right), \quad \text{where} \quad \Gamma_\eta = G(\mathbb{Q}) \cap \left(\begin{smallmatrix}\eta & 0\\ 0 & 1\end{smallmatrix}\right)K\left(\begin{smallmatrix}\eta & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}G_\infty^+.$$

In what follows we assume that K is sufficiently small in the sense that for all $g \in G(\mathbb{A})$:

$$G(\mathbb{Q}) \cap gKK_\infty^+g^{-1} = F^\times \cap KF_\infty^\times. \tag{1.1}$$

It is equivalent to ask that Γ_η modulo its center $\Gamma_\eta \cap F^\times$ is torsion-free, this property being independent of the choice of the representative η. Then $Y_K[\eta]$ is a complex manifold admitting G_∞^+/K_∞^+ as a universal covering space with group $\Gamma_\eta/(\Gamma_\eta \cap F^\times)$.

1.2. Local systems and cohomology

We will now describe two natural constructions of local systems on Y_K. In §1.3 we will apply these constructions to attach local systems to algebraic representations of G. Consider first a left $G(\mathbb{Q})$-module V such that

$$F^\times \cap KF_\infty^\times \text{ acts trivially on } V. \tag{1.2}$$

By (1.1) and (1.2) the group $G(\mathbb{Q}) \cap gKK_\infty^+ g^{-1} = F^\times \cap KF_\infty^\times$ acts trivially on V. Therefore

$$G(\mathbb{Q})\backslash(G(\mathbb{A}) \times V)/KK_\infty^+ \to Y_K$$

is a local system with left $G(\mathbb{Q})$-action and right KK_∞^+-action given by $\gamma(g, v)k = (\gamma gk, \gamma \cdot v)$.

Alternatively, given a left K-module V satisfying (1.2), one can consider the local system $\mathcal{V}_K$,

$$G(\mathbb{Q})\backslash(G(\mathbb{A}) \times V)/KK_\infty^+ \to Y_K,$$

with left $G(\mathbb{Q})$-action and right KK_∞^+-action given by $\gamma(g, v)k = (\gamma gk, k^{-1} \cdot v)$.

We will denote by $\mathcal{V}_K$ (or $\mathcal{V}$ if K is clear from the context) the corresponding sheaf of locally constant sections on Y_K and will consider the usual (resp. compactly supported) cohomology groups $\mathrm{H}^i(Y_K, \mathcal{V})$ (resp. $\mathrm{H}^i_c(Y_K, \mathcal{V})$). Although we use the same notation for both constructions, it will be generally clear from the context which one applies and otherwise we will name it explicitly. When the actions of $G(\mathbb{Q})$ and K on V extend compatibly to a left action of $G(\mathbb{A})$, the resulting two local systems are isomorphic by $(g, v) \mapsto (g, g^{-1} \cdot v)$, yielding an isomorphism of sheaves and their cohomology groups, thus justifying the abuse of notation.

1.3. Cohomological weights

Let B be the standard Borel subgroup of G consisting of upper triangular matrices, whose Levi subgroup T consists of the diagonal matrices.

The characters of the torus $\mathrm{Res}_{\mathbb{Q}}^F \mathbb{G}_{\mathrm{m}}$ can be identified with $\mathbb{Z}[\Sigma]$ as follows: for any $k = \sum_{\sigma\in\Sigma} k_\sigma \sigma \in \mathbb{Z}[\Sigma]$ and any $\mathbb{Q}$-algebra A splitting F, we consider the character

$$(F \otimes_{\mathbb{Q}} A)^\times \ni x \mapsto x^k = \prod_{\sigma\in\Sigma} \sigma(x)^{k_\sigma} \in A^\times. \tag{1.3}$$

The norm character $\mathrm{N}_{F/\mathbb{Q}} : \mathrm{Res}_{\mathbb{Q}}^F \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ then corresponds to the element $t = \sum_{\sigma\in\Sigma} \sigma$.

Integral weights of G are given by characters of the form $(a, d) \mapsto a^k d^{k'}$ for some $(k, k') \in \mathbb{Z}[\Sigma]^2$. Characters such that $k_\sigma \geq k'_\sigma$ for all $\sigma \in \Sigma$ are called *dominant* with respect to B and parametrize the irreducible algebraic representations of G, explicitly given by

$$\bigotimes_{\sigma\in\Sigma} (\mathrm{Sym}_\sigma^{k_\sigma - k'_\sigma} \otimes \mathrm{Det}_\sigma^{k'_\sigma}).$$

Definition 1.1. We say that a dominant weight of G is *cohomological* if it is of the form

$$\left(\frac{(\mathsf{w}-2)t + k}{2}, \frac{(\mathsf{w}+2)t - k}{2}\right),$$

where $(k, \mathsf{w}) \in \mathbb{Z}[\Sigma] \times \mathbb{Z}$ is such that for all $\sigma \in \Sigma$ we have $k_\sigma \geq 2$ and $k_\sigma \equiv \mathsf{w} \pmod 2$. We will identify the cohomological weight with the tuple (k, w) defining it, and call w the *purity weight*.

A dominant integral weight is cohomological exactly when the central character of the corresponding G-representation factors through the norm $\mathrm{N}_{F/\mathbb{Q}}$.

Given a cohomological weight (k, w) and a $\mathbb{Q}$-algebra A splitting F, we consider the A-module $L_{k,\mathsf{w}}(A)$ of polynomials f of degree at most $k - 2t = (k_\sigma - 2)_{\sigma \in \Sigma}$ in the variables $z = (z_\sigma)_{\sigma \in \Sigma}$ with coefficients in A, endowed with the following right action of $G(A) \simeq \mathrm{GL}_2(A)^\Sigma$:

$$f_{|\gamma}(z) = \det(\gamma)^{((\mathsf{w}+2)t-k)/2}(cz + d)^{k-2t} f\left(\tfrac{az+b}{cz+d}\right), \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(A). \quad (1.4)$$

Then its dual $L^\vee_{k,\mathsf{w}}(A) = \mathrm{Hom}_A(L_{k,\mathsf{w}}(A), A)$ is endowed with a left action of $G(A)$ given by

$$(\gamma \cdot \mu)(f) = \mu(f_{|\det(\gamma)^{-1} \cdot \gamma}), \quad \text{where} \quad \gamma \in G(A),\ \mu \in L^\vee_{k,\mathsf{w}}(A),\ f \in L_{k,\mathsf{w}}(A), \quad (1.5)$$

and there is an isomorphism of left $G(A)$-modules

$$L^\vee_{k,\mathsf{w}}(A) \simeq \bigotimes_{\sigma \in \Sigma} (\mathrm{Sym}_\sigma^{k_\sigma - 2} \otimes \mathrm{Det}_\sigma^{(2-k_\sigma-\mathsf{w})/2})(A^2). \quad (1.6)$$

For (k, w) and A as above, the assumption (1.2) for the left $G(A)$-module $L^\vee_{k,\mathsf{w}}(A)$ reads

$$\mathrm{N}^{\mathsf{w}}_{F/\mathbb{Q}}(\varepsilon) = 1 \quad \text{for all } \varepsilon \in F^\times \cap KF^\times_\infty. \quad (1.7)$$

Under this condition, applying the construction of §1.2 yields a sheaf $\mathcal{L}^\vee_{k,\mathsf{w}}(A)$ whose cohomology groups $\mathrm{H}^i(Y_K, \mathcal{L}^\vee_{k,\mathsf{w}}(A))$ and $\mathrm{H}^i_c(Y_K, \mathcal{L}^\vee_{k,\mathsf{w}}(A))$ will play a prominent role in this paper.

1.4. *Cohomological cuspidal automorphic representations*

The aim of this section is to describe the cuspidal automorphic representations contributing to $\mathrm{H}^d(Y_K, \mathcal{L}^\vee_{k,\mathsf{w}}(\mathbb{C}))$ for (k, w) a cohomological weight as in Definition 1.1 and K satisfying (1.7), and to perform some archimedean computations which will be used to interpret cohomologically the special values of automorphic L-functions. While the general theory is well known, the applications we have in mind require an explicit version as in [41, §4.4].

Let $\mathfrak{g}_\infty$ (resp. $\mathfrak{k}_\infty$) be the complexified Lie algebra of G_∞ (resp. K_∞). Using the comparison between Betti cohomology over $\mathbb{C}$ and de Rham cohomology, and further reinterpreting the de Rham complex in terms of the complex computing relative Lie algebra cohomology, we obtain

$$\begin{aligned} \mathrm{H}^d_{\mathrm{cusp}}(Y_K, \mathcal{L}^\vee_{k,\mathsf{w}}(\mathbb{C})) &= \mathrm{H}^d\left(\mathfrak{g}_\infty, K^+_\infty, L^\vee_{k,\mathsf{w}}(\mathbb{C}) \otimes C^\infty_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K)\right) \\ &= \bigoplus_\pi \mathrm{H}^d(\mathfrak{g}_\infty, K^+_\infty, L^\vee_{k,\mathsf{w}}(\mathbb{C}) \otimes \pi_\infty) \otimes \pi_f^K, \end{aligned}$$

where π runs over the cuspidal automorphic representation of $G(\mathbb{A})$. By Künneth's formula,

$$\mathrm{H}^d(\mathfrak{g}_\infty, K_\infty^+, L_{k,\mathsf{w}}^\vee(\mathbb{C}) \otimes \pi_\infty) = \bigotimes_{\sigma\in\Sigma} \mathrm{H}^1(\mathfrak{g}_\sigma, K_\sigma^+, L_{k_\sigma,\mathsf{w}}^\vee(\mathbb{C}) \otimes \pi_\sigma),$$

where $\mathrm{H}^1(\mathfrak{g}_\sigma, K_\sigma^+, L_{k_\sigma,\mathsf{w}}^\vee(\mathbb{C}) \otimes \pi_\sigma) \neq 0$ if and only if π_σ is the irreducible infinite-dimensional representation $\pi_{k_\sigma,\mathsf{w}}$ of $\mathrm{GL}_2(\mathbb{R})$ whose Langlands parameter $\mathbb{C}^\times \rtimes \{1, j\} \to \mathrm{GL}_2(\mathbb{C})$ is given by

$$\mathbb{C}^\times \ni z \mapsto |z|^{\mathsf{w}/2} \begin{pmatrix} (\bar{z}/z)^{(k_\sigma-1)/2} & \\ & (z/\bar{z})^{(k_\sigma-1)/2} \end{pmatrix} \quad \text{and} \quad j \mapsto \begin{pmatrix} & 1 \\ (-1)^{k_\sigma-1} & \end{pmatrix}.$$

One can also describe $\pi_{k_\sigma,\mathsf{w}}$ as follows. Consider the unitary induction from B_σ to G_σ of the character which is trivial on the unipotent radical and given on T_σ by

$$(a, d) \mapsto a^{(k_\sigma+\mathsf{w}-2)/2} d^{(\mathsf{w}-k_\sigma+2)/2} |a/d|^{1/2}. \tag{1.8}$$

By Frobenius reciprocity it has a unique non-trivial finite-dimensional quotient given by $L_{k_\sigma,\mathsf{w}}(\mathbb{C})$ and the kernel turns out to be isomorphic to $\pi_{k_\sigma,\mathsf{w}}$. The fact that the extension is non-split implies the non-vanishing of $\mathrm{H}^1(\mathfrak{g}_\sigma, K_\sigma^+, L_{k_\sigma,\mathsf{w}}^\vee(\mathbb{C}) \otimes \pi_\sigma)$.

Definition 1.2. We say that an automorphic representation π of $G(\mathbb{A})$ has *weight* (k, w) if $\pi_\infty = \otimes_{\sigma\in\Sigma} \pi_{k_\sigma,\mathsf{w}}$. The integer w is the *purity weight* of π, i.e., the weight of its central character.

A cuspidal automorphic representation π of $G(\mathbb{A})$ contributes to $\mathrm{H}^d(Y_K, \mathscr{L}_{k,\mathsf{w}}^\vee(\mathbb{C}))$ if and only if it has weight (k, w) and $\pi_f^K \neq 0$. If such π exist then condition (1.7) is always satisfied. Indeed, if $\phi \in \pi^K$, then for each $\varepsilon \in F^\times \cap K F_\infty^\times$ and $g \in G(\mathbb{A})$ we have

$$\phi(g) = \phi\big(\big(\begin{smallmatrix} \varepsilon & \\ & \varepsilon \end{smallmatrix}\big) g\big) = \phi\big(g \big(\begin{smallmatrix} \varepsilon_\infty & \\ & \varepsilon_\infty \end{smallmatrix}\big)\big) = \mathrm{N}_{F/\mathbb{Q}}^{\mathsf{w}}(\varepsilon)\phi(g).$$

We will now specify a basis of

$$\mathrm{H}^1(\mathfrak{g}_\sigma, K_\sigma^+, L_{k_\sigma,\mathsf{w}}^\vee(\mathbb{C}) \otimes \pi_{k_\sigma,\mathsf{w}}) = \mathrm{Hom}_{C_\sigma^+}(\mathfrak{g}_\sigma/\mathfrak{k}_\sigma, L_{k_\sigma,\mathsf{w}}^\vee(\mathbb{C}) \otimes \pi_{k_\sigma,\mathsf{w}}), \tag{1.9}$$

where one considers the adjoint action of $C_\sigma^+ = \big\{r(\theta) = \big(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\big) \big| \theta \in \mathbb{R}\big\}$ on $\mathfrak{g}_\sigma/\mathfrak{k}_\sigma$.

Let $(w_\sigma^*, \bar{w}_\sigma^*)$ be the dual basis of the basis $w_\sigma = \frac{1}{4}\big(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\big)$, $\bar{w}_\sigma = \frac{1}{4}\big(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\big)$ of $\mathfrak{g}_\sigma/\mathfrak{k}_\sigma$.

Consider $\mathrm{eval}_{\pm i} \in L_{k_\sigma,\mathsf{w}}^\vee(\mathbb{C})$ defined by $\mathrm{eval}_{\pm i}(P) = P(\pm i)$. From (1.5) one finds that

$$\mathrm{Ad}(r(\theta))(w_\sigma^*) = e^{2i\theta} w_\sigma^*,\ \mathrm{Ad}(r(\theta))(\bar{w}_\sigma^*) = e^{-2i\theta} \bar{w}_\sigma^*,\ r(\theta) \cdot \mathrm{eval}_{\pm i} = e^{\pm i\theta(k_\sigma-2)} \mathrm{eval}_{\pm i}.$$

Since the C_σ^+-types $r(\theta) \mapsto e^{\pm i\theta k_\sigma}$ do not occur in $L_{k_\sigma,\mathsf{w}}^\vee(\mathbb{C})$ but do occur in the induced representation from (1.8), one deduces that there is a unique function $\phi_\sigma \in \pi_{k_\sigma,\mathsf{w}}$ such that $\phi_\sigma(\cdot\, r(\theta)) = e^{-i\theta k_\sigma}\phi_\sigma$ for all $\theta \in \mathbb{R}$ and normalized so that $\phi_\sigma(1) = 1$ (here we use the fact that the characters $r(\theta) \mapsto e^{\pm i\theta k_\sigma}$ and (1.8) agree on $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ since $k_\sigma \equiv \mathsf{w} \pmod 2$). Hence

$$\mathrm{Hom}_{C_\sigma^+}(\mathfrak{g}_\sigma/\mathfrak{k}_\sigma, L^\vee_{k_\sigma,\mathsf{w}}(\mathbb{C}) \otimes \pi_{k_\sigma,\mathsf{w}}) = \mathbb{C}(w_\sigma^* \otimes \mathrm{eval}_i \otimes \phi_\sigma) \oplus \mathbb{C}(\bar{w}_\sigma^* \otimes \mathrm{eval}_{-i} \otimes \overline{\phi_\sigma}).$$

Since $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \cdot w_\sigma^* = \bar{w}_\sigma^*$, $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \cdot \mathrm{eval}_i = (-1)^{(\mathsf{w}+k_\sigma-2)/2}\mathrm{eval}_{-i}$ and $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \cdot \phi_\sigma = (-1)^{(\mathsf{w}+k_\sigma-2)/2}\overline{\phi_\sigma}$, we see that

$$\Xi_{\pi_\sigma}^{\epsilon_\sigma} = i^{(2-\mathsf{w}-k_\sigma)/2}(w_\sigma^* \otimes \mathrm{eval}_i \otimes \phi_\sigma + \epsilon_\sigma(-1)\bar{w}_\sigma^* \otimes \mathrm{eval}_{-i} \otimes \overline{\phi_\sigma})$$

is an eigenbasis of (1.9) for the action of $C_\sigma/C_\sigma^+ = K_\sigma/K_\sigma^+ \xrightarrow[\sim]{\det} F_\sigma^\times/F_\sigma^{\times +} = \{\pm 1\}$.

Letting $w_\infty^* = \bigotimes_{\sigma\in\Sigma} w_\sigma^*$ and $\phi_\infty = \bigotimes_{\sigma\in\Sigma}\phi_\sigma$, the space $\mathrm{H}^d(\mathfrak{g}_\infty, K_\infty^+, \pi_\infty \otimes L^\vee_{k,\mathsf{w}}(\mathbb{C}))$ has the following basis indexed by the characters $\epsilon : K_\infty/K_\infty^+ = \{\pm1\}^\Sigma \to \{\pm 1\}$:

$$\Xi_{\pi_\infty}^{\epsilon} = \bigotimes_{\sigma\in\Sigma} \Xi_{\pi_\sigma}^{\epsilon_\sigma} = i^{((2-\mathsf{w})t-k)/2} \sum_{s_\infty \in \{\pm1\}^\Sigma} \epsilon(s_\infty)(s_\infty \cdot (w_\infty^* \otimes \mathrm{eval}_i \otimes \phi_\infty)). \tag{1.10}$$

1.5. *Automorphic representations of nearly finite slope*

In this section, we introduce the notion of a *nearly finite slope* cuspidal automorphic representation π. These are stable under twists and encompass both Coleman's finite slope and Hida's nearly ordinary cases. Furthermore, we define a p-refined line in such a π.

Definition 1.3. Let π be a cohomological cuspidal automorphic representation π of $G(\mathbb{A})$ and let $v \in S_p$ be a place of F.

(i) π_v has *finite slope* if either π_v is a principal series representation with at least one unramified character or π_v is an unramified twist of the Steinberg representation.

(ii) π_v has *nearly finite slope* if π_v is not supercuspidal, or equivalently if it is a twist of a finite slope representation by a finite order character.

(iii) A representation π_v is *regular* if either it is a twist of the Steinberg representation or it is a principal series representation with distinct characters $\chi_{1,v} \neq \chi_{2,v}$.

(iv) A *refinement* of a nearly finite slope representation π_v of $\mathrm{GL}_2(F_v)$ is a 1-dimensional subrepresentation ν_v of the Weil–Deligne representation attached to π_v via the local Langlands correspondence for $\mathrm{GL}_2(F_v)$.

(v) For $S \subset S_p$, a [*regular*] S-*refinement* of an automorphic representation π is a pair $\tilde{\pi}_S = (\pi, \{\nu_v\}_{v\in S})$ such that ν_v is a [regular] refinement of π_v for all $v \in S$. When $S = S_p$ we call $\tilde{\pi} = \tilde{\pi}_{S_p}$ a p-*refinement*.

Suppose (π_v, ν_v) is a refined regular nearly finite slope representation.

If π_v is a twist of the Steinberg representation, we let K_v be the Iwahori group

$$I_v = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_v) \;\middle|\; c \in \varpi_v\mathcal{O}_v\right\}, \tag{1.11}$$

whereas if π_v is a principal series representation, we let $K_v = I_v \cap K_1(v^{m_v})$ where m_v is the conductor of $\chi_{1,v}/\chi_{2,v}$. Let

$$K'_v = K(\pi_v, \nu_v) = \ker(K_v \xrightarrow{\det} \mathcal{O}_v^\times \xrightarrow{\nu_v} \mathbb{C}^\times). \tag{1.12}$$

For a uniformizer $\varpi_v \in \mathcal{O}_v$ and $\delta \in \mathcal{O}_v^\times$ we define the Hecke operators

$$U_{\varpi_v} = \left[K'_v\begin{pmatrix}\varpi_v & \\ & 1\end{pmatrix}K'_v\right] \quad \text{and} \quad U_\delta = \left[K'_v\begin{pmatrix}\delta & \\ & 1\end{pmatrix}K'_v\right]. \tag{1.13}$$

Lemma 1.4. *For any refined regular nearly finite slope representation* (π_v, ν_v) *one has*

$$\dim\,(\pi_v^{K'_v})_{(U_{\varpi_v}-\nu_v(\varpi_v), U_\delta-\nu_v(\delta)\mid\delta\in\mathcal{O}_v^\times)} = 1.$$

Proof. Since the $\mathcal{O}_v^\times$-action is semisimple, by (1.12) there is an isomorphism

$$(\pi_v^{K'_v})_{(U_\delta-\nu_v(\delta)\mid\delta\in\mathcal{O}_v^\times)} = (\pi_v \otimes \nu_v^{-1})^{K_v}. \tag{1.14}$$

If π_v is either a twist of the Steinberg representation or a principal series representation with $\chi_{1,v}/\chi_{2,v}$ ramified, then (1.14) is 1-dimensional on which U_{ϖ_v} acts by $\nu_v(\varpi_v)$. If π_v is a principal series representation with $\chi_{1,v}/\chi_{2,v}$ unramified then (1.14) is 2-dimensional and we conclude that the U_{ϖ_v}-eigenspace for $\nu_v(\varpi_v)$ is a line by regularity. ■

Let $\tilde{\pi}_S = (\pi, \{\nu_v\}_{v\in S})$ be a regular S-refinement of a nearly finite slope cuspidal automorphic representation of $G(\mathbb{A})$ of cohomological weight (k, w).

Definition 1.5. Let $\mathfrak{u}$ be an unramified prime of F such that

(i) any open compact subgroup K of $\mathrm{GL}_2(\mathbb{A}_{F,f})$ such that $K_\mathfrak{u} = K_0(\mathfrak{u})$ satisfies (1.1),

(ii) $\pi_\mathfrak{u}$ is an unramified principal series representation with Hecke parameters $\alpha_\mathfrak{u} \neq \beta_\mathfrak{u}$.

The existence of $\mathfrak{u}$ satisfying (i) follows from [17, Lem. 2.1], while the fact that $\mathfrak{u}$ can be chosen to satisfy (ii) as well can be shown using the irreducibility of the Galois representation V_π. It is also a consequence of the Sato–Tate conjecture which is known to hold for Hilbert cusp forms. In this case $\pi_\mathfrak{u}^{K_0(\mathfrak{u})}$ is 2-dimensional on which $U_\mathfrak{u}$ acts with eigenvalues $\alpha_\mathfrak{u}$ and $\beta_\mathfrak{u}$.

Definition 1.6. Let E be a number field containing the Galois closure of F, the field of rationality of π_f, the Hecke parameters of $\pi_\mathfrak{u}$, and the values of the characters $(\nu_v)_{v\in S_p}$.

Let $\mathfrak{m}_\pi$ be the maximal ideal of the Hecke algebra $\mathbb{T} = E[T_v, S_v \mid v \nmid \mathfrak{n}\mathfrak{u}p]$ corresponding to π_f. For $S \subset S_p$, we consider the maximal ideal

$$\mathfrak{m}_{\tilde{\pi}_S} = (\mathfrak{m}_\pi, U_\mathfrak{u} - \alpha_\mathfrak{u}, U_{\varpi_v} - \nu_v(\varpi_v), U_\delta - \nu_v(\delta) \mid \delta \in \mathcal{O}_v^\times,\ v \in S)$$

of the Hecke algebra $\tilde{\mathbb{T}}_S = \mathbb{T}[U_\mathfrak{u}, U_{\varpi_v}, U_\delta \mid \delta \in \mathcal{O}_v^\times,\ v \in S]$, and we let $\mathfrak{m}_{\tilde{\pi}} = \mathfrak{m}_{\tilde{\pi}_{S_p}}$.

Definition 1.7. Let $K(\tilde{\pi}_S, \mathfrak{u}) = K_0(\mathfrak{u}) \prod_{v\notin S\cup\{\mathfrak{u}\}} K_1(v^{m_v}) \prod_{v\in S} K'_v$ where m_v is the conductor of π_v and K'_v is as in (1.12). The $(S, \alpha_\mathfrak{u})$-*refined* newline of a regular $\tilde{\pi}_S$ is given by

$$N_{\tilde{\pi}_S,\alpha_\mathfrak{u}} = \left(\pi_f^{K(\tilde{\pi}_S,\mathfrak{u})}\right)_{\mathfrak{m}_{\tilde{\pi}_S}} = \left(\pi_f^{K(\tilde{\pi}_S,\mathfrak{u})}\right)[\mathfrak{m}_{\tilde{\pi}_S}],$$

where $[\mathfrak{m}_{\tilde{\pi}_S}]$ denotes the subspace annihilated by $\mathfrak{m}_{\tilde{\pi}_S}$. While for $v \in S_p$ the U_{ϖ_v}-eigenvalue $\nu_v(\varpi_v)$ on $N_{\tilde{\pi},\alpha_\mathfrak{u}}$ depends on the choice of ϖ_v, its p-adic valuation is independent of it.

Definition 1.8. The *slope* $h_{\tilde{\pi}_v}$ of $\tilde{\pi}_v = (\pi_v, \nu_v)$ is defined as the p-adic valuation of

$$\nu_v(\varpi_v) \prod_{\sigma \in \Sigma_v} \sigma(\varpi_v)^{(k_\sigma + \mathsf{w} - 2)/2}.$$

We say that the refinement $\tilde{\pi}_v$ has *non-critical slope* if $e_v h_{\tilde{\pi}_v} < \min_{\sigma \in \Sigma_v}(k_\sigma - 1)$.

For $S \subset S_p$, we say that $\tilde{\pi}_S$ has *non-critical slope* if $\tilde{\pi}_v$ has non-critical slope for each $v \in S$.

Finally, we say that $\tilde{\pi} = \tilde{\pi}_{S_p}$ has *very non-critical slope* if

$$\sum_{v \in S_p} e_v h_{\tilde{\pi}_v} < \min_{\sigma \in \Sigma}(k_\sigma - 1). \tag{1.15}$$

1.6. Normalized $(S, \mathfrak{u})$-refined eigenforms and Whittaker functions

Let $\tilde{\pi}_S = (\pi, \{\nu_v\}_{v \in S})$ be a regular S-refinement of a nearly finite slope refinement of a cuspidal cohomological automorphic representation π of $G(\mathbb{A})$. In this section we use Whittaker models to choose a basis $\phi_{\tilde{\pi}_S,\alpha_\mathfrak{u}}$ of the line $N_{\tilde{\pi}_S,\alpha_\mathfrak{u}}$ from Definition 1.7 for which a suitable zeta integral yields the Jacquet–Langlands L-function of π. The cusp form $\phi_{\tilde{\pi}_S,\alpha_\mathfrak{u}}$, divided by a suitable complex period, will yield an overconvergent cohomology class to which we will attach a p-adic L-function in §4.1.

The global Whittaker model $\mathcal{W}(\pi, \psi)$ of π can be written as a restricted tensor product of local Whittaker models $\mathcal{W}(\pi_v, \psi_v)$, with respect to $W_v^\circ \in \mathcal{W}(\pi_v, \psi_v)$ for v outside a finite set of bad places, where $W_v^\circ \in \mathcal{W}(\pi_v, \psi_v)^{\mathrm{GL}_2(\mathcal{O}_v)}$ is such that $W_v^\circ(1) = 1$. To relate values of complex L-functions to Whittaker integrals we will use the isomorphism

$$\pi \xrightarrow{\sim} \mathcal{W}(\pi, \psi), \quad \phi \mapsto W_\phi(g) = \int_{\mathbb{A}_F/F} \phi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right) \psi(-x)\, dx, \tag{1.16}$$

whose inverse is given by the Fourier expansion $\phi(g) = \sum_{\xi \in F^\times} W_\phi\left(\left(\begin{smallmatrix} \xi & \\ & 1 \end{smallmatrix}\right) g\right)$.

Given any collection $W_v \in \mathcal{W}(\pi_v, \psi_v)$ such that $W_v = W_v^\circ$ for almost all v, the tensor $\otimes W_v$ lies in $\mathcal{W}(\pi, \psi)$ and therefore is of the form W_ϕ for some $\phi \in \pi$. We remark that for any choice of isomorphisms $\mathcal{W}(\pi_v, \psi_v) \simeq \pi_v$ sending W_v to ϕ_v such that $\phi_v = \phi_v^\circ$ for almost all v (ϕ_v° being the vector relative to which the restricted tensor product $\otimes' \pi_v$ is defined), ϕ and $\otimes \phi_v$ differ by a scalar, and hence ϕ is a pure tensor itself.

We now specify explicitly such a collection of Whittaker functions, beginning with places $v \notin S \cup \Sigma$ for which $\pi_v^{K_1(v^{m_v})}$ is a line. Let W_v^{new} be the generator of the line $\mathcal{W}(\pi_v, \psi_v)^{K_1(v^{m_v})}$ given by the following formulas (see [41, §3.3.1] and [12, Thm. 4.6.5]).

(i) If π_v is the unramified principal series representation with characters $\chi_{1,v}$ and $\chi_{2,v}$ then

$$W_v^{\mathrm{new}}\left(\begin{pmatrix} \varpi_v^{m-\delta_v} & \\ & 1\end{pmatrix}\right) = \begin{cases} q_v^{-m/2} \sum_{l=0}^m \chi_{1,v}(\varpi_v)^l \chi_{2,v}(\varpi_v)^{m-l}, & m \geq 0, \\ 0, & m < 0. \end{cases} \quad (1.17)$$

(ii) If π_v is a principal series representation with unramified $\chi_{1,v}$ and ramified $\chi_{2,v}$ then

$$W_v^{\mathrm{new}}\left(\begin{pmatrix} \varpi_v^{m-\delta_v} & \\ & 1\end{pmatrix}\right) = \begin{cases} q_v^{-m/2} \chi_{1,v}(\varpi_v)^m, & m \geq 0, \\ 0, & m < 0. \end{cases} \quad (1.18)$$

(iii) If π_v is a twist of the Steinberg representation by the unramified character χ_v then

$$W_v^{\mathrm{new}}\left(\begin{pmatrix} \varpi_v^{m-\delta_v} & \\ & 1\end{pmatrix}\right) = \begin{cases} q_v^{-m} \chi_v(\varpi_v)^m, & m \geq 0, \\ 0, & m < 0. \end{cases} \quad (1.19)$$

(iv) In all other cases $W_v^{\mathrm{new}}\left(\begin{pmatrix} \varpi_v^{m-\delta_v} & \\ & 1\end{pmatrix}\right) = \begin{cases} 1, & m \geq 0, \\ 0, & m < 0. \end{cases}$

Denoting $\{\alpha_{\mathfrak{u}}, \beta_{\mathfrak{u}}\} = \{\chi_{1,\mathfrak{u}}(\varpi_{\mathfrak{u}})\sqrt{q_{\mathfrak{u}}}, \chi_{2,\mathfrak{u}}(\varpi_{\mathfrak{u}})\sqrt{q_{\mathfrak{u}}}\}$ the Hecke parameters of $\pi_{\mathfrak{u}}$ we let

$$W_{\mathfrak{u}}^{\alpha}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) = W_{\mathfrak{u}}^{\mathrm{new}}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) - \beta_{\mathfrak{u}} q_{\mathfrak{u}}^{-1} W_{\mathfrak{u}}^{\mathrm{new}}\left(\begin{pmatrix} y\varpi_{\mathfrak{u}}^{-1} & \\ & 1\end{pmatrix}\right).$$

Then we have $W_{\mathfrak{u}}^{\alpha}\left(\begin{pmatrix} \varpi_{\mathfrak{u}}^{m} & \\ & 1\end{pmatrix}\right) = \begin{cases} q_{\mathfrak{u}}^{-m}\alpha_{\mathfrak{u}}^m, & m \geq 0, \\ 0, & m < 0. \end{cases}$

Finally, we specify v-refined Whittaker functions at $v \in S$. Recall that ν_v is a one-dimensional subrepresentation of the Weil–Deligne representation attached to π_v, which is assumed to be non-supercuspidal. Let $W_v' \in \mathcal{W}(\pi_v \otimes \nu_v^{-1}, \psi_v)$ be the new vector chosen as above. If $\pi_v \otimes \nu_v^{-1}$ is ramified we let

$$W_v^{\nu} = \nu_v(\varpi_v)^{\delta_v}(\nu_v \circ \det) \cdot W_v' \in \mathcal{W}(\pi_v, \psi_v).$$

When $\pi_v \otimes \nu_v^{-1}$ is an unramified principal series representation with characters $|\cdot|^{1/2} \neq |\cdot|^{1/2}\chi_{2,v}/\chi_{1,v}$ we let

$$W_v^{\nu} = \nu_v(\varpi_v)^{\delta_v}(\nu_v \circ \det) \cdot \left(W_v' - \frac{\chi_{2,v}(\varpi_v)}{q_v \chi_{1,v}(\varpi_v)} W_v'\left(\cdot \begin{pmatrix} \varpi_v^{-1} & \\ & 1\end{pmatrix}\right)\right) \in \mathcal{W}(\pi_v, \psi_v).$$

Formulas (1.17)–(1.19) then imply

$$W_v^{\nu}\left(\begin{pmatrix} \varpi_v^{m-\delta_v} & \\ & 1\end{pmatrix}\right) = \begin{cases} q_v^{-m} \nu_v(\varpi_v)^m, & m \geq 0, \\ 0, & m < 0. \end{cases} \quad (1.20)$$

Lemma 1.9. *The image of* $W_{\tilde{\pi}_S, \alpha_{\mathfrak{u}}, f} = W_{\mathfrak{u}}^{\alpha} \otimes \bigotimes_{v \notin S \cup \{\mathfrak{u}\}} W_v^{\mathrm{new}} \otimes \bigotimes_{v \in S} W_v^{\nu}$ *in* π_f *is a basis of* $N_{\tilde{\pi}_S, \alpha_{\mathfrak{u}}}$.

Proof. The statement is clear when $v \nmid p\mathfrak{u}$ since any isomorphism $\mathcal{W}(\pi_v, \psi_v) \simeq \pi_v$ matches the new lines, as well as in the case $v = \mathfrak{u}$ because by construction $W_{\mathfrak{u}}^{\alpha}$ has $U_{\mathfrak{u}}$-eigenvalue $\alpha_{\mathfrak{u}}$. Suppose that $v \in S_p$. The function W_v^{ν} is K_v'-invariant as both W_v' and $\nu_v \circ \det$ are. The fact that $U_\delta W_v^{\nu} = \nu_v(\delta) W_v^{\nu}$ for $\delta \in \mathcal{O}_v^{\times}$ then follows from the K_v-invariance of W_v'. Finally, U_{ϖ_v} fixes the refined new vector in $\mathcal{W}(\pi_v \otimes \nu_v^{-1}, \psi_v)$, hence acts as $\nu_v(\varpi_v)$ on W_v^{ν}. ∎

For $\sigma \in \Sigma$ we choose $W_\sigma \in \mathcal{W}(\pi_\sigma, \psi_\sigma)$ such that $W_\sigma(\cdot\, r(\theta)) = e^{-ik_\sigma \theta} W_\sigma$ and $W_\sigma\big(\big(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\big)\big) = y^{(k_\sigma + \mathrm{w})/2} e^{-2\pi y}$ for all $y > 0$ (see [12, 41]). Since $\pi_\sigma \simeq \pi_\sigma \otimes \mathrm{sign}_\sigma$, we have:

Lemma 1.10. *W_σ has support in G_σ^+ and its image in π_σ belongs to the line generated by ϕ_σ.*

Definition 1.11. We define the *normalized* $(S, \alpha_{\mathfrak{u}})$*-refined newform* as the cusp form $\phi_{\tilde{\pi}_S, \alpha_{\mathfrak{u}}} \in \pi$ whose image under the isomorphism (1.16) corresponds to the pure tensor

$$W_{\tilde{\pi}_S, \alpha_{\mathfrak{u}}} = W_{\tilde{\pi}_S, \alpha_{\mathfrak{u}}, f} \otimes \bigotimes_{\sigma \in \Sigma} W_\sigma.$$

We end this section by computing the local Whittaker integrals that yield the local Euler factors of the complex L-function. The following proposition shows how the choice of level $\mathfrak{f}$ in the automorphic symbols in §3 reflects in the local Whittaker integrals.

Proposition 1.12. *Suppose $v \in S$ and χ_v is a finite order character of $F_v^{\times}$. Then*

$$\begin{aligned} Z_v &= \int_{F_v^{\times}} \chi_v(y) W_v^{\nu}\big(\big(\begin{smallmatrix} y\varpi_v^{n_v} & y \\ & 1 \end{smallmatrix}\big)\big) |y|_v^{s-1}\, d^{\times} y \\ &= q_v^{\delta_v(s-1)} \chi_v(\varpi_v^{-\delta_v}) (q_v^{-1} \nu_v(\varpi_v))^{n_v} \frac{q_v}{q_v - 1} Q(\chi_v \nu_v, s) \end{aligned}$$

for $\mathrm{Re}(s)$ *sufficiently large, where c_v denotes the conductor of $\chi_v \nu_v$, and*

$$Q(\chi_v\nu_v, s) = \begin{cases} q_v^{sc_v} (\chi_v\nu_v)(\varpi_v^{\delta_v}) \tau(\chi_v\nu_v, \psi_v, d_{\chi_v\nu_v}) & \text{if } n_v \geq c_v \geq 1, \\ \big(1 - \frac{(\chi_v\nu_v)(\varpi_v)}{q_v^s}\big)^{-1} \big(1 - \frac{q_v^{s-1}}{(\chi_v\nu_v)(\varpi_v)}\big) & \text{if } n_v > c_v = 0, \\ \big(1 - \frac{(\chi_v\nu_v)(\varpi_v)}{q_v^s}\big)^{-1} \big(1 - \frac{1}{q_v}\big) & \text{if } n_v = c_v = 0. \end{cases}$$

Proof. Since $W_v^{\nu} \in \mathcal{W}(\pi_v, \psi_v)$ we have $W_v^{\nu}\big(\big(\begin{smallmatrix} y\varpi_v^{n_v} & y \\ & 1 \end{smallmatrix}\big)\big) = \psi_v(y) W_v^{\nu}\big(\big(\begin{smallmatrix} y\varpi_v^{n_v} & \\ & 1 \end{smallmatrix}\big)\big)$. By (1.20),

$$\begin{aligned} Z_v &= \nu_v(\varpi_v)^{n_v + \delta_v} \\ &\quad \times \int_{F_v^{\times}} (\chi_v\nu_v)(y) \psi_v(y) (\nu_v(\varpi_v^{-\delta_v}) (\nu_v^{-1} \circ \det) \cdot W_v^{\nu}) \big(\big(\begin{smallmatrix} y\varpi_v^{n_v} & \\ & 1 \end{smallmatrix}\big)\big) |y|_v^{s-1}\, d^{\times} y \\ &= \nu_v(\varpi_v)^{n_v + \delta_v} \sum_{m \geq -n_v - \delta_v} q_v^{-n_v - \delta_v - ms} \int_{\mathcal{O}_v^{\times}} (\chi_v\nu_v)(u\varpi_v^m) \psi_v(u\varpi_v^m)\, d^{\times} u. \end{aligned}$$

If $\chi_v\nu_v$ is ramified the above integral vanishes except for $m = -\delta_v - c_v \geq -\delta_v - n_v$, in which case its value is the Gauss sum $\tau(\chi_v\nu_v, \psi_v, d^\times)) = \frac{q_v-1}{q_v}\tau(\chi_v\nu_v, \psi_v, d_{\chi_v\nu_v})$ (see (0.1)).

If $\chi_v\nu_v$ is unramified ($c_v = 0$) then

$$\frac{((\chi_v\nu_v)(\varpi_v)q_v^{-s})^{\delta_v}}{(q_v^{-1}\nu_v(\varpi_v))^{n_v+\delta_v}} \cdot Z_v = Q(\chi_v\nu_v, s)$$
$$= \sum_{m\geq -n_v} ((\chi_v\nu_v)(\varpi_v)q_v^{-s})^m \int_{\mathcal{O}_v^\times} \psi_v(u\varpi_v^{m-\delta_v})\, d^\times u,$$

which is computed using the formula

$$\int_{\mathcal{O}_v^\times} \psi_v(u\varpi_v^{m-\delta_v})\, d^\times u = \begin{cases} 1 & \text{if } m \geq 0, \\ \frac{1}{1-q_v} & \text{if } m = -1, \\ 0 & \text{if } m < -1. \end{cases} \qquad \blacksquare$$

1.7. *Periods for* $(S, \mathfrak{u})$*-refined newforms*

The normalized $(S, \mathfrak{u})$-refined newform $\phi_{\tilde{\pi}_S,\alpha_\mathfrak{u}}$ of the previous section is a Hilbert cusp form in the following sense:

Definition 1.13. A *holomorphic Hilbert automorphic form* of level K and weight (k, w) is a function $\phi : G(\mathbb{Q})\backslash G(\mathbb{A})/K \to \mathbb{C}$ such that for all $\sigma \in \Sigma$, $z \in F_\sigma^\times$ and $r(\theta) \in K_\sigma^+$,

$$\phi\big(\cdot \left(\begin{smallmatrix} z & \\ & z \end{smallmatrix}\right) r(\theta)\big) = z^{\mathsf{w}} e^{-i\theta k_\sigma}\phi$$

and, for all $g_f \in G(\mathbb{A}_f)$, the function $\phi\big(g_f\left(\begin{smallmatrix} y_\infty & x_\infty \\ 0 & 1 \end{smallmatrix}\right)\big)$ is holomorphic in $x_\sigma + iy_\sigma$ in the upper half-plane for every $\sigma \in \Sigma$. It is a *cusp form* if $\int_{\mathbb{A}_F/F} \phi\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)g\big)\, dx = 0$ for all $g \in G(\mathbb{A})$.

Note that a classical weight k modular form for $F = \mathbb{Q}$ has $\mathsf{w} = 2 - k$. The restriction to $G(\mathbb{A}_f)G_\infty^+$ of the Fourier expansion of ϕ as above is supported on totally positive elements, i.e.,

$$\phi(g) = \sum_{\xi \in F_+^\times} W_\phi\big(\left(\begin{smallmatrix} \xi & \\ & 1 \end{smallmatrix}\right)g\big) \quad \text{for all } g \in G(\mathbb{A}_f)G_\infty^+. \tag{1.21}$$

By Lemma 1.10 the normalized holomorphic cusp form $\phi_{\tilde{\pi}_S,\alpha_\mathfrak{u}}$ can be written as a pure tensor of the form $\phi_{\tilde{\pi}_S,\alpha_\mathfrak{u}} = \phi_{\tilde{\pi}_S,\alpha_\mathfrak{u},f} \otimes \bigotimes_{\sigma\in\Sigma} \phi_\sigma$. Recall that for a character $\epsilon : \{\pm 1\}^\Sigma \to \{\pm 1\}$ we constructed in (1.10) a cohomology class $\Xi_{\pi_\infty}^\epsilon \in \mathrm{H}^d(\mathfrak{g}_\infty, K_\infty^+, L_{k,\mathsf{w}}^\vee(\mathbb{C}) \otimes \pi_\infty)$ yielding a map

$$\Theta_\pi^\epsilon : \pi_f^K \xrightarrow{\Xi_{\pi_\infty}^\epsilon \otimes} \mathrm{H}^d(\mathfrak{g}_\infty, K_\infty^+, L_{k,\mathsf{w}}^\vee(\mathbb{C}) \otimes \pi^K)^\epsilon \hookrightarrow \mathrm{H}^d_{\mathrm{cusp}}(Y_K, \mathcal{L}_{k,\mathsf{w}}^\vee(\mathbb{C}))^\epsilon \tag{1.22}$$

where $K = K(\tilde{\pi}_S, \mathfrak{u})$ is as in Definition 1.7. For E and $\mathfrak{m}_{\tilde{\pi}_S}$ as in Definition 1.6 we consider the line

$$\mathrm{H}^d_{\mathrm{cusp}}(Y_K, \mathcal{L}_{k,\mathsf{w}}^\vee(E))^\epsilon_{\mathfrak{m}_{\tilde{\pi}_S}}. \tag{1.23}$$

Definition 1.14. Given a basis $b^{\epsilon}_{\tilde{\pi}_S,\alpha_{\mathfrak{u}}}$ of the E-line (1.23), we let $\Omega^{\epsilon}_{\tilde{\pi}_S} \in \mathbb{C}^{\times}$ be such that $\Theta^{\epsilon}_{\pi}(\phi_{\tilde{\pi}_S,\alpha_{\mathfrak{u}},f}) = \Omega^{\epsilon}_{\tilde{\pi}_S} b^{\epsilon}_{\tilde{\pi}_S,\alpha_{\mathfrak{u}}}$. When ϵ is the trivial character we denote this period simply by $\Omega_{\tilde{\pi}_S}$.

Since $\alpha_{\mathfrak{u}}, \beta_{\mathfrak{u}} \in E$ the period can be taken the same for either choice of Hecke parameter at $\mathfrak{u}$. The precise choice of a p-refined automorphic newform in §1.6 allows us to prove the following formula describing the behavior of the periods $\Omega^{\epsilon}_{\tilde{\pi}}$ under twisting by characters. Here $S = S_p$.

Proposition 1.15. *One can choose the bases $b^{\epsilon}_{\tilde{\pi},\alpha_{\mathfrak{u}}}$ in Definition 1.14 such that for every algebraic Hecke character χ of weight w and p-power conductor, one has*

$$\Omega^{\epsilon\chi_\infty}_{\widetilde{\pi\otimes\chi}} = i^{-dw}\chi_f(\varpi_{\mathfrak{d}})\Omega^{\epsilon\omega^{w}_{p,\infty}}_{\tilde{\pi}}$$

for any character $\epsilon : \{\pm 1\}^{\Sigma} \to \{\pm 1\}$, where $\widetilde{\pi\otimes\chi} = (\pi\otimes\chi, \{\nu_v \otimes \chi_v\}_{v\in S_p})$.

Proof. We drop $\alpha_{\mathfrak{u}}$ to avoid cumbersome notation. By (1.20) and Definition 1.11 we have $W_{\widetilde{\pi\otimes\chi},f} = \chi(\varpi_{\mathfrak{d}})W_{\tilde{\pi},f}\cdot\chi_f\circ\det$ (see also [40, Thm. 1.1]), hence $\phi_{\widetilde{\pi\otimes\chi}} = \chi(\varpi_{\mathfrak{d}})\phi_{\tilde{\pi}}\cdot\chi\circ\det$.

Since $\left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right)\cdot \mathrm{eval}_i^{k,\mathrm{w}+2w} = (-1)^w\left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right)\cdot\mathrm{eval}_i^{k,\mathrm{w}}$, definition (1.10) implies that

$$\begin{aligned}\Xi^{\epsilon}_{\pi_\infty\otimes\chi_\infty} &\otimes \phi_{\widetilde{\pi\otimes\chi},f}\\ &= i^{((2-\mathrm{w}-2w)t-k)/2}\sum_{s_\infty\in\{\pm1\}^{\Sigma}}\epsilon(s_\infty)(s_\infty\cdot(w^*_\infty\otimes\phi_{\widetilde{\pi\otimes\chi}}\otimes\mathrm{eval}_i^{k,\mathrm{w}+2w}))\\ &= i^{-dw}\chi(\varpi_{\mathfrak{d}})\Xi^{\epsilon\chi_\infty\omega^{w}_{p,\infty}}_{\pi_\infty}\otimes(\phi_{\tilde{\pi},f}\cdot\chi_f\circ\det).\end{aligned}\tag{1.24}$$

For $v\in S_p$ let K'_v be as in (1.12) and let K''_v be the analogous subgroup for $(\pi_v\otimes\chi_v, \nu_v\otimes\chi_v)$. Letting $K'' = K^p\prod_{v\in S_p}(K'_v\cap K''_v)$, we see that χ_f factors through $\pi_0(Y_{K''}) \simeq \mathbb{A}^{\times}_{F,f}/F^{\times}_{+}\det(K'')$. Let

$$\mathrm{pr}^* b^{\epsilon}_{\tilde{\pi}} = (c_{[\eta]})_{[\eta]\in\pi_0(Y_{K''})} \in \bigoplus_{[\eta]\in\pi_0(Y_{K''})} \mathrm{H}^d_{\mathrm{cusp}}(Y_{K''}[\eta], \mathscr{L}^{\vee}_{k,\mathrm{w}}(E)_{|Y_{K''}[\eta]}),$$

where $\mathrm{pr} : Y_{K''}\to Y_K$ is the natural projection. One sees that $(\chi_f([\eta])c_{[\eta]})_{[\eta]\in\pi_0(Y_{K''})}$ is E-rational as well, since the rational structure on Betti cohomology is imposed componentwise. Identifying the local systems $\mathscr{L}^{\vee}_{k,\mathrm{w}}(E)$ and $\mathscr{L}^{\vee}_{k,\mathrm{w}+2w}(E)$, and then choosing the basis $b^{\epsilon\chi_\infty\omega^{w}_{p,\infty}}_{\widetilde{\pi\otimes\chi}}$ to correspond to $(\chi_f([\eta])c_{[\eta]})_{[\eta]\in\pi_0(Y_{K''})}$, yields the desired relation in view of (1.24) and Definition 1.14. ∎

2. Overconvergent cohomology and partial nearly finite slope families

In this section we introduce overconvergent cohomology spaces for individual weights and in families which naturally interpolate the spaces $\mathrm{H}^i_c(Y_K, \mathscr{L}^{\vee}_{k,\mathrm{w}}(L))$ for cohomological weights (k,w). Moreover, we establish, in Theorem 2.7, a classicality criterion from which we deduce in Corollary 2.13 that cohomological $\tilde{\pi}$ of non-critical slope are non-critical. Furthermore, we construct a cohomological cuspidal Hilbert eigenvariety in a neighborhood of such a non-critical $\tilde{\pi}$ and show that it is etale over the weight space. We remark that our local families do not *a priori* fit Buzzard's eigenvarieties machine because one cannot guarantee the projectivity of overconvergent cohomology groups beyond H^0 and H^1. Instead we adapt Hida's axiomatic construction of nearly ordinary families to the rigid analytic context and prove equidimensionality and etaleness using the fact that cuspidal cohomology is supported in middle degree.

Finally, we construct partial nearly finite slope p-adic families that impose no restriction on the local representation at a set of places $S \subset S_p$. These results are crucial to carry out the construction of p-adic L-functions for families, and to control the behavior of the local representation at $v \in S_p \setminus S$ in partial families with fixed weights at Σ_v.

2.1. *Weight spaces*

Let $\mathcal{X}$ be the $(d+1)$-dimensional rigid analytic space over $\mathbb{Q}_p$ such that

$$\mathcal{X}(\mathbb{C}_p) = \left\{\lambda \in \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p), \mathbb{C}_p^\times) \;\middle|\; \exists \mathsf{w}_\lambda \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times),\ \lambda\left(\left(\begin{smallmatrix} z & \\ & z\end{smallmatrix}\right)\right) = \mathsf{w}_\lambda(z^t)\right\}. \tag{2.1}$$

Letting $k_\lambda(z) = \lambda\left(\left(\begin{smallmatrix} z & \\ & z^{-1}\end{smallmatrix}\right)\right) \cdot z^{2t}$ for $z \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ there is a finite morphism

$$\mathcal{X}(\mathbb{C}_p) \to \mathrm{Hom}_{\mathrm{cont}}((\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times \mathbb{Z}_p^\times, \mathbb{C}_p^\times), \qquad \lambda \mapsto (k_\lambda, \mathsf{w}_\lambda). \tag{2.2}$$

The cohomological weights of G (see Definition 1.1) all belong to $\mathcal{X}$ and are very Zariski dense in it.

Given an affinoid $\mathcal{U} \subset \mathcal{X}$ we let $\mathcal{O}(\mathcal{U})$ denote the ring of its rigid analytic functions and consider the universal locally analytic character (see [50, Lem. 3.4.6])

$$\langle\cdot\rangle_{\mathcal{U}} : T(\mathbb{Z}_p) \to \mathcal{O}(\mathcal{U})^\times, \qquad t \mapsto (\lambda \mapsto \lambda(t)).$$

Definition 2.1. Fix a cohomological weight (k, w). Given a subset S of S_p we let $\mathcal{X}_S$, resp. $\mathcal{X}'_S$, denote the rigid analytic subspace of $\mathcal{X}$ parametrizing weights which agree with (k, w) on

$$\prod_{v \in S_p \setminus S} \left(\begin{smallmatrix} \mathcal{O}_v^\times & 0 \\ 0 & \mathcal{O}_v^\times \end{smallmatrix}\right), \quad \text{resp.} \quad \prod_{v \in S_p \setminus S} \left(\begin{smallmatrix} \mathcal{O}_v^\times & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

We have $\mathcal{X}_{S_p} = \mathcal{X}'_{S_p} = \mathcal{X}$ and $\dim \mathcal{X}_S = \dim \mathcal{X}'_S - 1 = |\Sigma_S|$ for any $S \subsetneq S_p$. The space $\mathcal{X}'_S$ is the natural place to consider partially improved p-adic L-functions (see §3.5 and §4.3), whereas the partially finite slope families of Hilbert modular cusp forms live on its subspace $\mathcal{X}_S$. We thank the referees for their suggestion to highlight the difference between these spaces, which is important for understanding the results of this paper.

2.2. *Overconvergent modules*

Following [50] we introduce certain modules that will be later used to define overconvergent sheaves on the Hilbert modular variety.

Let L be a finite extension of $\mathbb{Q}_p$ and X be an open compact subset of a finite-dimensional $\mathbb{Q}_p$-vector space. Given $n \in \mathbb{Z}_{\geq 0}$, we let $A_n(X, L)$ denote the Banach L-vector space of n-locally analytic functions on X and $D_n(X, L)$ its Banach dual (see [50, §3.2.1]). More generally, given an admissible affinoid $\mathcal{U} \subset \mathcal{X}$ we consider the orthonormalizable Banach $\mathcal{O}(\mathcal{U})$-modules $A_n(X, \mathcal{O}(\mathcal{U})) = A_n(X, L) \widehat{\otimes}_L \mathcal{O}(\mathcal{U})$ and $D_n(X, \mathcal{O}(\mathcal{U})) = D_n(X, L) \widehat{\otimes}_L \mathcal{O}(\mathcal{U})$ (see [22, §2.2]). The space $A(X, \mathcal{O}(\mathcal{U})) = \bigcup_{n \in \mathbb{Z}_{\geq 0}} A_n(X, \mathcal{O}(\mathcal{U}))$ of locally analytic $\mathcal{O}(\mathcal{U})$-valued functions of X, endowed with the inductive limit topology, is a Fréchet $\mathcal{O}(\mathcal{U})$-module. The natural maps $D_{n+1}(X, \mathcal{O}(\mathcal{U})) \to D_n(X, \mathcal{O}(\mathcal{U}))$ are compact, and $D(X, \mathcal{O}(\mathcal{U})) = \varprojlim D_n(X, \mathcal{O}(\mathcal{U}))$ is a compact Fréchet $\mathcal{O}(\mathcal{U})$-module. There is a functorial (in $\mathcal{O}(\mathcal{U})$) pairing

$$\langle \cdot, \cdot \rangle : D(X, \mathcal{O}(\mathcal{U})) \times A(X, \mathcal{O}(\mathcal{U})) \to \mathcal{O}(\mathcal{U}), \tag{2.3}$$

yielding $D(X, \mathcal{O}(\mathcal{U})) \hookrightarrow \mathrm{Hom}_{\mathcal{O}(\mathcal{U})}(A(X, \mathcal{O}(\mathcal{U})), \mathcal{O}(\mathcal{U}))$, but, as observed in [4, Rem. 3.1], the natural injective maps $D_n(X, \mathcal{O}(\mathcal{U})) \hookrightarrow \mathrm{Hom}_{\mathcal{O}(\mathcal{U})}(A_n(X, \mathcal{O}(\mathcal{U})), \mathcal{O}(\mathcal{U}))$ need not be surjective. The above construction applies to $X = \mathcal{O}_F \otimes \mathbb{Z}_p$ considered as an open compact subset of $\mathbb{Q}_p^d$.

We fix a cohomological weight (k, w) and a subset S of S_p. We will now introduce certain *partial* overconvergent distributions over an admissible affinoid $\mathcal{U}_S \subset \mathcal{X}_S$ containing (k, w). These distributions will allow us to construct p-adic families parametrized by $\mathcal{U}_S$ containing π even when its local components π_v for $v \in S_p \setminus S$ have critical slope, e.g. are supercuspidal, and to attach to them what appears to be a genuinely new kind of p-adic L-function (see §4.4).

We consider the semigroup

$$\Lambda_S = \prod_{v \in S_p \setminus S} \mathrm{GL}_2(F_v) \prod_{v \in S} \mathrm{GL}_2(F_v) \cap \left(F_v^\times \cdot \left(\begin{smallmatrix} \mathcal{O}_v & \mathcal{O}_v \\ \varpi_v \mathcal{O}_v & \mathcal{O}_v^\times \end{smallmatrix}\right)\right) \tag{2.4}$$

and we define the partial Iwahori subgroup $I_S = \Lambda_S \cap G(\mathbb{Z}_p) = \prod_{v \in S_p \setminus S} \mathrm{GL}_2(\mathcal{O}_v) \cdot \prod_{v \in S} I_v$.

Let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup satisfying (1.1) such that $K_p \subset I_S$. In particular, we allow K_v to be the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_v)$ at places $v \in S_p \setminus S$. We let

$$A_{S, \mathcal{U}_S} = A(\mathcal{O}_{F,S}, \mathcal{O}(\mathcal{U}_S)) \otimes_L \bigotimes_{\sigma \in \Sigma_{S_p \setminus S}} L_{k_\sigma, \mathsf{w}}(L) \tag{2.5}$$

be the subspace of $A(\mathcal{O}_F \otimes \mathbb{Z}_p, \mathcal{O}(\mathcal{U}_S))$ consisting of functions which are polynomial of degree at most $(k_\sigma - 2)_{\sigma \in \Sigma_v}$ in the variables $(z_\sigma)_{\sigma \in \Sigma_v}$ for all $v \in S_p \setminus S$. For $\lambda \in \mathcal{U}_S(L)$ we let $A_{S,\lambda} = A_{S,\{\lambda\}}$. For $n \in \mathbb{Z}_{\geq 0}$ we let $A_{S,\lambda,n} = A_{S,\lambda} \cap A_n(\mathcal{O}_F \otimes \mathbb{Z}_p, L)$ and we denote by $D_{S,\lambda,n}$ its topological dual. Finally, we consider the Banach $\mathcal{O}(\mathcal{U}_S)$-module $D_{S,\mathcal{U}_S,n} = D_{S,\lambda,n} \widehat{\otimes}_L \mathcal{O}(\mathcal{U}_S)$ and the compact Fréchet $\mathcal{O}(\mathcal{U}_S)$-module

$$D_{S,\mathcal{U}_S} = \varprojlim D_{S,\mathcal{U}_S,n}.$$

Definition 2.2. We consider the following continuous right action of $\gamma \in I_S$ on $f \in A_{S,\mathcal{U}_S}$:

$$f_{|\gamma}(z) = f\left(\tfrac{az+b}{cz+d}\right)\left\langle\begin{pmatrix} (cz+d) & 0 \\ 0 & \det(\gamma)\cdot(cz+d)^{-1}\end{pmatrix}\right\rangle_{\mathcal{U}_S}, \quad \text{where} \quad z \in \mathcal{O}_F \otimes \mathbb{Z}_p, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right). \tag{2.6}$$

Furthermore, if for all $v \in S$ and all integers $r \geq s$ we let $f_{\left|\left(\begin{smallmatrix} \varpi_v^r & 0 \\ 0 & \varpi_v^s \end{smallmatrix}\right)\right.}(z) = f(\varpi_v^{r-s} z)$.

A direct computation shows that the above actions uniquely extend to a continuous right action of Λ_S on $A_{S,\mathcal{U}_S}$ inducing, via the pairing (2.3), a continuous left action of Λ_S on $D_{S,\mathcal{U}_S}$:

$$(\gamma \cdot \mu)(f) = \mu(f_{|\det(\gamma)^{-1}\cdot\gamma}) \quad \text{for all } \mu \in D_{S,\mathcal{U}_S},\ f \in A_{S,\mathcal{U}_S}. \tag{2.7}$$

For $v \in S$ one has $\left(\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \mu\right)(f) = \mu(f(\varpi_v \cdot))$ for all $\mu \in D_{S,\mathcal{U}_S}$, $f \in A_{S,\mathcal{U}_S}$. Thus the element $\prod_{v \in S}\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Lambda_S$ induces a compact endomorphism on $D_{S,\mathcal{U}_S}$ (see [50, §3.4.12]).

For $\lambda \in \mathcal{U}_S(L)$ we consider the natural Λ_S-equivariant specialization map

$$D_{S,\mathcal{U}_S} \to D_{S,\mathcal{U}_S} \otimes_{\mathcal{O}(\mathcal{U}_S),\lambda} L = D_{S,\lambda}. \tag{2.8}$$

Let (k, w) be a cohomological weight (see Definition 1.1). Using (1.4) and (1.5) one sees that the natural injection $L_{k,\mathsf{w}}(L) \hookrightarrow A_{S,(k,\mathsf{w})}$ is equivariant for the right I_S-action, yielding a natural homomorphism of left I_S-modules

$$\vartheta_S : D_{S,(k,\mathsf{w})} \to L^\vee_{k,\mathsf{w}}(L). \tag{2.9}$$

However, ϑ_S is not Λ_S-equivariant, since for all $v \in S$ and $\mu \in D_{S,(k,\mathsf{w})}$, one has

$$\vartheta_S\left(\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \mu\right) \prod_{\sigma \in \Sigma_v} \sigma(\varpi_v)^{(2-\mathsf{w}-k_\sigma)/2} = \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \vartheta_S(\mu). \tag{2.10}$$

When $S = S_p$ we will drop it from the notations, e.g., $A_{\mathcal{U}} = A_{S_p,\mathcal{U}_{S_p}}$, $D_{\mathcal{U}} = D_{S_p,\mathcal{U}_{S_p}}$, $\Lambda = \Lambda_{S_p}$.

2.3. *Slope decomposition for overconvergent cohomology*

Let $\mathcal{U}$ be an L-affinoid. We consider a compact Fréchet $\mathcal{O}(\mathcal{U})$-module $M = \varprojlim M_n$ such that for all $n \in \mathbb{Z}_{\geq 0}$ the Banach $\mathcal{O}(\mathcal{U})$-module M_n is orthonormalizable, endowed with a compact endomorphism $U : M \to M$, i.e., a system of maps $U_n : M_n \to M_n$ factoring through the natural projections $M_n \to M_{n-1}$ which are compact. For $h \in \mathbb{Q}_{\geq 0}$, if M admits a slope $\leq h$ decomposition with respect to U written as $M^{\leq h} \oplus M^{>h}$ (see [22, Def. 2.3.1]), then $M^{\leq h}$ is a finitely generated Banach $\mathcal{O}(\mathcal{U})$-module.

The following result is a generalization to compact Fréchet $\mathcal{O}(\mathcal{U})$-modules of a well-known proposition about Banach $\mathcal{O}(\mathcal{U})$-modules. If $\mathcal{U}' \subset \mathcal{U}$ is a subaffinoid, we let $U_{\mathcal{U}'}$ denote the endomorphism of $M_{\mathcal{U}'} = M \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U}')$ induced by U.

Proposition 2.3. *Given any $\lambda \in \mathcal{U}(L)$ and any $h \in \mathbb{Q}_{\geq 0}$, there exists an admissible L-affinoid neighborhood $\mathcal{U}' \subset \mathcal{U}$ of λ such that $M_{\mathcal{U}'}$ admits a slope $\leq h$ decomposition with respect to $U_{\mathcal{U}'}$.*

Proof. When $\mathcal{U}$ is a point, i.e., $\mathcal{O}(\mathcal{U})$ is a p-adic field, then this is proven in [50, Lem. 2.3.13].

By [22, Prop. 2.3.3] applied to the orthonormalizable Banach $\mathcal{O}(\mathcal{U})$-module M_n endowed with the compact endomorphism $U_n : M_n \to M_n$, there exists an L-affinoid neighborhood $\mathcal{U}' \subset \mathcal{U}$ of λ such that $M_{n,\mathcal{U}'}$ admits a slope $\leq h$ decomposition with respect to $U_{n,\mathcal{U}'}$ as $M_{n,\mathcal{U}'}^{\leq h} \oplus M_{n,\mathcal{U}'}^{>h}$. Since the Fredholm determinant $\det(1 - x \cdot U_n \mid M_n)$ is independent of n (see [13, Lem. 2.7]), one can take the same $\mathcal{U}'$ for all n. Passing to the limit we obtain a direct sum decomposition $M_{\mathcal{U}'} = \varprojlim M_{n,\mathcal{U}'}^{\leq h} \oplus \varprojlim M_{n,\mathcal{U}'}^{>h}$. It remains to see that this is a slope $\leq h$ decomposition. Since the endomorphism U is compact, applying exactly the same steps as in the proof of [50, Cor. 2.3.4] shows that the natural maps $M_{n,\mathcal{U}'}^{\leq h} \to M_{n-1,\mathcal{U}'}^{\leq h}$ are all isomorphisms, hence $M_{\mathcal{U}'}^{\leq h} \simeq M_{n,\mathcal{U}'}^{\leq h}$ is a finitely generated Banach $\mathcal{O}(\mathcal{U}')$-module for all n. ∎

Let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup satisfying (1.1) and $\Lambda \subset G(\mathbb{A}_f)$ a semigroup containing K. Given a Fréchet $\mathcal{O}(\mathcal{U})$-module M as above, we suppose that it is endowed with a continuous left action of Λ such that (1.2) holds, and we let $\mathcal{M}$ denote the associated sheaf on Y_K (see §1.2).

Proposition 2.4. *Suppose that $x \in \Lambda$ induces a compact endomorphism on M. For $\lambda \in \mathcal{U}(L)$ and $h \in \mathbb{Q}_{\geq 0}$ there is an admissible L-affinoid neighborhood $\mathcal{U}' \subset \mathcal{U}$ of λ such that $\mathrm{H}_c^\bullet(Y_K, \mathcal{M}_{\mathcal{U}'})$ admits a slope $\leq h$ decomposition with respect to the Hecke operator $[KxK]$.*

Proof. By Proposition 2.3 there exists an admissible L-affinoid $\mathcal{U}' \subset \mathcal{U}$ containing λ such that $M_{\mathcal{U}'}$ admits a slope $\leq h$ decomposition with respect to the endomorphism induced by x. By [2, Lem. 2] the cohomology $\mathrm{H}_c^\bullet(Y_K, \mathcal{M})$ can be computed by a bounded complex $\mathrm{R}\Gamma_c^\bullet(K, M_{\mathcal{U}'})$ whose terms are compact Fréchet $\mathcal{O}(\mathcal{U}')$-modules on which the Hecke operator $[KxK]$ acts compactly. Thus $\mathrm{R}\Gamma_c^\bullet(K, M_{\mathcal{U}'})$ admits a slope $\leq h$ decomposition with respect to $[KxK]$ and the proposition follows from [22, Prop. 2.3.2]. ∎

Definition 2.5. Let $(U_i)_{i\in I}$ be a family of $\mathcal{O}(\mathcal{U})$-linear endomorphisms of an $\mathcal{O}(\mathcal{U})$-module M. Given $h_I = (h_i)_{i\in I} \in \mathbb{Q}_{\geq 0}^I$ we let $M^{\leq h_I}$ denote the subspace consisting of elements having slope $\leq h_i$ with respect to U_i for all $i \in I$.

We fix $S \subset S_p$ such that $K_p \subset I_S$. When condition (1.2) is satisfied by (k, w), it is also satisfied by all weights in $\mathcal{U}_S \subset \mathcal{X}_S$ sufficiently small containing (k, w), yielding a sheaf $\mathcal{D}_{S,\mathcal{U}_S}$ on Y_K.

Considering the family $\left(\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)_{v\in S}$ of mutually commuting endomorphisms of $D_{S,\mathcal{U}_S}$ and applying Proposition 2.4 to their product, which is compact, has the following consequence.

Corollary 2.6. *For any $h_S \in \mathbb{Q}_{\geq 0}^S$ and any cohomological weight (k, w) satisfying (1.2), there exists an admissible affinoid $\mathcal{U}_S \subset \mathcal{X}_S$ containing (k, w) such that $\mathrm{H}_c^\bullet(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\leq h_S}$ is a finitely generated $\mathcal{O}(\mathcal{U}_S)$-module, where the slope condition is with respect to the family $(U_{\varpi_v})_{v \in S}$.*

2.4. Classicality

For $S \subset S_p$ and $K \subset G(\mathbb{A}_f)$ as in §2.3, the map resulting from (2.9),

$$\vartheta_S : \mathrm{H}_c^\bullet(Y_K, \mathcal{D}_{S,(k,\mathsf{w})}) \to \mathrm{H}_c^\bullet(Y_K, \mathcal{L}_{k,\mathsf{w}}^\vee(L)), \tag{2.11}$$

intertwines for $v \in S$ the U_{ϖ_v}-action on $\mathrm{H}_c^\bullet(Y_K, \mathcal{D}_{S,(k,\mathsf{w})})$ with the action of the normalized

$$U_{\varpi_v}^\circ = \Big(\prod_{\sigma \in \Sigma_v} \sigma(\varpi_v)^{(k_\sigma + \mathsf{w} - 2)/2}\Big) \cdot U_{\varpi_v} \tag{2.12}$$

on $\mathrm{H}_c^\bullet(Y_K, \mathcal{L}_{k,\mathsf{w}}^\vee(L))$. We remark that the $U_{\varpi_v}^\circ$-action is independent of w.

Theorem 2.7. *Let $h_S = (h_v)_{v \in S} \in \mathbb{Q}_{\geq 0}^S$ be such that $e_v h_v < \min_{\sigma \in \Sigma_v}(k_\sigma - 1)$ for all $v \in S$. Then (2.11) induces an isomorphism of slope $\leq h_S$ subspaces in the sense of Definition 2.5:*

$$\vartheta_S : \mathrm{H}_c^\bullet(Y_K, \mathcal{D}_{S,(k,\mathsf{w})})^{\leq h_S} \xrightarrow{\sim} \mathrm{H}_c^\bullet(Y_K, \mathcal{L}_{k,\mathsf{w}}^\vee(L))^{\leq h_S}, \tag{2.13}$$

where we consider $\{U_{\varpi_v}, v \in S\}$ on the left hand side and $\{U_{\varpi_v}^\circ, v \in S\}$ on the right hand side.

Proof. If $S = S_p$ then this follows from [3, Thm. 8.7]. For a general S we will use a partial version of the locally analytic BGG resolution. For $\sigma \in \Sigma_S$ the image of (k, w) by a generator of the Weyl group of G_σ yields a cohomological weight (k^σ, w), where $k_\sigma^\sigma = 2 - k_\sigma$ and $k_{\sigma'}^\sigma = k_{\sigma'}$ for all $\sigma' \in \Sigma \setminus \{\sigma\}$. The restriction to $A_{S,(k,\mathsf{w})}$ of the map introduced in [50, Prop. 3.2.11] yields an I_S-equivariant map $\Theta_{S,\sigma} : A_{S,(k,\mathsf{w})} \to A_{S,(k^\sigma,\mathsf{w})}$, whose dual $\Theta_{S,\sigma}^\vee : D_{S,(k^\sigma,\mathsf{w})} \to D_{S,(k,\mathsf{w})}$ is I_S-equivariant as well. From *loc. cit.* the cokernel of the map

$$\sum_{\sigma \in \Sigma_S} \Theta_{S,\sigma}^\vee : \bigoplus_{\sigma \in \Sigma_S} D_{S,(k^\sigma,\mathsf{w})} \to D_{S,(k,\mathsf{w})} \tag{2.14}$$

is given by the continuous dual of the subspace of locally algebraic functions in $A_{S,(k,\mathsf{w})}$. The proof of the theorem then proceeds exactly as in [3, Thm. 8.7] using the fact that the finite slope parts of the cohomology of algebraic and locally algebraic distributions coincide (see [50, Lem.4.3.8]), and the following computation for $v \in S$ and $\mu \in D_{S,(k^\sigma,\mathsf{w})}$:

$$\Theta_{S,\sigma}^\vee\big(\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \mu\big) = \begin{cases} \varpi_v^{-k_\sigma+1} \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \Theta_{S,\sigma}^\vee(\mu) & \text{if } \sigma \in \Sigma_v, \\ \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \Theta_{S,\sigma}^\vee(\mu) & \text{if } \sigma \in \Sigma_S \setminus \Sigma_v. \end{cases} \qquad \blacksquare$$

Remark 2.8. The map (2.14) is a locally-analytic analogue of Lepowsky's generalized BGG resolution for G relative to the parabolic subgroup given by GL_2 at the places outside S and the upper triangular matrices at the places in S (see [29, Thm. 4.3]). On the other hand, in [31] the author uses the general Lepowsky generalized BGG resolution to prove classicality results for his construction of eigenvarieties for reductive algebraic groups whose real points are compact modulo their center.

2.5. Axiomatic control and freeness

We will generalize a strategy due to Hida, establishing an exact control theorem and freeness results of the overconvergent cohomology and the Hecke algebra acting on it. The motivating principle is that while, in general, one cannot establish torsion-freeness for cohomology, this can be done when the cohomology is supported in a single degree. We begin with an adaptation of [24, Lem. 7.1] to the setting of analytic families.

Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and let $\mathcal{C}$ be a subcategory of the category of A-modules such that if M is in $\mathcal{C}$ and I is an ideal of A then $M \otimes_A A/I$ is also in $\mathcal{C}$. Henceforth $\mathcal{H}^\bullet$ will denote a cohomology functor on $\mathcal{C}$ such that

(i) $\mathcal{H}^\bullet$ sends short exact sequences to long exact ones, and

(ii) $\mathcal{H}^\bullet(M)$ is a finitely generated A-module for every M in $\mathcal{C}$.

Lemma 2.9. *Suppose M in $\mathcal{C}$ is A-flat and $\mathcal{H}^\bullet(M \otimes_A A/\mathfrak{m})$ is supported in degree d. Then*

(i) *$\mathcal{H}^\bullet(M)$ is supported in degree d and $\mathcal{H}^d(M) \otimes_A A/\mathfrak{m} \simeq \mathcal{H}^d(M \otimes_A A/\mathfrak{m})$,*

(ii) *$\mathcal{H}^d(M)$ is A-torsion-free; in particular, if A is a discrete valuation ring then $\mathcal{H}^d(M)$ is a free A-module of rank $\dim_{A/\mathfrak{m}} \mathcal{H}^d(M \otimes_A A/\mathfrak{m})$.*

Proof. Let $T_1, \dots, T_k$ be a regular sequence in A and consider the filtration

$$I_0 = 0 \subset I_1 = (T_1) \subset I_2 = (T_1, T_2) \subset \cdots \subset I_k = (T_1, \dots, T_k) = \mathfrak{m}.$$

For $i \neq d$ we will prove by descending induction on r that $\mathcal{H}^i(M \otimes_A A/I_r) = 0$. The base case $r = k$ follows from the hypothesis. By flatness, we have a short exact sequence

$$0 \to M \otimes A/I_{r-1} \xrightarrow{\cdot T_r} M \otimes A/I_{r-1} \to M \otimes A/I_r \to 0.$$

The corresponding long exact sequence yields an injection

$$\mathcal{H}^i(M \otimes A/I_{r-1}) \otimes_{A/I_{r-1}} A/I_r \hookrightarrow \mathcal{H}^i(M \otimes A/I_r).$$

By the inductive hypothesis we have $\mathcal{H}^i(M \otimes A/I_{r-1}) \otimes_{A/I_{r-1}} A/I_r = 0$. Since the A/I_{r-1}-module $\mathcal{H}^i(M \otimes A/I_{r-1})$ is finitely generated, Nakayama's lemma yields $\mathcal{H}^i(M \otimes A/I_{r-1}) = 0$. Finally, let $i = d$. The long exact sequence and the vanishing result in degree $d+1$ yield

$$\mathcal{H}^d(M \otimes A/I_{r-1}) \otimes_{A/I_{r-1}} A/I_r \simeq \mathcal{H}^d(M \otimes A/I_r),$$

and concatenating these isomorphisms for $1 \le r \le k$ yields part (i). For (ii), note that it suffices to show that $\mathcal{H}^d(M)$ has no T-torsion where the non-zero divisor T can be assumed to be T_1. The arguments of (i) imply, by descending induction on r, that multiplication by T_r is injective on $\mathcal{H}^d(M \otimes A/I_{r-1})$. The case $r = 1$ shows that multiplication by T is injective on $\mathcal{H}^d(M)$ as desired. Finally, when A is a discrete valuation ring the module $\mathcal{H}^d(M)$ is free of rank

$$\dim_{A/\mathfrak{m}} \mathcal{H}^d(M) \otimes A/\mathfrak{m} = \dim_{A/\mathfrak{m}} \mathcal{H}^d(M \otimes A/\mathfrak{m}). \qquad \blacksquare$$

We will now apply the abstract paradigm of Lemma 2.9 to the setting of overconvergent sheaves and Hecke algebras. The rigid localization of an admissible L-affinoid $\mathcal{U} \subset \mathcal{X}$ at a point $\lambda \in \mathcal{U}(L)$ is defined as $\mathcal{O}(\mathcal{U})_\lambda = \varinjlim_{\lambda \in \mathcal{U}' \subset \mathcal{U}} \mathcal{O}(\mathcal{U}')$, where the limit is taken over all admissible open subaffinoids $\mathcal{U}'$ in $\mathcal{U}$ containing λ. It is a local ring which contains the algebraic localization of $\mathcal{O}(\mathcal{U})$ at the maximal ideal $\mathfrak{m}_\lambda$ at λ. For an $\mathcal{O}(\mathcal{U})$-module $\mathcal{F}$ we let

$$\mathcal{F}_\lambda = \mathcal{F} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U})_\lambda = \varinjlim_{\lambda \in \mathcal{U}' \subset \mathcal{U}} (\mathcal{F} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U}')). \tag{2.15}$$

Lemma 2.10. *Let $\mathcal{F}$ and $\mathcal{G}$ be finitely generated $\mathcal{O}(\mathcal{U})$-modules. If there exists an $\mathcal{O}(\mathcal{U})_\lambda$-linear isomorphism $\mathcal{F}_\lambda \xrightarrow{\sim} \mathcal{G}_\lambda$, then there exists an admissible open subaffinoid $\mathcal{U}' \subset \mathcal{U}$ containing λ such that $\mathcal{F} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U}') \simeq \mathcal{G} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U}')$. In particular, if $\mathcal{F}_\lambda$ is free over $\mathcal{O}(\mathcal{U})_\lambda$, then there exists an admissible open affinoid $\mathcal{U}' \subset \mathcal{U}$ containing λ such that $\mathcal{F} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U}')$ is $\mathcal{O}(\mathcal{U}')$-free.*

Proof. Suppose $f_1, \dots, f_M$ generate $\mathcal{F}$ as an $\mathcal{O}(\mathcal{U})$-module. Let $\phi : \mathcal{F}_\lambda \to \mathcal{G}_\lambda$ be the given isomorphism. The map ϕ is uniquely determined by the elements $\phi(f_1), \dots, \phi(f_M) \in \mathcal{G}_\lambda$. Let $\mathcal{U}'$ be an admissible neighborhood of λ in $\mathcal{U}$ over which the elements $\phi(f_i)$ are all defined. By $\mathcal{O}(\mathcal{U}')$-linearity we get a homomorphism $\phi_{\mathcal{U}'} : \mathcal{F} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U}') \to \mathcal{G} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U}')$. Since $\mathcal{O}(\mathcal{U}')$ is noetherian, $\ker(\phi_{\mathcal{U}'})$ and $\operatorname{coker}(\phi_{\mathcal{U}'})$ are finitely generated $\mathcal{O}(\mathcal{U}')$-modules, whose localizations at λ vanish. It follows that, by shrinking $\mathcal{U}'$ further, one may ensure that generators for $\ker(\phi_{\mathcal{U}'})$ and $\operatorname{coker}(\phi_{\mathcal{U}'})$ vanish, proving that $\phi_{\mathcal{U}'}$ is an isomorphism. The last statement follows by taking $\mathcal{G}$ to be the free $\mathcal{O}(\mathcal{U})$-module of rank equal to the rank of $\mathcal{F}_\lambda$. $\blacksquare$

Our final abstract lemma is an application of Lemma 2.9 to rigid spaces. Fix $S \subset S_p$.

Lemma 2.11. *Let $A = \mathcal{O}(\mathcal{U})_\lambda$, where $\mathcal{U} \subset \mathcal{X}_S$ is an admissible neighborhood of λ, and suppose that $\mathcal{H}^\bullet(\mathcal{D}_{S,\lambda})$ is supported in degree d. Then, after possibly shrinking $\mathcal{U}$, $\mathcal{H}^d((\mathcal{D}_{S,\mathcal{U}})_\lambda)$ is $\mathcal{O}(\mathcal{U})_\lambda$-free.*

Proof. Since $(\mathcal{D}_{S,\mathcal{U}})_\lambda$ is $\mathcal{O}(\mathcal{U})_\lambda$-flat, Lemma 2.9(i) yields an isomorphism of $\mathcal{O}(\mathcal{U})_\lambda$-modules $\mathcal{H}^d((\mathcal{D}_{S,\mathcal{U}})_\lambda) \otimes_{\mathcal{O}(\mathcal{U})_\lambda} L \xrightarrow{\sim} \mathcal{H}^d(\mathcal{D}_{S,\lambda})$. It then follows from Nakayama's lemma that the module $\mathcal{H}^d((\mathcal{D}_{S,\mathcal{U}})_\lambda)$ can be generated over $\mathcal{O}(\mathcal{U})_\lambda$ by $r = \dim_L \mathcal{H}^d(\mathcal{D}_{S,\lambda})$ generators $m_1, \dots, m_r$.

Suppose there exists a relation $f_1 m_1 + \cdots + f_r m_r = 0$ with $f_1, \ldots, f_r \in \mathcal{O}(\mathcal{U})_\lambda$ not all 0. Then, for example, $f_1 \in \mathcal{O}(\mathcal{U}') \setminus \{0\}$ for some closed polydisc $\mathcal{U}' \subset \mathcal{U}$ containing λ. Since f_1 is analytic, there exists a 1-dimensional disk $\mathcal{V} \subset \mathcal{U}'$ such that the image of f_1 in $\mathcal{O}(\mathcal{V})$ is non-zero, yielding a dependence relation between $m_1, \ldots, m_r$ over the discrete valuation ring $\mathcal{O}(\mathcal{V})_\lambda$. This contradicts Lemma 2.9(ii) since the $\mathcal{O}(\mathcal{V})_\lambda$-rank of $\mathcal{H}^d(\mathcal{D}_{S,\mathcal{V}} \otimes \mathcal{O}(\mathcal{V})_\lambda)$ would be $\leq r-1$. ∎

2.6. *Etaleness at non-critical points*

Our main interest is in compactly supported cohomology, which is not supported in middle degree. To account for this, we will localize at the maximal ideal defined by $\tilde{\pi}$ and will obtain etaleness at non-critical points in both full and partial p-adic families.

Let $S \subset S_p$ and let $\tilde{\pi}_S = (\pi, \{\nu_v\}_{v \in S})$ be a regular S-refinement of π (see Definition 1.3).

Henceforth we let $K = K(\tilde{\pi}_S, \mathfrak{n})$ (see Definition 1.7). By Corollary 2.6 for any $h_S \in \mathbb{Q}^S_{\geq 0}$ there exists an L-affinoid neighborhood $\mathcal{U}_S$ of (k, w) in $\mathcal{X}_S$ such that $\mathrm{H}^d_c(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\leq h_S}$ is a finitely generated $\mathcal{O}(\mathcal{U}_S)$-module. Since $\mathcal{O}(\mathcal{U}_S)$ is noetherian, the $\mathcal{O}(\mathcal{U}_S)$-algebra $\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S}$, generated by the image of $\tilde{\mathbb{T}}_S$ (see Definition 1.6) in $\mathrm{End}_{\mathcal{O}(\mathcal{U}_S)}(\mathrm{H}^d_c(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\leq h_S})$, is finite. For *any* subaffinoid $\mathcal{U}'_S \subset \mathcal{U}_S$ containing (k, w) consider the maximal ideal of $\tilde{\mathbb{T}}_S \otimes_{E,\iota_p} \mathcal{O}(\mathcal{U}'_S)$ generated by $\mathfrak{m}_{\tilde{\pi}_S}$ and by $\mathfrak{m}_{(k,\mathsf{w})}$ and, by an abuse of notation, let $\mathfrak{m}_{\tilde{\pi}_S}$ denote its image in $\mathbb{T}^{\leq h_S}_{S,\mathcal{U}'_S}$, as well as the corresponding maximal ideal of the rigid analytic localization $(\mathbb{T}^{\leq h_S}_{S,\mathcal{U}'_S})_{(k,\mathsf{w})}$ (see (2.15)). The rigid localization of $\mathrm{Sp}(\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S})$ at the point $\tilde{\pi}_S$ corresponding to the maximal ideal $\mathfrak{m}_{\tilde{\pi}_S} \subset \mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S}$ is given by the limit

$$(\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S})_{\tilde{\pi}_S} \simeq \varinjlim_{\tilde{\pi}_S \in \mathcal{V}} \mathcal{O}(\mathcal{V})$$

over all admissible neighborhoods $\mathcal{V}$ of $\tilde{\pi}_S$ in $\mathrm{Sp}(\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S})$. The weight map $\kappa : \mathrm{Sp}(\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S}) \to \mathcal{U}_S$ induces a ring homomorphism $\mathcal{O}(\mathcal{U}_S)_{(k,\mathsf{w})} \to (\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S})_{\tilde{\pi}_S}$. More generally, given a $\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S}$-module $\mathcal{F}$ we let $\mathcal{F}_{\tilde{\pi}_S} = \mathcal{F} \otimes_{\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S}} (\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S})_{\tilde{\pi}_S}$. The natural map $\mathcal{F}_{(k,\mathsf{w})} \to \mathcal{F}_{\tilde{\pi}_S}$ induces an isomorphism

$$(\mathcal{F}_{(k,\mathsf{w})})_{\mathfrak{m}_{\tilde{\pi}_S}} \xrightarrow{\sim} \mathcal{F}_{\tilde{\pi}_S}. \tag{2.16}$$

Definition 2.12. We say that $\tilde{\pi}_S$ is *non-critical* if the $\mathfrak{m}_{\tilde{\pi}_S}$-localization

$$\vartheta_S : \mathrm{H}^\bullet_c(Y_K, \mathcal{D}_{S,(k,\mathsf{w})})_{\mathfrak{m}_{\tilde{\pi}_S}} \to \mathrm{H}^\bullet_c(Y_K, \mathcal{L}^\vee_{k,\mathsf{w}}(L))_{\mathfrak{m}_{\tilde{\pi}_S}}$$

of (2.11) is an isomorphism. When $S = S_p$ we let $\mathbb{T}_\mathcal{U} = \mathbb{T}_{S_p,\mathcal{U}_{S_p}}$ and will say that $\tilde{\pi}$ is *non-critical*.

Theorem 2.7 applied to $h_S = (h_{\tilde{\pi}_v})_{v \in S}$ (see Definition 1.8) has the following direct consequence.

Corollary 2.13. *If $\tilde{\pi}_S$ has non-critical slope, then $\tilde{\pi}_S$ is non-critical.*

The main result of this section is the following control and freeness theorem for compactly supported cohomology. Fix a character ϵ of $\{\pm 1\}^{\Sigma}$.

Theorem 2.14. *Suppose that $\tilde{\pi}_S$ is non-critical. Then, after possibly shrinking $\mathcal{U}_S \subset \mathcal{X}_S$:*

(i) $\mathrm{H}_c^{\bullet}(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon}_{\tilde{\pi}_S}$ *is a free $\mathcal{O}(\mathcal{U}_S)_{(k,\mathrm{w})}$-module of rank 1 and is supported in degree d.*

(ii) *The weight map $\kappa : \mathrm{Sp}(\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S}) \to \mathcal{U}_S$ is etale at $\tilde{\pi}_S$, i.e., there exists an irreducible component $\mathcal{V}_S$ of $\mathrm{Sp}(\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S})$ containing $\tilde{\pi}_S$ such that $\kappa : \mathcal{V}_S \to \mathcal{U}_S$ is an isomorphism of affinoids. Moreover $\mathrm{H}_c^{\bullet}(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon,\leq h_S} \otimes_{\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S}} \mathcal{O}(\mathcal{V}_S)$ is supported in degree d and is free of rank 1 over $\mathcal{O}(\mathcal{U}_S)$.*

(iii) *For any cohomological weight $\lambda \in \mathcal{U}_S$, $\kappa^{-1}(\lambda) \in \mathcal{V}_S$ corresponds to a non-critically S-refined weight λ cuspidal automorphic representation $\tilde{\pi}_{\lambda,S}$ of $G(\mathbb{A})$.*

Proof. (i) By non-criticality and cuspidality,

$$\mathrm{H}_c^{\bullet}(Y_K, \mathcal{D}_{S,(k,\mathrm{w})})^{\epsilon}_{\mathfrak{m}_{\tilde{\pi}_S}} \xrightarrow{\sim} \mathrm{H}_c^{\bullet}(Y_K, \mathcal{L}^{\vee}_{k,\mathrm{w}}(L))^{\epsilon}_{\mathfrak{m}_{\tilde{\pi}_S}}$$

is supported in degree d and has dimension equal to $\dim(\pi_f^K)_{\mathfrak{m}_{\tilde{\pi}_S}} = 1$ (see Lemma 1.4). Applying Lemmas 2.9(i) and 2.11 to the cohomology functor $\mathcal{H}^{\bullet}(-) = \mathrm{H}_c^{\bullet}(Y_K, -)^{\epsilon}_{\mathfrak{m}_{\tilde{\pi}_S}}$ shows that

$$\mathrm{H}_c^{\bullet}(Y_K, (\mathcal{D}_{S,\mathcal{U}_S})_{(k,\mathrm{w})})^{\epsilon}_{\mathfrak{m}_{\tilde{\pi}_S}} \xrightarrow{\sim} \mathrm{H}_c^{\bullet}(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon}_{\tilde{\pi}_S}$$

is also supported in degree d and is $\mathcal{O}(\mathcal{U}_S)_{(k,\mathrm{w})}$-free of rank 1 (see (2.16))

(ii) It follows from (i) that the natural map $\mathcal{O}(\mathcal{U}_S)_{(k,\mathrm{w})} \to (\mathbb{T}_{S,\mathcal{U}_S})_{\tilde{\pi}_S}$ is an isomorphism and that $\mathrm{H}_c^{\bullet}(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon}_{\tilde{\pi}_S}$ has rank 1 and is supported in degree d. Both claims then follow straightforwardly using Lemma 2.10.

(iii) After shrinking $\mathcal{U}_S$, we may assume that any cohomological $\lambda \in \mathcal{U}_S \setminus \{(k,\mathrm{w})\}$ is non-constant, i.e. $k_\lambda \neq 2t$, and such that $e_v h_{\tilde{\pi}_v} < \min_{\sigma \in \Sigma_v}(k_{\lambda,\sigma} - 1)$ for all $v \in S$. Let $\mathfrak{m} \subset \tilde{\mathbb{T}}_S$ be the maximal ideal corresponding to the map $\tilde{\mathbb{T}}_S \to \mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S} \to \mathcal{O}(\mathcal{V}_S) \to L$ induced by $\kappa^{-1}(\lambda)$. Using the same abuse of notation for $\mathfrak{m}$ as we did for $\mathfrak{m}_{\tilde{\pi}_S}$ (see the paragraph above (2.16)), (ii) yields an isomorphism

$$\mathrm{H}_c^{\bullet}(Y_K, (\mathcal{D}_{S,\mathcal{U}_S})_\lambda)^{\epsilon}_{\mathfrak{m}} \xrightarrow{\sim} \mathrm{H}_c^{\bullet}(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon,\leq h_S} \otimes_{\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S}} \mathcal{O}(\mathcal{V}_S)_{\kappa^{-1}(\lambda)} = \mathrm{H}_c^{\bullet}(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon}_{\kappa^{-1}(\lambda)}$$

of free $\mathcal{O}(\mathcal{U}_S)_\lambda$-modules of rank 1 supported in degree d. By using the long exact sequences from the proof of Lemma 2.9 for the functor $\mathcal{H}^{\bullet}(-) = \mathrm{H}_c^{\bullet}(Y_K, -)^{\epsilon}_{\mathfrak{m}}$ one can perform this time an *ascending* induction showing that the L-vector space

$\mathrm{H}_c^\bullet(Y_K, \mathcal{D}_{S,\lambda})_{\mathfrak{m}}^\epsilon$ is 1-dimensional and supported in degree d. The non-critical slope assumption on λ implies, via Theorem 2.7, that

$$\mathrm{H}_c^\bullet(Y_K, \mathcal{D}_{S,\lambda})_{\mathfrak{m}}^\epsilon \xrightarrow{\sim} \mathrm{H}_c^\bullet(Y_K, \mathcal{L}_\lambda^\vee(L))_{\mathfrak{m}}^\epsilon$$

is concentrated in degree d and is 1-dimensional, and therefore $\kappa^{-1}(\lambda)$ corresponds to a non-critically S-refined automorphic representation $\tilde{\pi}_{\lambda,S}$ of $G(\mathbb{A})$ of weight λ (see Corollary 2.13). Finally, π_λ is necessarily cuspidal as otherwise it would necessarily be an Eisenstein series and contribute to the usual cohomology in all degrees between d and $2d-1$, hence in all degrees between 1 and d of the compactly supported one (see, for example, [34]). ■

Remark 2.15. Our construction of partial eigenvarieties will be crucially used in §7 for the calculations of higher order derivatives of p-adic L-functions. In the literature we find other examples of the use of partial eigenvarieties in arithmetic applications. For example in [27] the authors use partial eigenvarieties for definite quaternion algebras to study the parity conjecture for Hilbert modular forms. Another example is [15] where partial eigenvarieties for definite unitary groups are used to attach Galois representations to conjugate self-dual automorphic representations of GL_n over CM fields.

3. Automorphic symbols and p-adic distributions

In this section we use the automorphic cycles introduced in [18] to construct, for any ideal $\mathfrak{f} \subset \mathcal{O}_F$ supported in $S \subset S_p$, natural evaluations on the cohomology of the Hilbert modular variety with general coefficients, and show that they satisfy certain distribution relations as $\mathfrak{f}$ varies. When applied to a finite slope U_p-eigenclass in overconvergent cohomology, the construction yields a distribution of controlled growth on $\mathrm{Gal}_{p\infty}$ in a canonical way, i.e., independent of choices of uniformizers or representatives in idele class groups. This will allow us in §4 to attach p-adic L-functions to nearly finite slope families of Hilbert cusp forms, generalizing [2].

When the support of the ideal $\mathfrak{f}$ is small, a case not previously considered, the relations established when S varies will be used in §4.3 to construct partially 'improved' p-adic L-functions.

For $S \subsetneq S_p$ our construction allows us to attach to a family $\mathcal{V}_S \xrightarrow{\sim} \mathcal{U}_S$ containing a non-critical S-refinement $\tilde{\pi}_S$ (see Theorem 2.14), a rather mysterious $\mathcal{O}(\mathcal{U}_S)$-valued distribution on $\mathrm{Gal}_{S\infty}$.

3.1. *Fundamental classes and automorphic cycles*

For an ideal $\mathfrak{f}$ the d-dimensional open manifold $X_{\mathfrak{f}}^+ = E(\mathfrak{f}) \setminus F_\infty^{\times +}$ is orientable with orientation induced by an orientation on $F_\infty^{\times +}$. The Borel–Moore homology $\mathrm{H}_i^{\mathrm{BM}}(X_{\mathfrak{f}}^+)$ can be computed as the relative homology $\mathrm{H}_i(\overline{X}_{\mathfrak{f}}^+, \overline{X}_{\mathfrak{f}}^+ \setminus X_{\mathfrak{f}}^+)$ where $\overline{X}_{\mathfrak{f}}^+$ is the two-point compactification of $X_{\mathfrak{f}}^{+}$ [18, Def. 1.5]. When $i = d$, $\mathrm{H}_d^{\mathrm{BM}}(X_{\mathfrak{f}}^{+}) \simeq \mathbb{Z}$ and one can choose a fundamental class $\theta_{\mathfrak{f}} = [X_{\mathfrak{f}}^{+}] \in \mathrm{H}_d^{\mathrm{BM}}(X_{\mathfrak{f}}^{+})$ in a compatible way as $\mathfrak{f}$ varies, such that for all $\mathfrak{f} \mid \mathfrak{f}'$ the finite map $\pi : X_{\mathfrak{f}'}^{+} \to X_{\mathfrak{f}}^{+}$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_d^{\mathrm{BM}}(X_{\mathfrak{f}}^{+}) & \xrightarrow{\ \sim\ } & \mathbb{Z} \\ {\scriptstyle \pi^*}\downarrow\uparrow{\scriptstyle \pi_*} & & {\scriptstyle \mathrm{id}}\downarrow\uparrow{\scriptstyle [E(\mathfrak{f}):E(\mathfrak{f}')]\cdot} \\ \mathrm{H}_d^{\mathrm{BM}}(X_{\mathfrak{f}'}^{+}) & \xrightarrow{\ \sim\ } & \mathbb{Z} \end{array}$$

In top degree the map $\mathrm{H}_d^{\mathrm{BM}}(X_{\mathfrak{f}}^{+}) \to \mathrm{H}_c^d(X_{\mathfrak{f}}^{+})^{\vee}$ sending θ to $c \mapsto c \cap \theta = \int_\theta c$ is an isomorphism. When $d \geq 2$ one has $\mathrm{H}_d^{\mathrm{BM}}(X_{\mathfrak{f}}^{+}) = \mathrm{H}_d(\overline{X}_{\mathfrak{f}}^{+}, \overline{X}_{\mathfrak{f}}^{+} \setminus X_{\mathfrak{f}}^{+}) \simeq \mathrm{H}_d(\overline{X}_{\mathfrak{f}}^{+})$. We let

$$X_{\mathfrak{f}} = \mathbb{A}_F^{\times}/F^{\times}U(\mathfrak{f}).$$

For any $\eta \in \mathbb{A}_{F,f}^{\times}$ representing a class $[\eta] \in \mathscr{C}\ell_F^{+}(\mathfrak{f}) \simeq \pi_0(X_{\mathfrak{f}})$ we denote by $X_{\mathfrak{f}}[\eta]$ the connected component of $X_{\mathfrak{f}}$ attached to $[\eta]$. The map $X_{\mathfrak{f}}^{+} \xrightarrow{\cdot\eta} X_{\mathfrak{f}}[\eta]$ yields an isomorphism $\mathrm{H}_d^{\mathrm{BM}}(X_{\mathfrak{f}}^{+}) \xrightarrow{\eta_*} \mathrm{H}_d^{\mathrm{BM}}(X_{\mathfrak{f}}[\eta])$ independent of the choice of the representative η. Hence $\theta_{\mathfrak{f}}$ yields a fundamental class

$$\theta_{\mathfrak{f},[\eta]} = \eta_*(\theta_{\mathfrak{f}}) \in \mathrm{H}_d^{\mathrm{BM}}(X_{\mathfrak{f}}[\eta]). \tag{3.1}$$

In this entire section $S \subset S_p$, $\mathfrak{f} \subset \mathcal{O}_F$ is an ideal supported in S and $K \subset G(\mathbb{A}_f)$ is an open compact subgroup containing $\left(\begin{smallmatrix} U(\mathfrak{f}) & \widehat{\mathcal{O}}_F \\ 0 & 1 \end{smallmatrix}\right)$ such that the image K_p of K in $G(\mathbb{Q}_p)$ is contained in I_S.

We define the *automorphic cycle of level* $\mathfrak{f}$ as

$$C_{\varpi_{\mathfrak{f}},K} : X_{\mathfrak{f}} \to Y_K, \qquad y \mapsto \left[\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \varpi_{\mathfrak{f}} & 1_{\mathfrak{f}} \\ 0 & 1 \end{smallmatrix}\right)\right],$$

which is a well-defined, continuous, finite map (see [19, Cor. 1.23]). When K is clear from context it will be dropped from the notation of the automorphic cycle.

3.2. Evaluations

We will axiomatize and generalize some constructions from [18] and [2] on the evaluations of automorphic cycles on the cohomology of the Hilbert modular varieties.

Let M be a left Λ_S-module (see (2.4)) and let $\mathcal{M}$ be the sheaf on Y_K attached as in §1.2 to M on which K acts via $K_p \subset I_S \subset \Lambda_S$. The pullback by $C_{\varpi_{\mathfrak{f}}}$ induces a homomorphism

$$C_{\varpi_{\mathfrak{f}}}^{*} : \mathrm{H}_c^d(Y_K, \mathcal{M}) \to \mathrm{H}_c^d(X_{\mathfrak{f}}, C_{\varpi_{\mathfrak{f}}}^{*}\mathcal{M}). \tag{3.2}$$

Since $\left(\begin{smallmatrix} \varpi_{\mathfrak{f}} & 1_{\mathfrak{f}} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} u & (u-1)1_{\mathfrak{f}}\varpi_{\mathfrak{f}}^{-1} \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \varpi_{\mathfrak{f}} & 1_{\mathfrak{f}} \\ 0 & 1 \end{smallmatrix}\right)$ one sees that $C_{\varpi_{\mathfrak{f}}}^{*}\mathcal{M}$ is the sheaf of locally constant sections of the local system

$$C_{\varpi_{\mathfrak{f}}}^{*}M = F^{\times}\backslash(\mathbb{A}_F^{\times} \times M)/U(\mathfrak{f}) \to X_{\mathfrak{f}}, \tag{3.3}$$

where $\xi(y,m)u = \left(\xi y u, \left(\begin{smallmatrix} u & (u-1)1_{\mathfrak{f}}\varpi_{\mathfrak{f}}^{-1} \\ 0 & 1 \end{smallmatrix}\right)^{-1} \cdot m\right)$ for $\xi \in F^{\times}$, $y \in \mathbb{A}_F^{\times}$, $m \in M$ and $u \in U(\mathfrak{f})$.

Let $\mathcal{M}_{\mathfrak{f}}$ be the sheaf of locally constant sections of the local system

$$F^\times\backslash(\mathbb{A}_F^\times \times M)/U(\mathfrak{f}) \to X_{\mathfrak{f}}, \tag{3.4}$$

where $\xi(y,m)u = \left(\xi y u, \left(\begin{smallmatrix} u & 0 \\ 0 & 1\end{smallmatrix}\right)^{-1} \cdot m\right)$ for $\xi \in F^\times$, $y \in \mathbb{A}_F^\times$, $m \in M$ and $u \in U(\mathfrak{f})$.

Since the image of $\left(\begin{smallmatrix} \varpi_{\mathfrak{f}} & 1_{\mathfrak{f}} \\ 0 & 1\end{smallmatrix}\right)$ in $G(\mathbb{Q}_p)$ belongs to Λ_S, one can consider the map

$$\mathbb{A}_F^\times \times M \to \mathbb{A}_F^\times \times M, \quad (y,m) \mapsto \left(y, \left(\begin{smallmatrix} \varpi_{\mathfrak{f}} & 1_{\mathfrak{f}} \\ 0 & 1\end{smallmatrix}\right) \cdot m\right),$$

which sends the local system (3.3) to the one from (3.4). The resulting homomorphism of sheaves $\mathrm{tw}_{\varpi_{\mathfrak{f}}} : C_{\varpi_{\mathfrak{f}}}^* \mathcal{M} \to \mathcal{M}_{\mathfrak{f}}$ over $X_{\mathfrak{f}}$ yields a homomorphism of cohomology groups

$$\mathrm{tw}_{\varpi_{\mathfrak{f}}} : \mathrm{H}_c^d(X_{\mathfrak{f}}, C_{\varpi_{\mathfrak{f}}}^* \mathcal{M}) \to \mathrm{H}_c^d(X_{\mathfrak{f}}, \mathcal{M}_{\mathfrak{f}}). \tag{3.5}$$

Let $M_{E(\mathfrak{f})}$ denote the $E(\mathfrak{f})$-coinvariants of M. Consider the sheaf $\mathcal{M}_{E(\mathfrak{f})}$ attached to the local system $F^\times\backslash(\mathbb{A}_F^\times \times M_{E(\mathfrak{f})})/U(\mathfrak{f})$ with $\xi(y,m)u = \left(\xi y u, \left(\begin{smallmatrix} u & \\ & 1\end{smallmatrix}\right)^{-1} \cdot m\right)$. There is a natural map

$$\mathrm{coinv}_{\mathfrak{f}} : \mathrm{H}_c^d(X_{\mathfrak{f}}, \mathcal{M}_{\mathfrak{f}}) \to \mathrm{H}_c^d(X_{\mathfrak{f}}, \mathcal{M}_{E(\mathfrak{f})}).$$

Trivializing the sheaf $\mathcal{M}_{E(\mathfrak{f})}$ requires choosing a representative $\eta \in \mathbb{A}_{F,f}^\times$ of $[\eta] \in \mathcal{C}\ell_F^+(\mathfrak{f})$. Then

$$\mathrm{triv}_\eta : X_{\mathfrak{f}}[\eta] \times M_{E(\mathfrak{f})} \to (\mathcal{M}_{E(\mathfrak{f})})_{|X_{\mathfrak{f}}[\eta]}, \quad (a\eta u_\infty u, m) \mapsto \left(a\eta u_\infty u, \left(\begin{smallmatrix} u^{-1} & \\ & 1\end{smallmatrix}\right) \cdot m\right),$$

is an isomorphism of local systems yielding the desired trivialization map

$$\mathrm{triv}_\eta^* : \mathrm{H}_c^d(X_{\mathfrak{f}}[\eta], \mathcal{M}_{E(\mathfrak{f})}) \to \mathrm{H}_c^d(X_{\mathfrak{f}}[\eta], \mathbb{Z}) \otimes M_{E(\mathfrak{f})}.$$

Capping with $\theta_{\mathfrak{f},[\eta]}$ from (3.1) yields an isomorphism $\mathrm{H}_c^d(X_{\mathfrak{f}}[\eta], \mathbb{Z}) \otimes M_{E(\mathfrak{f})} \simeq M_{E(\mathfrak{f})}$. We define

$$\mathrm{ev}_{\varpi_{\mathfrak{f}}}^\eta(M) = (- \cap \theta_{\mathfrak{f},[\eta]}) \circ \mathrm{triv}_\eta^* \circ \mathrm{coinv}_{\mathfrak{f}} \circ \mathrm{tw}_{\varpi_{\mathfrak{f}}} \circ C_{\varpi_{\mathfrak{f}},K}^* : \mathrm{H}_c^d(Y_K, \mathcal{M}) \to M_{E(\mathfrak{f})}. \tag{3.6}$$

Where M is clear from context we will drop it from the notation of the evaluation map.

Lemma 3.1. *The evaluation maps $\mathrm{ev}_{\varpi_{\mathfrak{f}}}^\eta$ are covariant in M, in the sense that if $\vartheta : M \to M'$ is a morphism of left Λ_S-modules then $\vartheta \circ \mathrm{ev}_{\varpi_{\mathfrak{f}}}^\eta(M) = \mathrm{ev}_{\varpi_{\mathfrak{f}}}^\eta(M') \circ \vartheta$.*

It follows from (2.12) that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_c^d(Y_K, \mathcal{D}_{S,(k,\mathrm{w})}) & \xrightarrow{\varpi_{\mathfrak{f}}^{((2-\mathrm{w})t-k)/2} \cdot \mathrm{ev}_{\varpi_{\mathfrak{f}}}^\eta(D_{S,(k,\mathrm{w})})} & (D_{S,(k,\mathrm{w})})_{E(\mathfrak{f})} \\ \downarrow \vartheta_S & & \downarrow \vartheta_S \\ \mathrm{H}_c^d(Y_K, \mathcal{L}_{k,\mathrm{w}}^\vee(L)) & \xrightarrow{\mathrm{ev}_{\varpi_{\mathfrak{f}}}^\eta(L_{k,\mathrm{w}}^\vee(L))} & L_{k,\mathrm{w}}^\vee(L)_{E(\mathfrak{f})} \end{array} \tag{3.7}$$

Lemma 3.2. *For $\eta \in \mathbb{A}_{F,f}^\times$, $u \in U(\mathfrak{f})$ and $\eta' \in F^\times \eta u F_\infty^{\times +}$ we have* $\mathrm{ev}_{\varpi_\mathfrak{f}}^{\eta'} = \left(\begin{smallmatrix} u_p^{-1} & \\ & 1\end{smallmatrix}\right) \cdot \mathrm{ev}_{\varpi_\mathfrak{f}}^{\eta}$.

Proof. By (3.6) , it suffices to check that $\mathrm{triv}_{\eta'}^* = \left(\mathrm{id} \otimes \left(\begin{smallmatrix} u_p^{-1} & \\ & 1\end{smallmatrix}\right)\right) \cdot \mathrm{triv}_\eta^*$. This follows from the fact that $\mathrm{triv}_{\eta'}(y, m) = \mathrm{triv}_\eta\left(y, \left(\begin{smallmatrix} u_p^{-1} & \\ & 1\end{smallmatrix}\right) \cdot m\right)$. ∎

Lemma 3.3. *If $v \mid \mathfrak{f}$ then for all $\delta \in \mathcal{O}_v^\times$ we have* $\mathrm{ev}_{\varpi_\mathfrak{f}\delta}^\eta = \mathrm{ev}_{\varpi_\mathfrak{f}}^\eta \circ U_\delta$.

3.3. Relations

We will prove a fundamental relation between the evaluations defined in §3.2, which will later be used to prove an interpolation property and a growth property of certain p-adic distributions on Galois groups, as well as a relation between the corresponding p-adic L-functions and their improved counterparts.

Proposition 3.4. *For $v \in S$ let $\mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}} : \mathcal{C}\ell_F^+(\mathfrak{f}v) \to \mathcal{C}\ell_F^+(\mathfrak{f})$ be the natural projection. Choose a representative $\eta \in \mathbb{A}_{F,f}^\times$ for $[\eta] \in \mathcal{C}\ell_F^+(\mathfrak{f})$ and for each $[\delta] \in \mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}}^{-1}([\eta])$ let $\delta \in \mathbb{A}_F^\times$ and $u_\delta \in U(\mathfrak{f})$ be such that $\delta \in F^\times \eta u_\delta F_\infty^{\times +}$. Then*

$$\sum_{[\delta]\in \mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}}^{-1}([\eta])} \begin{pmatrix} u_\delta & \\ & 1\end{pmatrix} \cdot \mathrm{ev}_{\varpi_{\mathfrak{f}v}}^\delta = \begin{cases} \mathrm{ev}_{\varpi_\mathfrak{f}}^\eta \circ U_{\varpi_v} & \text{if } v \mid \mathfrak{f}, \\ \mathrm{ev}_{\varpi_\mathfrak{f}}^\eta \circ U_{\varpi_v} - \begin{pmatrix} \varpi_v & 0 \\ 0 & 1\end{pmatrix} \cdot \mathrm{ev}_{\varpi_\mathfrak{f}}^{\eta\varpi_v} & \text{if } v \nmid \mathfrak{f}. \end{cases}$$

Proof. We first recall the definition of U_{ϖ_v}. Let $\gamma = \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1\end{smallmatrix}\right)$ and consider the natural projections $\mathrm{pr}_1 : Y_{K_0(v)} \to Y_K$ and $\mathrm{pr}_2 : Y_{K^0(v)} \to Y_K$ where $K^0(v) = K \cap \gamma K \gamma^{-1}$ and $K_0(v) = K \cap \gamma^{-1} K \gamma$. For clarity we will denote by $\mathcal{M}_K$ the local system on Y_K, in which case $\mathrm{pr}_1^* \mathcal{M}_K = \mathcal{M}_{K_0(v)}$ and $\mathrm{pr}_2^* \mathcal{M}_K = \mathcal{M}_{K^0(v)}$. Define

$$U_{\varpi_v} = \mathrm{Tr}(\mathrm{pr}_2) \circ [\gamma] \circ \mathrm{pr}_1^* : \mathrm{H}_c^d(Y_K, \mathcal{M}_K) \to \mathrm{H}_c^d(Y_K, \mathcal{M}_K), \tag{3.8}$$

where $[\gamma] : \mathrm{H}_c^d(Y_{K_0(v)}, \mathcal{M}_{K_0(v)}) \to \mathrm{H}_c^d(Y_{K^0(v)}, \mathcal{M}_{K^0(v)})$ is induced by the morphism of local systems given by $(g, m) \mapsto (g\gamma^{-1}, \gamma \cdot m)$ (note that the image of γ in $G(\mathbb{Q}_p)$ belongs to Λ_S), $\mathrm{pr}_1^* : \mathrm{H}_c^d(Y_K, \mathcal{M}_K) \to \mathrm{H}_c^d(Y_{K_0(v)}, \mathcal{M}_{K_0(v)})$ is the pullback, and $\mathrm{Tr}(\mathrm{pr}_2) : \mathrm{H}_c^d(Y_{K^0(v)}, \mathcal{M}_{K^0(v)}) = \mathrm{H}_c^d(Y_K, p_{2*}\mathrm{pr}_2^* \mathcal{M}_K) \to \mathrm{H}_c^d(Y_K, \mathcal{M}_K)$ is the trace attached to the finite map pr_2.

We will now define analogous maps on $X_\mathfrak{f}$. The map

$$\widetilde{C}_{\mathfrak{f},K^0(v)} : X_{\mathfrak{f}v} \to Y_{K^0(v)}, \qquad [y] \mapsto \left[\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \varpi_\mathfrak{f} & 1_{\mathfrak{f}v} \\ 0 & 1\end{pmatrix}\right],$$

is well-defined since for all $\xi \in F^\times$ and $u \in U(\mathfrak{f}v)$, we have $\left(\begin{smallmatrix} u & (u-1)1_{\mathfrak{f}v}\varpi_\mathfrak{f}^{-1} \\ 0 & 1\end{smallmatrix}\right) \in K^0(v)$ and

$$\begin{pmatrix} \xi y u & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \varpi_\mathfrak{f} & 1_{\mathfrak{f}v} \\ 0 & 1\end{pmatrix} = \begin{pmatrix} \xi & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \varpi_\mathfrak{f} & 1_{\mathfrak{f}v} \\ 0 & 1\end{pmatrix}\begin{pmatrix} u & (u-1)1_{\mathfrak{f}v}\varpi_\mathfrak{f}^{-1} \\ 0 & 1\end{pmatrix}. \tag{3.9}$$

Case 1: $v \mid \mathfrak{f}$. In this case $1_{\mathfrak{f}v} = 1_{\mathfrak{f}}$. Denoting by $\cdot\gamma$ the right translation, one checks that the following diagram commutes:

$$\begin{array}{ccccccc}
Y_K & \xleftarrow{\ \mathrm{pr}_2\ } & Y_{K^0(v)} & \xrightarrow{\ \cdot\gamma\ } & Y_{K_0(v)} & \xrightarrow{\ \mathrm{pr}_1\ } & Y_K \\
{\scriptstyle C_{\varpi_{\mathfrak{f}}}}\uparrow & & \uparrow{\scriptstyle \widetilde{C}_{\mathfrak{f},K^0(v)}} & & \uparrow{\scriptstyle C_{\varpi_{\mathfrak{f}v},K_0(v)}} & & \uparrow{\scriptstyle C_{\varpi_{\mathfrak{f}v},K}} \\
X_{\mathfrak{f}} & \xleftarrow{\ \mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}}\ } & X_{\mathfrak{f}v} & = & X_{\mathfrak{f}v} & = & X_{\mathfrak{f}v}
\end{array} \tag{3.10}$$

We consider the local system $\widetilde{C}^*_{\mathfrak{f},K^0(v)}\mathcal{M} = F^\times\backslash(\mathbb{A}_F^\times \times M)/U(\mathfrak{f}v)$ on $X_{\mathfrak{f}v}$, where $u \in U(\mathfrak{f}v)$ acts on M by $\left(\begin{smallmatrix} u & (u-1)1_{\mathfrak{f}}\varpi_{\mathfrak{f}}^{-1} \\ 0 & 1\end{smallmatrix}\right)$. Since

$$\begin{pmatrix} \varpi_v & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} u & (u-1)1_{\mathfrak{f}v}\varpi_{\mathfrak{f}v}^{-1} \\ 0 & 1\end{pmatrix} = \begin{pmatrix} u & (u-1)1_{\mathfrak{f}}\varpi_{\mathfrak{f}}^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} \varpi_v & 0 \\ 0 & 1\end{pmatrix},$$

one has a morphism of local systems

$$C^*_{\varpi_{\mathfrak{f}v},K}\mathcal{M} = C^*_{\varpi_{\mathfrak{f}v},K_0(v)}\mathcal{M} \to \widetilde{C}^*_{\mathfrak{f},K^0(v)}, \qquad [(y,m)] \mapsto \left[\left(y, \begin{pmatrix} \varpi_v & 0 \\ 0 & 1\end{pmatrix}\cdot m\right)\right],$$

inducing a homomorphism on the cohomology:

$$[\varpi_v] : \mathrm{H}^d_c(X_{\mathfrak{f}v}, C^*_{\varpi_{\mathfrak{f}v},K_0(v)}\mathcal{M}) \to \mathrm{H}^d_c(X_{\mathfrak{f}v}, \widetilde{C}^*_{\mathfrak{f},K^0(v)}\mathcal{M}).$$

Pulling back the U_{ϖ_v} defined in (3.8) by the vertical maps in (3.10), and noticing that the etale maps pr_2 and $\mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}}$ have the same degree, yields a homomorphism

$$\begin{aligned} U_{\varpi_v} &= \mathrm{Tr}(\mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}}) \circ [\varpi_v] : \\ &\mathrm{H}^d_c(X_{\mathfrak{f}v}, C^*_{\varpi_{\mathfrak{f}v},K}\mathcal{M}) = \mathrm{H}^d_c(X_{\mathfrak{f}v}, C^*_{\varpi_{\mathfrak{f}v},K_0(v)}\mathcal{M}) \to \mathrm{H}^d_c(X_{\mathfrak{f}}, C^*_{\varpi_{\mathfrak{f}}}\mathcal{M}). \end{aligned} \tag{3.11}$$

Next, we pull back the U_{ϖ_v} action by the twisting operators. By (3.9) and the fact that $\left(\begin{smallmatrix} \varpi_{\mathfrak{f}} & 1_{\mathfrak{f}} \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1_{\mathfrak{f}}\varpi_{\mathfrak{f}}^{-1} \\ & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} \varpi_{\mathfrak{f}} & \\ & 1\end{smallmatrix}\right)$ belongs to the torus, we have a morphism of local systems

$$\widetilde{\mathrm{tw}}_{\mathfrak{f}} : \widetilde{C}^*_{K^0(v),\mathfrak{f}}\mathcal{M} \to \mathcal{M}_{\mathfrak{f}v}, \qquad (y,m) \mapsto \left(y, \begin{pmatrix} \varpi_{\mathfrak{f}} & 1_{\mathfrak{f}} \\ & 1\end{pmatrix}\cdot m\right).$$

Moreover, as $v \mid \mathfrak{f}$ we have

$$\begin{pmatrix} \varpi_{\mathfrak{f}} & 1_{\mathfrak{f}} \\ 0 & 1\end{pmatrix}\begin{pmatrix} \varpi_v & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} \varpi_{\mathfrak{f}v} & 1_{\mathfrak{f}v} \\ 0 & 1\end{pmatrix},$$

hence the diagram

$$\begin{array}{ccccc}
\mathrm{H}^d_c(X_{\mathfrak{f}v}, C^*_{\varpi_{\mathfrak{f}v},K}\mathcal{M}) & \xrightarrow{\ [\varpi_v]\ } & \mathrm{H}^d_c(X_{\mathfrak{f}v}, \widetilde{C}^*_{\mathfrak{f},K^0(v)}\mathcal{M}) & \xrightarrow{\ \mathrm{Tr}(\mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}})\ } & \mathrm{H}^d_c(X_{\mathfrak{f}}, C^*_{\varpi_{\mathfrak{f}}}\mathcal{M}) \\
{\scriptstyle \mathrm{tw}_{\mathfrak{f}v}}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathrm{tw}}_{\mathfrak{f}}} & & \downarrow{\scriptstyle \mathrm{tw}_{\mathfrak{f}}} \\
\mathrm{H}^d_c(X_{\mathfrak{f}v}, \mathcal{M}_{\mathfrak{f}v}) & = & \mathrm{H}^d_c(X_{\mathfrak{f}v}, \mathcal{M}_{\mathfrak{f}v}) & \xrightarrow{\ \mathrm{Tr}(\mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}})\ } & \mathrm{H}^d_c(X_{\mathfrak{f}}, \mathcal{M}_{\mathfrak{f}})
\end{array} \tag{3.12}$$

is commutative. Taking coinvariants yields

$$\begin{array}{ccc}
\bigoplus_{[\delta]\in \mathrm{pr}^{-1}_{\mathfrak{f}v,\mathfrak{f}}([\eta])} \mathrm{H}^d_c(X_{\mathfrak{f}v}[\delta], \mathcal{M}_{E(\mathfrak{f}v)}) & \longrightarrow & \mathrm{H}^d_c(X_{\mathfrak{f}}[\eta], \mathcal{M}_{E(\mathfrak{f})}) \\
\Big\downarrow {\scriptstyle \oplus\, \mathrm{triv}^*_\delta} & & \Big\downarrow {\scriptstyle \mathrm{triv}^*_\eta} \\
\bigoplus_{[\delta]} \mathrm{H}^d_c(X_{\mathfrak{f}v}[\delta]) \otimes M_{E(\mathfrak{f}v)} & \xrightarrow{\ \oplus \left(\begin{smallmatrix} u_\delta & \\ & 1 \end{smallmatrix}\right)\cdot\ } & \mathrm{H}^d_c(X_{\mathfrak{f}}[\eta]) \otimes M_{E(\mathfrak{f})} \\
\Big\downarrow {\scriptstyle \oplus - \cap \theta_{\mathfrak{f}v,[\delta]}} & & \Big\downarrow {\scriptstyle - \cap \theta_{\mathfrak{f},[\eta]}} \\
\bigoplus_{[\delta]\in \mathrm{pr}^{-1}_{\mathfrak{f}v,\mathfrak{f}}([\eta])} M_{E(\mathfrak{f}v)} & \xrightarrow{\ (m_{[\delta]})_{[\delta]} \mapsto \sum_{[\delta]} m_{[\delta]}\ } & M_{E(\mathfrak{f})}
\end{array} \tag{3.13}$$

where the map $M_{E(\mathfrak{f}v)} \to M_{E(\mathfrak{f})}$ is the canonical projection. The commutativity of the upper square follows from the proof of Lemma 3.2, while the commutativity of the bottom square follows from the compatible choice of fundamental classes $\theta_{\mathfrak{f},[\eta]}$ and $\theta_{\mathfrak{f}v,[\delta]}$ in §3.1. The proposition then follows from (3.6), (3.12), and (3.13).

Case 2: $v \nmid \mathfrak{f}$. The extra term comes from the fact $\mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}}$ has degree 1 less than the degree of pr_2. Instead of (3.10) we consider the commutative diagram

$$\begin{array}{ccccccc}
Y_K & \xleftarrow{\ \mathrm{pr}_2\ } & Y_{K^0(v)} & \xrightarrow{\ \cdot\gamma\ } & Y_{K_0(v)} & \xrightarrow{\ \mathrm{pr}_1\ } & Y_K \\
{\scriptstyle C_{\varpi_{\mathfrak{f}}}} \Big\uparrow & & {\scriptstyle C_{\varpi_{\mathfrak{f}},K^0(v)}} \Big\uparrow {\scriptstyle \tilde{C}_{\mathfrak{f},K^0(v)}} & & {\scriptstyle C_{\varpi_{\mathfrak{f}},K_0(v)}} \Big\uparrow {\scriptstyle C_{\varpi_{\mathfrak{f}v},K_0(v)}} & & {\scriptstyle C_{\varpi_{\mathfrak{f}}}} \Big\uparrow {\scriptstyle C_{\varpi_{\mathfrak{f}v},K}} \\
X_{\mathfrak{f}} & \xleftarrow{\ \mathrm{id} \sqcup \mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}}\ } & X_{\mathfrak{f}} \sqcup X_{\mathfrak{f}v} & \xrightarrow{\ \cdot\varpi_v \sqcup \mathrm{id}\ } & X_{\mathfrak{f}} \sqcup X_{\mathfrak{f}v} & = & X_{\mathfrak{f}} \sqcup X_{\mathfrak{f}v}
\end{array} \tag{3.14}$$

As in (3.11), pulling back the U_{ϖ_v} defined in (3.8) by the vertical maps in (3.14) yields

$$U_{\varpi_v} = U_{v,1} + U_{v,2} : \mathrm{H}^d_c(X_{\mathfrak{f}}, C^*_{\varpi_{\mathfrak{f}}}\mathcal{M}) \oplus \mathrm{H}^d_c(X_{\mathfrak{f}v}, C^*_{\varpi_{\mathfrak{f}v},K}\mathcal{M}) \to \mathrm{H}^d_c(X_{\mathfrak{f}}, C^*_{\varpi_{\mathfrak{f}}}\mathcal{M}),$$

where $U_{v,2}$ is given by the same formulas as U_{ϖ_v} in Case 1, whereas $U_{v,1}$ comes from the map $(y,m) \mapsto \left(\varpi_v^{-1} y, \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right) \cdot m\right)$. Applying $\mathrm{coinv}_{\mathfrak{f}} \circ \mathrm{tw}_{\varpi_{\mathfrak{f}}}$ to both sides of the map $U_{v,1}$, one completes the proof by checking the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{H}^d_c(X_{\mathfrak{f}}[\eta\varpi_v], \mathcal{M}_{E(\mathfrak{f})}) & \xrightarrow{\ U_{v,1}\ } & \mathrm{H}^d_c(X_{\mathfrak{f}}[\eta], \mathcal{M}_{E(\mathfrak{f})}) \\
{\scriptstyle (-\cap\theta_{\mathfrak{f},[\eta\varpi_v]})\circ \mathrm{triv}^*_{\eta\varpi_v}} \Big\downarrow & & \Big\downarrow {\scriptstyle (-\cap\theta_{\mathfrak{f},[\eta]})\circ \mathrm{triv}^*_{\eta}} \\
M_{E(\mathfrak{f})} & \xrightarrow{\ \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right)\ } & M_{E(\mathfrak{f})}
\end{array}$$

■

Remark 3.5. This proposition completes and generalizes [2, Lem. 5.1]. The second part of this proposition generalizes [21, Prop. 5.8(i)] used to obtain a relation between the standard and improved p-adic L-function in the context of modular curves. Such relations will be vastly generalized in Proposition 3.17.

3.4. Distributions on Galois groups

The evaluation map $\mathrm{ev}^{\eta}_{\varpi_{\mathfrak f}}(M)$ constructed in §3.2 depends on a representative $\eta \in \mathbb{A}^{\times}_{F,f}$ of the class $[\eta] \in \mathscr{C}\ell^{+}_{F}(\mathfrak f)$ and on the choice of uniformizers. In this section we will focus on the case where $M = D_{\mathcal U}$ (see §2.2), with $\mathcal U$ an L-affinoid of the weight space $\mathcal X$ (see §2.1), and produce distributions on Galois groups which are independent of the above choices. These in turn will be used in §4 to construct p-adic L-functions.

By class field theory, for any integral ideal $\mathfrak f$ supported in S_p there is an exact sequence

$$1 \to U(\mathfrak f)_p/\overline{E(\mathfrak f)} \xrightarrow{\iota_{\mathfrak f}} \mathrm{Gal}_{p\infty} \to \mathscr{C}\ell^{+}_{F}(\mathfrak f) \to 1, \tag{3.15}$$

where $U(\mathfrak f)_p = (\mathcal O_F \otimes \mathbb Z_p)^{\times} \cap (1 + \mathfrak f(\mathcal O_F \otimes \mathbb Z_p))$ and $\overline{E(\mathfrak f)}$ is the p-adic closure of $E(\mathfrak f)$ in $U(\mathfrak f)_p$. We have $(D_{\mathcal U})_{E(\mathfrak f)} \subset \mathrm{Hom}_{\mathcal O(\mathcal U)}(A^{E(\mathfrak f)}_{\mathcal U}, \mathcal O(\mathcal U))$, where $A^{E(\mathfrak f)}_{\mathcal U}$ consists of $f \in A_{\mathcal U}$ such that $f_{|\left(\begin{smallmatrix} e & \\ & 1\end{smallmatrix}\right)} = f$ for all $e \in E(\mathfrak f)$ (see Definition 2.2). As in [2] we define an 'extension by zero' morphism

$$A(U(\mathfrak f)_p/\overline{E(\mathfrak f)}, \mathcal O(\mathcal U)) \to A^{E(\mathfrak f)}_{\mathcal U}, \quad f \mapsto f^{\times}(z) = \begin{cases} \left\langle \left(\begin{smallmatrix} z & 0 \\ 0 & 1\end{smallmatrix}\right)\right\rangle_{\mathcal U} f(z) & \text{if } z \in U(\mathfrak f)_p, \\ 0 & \text{if } z \notin U(\mathfrak f)_p. \end{cases} \tag{3.16}$$

Dualizing, we get a map $(\mathcal D_{\mathcal U})_{E(\mathfrak f)} \to D(U(\mathfrak f)_p/\overline{E(\mathfrak f)}, \mathcal O(\mathcal U))$, which we denote by $\mu \mapsto \mu^{\times}$.

Let $\mathrm{Gal}_{p\infty}[\eta]$ denote the pre-image of $[\eta] \in \mathscr{C}\ell^{+}_{F}(\mathfrak f)$ in $\mathrm{Gal}_{p\infty}$. Multiplication by the image of $\eta \in \mathbb{A}^{\times}_{F,f}$ in $\mathrm{Gal}_{p\infty}$ under the Artin recipocity map yields a bijection

$$\iota_\eta : U(\mathfrak f)_p/\overline{E(\mathfrak f)} \xrightarrow{\sim} \mathrm{Gal}_{p\infty}[\eta], \qquad u_p \mapsto \eta\iota_{\mathfrak f}(u_p). \tag{3.17}$$

Dualizing, we obtain a map $\iota^{*}_{\eta} : D(U(\mathfrak f)_p/\overline{E(\mathfrak f)}, \mathcal O(\mathcal U)) \xrightarrow{\sim} D(\mathrm{Gal}_{p\infty}[\eta], \mathcal O(\mathcal U))$. Explicitly, for all $\mu \in D(U(\mathfrak f)_p/\overline{E(\mathfrak f)}, \mathcal O(\mathcal U))$ and $f \in A(\mathrm{Gal}_{p\infty}[\eta], \mathcal O(\mathcal U))$ we have $\langle \iota^{*}_{\eta}\mu, f\rangle = \langle \mu, f \circ \iota_\eta\rangle$.

Lemma 3.6. *The following map does not depend on the representative η of $[\eta] \in \mathscr{C}\ell^{+}_{F}(\mathfrak f)$:*

$$\mathrm{ev}^{[\eta]}_{\varpi_{\mathfrak f}} = \iota^{*}_{\eta} \circ \mathrm{ev}^{\eta,\times}_{\varpi_{\mathfrak f}} : \mathrm{H}^{d}_{c}(Y_K, \mathcal D_{\mathcal U}) \to D(\mathrm{Gal}_{p\infty}[\eta], \mathcal O(\mathcal U)). \tag{3.18}$$

We omit the proof, which is a consequence of Lemma 3.2. The following result shows that when v divides $\mathfrak f$ for all $v \in S_p$, the passage to $\mathrm{ev}^{\eta,\times}_{\varpi_{\mathfrak f}}$ does not make one loose information.

Lemma 3.7. *Given $\Phi \in \mathrm{H}^{d}_{c}(Y_K, \mathcal D_{\mathcal U})$, one has $\mathrm{ev}^{\eta}_{\varpi_{\mathfrak f}}(\Phi) \in \left(\begin{smallmatrix} \varpi_{\mathfrak f} & 1_{\mathfrak f} \\ & 1\end{smallmatrix}\right) \cdot D_{\mathcal U}$. In particular, for any $f \in A_{\mathcal U}$ one has $\langle \mathrm{ev}^{\eta}_{\varpi_{\mathfrak f}}(\Phi), f\rangle = \langle \mathrm{ev}^{\eta}_{\varpi_{\mathfrak f}}(\Phi), f_{|1+\mathfrak f(\mathcal O_F \otimes \mathbb Z_p)}\rangle$.*

Definition 3.8. We define

$$\mathrm{ev}_{\varpi_{\mathfrak{f}}} = \bigoplus_{[\eta]\in \mathscr{C}\ell_F^+(\mathfrak{f})} \mathrm{ev}_{\varpi_{\mathfrak{f}}}^{[\eta]} : \mathrm{H}_c^d(Y_K, \mathcal{D}_{\mathcal{U}}) \to D(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U})),$$

$$\langle \mathrm{ev}_{\varpi_{\mathfrak{f}}}, f\rangle = \sum_{[\eta]\in \mathscr{C}\ell_F^+(\mathfrak{f})} \langle \mathrm{ev}_{\varpi_{\mathfrak{f}}}^{[\eta]}, f_{|\mathrm{Gal}_{p\infty}[\eta]}\rangle.$$

Proposition 3.9. *Let* $[\eta] \in \mathscr{C}\ell_F^+(\mathfrak{f})$. *Then for any* $v \in S_p$ *we have*

$$\mathrm{ev}_{\varpi_{\mathfrak{f}}}^{[\eta]} \circ U_{\varpi_v} = \sum_{[\delta]\in \mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}}^{-1}([\eta])} \mathrm{ev}_{\varpi_{\mathfrak{f}v}}^{[\delta]} \quad \textit{and} \quad \mathrm{ev}_{\varpi_{\mathfrak{f}}} \circ U_{\varpi_v} = \mathrm{ev}_{\varpi_{\mathfrak{f}v}}.$$

Proof. Let $\Phi \in \mathrm{H}_c^d(Y_K, \mathcal{D}_{\mathcal{U}})$ and $f \in A(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$. Using Proposition 3.4 it suffices to show that $\langle \left(\begin{smallmatrix}\varpi_v & \\ & 1\end{smallmatrix}\right) \cdot \mathrm{ev}_{\varpi_{\mathfrak{f}}}^{\eta\varpi_v}(\Phi), (f\circ\iota_\eta)^\times\rangle = 0$ when $v \nmid \mathfrak{f}$ and $\langle \left(\begin{smallmatrix}u_\delta & \\ & 1\end{smallmatrix}\right)\mathrm{ev}_{\varpi_{\mathfrak{f}v}}^{\delta}, (f\circ\iota_\eta)^\times\rangle = \langle \mathrm{ev}_{\varpi_{\mathfrak{f}v}}^{\delta}, (f\circ\iota_\delta)^\times\rangle$ for all v, where $\delta \in F^\times \eta u_\delta F_\infty^{\times +}$ and $u_\delta \in U(\mathfrak{f})$ are as in Proposition 3.4. The former follows from (3.16) since for all $u_p \in U(\mathfrak{f})_p$ we have $(f\circ\iota_\eta)^\times(\varpi_v u_p) = 0$. The latter follows from the fact that if $u_p \in U(\mathfrak{f})_p$ then applying Definition 2.2 and (3.16) we get

$$(f\circ\iota_\eta)^\times_{|\left(\begin{smallmatrix}1 & \\ & u_\delta^{-1}\end{smallmatrix}\right)}(u_p) = \langle\left(\begin{smallmatrix}u_\delta^{-1} & \\ & 1\end{smallmatrix}\right)\rangle_{\mathcal{U}}(f\circ\iota_\eta)^\times(u_\delta u_p) = \langle\left(\begin{smallmatrix}u_p & \\ & 1\end{smallmatrix}\right)\rangle_{\mathcal{U}}(f\circ\iota_\eta)(u_\delta u_p)$$
$$= \langle\left(\begin{smallmatrix}u_p & \\ & 1\end{smallmatrix}\right)\rangle_{\mathcal{U}} f(u_\delta \eta u_p) = \langle\left(\begin{smallmatrix}u_p & \\ & 1\end{smallmatrix}\right)\rangle_{\mathcal{U}} f(\delta u_p) = \langle\left(\begin{smallmatrix}u_p & \\ & 1\end{smallmatrix}\right)\rangle_{\mathcal{U}}(f\circ\iota_\delta)(u_p) = (f\circ\iota_\delta)^\times(u_p). \quad \blacksquare$$

Suppose that $\Phi \in \mathrm{H}_c^d(Y_K, \mathcal{D}_{\mathcal{U}})$ is such that $U_{\varpi_{\mathfrak{f}}}\Phi = \alpha_{\mathfrak{f}}^\circ \Phi$ with $\alpha_{\mathfrak{f}}^\circ \in \mathcal{O}(\mathcal{U})^\times$. By Lemma 3.3,

$$\mathrm{ev}(\Phi) = (\alpha_{\mathfrak{f}}^\circ)^{-1}\mathrm{ev}_{\varpi_{\mathfrak{f}}}(\Phi) \in D(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U})) \tag{3.19}$$

is independent of the choice of uniformizers and by Proposition 3.9 it is independent of $\mathfrak{f}$ as well.

Our final result in this subsection concerns the growth of the distributions $\mathrm{ev}(\Phi)$ on $\mathrm{Gal}_{p\infty}$. This will be used in §4 to uniquely characterize by interpolation property the p-adic L-functions attached to non-critical nearly finite slope Hilbert cusp forms. Using the notations from §2.2, $A(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$ is a union of orthonormalizable Banach $\mathcal{O}(\mathcal{U})$-modules $A_n(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$, $n \in \mathbb{Z}_{\geq 0}$, and $D(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U})) = \varprojlim D_n(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$. The restriction of $\mu \in D(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$ to $A_n(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$ belongs to the orthonormalizable Banach $\mathcal{O}(\mathcal{U})$-module $D_n(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$, and its norm is denoted by $\|\mu\|_n$. The following definition generalizes the notion of growth introduced by Amice-Vélu and Vishik (see [2, Def. 4.1]).

Definition 3.10. We say that a distribution $\mu \in D(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$ has *growth at most* $h \in \mathbb{Q}_{\geq 0}$ if there exists $C \geq 0$ such that for each $n \in \mathbb{Z}_{\geq 0}$ we have $\|\mu\|_n \leq p^{nh}C$.

Proposition 3.11. *Suppose* $\Phi \in \mathrm{H}_c^d(Y_K, \mathcal{D}_{\mathcal{U}})$ *is such that* $U_p\Phi = \alpha_p^\circ\Phi$ *with* $\alpha_p^\circ \in \mathcal{O}(\mathcal{U})^\times$. *Then* $\mathrm{ev}(\Phi) \in D(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$ *has growth at most* h_p, *where* h_p *is the* p*-adic valuation of* α_p°.

Proof. This is proved in [2, Prop. 5.10] when $\mathcal{U} = \{\lambda\}$. Recall that from [50, Lem. 3.4.6] we know that there exists $m \in \mathbb{Z}_{\geq 0}$ such that the universal character $\langle \cdot \rangle_{\mathcal{U}}$ is m-locally $\mathcal{O}(\mathcal{U})$-analytic, and we may further assume that $K \supset \left(\begin{smallmatrix} U(p^m) & \widehat{\mathcal{O}}_F \\ 0 & 1 \end{smallmatrix}\right)$. Let $\mathcal{O}(\mathcal{U})^\circ \subset \mathcal{O}(\mathcal{U})$ be the ring of rigid functions bounded by 1 and denote by $D^\circ_{\mathcal{U},m}$ the $\mathcal{O}(\mathcal{U})^\circ$-lattice in the $\mathcal{O}(\mathcal{U})$-Banach space $D_{\mathcal{U},m}$. After rescaling Φ we may assume that its image Φ_m under the natural restriction map belongs to $\mathrm{H}^d_c(Y_K, \mathcal{D}^\circ_{\mathcal{U},m})$. By (3.18) and (3.19) for $f \in A_n(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U}))$,

$$\langle (\alpha_p^\circ)^n \cdot \mathrm{ev}(\Phi), f \rangle = \langle \mathrm{ev}_{\varpi_p^n}(\Phi), f \rangle = \sum_{[\eta] \in \mathcal{C}\ell_F^+(p^n)} \langle \mathrm{ev}^\eta_{\varpi_p^n}(\Phi), (f \circ \iota_\eta)^\times \rangle.$$

In view of (3.16) and the fact that $|\langle T(\mathbb{Z}_p)\rangle_{\mathcal{U}}|_p = 1$, to prove the proposition it suffices to bound $|\langle \mathrm{ev}^\eta_{\varpi_p^n}(\Phi), g\rangle|_p$ for each $g \in A^{E(p^n)}_{\mathcal{U},n}$ such that $\|g\|_n \leq 1$, for all $n \geq m$ and $\eta \in \mathbb{A}^\times_{F,f}$. By Lemma 3.7, there exist $\mu \in (D_{\mathcal{U}})_{E_n}$ and $\mu' \in (D^\circ_{\mathcal{U},m})_{E_n}$, where $E_n = \left(\begin{smallmatrix} \varpi_p^n & 1_p \\ 0 & 1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} E(p^n) & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \varpi_p^n & 1_p \\ 0 & 1 \end{smallmatrix}\right) \subset K$, such that $\mathrm{ev}^\eta_{\varpi_p^n}(\Phi) = \left(\begin{smallmatrix} \varpi_p^n & 1_p \\ 0 & 1 \end{smallmatrix}\right) \cdot \mu$ and $\mathrm{ev}^\eta_{\varpi_p^n}(\Phi_m) = \left(\begin{smallmatrix} \varpi_p^n & 1_p \\ 0 & 1 \end{smallmatrix}\right) \cdot \mu'$. By functoriality of the evaluation maps (see Lemma 3.1), μ and μ' have the same restrictions to $(A^{E(p^n)}_{\mathcal{U},m})_{|\left(\begin{smallmatrix} \varpi_p^n & 1_p \\ 0 & 1 \end{smallmatrix}\right)}$. Using this, a direct computation shows that μ and μ' also agree over $(A^{E(p^n)}_{\mathcal{U},n})_{|\left(\begin{smallmatrix} \varpi_p^n & 1_p \\ 0 & 1 \end{smallmatrix}\right)}$. Thus, if $g \in A^{E(p^n)}_{\mathcal{U},n}$ is such that $\|g\|_n \leq 1$ then

$$|\langle \mathrm{ev}^\eta_{\varpi_p^n}(\Phi), g\rangle|_p = \left|\left\langle \left(\begin{smallmatrix} \varpi_p^n & 1_p \\ 0 & 1 \end{smallmatrix}\right) \cdot \mu, g \right\rangle\right|_p = |\langle \mu, g_{|\left(\begin{smallmatrix} \varpi_p^n & 1_p \\ 0 & 1 \end{smallmatrix}\right)} \rangle|_p = |\langle \mu', g_{|\left(\begin{smallmatrix} \varpi_p^n & 1_p \\ 0 & 1 \end{smallmatrix}\right)} \rangle|_p \leq 1. \qquad \blacksquare$$

3.5. *Distributions evaluated at norm maps*

To compute higher derivatives of p-adic L-functions at central trivial zeros we need to construct partially improved p-adic L-functions. These will be obtained by evaluating the distributions $\mathrm{ev}^\eta_{\varpi_\mathfrak{f}}(D_{\mathcal{U}})$ on certain partially polynomial functions in $A_{\mathcal{U}}$ for certain well-chosen subaffinoids $\mathcal{U}$ of $\mathcal{X}$. The improvement comes from the fact that if $\mathfrak{f}$ is only divisible by certain primes above p, then the support of $\mathrm{ev}^\eta_{\varpi_\mathfrak{f}}(D_{\mathcal{U}})$ need no longer be contained in $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ (see Lemma 3.7). Part of the construction will also be used to attach a new kind of p-adic L-function to the partial families from Theorem 2.14.

Given an L-affinoid $\mathcal{U} \subset \mathcal{X}$ containing the cohomological weight (k, w) and a subset $S \subset S_p$, we let $\mathcal{U}'_S = \mathcal{U} \cap \mathcal{X}'_S$ (see Definition 2.1). Henceforth we fix an integer r such that

$$j_\sigma := r - 1 + \frac{\mathsf{w} - 2 + k_\sigma}{2} \geq 0 \quad \text{for all } \sigma \in \Sigma_{S_p \setminus S}, \tag{3.20}$$

and for $z_{S_p\setminus S} \in \mathcal{O}_{F,S_p\setminus S}$ we let $z^j_{S_p\setminus S} = \prod_{v \in S_p\setminus S} \prod_{\sigma \in \Sigma_v} \sigma(z_v)^{j_\sigma}$. For the remainder of this section we will only consider ideals $\mathfrak{f} \subset \mathcal{O}_F$ whose support in contained in S. We let $\overline{E(\mathfrak{f})}$ denote the p-adic closure of $E(\mathfrak{f})$ in $U(\mathfrak{f})_S = \mathcal{O}^\times_{F,S} \cap (1 + \mathfrak{f}\mathcal{O}_{F,S})$. Similarly to (3.16) one considers the map

$$A(U(\mathfrak{f})_S/\overline{E(\mathfrak{f})}, \mathcal{O}(\mathcal{U}'_S)) \to A^{E(\mathfrak{f})}_{\mathcal{U}'_S}, \qquad f \mapsto f^\times_{S,r},$$

where for $z = (z_S, z_{S_p\setminus S}) \in \mathcal{O}_{F,S} \times \mathcal{O}_{F,S_p\setminus S}$,

$$f^{\times}_{S,r}(z) = \begin{cases} f(z_S)\cdot z^{j}_{S_p\setminus S}\cdot \prod_{v\in S}\left\langle\begin{pmatrix} z_v & \\ & 1\end{pmatrix}\right\rangle_{\mathcal{U}'_S} \mathrm{N}^{r-1}_{F_v/\mathbb{Q}_p}(z_v) & \text{if } z_S \in U(\mathfrak{f})_S, \\ 0 & \text{if } z_S \notin U(\mathfrak{f})_S. \end{cases} \tag{3.21}$$

Dualizing we obtain a map $(\mathcal{D}_{\mathcal{U}'_S})_{E(\mathfrak{f})} \to D(U(\mathfrak{f})_S/E(\mathfrak{f}), \mathcal{O}(\mathcal{U}'_S))$, denoted $\mu \mapsto \mu^{\times}_{S,r}$.

Note that for all $v \in S_p \setminus S$ and $z_v \in \mathcal{O}_v^{\times}$ one has

$$\prod_{\sigma\in\Sigma_v} \sigma(z_v)^{j_\sigma} = \left\langle\begin{pmatrix} z_v & \\ & 1\end{pmatrix}\right\rangle_{\mathcal{U}'_S} \mathrm{N}^{r-1}_{F_v/\mathbb{Q}_p}(z_v).$$

Definition 3.12. We say that $r \in \mathbb{Z}$ is S*-critical* for the cohomological weight (k, w) if

$$0 \le r - 1 + \frac{\mathsf{w} - 2 + k_\sigma}{2} \le k_\sigma - 2 \quad \text{for all } \sigma \in \Sigma_{S_p\setminus S}.$$

When $S = \emptyset$ we say that r is *critical*.

Remark 3.13. (i) The inequality (3.20) holds for any cohomological weight in $\mathcal{X}'_S$.

(ii) If r is S-critical for (k, w), then it is S-critical as well for any cohomological weight in $\mathcal{X}_S$. Furthermore if $\mathcal{U}_S \subset \mathcal{X}_S$ is an L-affinoid containing (k, w), then for all $f \in A(U(\mathfrak{f})_S/\overline{E(\mathfrak{f})}, \mathcal{O}(\mathcal{U}_S))$ one has $f^{\times}_{S,r} \in A^{E(\mathfrak{f})}_{S,\mathcal{U}_S}$.

(iii) Note that $r \in \mathbb{Z}$ is critical for (k, w) if and only if $r - 1/2$ is a critical point for the L-function of an automorphic representation π of cohomological weight (k, w) in the sense of Deligne. Moreover, the central point $(1 - \mathsf{w})/2$ is critical if and only if w is even. Using (1.4) and (1.5) one checks that the space of linear forms on $(L^{\vee}_{k,\mathsf{w}}(L))_{E(\mathcal{O}_F)}$ has a basis $\mu \mapsto \mu(z^j)$ indexed by $j = (k + (\mathsf{w} - 2)t)/2 + (r - 1)t$, where r ranges over all critical integers for (k, w).

Moreover

$$\vartheta_{S,v} \circ (f \circ (z_S \mapsto z_{S\setminus\{v\}})^{\times}_{S,r} = (\vartheta_{S,v} \circ f)^{\times}_{S\setminus\{v\},r}$$

for $f \in A(U(\mathfrak{f})_{S\setminus\{v\}}/\overline{E(\mathfrak{f})}, \mathcal{O}(\mathcal{U}'_S))$ and $v \in S$, where $\vartheta_{S,v} : \mathcal{O}(\mathcal{U}'_S) \to \mathcal{O}(\mathcal{U}'_{S\setminus\{v\}})$ is the restriction map. Applying Lemma 3.1 yields

$$\vartheta_{S,v} \circ \mathrm{ev}^{\eta}_{\varpi_\mathfrak{f}}(D_{\mathcal{U}'_S}) = \mathrm{ev}^{\eta}_{\varpi_\mathfrak{f}}(D_{\mathcal{U}'_{S\setminus\{v\}}}) \circ \vartheta_{S,v}. \tag{3.22}$$

As in (3.17), for $\eta \in \mathbb{A}^{\times}_{F,f}$, there is an isomorphism $\iota^*_\eta : D(U(\mathfrak{f})_S/E(\mathfrak{f}), \mathcal{O}(\mathcal{U})) \xrightarrow{\sim} D(\mathrm{Gal}_{S\infty}[\eta], \mathcal{O}(\mathcal{U}))$.

Lemma 3.14. *The following map does not depend on the representative $\eta \in \mathbb{A}^{\times}_{F,f}$ of $[\eta] \in \mathcal{C}\ell^+_F(\mathfrak{f})$:*

$$\mathrm{ev}^{[\eta],r}_{\varpi_\mathfrak{f},S} = \chi^{r-1}_{\mathrm{cyc}}(\eta)[\iota^*_\eta \circ (\mathrm{ev}^{\eta}_{\varpi_\mathfrak{f}})^{\times}_{S,r}] : \mathrm{H}^d_c(Y_K, \mathcal{D}_{\mathcal{U}'_S}) \to D(\mathrm{Gal}_{S\infty}[\eta], \mathcal{O}(\mathcal{U}'_S)). \tag{3.23}$$

Proof. Suppose $\eta' \in F^\times \eta u F_\infty^{\times +}$ with $u \in U(\mathfrak{f})$. By Lemma 3.2 we have

$$\langle \iota_\eta^* \circ (\mathrm{ev}^\eta_{\varpi_\mathfrak{f}})^\times_{S,r}, f\rangle = \langle \left(\begin{smallmatrix} u_p & \\ & 1\end{smallmatrix}\right) \mathrm{ev}^{\eta'}_{\varpi_\mathfrak{f}}, (f\circ\iota_\eta)^\times_{S,r}\rangle = \left\langle \mathrm{ev}^{\eta'}_{\varpi_\mathfrak{f}}, (f\circ\iota_\eta)^\times_{S,r}|\left(\begin{smallmatrix} 1 & \\ & u_p^{-1}\end{smallmatrix}\right)\right\rangle.$$

Using Definition 2.2 and the fact that $\iota_\eta(u_p\,\cdot) = \iota_{\eta'}$ we find

$$(f\circ\iota_\eta)^\times_{S,r}|\left(\begin{smallmatrix} 1 & \\ & u_p^{-1}\end{smallmatrix}\right) = \langle\left(\begin{smallmatrix} u_p^{-1} & \\ & 1\end{smallmatrix}\right)\rangle_{\mathcal{U}'_S}(f\circ\iota_\eta)^\times_{S,r}(u_p\,\cdot) = \chi_{\mathrm{cyc}}^{r-1}(u_p)(f\circ\iota_{\eta'})^\times_{S,r}, \tag{3.24}$$

hence $\langle \iota_\eta^* \circ (\mathrm{ev}^\eta_{\varpi_\mathfrak{f}})^\times_{S,r}, f\rangle = \chi_{\mathrm{cyc}}^{r-1}(u_p)\langle \mathrm{ev}^{\eta'}_{\varpi_\mathfrak{f}}, (f\circ\iota_{\eta'})^\times_{S,r}\rangle = \chi_{\mathrm{cyc}}^{r-1}(u_p)\langle \iota_{\eta'}^* \circ (\mathrm{ev}^{\eta'}_{\varpi_\mathfrak{f}})^\times_{S,r}, f\rangle$. ■

The above lemma allows one to introduce the following notation analogous to Definition 3.8:

$$\mathrm{ev}^r_{\varpi_\mathfrak{f},S} : \mathrm{H}^d_c(Y_K, \mathcal{D}_{\mathcal{U}'_S}) \to D(\mathrm{Gal}_{S\infty}, \mathcal{O}(\mathcal{U}'_S)).$$

We first state a distribution relation extending Proposition 3.9, whose proof is very similar and uses the single additional fact that $f^\times_{S,r}|\left(\begin{smallmatrix} \varpi_v & \\ & 1\end{smallmatrix}\right) = 0$ for all $v \in S$.

Proposition 3.15. *For $v \in S$ let $\mathrm{pr}_{\mathfrak{f}v,\mathfrak{f}} : \mathcal{C}\ell^+_F(\mathfrak{f}v) \to \mathcal{C}\ell^+_F(\mathfrak{f})$ be the natural projection. Then*

$$\mathrm{ev}^{[\eta],r}_{\varpi_\mathfrak{f},S} \circ U_{\varpi_v} = \sum_{[\delta]\in \mathrm{pr}^{-1}_{\mathfrak{f}v,\mathfrak{f}}([\eta])} \mathrm{ev}^{[\delta],r}_{\varpi_{\mathfrak{f}v},S} \quad \textit{and} \quad \mathrm{ev}^r_{\varpi_\mathfrak{f},S} \circ U_{\varpi_v} = \mathrm{ev}^r_{\varpi_{\mathfrak{f}v},S}.$$

The next result will be used in §4 to compare p-adic L-functions and improved ones.

Lemma 3.16. *We have $\langle \mathrm{ev}^r_{\varpi_\mathfrak{f},S_p}, \cdot\rangle = \langle \mathrm{ev}_{\varpi_\mathfrak{f}}, \chi_{\mathrm{cyc}}^{r-1}\,\cdot\rangle$.*

Proof. Note that $\mathcal{U}'_{S_p} = \mathcal{U}$. By definition,

$$f^\times_{S_p,r}(z) = f^\times(z) z^{t(r-1)} \quad \text{for } f \in A(U(\mathfrak{f})_p/\overline{E(\mathfrak{f})}, \mathcal{O}(\mathcal{U})).$$

Using (3.23) and Definition 3.8 we find that for any $[\eta] \in \mathcal{C}\ell^+_F(\mathfrak{f})$ and any $f \in A(\mathrm{Gal}_{p\infty}[\eta], \mathcal{O}(\mathcal{U}))$,

$$\begin{aligned}\langle \mathrm{ev}^{[\eta],r}_{\varpi_\mathfrak{f},S_p}, f\rangle &= \chi_{\mathrm{cyc}}^{r-1}(\eta)\langle \mathrm{ev}^\eta_{\varpi_\mathfrak{f}}, (f\circ\iota_\eta)^\times_{S_p,r}\rangle = \langle \mathrm{ev}^\eta_{\varpi_\mathfrak{f}}, (f\circ\iota_\eta)^\times(\chi_{\mathrm{cyc}}^{r-1}\circ\iota_\eta)\rangle \\ &= \langle \mathrm{ev}^{[\eta]}_{\varpi_\mathfrak{f}}, f\chi_{\mathrm{cyc}}^{r-1}\rangle.\end{aligned}$$ ■

Finally, we relate the improved evaluations when S varies.

Proposition 3.17. *For any $v \in S$ with $v \nmid \mathfrak{f}$ we have*

$$\begin{aligned}&(\mathrm{ev}^{[\eta],r}_{\varpi_\mathfrak{f},S\setminus\{v\}} \circ \vartheta_{S,v} - \vartheta_{S,v} \circ \mathrm{ev}^{[\eta],r}_{\varpi_\mathfrak{f},S}) \circ U_{\varpi_v} \\ &\qquad = \left(q_v^{r-1} \prod_{\sigma\in\Sigma_v} \sigma(\varpi_v)^{(\mathrm{w}-2+k_\sigma)/2}\right) \iota^*_{\varpi_v^{-1}} \circ \mathrm{ev}^{[\eta\varpi_v],r}_{\varpi_\mathfrak{f},S\setminus\{v\}} \circ \vartheta_{S,v},\end{aligned}$$

where $\iota_{\varpi_v^{-1}}: \mathrm{Gal}_{(S\setminus\{v\})\infty}[\eta\varpi_v] \xrightarrow{\cdot\varpi_v^{-1}} \mathrm{Gal}_{(S\setminus\{v\})\infty}[\eta]$ *and* $\vartheta_{S,v}: \mathcal{O}(\mathcal{U}'_S) \to \mathcal{O}(\mathcal{U}'_{S\setminus\{v\}})$ *is the restriction.*

Proof. Using

$$f^{\times}_{S\setminus\{v\},r}\big|\begin{pmatrix}\varpi_v & \\ & 1\end{pmatrix} = \prod_{\sigma\in\Sigma_v}\sigma(\varpi_v)^{j_\sigma} f^{\times}_{S\setminus\{v\},r} \quad \text{and} \quad \chi_{\mathrm{cyc}}(\varpi_v)q_v = \mathrm{N}_{F_v/\mathbb{Q}_p}(\varpi_v)$$

we obtain

$$\begin{aligned}
&\chi_{\mathrm{cyc}}^{r-1}(\eta)\big\langle\begin{pmatrix}\varpi_v & 0\\ 0 & 1\end{pmatrix}\cdot \mathrm{ev}^{\eta\varpi_v}_{\varpi_{\mathfrak{f}}}, (f\circ\iota_\eta)^{\times}_{S\setminus\{v\},r}\big\rangle\\
&\quad = \chi_{\mathrm{cyc}}^{r-1}(\eta)\prod_{\sigma\in\Sigma_v}\sigma(\varpi_v)^{j_\sigma}\langle \mathrm{ev}^{\eta\varpi_v}_{\varpi_{\mathfrak{f}}}, (f\circ\iota_\eta)^{\times}_{S\setminus\{v\},r}\rangle\\
&\quad = \chi_{\mathrm{cyc}}^{r-1}(\eta\varpi_v)\langle\iota_\eta^*(\mathrm{ev}^{\eta\varpi_v}_{\varpi_{\mathfrak{f}}})^{\times}_{S\setminus\{v\},r}, f\rangle = q_v^{r-1}\prod_{\sigma\in\Sigma_v}\sigma(\varpi_v)^{j_\sigma-r+1}\langle\iota^*_{\varpi_v^{-1}}\mathrm{ev}^{[\eta\varpi_v],r}_{\varpi_{\mathfrak{f}},S\setminus\{v\}}, f\rangle.
\end{aligned}$$

Since $j_\sigma - r + 1 = \frac{\mathrm{w}-2+k_\sigma}{2}$, Proposition 3.4 applied to the case $v\nmid\mathfrak{f}$ yields

$$\begin{aligned}
&\chi_{\mathrm{cyc}}^{1-r}(\eta)\Big\langle \mathrm{ev}^{[\eta],r}_{\varpi_{\mathfrak{f}},S\setminus\{v\}}\circ U_{\varpi_v} - q_v^{r-1}\prod_{\sigma\in\Sigma_v}\sigma(\varpi_v)^{(\mathrm{w}-2+k_\sigma)/2}\iota^*_{\varpi_v^{-1}}\mathrm{ev}^{[\eta\varpi_v],r}_{\varpi_{\mathfrak{f}},S\setminus\{v\}}, f\Big\rangle\\
&\qquad = \big\langle \mathrm{ev}^{\eta}_{\varpi_{\mathfrak{f}}}\circ U_{\varpi_v} - \begin{pmatrix}\varpi_v & 0\\ 0 & 1\end{pmatrix}\cdot\mathrm{ev}^{\eta\varpi_v}_{\varpi_{\mathfrak{f}}}, (f\circ\iota_\eta)^{\times}_{S\setminus\{v\},r}\big\rangle\\
&\qquad = \sum_{[\delta]\in\mathrm{pr}^{-1}_{\mathfrak{f}v,\mathfrak{f}}([\eta])}\big\langle\begin{pmatrix}u_\delta & \\ & 1\end{pmatrix}\cdot\mathrm{ev}^{\delta}_{\varpi_{\mathfrak{f}v}}, (f\circ\iota_\eta)^{\times}_{S\setminus\{v\},r}\big\rangle.
\end{aligned}$$

By (3.22) and a computation similar to the proof of Proposition 3.9 we obtain

$$\big\langle\begin{pmatrix}u_\delta & \\ & 1\end{pmatrix}\cdot\mathrm{ev}^{\delta}_{\varpi_{\mathfrak{f}v}}\circ\vartheta_{S,v}, (f\circ\iota_\eta)^{\times}_{S\setminus\{v\},r}\big\rangle = \chi_{\mathrm{cyc}}^{1-r}(\eta)\cdot\vartheta_{S,v}(\langle\mathrm{ev}^{[\delta],r}_{\varpi_{\mathfrak{f}v},S}, f\rangle).$$

Finally, by Proposition 3.15 we find

$$\sum_{[\delta]\in\mathrm{pr}^{-1}_{\mathfrak{f}v,\mathfrak{f}}([\eta])}\vartheta_{S,v}\circ\mathrm{ev}^{[\delta],r}_{\varpi_{\mathfrak{f}v},S} = \vartheta_{S,v}\circ\mathrm{ev}^{[\eta],r}_{\varpi_{\mathfrak{f}},S}\circ U_{\varpi_v}. \qquad \blacksquare$$

Let $\Phi\in\mathrm{H}^d_c(Y_K,\mathcal{D}_{\mathcal{U}'_S})$ be such that for all $v\in S$ we have $U_{\varpi_v}\Phi = \alpha_v^\circ\Phi$ with $\alpha_v^\circ\in\mathcal{O}(\mathcal{U}'_S)^\times$. Letting $\alpha_{\mathfrak{f}}^\circ = \prod_{v\in S}(\alpha_v^\circ)^{n_v}$, where n_v denotes the valuation of $\mathfrak{f}$ at v, the distribution

$$\mathrm{ev}^r_S(\Phi) = (\alpha^\circ_{\mathfrak{f}})^{-1}\mathrm{ev}^r_{\varpi_{\mathfrak{f}},S}(\Phi)\in D(\mathrm{Gal}_{S\infty},\mathcal{O}(\mathcal{U}'_S)) \tag{3.25}$$

is independent of the choice of uniformizers (see Lemma 3.3) as well of the ideal $\mathfrak{f}$ (see Proposition 3.15). Lemma 3.16 and (3.19) then imply that

$$\langle\mathrm{ev}^r_{S_p}(\Phi), f\rangle = \langle\mathrm{ev}(\Phi), \chi_{\mathrm{cyc}}^{r-1}f\rangle \quad \text{for all } f\in A(\mathrm{Gal}_{p\infty},\mathcal{O}(\mathcal{U}'_{S_p})) = A(\mathrm{Gal}_{p\infty},\mathcal{O}(\mathcal{U})). \tag{3.26}$$

The following important consequence of Proposition 3.17 will be used in §4.3.

Corollary 3.18. *For $\Phi \in \mathrm{H}_c^d(Y_K, \mathcal{D}_{\mathcal{U}'_S})$ as above, let $\alpha_v = \alpha_v^\circ \prod_{\sigma\in\Sigma_v} \sigma(\varpi_v)^{(2-\mathrm{w}-k_\sigma)/2}$. Then for any continuous character $\chi : \mathrm{Gal}_{(S\setminus\{v\})\infty} \to \mathcal{O}(\mathcal{U}'_{S\setminus\{v\}})^\times$ we have*

$$\langle \vartheta_{S,v}(\mathrm{ev}_S^r(\Phi)), \chi\rangle = \langle \mathrm{ev}_{S\setminus\{v\}}^r(\vartheta_{S,v}(\Phi)), \chi\rangle \left(1 - \frac{q_v^{r-1}}{\vartheta_{S,v}(\alpha_v)\chi(\varpi_v)}\right).$$

4. p-adic L-functions

In this section we use the distribution valued maps from §3 to attach cyclotomic p-adic L-functions to rigid analytic families of non-critically refined Hilbert cusp forms, which are uniquely determined by an interpolation property (see Theorem 4.7). We also construct improved p-adic L-functions, as well as 'partial' p-adic L-functions for families of S-refined cusp forms, which do not appear to have been previously brought into light.

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ of central character ω_π and cohomological weight (k, w) (see Definition 1.1). Throughout this section we assume that π_v has nearly finite slope for all $v \in S_p$, except in §4.4 where we only assume this at $S \subsetneq S_p$. For $\tilde{\pi} = (\pi, \{\nu_v\}_{v\in S_p})$ a (regular) non-critical p-refinement (see Definitions 1.3 and 2.12) we consider the neat open compact subgroup $K = K(\tilde{\pi}, \mathfrak{u}) \subset G(\mathbb{A}_f)$ from Definition 1.7 and the $(p, \mathfrak{u})$-refined newforms $\phi_{\tilde{\pi},\alpha_\mathfrak{u}}, \phi_{\tilde{\pi},\beta_\mathfrak{u}}$ from Definition 1.11.

4.1. *p-adic L-functions for nearly finite slope Hilbert cusp forms*

Let $L/\mathbb{Q}_p$ be a finite extension containing the image by ι_p of the number field E from Definition 1.6. By cuspidality and non-criticality of $\tilde{\pi}$, for each character $\epsilon : \{\pm 1\}^\Sigma \to \{\pm 1\}$, the basis $\iota_p(b^\epsilon_{\tilde{\pi},\alpha_\mathfrak{u}})$ of $\mathrm{H}^d_{\mathrm{cusp}}(Y_K, \mathcal{L}^\vee_{k,\mathrm{w}}(L))^\epsilon_{\mathfrak{m}_{\tilde{\pi}}} = \mathrm{H}^\bullet_c(Y_K, \mathcal{L}^\vee_{k,\mathrm{w}}(L))^\epsilon_{\mathfrak{m}_{\tilde{\pi}}}$ (see (1.23)) lifts canonically to a basis $\Phi^\epsilon_{\tilde{\pi},\alpha_\mathfrak{u}}$ of $\mathrm{H}^d_c(Y_K, \mathcal{D}_{(k,\mathrm{w})})^\epsilon_{\mathfrak{m}_{\tilde{\pi}}}$ having the same U_{ϖ_v}-eigenvalue $\alpha_v^\circ \in L^\times$, $v \in S_p$. For $\mathfrak{f} = \prod_{v\in S_p} v^{n_v}$ we let

$$\alpha_\mathfrak{f}^\circ = \prod_{v\mid p} (\alpha_v^\circ)^{n_v}, \quad \alpha_\mathfrak{f} = \prod_{v\mid p} \alpha_v^{n_v}, \quad \alpha_v = \nu_v(\varpi_v) = \alpha_v^\circ \prod_{\sigma\in\Sigma_v} \sigma(\varpi_v)^{(2-\mathrm{w}-k_\sigma)/2}. \tag{4.1}$$

Consider the distribution $\mathrm{ev}(\Phi^\epsilon_{\tilde{\pi},\alpha_\mathfrak{u}}) \in D(\mathrm{Gal}_{p\infty}, L)$ defined in (3.19). In order to attach a p-adic L-function to $\tilde{\pi}$ without missing Euler factors at $\mathfrak{u}$ we need to also consider the distribution $\mathrm{ev}(\Phi^\epsilon_{\tilde{\pi},\beta_\mathfrak{u}}) \in D(\mathrm{Gal}_{p\infty}, L)$ using the other Hecke parameter $\beta_\mathfrak{u} \neq \alpha_\mathfrak{u}$ of $\pi_\mathfrak{u}$ (see Definition 1.5). We let

$$\mathcal{L}_p(\tilde{\pi}) = \sum_{\epsilon:\{\pm1\}^\Sigma\to\{\pm1\}} \frac{\alpha_\mathfrak{u}\mathrm{ev}(\Phi^\epsilon_{\tilde{\pi},\alpha_\mathfrak{u}}) - \beta_\mathfrak{u}\mathrm{ev}(\Phi^\epsilon_{\tilde{\pi},\beta_\mathfrak{u}})}{\alpha_\mathfrak{u} - \beta_\mathfrak{u}} \in D(\mathrm{Gal}_{p\infty}, L). \tag{4.2}$$

For any $f \in A(\mathrm{Gal}_{p\infty}, L)$, we let $\mathcal{L}_p(\tilde{\pi}, f) = \mathcal{L}_p(\tilde{\pi})(f)$.

By Proposition 3.11 the distribution $\mathcal{L}_p(\tilde{\pi})$ has growth at most $\sum_{v\in S_p} e_v h_{\tilde{\pi}_v}$ (see Definition 1.8), and we will next show that it interpolates critical values of the archimedean L-function of π and its twists.

We let r be a critical integer for the weight (k, w) and $j = (r-1)t + \frac{(\mathsf{w}-2)t+k}{2} \geq 0$ (see Definition 3.12). As observed in Remark 3.13(ii) we have $z^j \in L_{k,\mathsf{w}}(L)^{E(\mathcal{O}_F)} \subset A_{(k,\mathsf{w})}(L)^{E(\mathcal{O}_F)}$. Let $\Omega_{\tilde{\pi}}^{\epsilon} \in \mathbb{C}^\times$ be the period from Definition 1.14. The following key proposition allows us to relate the values at z^j of the distributions constructed in §3 to certain adelic integrals.

Proposition 4.1. *Let $\mathfrak{f} \mid p^\infty$ be an integral ideal, $\eta \in \mathbb{A}_{F,f}^\times$, and let $\mathrm{ev}_{\varpi_\mathfrak{f}}^{\eta} = \mathrm{ev}_{\varpi_\mathfrak{f}}^{\eta}(D_{(k,\mathsf{w})})$. Then*

$$\frac{\alpha_\mathfrak{f}\,\chi_{\mathrm{cyc}}^{r-1}(\eta)}{\alpha_\mathfrak{f}^{\circ}} \langle \mathrm{ev}_{\varpi_\mathfrak{f}}^{\eta}(\Phi_{\tilde{\pi},\alpha_\mathfrak{u}}^{\epsilon}), z^j \rangle = \frac{i^{(r-1)d}}{\Omega_{\tilde{\pi}}^{\epsilon}} \sum_{s_\infty \in \{\pm 1\}^{\Sigma}} s_\infty^{(r-1)t} \epsilon(s_\infty) \int_{X_\mathfrak{f}[\eta s_\infty]} \phi_{\tilde{\pi},\alpha_\mathfrak{u}}\left(\begin{smallmatrix} y\varpi_\mathfrak{f} & y 1_\mathfrak{f} \\ & 1 \end{smallmatrix}\right) |y|_F^{r-1}\, d^\times y.$$

Proof. Since the right hand side in the above formula is in $\mathbb{C}$, we will first prove that the left hand side, which is a priori a p-adic number, belongs in fact to $\iota_p(E)$. Denote by $\mathcal{L}_{k,\mathsf{w}}^{\vee}(E)$ the $G(\mathbb{Q})$-construction of a local system attached to $L_{k,\mathsf{w}}^{\vee}(E)$. Recall that $\mathcal{L}_{k,\mathsf{w}}^{\vee}(L)$ denotes the local system attached to $L_{k,\mathsf{w}}^{\vee}(L)$ by the K_p-construction. The following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}_c^d(Y_K, \mathcal{L}_{k,\mathsf{w}}^{\vee}(E)) & \xrightarrow{(g,v)\mapsto(g,g^{-1}\cdot v)} & \mathrm{H}_c^d(Y_K, \mathcal{L}_{k,\mathsf{w}}^{\vee}(L)) \\
\downarrow{\scriptstyle \mathcal{T}_\mathfrak{f}} & & \downarrow{\scriptstyle \mathrm{tw}_{\varpi_\mathfrak{f}} \circ C_{\varpi_\mathfrak{f}}^{*}} \\
\mathrm{H}_c^d(X_\mathfrak{f}, \mathcal{L}_{k,\mathsf{w}}^{\vee}(E)) & \xrightarrow{(y,v)\mapsto\left(y,\left(\begin{smallmatrix} y^{-1} & \\ & 1\end{smallmatrix}\right)\cdot v\right)} & \mathrm{H}^d(X_\mathfrak{f}, \mathcal{L}_{k,\mathsf{w}}^{\vee}(L)) \\
\downarrow{\scriptstyle \mathrm{triv}_\xi^{*} \circ \mathrm{coinv}_\mathfrak{f}} & & \downarrow{\scriptstyle \mathrm{triv}_\eta^{*} \circ \mathrm{coinv}_\mathfrak{f}} \\
\mathrm{H}_c^d(X_\mathfrak{f}[\eta]) \otimes L_{k,\mathsf{w}}^{\vee}(E)_{E(\mathfrak{f})} & \xrightarrow{\mathrm{id}\otimes\left(\begin{smallmatrix} \eta^{-1} & \\ & 1\end{smallmatrix}\right)_p} & \mathrm{H}_c^d(X_\mathfrak{f}[\eta]) \otimes L_{k,\mathsf{w}}^{\vee}(L)_{E(\mathfrak{f})} \\
\downarrow{\scriptstyle \langle - \cap \theta_{\mathfrak{f},[\eta]}, z^j\rangle} & & \downarrow{\scriptstyle \langle - \cap \theta_{\mathfrak{f},[\eta]}, z^j\rangle} \\
E & \xrightarrow{(\eta_p^{-1})^{(r-1)t}} & L
\end{array} \tag{4.3}$$

where the horizontal maps are induced from the morphisms of local systems written above them, the map $\mathcal{T}_\mathfrak{f}$ is induced from the morphisms of local systems $(y, v) \mapsto \left(\left(\begin{smallmatrix} y\varpi_\mathfrak{f} & y1_\mathfrak{f} \\ & 1 \end{smallmatrix}\right), v\right)$ and, for $\xi \in F^\times$, triv_ξ is induced from the morphisms of local systems:

$$X_\mathfrak{f}[\eta] \times L_{k,\mathsf{w}}^{\vee}(E)_{E(\mathfrak{f})} \to (\mathcal{L}_{k,\mathsf{w}}^{\vee}(E)_{E(\mathfrak{f})})_{|X_\mathfrak{f}[\eta]}, \qquad (y, v) \mapsto \left(y, \left(\begin{smallmatrix} \xi & \\ & 1 \end{smallmatrix}\right)^{-1} \cdot v\right).$$

By definition of the evaluations in §3 and by the functoriality relation (3.7), the composition of the maps in the right column sends $\iota_p(b_{\tilde{\pi},\alpha_\mathfrak{u}}^{\epsilon})$ to $\langle \mathrm{ev}_{\varpi_\mathfrak{f}}^{\eta}(L_{k,\mathsf{w}}^{\vee})(b_{\tilde{\pi},\alpha_\mathfrak{u}}^{\epsilon}), z^j \rangle = \frac{\alpha_\mathfrak{f}}{\alpha_\mathfrak{f}^{\circ}} \langle \mathrm{ev}_{\varpi_\mathfrak{f}}^{\eta}(\Phi_{\tilde{\pi},\alpha_\mathfrak{u}}^{\epsilon}), z^j \rangle$. The commutativity of the diagram then yields

$$\langle (\mathrm{triv}_\xi^{*} \circ \mathrm{coinv}_\mathfrak{f} \circ \mathcal{T}_\mathfrak{f})(b_{\tilde{\pi},\alpha_\mathfrak{u}}^{\epsilon}) \cap \theta_{\mathfrak{f},[\eta]}, z^j \rangle = \frac{\alpha_\mathfrak{f}}{\alpha_\mathfrak{f}^{\circ}} \eta_p^{(r-1)t} \langle \mathrm{ev}_{\varpi_\mathfrak{f}}^{\eta}(\Phi_{\tilde{\pi},\alpha_\mathfrak{u}}^{\epsilon}), z^j \rangle \in E. \tag{4.4}$$

Since the left column in (4.3) can be reproduced with $\mathcal{L}^\vee_{k,\mathrm{w}}(\mathbb{C})$ instead of $\mathcal{L}^\vee_{k,\mathrm{w}}(E)$, it follows that the left hand side of (4.4) can be computed via analytic methods, namely the comparison between Betti and de Rham cohomology over $\mathbb{C}$. Since $\chi_{\mathrm{cyc}}^{r-1}(\eta)\eta_p^{(1-r)t} = |\eta|_F^{r-1}$ one has to show

$$\begin{aligned}|\eta|_F^{r-1}&\langle(\mathrm{triv}_\xi^* \circ \mathrm{coinv}_\mathfrak{f} \circ \mathcal{T}_\mathfrak{f})(\Theta_\pi^\epsilon(\phi_{\tilde{\pi},\alpha_\mathfrak{u},f})) \cap \theta_{\mathfrak{f},[\eta]}, z^j\rangle \\ &= i^{(r-1)d} \sum_{s_\infty\in\{\pm1\}^\Sigma} s_\infty^{(r-1)t}\epsilon(s_\infty)\int_{X_\mathfrak{f}[\eta]} \phi_{\tilde{\pi},\alpha_\mathfrak{u}}\left(\begin{pmatrix} ys_\infty\varpi_\mathfrak{f} & y1_\mathfrak{f} \\ & 1\end{pmatrix}\right)|y|_F^{r-1}\, d^\times y. \qquad (4.5)\end{aligned}$$

Explicitly, $\Theta_\pi^\epsilon(\phi_{\tilde{\pi},\alpha_\mathfrak{u},f}) \in \mathrm{H}^d_{\mathrm{cusp}}(Y_K, \mathcal{L}^\vee_{k,\mathrm{w}}(\mathbb{C})) = \mathrm{H}^d_{\mathrm{dR},!}(Y_K, \mathcal{L}^\vee_{k,\mathrm{w}}(\mathbb{C}))$ is obtained as follows. For each $\eta \in \mathbb{A}^\times_{F,f}$ the relative Lie algebra differential

$$\bigotimes_{\sigma\in\Sigma} w_\sigma^* \otimes \mathrm{eval}_i \otimes \phi_\sigma \in \mathrm{H}^d(\mathfrak{g}_\infty, K_\infty^+, L^\vee_{k,\mathrm{w}}(\mathbb{C}) \otimes \pi_\infty)$$

yields a left-invariant d-form $\phi_{\tilde{\pi},\alpha_\mathfrak{u}}\left(\begin{pmatrix}\eta & \\ & 1\end{pmatrix} g_\infty\right)(g_\infty \cdot \mathrm{eval}_i)(g_\infty^{-1})^*(\wedge_\sigma w_\sigma^*)$ on G_∞^+/K_∞^+ which descends to $\Gamma_\eta\backslash G_\infty^+/K_\infty^+$ and, once translated by $\begin{pmatrix}\eta & \\ & 1\end{pmatrix}$, yields a d-form on $Y_K[\eta]$:

$$\phi_{\tilde{\pi},\alpha_\mathfrak{u}}(g)(g_\infty \cdot \mathrm{eval}_i)(g^{-1})^*(\wedge_\sigma w_\sigma^*).$$

Then

$$\begin{aligned}\Theta_\pi^\epsilon&(\phi_{\tilde{\pi},\alpha_\mathfrak{u},f}) \\ &= i^{\sum_{\sigma\in\Sigma}(2-\mathrm{w}-k_\sigma)/2} \sum_{s_\infty\in\begin{pmatrix}\pm1 & \\ & 1\end{pmatrix}^\Sigma} \epsilon(s_\infty)\phi_{\tilde{\pi},\alpha_\mathfrak{u}}(gs_\infty)(g_\infty s_\infty \cdot \mathrm{eval}_i)((gs_\infty)^{-1})^*(\wedge_\sigma w_\sigma^*).\end{aligned}$$

By [19] there exists $v \in F$ and a commutative diagram

$$\begin{array}{ccc} \Gamma_\eta\backslash G_\infty^+/K_\infty^+ & \xrightarrow{\begin{pmatrix} y & \\ & 1\end{pmatrix}} & Y_K[\eta] \\ {\scriptstyle u_\infty\mapsto\begin{pmatrix}u_\infty & -v_\infty \\ & 1\end{pmatrix}}\uparrow & & \uparrow{\scriptstyle C_{\varpi_\mathfrak{f}}} \\ E(\mathfrak{f})\setminus F_\infty^{\times+} & \xrightarrow{\cdot y} & X_\mathfrak{f}[\eta] \end{array}$$

inducing for $g = C_{\varpi_\mathfrak{f}}(y)$ and $\mathfrak{h}$ the Lie algebra of GL_1 a commutative diagram

$$\begin{array}{ccc} (\mathfrak{g}/\mathfrak{k})^* & \xrightarrow{(g^{-1})^*} & (T_g Y_K[\eta])^* \\ {\scriptstyle \iota^*}\downarrow & & \downarrow{\scriptstyle C^*_{\varpi_\mathfrak{f}}} \\ \mathfrak{h}^* & \xrightarrow{(y^{-1})^*} & (T_y X_\mathfrak{f}[\eta])^* \end{array}$$

While $\mathfrak{g}/\mathfrak{k}$ denotes the tangent space of G_∞^+/K_∞^+ at $\begin{pmatrix}1 & -v_\infty \\ & 1\end{pmatrix}$ and not at the identity, the map $\iota : \mathfrak{h} \to \mathfrak{g}/\mathfrak{k}$ is still given by $u_\sigma = 1 \mapsto \left(\begin{smallmatrix}1 & 0 \\ 0 & 0\end{smallmatrix}\right) = w_\sigma + \bar{w}_\sigma$ since horizontal translations in the upper half-plane do not change dy. Hence

$$\mathcal{T}_{\mathfrak{f}}(\phi_{\tilde{\pi},\alpha_{\mathfrak{u}}}(g)(g_\infty\cdot\mathrm{eval}_i)(g^{-1})^*(\wedge_\sigma w_\sigma^*)) \\ = \phi_{\tilde{\pi},\alpha_{\mathfrak{u}}}\left(\begin{smallmatrix} y\varpi_{\mathfrak{f}} & y1_{\mathfrak{f}} \\ & 1\end{smallmatrix}\right)\left(\left(\begin{smallmatrix} y_\infty & \\ & 1\end{smallmatrix}\right)\cdot\mathrm{eval}_i\right)(y^{-1})^*(\wedge_\sigma u_\sigma^*).$$

Since $y=\xi\eta u u_\infty$, we have $y_\infty=\xi_\infty u_\infty$ and

$$\mathrm{triv}_\xi^*\circ\mathrm{coinv}_{\mathfrak{f}}(\mathcal{T}_{\mathfrak{f}}(\phi_{\tilde{\pi},\alpha_{\mathfrak{u}}}(g)(g_\infty\cdot\mathrm{eval}_i)(g^{-1})^*(\wedge_\sigma w_\sigma^*))) \\ = (\phi_{\tilde{\pi},\alpha_{\mathfrak{u}}}\left(\begin{smallmatrix} y\varpi_{\mathfrak{f}} & y1_{\mathfrak{f}} \\ & 1\end{smallmatrix}\right)\left(\left(\begin{smallmatrix} u_\infty & \\ & 1\end{smallmatrix}\right)\cdot\mathrm{eval}_i\right)(y^{-1})^*(\wedge_\sigma u_\sigma^*)).$$

Now, a top degree invariant differential is a Haar measure, hence $(y^{-1})^*(\wedge_\sigma u_\sigma^*)=d^\times y$. By (1.5) we have $\left(\left(\begin{smallmatrix} u_\infty & \\ & 1\end{smallmatrix}\right)\cdot\mathrm{eval}_i\right)(z^j)=u_\infty^{(r-1)t}i^j$, and since $|y|_F=|\eta|_F u_\infty^t$ we deduce that

$$|\eta|_F^{r-1}\langle(\mathrm{triv}_\xi^*\circ\mathrm{coinv}_{\mathfrak{f}}\circ\mathcal{T}_{\mathfrak{f}})(\phi_{\tilde{\pi},\alpha_{\mathfrak{u}}}(g)(g_\infty\cdot\mathrm{eval}_i)(g^{-1})^*(\wedge_\sigma w_\sigma^*))\cap\theta_{\mathfrak{f},[\eta]},z^j\rangle \\ = i^j\int_{X_{\mathfrak{f}}[\eta]}\phi_{\tilde{\pi},\alpha_{\mathfrak{u}}}\left(\left(\begin{smallmatrix} y\varpi_{\mathfrak{f}} & y1_{\mathfrak{f}} \\ & 1\end{smallmatrix}\right)\right)|y|_F^{r-1}\,d^\times y.$$

Since $j_\sigma+\frac{2-\mathsf{w}-k_\sigma}{2}=r-1$ for all $\sigma\in\Sigma$, and since right translating g by elements of $\left(\begin{smallmatrix}\pm1 & \\ & 1\end{smallmatrix}\right)^\Sigma$ amounts to translating u_∞ by elements of $\{\pm1\}^\Sigma$, one obtains (4.5) and hence the claim. ∎

We now prove the main interpolation relation between p-adic and complex L-functions.

Theorem 4.2. *Let χ be a finite order character of $\mathrm{Gal}_{p\infty}$ and for v dividing p denote by c_v the conductor of $\chi_v\nu_v$. Let $r\in\mathbb{Z}$ be a critical integer for (k,w). Letting $\mathrm{N}_{F/\mathbb{Q}}(i)=i^d$, one has*

$$\mathcal{L}_p(\tilde{\pi},\chi\cdot\chi_{\mathrm{cyc}}^{r-1})=\frac{\mathrm{N}_{F/\mathbb{Q}}^{r-1}(i\mathfrak{d})\chi(\varpi_{\mathfrak{d}}^{-1})}{\Omega_{\tilde{\pi}}^{\chi_\infty\omega_{p,\infty}^{r-1}}}L(\pi\otimes\chi,r-1/2)\prod_{v\in S_p}E(\tilde{\pi}_v,\chi_v,r),\qquad(4.6)$$

where $E(\tilde{\pi}_v,\chi_v,s)$ equals

$$\begin{cases} q_v^{sc_v}(\chi_v\nu_v)(\varpi_v^{\delta_v})\tau(\chi_v\nu_v,\psi_v,d_{\chi_v\nu_v}) & \text{if } c_v\ge1 \text{ and } \chi_v\omega_\pi\nu_v^{-1} \text{ is ramified},\\ \left(1-\frac{(\chi_v\omega_\pi\nu_v^{-1})(\varpi_v)}{q_v^{s-1}}\right)\frac{\tau(\chi_v\nu_v,\psi_v,d_{\chi_v\nu_v})}{q_v^{-sc_v}(\chi_v\nu_v)(\varpi_v^{-\delta_v})} & \text{if } c_v\ge1 \text{ and } \chi_v\omega_\pi\nu_v^{-1} \text{ is unramified},\\ \left(1-\frac{(\chi_v\omega_\pi\nu_v^{-1})(\varpi_v)}{q_v^{s-1}}\right)\left(1-\frac{q_v^{s-1}}{(\chi_v\nu_v)(\varpi_v)}\right) & \text{if } \pi_v\otimes\chi_v \text{ is unramified},\\ 1-\frac{q_v^{s-1}}{(\chi_v\nu_v)(\varpi_v)} & \text{otherwise}.\end{cases}$$

Proof. We will evaluate $\mathcal{L}_p(\tilde{\pi},\chi\cdot\chi_{\mathrm{cyc}}^{r-1})$ using automorphic symbols of level $\mathfrak{f}=\prod_{v\in S_p}v^{n_v}$ such that $n_v\ge\max(c_v,1)$ and will see that the result does not depend on $\mathfrak{f}$, as expected.

Using the notations from (4.1), (3.19), Definition 3.8, (3.18) and (3.16), one gets

$$\begin{aligned}\langle \mathrm{ev}(\Phi^{\epsilon}_{\tilde{\pi},\alpha_{\mathfrak{u}}}), \chi\cdot\chi_{\mathrm{cyc}}^{r-1}\rangle &= (\alpha_{\mathfrak{f}}^{\circ})^{-1}\langle \mathrm{ev}_{\varpi_{\mathfrak{f}}}(\Phi^{\epsilon}_{\tilde{\pi},\alpha_{\mathfrak{u}}}), \chi\cdot\chi_{\mathrm{cyc}}^{r-1}\rangle \\ &= (\alpha_{\mathfrak{f}}^{\circ})^{-1}\sum_{[\eta]\in\mathscr{C}\ell_F^+(\mathfrak{f})}\chi(\eta)\langle \mathrm{ev}^{\eta}_{\varpi_{\mathfrak{f}}}(\Phi^{\epsilon}_{\tilde{\pi},\alpha_{\mathfrak{u}}}), (\chi_{\mathrm{cyc}}^{r-1}\circ\iota_\eta)^{\times}\rangle \\ &= \alpha_{\mathfrak{f}}^{-1}\sum_{[\eta]\in\mathscr{C}\ell_F^+(\mathfrak{f})}\chi(\eta)\frac{\alpha_{\mathfrak{f}}\chi_{\mathrm{cyc}}^{r-1}(\eta)}{\alpha_{\mathfrak{f}}^{\circ}}\langle \mathrm{ev}^{\eta}_{\varpi_{\mathfrak{f}}}(\Phi^{\epsilon}_{\tilde{\pi},\alpha_{\mathfrak{u}}}), z^{j}\rangle.\end{aligned}$$

By Proposition 4.1 the latter sum vanishes unless $\epsilon = \chi_\infty\omega_{p,\infty}^{r-1}$, in which case it equals

$$\mathscr{I} = \frac{2^d i^{(r-1)d}}{\Omega^{\epsilon}_{\tilde{\pi}}}\int_{X_{\mathfrak{f}}}\chi(y)\phi_{\tilde{\pi},\alpha_{\mathfrak{u}}}\left(\begin{smallmatrix} y\varpi_{\mathfrak{f}} & y1_{\mathfrak{f}}\\ & 1\end{smallmatrix}\right)|y|_F^{r-1}\,d^{\times}y.$$

Since $X_{\mathfrak{f}} = F_+^{\times}\backslash\mathbb{A}^{\times}_{F,f}F_\infty^{\times +}/U(\mathfrak{f})$ and the Haar measure on $\mathbb{A}_F^{\times}$ gives $U(1)$ volume 1, we have

$$\mathscr{I} = \frac{2^d i^{(r-1)d}}{\Omega^{\epsilon}_{\tilde{\pi}}}\prod_{v\in S_p} q_v^{n_v}\left(1-\frac{1}{q_v}\right)\cdot\int_{F_+^{\times}\backslash\mathbb{A}^{\times}_{F,f}F_\infty^{\times +}}\chi(y)\phi_{\tilde{\pi},\alpha_{\mathfrak{u}}}\left(\begin{smallmatrix} y\varpi_{\mathfrak{f}} & y1_{\mathfrak{f}}\\ & 1\end{smallmatrix}\right)|y|_F^{r-1}\,d^{\times}y.$$

Since $\left(\begin{smallmatrix} y\varpi_{\mathfrak{f}} & y1_{\mathfrak{f}}\\ & 1\end{smallmatrix}\right)\in G(\mathbb{A}_f)G_\infty^+$, using the Fourier expansion formula (1.21) we further compute

$$\begin{aligned}&\int_{F_+^{\times}\backslash\mathbb{A}^{\times}_{F,f}F_\infty^{\times +}}\chi(y)\phi_{\tilde{\pi},\alpha_{\mathfrak{u}}}\left(\begin{smallmatrix} y\varpi_{\mathfrak{f}} & y1_{\mathfrak{f}}\\ & 1\end{smallmatrix}\right)|y|_F^{r-1}\,d^{\times}y \\ &\qquad = \int_{\mathbb{A}^{\times}_{F,f}F_\infty^{\times +}}\chi(y)W_{\tilde{\pi},\alpha_{\mathfrak{u}}}\left(\begin{smallmatrix} y\varpi_{\mathfrak{f}} & y1_{\mathfrak{f}}\\ & 1\end{smallmatrix}\right)|y|_F^{r-1}\,d^{\times}y = \prod_v Z_v.\end{aligned}$$

A standard calculation (see, e.g., [18, (16)]) shows that

$$\begin{aligned} Z_\sigma &= \int_{\mathbb{R}_+^{\times}} W_\sigma\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)y^{r-2}\,dy = \int_0^{\infty} e^{-2\pi y}y^{j_\sigma}\,dy = \frac{j_\sigma!}{(2\pi)^{j_\sigma+1}} \\ &= \tfrac{1}{2}L(\pi_\sigma, r-1/2)\qquad \text{if } \sigma\mid\infty.\end{aligned}$$

A straightforward generalization of [41, Prop. 3.5] from χ_v trivial to χ_v unramified yields

$$\begin{aligned} Z_v &= \int_{F_v^{\times}}\chi_v(y)W_v^{\mathrm{new}}\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)|y|_v^{r-1}\,d^{\times}y \\ &= (q_v^{1-r}\chi_v(\varpi_v))^{-\delta_v}L(\pi_v\otimes\chi_v, r-1/2)\qquad \text{if } v\nmid p\mathfrak{u}\infty, \\ Z_{\alpha_{\mathfrak{u}}} &= \int_{F_{\mathfrak{u}}^{\times}}\chi_{\mathfrak{u}}(y)W_{\mathfrak{u}}^{\alpha}\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)|y|_{\mathfrak{u}}^{r-1}\,d^{\times}y_{\mathfrak{u}} = L(\pi_{\mathfrak{u}}\otimes\chi_{\mathfrak{u}}, r-1/2)\big(1-\tfrac{\chi_{\mathfrak{u}}(\varpi_{\mathfrak{u}})\beta_{\mathfrak{u}}}{q_{\mathfrak{u}}^r}\big),\end{aligned}$$

hence $\frac{\alpha_{\mathfrak{u}}Z_{\alpha_{\mathfrak{u}}}-\beta_{\mathfrak{u}}Z_{\beta_{\mathfrak{u}}}}{\alpha_{\mathfrak{u}}-\beta_{\mathfrak{u}}} = L(\pi_{\mathfrak{u}}\otimes\chi_{\mathfrak{u}}, r-1/2)$. Finally, for $v\in S_p$ the integral Z_v is computed in Proposition 1.12 and $E(\tilde{\pi}_v, \chi_v, s) = Q(\chi_v\nu_v, s)/L(\pi_v\otimes\chi_v, s)$. Putting everything together yields the desired formula. ∎

Remark 4.3. The interpolation formula (4.6) is independent of the choice of uniformizers ϖ_v at $v \in S_p$, since $\Omega_{\tilde{\pi}}^{\epsilon} \prod_{v\in S_p} \nu_v(\varpi_v^{-\delta_v})$ is independent of that choice by Proposition 1.15.

We end this subsection by classifying the trivial zeros of $\mathcal{L}_p(\tilde{\pi})$, i.e., by determining when $\mathcal{L}_p(\tilde{\pi}, \chi \cdot \chi_{\mathrm{cyc}}^{r-1})$ in (4.6) vanishes regardless of the value of $L(\pi \otimes \chi, r - 1/2)$.

Proposition 4.4. *Given a finite order character χ_v of $F_v^\times$ and $r \in \mathbb{Z}$, one has $E(\tilde{\pi}_v, \chi_v, r) = 0$ if and only if either*

(i) *$\pi_v \otimes \nu_v^{-1}$ is the Steinberg representation, $r = \frac{2-\mathsf{w}}{2}$, and $\chi_v = \nu_v^{-1} \cdot \mathrm{unr}(q_v^{-\mathsf{w}/2})$, or*

(ii) *π_v is a principal series representation, $r = \frac{3-\mathsf{w}}{2}$ (resp. $r = \frac{1-\mathsf{w}}{2}$) and $\chi_v = \nu_v^{-1} \cdot \mathrm{unr}(q_v^{(1-\mathsf{w})/2})$ (resp. $\chi_v = \nu_v \omega_\pi^{-1} \cdot \mathrm{unr}(q_v^{-(1+\mathsf{w})/2})$.*

In the first case w is necessarily even and the trivial zero occurs at a central critical point, while in the second case w is odd and the trivial zero occurs at a nearly central critical point.

Proof. If π_v is a twist (necessarily by ν_v) of the unitary Steinberg representation, then $E(\tilde{\pi}_v, \chi_v, r)$ is non-zero, unless $c_v = 0$ in which case $E(\tilde{\pi}_v, \chi_v, r) = 1 - \frac{q_v^{r-1}}{(\chi_v\nu_v)(\varpi_v)}$. By local-global compatibility, $\nu_v(\varpi_v)$ is a Weil number of weight $-\mathsf{w}$, i.e., an algebraic number whose absolute values are all equal to $q_v^{-\mathsf{w}/2}$. Hence $E(\tilde{\pi}_v, \chi_v, r)$ vanishes precisely as stated.

If π_v is a principal series representation, then

$$E(\tilde{\pi}_v, \chi_v, r) = (1 - (\chi_v\nu_v)^{-1}(\varpi_v)q_v^{r-1})(1 - (\chi_v\omega_\pi\nu_v^{-1})(\varpi_v)q_v^{1-r}),$$

where the first (resp. second) factor is dropped if $\chi_v\nu_v$ (resp. $\chi_v\omega_\pi\nu_v^{-1}$) is ramified. The Ramanujan conjecture for the cuspidal automorphic representation π, proven in [10, Thm. 1], implies that $\nu_v(\varpi_v)$ is a Weil number of weight $1 - \mathsf{w}$. Since ω_π has purity weight w, the first (resp. second) factor can only vanish for $r = \frac{3-\mathsf{w}}{2}$ (resp. $r = \frac{1-\mathsf{w}}{2}$) precisely as stated. ∎

4.2. Multi-variable p-adic L-functions

Since the construction of $\mathcal{O}(\mathcal{U})$-valued distributions over $\mathrm{Gal}_{p\infty}$ presented in §3.4 is functorial in the L-affinoid $\mathcal{U}$, following the same steps as in §4.1 would allow us to attach a p-adic L-function to a rigid analytic family containing $\tilde{\pi}$. In addition to the cyclotomic variable, this function will have several weight variables. In conjunction with its improvements which will be constructed in the next subsection, this multi-variable p-adic L-function will allow us to prove the Trivial Zero Conjecture for π at the central point in the second part of this paper.

Let $\tilde{\pi}$ be a regular non-critical p-refinement of a cuspidal automorphic representation π of $G(\mathbb{A})$ of cohomological weight (k, w). Letting $h = (h_{\tilde{\pi}_v})_{v\in S_p}$ (see Definition 1.8), by Theorem 2.14 there exists an L-affinoid neighborhood $\mathcal{U}$ of (k, w) in $\mathcal{X}$ and a connected component $\mathcal{V}$ of $\mathrm{Sp}(\mathbb{T}_{\mathcal{U}}^{\leq h})$ containing $\tilde{\pi}$ such that the weight map $\kappa : \mathcal{V} \xrightarrow{\sim} \mathcal{U}$ is an isomorphism. By shrinking $\mathcal{U}$ one can assume that $\mathsf{w}_\lambda \circ \omega_p = \omega_p^{\mathsf{w}}$ and

$$\alpha_{\mathfrak{u}}(\kappa^{-1}(\lambda))^2 \omega_p^{-\mathsf{w}}(\varpi_{\mathfrak{u}})\mathsf{w}_\lambda^{-1}(\langle \varpi_{\mathfrak{u}} \rangle) \neq q_{\mathfrak{u}}^i \quad \text{for } i \in \{0,1,2\} \text{ and } \lambda \in \mathcal{U}, \tag{4.7}$$

since the left hand side is an analytic function on $\lambda \in \mathcal{U}$ and (4.7) holds at (k, w).

Definition 4.5. Given a character $\epsilon : \{\pm 1\}^\Sigma \to \{\pm 1\}$ we let $\Phi^\epsilon_{\mathcal{U},\alpha_{\mathfrak{u}}} \in \mathrm{H}^d_c(Y_K, \mathcal{D}_{\mathcal{U}})^{\leq h}$ be a basis of the free rank 1 $\mathcal{O}(\mathcal{U})$-module $\mathrm{H}^d_c(Y_K, \mathcal{D}_{\mathcal{U}})^{\epsilon,\leq h} \otimes_{\mathbb{T}_{\mathcal{U}}^{\leq h}} \mathcal{O}(\mathcal{V})$ (see Theorem 2.14(ii)) such that $(k, \mathsf{w}) \circ \Phi^\epsilon_{\mathcal{U},\alpha_{\mathfrak{u}}} = \Phi^\epsilon_{\tilde{\pi},\alpha_{\mathfrak{u}}}$. For $v \in S_p$ we let $\alpha_v^\circ \in \mathcal{O}(\mathcal{U})^\times$ denote the U_{ϖ_v}-eigenvalue on $\Phi^\epsilon_{\mathcal{U},\alpha_{\mathfrak{u}}}$.

When $\lambda \in \mathcal{U}(L)$ is cohomological, by Theorem 2.14(iii) there exists a p-refined nearly finite slope cuspidal automorphic representation $\tilde{\pi}_\lambda$ of weight λ whose system of Hecke eigenvalues corresponds to $\kappa^{-1}(\lambda) \in \mathcal{V}$. Using the specialization map at such a cohomological weight λ, we obtain a class $\lambda \circ \Phi^\epsilon_{\mathcal{U},\alpha_{\mathfrak{u}}} \in \mathrm{H}^d_c(Y_K, \mathcal{D}_\lambda)^{\leq h}$ which generates the same line as the class $\Phi^\epsilon_{\tilde{\pi}_\lambda,\alpha_{\mathfrak{u}}(\lambda)}$ from §4.1.

Definition 4.6. Let $C^\epsilon_\lambda \in L^\times$ be such that $\lambda \circ \Phi^\epsilon_{\mathcal{U},\alpha_{\mathfrak{u}}} = C^\epsilon_\lambda \cdot \Phi^\epsilon_{\tilde{\pi}_\lambda,\alpha_{\mathfrak{u}}(\lambda)}$.

Note that whereas $C^\epsilon_{(k,\mathsf{w})} = 1$ by definition, $C^\epsilon_\lambda \in L^\times$ is a p-adic period analogous to those considered in [21], and cannot in general be rescaled to be 1, since the individual periods $\Omega^\epsilon_{\tilde{\pi}_\lambda}$ for λ cohomological are well-defined up to $\overline{\mathbb{Q}}^\times$. Since $\alpha_{\mathfrak{u}}, \beta_{\mathfrak{u}} \in \mathcal{O}(\mathcal{U})^\times$, one can analogously consider $\Phi^\epsilon_{\mathcal{U},\beta_{\mathfrak{u}}}$ and rescale it so that it yields the same p-adic periods C^ϵ_λ.

The multi-variable p-adic L-function is defined as the distribution (see (3.19))

$$\mathcal{L}_p = \sum_{\epsilon : \{\pm 1\}^\Sigma \to \{\pm 1\}} \frac{\alpha_{\mathfrak{u}} \mathrm{ev}(\Phi^\epsilon_{\mathcal{U},\alpha_{\mathfrak{u}}}) - \beta_{\mathfrak{u}} \mathrm{ev}(\Phi^\epsilon_{\mathcal{U},\beta_{\mathfrak{u}}})}{\alpha_{\mathfrak{u}} - \beta_{\mathfrak{u}}} \in D(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{U})). \tag{4.8}$$

For any finite order character $\chi : \mathrm{Gal}_{p\infty} \to L^\times$, the natural projection $\mathrm{Gal}_{p\infty} \twoheadrightarrow \mathrm{Gal}_{\mathrm{cyc}}$ yields

$$\mathcal{L}_{p,\chi} = \frac{\alpha_{\mathfrak{u}} \langle \mathrm{ev}(\Phi^{\chi_\infty}_{\mathcal{U},\alpha_{\mathfrak{u}}}), \chi \cdot \rangle - \beta_{\mathfrak{u}} \langle \mathrm{ev}(\Phi^{\chi_\infty}_{\mathcal{U},\beta_{\mathfrak{u}}}), \chi \cdot \rangle}{\alpha_{\mathfrak{u}} - \beta_{\mathfrak{u}}} \in D(\mathrm{Gal}_{\mathrm{cyc}}, \mathcal{O}(\mathcal{U})). \tag{4.9}$$

Theorem 4.7. *Let $\tilde{\pi}$ and $\mathcal{U}$ be as above. Fix a finite order character $\chi : \mathrm{Gal}_{p\infty} \to L^\times$.*

(i) *For any $\lambda \in \mathcal{U}(L)$ cohomological, $\mathcal{L}_{p,\chi}(\lambda) = C^{\chi_\infty}_\lambda \mathcal{L}_p(\tilde{\pi}_\lambda, \chi \cdot)$ in $D(\mathrm{Gal}_{\mathrm{cyc}}, L)$.*

(ii) *$\mathcal{L}_{p,\chi}$ has order of growth at most $\sum_{v \in S_p} e_v h_{\tilde{\pi}_v}$ and is uniquely determined by its values*

$$\mathcal{L}_{p,\chi}(\lambda, \chi' \chi_{\mathrm{cyc}}^{r-1}) = \mathcal{L}_p(\lambda, \chi\chi' \chi_{\mathrm{cyc}}^{r-1}),$$

at finite order characters χ' of $\mathrm{Gal}_{\mathrm{cyc}}$ and $r \in \mathbb{Z}$ critical for λ cohomological (see (4.6)).

Proof. (i) follows from the definition and the functoriality of ev.

(ii) For $\lambda \in \mathcal{U}$ cohomological such that $\tilde{\pi}_\lambda$ has very non-critical slope (see (1.15)), the distribution $\mathcal{L}_{p,\chi}(\lambda)$ on $\mathrm{Gal}_{\mathrm{cyc}}$ has order of growth strictly less than the number $\min_{\sigma\in\Sigma}(k_{\lambda,\sigma}-1)$ of critical integers for λ. A well-known result of Vishik [51, Thm. 2.3, Lem. 2.10], proven independently by Amice-Vélu, implies that $\mathcal{L}_{p,\chi}(\lambda)$ is uniquely determined by its values at $\chi'\chi_{\mathrm{cyc}}^{r-1}$, where χ' is a finite order character of $\mathrm{Gal}_{\mathrm{cyc}}$ and $r \in \mathbb{Z}$ is critical for λ. The claim is deduced by noticing that such λ form a very Zariski dense subset of $\mathcal{U}$. ■

Remark 4.8. When Leopoldt's conjecture holds for F at p, the kernel of the natural projection $\mathrm{Gal}_{p\infty} \twoheadrightarrow \mathrm{Gal}_{\mathrm{cyc}}$ is a finite abelian group and $\mathcal{L}_p$ is merely a collection of $\mathcal{L}_{p,\chi}$ with χ running over the characters of that group.

Remark 4.9. If $\tilde{\pi}$ is non-critical, but has critical slope, then interpolation formula (4.6) does not suffice to determine $\mathcal{L}_p(\tilde{\pi})$ uniquely, and we are indebted to J. Bellaïche for having explained to one of us how the smoothness of the eigenvariety can be used to palliate this indeterminacy. When π is Iwahori spherical at all places above p, a similar approach has also been successfully used by Bergdall and Hansen [8] who construct $\mathcal{L}_p$ for regular $\tilde{\pi}$ which are either non-critical, or such that $\mathrm{H}_c^\bullet(Y_K, \mathcal{D}_{k,\mathrm{w}})_{\mathfrak{m}_{\tilde{\pi}}}$ is concentrated in degree d and the adjoint Bloch–Kato Selmer group $\mathrm{H}^1_f(F, \mathrm{ad}(V_\pi))$ vanishes.

It will be essential in §7.2 to control $\mathcal{L}_{p,\chi}$ under simultaneous variation in w_λ and the cyclotomic variable. This, however, can only be achieved after a renormalization of the p-adic periods, whose variations in $\mathcal{U}$ are *a priori* well-defined up to an invertible analytic function. This is equivalent to rescaling the basis $\Phi^\epsilon_{\mathcal{U},\alpha_{\mathfrak{u}}}$ by an invertible element of $\mathcal{O}(\mathcal{U})$.

Proposition 4.10. *For any finite order character $\chi : \mathrm{Gal}_{p\infty} \to L^\times$, the p-adic periods $C_\lambda^{\chi\infty}$ of Definition 4.6 can be renormalized so that for $z \in \mathcal{O}_{\mathbb{C}_p}$ and $\lambda \in \mathcal{U}$ such that $\lambda(z) = (k_\lambda, \mathrm{w}_\lambda\langle\cdot\rangle^{2z}) \in \mathcal{U}$, one has*

$$\mathcal{L}_{p,\chi}(\lambda(z)) = \mathcal{L}_{p,\chi}(\lambda, \langle\cdot\rangle_p^z\,\cdot). \tag{4.10}$$

Proof. By analyticity it suffices to check (4.10) for $z \in \mathbb{Z}$ and for $\lambda \in \mathcal{U}$ cohomological such that $\tilde{\pi}_\lambda$ has very non-critical slope, since such pairs (λ, z) are very Zariski dense in $\mathcal{U} \times \mathcal{O}_{\mathbb{C}_p}$. By Theorem 2.14 the weight map $\kappa : \mathcal{V} \to \mathcal{U}$ is etale at λ and so, by the assumption on z, if $\kappa^{-1}(\lambda) = \tilde{\pi}_\lambda$ then $\kappa^{-1}(\lambda(z)) = \widetilde{\pi_\lambda \otimes |\cdot|^z}$. Noting that an integer r is critical for π if and only if $r - z$ is critical for $\pi \otimes |\cdot|^z$, Propositions 1.15 and 4.2 imply

$$\mathcal{L}_p(\widetilde{\pi_\lambda \otimes |\cdot|^z}, \langle\cdot\rangle_p^{-z}\chi\,\cdot) = \mathcal{L}_p(\tilde{\pi}_\lambda, \chi\,\cdot) \quad \text{in } D(\mathrm{Gal}_{\mathrm{cyc}}, L).$$

As a result, $C_{\lambda(z)}^{\chi\infty} = C_\lambda^{\chi\infty}\frac{\mathcal{L}_{p,\chi}(\lambda(z),\langle\cdot\rangle_p^z\cdot)}{\mathcal{L}_{p,\chi}(\lambda)}$ for $\lambda \in \mathcal{U}$ cohomological and for $z \in \mathbb{Z}$ sufficiently small (p-adically). For $\mathrm{w}_\lambda = \langle\cdot\rangle^{2z}\mathrm{w}$, the function $\lambda \mapsto \mathcal{L}_{p,\chi}(\lambda, \langle\cdot\rangle_p^z\cdot)/\mathcal{L}_{p,\chi}((k_\lambda, \mathrm{w}))$ belongs to $\mathcal{O}(\mathcal{U})^\times$, for $\mathcal{U}$ sufficiently small, and interpolates $C_\lambda^{\chi\infty}/C_{(k_\lambda,\mathrm{w})}^{\chi\infty}$ for λ cohomological. We may therefore renormalize the periods $C_\lambda^{\chi\infty}$ to guarantee (4.10). ■

Remark 4.11. As in Proposition 4.10, for any finite order characters χ of $\mathrm{Gal}_{p\infty}$ one has

$$\mathcal{L}_p(\pi \mathbin{\widetilde{\otimes}} \chi, \cdot) = \mathcal{L}_p(\widetilde{\pi}, \chi\, \cdot) \quad \text{in } D(\mathrm{Gal}_{\mathrm{cyc}}, L). \tag{4.11}$$

If there exists a finite order character χ of $\mathrm{Gal}_{p\infty}$ such that $\chi_{|\mathcal{O}_v^\times} = \nu_v^{-1}{}_{|\mathcal{O}_v^\times}$ for all $v \in S_p$, then $\pi \otimes \chi$ has finite slope. Such a character always exists when $F = \mathbb{Q}$, allowing one to reduce to the finite slope case. For general F, such a character may have auxiliary ramification forcing the twisted finite slope form to have a tame level different from the original one.

4.3. *Improved p-adic L-functions*

When $\mathcal{L}_p(\widetilde{\pi})$ has a trivial zero at a critical integer r the interpolation formula for $\mathcal{L}_p(\widetilde{\pi}, \chi\chi_{\mathrm{cyc}}^r)$ 'misses' the special L-value $L(\pi \otimes \chi, r - 1/2)$. An idea due to Greenberg and Stevens [21] is to then construct a so-called improved p-adic L-function having only weight variables and interpolating, with non-vanishing extra factors, the critical L-value. In order to retrieve the 'missed' L-value even in the case where several local factors $E(\widetilde{\pi}_v, \chi_v, r)$ simultaneously vanish we will construct for any $S \subset S_p$ a rigid analytic function $L_S(\lambda, \chi, r)$ over an $(|\Sigma_S| + 1)$-dimensional affinoid $\mathcal{U}'_S = \mathcal{X}'_S \cap \mathcal{U}$, where $\mathcal{U}$ is as in §4.2.

Consider the subset $S \subset S_p$ containing all places $v \in S_p$ such that ν_v is ramified (note that this is always the case if π has finite slope). For each character $\epsilon : \{\pm 1\}^\Sigma \to \{\pm 1\}$ let $\Phi^\epsilon_{\mathcal{U},\alpha_{\mathfrak{u}}}$ be as in §4.2 and denote by α_v° its U_{ϖ_v}-eigenvalue. Denote by $\Phi^\epsilon_{\mathcal{U}'_S,\alpha_{\mathfrak{u}}}$ the image of $\Phi^\epsilon_{\mathcal{U},\alpha_{\mathfrak{u}}}$ in $\mathrm{H}^d_c(Y_K, \mathcal{D}_{\mathcal{U}'_S})^{\leq h}$. By Definition 2.1 and (4.1), for all $v \in S_p \setminus S$ we have

$$\alpha_v(\lambda) = \alpha_v^\circ(\lambda) \prod_{\sigma \in \Sigma_v} \sigma(\varpi_v)^{(2-\mathrm{w}-k_\sigma)/2} \in \mathcal{O}(\mathcal{U}'_S)^\times.$$

For any $v \in S_p \setminus S$, the rigid analytic function $1 - \frac{q_v^{r-1}}{\chi_v(\varpi_v)\alpha_v(\cdot)} \in \mathcal{O}(\mathcal{U}'_S)$ specializes at λ cohomological to the interpolation factor $1 - \frac{q_v^{r-1}}{(\chi_v\nu_{\lambda,v})(\varpi_v)}$ from Theorem 4.2. Our aim is to show that the meromorphic quotient $\mathcal{L}_p(\lambda, \chi\chi_{\mathrm{cyc}}^r) \prod_{S_p \setminus S}\big(1 - \frac{q_v^{r-1}}{\chi_v(\varpi_v)\alpha_v(\cdot)}\big)$ is in fact analytic and to compute its value at (k, w). Achieving this requires to take a step back and define the improved p-adic L-functions using the tools developed in §3.5.

Definition 4.12. For $\chi : \mathrm{Gal}_{S\infty} \to L^\times$ a finite order character, we define the improved p-adic L-function as

$$L_S(\cdot, \chi, r) = \frac{\alpha_{\mathfrak{u}} \langle \mathrm{ev}_S^r(\Phi^{\chi_\infty \omega_{p,\infty}^{r-1}}_{\mathcal{U}'_S,\alpha_{\mathfrak{u}}}), \chi\rangle - \beta_{\mathfrak{u}} \langle \mathrm{ev}_S^r(\Phi^{\chi_\infty \omega_{p,\infty}^{r-1}}_{\mathcal{U}'_S,\beta_{\mathfrak{u}}}), \chi\rangle}{\alpha_{\mathfrak{u}} - \beta_{\mathfrak{u}}} \in \mathcal{O}(\mathcal{U}'_S).$$

It follows from (3.26) that $L_{S_p}(\lambda, \chi, r) = \mathcal{L}_p(\lambda, \chi\chi_{\mathrm{cyc}}^{r-1})$.

Theorem 4.13. (i) *For $v \in S$, we have the following equality in $\mathcal{O}(\mathcal{U}'_{S\setminus\{v\}})$:*

$$\vartheta_{S,v}(L_S(\cdot, \chi, r)) = L_{S\setminus\{v\}}(\cdot, \chi, r)\left(1 - \frac{q_v^{r-1}}{\alpha_v(\cdot)\chi(\varpi_v)}\right).$$

(ii) *For any cohomological weight $\lambda \in \mathcal{U}'_S(L)$ and any integer r critical for λ, we have*

$$L_S(\lambda,\chi,r) = \frac{\mathrm{N}_{F/\mathbb{Q}}^{r-1}(i\mathfrak{d})\chi(\varpi_{\mathfrak{d}}^{-1})}{\Omega_{\tilde{\pi}_\lambda}^{\epsilon}} \cdot C_\lambda^{\epsilon} \cdot L(\pi_\lambda \otimes \chi, r-1/2)$$
$$\cdot \prod_{v\in S} E(\tilde{\pi}_v,\chi_v,r) \prod_{\substack{v\in S_p\setminus S\\ \pi_v \text{ unram.}}} \left(1 - \frac{\chi_v\omega_\pi\nu_v^{-1}(\varpi_v)}{q_v^{r-1}}\right),$$

where $\epsilon = \chi_\infty\omega_{p,\infty}^{r-1}$ and C_λ^{ϵ} is the p-adic period introduced in Definition 4.6.

Proof. (i) It suffices to apply Corollary 3.18 to $\Phi^{\epsilon}_{\mathcal{U}'_S,\alpha_{\mathfrak{u}}}$ and to $\Phi^{\epsilon}_{\mathcal{U}'_S,\beta_{\mathfrak{u}}}$.

(ii) Let $\mathfrak{f} = \prod_{v\in S} v^{n_v}$ be such that $n_v \geq \max(c_v,1)$. Using the definition of ev_S^r we obtain

$$\langle \mathrm{ev}_S^r(\Phi^{\epsilon}_{\mathcal{U}'_S,\alpha_{\mathfrak{u}}}),\chi\rangle(\lambda) = (\alpha_{\mathfrak{f}}^{\circ}(\lambda))^{-1} \sum_{[\eta]\in \mathscr{C}\ell_F^+(\mathfrak{f})} \chi(\eta)\chi_{\mathrm{cyc}}^{r-1}(\eta)\langle \mathrm{ev}^{\eta}_{\varpi_{\mathfrak{f}}}(\Phi^{\epsilon}_{\mathcal{U}'_S,\alpha_{\mathfrak{u}}}), 1^{\times}_{S,r}\rangle(\lambda).$$

Letting $j_\lambda = \left(r-1+\frac{\mathsf{w}_\lambda - 2 + k_{\lambda,\sigma}}{2}\right)_{\sigma\in\Sigma}$, we remark that $\lambda \circ 1^{\times}_{S,r}$ and z^{j_λ} agree on the support of $\mathrm{ev}^{\eta}_{\varpi_{\mathfrak{f}}}(\Phi^{\epsilon}_{\tilde{\pi}_\lambda,\alpha_{\mathfrak{u}}})$. Together with the definition of C_λ^{ϵ} this implies that

$$\langle \mathrm{ev}_S^r(\Phi^{\epsilon}_{\mathcal{U}'_S,\alpha_{\mathfrak{u}}}),\chi\rangle(\lambda) = C_\lambda^{\epsilon}(\alpha_{\mathfrak{f}}^{\circ}(\lambda))^{-1} \sum_{[\eta]\in \mathscr{C}\ell_F^+(\mathfrak{f})} \chi(\eta)\chi_{\mathrm{cyc}}^{r-1}(\eta)\langle \mathrm{ev}^{\eta}_{\varpi_{\mathfrak{f}}}(\Phi^{\epsilon}_{\tilde{\pi}_\lambda,\alpha_{\mathfrak{u}}(\lambda)}), z^{j_\lambda}\rangle.$$

The rest of the proof follows from Proposition 4.1 as in the proof of Theorem 4.2. ∎

4.4. *Partial p-adic L-functions*

We will now explain how the construction of the previous subsection can be adapted to the partial families constructed in Theorem 2.14. We are indebted to the referees for their insight and encouragement to present this construction.

Henceforth we fix $S \subsetneq S_p$ and we suppose we are given a regular non-critical S-refinement $\tilde{\pi}_S = (\pi, \{\nu_v\}_{v\in S})$ of π (see Definitions 1.3 and 2.12). We consider the neat open compact subgroup $K = K(\tilde{\pi}_S, \mathfrak{u}) \subset G(\mathbb{A}_f)$ from Definition 1.7. By non-criticality of $\tilde{\pi}_S$, for each character $\epsilon : \{\pm 1\}^{\Sigma} \to \{\pm 1\}$, the basis $\iota_p(b^{\epsilon}_{\tilde{\pi}_S,\alpha_{\mathfrak{u}}})$ of $\mathrm{H}_c^d(Y_K, \mathcal{L}^{\vee}_{k,\mathsf{w}}(L))^{\epsilon}_{\mathfrak{m}_{\tilde{\pi}_S}}$ (see Definition 1.14) lifts canonically to a basis of $\Phi^{\epsilon}_{\tilde{\pi}_S,\alpha_{\mathfrak{u}}}$ of $\mathrm{H}_c^d(Y_K, \mathcal{D}_{S,(k,\mathsf{w})})^{\epsilon}_{\mathfrak{m}_{\tilde{\pi}_S}}$. Letting $h_S = (h_{\tilde{\pi}_v})_{v\in S}$, by Theorem 2.14 there exists an L-affinoid neighborhood $\mathcal{U}_S$ of (k,w) in $\mathcal{X}_S$ and a connected component $\mathcal{V}_S$ of $\mathrm{Sp}(\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S})$ containing $\tilde{\pi}_S$ such that the weight map induces an isomorphism $\kappa : \mathcal{V}_S \xrightarrow{\sim} \mathcal{U}_S$. We can choose $\mathcal{U}_S$ sufficiently small so that it satisfies the technical properties stated in §4.2.

Definition 4.14. Fix a basis $\Phi^{\epsilon}_{\mathcal{U}_S,\alpha_{\mathfrak{u}}}$ of the free $\mathcal{O}(\mathcal{U}_S)$-module $\mathrm{H}_c^d(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon,\leq h_S} \otimes_{\mathbb{T}^{\leq h_S}_{S,\mathcal{U}_S}} \mathcal{O}(\mathcal{V}_S)$ of rank 1 such that $(k,\mathsf{w}) \circ \Phi^{\epsilon}_{\mathcal{U}_S,\alpha_{\mathfrak{u}}} = \Phi^{\epsilon}_{\tilde{\pi}_S,\alpha_{\mathfrak{u}}}$. For $v \in S$ we let $\alpha_v^{\circ} \in \mathcal{O}(\mathcal{U}_S)^{\times}$ denote the U_{ϖ_v}-eigenvalue on $\Phi^{\epsilon}_{\mathcal{U}_S,\alpha_{\mathfrak{u}}}$.

Given an integer r which is S-critical for (k, w) (see Definition 3.12) and an integral ideal $\mathfrak{f}$ supported at primes in S, Remark 3.13(ii) yields a map $(\mathcal{D}_{S,\mathcal{U}_S})_{E(\mathfrak{f})} \to D(U(\mathfrak{f})_S/E(\mathfrak{f}), \mathcal{O}(\mathcal{U}_S))$. The construction performed in §3.5 applies *mutatis mutandis* and yields a well-defined map

$$\mathrm{ev}^r_{\varpi_\mathfrak{f},S} : \mathrm{H}^d_c(Y_K, \mathcal{D}_{S,\mathcal{U}_S}) \to D(\mathrm{Gal}_{S\infty}, \mathcal{O}(\mathcal{U}_S)).$$

For $\chi : \mathrm{Gal}_{S\infty} \to L^\times$ a finite order character, we define the partial p-adic L-function

$$L^S(\cdot, \chi, r) = \frac{\alpha_\mathfrak{u}\langle \mathrm{ev}^r_S(\Phi^{\chi_\infty\omega^{r-1}_{p,\infty}}_{\mathcal{U}_S,\alpha_\mathfrak{u}}), \chi\rangle - \beta_\mathfrak{u}\langle \mathrm{ev}^r_S(\Phi^{\chi_\infty\omega^{r-1}_{p,\infty}}_{\mathcal{U}_S,\beta_\mathfrak{u}}), \chi\rangle}{\alpha_\mathfrak{u} - \beta_\mathfrak{u}} \in \mathcal{O}(\mathcal{U}_S).$$

For any cohomological weight $\lambda \in \mathcal{U}_S(L)$, Theorem 2.14(iii) yields an S-refined cuspidal automorphic representation $\tilde{\pi}_{\lambda,S}$ of weight λ whose system of Hecke eigenvalues corresponds to $\kappa^{-1}(\lambda) \in \mathcal{V}_S$. Given any character $\epsilon : \{\pm 1\}^\Sigma \to \{\pm 1\}$, the specialization $\lambda \circ \Phi^\epsilon_{\mathcal{U}_S,\alpha_\mathfrak{u}}$ of $\Phi^\epsilon_{\mathcal{U}_S,\alpha_\mathfrak{u}}$ at λ generates the same line in $\mathrm{H}^d_c(Y_K, \mathcal{D}_{S,\lambda})^{\epsilon,\leq h_S}$ as $\Phi^\epsilon_{\tilde{\pi}_{\lambda,S},\alpha_\mathfrak{u}(\lambda)}$, hence there exists

$$C^\epsilon_{\lambda,S} \in L^\times \quad \text{such that} \quad \lambda \circ \Phi^\epsilon_{\mathcal{U}_S,\alpha_\mathfrak{u}} = C^\epsilon_{\lambda,S} \cdot \Phi^\epsilon_{\tilde{\pi}_{\lambda,S},\alpha_\mathfrak{u}(\lambda)}.$$

Theorem 4.15. *Given a finite order character χ of $\mathrm{Gal}_{S\infty}$, and given a cohomological weight $\lambda \in \mathcal{U}_S(L)$ for which r is critical, we have*

$$L^S(\lambda, \chi, r) = \frac{\mathrm{N}^{r-1}_{F/\mathbb{Q}}(i\mathfrak{d})\chi(\varpi_\mathfrak{d}^{-1})}{\Omega^{\chi_\infty\omega^{r-1}_{p,\infty}}_{\tilde{\pi}_{\lambda,S}}} C^{\chi_\infty\omega^{r-1}_{p,\infty}}_{\lambda,S} L(\pi_\lambda \otimes \chi, r - 1/2) \prod_{v\in S} E(\tilde{\pi}_v, \chi_v, r).$$

Proof. The proof is based on Proposition 4.1 in the same way as the proof of Theorem 4.2. ∎

One may also define partial p-adic L-functions which are improved at places lying in a subset of S. We leave the details of the construction to the interested reader.

Part II

The Trivial Zero Conjecture at the central critical point

Throughout this part $\tilde{\pi} = (\pi, \{\nu_v\}_{v\in S_p})$ will be a regular non-critical p-refinement of a cuspidal automorphic representation π of $G(\mathbb{A})$ of cohomological weight (k, w) and tame conductor $\mathfrak{n}$ satisfying the following assumption:

$$\begin{aligned} &\pi \text{ has central character } \omega_\pi = |\cdot|^{\mathsf{w}}_F \text{ with } \mathsf{w} \text{ even, and} \\ &\pi_v \text{ is Iwahori spherical for all } v \in S_p. \end{aligned} \tag{4.12}$$

It follows that ν_v is unramified for all $v \in S_p$ and we let $\alpha_v = \nu_v(\varpi_v)$. The set S_p is then partitioned into St_p consisting of v such that π_v is an unramified twist (by ν_v) of the unitary Steinberg representation and its complement $S_p \setminus \mathrm{St}_p$ consisting of v such that π_v is unramified.

5. Galois representations and arithmetic $\mathscr{L}$-invariants

The Trivial Zero Conjecture stated in the introduction posits a relationship between the special values of p-adic L-functions and arithmetic $\mathscr{L}$-invariants. In this section we turn to the Galois representation side of p-adic families and explain the connection between two types of arithmetic $\mathscr{L}$-invariants associated to Hilbert modular forms. Moreover we will express them in terms of derivatives of Hecke eigenvalues, which will allow relating them to p-adic L-functions in the last section.

5.1. Galois representations for Hilbert modular forms

By Theorem 2.14 there exists an L-affinoid neighborhood $\mathcal{U}$ of (k, w) in $\mathcal{X}$ and a family $\kappa : \mathcal{V} \xrightarrow{\sim} \mathcal{U}$ containing $\tilde{\pi}$ such that for any $\lambda \in \mathcal{U}$ cohomological, $\kappa^{-1}(\lambda) \in \mathcal{V}$ corresponds to p-refined cuspidal automorphic representation $\tilde{\pi}_\lambda$ of weight λ. The cohomological points being Zariski dense in $\mathcal{U}$, there exists a unique 2-dimensional pseudo-character $\mathrm{G}_F \to \mathcal{O}(\mathcal{U})$ interpolating the traces of the corresponding p-adic Galois representations $\rho_{\pi_\lambda} : \mathrm{G}_F \to \mathrm{Aut}_L(V_{\pi_\lambda})$ attached to π_λ. Since V_π is absolutely irreducible, using a result of Nyssen [35] and Rouquier [44] one shows that, after possibly further shrinking $\mathcal{U}$, there exists a continuous Galois representation

$$\rho_{\mathcal{U}} : \mathrm{G}_F \to \mathrm{GL}_2(\mathcal{O}(\mathcal{U})) \tag{5.1}$$

whose specialization at every cohomological weight $\lambda \in \mathcal{U}$ is isomorphic to V_{π_λ}. Further shrinking $\mathcal{U}$ one can assume that the map $\lambda \mapsto (k_\lambda, \mathsf{w}_\lambda)$ defined in (2.2) is injective on $\mathcal{U}$, that $\mathsf{w}_\lambda \circ \omega_p = \omega_p^{\mathsf{w}}$, and that (4.7) holds. When $\mathcal{U}$ satisfies all these assumptions and in addition the tame conductor of π_λ equals $\mathfrak{n}$ for every cohomological weight $\lambda \in \mathcal{U}$ (see Lemma 5.1), we denote

$$\mathcal{X}(\tilde{\pi}) = \mathcal{U}. \tag{5.2}$$

Lemma 5.1. *The tame conductor of π_λ equals $\mathfrak{n}$ for all cohomological weights λ sufficiently close to (k, w).*

Proof. By construction, for any cohomological weight $\lambda \in \mathcal{U}$ the tame level $\mathfrak{n}_\lambda$ of π_λ divides $\mathfrak{n}\mathfrak{u}$. We will use the Galois representation $\rho_{\mathcal{U}}$ to show that, after shrinking $\mathcal{U}$, one has $\mathfrak{n}_\lambda = \mathfrak{n}$. The local-global compatibility at a finite place $v \notin S_p$, established by Carayol [14] and Taylor [49], asserts that the Frobenius semisimplification of the Weil–Deligne representation $(r_{v,\lambda}, N_{v,\lambda})$ attached to $\rho_{\pi_\lambda|\mathrm{G}_{F_v}}$ corresponds, via the local Langlands correspondence, to $\pi_{\lambda,v} \otimes |\cdot|_v^{-1/2}$. On the other hand, by [6, Lem. 7.8.14], one can attach to $\rho_{\mathcal{U}|\mathrm{G}_{F_v}}$ a Weil–Deligne representation $(r_{v,\mathcal{U}}, N_{v,\mathcal{U}})$. By [6, Lem. 7.8.17], after possibly shrinking $\mathcal{U}$, the restriction of $r_{v,\mathcal{U}}$ to the inertia subgroup at v has finite image whose specialization at any cohomological weight $\lambda \in \mathcal{U}$ is isomorphic to the restriction of $r_{v,\lambda}$ to the same inertia subgroup. Therefore, to conclude that $\mathfrak{n}_\lambda = \mathfrak{n}$ it suffices to show that $N_{v,\lambda} = N_{v,\mathcal{U}}$. From [6, Prop. 7.8.19] it follows that $N_{v,\lambda}$ lies in the p-adic closure of the conjugacy class of $N_{v,\mathcal{U}}$, which together with the fact that $\mathfrak{n}_\lambda$ divides $\mathfrak{n}\mathfrak{u}$ implies equality except possibly when $v = \mathfrak{u}$. Finally, $\pi_{\lambda,\mathfrak{u}}$ is unramified for all cohomological $\lambda \in \mathcal{U}$, since being an unramified twist of the Steinberg representation is excluded by (4.7). ■

To compute derivatives of analytic functions on $\mathcal{X}(\tilde{\pi})$ in the following sections, we will consider a subset of $\mathcal{X}(\tilde{\pi})$ which can be parametrized with the variables $((k_{\lambda,\sigma})_{\sigma\in\Sigma}, \mathsf{w}_\lambda)$ corresponding via (2.2) to characters of the form

$$\begin{aligned} &\prod_{v\in S_p} \mathcal{O}_v^\times \times \mathbb{Z}_p^\times \to \mathbb{C}_p^\times, \\ &z = ((z_v)_{v\in S_p}, z_0) \mapsto (k,\mathsf{w})(z)\cdot \langle z_0\rangle_p^{\mathsf{w}_\lambda - \mathsf{w}} \prod_{v\in S_p}\prod_{\sigma\in\Sigma_v} \sigma(\langle z_v\rangle_v)^{k_{\lambda,\sigma}-k_\sigma}, \end{aligned} \tag{5.3}$$

where $\langle\cdot\rangle_v : \mathcal{O}_v^\times \to 1 + (\varpi_v)$ is the natural projection map. This allows us to parametrize $\mathcal{X}(\tilde{\pi})$ by a neighborhood, denoted $\mathcal{X}^{\mathrm{an}}(\tilde{\pi})$, of (k,w) in the space $\prod_{\sigma\in\Sigma}(k_\sigma + 2p\mathcal{O}_{\mathbb{C}_p}) \times (\mathsf{w} + \mathcal{O}_{\mathbb{C}_p})$ of analytic weights. If we impose the weights to vary only in parallel direction per place above p, i.e., if $k_{\lambda,\sigma} - k_\sigma = x_v \in \mathcal{O}_{\mathbb{C}_p}$ for all $\sigma\in\Sigma_v$, then (5.3) becomes

$$\prod_{v\in S_p} \mathcal{O}_v^\times \times \mathbb{Z}_p^\times \to \mathbb{C}_p^\times, \quad z = ((z_v)_{v\in S_p}, z_0) \mapsto (k,\mathsf{w})(z)\cdot\langle z_0\rangle_p^{\mathsf{w}_\lambda-\mathsf{w}} \prod_{v\in S_p} \langle \mathrm{N}_{F_v/\mathbb{Q}_p}(z_v)\rangle_p^{x_v}. \tag{5.4}$$

5.2. *Fontaine–Mazur $\mathscr{L}$-invariants*

Consider the 2-dimensional L-vector space $V = V_\pi\left(\frac{2-\mathsf{w}}{2}\right)$ endowed with a continuous action of G_F. The local-global compatibility is also known at places $v\in S_p$ from the work of Saito [45], Blasius–Rogawski [11] and Skinner [46]. Letting $V_v = V_{|\mathrm{G}_{F_v}}$, $\mathcal{D}_{\mathrm{st}}(V_v) = (V_v \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}})^{\mathrm{G}_{F_v}}$ is a free $L\otimes F_{v,0}$-module of rank 2 ($F_{v,0}$ is the maximal unramified subfield of F_v) carrying a semilinear Frobenius φ_v and a nilpotent linear map N_v both inherited from Fontaine's ring B_{st} and such that $N_v\circ\varphi_v = p\varphi_v\circ N_v$. To be more precise, the linear map $\varphi_v^{f_v}$, where f_v is the inertial index at v, has eigenvalues $q_v^{(\mathsf{w}-2)/2}\alpha_v$ and $q_v^{-\mathsf{w}/2}\alpha_v^{-1}$, and the monodromy matrix vanishes if and only if π_v is unramified. Moreover the $L\otimes_{\mathbb{Q}_p} F_v$-module $\mathcal{D}_{\mathrm{st}}(V_v)\otimes_{F_{v,0}} F_v$ is endowed with a decreasing de Rham filtration $\mathrm{Fil}^\bullet(\mathcal{D}_{\mathrm{st}}(V_v)\otimes_{F_{v,0}} F_v)$ whose jumps, called the labeled Hodge–Tate weights, occur precisely at the integers $((k_\sigma-2)/2, -k_\sigma/2)_{\sigma\in\Sigma_v}$ (we recall that we are using the convention in which the cyclotomic character has Hodge–Tate weight -1). In particular $\mathrm{Fil}^0(\mathcal{D}_{\mathrm{st}}(V_v)\otimes_{F_{v,0}} F_v)$ is free of rank 1 over $L\otimes_{\mathbb{Q}_p} F_v$.

When $\pi_v\otimes|\cdot|^{-\mathsf{w}/2}$ is the Steinberg representation, there is a unique (up to a scalar) basis (e_1, e_2) of $\mathcal{D}_{\mathrm{st}}(V_v)$ such that $\varphi_v^{f_v}(e_1) = q_v^{(\mathsf{w}-2)/2}\alpha_v\cdot e_1$, $\varphi_v^{f_v}(e_2) = q_v^{-\mathsf{w}/2}\alpha_v^{-1}\cdot e_2$ and $N_v(e_2) = e_1$. When the unique refinement of V_v is non-critical in the sense of [30, Def. 5.29], the *Fontaine–Mazur $\mathscr{L}$-invariant* is the unique $\mathscr{L}_{\mathrm{FM}}(V_v)\in L\otimes_{\mathbb{Q}_p} F_v$ such that

$$\mathrm{Fil}^0(\mathcal{D}_{\mathrm{st}}(V_v)\otimes_{F_{v,0}} F_v) = (L\otimes_{\mathbb{Q}_p} F_v)\cdot(e_1 + \mathscr{L}_{\mathrm{FM}}(V_v)\cdot e_2).$$

We next relate $\mathscr{L}_{\mathrm{FM}}(V_v)$ to derivatives of Hecke eigenvalues. We will consider the $U^{\circ}_{\varpi_v}$-eigenvalues $\alpha^{\circ}_v \in \mathcal{O}(\mathcal{X}(\tilde{\pi}))$ (see Definition 4.5) as functions $\alpha^{\circ}_v((k_{\lambda,\sigma})_{\sigma\in\Sigma}, \mathsf{w}_\lambda)$ by restriction to $\mathcal{X}^{\mathrm{an}}(\tilde{\pi})$, and we will denote by $\mathrm{dlog}_u\,\alpha^{\circ}_v$ the logarithmic derivative at (k, w) in the direction $u = ((u_\sigma)_{\sigma\in\Sigma}, u_0)$, i.e., $\mathrm{dlog}_u\,\alpha^{\circ}_v = \frac{1}{\alpha^{\circ}_v(k,\mathsf{w})}\frac{d}{dx}\alpha^{\circ}_v((k,\mathsf{w}) + x\cdot u)\big|_{x=0}$.

Proposition 5.2. *If $u_\sigma = 1 = -u_0$ for all $\sigma \in \Sigma_v$, then $e_v^{-1}\cdot \mathrm{Tr}_{F_v/\mathbb{Q}_p}(\mathscr{L}_{\mathrm{FM}}(V_v)) = -2\,\mathrm{dlog}_u\,\alpha^{\circ}_v$, where e_v is the ramification index at v.*

Proof. The line defined by the direction u lies in the space $\mathcal{X}'_{S_p\setminus\{v\}}$, and we denote by $\mathcal{U}'$ the portion of this line inside the ball $\mathcal{X}^{\mathrm{an}}(\tilde{\pi})$. We will write $\rho_{\mathcal{U}'}$ for the restriction to $\mathcal{U}'$ of the analytic Galois representation on $\mathcal{X}^{\mathrm{an}}(\tilde{\pi})$ obtained from $\rho_{\mathcal{X}(\tilde{\pi})}$ (see (5.1)), and note that on this line $\alpha_v = \alpha^{\circ}_v \prod_{\sigma\in\Sigma_v}\sigma(\varpi_v)^{(2-k_\sigma-\mathsf{w})/2}$ is an analytic function. By [28], $\mathcal{D}_{\mathrm{st}}(\rho_{\mathcal{U}'})^{\varphi_v^{f_v}=\alpha_v}$ is an $\mathcal{O}(\mathcal{U}')\otimes_{\mathbb{Q}_p} F_{v,0}$-module of rank 1 and therefore we may apply [52, Thm. 1.1]. Since $\det\rho_{\mathcal{U}'} = \chi_{\mathrm{cyc}}^{\mathsf{w}_\lambda-1} = \chi_{\mathrm{cyc}}^{\mathsf{w}-1-xu_0}$ we have the following equality of differentials at $x = 0$:

$$\frac{d\alpha_v((k,\mathsf{w}) + xu)}{\alpha_v(k,\mathsf{w})} = \frac{1}{2e_v}\mathrm{Tr}_{F_v/\mathbb{Q}_p}(\mathscr{L}_{\mathrm{FM}}(V_v)d(xu_0)),$$

which immediately implies the formula, as $u_0 = -1$. ∎

The fact that $\mathrm{dlog}_u\,\alpha^{\circ}_v$ does not depend on u as long as $u_\sigma = 1 = -u_0$ for $\sigma \in \Sigma_v$ will be used in the final section.

Definition 5.3. We define $\mathscr{L}(\tilde{\pi}) = \prod_{v\in E} e_v^{-1}\,\mathrm{Tr}_{F_v/\mathbb{Q}_p}(\mathscr{L}_{\mathrm{FM}}(V_v))$, where $E \subset \mathrm{St}_p$ consists of those places v for which $\pi_v \otimes |\cdot|^{-\mathsf{w}/2}$ is the Steinberg representation.

5.3. Greenberg–Benois $\mathscr{L}$-invariants

The connection between the analytic Galois representation and the p-adic family runs deeper than the above description, and in order to state this connection precisely, we introduce the category of (φ,Γ)-modules over Robba rings (we refer to [7, §1] for more details). For $v \in S_p$ the Robba ring $\mathcal{R}_{F_v}$ is an F_v-algebra endowed with a continuous map φ_v and a continuous action of $\Gamma_v = \mathrm{Gal}(F_v(\mu_{p^\infty})/F_v)$. Writing $\mathcal{U} = \mathcal{X}(\tilde{\pi})$, a (φ_v,Γ_v)-module over $\mathcal{R}_{F_v,\mathcal{U}} = \mathcal{R}_{F_v}\widehat{\otimes}_{\mathbb{Q}_p}\mathcal{O}(\mathcal{U})$ is a coherent, locally free sheaf D over $\mathcal{R}_{F_v,\mathcal{U}}$ of finite rank endowed with a φ_v-semilinear map φ_D which gives an isomorphism $\varphi_D^* D \cong D$, and a semilinear action of Γ_v commuting with φ_D. Furthermore, there exists a functor $\mathcal{D}^{\dagger}_{\mathrm{rig}}$ associating to a continuous $\mathcal{O}(\mathcal{U})$-linear representation of G_{F_v} a (φ_v,Γ_v)-module over $\mathcal{R}_{F_v,\mathcal{U}}$. Since for $\lambda \in \mathcal{U}$ cohomological $\rho_{\pi_\lambda|\mathrm{G}_{F_v}}$ has labeled Hodge–Tate weights $((k_{\lambda,\sigma} - \mathsf{w}_\lambda)/2, (2 - \mathsf{w}_\lambda - k_{\lambda,\sigma})/2)_{\sigma\in\Sigma_v}$, we deduce from [30, Thm. 1.8] (taking into account that [30, Def. 1.7(f)] has a typo, the exponents should be $\kappa_i(z)_\tau$) that there exists a triangulation

$$\mathcal{D}^{\dagger}_{\mathrm{rig}}(\rho_{\mathcal{U}|\mathrm{G}_{F_v}}) \sim \begin{pmatrix}\psi_{v,1} & * \\ & \psi_{v,2}\end{pmatrix}, \tag{5.5}$$

where $\psi_{v,1}, \psi_{v,2} : F_v^\times \to \mathcal{O}(\mathcal{U})^\times$ are continuous characters such that $\psi_{v,1}(\varpi_v) = \alpha_v^\circ$ is the analytically varying renormalized Hecke eigenvalue, $\psi_{v,2}(\varpi_v) = \chi_{\mathrm{cyc}}^{\mathrm{w}_\lambda - 1}(\varpi_v)(\alpha_v^\circ)^{-1}$, and for $z_v \in \mathcal{O}_v^\times$,

$$\psi_{v,1}(z_v)(\lambda) = \prod_{\sigma\in\Sigma_v} \sigma(z_v)^{(\mathrm{w}_\lambda + k_{\lambda,\sigma}-2)/2}, \qquad \psi_{v,2}(z_v)(\lambda) = \prod_{\sigma\in\Sigma_v} \sigma(z_v)^{(\mathrm{w}_\lambda - k_{\lambda,\sigma})/2}.$$

Suppose the Galois representation $V = V_\pi\left(\frac{2-\mathrm{w}}{2}\right)$ satisfies $\mathrm{H}^1_f(F, V) = 0$, as predicted by the Bloch–Kato conjecture when $L\left(\pi, \frac{1-\mathrm{w}}{2}\right) \neq 0$. For $v \in S_p$, the (φ_v, N_v)-submodule $D_v = \mathcal{D}_{\mathrm{st}}(\nu_v(\frac{2-\mathrm{w}}{2})) \subset \mathcal{D}_{\mathrm{st}}(V_v)$ is regular in the sense of Perrin-Riou [38, §3.1.2] (see also [37]). In this context, the technical conditions of Greenberg [20] and Benois [7] mentioned in the introduction are all satisfied and there is a well-defined arithmetic $\mathscr{L}$-invariant $\mathscr{L}_{\mathrm{GB}}(V, \{D_v\})$ (this also uses [23, 43] extending the construction to an arbitrary F). We will not recall its intricate construction, but will instead show that it can be computed in this instance by a formula similar to that of Proposition 5.2.

Proposition 5.4. *Assume that* $\mathrm{H}^1_f(F, V) = 0$*. Then* $\mathscr{L}_{\mathrm{GB}}(V, \{D_v\}) = \mathscr{L}(\tilde{\pi}) \cdot \prod_{v\in E} f_v^{-1}$.

Proof. Consider the triangulation of $\rho_{\mathcal{U}}(\frac{2-\mathrm{w}}{2})$ induced from (5.5) restricted to the 1-dimensional affinoid $\mathcal{X}(\tilde{\pi}) \cap \mathcal{X}'_\emptyset$ (see Definition 2.1). Proposition 4.4 implies that V has a trivial zero contribution exactly from places $v \in E$. Therefore we can apply [43, Thm. 4.1, Prop. 4.13] to compute the $\mathscr{L}$-invariant in terms of dlog in the parallel direction $u_\sigma = 1 = -u_0$ for all $\sigma \in \Sigma$ (the theorem in *loc. cit.* assumes parallel weights, but its proof applies verbatim to general weights deforming in a parallel direction). Thus

$$\begin{aligned}\mathscr{L}_{\mathrm{GB}}(V, \{D_v\}) &= \prod_{v\in E} \frac{f_v^{-1}\,\mathrm{dlog}_u((\psi_{1,v}\psi_{2,v}^{-1})(\varpi_v))}{-\,\mathrm{dlog}_u((\psi_{1,v}\psi_{2,v}^{-1})(z_v))/\log_p(\mathrm{N}_{F_v/\mathbb{Q}_p}(z_v))}\\ &= \prod_{v\in E} -2 f_v^{-1}\,\mathrm{dlog}\,\alpha_v^\circ\end{aligned}$$

for any units $z_v \in \mathcal{O}_v^\times$, $v \in E$. The desired formula then follows from the equality $\mathrm{dlog}(\psi_{1,v}\psi_{2,v}^{-1}(z_v)) = \sum_{\sigma\in\Sigma_v} u_\sigma \log_p \sigma(z_v) = \log_p \mathrm{N}_{F_v/\mathbb{Q}_p}(z_v)$. ■

Finally, we remark that $D = \bigoplus_{v\mid p} \mathrm{Ind}_{\mathrm{G}_{F_v}}^{G_{\mathbb{Q}_p}} D_v$ is a regular submodule of $(\mathrm{Ind}_{\mathrm{G}_F}^{\mathrm{G}_\mathbb{Q}} V)_{|\mathrm{G}_{\mathbb{Q}_p}}$. In the Main Theorem we are proving the Trivial Zero Conjecture for the pair $(\mathrm{Ind}_{\mathrm{G}_F}^{\mathrm{G}_\mathbb{Q}} V, D)$.

6. Functional equations

In this section we establish functional equations for the p-adic L-functions constructed in §4 based on their growth, interpolation properties, and the Godement–Jacquet functional equation of the corresponding complex L-functions. These results will be used in the proof of the Trivial Zero Conjecture at the central point in §7.

6.1. Functional equations for archimedean L-functions

The Jacquet–Langlands global L-function and ε-factor are products over all places of F,

$$L(\pi, s) = \prod_v L(\pi_v, s) \quad \text{and} \quad \varepsilon(\pi, s) = \prod_v \varepsilon(\pi_v, \psi_v, dx_v, s),$$

where ψ_v and dx_v are as in §0 and will henceforth be dropped for the notation. Under the assumption (4.12), $\pi \otimes |\cdot|_F^{-\mathrm{w}/2}$ is unitary and self-dual, and its root number is given by

$$\varepsilon_\pi = \varepsilon\big(\pi, \tfrac{1-\mathrm{w}}{2}\big) \in \{\pm 1\}. \tag{6.1}$$

Let $\mathfrak{c}_\pi = \mathfrak{n} \prod_{v \in \mathrm{St}_p} v$ be the conductor of π. The functional equation states (see [26, Thm. 11.1]) that

$$L(\pi, s) = \varepsilon(\pi, s) L(\pi^\vee, 1-s), \quad \text{where} \quad \varepsilon(\pi, s) = \varepsilon_\pi \cdot (\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}^2 \mathfrak{c}_\pi))^{\frac{1-\mathrm{w}}{2} - s}. \tag{6.2}$$

Proposition 6.1. *Let χ be a character of $\mathrm{Gal}_{p\infty}$ of conductor $\mathfrak{c}_\chi = \prod_{v \in S_p} v^{c_v}$. Then*

$$\mathfrak{c}_{\pi\otimes\chi} = \mathfrak{n}_\chi \mathfrak{c}_\chi^2 \quad \textit{and} \quad \varepsilon_{\pi\otimes\chi} = \varepsilon_\pi \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{c}_\chi) \chi(\varpi_{\mathfrak{n}_\chi}) \tau(\overline{\chi})^2 \prod_{v \in \mathrm{St}_p, c_v > 0} \varepsilon\big(\pi_v, \tfrac{1-\mathrm{w}}{2}\big),$$

where $\tau(\chi)$ is the Gauss sum defined in (0.2) and $\mathfrak{n}_\chi = \mathfrak{n} \cdot \prod_{v \in \mathrm{St}_p, c_v = 0} v$.

Proof. The conductor formula follows from (4.12). The formula involving the ε-factors is checked by decomposing both sides as products of local terms and using [26, §1.3].

If either χ_v or π_v is unramified then using $|\omega_\pi|_F^{\mathrm{w}} = q_v^{-\mathrm{w}}$ one has

$$\varepsilon\big(\pi_v \otimes \chi_v, \tfrac{1-\mathrm{w}}{2}\big) = \varepsilon\big(\pi_v, \tfrac{1-\mathrm{w}}{2}\big) q_v^{c_v} \chi_v(\varpi_v)^{\mathfrak{c}_{\pi_v}} \tau(\overline{\chi}_v, \psi_v, d_{\chi_v})^2.$$

If $v \in \mathrm{St}_p$ and χ_v is ramified, then using $\varepsilon\big(\pi_v, \frac{1-\mathrm{w}}{2}\big) = -\nu_v(\varpi_v)^{-1} q_v^{-\mathrm{w}/2}$ one has

$$\varepsilon\big(\pi_v \otimes \chi_v, \tfrac{1-\mathrm{w}}{2}\big) = \varepsilon\big(\pi_v, \tfrac{1-\mathrm{w}}{2}\big) q_v^{c_v} \tau(\overline{\chi}_v, \psi_v, d_{\chi_v})^2 \varepsilon\big(\pi_v, \tfrac{1-\mathrm{w}}{2}\big). \qquad \blacksquare$$

6.2. Interpolation formulas

Under the assumption (4.12), Theorem 4.13 and Proposition 4.4 take the following more familiar form, involving the global Gauss sum $\tau(\chi)$ (see (0.2)).

Corollary 6.2. *For any $S \subset S_p$, $r \in \mathbb{Z}$ critical for (k, w) and χ of conductor $\prod_{v \in S} v^{c_v}$,*

$$L_S(\tilde{\pi}, \chi, r) = L_S((k, \mathrm{w}), \chi, r) = \frac{\mathrm{N}_{F/\mathbb{Q}}^{r-1}(i\,\mathfrak{d})}{\Omega_{\tilde{\pi}}^{\chi_\infty \omega_{p,\infty}^{r-1}}} L(\pi \otimes \chi, r - 1/2) \tau(\chi) \prod_{v \in S_p} E_S(\tilde{\pi}_v, \chi_v, r),$$

where

$$\prod_{v \in S_p} E_S(\tilde{\pi}_v, \chi_v, r) = \prod_{c_v > 0} \frac{q_v^{r c_v}}{\alpha_v^{c_v}} \prod_{\substack{c_v = 0 \\ v \notin \mathrm{St}_p}} \left(1 - \frac{\chi_v(\varpi_v)}{\alpha_v q_v^{r+\mathrm{w}-1}}\right) \prod_{\substack{c_v = 0 \\ v \in S}} \left(1 - \frac{q_v^{r-1}}{\alpha_v \chi_v(\varpi_v)}\right).$$

Moreover $E_S(\tilde{\pi}_v, \chi_v, r)$ vanishes if and only if $r = \frac{2-\mathrm{w}}{2}$ is the central critical point, $v \in S \cap \mathrm{St}_p$ and $\chi_v(\varpi_v) = \alpha_v^{-1} q_v^{-\mathrm{w}/2} = -\varepsilon\big(\pi_v, \frac{1-\mathrm{w}}{2}\big) \in \{\pm 1\}$.

In §7 we will prove that the order of vanishing of $L_p(\tilde{\pi}, s)$ at $s = \frac{2-\mathsf{w}}{2}$ is at least as large as the number of places v where $E_S\big(\tilde{\pi}_v, \chi_v, \frac{2-\mathsf{w}}{2}\big) = 0$. We remark that the interpolation formula from Corollary 6.2 only gives information about the vanishing of the p-adic L-function, and leaves completely unanswered the question of higher orders of vanishing.

The next result will be used in §6.3 to prove the functional equation for p-adic L-functions.

Corollary 6.3. *Suppose $\tilde{\pi}$ satisfies (4.12) and let $\tilde{\varepsilon}_\pi = \varepsilon_\pi \cdot \prod_{v \in \mathrm{St}_p} \varepsilon\big(\pi_v, \frac{1-\mathsf{w}}{2}\big) \in \{\pm 1\}$. Then*

$$\mathcal{L}_p(\tilde{\pi}, \chi \langle\cdot\rangle_p^{r-1}) = \tilde{\varepsilon}_\pi \cdot (\chi \omega_p^{\mathsf{w}/2})(-\varpi_{\mathfrak{n}}) \cdot \langle \mathfrak{n} \rangle_p^{r-1+\mathsf{w}/2} \mathcal{L}_p(\tilde{\pi}, \chi^{-1} \omega_p^{-\mathsf{w}} \langle\cdot\rangle_p^{1-\mathsf{w}-r}) \tag{6.3}$$

for any finite order characters $\chi : \mathrm{Gal}_{p\infty} \to L^\times$ and any integer r critical for (k, w).

Proof. Since $\pi^\vee = \pi \otimes |\cdot|_F^{-\mathsf{w}}$, the archimedean functional equation (6.2) for $\pi \otimes \chi$, Proposition 6.1 and Corollary 6.2 for $S = S_p$ yield

$$\frac{\Omega_{\tilde{\pi}}^{\chi_\infty \omega_{p,\infty}^{r-1}} \mathcal{L}_p(\tilde{\pi}, \chi \chi_{\mathrm{cyc}}^{r-1})}{\mathrm{N}_{F/\mathbb{Q}}^{r-1}(i\mathfrak{d}) \tau(\chi) \prod_{v \in S_p} E(\tilde{\pi}_v, \chi_v, r)} = \varepsilon_\pi \prod_{v \in \mathrm{St}_p, c_v > 0} \varepsilon\big(\pi_v, \tfrac{1-\mathsf{w}}{2}\big) \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{c}_\chi) \tau(\overline{\chi})^2$$
$$\times \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}^2 \mathfrak{n}_\chi \mathfrak{c}_\chi^2)^{1-\mathsf{w}/2-r} \chi(\varpi_{\mathfrak{n}_\chi}) \frac{\Omega_{\tilde{\pi}}^{\chi_\infty \omega_{p,\infty}^{r-1+\mathsf{w}}} \mathcal{L}_p(\tilde{\pi}, \chi^{-1} \chi_{\mathrm{cyc}}^{1-\mathsf{w}-r})}{\mathrm{N}_{F/\mathbb{Q}}^{1-\mathsf{w}-r}(i\mathfrak{d}) \tau(\overline{\chi}) \prod_{v \in S_p} E(\tilde{\pi}_v, \chi_v^{-1}, 2-\mathsf{w}-r)}.$$

Using the global Gauss sum identity $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{c}_\chi) \tau(\chi) \tau(\overline{\chi}) = \chi_f(-1) = \chi_\infty(-1)$ we get

$$\mathcal{L}_p(\tilde{\pi}, \chi \chi_{\mathrm{cyc}}^{r-1})$$
$$= \varepsilon_\pi \prod_{v \in \mathrm{St}_p, c_v > 0} \varepsilon\big(\pi_v, \tfrac{1-\mathsf{w}}{2}\big) \cdot \frac{\chi(-\varpi_{\mathfrak{n}_\chi}) \mathcal{L}_p(\tilde{\pi}, \chi^{-1} \chi_{\mathrm{cyc}}^{1-\mathsf{w}-r})}{\mathrm{N}_{F/\mathbb{Q}}(-\mathfrak{n}_\chi \mathfrak{c}_\chi^2)^{r+\mathsf{w}/2-1}} \prod_{v \in S_p} \frac{E(\tilde{\pi}_v, \chi_v, r)}{E(\tilde{\pi}_v, \chi_v^{-1}, 2-\mathsf{w}-r)}.$$

If $c_v = 0$ and π_v is unramified then $\alpha_v \beta_v = q_v^{1-\mathsf{w}}$ and so $E(\tilde{\pi}_v, \chi_v, r) = E(\tilde{\pi}_v, \chi_v^{-1}, 2-\mathsf{w}-r)$. If $c_v > 0$ then $\frac{E(\tilde{\pi}_v, \chi_v, r)}{E(\tilde{\pi}_v, \chi_v^{-1}, 2-\mathsf{w}-r)} = q_v^{(2r+\mathsf{w}-2)c_v}$. Finally, if $c_v = 0$ and $v \in \mathrm{St}_p$ then

$$\frac{E(\tilde{\pi}_v, \chi_v, r)}{E(\tilde{\pi}_v, \chi_v^{-1}, 2-\mathsf{w}-r)} = -\alpha_v q_v^{r-1+\mathsf{w}} \chi_v(\varpi_v)^{-1} = \varepsilon\big(\pi_v, \tfrac{1-\mathsf{w}}{2}\big) q_v^{r-1+\mathsf{w}/2} \chi_v(\varpi_v)^{-1}.$$

Putting everything together and using $\mathfrak{n}_\chi = \mathfrak{n} \cdot \prod_{v \in \mathrm{St}_p, c_v = 0} v$ we obtain

$$\mathcal{L}_p(\tilde{\pi}, \chi \chi_{\mathrm{cyc}}^{r-1}) = \tilde{\varepsilon}_\pi \chi(-\varpi_{\mathfrak{n}}) \mathrm{N}_{F/\mathbb{Q}}(-\varpi_{\mathfrak{n}})^{1-r-\mathsf{w}/2} \mathcal{L}_p(\tilde{\pi}, \chi^{-1} \chi_{\mathrm{cyc}}^{1-\mathsf{w}-r}).$$

Replacing χ by $\chi \omega_p^{1-r}$, and noting that $\chi_{\mathrm{cyc}} \omega_p^{-1} = \langle\cdot\rangle_p$ is an even character, yields (6.3). ∎

When $\tilde{\pi}$ has very non-critical slope, Corollary 6.3 provides enough relations to establish a functional equation for $\mathcal{L}_p(\tilde{\pi}, \chi\,\cdot) \in D(\mathrm{Gal}_{\mathrm{cyc}}, \mathcal{O}(\mathcal{U}))$ given its growth. If $\tilde{\pi}$ has critical slope, but is still non-critical, then the functional equation will be proven in the next subsection using the unique p-adic family containing $\tilde{\pi}$ from Theorem 2.14.

6.3. Functional equations for p-adic L-functions

We recall that the multi-variable p-adic L-function $\mathcal{L}_p \in D(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{X}(\tilde{\pi})))$ from (4.8) interpolates the p-adic L-functions $\mathcal{L}_p(\tilde{\pi}_\lambda, \cdot) \in D(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{X}(\tilde{\pi})))$ for all cohomological weights $\lambda \in \mathcal{X}(\tilde{\pi})$.

The cyclotomic (resp. multi-variable) p-adic L-function attached to $\tilde{\pi}$ is defined as

$$L_p(\tilde{\pi}, s) = \mathcal{L}_p(\tilde{\pi}, \omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{s-1}), \quad \text{resp.} \quad L_p(\lambda, s) = \mathcal{L}_p(\lambda, \omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{s-1}), \tag{6.4}$$

for $s \in \mathcal{O}_{\mathbb{C}_p}$, $\lambda \in \mathcal{X}(\tilde{\pi})$. By Proposition 4.10 for $z \in \mathcal{O}_{\mathbb{C}_p}$ and $(k_\lambda, \mathrm{w}_\lambda) \in \mathcal{X}^{\mathrm{an}}(\tilde{\pi})$ such that $(k_\lambda, \mathrm{w}_\lambda\langle\cdot\rangle^{2z}) \in \mathcal{X}^{\mathrm{an}}(\tilde{\pi})$ one has

$$L_p((k_\lambda, \mathrm{w}_\lambda + 2z), s) = L_p((k_\lambda, \mathrm{w}_\lambda), s + z). \tag{6.5}$$

Theorem 6.4. *The sign $\tilde{\varepsilon}_{\pi_\lambda}$ of $\tilde{\pi}_\lambda$ is independent of the cohomological weight $\lambda \in \mathcal{X}(\tilde{\pi})$. For any $\lambda \in \mathcal{X}(\tilde{\pi})$, for any multiplicative continuous character $f \in A(\mathrm{Gal}_{\mathrm{cyc}}, L)$ and for any finite order character $\chi : \mathrm{Gal}_{p\infty} \to L^\times$, one has*

$$\mathcal{L}_p(\lambda, \chi \cdot f) = \tilde{\varepsilon}_\pi \cdot (\chi\omega_p^{\mathrm{w}/2} f)(-\varpi_{\mathfrak{n}})\langle\mathfrak{n}\rangle_p^{\mathrm{w}_\lambda/2}\mathcal{L}_p(\lambda, \chi_{\mathrm{cyc}}^{-\mathrm{w}_\lambda}((\chi \cdot f) \circ (\cdot)^{-1})). \tag{6.6}$$

In particular, $L_p(\tilde{\pi}_\lambda, s) = \tilde{\varepsilon}_\pi \cdot \langle\mathfrak{n}\rangle_p^{s-1+\mathrm{w}_\lambda/2} L_p(\tilde{\pi}_\lambda, 2 - \mathrm{w}_\lambda - s)$ as analytic functions in s.

Proof. We remark that the slope is constant in the family and equals $\sum_{v\in S_p} e_v h_{\tilde{\pi}_v}$ (see Definition 1.8). Consider a cohomological weight $\lambda' \in \mathcal{X}(\tilde{\pi})$ having very non-critical slope (see (1.15)). Since the elements $\mathcal{L}_p(\lambda', \chi\cdot)$ and $\mathcal{L}_p(\lambda', \chi_{\mathrm{cyc}}^{-\mathrm{w}_{\lambda'}}\chi^{-1}\cdot)$ of $D(\mathrm{Gal}_{\mathrm{cyc}}, \mathcal{O}(\mathcal{U}))$ both have growth at most $\sum_{v\in S_p} e_v h_{\tilde{\pi}_v}$, by [51, Thm. 2.3, Lem. 2.10] it suffices to check (6.6) for all $f = \chi'\langle\cdot\rangle_p^{r-1}$, where r is a critical integer for (k, w) and χ' is a finite order character of $\mathrm{Gal}_{\mathrm{cyc}}$. This is precisely formula (6.3) applied to $\chi\chi'$, except that there is $\tilde{\varepsilon}_{\pi_{\lambda'}}$ instead of $\tilde{\varepsilon}_\pi$.

We now turn to the problem of showing that the sign is generically constant in the family. Choose $\lambda' \in \mathcal{X}(\tilde{\pi})$ as above for which $2 - \mathrm{w}_{\lambda'}/2$ is a critical integer. The absolute convergence of $L(\pi_{\lambda'} \otimes \chi', s)$ for $\mathrm{Re}(s) > 1 - \mathrm{w}_{\lambda'}/2$ implies that $L(\pi_{\lambda'} \otimes \chi\omega_p^{-1}, (3 - \mathrm{w}_{\lambda'})/2) \neq 0$. Moreover Proposition 4.4 implies that $E(\tilde{\pi}_{\lambda',v}, \chi_v\omega_p^{-\mathrm{w}/2}, 2 - \mathrm{w}_{\lambda'}/2) \neq 0$ for any $v \in S_p$, and therefore $\mathcal{L}_p(\lambda', \chi'\chi_{\mathrm{cyc}}^{1-\mathrm{w}_{\lambda'}/2}) \neq 0$ by Corollary 6.2. Letting $f = \omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{s-1}$, the quotient

$$\varepsilon(\lambda, s) = \frac{\langle\mathfrak{n}\rangle_p^{1-s-\mathrm{w}_\lambda/2}\mathcal{L}_p(\lambda, \chi\omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{s-1})}{\chi(-\varpi_{\mathfrak{n}})\mathcal{L}_p(\lambda, \chi^{-1}\omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{1-\mathrm{w}_\lambda-s})},$$

is a well-defined, non-identically-zero meromorphic function in the variables $(\lambda, s) \in \mathcal{X}(\tilde{\pi}) \times \mathcal{O}_{\mathbb{C}_p}$. Indeed, we have shown that its numerator does not vanish at $(\lambda', 2 - \mathrm{w}_{\lambda'}/2)$, and $\varepsilon(\lambda', 2 - \mathrm{w}_{\lambda'}/2) = \tilde{\varepsilon}_{\pi_{\lambda'}}$ by the functional equation. Similarly $\varepsilon(\lambda, s) = \tilde{\varepsilon}_{\pi_\lambda} \in \{\pm 1\}$ for all cohomological weights $\lambda \in \mathcal{X}(\tilde{\pi})$ having very non-critical slope and such that $\varepsilon(\lambda, s)$ is well-defined, and the Zariski density of such points implies that $\varepsilon(\lambda, s)$ is constant with value $\tilde{\varepsilon} \in \{\pm 1\}$, independent of χ.

To finish the proof of (6.6) it suffices to check that $\widetilde{\varepsilon} = \widetilde{\varepsilon}_{\pi_\lambda}$ for any cohomological $\lambda \in \mathcal{X}(\widetilde{\pi})$. By a theorem of Rohrlich [42] applied at the central critical point, there exists a finite order character χ' of $\mathrm{Gal}_{p\infty}$ such that $L\big(\pi_\lambda \otimes \chi', \frac{1-\mathsf{w}_\lambda}{2}\big) \neq 0$ and $E(\widetilde{\pi}_{\lambda,v}, \chi'_v \omega_p^{-\mathsf{w}/2}, 1-\mathsf{w}_\lambda/2) \neq 0$ for all $v \in S_p$. Then Corollary 6.2 implies that $\mathcal{L}_p(\lambda, \chi' \chi_{\mathrm{cyc}}^{-\mathsf{w}_\lambda/2}) \neq 0$, and it then follows from Corollary 6.3 that $\widetilde{\varepsilon}_{\pi_\lambda} = \widetilde{\varepsilon}$. ∎

7. The Trivial Zero Conjecture

By Corollary 6.2, the set $E \subset S_p$ of places at which the local interpolation factor of $L_p(\widetilde{\pi}, s)$ vanishes at the central point $\frac{2-\mathsf{w}}{2}$ consists precisely of $v \in \mathrm{St}_p$ such that $\varepsilon\big(\pi_v, \frac{1-\mathsf{w}}{2}\big) = -1$.

Theorem 7.1 (Trivial Zero Conjecture at the central critical point). *The p-adic L-function $L_p(\widetilde{\pi}, s)$ has order of vanishing at least $e = |E|$ at $\frac{2-\mathsf{w}}{2}$ and*

$$\frac{L_p^{(e)}\big(\widetilde{\pi}, \frac{2-\mathsf{w}}{2}\big)}{e!} = \mathscr{L}(\widetilde{\pi}) \frac{L\big(\pi, \frac{1-\mathsf{w}}{2}\big)}{\mathrm{N}_{F/\mathbb{Q}}^{\mathsf{w}/2}(i\,\mathfrak{d})\Omega_{\widetilde{\pi}}^{\omega_{p,\infty}^{\mathsf{w}/2}}} \cdot 2^{|\mathrm{St}_p \setminus E|} \prod_{v \in S_p \setminus \mathrm{St}_p} (1 - \alpha_v^{-1} q_v^{-\mathsf{w}/2})^2.$$

Moreover, if the Greenberg–Benois arithmetic $\mathscr{L}$-invariant is defined, the Trivial Zero Conjecture of the introduction holds for the Galois representation $\mathrm{Ind}_{\mathrm{G}_F}^{\mathrm{G}_\mathbb{Q}} V_\pi\big(\frac{2-\mathsf{w}}{2}\big)$ with the choice of regular submodule as in §5.3.

The proof of Theorem 7.1 will occupy the remainder of this section.

7.1. Local behavior in partial families

Crucial for the computation of Taylor coefficients of our p-adic L-functions is the following technical lemma.

Lemma 7.2. *Let $S = S_p \setminus \{v\}$ for some $v \in \mathrm{St}_p$. After possibly shrinking $\mathcal{X}(\widetilde{\pi})$, for any cohomological $\lambda \in \mathcal{X}_S \cap \mathcal{X}(\widetilde{\pi})$ the local representation $\pi_{\lambda,v}$ is an unramified twist of the Steinberg representation* (*see Definition* 2.1 *and* (5.2)).

Proof. Let us first present an argument using p-adic Hodge theory and the existence of the analytic Galois representation $\rho_{\mathcal{U}}$ from (5.1). Let $\mathcal{Z} \subset \mathcal{X}(\widetilde{\pi}) \cap \mathcal{X}_S$ be the Zariski closure of the subset of cohomological points λ such that $\pi_{\lambda,v}$ is unramified, i.e., $\rho_{\pi_\lambda,v}$ is crystalline. Since the labeled Hodge–Tate weights of $\rho_{\pi_\lambda,v}$ are constant for all cohomological weights $\lambda \in \mathcal{X}_S$, a theorem of Berger and Colmez [9] (see also [15, Prop. 3.17]) implies that $(k, \mathsf{w}) \notin \mathcal{Z}$, hence the claim.

We now give a proof based on the theory of partially finite slope families developed in §2.6, which we believe is of independent interest. Let $K = K(\widetilde{\pi}, \mathfrak{u})$ (which, by assumption (4.12), is the same as $K(\widetilde{\pi}_S, \mathfrak{u})$), $h = (h_{\widetilde{\pi}_w})_{w \in S_p}$ and $h_S = (h_{\widetilde{\pi}_w})_{w \in S}$. Writing $\mathcal{U}_S = \mathcal{X}_S \cap \mathcal{X}(\widetilde{\pi})$, the natural restriction map $\mathcal{D}_{\mathcal{U}_S} \to \mathcal{D}_{S,\mathcal{U}_S}$ yields a $\widetilde{\mathbb{T}}_S$-equivariant map $\mathrm{H}_c^d(Y_K, \mathcal{D}_{\mathcal{U}_S})^{\epsilon, \leq h} \to \mathrm{H}_c^d(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon, \leq h_S}$ such that the action of U_{ϖ_v} on the left agrees with that of $U^{\circ}_{\varpi_v}$ on the right. Under this map, the localization $\mathrm{H}^d_c(Y_K, \mathcal{D}_{\mathcal{U}_S})^{\epsilon,\le h}_{\tilde{\pi}}$ maps to $\mathrm{H}^d_c(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon,\le h_S}_{\tilde{\pi}_S}$ further localized at the ideal generated by $U^{\circ}_{\varpi_v} - \alpha^{\circ}_v$ and $U_\delta - \nu_v(\delta)$ for $\delta \in \mathcal{O}_v^\times$, which does not vanish. Therefore, Theorem 2.14(ii) applied to both S_p and S implies that, after possibly shrinking $\mathcal{X}(\tilde{\pi})$, there exist components $\mathcal{V}$ of $\mathbb{T}^{\le h}_{\mathcal{U}_S}$ and $\mathcal{V}_S$ of $\mathbb{T}^{\le h_S}_{S,\mathcal{U}_S}$ such that the weight maps $\kappa : \mathcal{V} \xrightarrow{\sim} \mathcal{U}_S$ and $\kappa_S : \mathcal{V}_S \xrightarrow{\sim} \mathcal{U}_S$ are isomorphisms and there exists an $\mathcal{O}(\mathcal{U}_S)$-linear natural map between free rank 1 $\mathcal{O}(\mathcal{U}_S)$-modules

$$\mathrm{H}^\bullet_c(Y_K, \mathcal{D}_{\mathcal{U}_S})^{\epsilon,\le h} \otimes_{\mathbb{T}^{\le h}_{\mathcal{U}_S}} \mathcal{O}(\mathcal{V}) \xrightarrow{\sim} \mathrm{H}^\bullet_c(Y_K, \mathcal{D}_{S,\mathcal{U}_S})^{\epsilon,\le h_S} \otimes_{\mathbb{T}^{\le h_S}_{S,\mathcal{U}_S}} \mathcal{O}(\mathcal{V}_S). \tag{7.1}$$

After fixing bases, the above isomorphism is necessarily given by multiplication by some $g \in \mathcal{O}(\mathcal{U}_S)$. Specializing at (k, w), the classicality Theorem 2.7 implies that $g(k, \mathsf{w}) \neq 0$, since multiplication by $g(k, \mathsf{w})$ takes the class corresponding to $\tilde{\pi}^K_f$ to the class corresponding to $\tilde{\pi}^K_{S,f}$. We then shrink $\mathcal{X}(\tilde{\pi})$ so as to ensure that g is nowhere vanishing on $\mathcal{U}_S$.

By the proof of Theorem 2.14(iii), specializing (7.1) at $\lambda \in \mathcal{U}_S(L)$ cohomological yields

$$\mathrm{H}^\bullet_c(Y_K, \mathcal{L}^\vee_\lambda(L))^{\epsilon,\le h} \otimes_{\mathbb{T}^{\le h}_{\kappa^{-1}(\lambda)}} L \xrightarrow{\sim} \mathrm{H}^\bullet_c(Y_K, \mathcal{L}^\vee_\lambda(L))^{\epsilon,\le h_S} \otimes_{\mathbb{T}^{\le h_S}_{S,\kappa_S^{-1}(\lambda)}} L,$$

which is an isomorphism since multiplication by $g(\lambda)$ is a non-zero map between L-lines. We conclude that $\tilde{\pi}_\lambda$ and $\tilde{\pi}_{\lambda,S}$ have the same Hecke eigenvalues away from a bad set of primes, and therefore $\pi_\lambda \simeq \pi_{\lambda,S}$ by Strong Multiplicity One for GL_2.

It therefore suffices to show that $\pi_{\lambda,S,v}$ is Steinberg for all $\lambda \in \mathcal{U}_S$. However, Theorem 2.14(iii) and its proof imply that $\tilde{\pi}_{\lambda,S}$ is a non-critical S-refinement, hence

$$1 = \dim \mathrm{H}^\bullet_c(Y_K, \mathcal{D}_{S,\lambda})^{\epsilon}_{\mathfrak{m}_{\tilde{\pi}_{\lambda,S}}} = \dim\, (\pi^K_{\lambda,S,f})_{\mathfrak{m}_{\tilde{\pi}_{\lambda,S}}}.$$

Decomposing as tensor product we deduce that $\dim(\pi^{I_v}_{\lambda,S,v}) = 1$, as we are not localizing using Hecke operators at $v \notin S$ (see Definition 1.6). Hence $\pi_{\lambda,v} \simeq \pi_{\lambda,S,v}$ is an unramified twist of the Steinberg representations, as unramified representations have 2-dimensional I_v-invariants. ■

7.2. Taylor coefficients

In this section we use the interplay between properties of partially improved p-adic L-functions and the variation of the root number in partial finite slope families to establish the vanishing of many Taylor coefficients of the p-adic L-function of the family.

Definition 7.3. Let $u \in 4\mathcal{O}_{\mathbb{C}_p}$ and $x = (x_v)_{v\in E} \in (2p\mathcal{O}_{\mathbb{C}_p})^E$. For any subset $S \subset E$ we denote $x_S = (x_v)_{v\in S}$ and define $\lambda^S_{x,u} = (k_\lambda, \mathsf{w}_\lambda) \in \mathcal{X}^{\mathrm{an}}(\tilde{\pi})$ by

$$\mathsf{w}_\lambda = \mathsf{w} - u \quad \text{and} \quad k_{\lambda,\sigma} = \begin{cases} k_\sigma & \text{for } \sigma \in \Sigma_{S_p\setminus E}, \\ k_\sigma + u & \text{for } \sigma \in \Sigma_{E\setminus S}, \\ k_\sigma + x_v & \text{for } \sigma \in \Sigma_S. \end{cases}$$

Writing $\mathbb{L}_p(x,u) = \langle \mathfrak{n} \rangle_p^{u/4} L_p\big(\lambda^E_{x,u}, \frac{2-\mathsf{w}}{2}\big)$, Theorem 6.4 implies that

$$\mathbb{L}_p(x,-u) = \widetilde{\varepsilon} \cdot \mathbb{L}_p(x,u) \quad \text{with} \quad \widetilde{\varepsilon} = (-1)^e \varepsilon_\pi. \tag{7.2}$$

In particular we may write $\mathbb{L}_p(x,u) = \sum_{i\geq 0} A_i(x)u^i$, where $A_i(x)$ is p-adic analytic in $(x_v)_{v\in E}$ and the sum runs over i even (resp. odd) when $\widetilde{\varepsilon} = 1$ (resp. $\widetilde{\varepsilon} = -1$). By (6.5) we see that

$$L_p(\widetilde{\pi}, s) = \langle \mathfrak{n} \rangle_p^{(2s+\mathsf{w}-2)/4} \mathbb{L}_p((0)_{v\in E}, 2-\mathsf{w}-2s).$$

By definition, $\lambda^S_{x,u} \in \mathcal{X}^{\mathrm{an}}(\widetilde{\pi}) \cap \mathcal{X}'_{S\sqcup(S_p\setminus E)}$ and we can define

$$\mathbb{L}_S(x_S,u) = \langle \mathfrak{n} \rangle_p^{u/4} L_{S\sqcup(S_p\setminus E)}(\lambda^S_{x,u}, \mathbb{1}, \tfrac{2-\mathsf{w}}{2}), \tag{7.3}$$

where $L_{S\sqcup(S_p\setminus E)}$ is the improved p-adic L-function from §4.3.

By definition we have $\mathbb{L}_p(x,u) = \mathbb{L}_E(x,u)$ and by Theorem 4.13 for all $S \subset E$,

$$\mathbb{L}_p((x_S,(u)_{v\in E\setminus S}),u) = \mathbb{L}_S(x_S,u) \prod_{v\in E\setminus S} \left(1 - \alpha^\circ_v(\lambda^S_{x,u})^{-1} \prod_{\sigma\in\Sigma_v} \sigma(\varpi_v)^{(k_\sigma-2)/2}\right). \tag{7.4}$$

We now turn to the Taylor expansions of the functions $A_i(x)$. For a multi-index $n = (n_v)_{v\in E}$ of non-negative integers we denote $x^n = \prod_{v\in E} x_v^{n_v}$. For each i write the power series expansion

$$A_i(x) = \sum_{n\in\mathbb{Z}^E_{\geq 0}} a_i(n)x^n.$$

We will prove that a large number of such coefficients vanish, a fact which is not implied by the Trivial Zero Conjecture. More precisely, all coefficients of total degree $< e$ and most of the coefficients in degree e vanish. For $S \subset E$ we will write $n_S = (n_v)_{v\in S}$ and $n = (n_S, n_{E\setminus S})$.

For convenience we denote $|n| = \sum_{v\in E} n_v$ and $\|n\| = |\{v \in E \mid n_v \neq 0\}|$, so that $\|n\| \leq |n|$.

Our first technical result concerns the vanishing of certain linear combinations of the Taylor coefficients $a_i(n)$. Recall that the function $\mathbb{L}_p(x,u)$ is even in u if $\widetilde{\varepsilon} = 1$ and odd in u if $\widetilde{\varepsilon} = -1$.

Lemma 7.4. *Let $S \subset E$ be such that $(-1)^{|E\setminus S|} = -\widetilde{\varepsilon}$. For $n_S \in \mathbb{Z}^S_{\geq 0}$ and $\ell \leq e - |S|$ we have*

$$\sum_{n_{E\setminus S}} a_{\ell-|n_{E\setminus S}|}(n_S, n_{E\setminus S}) = 0.$$

Proof. For $x_S \in \mathbb{Z}^S_{>0}$ p-adically close to 0, $\lambda = \lambda^S_{x_S,0} \in \mathcal{X}^{\mathrm{an}}(\widetilde{\pi}) \cap \mathcal{X}'_{S\sqcup(S_p\setminus E)}$ is a cohomological weight such that $k_{\lambda,\sigma} = k_\sigma$ for $\sigma \in \Sigma_{S_p\setminus S}$. By Lemma 7.2 and the analyticity of the eigenvalue $\alpha^\circ_v(\lambda)$ we conclude that $\pi_{\lambda,v} \otimes |\cdot|^{-\mathsf{w}/2}$ is the Steinberg representation for all $v \in E \setminus S$, in particular $\varepsilon\big(\pi_{\lambda,v}, \frac{1-\mathsf{w}}{2}\big) = -1$. From this and from Theorem 6.4 we deduce that

$$\varepsilon\big(\pi_\lambda, \tfrac{1-\mathsf{w}}{2}\big) = (-1)^{|E\setminus S|} \cdot \widetilde{\varepsilon} = -1,$$

which implies that $L(\pi_\lambda, \frac{1-\mathrm{w}}{2}) = 0$. Corollary 6.2 then implies that $\mathbb{L}_S(x_S, 0) = 0$. Moreover, the fact that $\varepsilon(\pi_{\lambda,v}, \frac{1-\mathrm{w}}{2}) = -1$ for $v \in E \setminus S$ implies that $\alpha_v^\circ(\lambda) = \prod_{\sigma\in\Sigma_v} \sigma(\varpi_v)^{(k_\sigma-2)/2}$ for $v \in E \setminus S$. By Zariski density these equalities are also true for all x_S. Thus each factor in (7.4) vanishes at $u = 0$ and so $u^{|E\setminus S|+1}$ divides the analytic function $\mathbb{L}_p((x_S, (u)_{v\in E\setminus S}), u)$. Expanding, we deduce that for all $n \in \mathbb{Z}_{\geq 0}^E$,

$$u^{|E\setminus S|+1} \text{ divides } \sum_{n_S} x_S^{n_S} \sum_i \sum_{n_{E\setminus S}} a_i(n) u^{|n_{E\setminus S}|+i}.$$

Collecting terms of the form $x_S^{n_S} u^\ell$ in the above divisibility yields the desired equality. ■

The following proposition proves that the Taylor expansion of $\mathbb{L}_p(x, u)$ contains only terms of degree $\geq e$.

Proposition 7.5. (i) *If $\|n\| < e - i$, then $a_i(n) = 0$.*

(ii) *For any given $i < e$, we have $\sum_{|n|=e-i} a_i(n) = 0$.*

Proof. If $(-1)^{|E\setminus S|} = (-1)^{i+1} \neq -\widetilde{\varepsilon}$ then $A_i(x) = 0$ by (7.2) and both claims are clear.

(i) We will prove this fact by induction on $(\|n\| + i, i)$ ordered lexicographically. The base case $i = \|n\| = 0$ follows from Lemma 7.4 applied to $n_S = (0)_{v\in S}$, $\ell = 0$ and to any $S \subset E$ satisfying its hypothesis. Suppose now that $a_j(m) = 0$ for all j and m such that either $\|m\| + j < \|n\| + i$ or $\|m\| + j = \|n\| + i$ and $j < i$. Since $\|n\| < e - i$ there exists $S \subset E$ such that $|S| = e - i - 1$, $n_v = 0$ for all $v \in E \setminus S$, and $\|n\| = \|n_S\|$. Then $(-1)^{|E\setminus S|} = (-1)^{i+1} = -\widetilde{\varepsilon}$ and Lemma 7.4 applied to $\ell = i < |E \setminus S| = i + 1$ yields

$$\sum_{m_{E\setminus S}} a_{i-|m_{E\setminus S}|}(n_S, m_{E\setminus S}) = 0.$$

Consider a term $a_j(n_S, m_{E\setminus S})$ in the above sum and write $m = (n_S, m_{E\setminus S})$. Then $\|m\| + j = \|n_S\| + \|m_{E\setminus S}\| + j \leq \|n\| + |m_{E\setminus S}| + j = \|n\| + i$. The inductive hypothesis then implies that $a_j(m) = 0$ whenever the previous inequality is strict, or when $j < i$. The sole surviving term in the sum is then $a_i(n_S, (0)_{v\in E\setminus S}) = a_i(n) = 0$, as desired.

(ii) Let $S \subset E$ be any subset with cardinality $e - i - 1 \geq 0$ and let $n_S = (1)_{v\in S}$. Since $(-1)^{|E\setminus S|} = -\widetilde{\varepsilon}$, Lemma 7.4 applied to $\ell = i + 1 = e - |S|$ yields

$$\sum_{n_{E\setminus S}} a_{i+1-|n_{E\setminus S}|}(n_S, n_{E\setminus S}) = 0. \tag{7.5}$$

Letting $n = (n_S, n_{E\setminus S})$ and noting that $|n| = |S| + |n_{E\setminus S}|$, we deduce from (i) that $a_{i+1-|n_{E\setminus S}|}(n) = a_{e-|n|}(n)$ vanishes unless $\|n\| = |n|$. Summing (7.5) over all such subsets $S \subset E$ yields

$$0 = \sum_{|S|=e-i-1} \sum_{|n_{E\setminus S}|=\|n_{E\setminus S}\|} a_{e-|n|}(n_S, n_{E\setminus S}) = \sum_{j=0}^{i+1} \binom{e-j}{e-i-1} \sum_{|n|=\|n\|=e-j} a_j(n). \tag{7.6}$$

Since $\sum_{|n|=e-i} a_i(n) = \sum_{\|n\|=|n|=e-i} a_i(n)$ by (i), it suffices to prove the vanishing of the latter, which is deduced from (7.6) by induction on i, using the fact that either $A_0(x) = 0$ or $A_1(x) = 0$. ∎

Remark 7.6. While Proposition 7.5 implies that $\mathbb{L}_p(x, u)$ only contains monomials of total degree $\geq e$, our methods do not imply that these monomials are multiples of u^e. For example, when $e = 4$ and $\varepsilon_\pi = 1$, one can only show that

$$\begin{aligned}&\mathbb{L}_p(x_1, x_2, x_3, x_4, u)\\&\quad = au^2(x_1 - x_2)(x_3 - x_4) + bu^2(x_1 - x_3)(x_2 - x_4) + cu^4 + (\text{degree} \geq 5 \text{ terms}).\end{aligned}$$

Similarly, for $\varepsilon_\pi = 1$ and $e = 2$ we have $\mathbb{L}_p(x_1, x_2, u) \in (x_1^2 x_2^2, u^2)$, while for $e = 3$,

$$\mathbb{L}_p(x_1, x_2, x_3, u) \in (ux_1^2(x_2 - x_3), ux_2^2(x_1 - x_3), ux_3^2(x_1 - x_2), ux_1x_2x_3, u^3).$$

7.3. Proof of the Trivial Zero Conjecture

In this section we prove Theorem 7.1.

Lemma 7.7. *Keeping the hypotheses and notations of Theorem 7.1, the analytic function $\mathbb{L}_p((u)_{v\in E}, u)$ vanishes at $u = 0$ to order at least e and*

$$\begin{aligned}&\frac{(-2)^e}{e!} \cdot \frac{d^e}{du^e} \mathbb{L}_p((u)_{v\in E}, u)\Big|_{u=0}\\&\qquad = \mathscr{L}(\widetilde{\pi}) \cdot \frac{L\left(\pi, \frac{1-\mathsf{w}}{2}\right)}{\mathrm{N}_{F/\mathbb{Q}}^{\mathsf{w}/2}(i\,\mathfrak{d})\Omega_{\widetilde{\pi}}^{\omega_{p,\infty}^{\mathsf{w}/2}}} \cdot 2^{|\mathrm{St}_p\setminus E|} \prod_{v\in S_p\setminus \mathrm{St}_p} \left(1 - \frac{q_v^{-\mathsf{w}/2}}{\alpha_v}\right)^2.\end{aligned}$$

Proof. By (7.4) we see that

$$\mathbb{L}_p((u)_{v\in E}, u) = \mathbb{L}_\emptyset(u) \prod_{v\in E} \Big(1 - \alpha_v^\circ(\lambda_{(u),u}^\emptyset)^{-1} \prod_{\sigma\in\Sigma_v} \sigma(\varpi_v)^{(k_\sigma-2)/2}\Big).$$

Since each interpolation factor vanishes at $u = 0$ it follows that the order of vanishing of $\mathbb{L}_p((u)_{v\in E}, u)$ at $u = 0$ is at least e. Differentiating e times at $u = 0$ we deduce from (7.3) that

$$\frac{d^e}{du^e} \mathbb{L}_p((u)_{v\in E}, u)\Big|_{u=0} = e! L_{S_p\setminus E}\left(\widetilde{\pi}, \mathbb{1}, \tfrac{2-\mathsf{w}}{2}\right) \prod_{v\in E} \operatorname{dlog} \alpha_v^\circ.$$

Moreover by Corollary 6.2 one has

$$E_{S_p\setminus E}\left(\widetilde{\pi}_v, \mathbb{1}, \tfrac{2-\mathsf{w}}{2}\right) = \begin{cases} (1 - \alpha_v^{-1} q_v^{-\mathsf{w}/2})^2 & \text{if } v \in S_p \setminus \mathrm{St}_p,\\ 1 + \varepsilon\left(\pi_v, \frac{1-\mathsf{w}}{2}\right) & \text{if } v \in \mathrm{St}_p\,.\end{cases}$$

The desired formula then follows from Corollary 6.2 and Proposition 5.2 noting that $\alpha_v^\circ(\lambda_{(0),0}^\emptyset) = \prod_{\sigma\in\Sigma_v} \sigma(\varpi_v)^{(k_\sigma-2)/2}$ for $v \in E$ and $\varepsilon\left(\pi_v, \frac{1-\mathsf{w}}{2}\right) = 1$ for $v \in \mathrm{St}_p \setminus E$. ∎

Proof of Theorem 7.1. Recall that $L_p(\widetilde{\pi}, s) = \langle \mathrm{u} \rangle_p^{(2s+\mathsf{w}-2)/4} \mathbb{L}_p((0)_{v\in E}, 2-\mathsf{w}-2s)$, hence

$$L_p^{(m)}(\widetilde{\pi}, s)|_{s=2-\mathsf{w}/2} = \sum_{k=0}^{m} \binom{m}{k} \left(\tfrac{1}{2}\log_p \langle \mathrm{u} \rangle\right)^{m-k} (-2)^k \frac{d^k}{du^k} \mathbb{L}_p((0)_{v\in E}, u)\Big|_{u=0}. \quad (7.7)$$

Differentiating $\mathbb{L}_p((0)_{v\in E}, u)$ we see that $\frac{d^k}{du^k}\mathbb{L}_p((0)_{v\in E}, u)\big|_{u=0} = k! A_k((0)_{v\in E})$. By Proposition 7.5 these derivatives vanish for $k < e$, which implies that the order of vanishing of $L_p(\widetilde{\pi}, s)$ at $s = \frac{2-\mathsf{w}}{2}$ is at least e. Differentiating the power series expansion of $\mathbb{L}_p((u)_{v\in E}, u)$ we see that

$$\frac{d^e}{du^e}\mathbb{L}_p((u)_{v\in E}, u)\Big|_{u=0} = e! \sum_i \sum_{|n|=e-i} a_i(n).$$

By Proposition 7.5 the interior sum above vanishes when $i < e$, hence

$$\frac{d^e}{du^e}\mathbb{L}_p((u)_{v\in E}, u)\Big|_{u=0} = e! a_e((0)_{v\in E}) = e! A_e((0)_{v\in E}) = \frac{d^e}{du^e}\mathbb{L}_p((0)_{v\in E}, u)\Big|_{u=0}.$$

Then, (7.7) implies that

$$L_p^{(e)}(\widetilde{\pi}, s)|_{s=(2-\mathsf{w})/2} = (-2)^e \frac{d^e}{du^e}\mathbb{L}_p((0)_{v\in E}, u)\Big|_{u=0} = (-2)^e \frac{d^e}{du^e}\mathbb{L}_p((u)_{v\in E}, u)\Big|_{u=0},$$

and Theorem 7.1 then follows from Lemma 7.7.

Finally, it remains to explain why the Trivial Zero Conjecture of the introduction holds for $\mathrm{Ind}_F^{\mathbb{Q}} V_\pi\left(\frac{2-\mathsf{w}}{2}\right)$. Let $V = V_\pi\left(\frac{2-\mathsf{w}}{2}\right)$ and $D_v \subset \mathcal{D}_{\mathrm{st}}(V_v)$ regular submodules as in §5.3. Then $D = \bigoplus_{v\in S_p} \mathrm{Ind}_{F_v}^{\mathbb{Q}_p} D_v \subset \mathcal{D}_{\mathrm{st}}(\mathrm{Ind}_F^{\mathbb{Q}} V|_{G_{\mathbb{Q}_p}})$ is a regular submodule. By Proposition 5.4 and [43, Cor.3.9], we have

$$\mathscr{L}_{\mathrm{GB}}\left(\mathrm{Ind}_F^{\mathbb{Q}} V_\pi\left(\tfrac{2-\mathsf{w}}{2}\right), D\right) = \mathscr{L}_{\mathrm{GB}}(\widetilde{\pi}) = \mathscr{L}(\widetilde{\pi}) \prod_{v\in E} f_v^{-1}.$$

The conjecture now follows from the main formula of this theorem and [25, p. 1398], which explains that the analytic $\mathscr{L}$-invariants of $\mathrm{Ind}_F^{\mathbb{Q}} V$ and V also differ by $\prod_{v\in E} f_v^{-1}$. ■

Acknowledgements. We are indebted to the referees for their numerous comments and suggestions on how to improve the exposition. We would also like to thank D. Benois, A. Betina, J. Bergdall, L. Gehrmann, D. Hansen, H. Hida, A. Iovita, D. N. Nguyen, A. Raghuram, D. Ramakrishnan, G. Rosso, J. Tilouine, E. Urban, and C. Williams for helpful discussions. The second author is grateful to the University of Notre Dame for its hospitality during his visit in 2017, while the third author is grateful to the CEMPI Lab at the University of Lille where he was a Visiting Associate in June 2016 and June 2019, and to the Institute of Mathematics at Sorbonne University, where part of this work was completed, for their excellent research environments.

Funding. The first author has received funding from the European Research Council grant n° 682152 and FONDECYT PAI 77180007, while the second author has benefited from the support of the Agence Nationale de la Recherche grants ANR-11-LABX-0007-01 and ANR-18-CE40-0029, and the third author was partially supported by the NSA Young Investigator grant H98230-16-1-0302.

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