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Wild solutions of the Navier–Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1

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Abstract. We prove non-uniqueness for a class of weak solutions to the Navier–Stokes equations which have bounded kinetic energy, integrable vorticity, and are smooth outside a fractal set of singular times with Hausdorff dimension strictly less than 1.

Keywords. Navier–Stokes equations, partial regularity, non-uniqueness, convex integration, wild solutions

1. Introduction

Throughout this paper we consider the *incompressible three-dimensional Navier–Stokes equations*:

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p - \Delta v = 0, \tag{1.1a}$$

$$\operatorname{div} v = 0, \tag{1.1b}$$

$$v|_{t=0} = v_0, \tag{1.1c}$$

on the torus $\mathbb{T}^3 = [-\pi, \pi]^3$. We consider solutions of zero mean, i.e. $\int_{\mathbb{T}^3} v(x,t)\,dx = 0$ for all $t \in [0,T]$. The notion of *weak solution* of (1.1) that we work with in this paper is that of distributional solution which has bounded kinetic energy, and is strongly continuous in time:

Definition 1.1 (Weak solution). Given any zero mean initial datum $v_0 \in L^2$, we say that $v \in C^0([0,T); L^2(\mathbb{T}^3))$ is a *weak* solution of the Cauchy problem for the Navier–Stokes

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equations (1.1) if the vector field $v(\cdot, t)$ is weakly divergence-free for all $t \in [0, T)$, has zero mean, and

$$\int_{\mathbb{T}^3} v_0 \cdot \varphi(\cdot, 0)\, dx + \int_0^T \int_{\mathbb{T}^3} v \cdot (\partial_t \varphi + (v \cdot \nabla)\varphi + \Delta\varphi)\, dx\, dt = 0$$

for any $\varphi \in C_0^\infty(\mathbb{T}^3 \times [0, T))$ such that $\varphi(\cdot, t)$ is divergence-free for all t.

In view of $C^0([0, T); L^2(\mathbb{T}^3))$ regularity, by [14] the above defined weak solutions are also *mild* or *Oseen* solutions (see also [30, Chapter 6]). That is, for $t \in [0, T)$ we have

$$v(\cdot, t) = e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P}_H \operatorname{div}(v(\cdot, s) \otimes v(\cdot, s))\, ds. \tag{1.2}$$

Here $\mathbb{P}_H$ is the Helmholtz projector and $e^{t\Delta} f$ is the heat extension of f. Our main result is as follows.

Theorem 1.1 (Main result). *There exists a $\beta > 0$ such that the following holds. For $T > 0$, let $u^{(1)}, u^{(2)} \in C^0([0, T]; \dot{H}^3(\mathbb{T}^3))$ be two strong solutions of the Navier–Stokes equations* (1.1a)–(1.1b) *on $[0, T]$, with data $u^{(1)}(0, x)$ and $u^{(2)}(0, x)$ of zero mean. There exists a weak solution v of the Cauchy problem for* (1.1) *on $[0, T]$ with initial datum $v|_{t=0} = u^{(1)}|_{t=0}$, which has the additional regularity*

$$v \in C^0([0, T]; H^\beta(\mathbb{T}^3) \cap W^{1,1+\beta}(\mathbb{T}^3)),$$

and such that

$$v \equiv u^{(1)} \quad \text{on } [0, T/3] \quad \text{and} \quad v \equiv u^{(2)} \quad \text{on } [2T/3, T].$$

Moreover, for every such v there exists a zero Lebesgue measure set $\Sigma_T \subset (0, T]$ with Hausdorff (in fact box-counting) dimension less than $1 - \beta$ such that

$$v \in C^\infty(((0, T] \setminus \Sigma_T) \times \mathbb{T}^3).$$

In particular, the weak solution v is almost everywhere smooth.

The outline of the proof of Theorem 1.1 is given in Section 2, while the detailed estimates are made in Sections 3–5.

Remark 1.2 (Non-uniqueness of weak solutions for strong initial data). Theorem 1.1 immediately implies that weak solutions of the Cauchy problem for the Navier–Stokes equation (1.1), in the sense of Definition 1.1, are not unique.

The cheap way to see this is to take any $T > 0$, $u^{(1)} \equiv 0$, and $u^{(2)}$ to be any non-trivial mean-zero solution of the Navier–Stokes equation on $[0, T]$ (e.g. a shear flow). Then the weak solution v given by Theorem 1.1 is non-trivial on $[0, T]$, and thus 0 is not the only weak solution with zero initial datum. Conversely, if we take $u^{(1)}$ to be any non-trivial solution to the Navier–Stokes equation, and $u^{(2)} \equiv 0$, Theorem 1.1 gives a counterexample to backward (in time) uniqueness for weak solutions of (1.1) in the sense of Definition 1.1.

More generally, we emphasize that Theorem 1.1 proves the non-uniqueness of weak solutions to the Cauchy problem for the Navier–Stokes equation (1.1) for *any strong initial datum*. To see this, consider any $v_0 \in \dot{H}^3$ and take $T = c\|v_0\|_{H^3}^{-1}$, where $c > 0$ is a sufficiently small universal constant (cf. Proposition 3.1). Then there exists a unique solution $u^{(1)} \in C^0([0,T]; H^3)$ to the Cauchy problem (1.3) below with datum v_0. Moreover $\|u^{(1)}(T)\|_{L^2} \leq \|v_0\|_{L^2}$. However, using Theorem 1.1 one can glue to this solution the shear flow $u^{(2)}(x_1, x_2, x_3, t) = (Ae^{-t}\sin(x_2), 0, 0)$. Then if A is chosen such that $Ae^{-T} > 2\|v_0\|_{L^2}$, we have $\|v(T)\|_{L^2} = \|u^{(2)}(T)\|_{L^2} > \|v_0\|_{L^2} \geq \|u^{(1)}(T)\|_{L^2}$. Therefore v is a weak solution to (1.3) with datum v_0, but v is not equal to the smooth solution $u^{(1)}$ at time T.

While for the above argument we have considered $v_0 \in \dot{H}^3$, it is clear that Theorem 1.1 also implies the non-uniqueness of weak solutions to the Cauchy problem for (1.1) for any initial datum for which one has unique local in time solvability of (1.1) (examples include $v_0 \in \dot{H}^{1/2}$ [16]; $v_0 \in L^3$ with zero mean [24]; $v_0 \in \mathrm{BMO}^{-1}$ which is small and has zero mean [26]; see [29] for further details). Indeed, for any such initial datum the unique local in time solution $u^{(1)}$ is smooth in positive time, and hence for any $\varepsilon > 0$ we have $u^{(1)}(\cdot, \varepsilon) \in \dot{H}^3$. We then apply Theorem 1.1 on the time interval $[\varepsilon, T]$, rather than $[0, T]$, in order to glue the strong solution to a shear flow with kinetic energy which is either strictly larger, or strictly less, at time T.

1.1. Background

We make a few comments concerning different notions of solution to the Navier–Stokes equation, other than in Definition 1.1 (see [30] for a more detailed discussion). The weakest notion of solution to the Cauchy problem for (1.1) is that of a *very weak solution*: these are distributional solutions of (1.1) which only lie in $C^0_{\mathrm{weak}}(0, T; L^2)$, and are weakly divergence-free. However, one typically proves the existence of solutions which are stronger than this.

Indeed, for any L^2 initial datum v_0, Leray [31] constructed a distributional solution $v \in C^0_{\mathrm{weak}}(0, \infty; L^2) \cap L^2(0, \infty; \dot{H}^1)$ which obeys the energy inequality $\|v(t)\|_{L^2}^2 + 2\int_s^t \|\nabla v(\tau)\|_{L^2}^2 \, d\tau \leq \|v(s)\|_{L^2}^2$ for a.e. $s \geq 0$ and all $t > s$. See also the work of Hopf [19] on bounded domains. These are the *Leray–Hopf weak solutions*. One nice feature of Leray–Hopf weak solutions is that they possess epochs of regularity, i.e. many time intervals on which they are smooth. In fact, already Leray [31] observed that these weak solutions are smooth almost everywhere in time, since the putative *singular set of times*, Σ_T, has Hausdorff dimension $\leq 1/2$. This fact follows directly from two ingredients: the fact that for $v_0 \in H^1$ the maximal time of existence of a unique smooth solution is bounded from below by $c\|v_0\|_{H^1}^{-4}$, and a Vitali-type covering lemma which may be combined with the $L_t^2 H_x^1$ information provided by the energy inequality. Scheffer [42] went further to prove that the 1/2-dimensional Hausdorff measure of Σ_T is 0. These results were strengthened to bounds on the box-counting dimension for Σ_T [27,41]. See [30,40] for further references.

Remark 1.3 (Weak solutions with partial regularity in time). We note that while the weak solutions constructed in Theorem 1.1 are not Leray–Hopf, they give the first example of a mild/weak solution to the Navier–Stokes equation whose singular set of times $\Sigma_T \subset (0, T]$ is both *non-empty* and has Hausdorff (in fact, box-counting) dimension *strictly less than* 1. This is in contrast with the prior work [5], where Σ_T has dimension 1. It is an interesting open problem to construct weak solutions to (1.1), in the sense of Definition 1.1, where the $1/2$-dimensional Hausdorff measure of the non-empty set of singular times is 0.

A fundamental step towards understanding the uniqueness and smoothness of weak solutions was to introduce the concept of a *suitable weak solution* by Scheffer [42] and Caffarelli–Kohn–Nirenberg [6]. Suitable weak solutions obey a localized in space-time version of the energy inequality, and they have partial regularity in space and time: the putative singular set of points in space-time has 1-dimensional parabolic Hausdorff measure 0. See the reviews [30, 40] for more recent extensions and further references.

The uniqueness of suitable weak solutions or of Leray–Hopf weak solutions is an outstanding open problem. The *weak-strong uniqueness* result of Prodi–Serrin [39, 43] states that if there exists a weak/mild solution $v \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x \cap L^p_t L^q_x$ of the Cauchy problem for (1.1) with $2/p + 3/q \leq 1$[1] and $p < \infty$, and if u is a Leray–Hopf weak solution with the same initial datum, then $u \equiv v$. This is a conditional uniqueness result within the class of Leray–Hopf weak solutions. Moreover, the solutions are smooth in positive time [28]. The $L^\infty_t L^3_x$ endpoint was established in [13]. Similar weak-strong uniqueness results hold within the class of mild solutions, except the $q = 3$ endpoint which requires continuity in time [14, 17, 34]. See [30, Chapter 12] for further references. A very interesting conjecture of Jia–Šverák [22,23] essentially states that the Prodi–Serrin uniqueness criteria are sharp, and that the non-uniqueness of Leray–Hopf weak solutions may already be expected in the regularity class $L^\infty_t L^{3,\infty}_x$. Compelling numerical evidence in support of this conjecture was recently provided by Guillod–Šverák [18]. A related interesting open problem is to establish the non-uniqueness of mild/weak solutions to (1.1) in the regularity class $C^0_t L^q_x \cap L^2_t H^1_x$, for any $q \in [2, 3)$.

We conclude this subsection by revisiting the non-uniqueness result of Remark 1.2, for rough initial data:

Remark 1.4 (Non-uniqueness of very weak solutions for any L^2 initial datum). If instead of the weak solutions of Definition 1.1 we consider *very weak solutions* of (1.1), so they only lie in $C^0_{\rm weak}(0, T; L^2)$, then Theorem 1.1 implies that non-uniqueness for the Cauchy problem holds for any L^2 initial datum of zero mean, within the class of very weak solutions. Indeed, for any such datum, by the work of Leray there exists a very weak solution u to the Cauchy problem for (1.1), which is in fact smooth most of the time. Pick any regular time $t_0 > 0$ of u, and let $v_0 = u(t_0) \in \dot{H}^3$. We then apply the argument of Remark 1.2

[1]The $L^p_t L^q_x$ norm, for $2/p + 3/q = 1$, is invariant under the Navier–Stokes scaling map $v(x, t) \mapsto v_\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t)$. Spaces that obey these properties are called *scaling critical* spaces. Since the Leray–Hopf energy space $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ obeys $2/\infty + 3/2 = 2/2 + 3/6 = 3/2 > 1$, we may call the system (1.1) *energy supercritical*.

on the time interval $[t_0, t_0 + T]$, with $u^{(1)}$ being the unique local in time smooth solution of (1.1) with initial datum v_0 at time t_0. Note that by weak-strong uniqueness, the Leray solution u is in fact equal to $u^{(1)}$ on $[t_0, t_0 + T]$. In view of Theorem 1.1 we can construct a very weak solution v which is equal to u on $[0, t_0 + T/3]$, and equal to a shear flow of our choice on $[t_0 + 2T/3, T]$. This solution v is smooth except for a set of times of Hausdorff dimension < 1, and is different from the Leray solution u.

1.2. The energy supercritical hyperdissipative Navier–Stokes equation

The proof of Theorem 1.1 uses essentially the fact that the kinetic energy space is supercritical with respect to the natural scaling invariance associated to (1.1). In fact, the proof applies *mutatis mutandis* to the energy supercritical α-hyperdissipative Navier–Stokes equation

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p + (-\Delta)^\alpha v = 0, \tag{1.3a}$$

$$\operatorname{div} v = 0, \tag{1.3b}$$

$$v|_{t=0} = v_0. \tag{1.3c}$$

Here we consider the *energy supercritical regime* $\alpha \in [1, 5/4)$. Indeed, (1.3) is invariant under the scaling map $v(x,t) \mapsto v_\lambda(x,t) = \lambda^{2\alpha-1} v(\lambda x, \lambda^{2\alpha} t)$, and the energy norm $L^\infty_t L^2_x$ is invariant under this map for $\alpha = 5/4$. Definition 1.1, with $\Delta\varphi$ replaced by $-(-\Delta)^\alpha \varphi$, gives the notion of a weak solution for (1.3). Our result is:

Theorem 1.5 (The hyperdissipative problem). *For $\alpha \in [1, 5/4)$ there exists $\beta = \beta(\alpha) > 0$ such that Theorem* 1.1*, and thus also Remark* 1.2*, holds with system* (1.1) *replaced by the more general system* (1.3)*.*

The system (1.3) was first considered by Lions [32,33] for α in the critical and subcritical regime $\alpha \geq 5/4$. He proved the existence and uniqueness of Leray weak solutions for any L^2 initial datum. These solutions are regular in positive time. In [45] it was proved that slightly below the critical threshold $\alpha = 5/4$ the existence of a globally regular solution still holds when the right-hand side of the first equation in (1.3) is replaced by a logarithmically supercritical operator. For $\alpha \in [3/4, 1)$ and $(1, 5/4)$ partial regularity results à la Caffarelli–Kohn–Nirenberg were established in [25, 44] and [8]. These works show the existence of a weak solution whose putative singular set (in space-time) has $(5 - 4\alpha)$-dimensional Hausdorff measure 0. In the opposite direction, the recent works [7,12] prove the non-uniqueness of Leray weak solutions to (1.3) in the parameter ranges $\alpha < 1/5$, respectively $\alpha < 1/3$. The non-uniqueness of weak solutions in the sense of Definition 1.1 is also shown to hold for $\alpha < 1/2$.

We note that very recently, by adapting the arguments in [5], Luo and Titi [35] demonstrated the non-uniqueness of very weak solutions for (1.3) in the parameter range $\alpha \in (1, 5/4)$. Compared to [35], the weak solutions constructed in this paper have the additional property that their set of singular times has Hausdorff dimension strictly less than 1. Together, the uniqueness result of [33], and the non-uniqueness results of [35] and of this work, confirm the well-posedness *criticality* of the exponent $\alpha = 5/4$ within the class of weak solutions defined in Definition 1.1.

We give the proof of Theorem 1.5 for general values of $\alpha < 5/4$. Theorem 1.1 follows by restricting to $\alpha = 1$.

2. Outline of the proof

The proof of Theorem 1.5 proceeds via a convex integration scheme based on the scheme introduced in [5], which is itself built on a long line of work initiated by De Lellis and Székelyhidi Jr. [10], culminating in the eventual resolution of Onsager's conjecture by Isett [20] (cf. [1–4, 9, 11, 21]). Such a scheme is used to inductively define a sequence of approximate solutions converging to a weak solution of (1.3). The principal new idea of this paper is to create *good regions* in time where the approximate solutions are strong solutions to (1.3) and are untouched in later inductive steps. This is achieved by employing the method of gluing introduced by Isett [20] (cf. [4]). Taking the countable union of the good regions over each inductive step, one forms a fractal set, whose complement has Hausdorff dimension strictly less than 1. This is explained in detail in Sections 2.1 and 2.2 below. The concept of good regions is partially inspired by similar concepts introduced in [1] (cf. [3]). An additional novelty of the present work is the introduction of *intermittent jets* which replace the intermittent Beltrami flows of [5] as the fundamental building blocks on which the convex integration scheme is based (see Sections 2.3 and 4.1).

2.1. Inductive estimates and main proposition

For every $q \in \mathbb{N}$ we will construct a solution $(v_q, \mathring{R}_q)$ to the Navier–Stokes–Reynolds system

$$\partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q + (-\Delta)^\alpha v_q = \operatorname{div} \mathring{R}_q, \tag{2.1a}$$

$$\operatorname{div} v_q = 0, \tag{2.1b}$$

where $\mathring{R}_q$ is a trace-free symmetric matrix. The pressure p_q is normalized to have zero mean on $\mathbb{T}^3$ and is explicitly given by the formula

$$p_q = \operatorname{div} \Delta^{-1} \operatorname{div}(\mathring{R}_q - v_q \otimes v_q). \tag{2.2}$$

Here we use the convention that for a 2-tensor $S = (S^{ij})_{i,j=1}^3$ the divergence contracts on the second component, i.e. $(\operatorname{div} S)^i = \partial_j S^{ij}$. The summation convention on repeated indices is used throughout.

Fix a sufficiently large integer $b = b(\alpha) > 0$.[2] Depending on this choice of b, fix a sufficiently small parameter $\beta = \beta(\alpha, b) > 0$.[3] In particular, $\beta b \ll 1$.

[2]For instance, it is sufficient to take $b(5 - 4\alpha) \geq 1000$, which satisfies (4.43).

[3]For instance, it is sufficient to require that $200\beta b^2 \leq 5 - 4\alpha$; this satisfies both (4.43) and (5.7).

The size of the Reynolds stress $\mathring{R}_q$ will be measured in terms of a size parameter

$$\delta_q = \lambda_1^{3\beta} \lambda_q^{-2\beta} \tag{2.3}$$

where λ_q is a frequency parameter defined by

$$\lambda_q = a^{(b^q)}$$

where $a \gg 1$ is a large real number to be chosen later. Note that $\delta_1 = \lambda_1^{\beta} = a^{\beta b}$ is large if a is sufficiently large.

For every $q \geq 0$ we assume that $\mathring{R}_q$ obeys the estimates

$$\|\mathring{R}_q\|_{L^1(\mathbb{T}^3)} \leq \lambda_q^{-\varepsilon_R} \delta_{q+1}, \tag{2.4a}$$

$$\|\mathring{R}_q\|_{H^3(\mathbb{T}^3)} \leq \lambda_q^7, \tag{2.4b}$$

for some $\varepsilon_R > 0$ to be chosen later, which depends only on the values of α, β, and b. For the approximate velocity field v_q, we assume that

$$\|v_q\|_{L^2(\mathbb{T}^3)} \leq 2\delta_0^{1/2} - \delta_q^{1/2}, \tag{2.5a}$$

$$\|v_q\|_{H^3(\mathbb{T}^3)} \leq \lambda_q^4. \tag{2.5b}$$

These inductive estimates will ensure that the approximate solutions v_q converge strongly in $C^0(0, T; L^2)$ to a weak solution v of the Navier–Stokes equations (1.3).

Consider $T > 0$ and fix the parameter sequences $\{\tau_q\}_{q \geq 0}$ and $\{\vartheta_q\}_{q \geq 1}$ defined in (2.7) and (2.8) below, which obey the heuristic bounds

$$\vartheta_{q+1} \ll \tau_q \ll \vartheta_q \ll 1. \tag{2.6}$$

In particular, for $q \geq 1$ we make the choices

$$\vartheta_q = \lambda_{q-1}^{-7} \delta_q^{1/2}, \tag{2.7}$$

$$\tau_q = \vartheta_q \lambda_{q-1}^{-\varepsilon_R/4} = \lambda_{q-1}^{-7-\varepsilon_R/4} \delta_q^{1/2}. \tag{2.8}$$

For the special case $q = 0$ we set $\tau_0 := T/15$. For ϑ_0 we do not need to assign a value.

In order to ensure that the singular set of times has Hausdorff dimension strictly less than 1, at every $q \geq 0$ we split the interval $[0, T]$ into a closed *good set* $\mathcal{G}^{(q)}$ and an open *bad set* $\mathcal{B}^{(q)} = [0, T] \setminus \mathcal{G}^{(q)}$, which obey the following properties:

(i) $\mathcal{G}^{(0)} = [0, T/3] \cup [2T/3, T]$.

(ii) $\mathcal{G}^{(q-1)} \subset \mathcal{G}^{(q)}$ for every $q \geq 1$.

(iii) $\mathcal{B}^{(q)}$ is a finite union of disjoint open intervals of length $5\tau_q$.[4]

[4]Observe that this condition is consistent with property (i) and the definition $\tau_0 = T/15$.

(iv) For $q \geq 1$, the bad sets obey

$$|\mathcal{B}^{(q)}| \leq |\mathcal{B}^{(q-1)}| 10\tau_q/\vartheta_q. \tag{2.9}$$

(v) The velocity fields obey

$$\text{if} \quad t \in \mathcal{G}^{(q')} \text{ for some } q' < q, \quad \text{then} \quad v_q(t) = v_{q'}(t). \tag{2.10}$$

(vi) The residual Reynolds stress obeys

$$\mathring{R}_q(t) = 0 \quad \text{for all } t \in [0,T] \text{ such that } \operatorname{dist}(t, \mathcal{G}^{(q)}) \leq \tau_q. \tag{2.11}$$

Due to (2.11) and the parabolic regularization of the Navier–Stokes equation (cf. (3.4) below), v_q is a C^∞ smooth exact solution of the Navier–Stokes equation on $\mathcal{G}^{(q)}$. In addition, (2.10) implies that $v = v_q$ on $\mathcal{G}^{(q)} \setminus \{0\}$, and thus the limiting solution v is C^∞ smooth on $(\mathcal{G}^{(q)} \setminus \{0\}) \times \mathbb{T}^3$. This justifies that the *singular set of times*, Σ_T, obeys

$$\Sigma_T \subset \bigcap_{q \geq 0} \mathcal{B}^{(q)}. \tag{2.12}$$

It thus follows from (2.9) and the definitions of τ_q and ϑ_q in (2.7) and (2.8) that

$$\begin{aligned} |\mathcal{B}^{(q)}| &\leq |\mathcal{B}^{(0)}| \prod_{q'=1}^{q} \frac{10\tau_{q'}}{\vartheta_{q'}} \leq 10^q T \prod_{q'=0}^{q-1} \lambda_{q'}^{-\varepsilon_R/4} \leq T 10^q a^{-\frac{\varepsilon_R(b^q-1)}{4(b-1)}} \\ &\leq T 10^q \lambda_q^{-\frac{\varepsilon_R}{8(b-1)}}. \end{aligned} \tag{2.13}$$

Here we have also used the definition of λ_q, and the fact that $b > 2$. To estimate the box-counting (Minkowski) dimension of Σ_T, we note that for every $q \geq 0$, the set Σ_T is covered by $\mathcal{B}^{(q)}$, which itself consists of disjoint intervals of length $5\tau_q$. Due to (2.13), the number of such intervals is at most

$$T 10^q \lambda_q^{-\frac{\varepsilon_R}{8(b-1)}} (5\tau_q)^{-1}.$$

By (2.12), and the superexponential growth of λ_q, we conclude that

$$\begin{aligned} \dim_{\text{box}}(\Sigma_T) &\leq \lim_{q\to\infty} \frac{\log(T) + q\log(10) - \frac{\varepsilon_R}{8(b-1)}\log(\lambda_q) - \log(5\tau_q)}{-\log(5\tau_q)} \\ &= 1 - \lim_{q\to\infty} \frac{\frac{\varepsilon_R b}{8(b-1)}\log(\lambda_{q-1})}{-\log(\tau_q)} \\ &= 1 - \frac{\varepsilon_R b}{8(b-1)(7 + \varepsilon_R/4 + \beta b)} < 1 - \frac{\varepsilon_R}{64} < 1. \end{aligned} \tag{2.14}$$

This implies that Σ_T also has box-counting dimension strictly less than 1.

Proposition 2.1 (Main Iteration Proposition). *There exists a sufficiently small parameter* $\varepsilon_R = \varepsilon_R(\alpha, b, \beta) \in (0, 1)$ *and a sufficiently large parameter* $a_0 = a_0(\alpha, b, \beta, \varepsilon_R) \geq 1$ *such that for any* $a \geq a_0$ *satisfying the technical condition* (2.24) *below, the following holds: Let* $(v_q, \mathring{R}_q)$ *be a pair solving the Navier–Stokes–Reynolds system* (2.1) *in* $\mathbb{T}^3 \times [0, T]$ *satisfying the inductive estimates* (2.4)–(2.5)*, and with the corresponding set* $\mathscr{G}^{(q)}$ *with the properties* (i)–(vi) *listed above. Then there exists a second pair* $(v_{q+1}, \mathring{R}_{q+1})$ *solving* (2.1) *and a set* $\mathscr{G}^{(q+1)}$ *which satisfy* (2.4)–(2.5) *and* (i)–(vi) *with* q *replaced by* $q + 1$*. In addition*

$$\|v_{q+1} - v_q\|_{L^2} \leq \delta_{q+1}^{1/2}. \tag{2.15}$$

2.2. *Gluing stage*

The first stage of proving Proposition 2.1 is to start with the approximate solution $(v_q, \mathring{R}_q)$ which obeys (2.4)–(2.5) and (2.11), and construct a new *glued* pair $(\bar{v}_q, \mathring{\bar{R}}_q)$, which solves (2.1), obeys bounds (2.4)–(2.5) up to a factor of 2, and has the advantage that $\mathring{\bar{R}}_q \equiv 0$ on $\mathbb{T}^3 \times \mathscr{B}^{(q+1)}$.

Specifically, the new velocity field $\bar{v}_q$ is defined as

$$\bar{v}_q(x, t) = \sum_i \eta_i(t) v_i(x, t),$$

where the η_i are certain cutoff functions with support in $[t_i, t_{i+1} + \tau_{q+1}]$ (with $t_i = \vartheta_{q+1} i$) that form a partition of unity (see (3.26) below), and the v_i are exact solutions of the Navier–Stokes equation with initial datum given by $v_i(t_{i-1}) = v_q(t_{i-1})$. Due to parabolic regularization, these exact solutions v_i are C^∞ smooth in space and time on the support of η_i, so that $\bar{v}_q$ inherits this C^∞ regularity. This is in contrast to $(v_q, \mathring{R}_q)$, which is only assumed to be H^3 smooth. Trivially, in the regions where a cutoff η_i is identically 1, $\bar{v}_q$ is an exact solution to (1.3).

Observe that property (2.11) ensures that v_q is already an exact solution of (1.3) on a large subset of $[0, T]$, namely the τ_q-neighborhood of $\mathscr{G}^{(q)}$. In particular if t_{i-1} and t_i both lie within this neighborhood, then by uniqueness of the Navier–Stokes equation in $C_t^0 H_x^3$, we have $v_i = v_{i+1} = v_q$ on the overlapping region $\operatorname{supp} \eta_i \eta_{i+1}$. Hence $\bar{v}_q = v_q$ is an exact solution there. In order to single out overlapping regions where $\bar{v}_q$ is not necessarily an exact solution of (1.3) we introduce the index set

$$\mathscr{C} = \{i \in \{1, \ldots, n_{q+1}\}\colon \text{there exists } t \in [t_{i-1}, t_{i+1} + \tau_{q+1}] \text{ with } \mathring{R}_q(t) \neq 0\}. \tag{2.16}$$

We then define

$$\mathscr{B}^{(q+1)} = \bigcup_{i \in \mathscr{C} \text{ or } i-1 \in \mathscr{C}} (t_i - 2\tau_{q+1}, t_i + 3\tau_{q+1}). \tag{2.17}$$

By the discussion above, it will follow that $\bar{v}_q$ is an exact solution on the complement of $\mathscr{B}^{(q+1)}$, that is, $\mathscr{G}^{(q+1)}$. We prove in Section 3 below that the above defined good and bad sets at level $q + 1$ obey the postulated properties (i)–(iv).

In Section 3 we prove the following proposition:

Proposition 2.2. *There exists a solution* $(\bar v_q, \mathring{\bar R}_q)$ *of* (2.1) *such that*

$$\bar v_q \equiv v_q \quad on\ \mathbb{T}^3 \times \mathscr{G}^{(q)}, \tag{2.18}$$

and moreover the velocity field $\bar v_q$ *satisfies*

$$\|\bar v_q\|_{L^2} \le 2\delta_0^{1/2} - \delta_q^{1/2}, \tag{2.19a}$$

$$\|\bar v_q\|_{H^3} \le 2\lambda_q^4, \tag{2.19b}$$

$$\|\bar v_q - v_q\|_{L^2} \le \vartheta_{q+1}\lambda_q^6 \le \tfrac{1}{4}\delta_{q+1}^{1/2}, \tag{2.19c}$$

$$\|\partial_t^M D^N \bar v_q\|_{L^\infty(T/3,2T/3;H^3)} \lesssim \tau_{q+1}^{-M}\vartheta_{q+1}^{-\frac{N}{2\alpha}}\lambda_q^4 \lesssim \tau_{q+1}^{-M-N}\lambda_q^4, \tag{2.19d}$$

and the stress tensor $\mathring{\bar R}_q$ *satisfies*

$$\mathring{\bar R}_q(t) = 0 \quad \textit{for all } t \in [0,T] \textit{ with } \mathrm{dist}(t, \mathscr{G}^{(q+1)}) \le 2\tau_{q+1}, \tag{2.20a}$$

$$\|\mathring{\bar R}_q\|_{L^1} \le \tau_{q+1}^{-1}\vartheta_{q+1}\lambda_q^{-\varepsilon_R/2}\delta_{q+1} \le \lambda_q^{-\varepsilon_R/4}\delta_{q+1}, \tag{2.20b}$$

$$\|\partial_t^M D^N \mathring{\bar R}_q\|_{H^3} \lesssim \tau_{q+1}^{-M-1}\vartheta_{q+1}^{-\frac{N}{2\alpha}}\lambda_q^4 \lesssim \tau_{q+1}^{-M-N-1}\lambda_q^4, \tag{2.20c}$$

for all $M, N \ge 0$.

2.3. *Convex integration stage*

In this step we start from the pair $(\bar v_q, \mathring{\bar R}_q)$, and construct a new pair $(v_{q+1}, \mathring R_{q+1})$ with $\mathring R_{q+1}$ obeying (2.11) at level $q+1$, and which obeys the bounds (2.4)–(2.5) at level $q+1$.

The perturbation $w_{q+1} := v_{q+1} - \bar v_q$ will be constructed to correct for $\mathring{\bar R}_q$. Moreover, w_{q+1} will be designed to have support outside a τ_{q+1}-neighborhood of $\mathscr{G}^{(q+1)}$ – this ensures properties (v) and (vi) in Section 2.1 will be satisfied. As in [5], the perturbation w_{q+1} will consist of three parts: the *principal part* $w_{q+1}^{(p)}$, the *divergence corrector* $w_{q+1}^{(c)}$, and the *temporal corrector* $w_{q+1}^{(c)}$.

The principal part $w_{q+1}^{(p)}$ will be constructed as a sum of *intermittent jets* $W_{(\xi)}$ (defined in (4.4), Section 4.1). The use of intermittent jets replaces the use of *intermittent Beltrami waves* in [5]. The principal difference between intermittent jets and intermittent Beltrami waves is that the definition of the former is in physical space rather than frequency space. Consequently, intermittent jets are comparatively simpler to define and they can be designed to have disjoint support, mimicking the advantageous support properties of Mikado flows, as introduced in [9]. We note that the intermittent variants of the $d-1$-dimensional Mikado flows found in [36, 37], lying in d-dimensional space, are insufficiently intermittent to be used as building blocks for a 3-D Navier–Stokes convex integration scheme.[5] Intermittent jets are inherently 3-dimensional (in space), with the trade-off

[5]For Navier–Stokes in dimensions greater than 3, they are however applicable, as demonstrated in [36].

that they are time dependent. We note in passing that it is likely that the convex integration results [37, 38] on the transport equation may be improved utilizing intermittent jets.

In the definition of $w_{q+1}^{(p)}$, the intermittent jets $W_{(\xi)}$ will be weighted by functions $a_{(\xi)}$:

$$w_{q+1}^{(p)} = \sum_{\xi} a_{(\xi)} W_{(\xi)},$$

where $a_{(\xi)}$ are constructed such that

$$\operatorname{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_q) \sim \frac{1}{\mu}\partial_t \mathbb{P}_H \mathbb{P}_{\neq 0}\Big(\sum_{\xi} a_{(\xi)}^2 |W_{(\xi)}|^2 \xi\Big) + (\text{pressure gradient}) + (\text{high frequency error}) \tag{2.21}$$

for some large parameter μ. As is typical in convex integration schemes, the high frequency error can be ignored since its contribution to $\mathring{R}_{q+1}$ can be bounded using the gain associated with solving the divergence equation. The temporal corrector $w_{q+1}^{(t)}$ is then defined to be

$$w_{q+1}^{(t)} := -\frac{1}{\mu}\mathbb{P}_H \mathbb{P}_{\neq 0}\Big(\sum_{\xi} a_{(\xi)}^2 |W_{(\xi)}|^2 \xi\Big),$$

where $\mathbb{P}_H$ is the Helmholtz projection, and $\mathbb{P}_{\neq 0}$ is the projection onto the functions with zero mean. That is, $\mathbb{P}_H f = f - \nabla(\Delta^{-1} \operatorname{div} f)$ and $\mathbb{P}_{\neq 0} f = f - \fint_{\mathbb{T}^3} f$. Hence

$$\operatorname{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_q) + \partial_t w_{q+1}^{(t)} \sim (\text{pressure gradient}) + (\text{high frequency error}).$$

Finally, the divergence corrector $w_{q+1}^{(c)}$ is designed so that $\operatorname{div}(w_{q+1}^{(p)} + w_{q+1}^{(c)}) \equiv 0$, and hence the perturbation

$$w_{q+1}^{(c)} := w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}$$

is divergence-free.

The intermittent jets will be defined to have support confined to $\sim (\ell_\perp \lambda_{q+1})^3$ cylinders of diameter $\sim \frac{1}{\lambda_{q+1}}$ and length $\sim \frac{\ell_\parallel}{\ell_\perp \lambda_{q+1}}$. In particular, the support of $w_{q+1}^{(p)}$ has measure $\sim \ell_\parallel \ell_\perp^2$. Using the heuristic that $\|w_{q+1}^{(p)}\|_{L^2}$ should be roughly $\|\mathring{R}_q\|_{L^1}^{1/2}$, by the L^p de-correlation result in Lemma 4.5 below, one would expect an L^p estimate

$$\|w_{q+1}^{(p)}\|_{L^p} \sim \delta_{q+1}^{1/2} \ell_\perp^{2/p-1} \ell_\parallel^{1/p-1/2}. \tag{2.22}$$

Indeed, we will prove estimate (2.22) for $p = 2$ and prove a slightly weaker estimate for $1 < p < 2$ (see Proposition 4.4). Utilizing (2.22), one may heuristically estimate the contribution of $(-\Delta)^\alpha w_{q+1}^{(p)}$ to the new Reynolds stress $\mathring{R}_{q+1}$:

$$\big\||\nabla|^{-1}(-\Delta)^\alpha (w_{q+1}^{(p)})\big\|_{L^1} \sim \|w_{q+1}^{(p)}\|_{W^{2\alpha-1,p}} \sim \delta_{q+1}^{1/2} \ell_\perp^{2/p-1} \ell_\parallel^{1/p-1/2} \lambda_{q+1}^{2\alpha-1},$$

with $p > 1$ arbitrarily close to 1. Here we see the necessity of the 3-dimensionality of the intermittent jets.

In order to ensure that an identity of the form (2.21) holds, the cylinder supports of the intermittent jets will be shifting at a speed $\ell_\perp \lambda_{q+1}\mu$. Heuristically, one would then expect that in order to ensure that the contribution of $\partial_t w_{q+1}^{(p)}$ to $\mathring{R}_{q+1}$ is small, one would need to impose an upper bound on the choice of μ. One then needs to choose μ carefully in order to balance different contributions to the Reynolds stress error. Explicitly, we will define the parameters μ, $\ell_\perp$ and $\ell_\|$ by

$$\mu = \frac{\lambda_{q+1}^{2\alpha-1}\ell_\|}{\ell_\perp}, \quad \ell_\perp := \lambda_{q+1}^{-\frac{20\alpha-1}{24}}, \quad \ell_\| := \lambda_{q+1}^{-\frac{20\alpha-13}{12}}. \tag{2.23}$$

With these choices, we have

$$\ell_\|^{-1} \ll \ell_\perp^{-1} \ll \lambda_{q+1}$$

since $\alpha < 5/4$. For technical reasons, we will require that $\lambda_{q+1}\ell_\perp \in \mathbb{N}$. This may be achieved by assuming that

$$a^{\frac{25-20\alpha}{24}} \in \mathbb{N}, \tag{2.24}$$

where we recall that we have previously assumed that $b \in \mathbb{N}$.

2.4. *Proof of Theorem* 1.5

Let $u^{(1)}$ and $u^{(2)}$ be two zero-mean solutions of the Navier–Stokes equations (with different, zero-mean initial data), as in the statement of the theorem. Also, let b, β, ϵ_R, and a_0 be as in Proposition 2.1. Let $\eta\colon [0,T] \to [0,1]$ be a smooth cutoff function such that $\eta = 1$ on $[0, 2T/5]$ and $\eta = 0$ on $[3T/5, T]$.

Define

$$\begin{aligned} v_0(x,t) &= \eta(t)u^{(1)}(x,t) + (1-\eta(t))u^{(2)}(x,t). \\ \mathring{R}_0 &= \partial_t \eta\, \mathcal{R}(u^{(1)} - u^{(2)}) - \eta(1-\eta)(u^{(1)} - u^{(2)}) \mathring{\otimes} (u^{(1)} - u^{(2)}), \end{aligned} \tag{2.25}$$

where $a \mathring{\otimes} b$ denotes the traceless part of the tensor $a \otimes b$, and $\mathcal{R}$ is a standard inverse divergence operator acting on vector fields v which have zero mean on $\mathbb{T}^3$ as

$$(\mathcal{R}v)^{k\ell} = (\partial_k \Delta^{-1} v^\ell + \partial_\ell \Delta^{-1} v^k) - \tfrac{1}{2}(\delta_{k\ell} + \partial_k \partial_\ell \Delta^{-1}) \operatorname{div} \Delta^{-1} v \tag{2.26}$$

for $k, \ell \in \{1,2,3\}$. The above inverse divergence operator has the property that $\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$, and $\mathcal{R}$ is a right inverse of the div operator, i.e. $\operatorname{div}(\mathcal{R}v) = v$. When v does not obey $\fint_{\mathbb{T}^3} v\, dx = 0$, we abuse notation and denote $\mathcal{R}v := \mathcal{R}(v - \fint_{\mathbb{T}^3} v\, dx)$. Note that $\nabla \mathcal{R}$ is a Calderón–Zygmund operator, and $\mathcal{R}$ obeys the same elliptic regularity estimates as $|\nabla|^{-1}$.

Observe that the pair $(v_0, \mathring{R}_0)$ obeys the Navier–Stokes–Reynolds system (2.1) for a suitable zero-mean pressure scalar p_0 which may be computed by solving a Poisson equation. Moreover, let a_0, β and b be as in Proposition 2.1. Then choosing $a \geq a_0$ sufficiently large, the pair $(v_0, \mathring{R}_0)$ satisfies (2.4)–(2.5). From the definition (2.25), it follows that $\mathring{R}_0$ is supported on the interval $[2T/5, 3T/5]$. Since by definition $\mathcal{G}^{(0)} = [0, T/3] \cup [2T/3, T]$ and $\tau_0 = T/15$, we obtain property (2.11).

For $q \geq 1$ we inductively apply Proposition 2.1. The bounds (2.5b) and (2.15) and interpolation yield

$$\sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_{\dot{H}^{\beta'}} \lesssim \sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_{L^2}^{1-\beta'/3} (\|v_{q+1}\|_{\dot{H}^3} + \|v_q\|_{\dot{H}^3})^{\beta'/3}$$
$$\lesssim \sum_{q=0}^{\infty} \lambda_{q+1}^{-\beta \frac{3-\beta'}{6}} \lambda_{q+1}^{4\beta'/3} \lesssim 1$$

for $0 \leq \beta' < \frac{3\beta}{8+\beta}$, where the implicit constant is universal (independent of a). Hence the limit

$$v := \lim_{q\to\infty} v_q \in H^{\beta'}$$

exists. Since $\|\mathring{R}_q\|_{L^1} \to 0$ as $q \to \infty$, and since $v_q \to v$ also in $L_t^\infty L^{2+\beta'''}$ for some $\beta''' > 0$, it is straightforward to show that v is a weak solution of the Navier–Stokes equation. Moreover, as a consequence of properties (i) and (v) from Section 2.1 and the definition of v_0 we have

$$v \equiv u^{(1)} \quad \text{on } [0, T/3] \quad \text{and} \quad v \equiv u^{(2)} \quad \text{on } [2T/3, T].$$

The argument leading to (2.14) implies that the singular set of times of v has box-counting dimension (and hence Hausdorff dimension) less than $\varepsilon_R/64$. Finally, the claimed $C_t^0 W_x^{1,1+\beta''}$ regularity on v, for some $\beta'' > 0$, follows from the maximal regularity of the heat equation (fractional heat equation if $\alpha > 1$), once we note that $\|\mathbb{P}_H(v \otimes v)\|_{L^{1+\beta''}} \lesssim \|v\|_{H^{\beta'}}^2$ if β'' is chosen suitably small. The theorem then holds with $\bar{\beta} = \min\{\beta'', \beta', \varepsilon_R/64\} > 0$.

3. Gluing step

3.1. Local in time estimates

It is well-known that Navier–Stokes equations are locally (in time) well-posed in H^3, which is a scaling subcritical space. Moreover, away from the initial time, parabolic regularization takes place. We summarize these facts in the form that is suitable for the applications in this paper.

Proposition 3.1. *Let $v_0 = v|_{t=t_0} \in H^3(\mathbb{T}^3)$ have zero mean on $\mathbb{T}^3$, and consider the Cauchy problem for (1.3) with this initial condition. There exists a universal constant $c \in (0, 1]$ such that if $t_1 > t_0$ is such that*

$$0 < t_1 - t_0 \leq \frac{c}{\|v_0\|_{H^3}}, \tag{3.1}$$

then there exists a unique strong solution to (1.3) *on* $[t_0, t_1)$*, and it obeys the estimates*

$$\sup_{t\in[t_0,t_1]} \|v(t)\|_{L^2}^2 + 2\int_{t_0}^{t_1} \|v(t)\|_{\dot H^\alpha}^2 \, dt \le \|v_0\|_{L^2}^2, \tag{3.2a}$$

$$\sup_{t\in[t_0,t_1]} \|v(t)\|_{H^3} \le 2\|v_0\|_{H^3}. \tag{3.2b}$$

Moreover, assuming that

$$0 < t_1 - t_0 \le \frac{c}{\|v_0\|_{H^3}(1+\|v_0\|_{L^2})^{\frac{1}{2\alpha-1}}}, \tag{3.3}$$

we have

$$\sup_{t\in(t_0,t_1]} |t-t_0|^{\frac{N}{2\alpha}+M} \|\partial_t^M D^N v(t)\|_{H^3} \lesssim \|v_0\|_{H^3} \tag{3.4}$$

for any $N \ge 0$ *and* $M \in \{0,1\}$*. The implicit constant may depend on* α, N, M*.*

Proof. The energy inequality gives a global in time control on $\|v(t)\|_{L^2}$:

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 \le -\|v\|_{\dot H^\alpha}^2.$$

From the Gagliardo–Nirenberg–Sobolev and the Poincaré inequalities, and using $\nabla \cdot v = 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|_{\dot H^3}^2 + \|v\|_{\dot H^{3+\alpha}}^2 \lesssim \|v\|_{\dot H^3}^2\|\nabla v\|_{L^\infty} + \|v\|_{\dot H^3}\|\Delta v\|_{L^4}^2 \lesssim \|v\|_{\dot H^3}^3,$$

which gives the bound (3.2b) for a time interval $[t_0, t_1]$ with t_1 that obeys (3.1). The bound (3.2b) is subcritical, in the sense that an $L_t^\infty H_x^3$ a priori estimate is sufficient to establish the uniqueness of the solution. The higher regularity claimed in (3.4) follows from the mild form of the solution

$$v(t) = e^{-(t-t_0)(-\Delta)^\alpha} v_0 + \int_{t_0}^{t} e^{-(t-s)(-\Delta)^\alpha} \mathbb{P}_H \operatorname{div}(v(s)\otimes v(s))\, ds, \tag{3.5}$$

and properties of the fractional heat equation which may be derived from Plancherel.

Let us first focus on the case $M = 0$. For $\alpha = 1$, estimate (3.4) is well-known, and follows from the instantaneous gain of analyticity of the solution [15], or a small modification of the argument below. For $\alpha > 1$ we briefly sketch the argument. Using Gallilean invariance, let us only consider the case $t_0 = 0$. From the inequality

$$\begin{aligned}\|uv\|_{H^3} &\lesssim \|u\|_{H^3}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^3}\\ &\lesssim \|u\|_{H^3}\|v\|_{L^2}^{1/2}\|v\|_{H^3}^{1/2} + \|u\|_{L^2}^{1/2}\|u\|_{H^3}^{1/2}\|v\|_{H^3},\end{aligned} \tag{3.6}$$

the formulation (3.5) and the boundedness of the Leray projector $\mathbb{P}_H$ on L^2, we obtain

$$\begin{aligned} t^{\frac{1}{2\alpha}}\|Dv(t)\|_{H^3} &\leq t^{\frac{1}{2\alpha}}\|De^{-t(-\Delta)^\alpha}\|_{L^2\to L^2}\|v_0\|_{H^3} \\ &\quad + t^{\frac{1}{2\alpha}}\int_0^t \|D^2 e^{-(t-s)(-\Delta)^\alpha}\|_{L^2\to L^2}\|v(s)\|_{H^3}^{3/2}\|v(s)\|_{L^2}^{1/2}\,ds \\ &\lesssim \|v_0\|_{H^3} + t^{\frac{1}{2\alpha}}\|v_0\|_{H^3}^{3/2}\|v_0\|_{L^2}^{1/2}\int_0^t \frac{ds}{(t-s)^{1/\alpha}} \\ &\lesssim \|v_0\|_{H^3}(1 + t^{1-\frac{1}{2\alpha}}\|v_0\|_{H^3}^{1/2}\|v_0\|_{L^2}^{1/2}) \\ &\lesssim \|v_0\|_{H^3}(1 + t^{\frac{2\alpha-1}{2\alpha}}\|v_0\|_{H^3}^{\frac{2\alpha-1}{2\alpha}}\|v_0\|_{L^2}^{\frac{1}{2\alpha}}), \end{aligned}$$

from which (3.4) with $N = 1$ and $M = 0$ follows in view of (3.3). In order to treat the case $N \geq 2$ and $M = 0$, we first note that for $1 \leq n \leq N - 1$ by induction on N we have

$$\begin{aligned} &\|D^n(v\otimes v)\|_{H^3} \lesssim \sum_{j=0}^{n}\|D^j v\otimes D^{n-j}v\|_{H^3} \lesssim \sum_{j=0}^{n}\|D^j v\|_{H^3}\|D^{n-j}v\|_{H^3}^{1/2}\|D^{n-j}v\|_{L^2}^{1/2} \\ &\quad\lesssim \sum_{j=0}^{n-3}\|D^j v\|_{H^3}\|D^{n-j}v\|_{H^3}^{1/2}\|D^{n-j-3}v\|_{H^3}^{1/2} + \|D^{n-2}v\|_{H^3}\|D^2 v\|_{H^3}^{1/2}\|v\|_{H^3}^{1/3}\|v\|_{L^2}^{1/6} \\ &\qquad + \|D^{n-1}v\|_{H^3}\|Dv\|_{H^3}^{1/2}\|v\|_{H^3}^{1/6}\|v\|_{L^2}^{1/3} + \|D^n v\|_{H^3}\|v\|_{H^3}^{1/2}\|v\|_{L^2}^{1/2} \\ &\quad\lesssim \|v_0\|_{H^3}^2 t^{-\frac{n}{2\alpha}+\frac{3}{4\alpha}} + \|v_0\|_{H^3}^{11/6} t^{-\frac{n}{2\alpha}+\frac{1}{2\alpha}}\|v_0\|_{L^2}^{1/6} \\ &\qquad + \|v_0\|_{H^3}^{5/3} t^{-\frac{n}{2\alpha}+\frac{1}{4\alpha}}\|v_0\|_{L^2}^{1/3} + \|v_0\|_{H^3}^{3/2} t^{-\frac{n}{2\alpha}}\|v_0\|_{L^2}^{1/2} \\ &\quad\lesssim \|v_0\|_{H^3}^{3/2} t^{-\frac{n}{2\alpha}}(\|v_0\|_{H^3}^{1/2} t^{\frac{3}{4\alpha}} + \|v_0\|_{L^2}^{1/2}). \end{aligned}$$

Using the above estimate with $n = N - 1$ we obtain

$$\begin{aligned} t^{\frac{N}{2\alpha}}\|D^N v(t)\|_{H^3} &\leq t^{\frac{N}{2\alpha}}\|D^N e^{-t(-\Delta)^\alpha}\|_{L^2\to L^2}\|v_0\|_{H^3} \\ &\quad + t^{\frac{N}{2\alpha}}\int_{t/2}^t \|D^2 e^{-(t-s)(-\Delta)^\alpha}\|_{L^2\to L^2}\|D^{N-1}(v(s)\otimes v(s))\|_{H^3} \\ &\quad + t^{\frac{N}{2\alpha}}\int_0^{t/2} \|D^{N+1} e^{-(t-s)(-\Delta)^\alpha}\|_{L^2\to L^2}\|v(s)\otimes v(s)\|_{H^3}\,ds \\ &\lesssim \|v_0\|_{H^3} + t^{\frac{N}{2\alpha}}\|v_0\|_{H^3}^{3/2}\int_{t/2}^t \frac{\|v_0\|_{H^3}^{1/2} s^{\frac{3}{4\alpha}} + \|v_0\|_{L^2}^{1/2}}{(t-s)^{\frac{1}{\alpha}} s^{\frac{N-1}{2\alpha}}}\,ds \\ &\quad + t^{\frac{N}{2\alpha}}\|v_0\|_{H^3}^{3/2}\|v_0\|_{L^2}^{1/2}\int_0^{t/2} \frac{ds}{(t-s)^{\frac{N+1}{2\alpha}}} \\ &\lesssim \|v_0\|_{H^3}(1 + t^{1+\frac{5}{4\alpha}}\|v_0\|_{H^3} + t^{1-\frac{1}{2\alpha}}\|v_0\|_{H^3}^{1/2}\|v_0\|_{L^2}^{1/2}), \end{aligned}$$

from which (3.4) follows in view of (3.3).

To obtain the desired bounds for $M = 1$, let us consider the case $N = 0$ first. Using the equation, the already established bounds for $M = 0$ and $N \geq 0$, the Gagliardo–Nirenberg–Sobolev inequalities, and the fact that the Leray projector is bounded on L^2, we find that

$$\begin{aligned} t\|\partial_t v(t)\|_{H^3} &\leq t\|(-\Delta)^\alpha v(t)\|_{H^3} + t\|\nabla v(t)\|_{H^3}\|v(t)\|_{H^3}^{1/2}\|v(t)\|_{L^2}^{1/2} \\ &\quad + t\|v(t)\|_{H^3}^{5/6}\|v(t)\|_{L^2}^{1/6}\|v(t)\|_{H^3} \\ &\lesssim \|v_0\|_{H^3} + t^{1-\frac{1}{2\alpha}}\|v_0\|_{H^3}^{3/2}\|v_0\|_{L^2}^{1/2} + t\|v_0\|_{H^3}^{11/6}\|v_0\|_{L^2}^{1/6}, \end{aligned}$$

and the desired bound follows from the assumption (3.3). The remaining cases $N \geq 1$ are treated in a similar manner, using the Leibniz rule. We omit the details. ■

3.2. Stability estimates

In this section we estimate the difference between an approximate solution v_q and an exact solution of the Navier–Stokes equation. Let $\mathcal{R}$ be the inverse divergence operator defined in (2.26). The main result is:

Proposition 3.2. *Fix $\alpha \in [1, 5/4)$ and an integrability index $p_0 \in (1, 5/4)$. Assuming the parameter δ_0 is sufficiently large, depending on p_0, the following holds.*

For $q \geq 0$, assume that $(v_q, \mathring{R}_q)$ is a $C_t^0 H_x^3$ smooth solution of (2.1) which obeys the estimates (2.4)–(2.5). Let $t_0 \in [0, T]$ and define

$$v_0 := v_q|_{t=t_0}.$$

Assume that $t_1 > t_0$ is such that $[t_0, t_1] \subset [0, T]$ and

$$0 < t_1 - t_0 \leq \delta_0^{-1}\lambda_q^{-4}. \tag{3.7}$$

Then, in view of (2.5b) and Proposition 3.1, there exists a unique $C_t^0 H_x^3$ smooth zero-mean solution v of the Cauchy problem for (1.3) on $[t_0, t_1]$, with initial datum v_0. Moreover, there exists a constant $C = C(p_0, \alpha) > 0$ such that for any $p \in [p_0, 2]$ and all $t \in (t_0, t_1]$,

$$\|v(t) - v_q(t)\|_{L^p} \leq C|t - t_0| \, \big\||\nabla|\mathring{R}_q\big\|_{L^\infty([t_0,t_1];L^p)}, \tag{3.8a}$$

$$\|\mathcal{R}v(t) - \mathcal{R}v_q(t)\|_{L^p} \leq C|t - t_0| \, \|\mathring{R}_q\|_{L^\infty([t_0,t_1];L^p)}. \tag{3.8b}$$

In particular, letting

$$p_0 = 1 + \frac{\varepsilon_R}{32} \in (1, 5/4), \tag{3.9}$$

from the bounds (3.8a)–(3.8b) we obtain the following stability estimate:

Corollary 3.3. *Fix $\alpha \in [1, 5/4)$. Assuming that $a \geq 1$ is sufficiently large, depending only on ε_R, if $t_1 \in (t_0, T]$ obeys (3.7), then*

$$\|v - v_q\|_{L^\infty([t_0,t_1];L^2)} \leq |t_1 - t_0|\lambda_q^5, \tag{3.10a}$$

$$\|\mathcal{R}(v - v_q)\|_{L^\infty([t_0,t_1];L^1)} \leq |t_1 - t_0|\lambda_q^{-3\varepsilon_R/4}\delta_{q+1}. \tag{3.10b}$$

Proof of Corollary 3.3. We show that estimates (3.8) imply (3.10). Recall that the stress $\mathring{R}_q$ has zero mean. For $p \in (1,2]$ and $\delta \in [0,1]$ by interpolation we have the inequalities $\||\nabla|^\delta f\|_{L^p} \lesssim \|f\|_{L^p}^{1-\delta/3}\|f\|_{\dot W^{3,p}}^{\delta/3}$ and $\|f\|_{L^p} \lesssim \|f\|_{L^1}^{1/p}\|f\|_{L^\infty}^{1-1/p}$. Moreover, since $H^3 \subset L^\infty$ and $H^3 \subset W^{3,p}$, we obtain the Gagliardo–Nirenberg-type inequality

$$\||\nabla|^\delta \mathring{R}_q\|_{L^p} \lesssim \|\mathring{R}_q\|_{L^1}^\gamma \|\mathring{R}_q\|_{H^3}^{1-\gamma} \quad \text{with} \quad \gamma = \frac{1}{p} - \frac{\delta}{3p}. \tag{3.11}$$

The implicit constant depends only on p and δ.

In order to prove (3.10a), we use (3.8a) and apply estimate (3.11) with $\delta = 1$ and $p = 2$. We deduce from (2.4) that

$$\||\nabla|\mathring{R}_q\|_{L^2} \lesssim (\lambda_q^{-\varepsilon_R}\delta_{q+1})^{\frac{1}{2}-\frac{1}{6}}\lambda_q^{7(\frac{1}{2}+\frac{1}{6})},$$

from which estimate (3.10a) follows, since $\delta_{q+1} \le \lambda_1^\beta$, and β is sufficiently small. The leftover power of λ_q may be used to absorb any constants.

Similarly, in order to prove (3.10b), we use (3.8b), the bound (3.11) with $\delta = 0$ and $p = p_0$, and the embedding $L^{p_0} \subset L^1$, to obtain

$$\begin{aligned}
\|\mathring{R}_q\|_{L^1} \lesssim \|\mathring{R}_q\|_{L^{p_0}} &\lesssim (\lambda_q^{-\varepsilon_R}\delta_{q+1})^{1/p_0}(\lambda_q^7)^{(p_0-1)/p_0} \\
&\lesssim (\lambda_q^{-\varepsilon_R}\delta_{q+1})^{1-(p_0-1)/p_0}\lambda_q^{7(p_0-1)/p_0} \\
&= \lambda_q^{-3\varepsilon_R/4}\delta_{q+1}(\lambda_q^{-\varepsilon_R/4}(\delta_{q+1}^{-1}\lambda_q^{\varepsilon_R+7})^{(p_0-1)/p_0}) \\
&\le \lambda_q^{-3\varepsilon_R/4}\delta_{q+1}\lambda_q^{-\varepsilon_R/4+(p_0-1)(\varepsilon_R+7+2\beta b)} \\
&\le \lambda_q^{-3\varepsilon_R/4}\delta_{q+1}\lambda_q^{-\varepsilon_R/4+8(p_0-1)}.
\end{aligned}$$

In the last inequality above we have used the definitions of δ_{q+1} and λ_q, and the fact that $p_0 \ge 1$. Estimate (3.10b) follows from the assumption (3.9) on p_0, upon using the leftover power of λ_q to absorb the implicit constants. ∎

Proof of Proposition 3.2. For simplicity, by temporal translation invariance it is sufficient to consider the case $t_0 = 0$.

In order to prove (3.8a) we let $u = v_q - v$ and $q = p_q - p$. Then $\operatorname{div} u = 0$, $u|_{t=0} = 0$, and u obeys the equation

$$\partial_t u + (-\Delta)^\alpha u = \mathbb{P}\operatorname{div}\mathring{R}_q - \mathbb{P}\operatorname{div}(v \otimes u + u \otimes v_q), \tag{3.12}$$

where $\mathbb{P}$ is the Leray projector. Then, since $u(0) = 0$, the solution of (3.12) may be written in integral form as

$$u(t) = \int_0^t e^{-(t-s)(-\Delta)^\alpha}\mathbb{P}\operatorname{div}(\mathring{R}_q - v \otimes u - u \otimes v_q)(s)\,ds. \tag{3.13}$$

Next, we use the fact that for $p \in [1,2]$, $t > 0$, and any periodic function ϕ of zero mean we have

$$\|e^{-t(-\Delta)^\alpha}\phi\|_{L^p} \lesssim \|\phi\|_{L^p}, \tag{3.14a}$$

$$\|\nabla e^{-t(-\Delta)^\alpha}\phi\|_{L^p} \lesssim \frac{1}{t^{\frac{1}{2\alpha}}}\|\phi\|_{L^p}, \tag{3.14b}$$

where the implicit constant only depends on α. These estimates follow from L^1 bounds for the Green's function of the fractional heat equation. We will also frequently use the Gagliardo–Nirenberg estimates

$$\|\nabla\phi\|_{L^\infty} \lesssim \|\phi\|_{L^2}^{1/6}\|\phi\|_{\dot H^3}^{5/6}, \tag{3.15a}$$

$$\|\phi\|_{L^\infty} \lesssim \|\phi\|_{L^2}^{1/2}\|\phi\|_{\dot H^3}^{1/2}, \tag{3.15b}$$

which hold for zero-mean periodic functions ϕ.

We return to (3.13) and obtain

$$\begin{aligned}
\|u(t)\|_{L^p} &\le \int_0^t \|e^{-(t-s)(-\Delta)^\alpha}\mathbb{P}\operatorname{div}(\mathring{R}_q - v\otimes u - u\otimes v_q)(s)\|_{L^p}\,ds\\
&\lesssim \int_0^t \left\||\nabla|\mathring{R}_q(s)\right\|_{L^p} + \frac{1}{(t-s)^{\frac{1}{2\alpha}}}\|(v\otimes u + u\otimes v_q)(s)\|_{L^p}\,ds\\
&\le C_1\int_0^t \left\||\nabla|\mathring{R}_q(s)\right\|_{L^p} + \frac{1}{(t-s)^{\frac{1}{2\alpha}}}(\|v(s)\|_{L^\infty} + \|v_q(s)\|_{L^\infty})\|u(s)\|_{L^p}\,ds,
\end{aligned} \tag{3.16}$$

for a suitable constant $C_1 > 0$ which only depends on p_0, since $p \in [p_0, 2]$ and $\alpha \in [1, 5/4]$.

Next, we claim that if $t_1 > 0$ is chosen sufficiently small, depending on $\|v\|_{L^\infty}$ and $\|v_q\|_{L^\infty}$, then

$$\|u(t)\|_{L^p} \le 2C_1 t\left\||\nabla|\mathring{R}_q(s)\right\|_{L^\infty([0,t_1];L^p)} \quad \text{for all } t \in (0, t_1]. \tag{3.17}$$

This estimate follows from Grönwall's inequality and the following bootstrap argument. Assuming that the bound (3.17) holds, we claim that the same estimate holds with the constant $2C_1$ replaced by the smaller constant $3C_1/2$. Indeed, inserting (3.17) in (3.16) we obtain

$$\begin{aligned}
\frac{\|u(t)\|_{L^p}}{2C_1 t\left\||\nabla|\mathring{R}_q(s)\right\|_{L^\infty([0,t_1];L^p)}} &\le \frac{1}{2} + \frac{1}{t}(\|v\|_{L^\infty} + \|v_q\|_{L^\infty})\int_0^t \frac{s\,ds}{(t-s)^{\frac{1}{2\alpha}}}\\
&\le \frac{1}{2} + \frac{2\alpha}{2\alpha-1}t^{1-\frac{1}{2\alpha}}(\|v\|_{L^\infty} + \|v_q\|_{L^\infty}).
\end{aligned} \tag{3.18}$$

Thus if we ensure that

$$4t_1^{\frac{1}{2}+\frac{\alpha-1}{2\alpha}}(\|v\|_{L^\infty} + \|v_q\|_{L^\infty}) \le 1/4, \tag{3.19}$$

then (3.18) shows that (3.17) holds with constant $3C_1/2$, as desired. However, by (3.15b) we know that

$$\|v\|_{L^\infty} + \|v_q\|_{L^\infty} \leq C_1(\|v\|_{L^2}^{1/2}\|v\|_{H^3}^{1/2} + \|v_q\|_{L^2}^{1/2}\|v_q\|_{H^3}^{1/2})$$

for some universal constant $C_1 > 0$, and further, using (2.5) and (3.2), we obtain

$$\|v\|_{L^\infty} + \|v_q\|_{L^\infty} \leq C_1(\|v_0\|_{L^2}^{1/2}(2\|v_0\|_{H^3})^{1/2} + \|v_q\|_{L^2}^{1/2}\|v_q\|_{H^3}^{1/2}) \leq 4C_1\delta_0^{1/4}\lambda_q^2.$$

To conclude, we use (3.7), which shows that the left side of (3.19) is bounded from above by

$$4(\delta_0^{-1}\lambda_q^{-4})^{\frac{1}{2}+\frac{\alpha-1}{2\alpha}} 4C_1\delta_0^{1/4}\lambda_q^2 = 16C_1\delta_0^{-1/4}(\delta_0\lambda_q^4)^{-\frac{\alpha-1}{2\alpha}} \leq 16C_1\delta_0^{-1/4} \leq 1/4,$$

by letting a, and hence δ_0, be sufficiently large. Here we have used $\alpha \geq 1$ and $\delta_0, \lambda_q \geq 1$. Thus, we have shown that (3.17) holds.

In order to prove (3.8b) we denote

$$z = \Delta^{-1} \operatorname{curl} u.$$

Since $\operatorname{div} u = 0$ we have $\operatorname{curl} z = -u$, and the Calderón–Zygmund inequality yields $\|\mathcal{R}u(t)\|_{L^p} \lesssim \|z(t)\|_{L^p}$. Thus our goal is to obtain L^p estimates for $z(t)$. We apply Δ^{-1} curl to the equation obeyed by u (it is convenient to rewrite (3.12) without Leray projectors, and add a pressure gradient term, which is then annihilated by the curl operator) and obtain

$$\begin{aligned}
&\partial_t z + v \cdot \nabla z + (-\Delta)^\alpha z \\
&\qquad = \Delta^{-1} \operatorname{curl} \operatorname{div} \mathring{R}_q + [\Delta^{-1} \operatorname{curl}, v \cdot \nabla] \operatorname{curl} z + \Delta^{-1} \operatorname{curl}(\operatorname{curl} z \cdot \nabla v_q) \\
&\qquad = \Delta^{-1} \operatorname{curl} \operatorname{div} \mathring{R}_q + \Delta^{-1} \operatorname{curl} \operatorname{div}((z \times \nabla)v) + \Delta^{-1}\nabla \operatorname{div}((z \cdot \nabla)v) \\
&\qquad\quad + \Delta^{-1} \operatorname{curl} \operatorname{div}(((z \times \nabla)v_q)^T).
\end{aligned} \tag{3.20}$$

For the last term on the right side of (3.20) we have used the identity

$$(\operatorname{curl} z \cdot \nabla)v_q = \operatorname{div}(((z \times \nabla)v_q)^T),$$

which written for the i^{th} component is

$$\begin{aligned}
((\operatorname{curl} z \cdot \nabla)v_q)^i &= \epsilon_{jkl}\partial_k z^l \partial_j v_q^i = \partial_k(\epsilon_{jkl} z^l \partial_j v_q^i) - \epsilon_{jkl} z^l \partial_j \partial_k v_q^i = \partial_k(\epsilon_{klj} z^l \partial_j v_q^i) \\
&=: \partial_k((z \times \nabla)v_q)^{ki}.
\end{aligned}$$

Here we have used the fact that the transposition of two indices in ϵ_{jkl} results in a (-1) factor. Moreover, we have also spelled out the commutator term on the right side of (3.20) as

$$[\Delta^{-1} \operatorname{curl}, v \cdot \nabla] \operatorname{curl} z = \Delta^{-1} \operatorname{curl} \operatorname{div}((z \times \nabla)v) + \Delta^{-1}\nabla \operatorname{div}((z \cdot \nabla)v),$$

which written for the i^{th} component is

$$
\begin{aligned}
([\Delta^{-1}\operatorname{curl}, v\cdot\nabla]\operatorname{curl} z)^i &= \epsilon_{ijk}\Delta^{-1}\partial_j(v^m\partial_m(\operatorname{curl} z)^k) + v^m\partial_m z^i \\
&= \epsilon_{ijk}\epsilon_{kln}\Delta^{-1}\partial_j(v^m\partial_m\partial_l z^n) + v^m\partial_m z^i \\
&= -\epsilon_{ijk}\epsilon_{kln}\Delta^{-1}\partial_j\partial_m(\partial_l v^m z^n) + \epsilon_{ijk}\epsilon_{kln}\Delta^{-1}\partial_j\partial_l(v^m\partial_m z^n) + v^m\partial_m z^i \\
&= \Delta^{-1}\epsilon_{ijk}\partial_j(\epsilon_{knl}\partial_m(\partial_l v^m z^n)) - \epsilon_{ijk}\epsilon_{nlk}\Delta^{-1}\partial_j\partial_l(v^m\partial_m z^n) + v^m\partial_m z^i \\
&= \Delta^{-1}\epsilon_{ijk}\partial_j(\partial_m(\epsilon_{knl}z^n\partial_l v^m)) + \Delta^{-1}\partial_i\partial_n(v^m\partial_m z^n) \\
&= \Delta^{-1}\epsilon_{ijk}\partial_j(\partial_m(\epsilon_{knl}z^n\partial_l v^m)) + \Delta^{-1}\partial_i\partial_m(z^n\partial_n v^m).
\end{aligned}
$$

Here we have also used the fact that $\epsilon_{ijk} = 0$ if two of the indices i, j, or k repeat, and that $\epsilon_{ijk}\epsilon_{nlk} = \delta_{in}\delta_{jl} - \delta_{il}\delta_{jn}$, where the δ's refer to the Kronecker symbol.

Using (3.20), upon placing the $v\cdot\nabla z = \operatorname{div}(v\otimes z)$ term on the right side, and using $z(t_0) = 0$, the solution to (3.20) may be written in integral form as

$$
\begin{aligned}
z(t) = &\int_0^t e^{-(t-s)(-\Delta)^\alpha}\big(\Delta^{-1}\operatorname{curl}\operatorname{div}\mathring{R}_q + \Delta^{-1}\operatorname{curl}\operatorname{div}(((z\times\nabla)v_q)^T\big) - \operatorname{div}(v\otimes z))(s)\,ds \\
&+ \int_0^t e^{-(t-s)(-\Delta)^\alpha}\big(\Delta^{-1}\operatorname{curl}\operatorname{div}((z\times\nabla)v) + \Delta^{-1}\nabla\operatorname{div}((z\cdot\nabla)v)\big)(s)\,ds.
\end{aligned}
\tag{3.21}
$$

From (3.14) and the boundedness of Calderón–Zygmund operators on L^p, similarly to (3.16) we conclude that

$$
\begin{aligned}
\|z(t)\|_{L^p} &\lesssim \int_0^t \Bigg(\|\mathring{R}_q(s)\|_{L^p} + \|((z\times\nabla)v_q)(s)\|_{L^p} + \frac{1}{(t-s)^{\frac{1}{2\alpha}}}\|(v\otimes z)(s)\|_{L^p} \\
&\qquad + \|((z\times\nabla)v)(s)\|_{L^p} + \|((z\cdot\nabla)v)(s)\|_{L^p}\Bigg)\,ds \\
&\le C_1 t\|\mathring{R}_q\|_{L^\infty([0,t_1];L^p)} + C_1(\|\nabla v_q\|_{L^\infty} + \|\nabla v\|_{L^\infty})\int_0^t \|z(s)\|_{L^p}\,ds \\
&\qquad + C_1\|v\|_{L^\infty}\int_0^t \frac{\|z(s)\|_{L^p}}{(t-s)^{\frac{1}{2\alpha}}}\,ds
\end{aligned}
\tag{3.22}
$$

where C_1 depends only on p_0 and α, since $p\in[p_0,\alpha]$. Next we claim that if t_1 is chosen sufficiently small, then

$$
\|z(t)\|_{L^p} \le 2C_1 t\|\mathring{R}_q\|_{L^\infty(0,t_1;L^p)} \qquad \text{for all } t\in(0,t_1]. \tag{3.23}
$$

The argument is similar to the one for $u(t)$, so we only sketch the details. Assume that (3.23) holds. Then from (3.22) we obtain

$$
\frac{\|z(t)\|_{L^p}}{2C_1 t\|\mathring{R}_q\|_{L^\infty(0,t_1;L^p)}} \le \frac{1}{2} + t(\|\nabla v_q\|_{L^\infty} + \|\nabla v\|_{L^\infty}) + 2t^{1-\frac{1}{2\alpha}}\|v\|_{L^\infty}. \tag{3.24}
$$

Therefore, if we ensure that t_1 is small enough that

$$t_1(\|\nabla v_q\|_{L^\infty} + \|\nabla v\|_{L^\infty}) + t_1^{1-\frac{1}{2\alpha}}\|v\|_{L^\infty} \leq \frac{1}{5}, \tag{3.25}$$

then (3.24) implies

$$\frac{\|z(t)\|_{L^p}}{2C_1 t\|\mathring{R}_q\|_{L^\infty(0,t_1;L^p)}} \leq \frac{1}{2} + \frac{2}{5} < 1$$

which shows that the bootstrap assumption was justified, and thus (3.23) holds on $[0,T]$. Denote by C_1 the universal constant in the Gagliardo–Nirenberg inequalities (3.15). By also appealing to (2.5), (3.2), and our assumption (3.7) for t_1, we find that the left side of (3.25) is bounded from above by

$$\begin{aligned}
&C_1 t_1(\|v_q\|_{L^2}^{1/6}\|v_q\|_{H^3}^{5/6} + \|v\|_{L^2}^{1/6}\|v\|_{H^3}^{5/6}) + C_1 t_1^{\frac{1}{2}+\frac{\alpha-1}{2\alpha}}\|v\|_{L^2}^{1/2}\|v\|_{H^3}^{1/2} \\
&\qquad \leq 4C_1 t_1 \delta_0^{1/12}\lambda_q^{10/3} + 2C_1 t_1^{\frac{1}{2}+\frac{(\alpha-1)}{2\alpha}}\delta_0^{1/4}\lambda_q^2 \\
&\qquad \leq 4C_1\delta_0^{-11/12}\lambda_q^{-2/3} + 2C_1\delta_0^{-1/4}(\delta_0\lambda_q^4)^{-\frac{\alpha-1}{2\alpha}} \leq 6C_1\delta_0^{-1/4} \leq 1/5
\end{aligned}$$

once we ensure that a, and hence δ_0, is sufficiently large. This concludes the proof of (3.23). ∎

3.3. *Proof of Proposition* 2.2

We first define a C^∞ smooth partition of unity $\{\eta_i\}_{i=0}^{n_{q+1}}$ such that $0 \leq \eta_i \leq 1$ and

$$\sum_{i=0}^{n_{q+1}} \eta_i(t) = 1 \qquad \text{for every } t \in [T/3, 2T/3]. \tag{3.26}$$

Denoting

$$t_i = \vartheta_{q+1} i,$$

this may be achieved by letting η_i also have the following properties:

(i) η_i has support in $[t_i, t_{i+1} + \tau_{q+1}]$,

(ii) η_i is identically 1 on $[t_i + \tau_{q+1}, t_{i+1}]$,

(iii) η_i satisfies the estimate

$$\|\partial_t^M \eta_i\|_{L^\infty} \lesssim \tau_{q+1}^{-M}, \tag{3.27}$$

where the implicit constant is independent of τ_{q+1}, ϑ_{q+1}, and i.

As a consequence of the above properties, we see that $\eta_i\eta_j = 0$ whenever $|i - j| > 1$, and

$$\operatorname{supp}(\eta_i\eta_{i-1}) \subset [t_i, t_i + \tau_{q+1}].$$

Having constructed the partition of unity $\{\eta_i\}_{i=0}^{n_{q+1}}$, we next construct exact solutions v_i of the Navier–Stokes equation for suitably defined data.

For every $1 \leq i \leq n_{q+1}$ we define $v_i(x,t)$ to be the unique smooth solution of the Cauchy problem for the Navier–Stokes equation (1.3) with initial condition equal to v_q at t_{i-1}:

$$\partial_t v_i + \operatorname{div}(v_i \otimes v_i) + \nabla p_i + (-\Delta)^\alpha v_i = 0, \tag{3.28a}$$
$$\operatorname{div} v_i = 0, \tag{3.28b}$$
$$v_i(t_{i-1}) = v_q(t_{i-1}). \tag{3.28c}$$

In view of (2.5), and Proposition 3.1, this solution v_i is uniquely defined and obeys the estimates

$$\|v_i(t)\|_{L^2} \leq \|v_q(t_{i-1})\|_{L^2} \leq 2\delta_0^{1/2} - \delta_q^{1/2}, \tag{3.29a}$$
$$\|v_i(t)\|_{H^3} \leq 2\|v_q(t_{i-1})\|_{H^3} \leq 2\lambda_q^4, \tag{3.29b}$$
$$|t - t_{i-1}|^{\frac{N}{2\alpha}+M} \|\partial_t^M D^N v_i(t)\|_{H^3}, \lesssim \lambda_q^4 \tag{3.29c}$$

for all $N \geq 0$, $M \in \{0,1\}$ and all

$$t > t_{i-1} \quad \text{such that} \quad t - t_{i-1} \leq \frac{c}{4\lambda_q^4 \delta_0^{1/2}} \leq \frac{c}{\lambda_q^4 (1 + 2\delta_0^{1/2})^{\frac{1}{2\alpha-1}}} \tag{3.30}$$

where $c \in (0,1)$ is the universal constant from (3.3), and $\alpha \geq 1$. Note that the definitions (2.3), (2.7), and the fact that $\beta \leq 1$, imply that

$$\vartheta_{q+1} = \frac{\delta_{q+1}^{1/2}}{\lambda_q^7} = \frac{1}{\lambda_q^4 \delta_0} \frac{\delta_0 \delta_{q+1}^{1/2}}{\lambda_q^3} \leq \frac{1}{\lambda_q^4 \delta_0} \frac{\lambda_1^{3\beta/2}}{\lambda_q^3} \leq \frac{1}{\lambda_q^4 \delta_0}. \tag{3.31}$$

Therefore, assuming that $\delta_0 = \lambda_1^{3\beta} \lambda_0^{-2\beta} \geq \lambda_0^\beta$ is sufficiently large, depending on the universal constant c, by (3.31) we find that

$$3\vartheta_{q+1} \leq \frac{c}{8\lambda_q^4 \delta_0^{1/2}},$$

which is consistent with (3.30). Therefore for all $1 \leq i \leq n_{q+1}$ the exact solutions $v_i(x,t)$ are smooth and well-defined for all $t \in (t_{i-1}, t_{i+2}] \supset \operatorname{supp}(\eta_i)$. Moreover, since

$$t \in \operatorname{supp}(\eta_i) \implies \vartheta_{q+1} \leq t - t_{i-1} \leq 3\vartheta_{q+1},$$

from (3.29c) we obtain the bound

$$\sup_{t \in \operatorname{supp}(\eta_i)} \|\partial_t^M D^N v_i(t)\|_{H^3} \lesssim \lambda_q^4 \vartheta_{q+1}^{-\frac{N}{2\alpha}-M} \qquad \text{for } 1 \leq i \leq n_{q+1}, \tag{3.32}$$

where the implicit constant depends only on $N \geq 0$ and $M \in \{0,1\}$.

At this stage we glue the solutions v_i together in order to construct $(\bar{v}_q, \mathring{\bar{R}}_q)$. We define the divergence-free (note that the cutoffs η_i are only functions of time) velocity and the interpolated pressure as

$$\begin{aligned} \bar{v}_q(x,t) &= \sum_{i=1}^{n_{q+1}} \eta_i(t) v_i(x,t) \quad \text{for all } t \in [T/3/,2T/3], \\ \bar{p}_q^{(1)}(x,t) &= \sum_{i=1}^{n_{q+1}} \eta_i(t) p_i(x,t) \quad \text{for all } t \in [T/3/,2T/3], \end{aligned} \tag{3.33}$$

where p_i is the pressure associated to the exact solution v_i. Also we let

$$\begin{aligned} \bar{v}_q(x,t) = v_q(x,t) &= v_0(x,t) \quad \text{for all } t \in [0,T/3] \cup [2T/3,T], \\ \bar{p}_q^{(1)}(x,t) = p_q(x,t) &= p_0(x,t) \quad \text{for all } t \in [0,T/3] \cup [2T/3,T]. \end{aligned} \tag{3.34}$$

Here we have used $[0,T/3] \cup [2T/3,T] = \mathscr{G}^{(0)}$, and the inductive assumption (2.10).

Having defined $\bar{v}_q$, we next prove that (2.18) holds. For $t \in \mathscr{G}^{(0)}$, this holds by construction. In view of (3.26), it suffices to show that if for some $i \in \{1,\dots,n_{q+1}\}$ we have $t \in \operatorname{supp}(\eta_i) \cap \mathscr{G}^{(q)}$, then $v_i(t) = v_q(t)$. For this purpose recall by (2.5b) and (2.11) that v_q is a strong solution of the Navier–Stokes equation for all t such that $\operatorname{dist}(t, \mathscr{G}^{(q)}) \le \tau_q$. Moreover, v_i solves the Cauchy problem (3.28), so by the uniqueness of solutions in $C_t^0 H_x^3$ of the Navier–Stokes equation, we only need to ensure that $\operatorname{dist}(t_{i-1}, \mathscr{G}^{(q)}) \le \tau_q$. This follows from the fact that $t \in \mathscr{G}^{(q)}$ and $0 < t - t_{i-1} \le 3\vartheta_{q+1} \le \tau_q$. The last inequality trivially holds by (2.7) and (2.8) for $q \ge 1$, and by taking a sufficiently large for $q = 0$. Thus, we have proven that (2.18) holds.

At this stage we show that the set $\mathscr{B}^{(q+1)}$ defined in (2.17), and hence implicitly $\mathscr{G}^{(q+1)} = [0,T] \setminus \mathscr{B}^{(q+1)}$, obey properties (ii)–(iv) with q replaced by $q+1$. In order to prove (ii), assume that $t \in \mathscr{G}^{(q)} \cap (t_i - 2\tau_{q+1}, t_i + 3\tau_{q+1})$ for some $i \in \{1,\dots,n_{q+1}\}$. Due to (2.11) we know that $\mathring{R}_q(t') = 0$ for all $|t - t'| \le \tau_q$. Since $\tau_q \ge 2\vartheta_{q+1} + 3\tau_{q+1}$, which holds by (2.7) and (2.8) for $q \ge 1$, and by taking a sufficiently large for $q = 0$, we find that $\mathring{R}_q \equiv 0$ on $[t_{i-2}, t_{i+1} + \tau_{q+1}]$. Hence, by the definition (2.16) we have $i, i-1 \notin \mathscr{C}$. Thus, $t \notin \mathscr{B}^{(q+1)}$ and so $t \in \mathscr{G}^{(q+1)}$ as desired. Property (iii) holds by definition (2.17), since τ_{q+1} is much smaller than ϑ_{q+1}. In order to prove property (iv), we need to estimate the cardinality of the set $\mathscr{C}$ defined in (2.16). By definition, if $i \in \mathscr{C}$, there exists $t \in [t_{i-1}, t_{i+1} + \tau_{q+1}]$ such that $\mathring{R}_q(t) \ne 0$, and thus by property (2.11) we have $\operatorname{dist}(t, \mathscr{G}^{(q)}) > \tau_q$. Therefore, $\mathscr{B}^{(q)} \supset (t - \tau_q, t + \tau_q) \supset [t_i, t_{i+1}]$. By the pigeonhole principle we obtain

$$\operatorname{card}(\mathscr{C}) \le \frac{|\mathscr{B}^{(q)}|}{\vartheta_{q+1}}.$$

Estimate (2.9) at level $q+1$ then follows from (2.17).

At this stage we remark that property (v) will also hold at the end of the convex integration stage. For this purpose, we remark that in the convex integration stage we do not add a perturbation to the solutions on the good set $\mathcal{G}^{(q+1)} \supset \mathcal{G}^{(q)}$, i.e. $v_{q+1}(t) = \bar{v}_q(t)$ for $t \in \mathcal{G}^{(q+1)} \supset \mathcal{G}^{(q)}$. Assuming for the moment this feature of our construction, property (2.18) established above and the inductive (2.10) shows that (2.10) holds at level $q+1$.

We now derive the formula for $\operatorname{supp}(\mathring{\bar{R}}_q)$. Note that on $[0, T/3] \supset [t_0, t_2]$ and on $[2T/3, T] \supset [t_{n_{q+1}-1}, t_{n_{q+1}}]$ the function $\bar{v}_q = v_q$ is a smooth solution of the Navier–Stokes equation, and hence automatically

$$\mathring{\bar{R}}_q = 0 \quad \text{on } [t_0, t_2] \cup [t_{n_{q+1}-1}, t_{n_{q+1}}].$$

For $i \geq 2$, on the interval $[t_i, t_{i+1}]$ we have

$$\bar{v}_q = (1-\eta_i)v_{i-1} + \eta_i v_i, \qquad \bar{p}_q^{(1)} = (1-\eta_i)p_{i-1} + \eta_i p_i,$$

and similarly to [4, Section 4.2], we obtain

$$\begin{aligned}
&\partial_t \bar{v}_q + \operatorname{div}(\bar{v}_q \otimes \bar{v}_q) + (-\Delta)^\alpha \bar{v}_q + \nabla \bar{p}_q^{(1)} \\
&\qquad = (1-\eta_i)\partial_t v_{i-1} + \eta_i \partial_t v_i + \partial_t \eta_i (v_i - v_{i-1}) \\
&\qquad\quad + (1-\eta_i)^2 \operatorname{div}(v_{i-1} \otimes v_{i-1}) + \eta_i^2 \operatorname{div}(v_i \otimes v_i) \\
&\qquad\quad + \eta_i(1-\eta_i) \operatorname{div}(v_{i-1} \otimes v_i + v_i \otimes v_{i-1})) \\
&\qquad\quad + (1-\eta_i)(-\Delta)^\alpha v_{i-1} + \eta_i (-\Delta)^\alpha v_i + (1-\eta_i)\nabla p_{i-1} + \eta_i \nabla p_i \\
&\qquad = \partial_t \eta_i (v_i - v_{i-1}) - \eta_i(1-\eta_i) \operatorname{div}((v_i - v_{i-1}) \otimes (v_i - v_{i-1})).
\end{aligned} \tag{3.35}$$

We observe that $v_i - v_{i-1}$ has zero mean because the exact solutions of the Navier–Stokes equations v_i, v_{i-1} preserve their average in time, and v_q has zero mean by assumption. Hence we can apply the inverse divergence operator $\mathcal{R}$ to $v_i - v_{i-1}$ and for $i \in \{2, \ldots, n_{q+1} - 1\}$ define the symmetric traceless 2-tensor

$$\mathring{\bar{R}}_q = \partial_t \eta_i \mathcal{R}(v_i - v_{i-1}) - \eta_i(1-\eta_i)(v_i - v_{i-1}) \mathring{\otimes} (v_i - v_{i-1}) \quad \text{for all } t \in [t_i, t_{i+1}], \tag{3.36}$$

where we denote by $a \mathring{\otimes} b$ the traceless part of the tensor $a \otimes b$. We also define the scalar pressure

$$\bar{p}_q = \bar{p}_q^{(1)} - \eta_i(1-\eta_i)\left(|v_i - v_{i-1}|^2 - \fint_{\mathbb{T}^3} |v_i - v_{i-1}|^2 \, dx\right) \quad \text{for all } t \in [t_i, t_{i+1}].$$

It follows from (3.35) that the pair $(\bar{v}_q, \mathring{\bar{R}}_q)$ defined by (3.33) and (3.36) solves the Navier–Stokes–Reynolds system (2.1) on $[0, T]$ with associated pressure $\bar{p}_q$.

Next, we prove that (2.20a) holds. Note that by construction, $\eta_i \equiv 1$ on $[t_i + \tau_{q+1}, t_{i+1}]$ for all $i \in \{0, \ldots, n_{q+1}\}$, and thus on these sets we have $\partial_t \eta_i = \eta_i(1-\eta_i) = 0$. Therefore, by (3.36) we have $\mathring{\bar{R}}_q(t) = 0$ whenever $t \in [t_i + \tau_{q+1}, t_{i+1}]$ for some i. Thus it suffices to consider sets of times of the form $(t_i, t_i + \tau_{q+1})$. If $i \in \mathcal{C}$ or $i - 1 \in \mathcal{C}$, then there is nothing to prove since by definition (2.17), $\operatorname{dist}((t_i, t_i + \tau_{q+1}), \mathcal{G}^{(q+1)}) > 2\tau_{q+1}$. Hence, consider the case $i, i-1 \notin \mathcal{C}$. Thus by the definition of $\mathcal{C}$, $\mathring{R}_q(t) = 0$ for all $t \in [t_{i-2}, t_{t+1} + \tau_{q+1}]$. Since $v_{i-1}(t_{i-2}) = v_q(t_{i-2})$ and $v_i(t_{i-1}) = v_q(t_{i-1})$, and since $\mathring{R}_q$ vanishes on $[t_{i-2}, t_{t+1} + \tau_{q+1}]$, it follows by the bounds (3.29b) and (2.5b) and the uniqueness of strong solutions to the Navier–Stokes equations that $v_{i-1} = v_i = v_q$ on $(t_i, t_i + \tau_{q+1})$. Thus by (3.36) we have $\mathring{R}_{q+1}(t) = 0$ for $(t_i, t_i + \tau_{q+1})$.

Since in the convex integration stage we do not change the stress on the set $\{t\colon \operatorname{dist}(t, \mathcal{G}^{(q+1)}) \le \tau_{q+1}\}$, it follows from (2.20a) that $\mathring{R}_{q+1}(t) = \mathring{\bar{R}}_q(t) = 0$ for all t such that $\operatorname{dist}(t, \mathcal{G}^{(q+1)}) \le \tau_{q+1}$. Thus (2.11), and hence property (vi), will automatically hold at the end of the convex integration step.

In order to conclude the proof of Proposition 2.2, it remains to prove estimates (2.19) for $\bar{v}_q$ and (2.20b)–(2.20c) for $\mathring{\bar{R}}_q$.

By (3.26), (3.29a), (3.29b), and the definition of $\bar{v}_q$ in (3.33), it follows that (2.19a) and (2.19b) hold for all $t \in [T/3, 2T/3]$. By (3.26), for all $t \in [T/3, 2T/3]$ we have

$$\overline{v}_q(x,t) - v_q(x,t) = \sum_{i=0}^{n_{q+1}} \eta_i(t)(v_i(x,t) - v_q(x,t)), \tag{3.37}$$

and at each time t at most two terms in the sum are non-zero. Since v_i solves (3.28), and since $t \in \operatorname{supp}(\eta_i)$ implies that (3.30) holds, we may appeal to Corollary 3.3, with t_0 replaced by t_{i-1}, and t_1 replaced by an arbitrary $t \in \operatorname{supp}(\eta_i)$. Here we note that condition (3.7) is satisfied on $\operatorname{supp}(\eta_i)$ due to (3.30). By (3.10a), we obtain

$$\sup_{t \in \operatorname{supp}(\eta_i)} \|v_i(t) - v_q(t)\|_{L^2} \le 4\vartheta_{q+1}\lambda_q^5.$$

Since at most two terms appear in (3.37), we may use the remaining power of λ_q^{-1} to absorb any constants, and (2.19c) follows on $[T/3, 2T/3]$. Moreover, estimates (2.19a)–(2.19c) hold trivially on $[0, T/3] \cup [2T/3, T]$ by the inductive assumptions and definition (3.34). Thus, we have proven (2.19a)–(2.19c) on $[0, T]$.

Lastly, (2.19d) follows from the definition (3.33), the Leibniz rule, estimate (3.27) for the time derivatives landing on the cutoff functions η_i, and estimate (3.32) for the space and time derivatives landing on the v_i. Here we have used $\tau_{q+1}^{-1} > \vartheta_{q+1}^{-1}$. Thus we have established all the desired bounds for $\bar{v}_q$.

In order to prove the claimed L^1 estimate for $\overline{\mathcal{B}}^{(q)}$, i.e. (2.20b), we appeal to the definition (3.36). For the first term, we use (3.27) and again appeal to Corollary 3.3, this time to estimate (3.10b), to obtain

$$\begin{aligned}
&\|\partial_t \eta_i \mathcal{R}(v_i - v_{i-1})\|_{L^1} \\
&\qquad \le \|\partial_t \eta_i\|_{L^\infty} \big(\|\mathcal{R}(v_i - v_q)\|_{L^\infty(\operatorname{supp}(\eta_i);L^1)} + \|\mathcal{R}(v_{i-1} - v_q)\|_{L^\infty(\operatorname{supp}(\eta_i);L^1)} \big) \\
&\qquad \lesssim \tau_{q+1}^{-1} \vartheta_{q+1} \lambda_q^{-3\varepsilon_R/4} \delta_{q+1} \le \tfrac{1}{2} \tau_{q+1}^{-1} \vartheta_{q+1} \lambda_q^{-\varepsilon_R/2} \delta_{q+1},
\end{aligned} \tag{3.38}$$

upon using the remaining power of $\lambda_q^{-\varepsilon_R/4}$ to absorb any constants. For the second term in (3.36), we use (3.10a) and obtain

$$\begin{aligned}\|\eta_i(1-\eta_i)(v_i - v_{i-1}) \mathring{\otimes} (v_i - v_{i-1})\|_{L^1} &\le \|v_i - v_{i-1}\|^2_{L^\infty(\mathrm{supp}(\eta_{i-1}\eta_i);L^2)} \\ &\le 4\|v_i - v_q\|^2_{L^\infty(\mathrm{supp}(\eta_i);L^2)} \lesssim (\vartheta_{q+1}\lambda_q^5)^2 \le \tfrac{1}{2}\tau_{q+1}^{-1}\vartheta_{q+1}\lambda_q^{-\varepsilon_R/2}\delta_{q+1}. \qquad (3.39)\end{aligned}$$

Here we have used $\tau_{q+1} \le \vartheta_{q+1}$, the definition (2.7), and $\varepsilon_R \le 1$, to conclude

$$\vartheta_{q+1}\tau_{q+1} \le \lambda_q^{-14}\delta_{q+1} \le \lambda_q^{-1}\lambda_q^{-10-\varepsilon_R/2}\delta_{q+1},$$

and using the leftover term λ_q^{-1} to absorb any implicit constants in (3.39). Combined, (3.38) and (3.39) prove (2.20b).

It remains to prove (2.20c). We return to (3.36). For the first term we use (3.32) and (3.27) to obtain

$$\begin{aligned}&\|\partial_t^M D^N(\partial_t\eta_i \mathcal{R}(v_i - v_{i-1}))\|_{H^3} \\ &\lesssim \sum_{M'=0}^{M} \|\partial_t^{M-M'+1}\eta_i\|_{L^\infty}(\|\partial_t^{M'}D^N v_i\|_{L^\infty(\mathrm{supp}(\eta_i);H^3)} + \|\partial_t^{M'}D^N v_{i-1}\|_{L^\infty(\mathrm{supp}(\eta_{i-1});H^3)}) \\ &\lesssim \sum_{M'=0}^{M} \tau_{q+1}^{-M+M'-1}\lambda_q^4\vartheta_{q+1}^{-\frac{N}{2\alpha}-M'} \lesssim \tau_{q+1}^{-M-1}\lambda_q^4\vartheta_{q+1}^{-\frac{N}{2\alpha}},\end{aligned}$$

since $\tau_{q+1} \le \vartheta_{q+1}$. This bound is consistent with (2.20c). For the second term in (3.36), since H^3 is an algebra we similarly deduce from (3.32) and (3.27) that

$$\begin{aligned}&\|\partial_t^M D^N(\chi_i(1-\chi_i)(v_i - v_{i-1}) \otimes (v_i - v_{i-1}))\|_{H^3} \\ &\qquad\lesssim \sum_{M'=0}^{M} \tau_{q+1}^{-M+M'}\|\partial_t^{M'}D^N((v_i - v_{i-1}) \otimes (v_i - v_{i-1}))\|_{L^\infty(\mathrm{supp}(\chi_{i-1}\chi_i);H^3)} \\ &\qquad\lesssim \sum_{M'=0}^{M} \tau_{q+1}^{-M+M'}\vartheta_{q+1}^{-M'}\lambda_q^8\vartheta_{q+1}^{-\frac{N}{2\alpha}} \lesssim \tau_{q+1}^{-M-1}\lambda_q^4\vartheta_{q+1}^{-\frac{N}{2\alpha}},\end{aligned}$$

where we have additionally used $\tau_{q+1} \le \vartheta_{q+1} \le \lambda_q^{-4}$, in view of (3.31).

To conclude the proof of Proposition 2.2, we note that the second inequality in (2.19d) and (2.20c), which bounds the cost of a spatial derivative by τ_{q+1}^{-1}, instead of $\vartheta_{q+1}^{-1/(2\alpha)}$, follows from the fact that $\alpha \in [1, 5/4)$ and $1 \le \vartheta_{q+1}^{-1} \le \tau_{q+1}^{-1}$.

4. Convex integration step: the perturbation

4.1. Intermittent jets

Let us recall the following result from [9]:

Lemma 4.1. *For $\alpha = 1, 2$, there exist subsets $\Lambda_\alpha \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ and smooth functions $\gamma_\xi : \mathcal{N} \to \mathbb{R}$ such that*

$$R = \sum_{\xi \in \Lambda_\alpha} \gamma_\xi^2(R)(\xi \otimes \xi)$$

for every symmetric matrix R satisfying $|R - \mathrm{Id}| \leq 1/2$.

For each $\xi \in \Lambda_\alpha$, let $A_\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ be a vector orthogonal to ξ. Then for each $\xi \in \Lambda_\alpha$, $\{\xi, A_\xi, \xi \times A_\xi\} \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ is an orthonormal basis for $\mathbb{R}^3$. Furthermore, since the index sets $\{\Lambda_\alpha\}_{\alpha=1,2}$ are finite, there exists a universal natural number N_Λ such that

$$\{N_\Lambda \xi, N_\Lambda A_\xi, N_\Lambda \xi \times A_\xi\} \subset N_\Lambda \mathbb{S}^2 \cap \mathbb{N}^3 \tag{4.1}$$

for every $\xi \in \Lambda_\alpha$.

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function with support in a ball of radius 1. Moreover, suppose Φ is normalized so that if $\phi = -\Delta\Phi$ then

$$\frac{1}{4\pi^2} \int \phi^2(x, y)\, dx\, dy = 1. \tag{4.2}$$

We remark that by definition ϕ has mean zero. Define $\psi : \mathbb{R} \to \mathbb{R}$ to be a smooth, mean zero function with support in the ball of radius 1 satisfying

$$\frac{1}{2\pi} \int \psi^2(z)\, dz = 1. \tag{4.3}$$

Let $\phi_{\ell_\perp}$, $\Phi_{\ell_\perp}$ and $\psi_{\ell_\|}$ be the rescalings

$$\phi_{\ell_\perp}(x, y) := \frac{\phi(x/\ell_\perp, y/\ell_\perp)}{\ell_\perp}, \quad \Phi_{\ell_\perp}(x, y) := \frac{\Phi(x/\ell_\perp, y/\ell_\perp)}{\ell_\perp}, \quad \psi_{\ell_\|}(z) := \frac{\psi(z/\ell_\|)}{\ell_\|^{1/2}}$$

so that $\phi_{\ell_\perp} = -\ell_\perp^2 \Delta\Phi_{\ell_\perp}$, where we will assume $\ell_\perp, \ell_\| > 0$ to be such that

$$\ell_\perp \ll \ell_\| \ll 1.$$

By an abuse of notation, let us periodize $\Phi_{\ell_\perp}$ and $\psi_{\ell_\|}$ so that the functions are treated as functions defined on $\mathbb{T}^2$ and $\mathbb{T}$ respectively. For a large *real number* λ such that $\lambda\ell_\perp \in \mathbb{N}$, we define $V_{\xi,\ell_\perp,\ell_\|,\lambda,\mu} : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} V_{(\xi)} &:= V_{\xi,\ell_\perp,\ell_\|,\lambda,\mu}(x, t) \\ &:= \frac{1}{\lambda^2 N_\Lambda^2} \psi_{\ell_\|}(N_\Lambda \ell_\perp \lambda(x \cdot \xi + \mu t)) \Phi_{\ell_\perp}\big(N_\Lambda \ell_\perp \lambda(x - \alpha_\xi) \cdot A_\xi, N_\Lambda \ell_\perp \lambda(x - \alpha_\xi) \cdot (\xi \times A_\xi)\big)\xi, \end{aligned}$$

where $\alpha_\xi \in \mathbb{R}^3$ are shifts that ensure that the functions $\{V_{\xi,\ell_\perp,\ell_\|,\lambda,\mu}\}_\xi$ have mutually disjoint supports. In order for such shifts α_ξ to exist, we require $\ell_\perp$ to be sufficiently small, depending on the finite sets Λ_α.

Our *intermittent jet* is then defined to be

$$\begin{aligned} W_{(\xi)} &:= W_{\xi,\ell_\perp,\ell_\|,\lambda,\mu}(x,t) \\ &:= \psi_{\ell_\|}(N_\Lambda \ell_\perp \lambda (x\cdot\xi + \mu t))\phi_{\ell_\perp}\big(N_\Lambda \ell_\perp\lambda(x-\alpha_\xi)\cdot A_\xi, N_\Lambda\ell_\perp\lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi)\big)\xi. \end{aligned} \tag{4.4}$$

From the definition, using (4.1) and $\ell_\perp\lambda\in\mathbb{N}$, we find that $W_{(\xi)}$ has zero mean, and $W_{(\xi)}$ is $(\mathbb{T}/\ell_\perp\lambda)^3$-periodic. Moreover, by our choice of α_ξ, the $W_{(\xi)}$ have mutually disjoint supports, i.e.

$$W_{(\xi)}\otimes W_{(\xi')}\equiv 0 \quad \text{whenever} \quad \xi\neq\xi'\in\bigcup_{\alpha\in\{1,2\}}\Lambda_\alpha. \tag{4.5}$$

Note that the intermittent jets $W_{(\xi)}$ are not divergence-free, but assuming $\ell_\perp\ll\ell_\|$ they can be corrected by a small term such that the sum with the corrector is divergence-free. To see this, let us adopt the shorthand notation

$$\begin{aligned} \psi_{(\xi)} &:= \psi_{\xi,\ell_\perp,\ell_\|,\lambda,\mu} := \psi_{\ell_\|}(N_\Lambda\ell_\perp\lambda(x\cdot\xi+\mu t)), \\ \Phi_{(\xi)} &:= \Phi_{\xi,\ell_\perp,\lambda,\mu} := \Phi_{\ell_\perp}\big(N_\Lambda\ell_\perp\lambda(x-\alpha_\xi)\cdot A_\xi, N_\Lambda\ell_\perp\lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi)\big) \\ \phi_{(\xi)} &:= \phi_{\xi,\ell_\perp,\lambda,\mu} := \phi_{\ell_\perp}\big(N_\Lambda\ell_\perp\lambda(x-\alpha_\xi)\cdot A_\xi, N_\Lambda\ell_\perp\lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi)\big), \end{aligned}$$

and compute

$$\begin{aligned} \operatorname{curl}\operatorname{curl} V_{(\xi)} &= W_{(\xi)} \\ &\quad + \underbrace{\frac{1}{\lambda^2N_\Lambda^2}\operatorname{curl}(\Phi_{(\xi)}\operatorname{curl}(\psi_{(\xi)}\xi))}_{\equiv 0} + \underbrace{\frac{1}{\lambda^2N_\Lambda^2}\nabla\psi_{(\xi)}\times\operatorname{curl}(\Phi_{(\xi)}\xi)}_{W^{(c)}_{(\xi)}}. \end{aligned} \tag{4.6}$$

Thus

$$\operatorname{div}(W_{(\xi)}+W^{(c)}_{(\xi)})\equiv 0.$$

Moreover, as long as $\ell_\perp\ll\ell_\|$, $W^{(c)}_{(\xi)}$ is small compared to $W_{(\xi)}$. Observe that as a consequence of the normalizations (4.2) and (4.3) we have

$$\fint_{\mathbb{T}^3} W_{(\xi)}(x)\otimes W_{(\xi)}(x)\,dx = \xi\otimes\xi.$$

We also note that by definition $W_{(\xi)}$ is mean zero. As a consequence, using Lemma 4.1 we have

$$\sum_{\xi\in\Lambda_\alpha}\gamma_\xi^2(R)\fint_{\mathbb{T}^3} W_{(\xi)}(x)\otimes W_{(\xi)}(x)\,dx = R \tag{4.7}$$

for every symmetric matrix R satisfying $|R-\mathrm{Id}|\le 1/2$.

By scaling and Fubini, we have

$$\|\nabla^N \partial_t^M \psi_{(\xi)}\|_{L^p} \lesssim \ell_{\|}^{1/p-1/2}\left(\frac{\ell_\perp \lambda}{\ell_\|}\right)^N \left(\frac{\ell_\perp \lambda \mu}{\ell_\|}\right)^M, \tag{4.8}$$

$$\|\nabla^N \phi_{(\xi)}\|_{L^p} + \|\nabla^N \Phi_{(\xi)}\|_{L^p} \lesssim \ell_\perp^{2/p-1}\lambda^N, \tag{4.9}$$

$$\|\nabla^N \partial_t^M W_{(\xi)}\|_{L^p} + \lambda^2\|\nabla^N \partial_t^M V_{(\xi)}\|_{L^p} \lesssim \ell_\perp^{2/p-1}\ell_\|^{1/p-1/2}\lambda^N\left(\frac{\ell_\perp \lambda \mu}{\ell_\|}\right)^M, \tag{4.10}$$

where again we have assumed $\ell_\|^{-1} \ll \ell_\perp^{-1} \ll \lambda$.

Finally, we note the essential identity

$$\operatorname{div}(W_{(\xi)} \otimes W_{(\xi)}) = 2(W_{(\xi)} \cdot \nabla \psi_{(\xi)})\phi_{(\xi)}\xi = \frac{1}{\mu}\phi_{(\xi)}^2 \partial_t \psi_{(\xi)}^2 \xi, \tag{4.11}$$

which follows from the fact that by construction $W_{(\xi)}$ is a scalar multiple of ξ,

$$(\xi \cdot \nabla)\psi_{(\xi)} = \frac{1}{\mu}\partial_t \psi_{(\xi)},$$

and $\phi_{(\xi)}$ is time-independent.

4.2. *The perturbation*

In this section we will construct the perturbation w_{q+1}.

4.2.1. Stress cutoffs. Because the Reynolds stress $\mathring{R}_q$ is not spatially homogeneous, we introduce stress cutoff functions. We let $0 \le \widetilde{\chi}_0, \widetilde{\chi} \le 1$ be bump functions adapted to the intervals $[0, 4]$ and $[1/4, 4]$, such that together they form a partition of unity:

$$\widetilde{\chi}_0^2(y) + \sum_{i \ge 1} \widetilde{\chi}_i^2(y) \equiv 1, \quad \text{where} \quad \widetilde{\chi}_i(y) = \widetilde{\chi}(4^{-i}y), \tag{4.12}$$

for any $y > 0$. We then define

$$\chi_{(i)}(x,t) = \chi_{i,q+1}(x,t) = \widetilde{\chi}_i\left(\left\langle \frac{\mathring{R}_q(x,t)}{\lambda_q^{-\varepsilon_R/4}\delta_{q+1}}\right\rangle\right) \tag{4.13}$$

for all $i \ge 0$. Here and throughout the paper we use the notation $\langle A \rangle = (1 + |A|^2)^{1/2}$ where $|A|$ denotes the Euclidean norm of the matrix A. By definition the cutoffs $\chi_{(i)}$ form a partition of unity,

$$\sum_{i \ge 0} \chi_{(i)}^2 \equiv 1, \tag{4.14}$$

and we will show in Lemma 4.2 below that there exists an index $i_{\max} = i_{\max}(q)$ such that $\chi_{(i)} \equiv 0$ for all $i > i_{\max}$, and moreover $4^{i_{\max}} \lesssim \tau_{q+1}^{-1}$.

4.2.2. The definition of the velocity increment. Recall from Lemma 4.1 that the functions $\gamma_{(\xi)}$ are well-defined and smooth in the $1/2$-neighborhood of the identity matrix. In view of (4.13), this motivates introducing the parameters ρ_i by

$$\rho_i := \lambda_q^{-\varepsilon_R/4}\delta_{q+1}4^{i+2} \qquad \text{for all } i \geq 0, \tag{4.15}$$

which have the property that

$$\frac{|\mathring{\bar{R}}_q|}{\rho_i} \leq \frac{1}{4} \quad \text{on the support of } \chi_{(i)} \text{ for all } i \geq 0.$$

For $i \geq 0$ we define the coefficient function $a_{\xi,i,q+1}$ by

$$a_{(\xi)} := a_{\xi,i,q+1}(x,t) := \theta(t)\rho_i^{1/2}\chi_{i,q+1}(x,t)\gamma_{(\xi)}\left(\mathrm{Id} - \frac{\mathring{\bar{R}}_q(x,t)}{\rho_i}\right), \tag{4.16}$$

where $\theta\colon [0,T] \to [0,1]$ is a smooth temporal cutoff function with the following properties:

(i) $\theta(t) = 1$ for all t such that $\mathrm{dist}(t, \mathcal{G}^{(q+1)}) \geq 2\tau_{q+1}$,

(ii) $\theta(t) = 0$ for all t such that $\mathrm{dist}(t, \mathcal{G}^{(q+1)}) \leq \tau_{q+1}$,

(iii) $\|\theta\|_{C^M} \lesssim \tau_{q+1}^{-M}$, where the implicit constant depends only on M.

To see that a choice for θ with property (iii) holding is possible, recall from (2.17) that the bad set $\mathcal{B}^{(q+1)}$ consists of a finite disjoint union of intervals of length $5\tau_{q+1}$. From (i) and (2.20a), we conclude that

$$t \in \mathrm{supp}(\mathring{\bar{R}}_q) \quad \text{implies} \quad \theta(t) = 1. \tag{4.17}$$

From (ii) we further see that

$$t \in \mathrm{supp}(\theta) \supset \mathrm{supp}(a_{(\xi)}) \quad \text{implies} \quad \mathrm{dist}(t, \mathcal{G}^{(q+1)}) > \tau_{q+1}. \tag{4.18}$$

We note that as a consequence of (4.7), (4.14), (4.16), and (4.17) we have

$$\sum_{i\geq 0}\sum_{\xi\in\Lambda_{(i)}} a_{(\xi)}^2 \fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi)}\, dx = \theta^2 \sum_{i\geq 0}\rho_i\chi_{(i)}^2\mathrm{Id} - \mathring{\bar{R}}_q, \tag{4.19}$$

which justifies the definition of the amplitude functions $a_{(\xi)}$. Note that $\theta = 1$ on the support of $\chi_{(i)}$ for any $i \geq 1$.

By a slight abuse of notation, let us now fix $\lambda, \sigma, \ell_\perp, \ell_\|$, and μ for the shorthand notation $W_{(\xi)}$, $V_{(\xi)}$, $\Phi_{(\xi)}$, $\phi_{(\xi)}$ and $\psi_{(\xi)}$ introduced in Section 4.1:

$$\begin{aligned}
&W_{(\xi)} := W_{\xi,\ell_\perp,\ell_\|,\lambda_{q+1},\mu}, \qquad V_{(\xi)} := V_{\xi,\ell_\perp,\ell_\|,\lambda_{q+1},\mu},\\
&\psi_{(\xi)} := \psi_{\xi,\ell_\perp,\ell_\|,\lambda_{q+1},\mu},\\
&\Phi_{(\xi)} := \Phi_{\xi,\ell_\perp,\lambda_{q+1},\mu}, \qquad \phi_{(\xi)} := \phi_{\xi,\ell_\perp,\lambda_{q+1},\mu},
\end{aligned}$$

where $\ell_\perp$, $\ell_\|$, and μ are defined in (2.23). Importantly, we see from (2.24) that $\lambda_{q+1}\ell_\perp \in \mathbb{N}$, which ensures the periodicity of $W_{(\xi)}$, $V_{(\xi)}$, $\Phi_{(\xi)}$, $\phi_{(\xi)}$ and $\psi_{(\xi)}$. Observe that as a consequence of our parameter choices we have the useful inequality

$$\mu^{-1}\ell_\perp^{-1}\ell_\|^{-1/2} = \lambda_{q+1}^{-\frac{5-4\alpha}{8}} \ll 1 \tag{4.20}$$

for all $\alpha < 5/4$.

The *principal part* of w_{q+1} is defined as

$$w_{q+1}^{(p)} := \sum_i \sum_{\xi \in \Lambda_{(i)}} a_{(\xi)} W_{(\xi)}, \tag{4.21}$$

where the sum is over $0 \le i \le i_{\max}(q)$. Here we write $\Lambda_{(i)} = \Lambda_{i \bmod 2}$. Note that $|i-j| \ge 2$ implies $\chi_i \chi_j \equiv 0$, and $\xi \ne \xi'$ implies $W_{(\xi)} \otimes W_{(\xi')} \equiv 0$. This implies that the summands in (4.21) have mutually disjoint supports. In order to fix the fact that $w_{q+1}^{(p)}$ is not divergence-free, we define an *incompressibility corrector* by

$$w_{q+1}^{(c)} := \sum_i \sum_{\xi \in \Lambda_{(i)}} \operatorname{curl}(\nabla a_{(\xi)} \times V_{(\xi)}) + \frac{1}{\lambda_{q+1}^2 N_\Lambda^2} \nabla(a_{(\xi)}\psi_{(\xi)}) \times \operatorname{curl}(\Phi_{(\xi)}\xi), \tag{4.22}$$

so that by a formula similar to (4.6),

$$w_{q+1}^{(p)} + w_{q+1}^{(c)} = \sum_i \sum_{\xi \in \Lambda_{(i)}} \operatorname{curl}\operatorname{curl}(a_{(\xi)} V_{(\xi)}), \tag{4.23}$$

and thus $\operatorname{div}(w_{q+1}^{(p)} + w_{q+1}^{(c)}) \equiv 0$.

In addition to the incompressibility corrector $w_{q+1}^{(c)}$, we introduce a *temporal corrector* $w_{q+1}^{(t)}$, which is defined by

$$w_{q+1}^{(t)} := -\frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}} \mathbb{P}_H \mathbb{P}_{\ne 0}(a_{(\xi)}^2 \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi). \tag{4.24}$$

Finally, we define the velocity increment w_{q+1} by

$$w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}, \tag{4.25}$$

which is by construction mean zero and divergence-free. The new velocity field v_{q+1} is then defined as

$$v_{q+1} = \bar{v}_q + w_{q+1}. \tag{4.26}$$

Observe that as a consequence of (4.18),

$$t \in \operatorname{supp}(w_{q+1}) \quad \text{implies} \quad \operatorname{dist}(t, \mathcal{G}^{(q+1)}) > \tau_{q+1}. \tag{4.27}$$

Hence $v_{q+1} = \bar{v}_q$ on $\mathcal{G}^{(q+1)}$, which we recall was required in Section 3.3 to deduce property (v) of Section 2.1 for $\mathcal{G}^{(q+1)}$. Moreover, property (vi) also follows as a consequence of (4.27) and (2.20a).

4.2.3. Estimates of the perturbation. This section closely mirrors [5, Section 4.4], and thus we omit most of the details where the estimates/proofs are *mutatis mutandis* those from [5]. There is an analogy between the mollification parameter ℓ in [5] and the time-scale τ_{q+1} in this paper, in view of parabolic smoothing.

First, similarly to [5, Lemmas 4.1 and 4.2] we state a useful lemma concerning the cutoffs $\chi_{(i)}$ defined in (4.13), summarizing their size and regularity:

Lemma 4.2. *For $q \geq 0$, there exists $i_{\max}(q) \geq 0$ such that*

$$\chi_{(i)} \equiv 0 \quad \textit{for all } i > i_{\max}. \tag{4.28}$$

Moreover, for all $0 \leq i \leq i_{\max}$,

$$\rho_i \leq \lambda_1^{\beta} 4^{i_{\max}} \leq \tau_{q+1}^{-2}, \tag{4.29}$$

and we have

$$\sum_{i=0}^{i_{\max}} \rho_i^{1/2} 2^{-i} \leq \lambda_q^{-\varepsilon_R/16} \delta_{q+1}^{1/2}. \tag{4.30}$$

Additionally, for $0 \leq i \leq i_{\max}$,

$$\|\chi_{(i)}\|_{L^2} \lesssim 2^{-i}, \tag{4.31}$$

$$\|\chi_{(i)}\|_{C^N_{x,t}} \lesssim \tau_{q+1}^{-3N}, \tag{4.32}$$

for all $N \geq 1$, where the implicit constant only depends on N.

Proof. The existence of $i_{\max}$ is a consequence of the bound

$$\|\mathring{R}_q\|_{L^\infty} \leq \lambda_q^8. \tag{4.33}$$

The bound (4.33) follows from (2.20b)–(2.20c) and the Gagliardo–Nirenberg inequality $\|f\|_{L^\infty} \lesssim \|f\|_{L^1}^{1/3} \|f\|_{\dot{H}^3}^{2/3}$, which holds for any zero-mean periodic function $f \in H^3$, the definition of τ_{q+1} in (2.8), and the fact that $\varepsilon_R \leq 1$. Indeed, we have

$$\begin{aligned}\|\mathring{R}_q\|_{L^\infty} &\lesssim (\lambda_q^{-\varepsilon_R/4}\delta_{q+1})^{1/3}(\tau_{q+1}^{-1}\lambda_q^4)^{2/3} = (\lambda_q^{-\varepsilon_R/4}\delta_{q+1})^{1/3}(\lambda_q^{7+\varepsilon_R/4}\delta_{q+1}^{-1/2}\lambda_q^4)^{2/3} \\ &= \lambda_q^{\varepsilon_R/12+22/3}\end{aligned}$$

and the remaining power of $\lambda_q^{-2/3}$ may be used to absorb $\lambda_q^{\varepsilon_R/12}$ and the implicit universal constant.

The first bound expressed in (4.29) follows from the definition of ρ_i in (4.15), the fact that by (2.3) we have $\delta_{q+1} \leq \lambda_1^{\beta}$, and the fact that a may be chosen sufficiently large to ensure that $4^5 \lambda_q^{-\varepsilon_R/4} \leq 1$. Next, we note that in view of the definition of $\chi_{(i)}$, for any $i \geq 1$, if (x,t) is such that

$$\langle \lambda_q^{\varepsilon_R/4} \delta_{q+1}^{-1} \mathring{R}_q(x,t) \rangle < 4^{i-1}$$

then $\chi_{(i)}(x,t) = 0$. Therefore, by the bound (4.33) and the fact that $\beta b \le 1/4$, if $i \ge 1$ is such that

$$\langle \lambda_q^{\varepsilon_R/4} \delta_{q+1}^{-1} \lambda_q^8 \rangle \le \lambda_q^9 < 4^{i-1}$$

then $\chi_{(i)} \equiv 0$. Therefore, in view of the parameter inequality

$$\lambda_q^9 \le 4^{-2} \lambda_1^{-\beta} \tau_{q+1}^{-2},$$

which holds in view of (2.8) and the fact that $\beta b \le 1/4$, upon taking a sufficiently large, we may thus define

$$i_{\max}(q) = \max\{i \ge 0 : \lambda_1^\beta \, 4^i \le \tau_{q+1}^{-2}\}.$$

With this choice of $i_{\max}$ the above argument yields (4.28). The bound on $i_{\max}$ claimed in the second inequality in (4.29) then follows from the above definition.

The bound (4.30) follows from the second estimate in (4.29) which gives an upper bound on $i_{\max}$, the definition (4.15), and using that $\lambda_q^{-\varepsilon_R/16} \log_4(\tau_{q+1}^{-2}) \le 8\lambda_q^{-\varepsilon_R/16} \log_4(\lambda_q)$ can be made arbitrarily small if a is chosen sufficiently large, depending on ε_R.

For $i = 0, 1$, the bound (4.31) follows from the fact that $\widetilde{\chi}_0, \widetilde{\chi} \le 1$. For $i \ge 2$, we appeal to the definition of $\chi_{(i)}$, Chebyshev's inequality, and the L^1 estimate on $\mathring{R}_q$ in (2.20b), to obtain $\|\chi_{(i)}\|_{L^1} \lesssim 4^{-i}$. The bound (4.31) follows by interpolation.

Estimate (4.32) is a consequence of (2.20c) and [2, Proposition C.1], applied to the composition with the smooth functions $\gamma_\xi(\cdot)$ and $\langle \cdot \rangle = \sqrt{1 + (\cdot)^2}$. Indeed, for any $i \ge 0$ we obtain

$$\begin{aligned}
\|\chi_{(i)}\|_{C^N_{x,t}} &\lesssim \|\langle \lambda_q^{\varepsilon_R/4} \delta_{q+1}^{-1} \mathring{R}_q \rangle\|_{C^N_{x,t}} + \|\langle \lambda_q^{\varepsilon_R/4} \delta_{q+1}^{-1} \mathring{R}_q \rangle\|_{C^1_{x,t}}^N \\
&\lesssim 1 + \lambda_q^{\varepsilon_R/4} \delta_{q+1}^{-1} \|\mathring{R}_q\|_{C^N_{x,t}} + (\lambda_q^{\varepsilon_R/4} \delta_{q+1}^{-1} \|\mathring{R}_q\|_{C^1_{x,t}})^N \\
&\lesssim 1 + \lambda_q^{\varepsilon_R/4} \delta_{q+1}^{-1} \tau_{q+1}^{-N-1} \lambda_q^4 + (\lambda_q^{\varepsilon_R/4} \delta_{q+1}^{-1} \tau_{q+1}^{-2} \lambda_q^4)^N \\
&\lesssim 1 + \tau_{q+1}^{-N-2} + \tau_{q+1}^{-3N} \lesssim \tau_{q+1}^{-3N}.
\end{aligned}$$

Here we have used (2.8) to get $\tau_{q+1} \le 1$ and $\lambda_q^{\varepsilon_R/4} \delta_{q+1}^{-1} \lambda_q^4 \le \lambda_q^{7+\varepsilon_R/4} \delta_{q+1}^{-1/2} = \tau_{q+1}^{-1}$. ∎

Next, we recall from [5, Lemma 4.3] the following bounds on the coefficients $a_{(\xi)}$.

Lemma 4.3. *The bounds*

$$\|a_{(\xi)}\|_{L^2} \lesssim \rho_i^{1/2} 2^{-i} \lesssim \delta_{q+1}^{1/2}, \tag{4.34}$$

$$\|a_{(\xi)}\|_{L^\infty} \lesssim \rho_i^{1/2} \lesssim \delta_{q+1}^{1/2} 2^i, \tag{4.35}$$

$$\|a_{(\xi)}\|_{C^N_{x,t}} \lesssim \tau_{q+1}^{-3N-1} \tag{4.36}$$

hold for all $0 \le i \le i_{\max}$ *and* $N \ge 1$.

Proof. The bound (4.35) follows directly from the definitions (4.16) and (4.15), and the boundedness of θ, $\chi_{(i)}$, and $\gamma_{(\xi)}$. Using also (4.31), the estimate (4.34) follows similarly. In order to prove (4.36), we apply derivatives to (4.16), use the bounds previously established in Lemma 4.2, use [2, Proposition C.1] and the bound (2.20c) for $\mathring{\bar{R}}_q$, combined with $\|\theta\|_{C^M} \lesssim \tau_{q+1}^{-M}$. The additional factor of τ_{q+1}^{-1} when compared to (4.32) is to absorb the factor of $\rho_i^{1/2}$ via (4.29). ∎

As a consequence of Lemma 4.3 and the definitions (4.21), (4.22), and (4.24), we obtain the following bounds:

Proposition 4.4. *The principal part of the velocity perturbation, the incompressibility, and the temporal correctors obey the bounds*

$$\|w_{q+1}^{(p)}\|_{L^2} \leq \tfrac{1}{2}\delta_{q+1}^{1/2}, \tag{4.37}$$

$$\|w_{q+1}^{(p)}\|_{W^{N,p}} \lesssim \tau_{q+1}^{-2}\ell_\perp^{2/p-1}\ell_\|^{1/p-1/2}\lambda_{q+1}^N, \tag{4.38}$$

$$\begin{aligned}\|w_{q+1}^{(c)}\|_{W^{N,p}} + \|w_{q+1}^{(t)}\|_{W^{N,p}} &\lesssim \mu^{-1}\tau_{q+1}^{-3}\ell_\perp^{2/p-2}\ell_\|^{1/p-1}\lambda_{q+1}^N \\ &\lesssim \lambda_{q+1}^{-\frac{5-4\alpha}{16}}\ell_\perp^{2/p-1}\ell_\|^{1/p-1/2}\lambda_{q+1}^N,\end{aligned} \tag{4.39}$$

for $N \in \mathbb{N}$ *and* $p > 1$.

From the second estimate in (4.39) it is clear that the incompressibility and temporal correctors obey better estimates than the principal corrector.

In order to establish the bound (4.37), it is essential to use the fact that $a_{(\xi)}$ oscillates at a frequency which is much smaller than that of $W_{(\xi)}$, which allows us to appeal to the L^p de-correlation lemma [5, Lemma 3.6], which we recall here for convenience:

Lemma 4.5. *Fix integers* $M, \kappa, \lambda \geq 1$ *such that*

$$\frac{2\pi\sqrt{3}\,\lambda}{\kappa} \leq \frac{1}{3} \quad \text{and} \quad \lambda^4 \frac{(2\pi\sqrt{3}\,\lambda)^M}{\kappa^M} \leq 1. \tag{4.40}$$

Let $p \in \{1, 2\}$*, and let* f *be a* $\mathbb{T}^3$*-periodic function such that there exists a constants* C_f *with*

$$\|D^j f\|_{L^p} \leq C_f \lambda^j$$

for all $1 \leq j \leq M + 4$*. In addition, let* g *be a* $(\mathbb{T}/\kappa)^3$*-periodic function. Then*

$$\|fg\|_{L^p} \lesssim C_f \|g\|_{L^p},$$

where the implicit constant is universal.

The bounds (4.37)–(4.39) follow by using Lemma 4.5 in the same spirit as [5, Proposition 4.5].

Proof of Proposition 4.4. In order to prove (4.37), we use (4.34) when $N = 0$, and (4.36) with $\lambda_q^{\varepsilon_R/8}\delta_{q+1}^{-1/2} \le \tau_{q+1}^{-1}$ for $N \ge 1$, to conclude that

$$\|D^N a_{(\xi)}\|_{L^2} \lesssim \rho_i^{1/2} 2^{-i} \tau_{q+1}^{-5N}, \tag{4.41}$$

where the implicit constant depends only on N. Since $\mathbb{W}_{(\xi)}$ is $(\mathbb{T}/\lambda_{q+1}\ell_\perp)^3$ periodic, in order to apply Lemma 4.5 with $\lambda = \tau_{q+1}^{-5}$ and $\kappa = \lambda_{q+1}\ell_\perp$, we first note that by (2.8) and (2.23),

$$\begin{aligned} 2\pi\sqrt{3}\,\lambda\kappa^{-1} &= 2\pi\sqrt{3}\,\tau_{q+1}^{-5}\lambda_{q+1}^{-1}\ell_\perp^{-1} = 2\pi\sqrt{3}\,\lambda_q^{35+\frac{5\varepsilon_R}{4}}\delta_{q+1}^{-5/2}\lambda_{q+1}^{-\frac{5(5-4\alpha)}{24}} \\ &\le \lambda_q^{36}\lambda_{q+1}^{-\frac{5(5-4\alpha)}{24}+5\beta} \le \lambda_{q+1}^{-\frac{5-4\alpha}{6}}, \end{aligned} \tag{4.42}$$

by using the fact that β is sufficiently small and b is sufficiently large, depending on α. For instance, we may take

$$5\beta \le \frac{5-4\alpha}{50} \quad \text{and} \quad \frac{36}{b} \le \frac{5-4\alpha}{50}. \tag{4.43}$$

In (4.42) we have also used $\lambda_1^{-15\beta/2}2\pi\sqrt{3} \le 1$, once a is chosen sufficiently large. Therefore, after a short computation we see that the assumptions of Lemma 4.5 hold with the aforementioned κ and λ, with $M = 4$ in (4.40). Therefore, we only care about $N \le 4$ in (4.41), which also fixes the implicit constant in this inequality, and we may take C_f to be proportional to $\rho_i^{1/2}2^{-i}$. It thus follows from Lemma 4.5 and estimate (4.10) with $M = N = 0$ and $p = 2$ that

$$\|a_{(\xi)}\mathbb{W}_{(\xi)}\|_{L^2} \lesssim \rho_i^{1/2}2^{-i}\|\mathbb{W}_{(\xi)}\|_{L^2} \lesssim \rho_i^{1/2}2^{-i}.$$

Upon summing over $i \in \{0,\dots,i_{\max}\}$, and appealing to (4.30), we obtain

$$\|w_{q+1}^{(p)}\|_{L^2} \lesssim \delta_{q+1}^{1/2}\lambda_q^{-\varepsilon_R/16} \le \tfrac{1}{2}\delta_{q+1}^{1/2},$$

by using the small negative power of λ_q to absorb the implicit constants in the first inequality.

Consider the estimate (4.38). Observe that by definition (4.21), estimate (4.10) with $M = 0$, and the bound (4.36), we have

$$\begin{aligned} \|w_{q+1}^{(p)}\|_{W^{N,p}} &\lesssim \sum_i \sum_{\xi\in\Lambda_{(i)}} \sum_{N'=0}^{N} \|a_{(\xi)}\|_{C^{N-N'}}\|\mathbb{W}_{(\xi)}\|_{W^{N',p}} \\ &\lesssim \sum_{i=0}^{i_{\max}} \sum_{\xi\in\Lambda_{(i)}} \sum_{N'=0}^{N} \tau_{q+1}^{-3(N-N')-1}\ell_\perp^{2/p-1}\ell_\parallel^{1/p-1/2}\lambda_{q+1}^{N'} \\ &\lesssim \tau_{q+1}^{-2}\ell_\perp^{2/p-1}\ell_\parallel^{1/p-1/2}\lambda_{q+1}^{N}. \end{aligned} \tag{4.44}$$

Here we have again used (4.29) in order to sum over i, and we have used the bound $\tau_{q+1}^{-3} \leq \lambda_{q+1}$ which holds since β is small and b is large.

For the analogous bound on $w_{q+1}^{(c)}$, by (4.8)–(4.10), estimate (4.36), the parameter estimates $\tau_{q+1}^{-3} \leq \lambda_{q+1}$ and $\ell_\perp \leq \ell_\|$, and Fubini (recall that $\psi_{(\xi)}$ and $\Phi_{(\xi)}$ are functions of one and respectively two variables which are orthogonal to each other), we have

$$
\begin{aligned}
&\left\| \operatorname{curl}(\nabla a_{(\xi)} \times V_{(\xi)}) + \frac{1}{\lambda_{q+1}^2 N_\Lambda^2} \nabla(a_{(\xi)}\psi_{(\xi)}) \times \operatorname{curl}(\Phi_{(\xi)}\xi) \right\|_{W^{N,p}} \\
&\quad \lesssim \sum_{N'=0}^{N+1} \|a_{(\xi)}\|_{C^{N+2-N'}} \|V_{(\xi)}\|_{W^{N',p}} \\
&\qquad + \frac{1}{\lambda_{q+1}^2} \sum_{N'=0}^{N} \sum_{N''=0}^{N-N'+1} \|a_{(\xi)}\|_{C^{N-N'+1-N''}} \|\psi_{(\xi)}\|_{W^{N'',p}} \|\Phi_{(\xi)}\|_{W^{N'+1,p}} \\
&\quad \lesssim \sum_{N'=0}^{N+1} \tau_{q+1}^{-3(N+2-N')-1} \lambda_{q+1}^{N'-2} \\
&\qquad + \sum_{N'=0}^{N} \sum_{N''=0}^{N-N'+1} \tau_{q+1}^{-3(N+1-N'-N'')-1} \ell_\|^{1/p-1/2} \left(\frac{\ell_\perp \lambda_{q+1}}{\ell_\|}\right)^{N''} \ell_\perp^{2/p-1} \lambda_{q+1}^{N'-1} \\
&\quad \lesssim \tau_{q+1}^{-1} \ell_\|^{1/p-1/2} \ell_\perp^{2/p-1} \lambda_{q+1}^{N} (\tau_{q+1}^{-3} \lambda_{q+1}^{-1}).
\end{aligned}
$$

Summing over $0 \leq i \leq i_{\max}$ loses an additional factor of τ_{q+1}^{-1}, which yields the desired bound for the first term on the left of (4.39). Similarly, to estimate the summands in the definition (4.24) of $w_{q+1}^{(t)}$ we use (4.8), (4.9), (4.36), the aforementioned parameter inequalities, and Fubini to obtain

$$
\begin{aligned}
&\|\mu^{-1} \mathbb{P}_H \mathbb{P}_{\neq 0} (a_{(\xi)}^2 \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi)\|_{W^{N,p}} \\
&\qquad \lesssim \mu^{-1} \sum_{N'=0}^{N} \sum_{N''=0}^{N'} \|a_{(\xi)}^2\|_{C^{N-N'}} \|\phi_{(\xi)}^2\|_{W^{N'',p}} \|\psi_{(\xi)}^2\|_{W^{N'-N'',p}} \\
&\qquad \lesssim \mu^{-1} \tau_{q+1}^{-3(N-N')-2} \ell_\perp^{2/p-2} \lambda_{q+1}^{N''} \ell_\|^{1/p-1} \left(\frac{\ell_\perp \lambda_{q+1}}{\ell_\|}\right)^{N'-N''} \\
&\qquad \lesssim \mu^{-1} \tau_{q+1}^{-2} \ell_\perp^{2/p-2} \ell_\|^{1/p-1} \lambda_{q+1}^{N}.
\end{aligned}
$$

Summing over i loses a factor of τ_{q+1}^{-1} (cf. (4.29)), and we obtain the bound for the second term on the left of (4.39).

For the proof of (4.39), we additionally note that (4.20) and (4.43) imply the parameter inequalities

$$
\tau_{q+1}^{-1} \leq \lambda_q^8 \leq \lambda_{q+1}^{\frac{5-4\alpha}{100}}, \quad \tau_{q+1}^{-3} \lambda_{q+1}^{-1} \leq \mu^{-1} \tau_{q+1}^{-1} \ell_\perp^{-1} \ell_\|^{-1/2}, \quad \mu^{-1} \tau_{q+1}^{-3} \ell_\perp^{-1} \ell_\|^{-1/2} \leq \lambda_{q+1}^{-\frac{5-4\alpha}{16}}, \tag{4.45}
$$

which concludes the proof of the proposition. ∎

The following bound shows that (2.15) holds, and collects a number of useful bounds for the cumulative velocity increment w_{q+1}, which in turn imply that (2.5a) and (2.5b) hold at level $q+1$.

Proposition 4.6. *The bounds*

$$\|w_{q+1}\|_{L^2} \le \tfrac{3}{4}\delta_{q+1}^{1/2}, \tag{4.46}$$

$$\|v_{q+1} - v_q\|_{L^2} \le \delta_{q+1}^{1/2}, \tag{4.47}$$

$$\|w_{q+1}\|_{W^{s,p}} \lesssim \tau_{q+1}^{-2}\ell_\perp^{2/p-1}\ell_\|^{1/p-1/2}\lambda_{q+1}^s, \tag{4.48}$$

$$\|\partial_t w_{q+1}\|_{H^2} \lesssim \lambda_{q+1}^5 \tag{4.49}$$

hold for $1 < p \le \infty$ *and* $s \ge 0$.

Before turning to the proof of the proposition, we note that estimate (4.47) and the inductive assumption (2.5a) at level q imply that

$$\|v_{q+1}\|_{L^2} \le 2\delta_0^{1/2} - \delta_q^{1/2} + \delta_{q+1}^{1/2} \le 2\delta_0^{1/2} - \delta_{q+1}^{1/2}, \tag{4.50}$$

which is a consequence of $2\lambda_q^\beta \le \lambda_{q+1}^\beta$. Thus, (2.5a) holds at level $q+1$. Similarly, from (2.19b) and (4.48) with $s = 3$ and $p = 2$, and the parameter inequality (4.45), we conclude

$$\|v_{q+1}\|_{H^3} \lesssim \lambda_q^4 + \tau_{q+1}^{-2}\lambda_{q+1}^3 \lesssim \lambda_{q+1}^{\frac{4}{b}+3+\frac{5-4\alpha}{50}} \lesssim \lambda_{q+1}^{7/2} \le \lambda_{q+1}^4, \tag{4.51}$$

where we have used the fact that b is large and $\alpha \in [1, 5/4)$. The remaining power of $\lambda_{q+1}^{-1/2}$ may be used to absorb the implicit constant, and thus (2.5b) holds also at level $q+1$.

Similarly to (4.51), we establish two bounds which will be useful in Section 5 for the proof of Corollary 5.2. First, from (4.48) with $s = 9/2$ and $p = 2$, and (2.19d) with $M = 0$ and $N = 2$, it follows that

$$\begin{aligned}\|v_{q+1}\|_{L^\infty(T/3,2T/3;H^{9/2})} &\le \|w_{q+1}\|_{H^{9/2}} + \|\bar v_q\|_{L^\infty(T/3,2T/3;H^5)} \\ &\lesssim \tau_{q+1}^{-2}\lambda_{q+1}^{9/2} + \tau_{q+1}^{-2}\lambda_q^4 \lesssim \lambda_{q+1}^5.\end{aligned} \tag{4.52}$$

Here we have also used the parameter inequality (4.45). Similarly, by (4.49) and the bound (2.19d) with $M = 1$ and $N = 0$ we obtain

$$\begin{aligned}\|\partial_t v_{q+1}\|_{L^\infty(T/3,2T/3;H^2)} &\le \|\partial_t w_{q+1}\|_{H^2} + \|\partial_t \bar v_q\|_{L^\infty(T/3,2T/3;H^3)} \\ &\lesssim \lambda_{q+1}^5 + \tau_{q+1}^{-1}\lambda_q^4 \lesssim \lambda_{q+1}^5.\end{aligned} \tag{4.53}$$

Proof of Proposition 4.6. The estimates (4.46) and (4.47) are direct consequences of the already established bounds and the definitions (4.25) and (4.26). Indeed, combining (4.37) with (4.39) with $p = 2$ and $N = 0$, we conclude that

$$\|w_{q+1}\|_{L^2}\delta_{q+1}^{-1/2} \le 1/2 + \lambda_{q+1}^{\beta-\frac{5-4\alpha}{16}} \le 3/4,$$

since β is sufficiently small (see (4.43)). From (4.46) and (2.19c) we obtain

$$\|v_{q+1} - v_q\|_{L^2} \leq \|\bar{v}_q - v_q\|_{L^2} + \|w_{q+1}\|_{L^2} \leq \delta_{q+1}^{1/2},$$

as desired. The estimate (4.48) with non-integer values of s follows by interpolation from the case $s \in \mathbb{N}$. Comparing (4.38) with the second inequality in (4.39), we see that the bound for the principal corrector is the worst, since $\lambda_{q+1}^{-\frac{5-4\alpha}{16}} \leq 1 \leq \tau_{q+1}^{-1}$, and thus (4.48) follows directly.

Thus it remains to prove (4.49). An estimate on $\partial_t w_{q+1}^{(p)}$ will clearly dominate an estimate on $\partial_t w_{q+1}^{(c)}$. Hence it suffices to estimate $\partial_t w_{q+1}^{(p)}$ and $\partial_t w_{q+1}^{(c)}$. First consider $\partial_t w_{q+1}^{(p)}$. From the bound (4.10) with $N = 2, M = 1, p = 2$, estimate (4.36) with $N = 3$, and the definition (2.23) of μ, we obtain

$$\begin{aligned}\|\partial_t w_{q+1}^{(p)}\|_{H^2} &\lesssim \sum_i \sum_{\xi \in \Lambda_{(i)}} \|a_{(\xi)}\|_{C^3_{x,t}} \|\partial_t W_{(\xi)}\|_{H^2} \\ &\lesssim \sum_{i=0}^{i_{\max}} \tau_{q+1}^{-10} \ell_\perp \ell_\parallel^{-1} \lambda_{q+1}^3 \mu \lesssim \tau_{q+1}^{-11} \lambda_{q+1}^{2\alpha+2} \lesssim \lambda_{q+1}^5.\end{aligned}$$

where in the last inequality we have used the fact that (4.45) provides an upper bound for τ_{q+1}^{-1}, and that $\alpha < 5/4$. In order to estimate $\partial_t w_{q+1}^{(t)}$ we use (4.8) and (4.9), Fubini, and (4.36) to obtain

$$\begin{aligned}\|\partial_t w_{q+1}^{(t)}\|_{H^2} &\lesssim \mu^{-1} \sum_i \sum_{\xi \in \Lambda_{(i)}} \left(\|a_{(\xi)}^2\|_{C^2_{x,t}} \|\phi_{(\xi)}^2 \partial_t \psi_{(\xi)}^2\|_{H^2} + \|a_{(\xi)}^2\|_{C^3_{x,t}} \|\phi_{(\xi)}^2 \psi_{(\xi)}^2\|_{H^2}\right) \\ &\lesssim \mu^{-1} \sum_{i=0}^{i_{\max}} \sum_{\xi \in \Lambda_{(i)}} \sum_{N=0}^{2} \left(\tau_{q+1}^{-8} \|\phi_{(\xi)}^2\|_{H^N} \|\partial_t \psi_{(\xi)}^2\|_{H^{2-N}} + \tau_{q+1}^{-11} \|\phi_{(\xi)}^2\|_{H^N} \|\psi_{(\xi)}^2\|_{H^{N-2}}\right) \\ &\lesssim \mu^{-1} \sum_{i=0}^{i_{\max}} \left(\tau_{q+1}^{-8} \ell_\perp^{-1} \ell_\parallel^{-1/2} \left(\frac{\ell_\perp \lambda_{q+1} \mu}{\ell_\parallel}\right) \lambda_{q+1}^2 + \tau_{q+1}^{-11} \ell_\perp^{-1} \ell_\parallel^{-1/2} \lambda_{q+1}^2\right) \\ &\lesssim \tau_{q+1}^{-9} \ell_\parallel^{-3/2} \lambda_{q+1}^3 + \tau_{q+1}^{-12} \mu^{-1} \ell_\perp^{-1} \ell_\parallel^{-1/2} \lambda_{q+1}^2 \lesssim \lambda_{q+1}^5.\end{aligned}$$

Here we have used explicitly the parameter choice (2.23), the parameter inequality (4.20), the first bound in (4.45), the bound $\ell_\parallel^{-1} \leq \ell_\perp^{-1} \leq \lambda_{q+1}$, and the inequality $i_{\max} \lesssim \tau_{q+1}^{-1}$. ■

5. Convex integration step: the Reynolds stress

The main result of this section may be summarized as follows:

Proposition 5.1. *There exists an $\varepsilon_R > 0$ sufficiently small, and a parameter $p > 1$ sufficiently close to 1, depending only on α, b, and β, such that the following holds: There* *exists a traceless symmetric* 2*-tensor* $\widetilde{R}$ *and a scalar pressure field* $\widetilde{p}$*, defined implicitly in* (5.5) *below, satisfying*

$$\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) + \nabla \widetilde{p} + (-\Delta)^\alpha v_{q+1} = \operatorname{div} \widetilde{R}, \tag{5.1a}$$

$$\operatorname{div} v_{q+1} = 0. \tag{5.1b}$$

Moreover $\widetilde{R}$ *obeys the bound*

$$\|\widetilde{R}\|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+2}, \tag{5.2}$$

where the constant depends on the choice of p *and* ε_R*, but is independent of* q*, and* $\widetilde{R}$ *has the support property*

$$\operatorname{supp}(\widetilde{R}) \subset \mathbb{T}^3 \times \{t \in [0,T] : \operatorname{dist}(t, \mathcal{G}^{(q+1)}) > \tau_{q+1}\}. \tag{5.3}$$

An immediate consequence of Proposition 5.1 is that the desired inductive estimates (2.4) and the support property (2.11) hold for the Reynolds stress $\mathring{R}_{q+1}$, which is defined as follows.

Corollary 5.2. *There exists a traceless symmetric* 2*-tensor* $\mathring{R}_{q+1}$ *and a scalar pressure field* p_{q+1} *such that the triple* $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ *solves the Navier–Stokes–Reynolds system* (2.1) *at level* $q+1$*. Moreover,*

$$\|\mathring{R}_{q+1}\|_{L^1} \leq \lambda_{q+1}^{-\varepsilon_R} \delta_{q+2}, \tag{5.4a}$$

$$\|\mathring{R}_{q+1}\|_{H^3} \leq \lambda_{q+1}^{7}, \tag{5.4b}$$

and $\mathring{R}_{q+1}(t) = 0$ *whenever* $\operatorname{dist}(t, \mathcal{G}^{(q+1)}) \leq \tau_{q+1}$.

Proof. With $\widetilde{R}$ and $\widetilde{p}$ defined in Proposition 5.1, we let

$$\mathring{R}_{q+1} = \mathcal{R}\mathbb{P}_H \operatorname{div} \widetilde{R}, \qquad p_{q+1} = \widetilde{p} - \Delta^{-1} \operatorname{div}\operatorname{div} \widetilde{R}.$$

It follows from (5.1) and the definitions of the inverse-divergence operator $\mathcal{R}$ and of the Helmholtz projection $\mathbb{P}_H$ that the $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ solve the Navier–Stokes–Reynolds system (2.1) at level $q+1$. Since the operator $\mathcal{R}\mathbb{P}_H \operatorname{div}$ is time-independent, the claimed support property for $\mathring{R}_{q+1}$, namely (2.11) at level $q+1$, follows directly from (5.3).

With the parameter $p > 1$ from Proposition 5.1, using $\|\mathcal{R}\mathbb{P}_H \operatorname{div}\|_{L^p \to L^p} \lesssim 1$, we directly bound

$$\|\mathring{R}_{q+1}\|_{L^1} \lesssim \|\mathring{R}_{q+1}\|_{L^p} \lesssim \|\widetilde{R}\|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+2}.$$

The estimate (5.4a) then follows since the residual factor $\lambda_{q+1}^{-\varepsilon_R}$ can absorb any constant if we assume a is sufficiently large. In order to prove (5.4b), we use equation (5.1), the support property of $\mathring{R}_{q+1}$ which implies that $\operatorname{supp}(\mathring{R}_{q+1}) \subset \mathbb{T}^3 \times [T/3, 2T/3]$, and the bounds (4.50)–(4.53). Combining these, we obtain

$$
\begin{aligned}
\|\mathring{R}_{q+1}\|_{H^3} &= \|\mathcal{R}\mathbb{P}_H(\operatorname{div}\widetilde{R})\|_{H^3} \\
&\lesssim \|\partial_t v_{q+1} + \operatorname{div}(v_{q+1}\otimes v_{q+1}) + (-\Delta)^\alpha v_{q+1}\|_{L^\infty(T/3,2T/3;H^2)} \\
&\lesssim \|\partial_t v_{q+1}\|_{L^\infty(T/3,2T/3;H^2)} + \|v_{q+1}\otimes v_{q+1}\|_{H^3} + \|v_{q+1}\|_{L^\infty(T/3,2T/3;H^{9/2})} \\
&\lesssim \|\partial_t v_{q+1}\|_{L^\infty(T/3,2T/3;H^2)} + \|v_{q+1}\|_{H^3}\|v_{q+1}\|_{L^\infty} + \|v_{q+1}\|_{L^\infty(T/3,2T/3;H^{9/2})} \\
&\lesssim \|\partial_t v_{q+1}\|_{L^\infty(T/3,2T/3;H^2)} + \|v_{q+1}\|_{H^3}^{3/2}\|v_{q+1}\|_{L^2}^{1/2} + \|v_{q+1}\|_{L^\infty(T/3,2T/3;H^{9/2})} \\
&\lesssim \lambda_{q+1}^5 + \lambda_{q+1}^6\delta_0^{1/4} + \lambda_{q+1}^5 \lesssim \lambda_{q+1}^{13/2}.
\end{aligned}
$$

For the dissipative term we have used $\alpha < 5/4$, so that $2\alpha + 2 < 9/2$. Using the residual power of $\lambda_{q+1}^{-1/2}$ we may absorb any constants and thus (5.4b) follows. ∎

5.1. *Proof of Proposition* 5.1

Recall that $v_{q+1} = w_{q+1} + \bar{v}_q$, where $\bar{v}_q$ is defined in Section 3.3 and $(\bar{v}_q, \mathring{\bar{R}}_q)$ solves (2.1). Using (4.25) we obtain

$$
\begin{aligned}
\operatorname{div}\widetilde{R} - \nabla\widetilde{p} = {} & (-\Delta)^\alpha w_{q+1} + \partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \operatorname{div}(\bar{v}_q \otimes w_{q+1} + w_{q+1}\otimes \bar{v}_q) \\
& + \operatorname{div}\big((w_{q+1}^{(c)} + w_{q+1}^{(t)})\otimes w_{q+1} + w_{q+1}^{(p)}\otimes(w_{q+1}^{(c)} + w_{q+1}^{(t)})\big) \\
& + \operatorname{div}(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)} + \mathring{\bar{R}}_q) + \partial_t w_{q+1}^{(t)} \\
=: {} & \operatorname{div}(\widetilde{R}_{\text{linear}} + \widetilde{R}_{\text{corrector}} + \widetilde{R}_{\text{oscillation}}) + \nabla q. \qquad (5.5)
\end{aligned}
$$

Here, the linear error and corrector errors are defined by applying $\mathcal{R}$ to the first and respectively second line of (5.5), while the oscillation error is defined in Section 5.1.3 below. The zero-mean pressure q is defined implicitly in a unique way.

Besides the already used inequalities between the parameters, $\ell_\perp$, $\ell_\|$ and λ_{q+1}, we shall use the fact that if p is sufficiently close to 1 then

$$
\begin{aligned}
&\tau_{q+1}^{-5}\lambda_{q+1}^{2\alpha-1}\ell_\perp^{2/p-1}\ell_\|^{1/p-1/2} + \tau_{q+1}^{-5}\lambda_{q+1}^{1-2\alpha}\ell_\perp^{2/p-2}\ell_\|^{1/p-5/2} \\
&\quad + \tau_{q+1}^{-6}\lambda_{q+1}^{-1}\ell_\perp^{2/p-3}\ell_\|^{1/p-1} + \tau_{q+1}^{-6}\lambda_{q+1}^{1-2\alpha}\ell_\perp^{2/p-1}\ell_\|^{1/p-2} \lesssim \lambda_{q+1}^{-2\varepsilon_R}\delta_{q+2}. \qquad (5.6)
\end{aligned}
$$

To see this, we appeal to the bound (4.45) for τ_{q+1}^{-1} and the parameter choices (2.23) to conclude that the left side of (5.6) is bounded from above as

$$
\begin{aligned}
&\ell_\perp^{2/p-2}\ell_\|^{1/p-1}(\tau_{q+1}^{-5}\lambda_{q+1}^{2\alpha-1}\ell_\perp\ell_\|^{1/2} + \tau_{q+1}^{-5}\lambda_{q+1}^{1-2\alpha}\ell_\|^{-3/2} + \tau_{q+1}^{-6}\lambda_{q+1}^{-1}\ell_\perp^{-1} \\
&\qquad\qquad + \tau_{q+1}^{-6}\lambda_{q+1}^{1-2\alpha}\ell_\perp\ell_\|^{-1}) \\
&\quad\lesssim \ell_\perp^{2/p-2}\ell_\|^{1/p-1}\tau_{q+1}^{-6}(\lambda_{q+1}^{2\alpha-1}\ell_\perp\ell_\|^{1/2} + \lambda_{q+1}^{1-2\alpha}\ell_\|^{-3/2} + \lambda_{q+1}^{-1}\ell_\perp^{-1} + \lambda_{q+1}^{1-2\alpha}\ell_\perp\ell_\|^{-1}) \\
&\quad\lesssim \ell_\perp^{2/p-2}\ell_\|^{1/p-1}\lambda_{q+1}^{\frac{3(5-4\alpha)}{50}}(\lambda_{q+1}^{-\frac{5-4\alpha}{12}} + \lambda_{q+1}^{-\frac{5-4\alpha}{8}} + \lambda_{q+1}^{-\frac{5(5-4\alpha)}{24}} + \lambda_{q+1}^{-\frac{28\alpha+1}{24}}) \\
&\quad\lesssim \ell_\perp^{2/p-2}\ell_\|^{1/p-1}\lambda_{q+1}^{-\frac{5-4\alpha}{50}} \lesssim \lambda_{q+1}^{-\frac{5-4\alpha}{100}},
\end{aligned}
$$

where in the last inequality we have chosen p sufficiently close to 1, depending only on α. To conclude the proof of (5.6), note that

$$\lambda_{q+1}^{-2\varepsilon_R}\delta_{q+2} \geq \lambda_{q+1}^{-2\varepsilon_R}\lambda_{q+2}^{-2\beta} \geq \lambda_q^{-2\varepsilon_R b - 2\beta b^2},$$

and therefore if we ensure that ε_R and β are sufficiently small, depending on α and b only, such that

$$2\varepsilon_R b + 2\beta b^2 \leq \frac{5-4\alpha}{100}, \tag{5.7}$$

then the three estimates above imply (5.6).

5.1.1. The linear error. In order to prove (5.2), we first estimate the contributions to $\widetilde{R}$ coming from $\widetilde{R}_{\text{linear}}$. Recalling (4.23), and the bounds (2.19b), (4.10), (4.36), and (4.48), we obtain

$$\begin{aligned}
\|\widetilde{R}_{\text{linear}}\|_{L^p} &\lesssim \|\mathcal{R}((-\Delta)^\alpha w_{q+1})\|_{L^p} + \|\mathcal{R}(\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}))\|_{L^p} \\
&\quad + \|\mathcal{R}\operatorname{div}(\bar{v}_q \otimes w_{q+1} + w_{q+1} \otimes \bar{v}_q)\|_{L^p} \\
&\lesssim \|w_{q+1}\|_{W^{2\alpha-1,p}} + \sum_i \sum_{\xi\in\Lambda_{(i)}} \|\partial_t \mathcal{R}\operatorname{curl}\operatorname{curl}(a_{(\xi)}V_{(\xi)})\|_{L^p} + \|\bar{v}_q\|_{L^\infty}\|w_{q+1}\|_{L^p} \\
&\lesssim (1 + \|\bar{v}_q\|_{L^\infty})\|w_{q+1}\|_{W^{2\alpha-1,p}} + \sum_i \sum_{\xi\in\Lambda_{(i)}} \|\partial_t \operatorname{curl}(a_{(\xi)}V_{(\xi)})\|_{L^p} \\
&\lesssim (1 + \|\bar{v}_q\|_{H^3})\|w_{q+1}\|_{W^{2\alpha-1,p}} \\
&\quad + \sum_i \sum_{\xi\in\Lambda_{(i)}} (\|a_{(\xi)}\|_{C^1_{x,t}}\|\partial_t V_{(\xi)}\|_{W^{1,p}} + \|a_{(\xi)}\|_{C^2_{x,t}}\|V_{(\xi)}\|_{W^{1,p}}) \\
&\lesssim \lambda_q^4 \tau_{q+1}^{-2}\lambda_{q+1}^{2\alpha-1}\ell_\perp^{2/p-1}\ell_\|^{1/p-1/2} + \tau_{q+1}^{-5}\ell_\perp^{2/p-1}\ell_\|^{1/p-1/2}\lambda_{q+1}^{-1}\left(\frac{\ell_\perp\lambda_{q+1}\mu}{\ell_\|}\right) \\
&\quad + \tau_{q+1}^{-8}\ell_\perp^{2/p-1}\ell_\|^{1/p-1/2}\lambda_{q+1}^{-1} \\
&\lesssim \tau_{q+1}^{-5}\lambda_{q+1}^{2\alpha-1}\ell_\perp^{2/p-1}\ell_\|^{1/p-1/2}.
\end{aligned} \tag{5.8}$$

Here we have used the definition of μ from (2.23), and the parameter inequalities $\lambda_q^4 \lesssim \tau_{q+1}^{-1} \lesssim \lambda_{q+1}^{\alpha/2}$. By (5.6), the above estimate is consistent with (5.2).

5.1.2. Corrector error. Next we turn to the errors involving correctors. Appealing to estimates (4.38) and (4.39) of Proposition 4.4, we have

$$\begin{aligned}
\|\widetilde{R}_{\text{corrector}}\|_{L^p} &\leq \left\|\mathcal{R}\operatorname{div}\left((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)})\right)\right\|_{L^p} \\
&\lesssim \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{L^{2p}}\|w_{q+1}\|_{L^{2p}} + \|w_{q+1}^{(p)}\|_{L^{2p}}\|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{L^{2p}} \\
&\lesssim \tau_{q+1}^{-5}\mu^{-1}\ell_\perp^{2/p-3}\ell_\|^{1/p-3/2} \lesssim \tau_{q+1}^{-5}\lambda_{q+1}^{1-2\alpha}\ell_\perp^{2/p-2}\ell_\|^{1/p-5/2}.
\end{aligned}$$

In the last inequality we have appealed to the definition (2.23). Due to (5.6) this estimate is sufficient for (5.2).

5.1.3. Oscillation error. In this section we estimate the remaining error, $\widetilde{R}_{\text{oscillation}}$, which obeys

$$\operatorname{div}(\widetilde{R}_{\text{oscillation}}) + \nabla P = \operatorname{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_q) + \partial_t w_{q+1}^{(t)}, \tag{5.9}$$

where P is a suitable pressure. From the definition of $w_{q+1}^{(p)}$ in (4.21) and of the coefficients $a_{(\xi)}$ in (4.16), using the disjoint support property of the intermittent jets (4.5), the fact that $\Lambda_{(1)} \cap \Lambda_{(2)} = \emptyset$, and appealing to the identity (4.19), we have

$$\begin{aligned}
\operatorname{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}) + \operatorname{div} \mathring{R}_q &= \sum_{i,j} \sum_{\xi \in \Lambda_{(i)}, \xi' \in \Lambda_{(j)}} \operatorname{div}(a_{(\xi)} a_{(\xi')} W_{(\xi)} \otimes W_{(\xi')}) + \operatorname{div} \mathring{R}_q \\
&= \sum_{i} \sum_{\xi \in \Lambda_{(i)}} \operatorname{div}(a_{(\xi)}^2 W_{(\xi)} \otimes W_{(\xi)}) + \operatorname{div} \mathring{R}_q \\
&= \sum_{i,j} \sum_{\xi \in \Lambda_{(i)}} \operatorname{div}\Big(a_{(\xi)}^2 \Big(W_{(\xi)} \otimes W_{(\xi)} - \fint_{\mathbb{T}^3} W_{(\xi)} \otimes W_{(\xi)}\, dx \Big)\Big) + \nabla\Big(\theta^2 \sum_{i \geq 0} \rho_i \chi_{(i)}^2\Big) \\
&= \sum_{i} \sum_{\xi \in \Lambda_{(i)}} \underbrace{\operatorname{div}(a_{(\xi)}^2 \mathbb{P}_{\geq \lambda_{q+1}\ell_\perp/2}(W_{(\xi)} \otimes W_{(\xi)}))}_{E_{(\xi)}} + \nabla\Big(\theta^2 \sum_{i \geq 0} \rho_i \chi_{(i)}^2\Big).
\end{aligned}$$

Here we use the fact that since $W_{(\xi)}$ is $(\mathbb{T}/\ell_\perp \lambda)^3$-periodic, the minimal separation between active frequencies of $W_{(\xi)} \otimes W_{(\xi)}$ and the 0 frequency is given by $\lambda_{q+1}\ell_\perp$. That is, $\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)}) = \mathbb{P}_{\geq \lambda_{q+1}\ell_\perp/2}(W_{(\xi)} \otimes W_{(\xi)})$. We further split

$$\begin{aligned}
E_{(\xi)} &= \mathbb{P}_{\neq 0}\big(\mathbb{P}_{\geq \lambda_{q+1}\ell_\perp/2}(W_{(\xi)} \otimes W_{(\xi)}) \nabla(a_{(\xi)}^2)\big) + \mathbb{P}_{\neq 0}\big(a_{(\xi)}^2 \operatorname{div}(W_{(\xi)} \otimes W_{(\xi)})\big) \\
&=: E_{(\xi,1)} + E_{(\xi,2)}.
\end{aligned}$$

The term $\mathcal{R} E_{(\xi,1)}$, which is the first contribution to $\widetilde{R}_{\text{oscillation}}$, is estimated by using the fact that the coefficient functions $a_{(\xi)}$ are essentially frequency localized inside of the ball of radius $\tau_{q+1}^{-3} \ll \lambda_{q+1}\ell_\perp$, in view of (4.36). More precisely, by Lemma 4.3 we are justified to use [5, Lemma B.1], with the parameter choices $\lambda = \tau_{q+1}^{-3}$, $C_a = \tau_{q+1}^{-5}$, $\kappa = \lambda_{q+1}\ell_\perp/2$, and L sufficiently large, to conclude

$$\begin{aligned}
\|\mathcal{R} E_{(\xi,1)}\|_{L^p} &\lesssim \big\| |\nabla|^{-1} E_{(\xi,1)} \big\|_{L^p} \\
&\lesssim \big\| |\nabla|^{-1} \mathbb{P}_{\neq 0}\big(\mathbb{P}_{\geq \lambda_{q+1}\ell_\perp/2}(W_{(\xi)} \otimes W_{(\xi)}) \nabla(a_{(\xi)}^2)\big) \big\|_{L^p} \\
&\lesssim \frac{\tau_{q+1}^{-5}}{\lambda_{q+1}\ell_\perp} \left(1 + \frac{\tau_{q+1}^{-6}}{(\tau_{q+1}^3 \lambda_{q+1}\ell_\perp)^{L-2}} \right) \|W_{(\xi)} \otimes W_{(\xi)}\|_{L^p} \\
&\lesssim \frac{\tau_{q+1}^{-5}}{\lambda_{q+1}\ell_\perp} \|W_{(\xi)}\|_{L^{2p}} \|W_{(\xi)}\|_{L^{2p}} \lesssim \tau_{q+1}^{-5} \lambda_{q+1}^{-1} \ell_\perp^{2/p-3} \ell_\parallel^{1/p-1}.
\end{aligned}$$

In the last inequality above we have used estimate (4.10), and in the second to last inequality we have used the fact that by taking L sufficiently large, for instance $L = 4$

is sufficient in view of the first inequality in (4.45) and the definition of $\ell_\perp$ in (2.23), we have $\tau_{q+1}^{-6}(\tau_{q+1}^3 \lambda_{q+1} \ell_\perp)^{2-L} \lesssim 1$. Summing these contributions over $0 \le i \le i_{\max}$ costs an additional factor of τ_{q+1}^{-1}, and from the third term in (5.6) we find that the bound for $\mathcal{R} E_{(\xi,1)}$ is consistent with (5.2).

It remains to estimate the contribution from the $E_{(\xi,2)}$ term. From identity (4.11) we see that

$$\begin{aligned} E_{(\xi,2)} &= \frac{1}{\mu} \mathbb{P}_{\neq 0}(a_{(\xi)}^2 \phi_{(\xi)}^2 \partial_t \psi_{(\xi)}^2 \xi) \\ &= \frac{1}{\mu} \partial_t \mathbb{P}_{\neq 0}(a_{(\xi)}^2 \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi) - \frac{1}{\mu} \mathbb{P}_{\neq 0}((\partial_t a_{(\xi)}^2) \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi). \end{aligned}$$

Hence, summing in ξ and i, pairing with the $\partial_t w_{q+1}^{(t)}$ present in (5.9), recalling the definition of $w_{q+1}^{(t)}$ in (4.24), and noting that $\mathrm{Id} - \mathbb{P}_H = \nabla(\Delta^{-1} \operatorname{div})$, we obtain

$$\begin{aligned} &\sum_i \sum_{\xi \in \Lambda_{(i)}} E_{(\xi,2)} + \partial_t w_{q+1}^{(t)} \\ &= \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}} (\mathrm{Id} - \mathbb{P}_H) \partial_t \mathbb{P}_{\neq 0}(a_{(\xi)}^2 \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi) - \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}} \mathbb{P}_{\neq 0}(\partial_t (a_{(\xi)}^2) \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi) \\ &= \nabla q - \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}} \mathbb{P}_{\neq 0}(\partial_t (a_{(\xi)}^2) \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi), \end{aligned} \tag{5.10}$$

where $q = \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}} \Delta^{-1} \operatorname{div} \partial_t \mathbb{P}_{\neq 0}(a_{(\xi)}^2 \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi)$ is a pressure term. Finally, we estimate the second contribution to $\widetilde{R}_{\text{oscillation}}$ by using (4.8), (4.9), Fubini, (4.29), and (4.36), to obtain

$$\begin{aligned} \Big\| \mathcal{R}\Big(\frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}} \mathbb{P}_{\neq 0}(\partial_t (a_{(\xi)}^2) \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi) \Big) \Big\|_{L^p} &\lesssim \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}} \| \partial_t (a_{(\xi)}^2) \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi \|_{L^p} \\ &\lesssim \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}} \| a_{(\xi)} \|_{C^1_{t,x}} \| a_{(\xi)} \|_{L^\infty} \| \phi_{(\xi)} \|_{L^{2p}}^2 \| \psi_{(\xi)} \|_{L^{2p}}^2 \\ &\lesssim \mu^{-1} \sum_{i=0}^{i_{\max}} \tau_{q+1}^{-5} \ell_\perp^{2/p-2} \ell_\parallel^{1/p-1} \lesssim \tau_{q+1}^{-6} \mu^{-1} \ell_\perp^{2/p-2} \ell_\parallel^{1/p-1} \\ &\lesssim \tau_{q+1}^{-6} \lambda_{q+1}^{1-2\alpha} \ell_\perp^{2/p-1} \ell_\parallel^{1/p-2}. \end{aligned} \tag{5.11}$$

In the last equality above we have used the definition of μ. Using the bound for the last term in (5.6), we conclude that the above estimate is consistent with (5.2), which shows that $\widetilde{R}_{\text{oscillation}}$ also obeys this inequality.

5.1.4. The temporal support of $\widetilde{R}$. In order to conclude the proof of Proposition 5.1, we need to show that (5.3) holds. From (5.5) it follows that

$$\operatorname{supp}(\widetilde{R}) \subset \operatorname{supp}(w_{q+1}^{(p)}) \cup \operatorname{supp}(w_{q+1}^{(c)}) \cup \operatorname{supp}(w_{q+1}^{(t)}) \cup \operatorname{supp}(\mathring{\bar{R}}_q).$$

By (2.20a) we know that $\mathring{\bar{R}}_q(t) = 0$ whenever $\operatorname{dist}(t, \mathcal{G}^{(q+1)}) \leq 2\tau_{q+1}$, while by (4.18) we see that $a_{(\xi)}(t) = 0$ whenever $\operatorname{dist}(t, \mathcal{G}^{(q+1)}) \leq \tau_{q+1}$. By their definitions, the principal (4.21), incompressibility (4.22), and temporal correctors (4.24) are composed only of terms which contain the coefficient functions $a_{(\xi)}$, and thus similarly to (4.27) we conclude that $w_{q+1}^{(p)}(t) = w_{q+1}^{(c)}(t) = w_{q+1}^{(t)}(t) = 0$ whenever $\operatorname{dist}(t, \mathcal{G}^{(q+1)}) \leq \tau_{q+1}$. This proves (5.3).

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